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APPLICATION OF THE DELAUNAY METHOD TO THE ARTIFICIAL SATELLITES OF THE MOON

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SUMMARY

This paper deals with the motion of three fictitious satellites of the Moon in the lunar equatorial plane, their orbits with eccentricities of 0.I8 being at a distance of 2,4 and 8 lunar radii from the Moon. The motion takes place in the gravitational field of the Moon under the influence of perturbations caused by the Earth (first problem), and of the Sun (second problem). The effect of the figure of the Moon is not taken into account. The Delaunay method, applied by him to Moon's artificial satellites, is used here for the solution of these problems.

The results obtained allow us to judge about the character of perturbations in the motion of these satellites, induced by the Earth and by the Sun.

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The questions linked with the study of the motion of Moon's artificial satellites (AMS) offer at present an unquestionable interest, for the materialization of the launching of AMS is a problem of immediate future. A series of Soviet as well as foreign works are devoted to these questions.

Observations of motions of AMS during prolonged time intervals will provide the possibility of making more precise the mass and the figure of the Moon. Since the Moon is devoid of atmosphere, all the irregularities in the motion of AMS, which could not find their explanation in the perturbing actions of the Earth and of the Sun, will be fully ascribed to the influence of Moon's nonsphericity and will allow us to draw conclusions about its true shape. Such conclusions may be attained only on the basis of motions of close Moon's satellites, since for remote satellites the perturbations due to Moon's shape are small by comparison with the perturbations due to the Earth.

It was shown in the work by Aksenov and Demin [I] that for close satellites (500 km from the surface of the Moon) the perturbing actions of the Earth and of the Moon's shape are almost identical.

However, the conditions of observation of close Moon's satellites from the Earth are beset with fairly great difficulties. It will be somewhat easier to observe the remote satellites, but by their motion one may only refine the mass of the Moon. But for the refining of the mass of the Moon by the motion of remote satellites, as well as for the determination of the true shape of the Moon by the motion of close satellites it is prerequisite to have a theory of their motion allowing one to account with great precision for the perturbations due to Earth and possible also those due to the Sun.

The study of the motion of AMS by the method of numerical integration was undertaken in the work by Brumberg, Kirpichnikov and Chebotarev in I96I, [2].

The present work aims at ascertaining the possibility of applying the lunar Delaunay method to the study of the motion of AMS and to determine by this method the character of perturbations induced by the influence of the Earth and of the Sun as a function of the distance of the satellite from the Sun.

I. STATEMENT OF THE PROBLEM

Let us consider the motion of an AMS of zero mass in the Moon's gravitational field under the action of perturbations induced by the Earth's influence. The influence of Moon's shape is not taken into account. All the three bodies are considered as material points with masses m (Moon), m' (Earth) and $m_0 = 0$ (satellite). It is assumed that the Earth moves relative to the Moon along an ellipse of which the major semi-axis is determined by the relation

$$a'^3 = \frac{k^2 (m + m')}{n'^2},$$

where n' is the mean daily motion of the Earth visible from the Moon, or, which is the same, - the mean daily motion of the Moon.

We shall consider the motion of the satellite in a selenocentrical rectangular system of coordinates, of which the basic plane is assumed to be the lunar orbit plane.

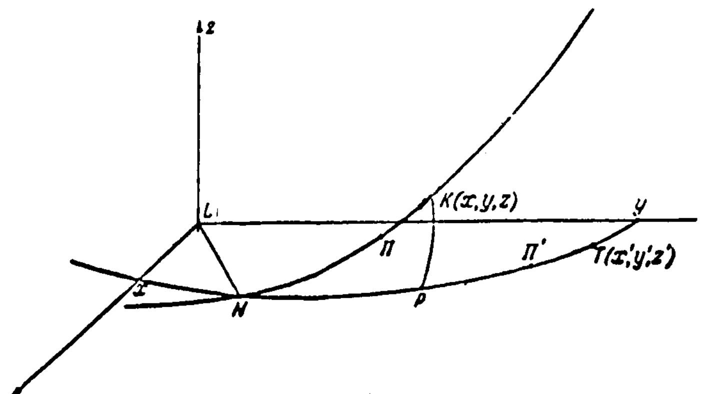

Fig.I.

Let x, y, z be the coordinates of the satellite and x', y' z' those of the Earth (Fig.I). The differential equations of motion of the Moon's satellite are as follows:

$$\left.\begin{aligned} \frac{d^2x}{dt^2}+\frac{mx}{r^3}&=\frac{\partial R}{\partial x}, \\ \frac{d^2y}{dt^2}+\frac{my}{r^3}&=\frac{\partial R}{\partial y}, \\ \frac{d^2z}{dt^2}+\frac{mz}{r^3}&=\frac{\partial R}{\partial z}, \end{aligned}\right\} \qquad (1)$$

where

$$R=-\frac{m'(xx'+yy'+zz')}{r'^3}+\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}. \qquad (2)$$

(For the sake of brevity we shall drop the factor K^2 in these formulas as well as subsequently). To the solution of this system we shall apply the method of arbitrary constants' variation. If we neglect at the outset the perturbation function, Eqs.(I) will represent an unperturbed motion. The solution of the problem of unperturbed elliptical motion is well known to us. It is given by the formulas

$$\left.\begin{aligned} x&=r(\cos\nu\cos h-\sin\nu\sin h\cos i), \\ y&=r(\cos\nu\sin h+\sin\nu\cos h\cos i), \\ z&=r\sin\nu\sin i, \\ E-e\sin E&=l=n(t+c), \quad n=\sqrt{\frac{m}{a^3}}, \\ r&=a(1-e\cos E), \\ \operatorname{tg}\frac{\nu-g}{2}&=\sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}, \end{aligned}\right\} \qquad (3)$$

where a, e, i, h, g, c are six integration constants which are elements of elliptical motion. If in Fig. I Π is the satellite's perigee point, N is the ascending node of the orbit then

g = NΠ is the distance from perigee to the node (ω)*,
h = xN is the longitude of the ascending node (Ω),
i is the inclination of the orbit to the basic plane,
ν = NK is the argument of the latitude (u),
l is the mean anomaly (M),
c = $-T$ is the moment of time corresponding to passage through perigee with opposite sign.

In order to integrate the total system of equations (I) we shall assume that the quantities a, e, i, h, g and c become variable in such a way, however, that in the real motion

*The denotations in parenthesis are after Subbotin I94I, [3].

$x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ they preserve the same expressions as in the elliptical motion. The latter are given by formulas [3] and formulas obtained as a result of their differentation.

Let us introduce instead of elliptical elements, the canonical elements of Delaunay

$$\left.\begin{array}{lll} L=\sqrt{ma}, & G=\sqrt{ma(1-e^2)}, & H=G\cos i, \\ l, & g, & h. \end{array}\right\} \quad (4)$$

These new variables must satisfy the canonical differential equations (Subbotin I937 [3]).

$$\left.\begin{array}{ll} \frac{dL}{dt}=\frac{\partial R}{\partial l}, & \frac{dl}{dt}=-\frac{\partial R}{\partial L}, \\ \frac{dG}{dt}=\frac{\partial R}{\partial g}, & \frac{dg}{dt}=-\frac{\partial R}{\partial G}, \\ \frac{dH}{dt}=\frac{\partial R}{\partial h}, & \frac{dh}{dt}=-\frac{\partial R}{\partial H}, \end{array}\right\} \quad (5)$$

where

$$R=\frac{m^2}{2L^2}-m'\left[\frac{xx'+yy'+zz'}{r'^3}-\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}\right]. \quad (6)$$

The integration of the system (5) gives the values of L, G, H, l, g, h in the form of function of time and six arbitrary constants.

We shall seek expressions for the selenocentrical longitude, latitude and radius-vector. If we denote the longitude xP by V, latitude PK by U (Fig.I.), by considering the triangle NKP, we shall obtain the relations

$$\left.\begin{array}{r} \operatorname{tg}(V-h)=\operatorname{tg}\nu\cos i, \\ \sin U=\sin\nu\sin i. \end{array}\right\} \quad (7)$$

We shall denote $\sin\frac{i}{2}=\gamma$ and expand V and U in series by powers γ to the seventh order relative to this quantity. We shall have [5] (I860)

$$\left.\begin{array}{l} V=h+\nu-(\gamma^2+\gamma^4+\gamma^6)\sin 2\nu+\left(\frac{1}{2}\gamma^4+\gamma^6\right)\sin 4\nu-\frac{1}{3}\gamma^6\sin 6\nu, \\ U=\left(2\gamma-\frac{1}{4}\gamma^5\right)\sin\nu-\left(\frac{1}{3}\gamma^3+\frac{1}{4}\gamma^5\right)\sin 3\nu+\frac{3}{20}\gamma^5\sin 5\nu. \end{array}\right\} \quad (8)$$

The expansion in series of coordinates of elliptical motion gives for ν and r expressions of the form [7]:

$$\left.\begin{array}{l} \frac{r}{a}=A_0+A_1\cos l+A_2\cos 2l+\ldots, \\ \nu=g+l+B_1\sin l+B_2\sin 2l+\ldots, \end{array}\right\} \quad (9)$$

in which the coefficients A_i and B_i are functions of e. Let us substitute the second of relations (9) into formulas (8) and expand in series, preserving the terms to the 7th order relative to e and γ. We shall thus obtain the expressions for the longitude and latitude in the elliptical motion. We shall also take advantage of the first formula in (9) for the formation of the value of $\frac{1}{r}$. All this gives

$$\left.\begin{aligned} V &= h+g+l+\sum A\sin(\alpha l+\beta g),\\ U &= \sum A'\sin(\alpha' l+\beta' g),\\ \frac{1}{r} &= \sum A''\cos\alpha'' l, \end{aligned}\right\} \qquad (10)$$

where A, A' and A" are functions of e and γ. If we substitute into the expressions (10) the values of $a, e, \gamma, l, g, h,$ obtained as a result of integration of the system of Eqs.(5), we shall have the coordinates of the satellite, taking into account the perturbing action of the Earth.

2. APPLICATION OF THE DELAUNAY METHOD

The statement of the problem in the form of the preceding section is fully identical to the problem set up and resolved by Delaunay in the case of motion of the Moon relative to the Earth, taking into account the perturbations from the Sun. This is why we may apply to the problem stated the Delaunay method utilized by him when constructing the theory of the motion of Moon.

Let us briefly pause at the essence of this method. In the Delaunay method the expansion of the function R was achieved as follows. The values of x, y, z, expressed by formulas of elliptical motion are substituted into formula (6) in the form of functions of time and six constants L, G, H, l, g, h. Introduced instead of L, G, H are a, e and $\gamma=\sin\frac{i}{2}$. Then R is already expressed as a function of time and constants a, e, γ, l, g, h. Delaunay considers that x', y' z', which are the coordinates of the Sun (in our problem they are the coordinates of the Earth), are expressed by formulas analogous to formulas (3).

If we denote the cosine of the angle between vectors r and r' by S, we shall have the relation

$$xx'+yy'+zz'=rr'S, \qquad (11)$$

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=r'^2+r^2-2rr'S. \qquad (12)$$

Taking this into account, function R may be written:

$$R=\frac{m}{2a}-m'\frac{r}{r'^2}S+\frac{m'}{\sqrt{r'^2+r^2-2rr'S}}.$$

Let us denote $\sqrt{r'^2+r^2-2rr'S}=\Delta$. The quantity $\frac{1}{\Delta}$ may be expanded in series by powers of small magnitude $\frac{r}{r'}$,

$$\frac{1}{\Delta}=\frac{1}{r'}\left[1+\left(\frac{r}{r'}\right)^2-2\frac{r}{r'}S\right]^{-1/2}=\frac{1}{r'}\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n P_n(S),$$

where P_n (S) are Legendre polynomials

$$P_0=1, \qquad P_2=\frac{3}{2}S^2-\frac{1}{2},$$
$$P_1=S, \qquad P_3=\frac{5}{2}S^3-\frac{3}{2}S\ldots$$

If in the perturbation function we reject the term $\frac{1}{r}$, which is not dependent on lunar orbit elements (in the Delaunay problem), R will have the form

$$R=\frac{m}{2a}+m'\frac{r^2}{r'^3}\left[P_2+\frac{r}{r'}P_3+\left(\frac{r}{r'}\right)^2P_4+\cdots\right]. \qquad (I3)$$

Since Delaunay chose for the basic plane the ecliptic plane (in our case it is the Moon's orbit plane), i=0 and consequently, z'=0. Taking into account the expression $\sin\frac{i}{2}=\gamma$, the coordinates x, y and x', y' may be written in the form

$$\left.\begin{aligned}\frac{x}{r}&=\cos(v+h)+2\gamma^2\sin v\sin h, & \frac{x'}{r'}&=\cos(v'+h'),\\ \frac{y}{r}&=\sin(v+h)-2\gamma^2\sin v\cos h, & \frac{y'}{r'}&=\sin(v'+h').\end{aligned}\right\} \qquad (I4)$$

Then, according to formula (11), S may be expressed through v, v', h, h' and γ. Substituting expressions (I4) into (II), we shall obtain after transformations

$$S=(1-\gamma^2)\cos(v+h-v'-h')+\gamma^2\cos(v-h+v'+h'). \qquad (I5)$$

In order to obtain the polynomials entering in (I3) it is necessary to obtain the powers S^2, S^3,...Transforming in S^2, S^3... the powers of cosines into cosines of multiple angles and neglecting the powers of the small parameter γ, beginning with a certain order, let us substitute them into the function R. Delaunay preserved in the expansion of R the small quantities to the eighth order inclusive Parameters e, γ, e' are considered as quantities of first order, but $\frac{r}{r'}$ and, consequently $\frac{a}{a'}$ as quantities of second order value. Inasmuch as the first periodical term in R (formula (I3)) has a multiplier $\frac{r^2}{r'^2}$, one may obviously neglect the quantity γ^6.

The expansion of R gives a series of terms of the form

$$\frac{r^p}{r'^{p+1}}\gamma^{2p}\cos[qv+q'(v'+h')+\sigma h]. \qquad (I6)$$

Now R has to be expanded by powers e and e'. To that effect, taking advantage of formulas of elliptical motion (9) we form expansion of the form

$$r^p{}^{\cos}_{\sin}(qv+\sigma h), \quad \frac{1}{r'^{p+1}}{}^{\cos}_{\sin}[q'(v'+h')]$$

and substitute them in expressions (I6).

Thus, Delaunay finally obtains the expansion of R in the form

$$R = -B - \sum A \cos[il + i'g + i^{II}h + i^{III}l' - i^{IV}(g' + h')], \qquad (17)$$

where A and B are polynomials, disposed by powers of the quantity $e, e', \gamma, \frac{a}{a'}$ and containing the multiplier $\frac{m'a^2}{a'^3}$; $i, i', i^{II}, i^{III}, i^{IV}$ are whole numbers, positive and negative.

The method of integration of differential equations consists in that Delaunay selects in sequence one of the periodical terms from the perturbation function. At the outset he integrates equations with such a perturbation function in which nonperiodical terms are preserved with only one periodical term. As a result of this integration L, G, H, l, g, h are obtained in the form of trigonometric series, of which the coefficients are dependent on six arbitrary constants. These arbitrary constants are assumed to be the new variables and the solution obtained is substituted in the initial differential equations. Then a variable substitution takes place so as to eliminate the appearance of t outside the signs of sines and cosines. At the same time the selected periodical terms in the perturbation function vanishes, while no new periodical terms appear. The whole process is repeated, whereupon the coefficients of periodical terms in R become of higher order relative to parameters $e, e', \gamma, \frac{a}{a'}$, as the number of such operations increases.

Delaunay conducted 497 such operations, as a result of which the perturbation function is reduced to a single nonperiodical term and a series of periodical terms of which the coefficients are so small that these terms can be neglected. The latter operation reduces the perturbation function to the following form:

$$R = \frac{\mu}{2a} + \frac{m'a^2}{a'^3} f\left(e^2, \gamma^2, \frac{a^2}{a'^2}, \frac{n'}{n}, e'^2\right). \qquad (18)$$

(For Delaunay, in this formula $\mu = k^2(M+m)$, where M is the mass of the Earth and m is the mass of the Moon).

The relations linking a, e, γ with the last variables L, G, H are reduced to the following form:

$$\left.\begin{aligned} L &= \sqrt{\mu a}\, F\left(e^2, \gamma^2, \frac{a^2}{a'^2}, \frac{n'}{n}, e'^2\right), \\ G &= \sqrt{\mu a}\, \Phi\left(e^2, \gamma^2, \frac{a^2}{a'^2}, \frac{n'}{n}, e'^2\right), \\ H &= \sqrt{\mu a}\, \Psi\left(e^2, \gamma^2, \frac{a^2}{a'^2}, \frac{n'}{n}, e'^2\right). \end{aligned}\right\} \qquad (19)$$

The first three of Eqs. (5) show that the L, G, H, and consequently a, e, γ too are constants, while the last three equations give

$$\frac{dl}{dt} = l_0, \quad \frac{dg}{dt} = g_0, \quad \frac{dh}{dt} = h_0, \qquad (20)$$

where l_0, g_0, h_0 are functions of $e^2, \gamma^2, \frac{a^2}{a'^2}, \frac{n'}{n}$ and e'^2, since

$$\left.\begin{aligned}\frac{\partial R}{\partial L}&=\frac{\partial R}{\partial a}\cdot\frac{\partial a}{\partial L}+\frac{\partial R}{\partial e}\cdot\frac{\partial e}{\partial L}+\frac{\partial R}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial L},\\ \frac{\partial R}{\partial G}&=\frac{\partial R}{\partial a}\cdot\frac{\partial a}{\partial G}+\frac{\partial R}{\partial e}\cdot\frac{\partial e}{\partial G}+\frac{\partial R}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial G},\\ \frac{\partial R}{\partial H}&=\frac{\partial R}{\partial a}\cdot\frac{\partial a}{\partial H}+\frac{\partial R}{\partial e}\cdot\frac{\partial e}{\partial H}+\frac{\partial R}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial H},\end{aligned}\right\} \qquad (21)$$

and quantities $\frac{\partial a}{\partial L}, \frac{\partial a}{\partial G}, \cdots \frac{\partial \gamma}{\partial H}$ are derived from formulas (I9).

The integration of Eqs (20) gives

$$l=(l)+l_0t; \quad g=(g)+g_0t; \quad h=(h)+h_0t, \qquad (22)$$

where (l), (g), (h) are three arbitrary constants. Delaunay introduced four basic arguments: $D=h+g+l-h'-g'-l'$ is the angular distance of the Moon from the Sun, $F=g+l$ is the mean angular distance of the Moon from its ascending node, l is the mean anomaly of the Moon; he denoted by m the ratio of mean daily motions of the Earth and of the Sun: $m=\frac{n'}{n}$.

Thus, Delaunay integrated Eqs.(5) by way of quite cumbersome algebric calculation. The quantities L, G, H constitute trigonometric series by multiple arguments D, F, l, l', and the quantities l, g, h still have one term proportional to time. This essential result, stemming from the theory obtained, gives analytical expressions for the mean motions of the node $\frac{d\Omega}{dt}=(h_0)$ and of the perigee $\frac{d\pi}{dt}=(h_0+g_0)$. They constitute series by powers $e, e', \gamma, \frac{a}{a'}$ and m.

The ultimate Delaunay's aim was to obtain expressions for three coordinates of the Moon (longitude V, latitude U and the quantity $\frac{1}{r}$) taking into account the perturbing action of the Sun. The elliptical values of V, U and $\frac{1}{r}$ were obtained in the form (I0). As a result of all the operations conducted Delaunay obtained a system of transformation formulas. Being successively applied to the function R, these formulas eliminated from it the periodical terms one after the other, and being applied to V, U, $\frac{1}{r}$, they introduced into them periodical terms or inequalities.

Delaunay brings forth at the end of his two volume work [5] the complete literal expressions for V, U, and $\frac{a}{r}$. They have the following form:

$$\left.\begin{aligned}V&=\text{const}+nt+\sum A\sin(iD\pm 2kF\pm jl\pm j'l'),\\ U&=\sum B\sin(iD\pm(2k+1)F\pm jl\pm j'l'),\\ \frac{a}{r}&=\sum C\cos(iD\pm 2kF\pm jl\pm jl'),\end{aligned}\right\} \qquad (23)$$

where

$$\left.\begin{aligned}A&=e^je'^{j'}\gamma^{2k}F_1\left(m,\ e^2,\ \gamma^2,\ \frac{a}{a'},\ e'^2\right),\\ B&=e^je'^{j'}\gamma^{2k+1}\Phi_1\left(m,\ e^2,\ \gamma^2,\ \frac{a}{a'},\ e'^2\right),\\ C&=e^je'^{j'}\gamma^{2k}\Psi_1\left(m,\ e^2,\ \gamma^2,\ \frac{a}{a'},\ e'^2\right),\end{aligned}\right\} \qquad (24)$$

i, j, j', k being whole numbers, positive or zero. The arguments are expressed through elliptical elements in the following manner:

$$\left.\begin{aligned} l &= M_0 + \left(n - \frac{d\pi}{dt}\right)t, \\ l' &= M_0' + n't, \\ D &= \varepsilon_0 - \varepsilon_0' + (n - n')t, \\ F &= \varepsilon_0 - \Omega_0 + \left(n - \frac{d\Omega}{dt}\right)t. \end{aligned}\right\} \quad (25)$$

The Subbotin's denotations were adopted in these formulas, whereupon the strokes refer to elements of the Sun in the theory of the Moon.

It remains to be noted that a correction is to be introduced into this final result. As a matter of fact, in the theory of Moon's motion the motion of the Sun is considered as a translation along an ellipse, whose major semiaxis is determined by the relation

$$a'^3 n'^2 = k^2 (M + m'),$$

where M is the sum of the masses of the Earth and of the Moon. However, in the equations of motion of the Moon it is assumed, for the sake of simplicity, that in the perturbation function $k^2m' = a'^3n'^2$, i.e., the masses of the Earth and of the Moon are neglected. This inaccuracy is fully taken into account if in the final result all coefficients of solar inequalities are multiplied by the quantity $(1 - M/m')$.

Let us return to our problem of motion of an AMS. If the latter is to move around the Moon along an orbit with elements, for which the values of the Delaunay parameters are close to the values of these parameters in the theory of motion of the Moon, we may utilize the ready literal Delaunay expansions and obtain formulas for the determination of selenocentrical coordinates of the AMS with a preassigned precision. Obviously, this precision is limited by the possibility of application of the given theory to a concrete satellite system. If optical observations of the lunar satellite are conducted from the Earth, no great precision is required of the selenocentrical coordinates for obtaining sufficiently good geocentric positions. And this is why the values of the parameters may be somewhat overrated by comparison with the parameters for the Moon, so that one may fully utilize the Delaunay formulas.

Thus, for example, if the theory gives us V and U with a precision to 0.0I and $\frac{a}{r}$ with a precision to 0.000I, we shall correspondingly obtain for a satellite, moving at the distance of 2,4 and 8 lunar radii, the geocentric coordinates α and δ with a precision to respectively 2,4 and 5 seconds of arc.

For the investigation of the character of inequalities, induced in the motion of AMS by the perturbing action of the Earth and also of inequalities induced by the Sun's perturbation, we considered three types of orbits of AMS. In all cases the motion takes place at the initial moment in the equatorial plane of the Moon along an ellipse with eccentricity e=0.18, and the major semiaxes are chosen respectively as follows:

I. a=2 lunar radii = 3473km;
II. a=4 lunar radii = 6947km;
III. a=8 lunar radii = 13894km.

If we appraise the series obtained by Delaunay for the coordinates by the basic rejected term of the form $\beta e^7 \sin 7l$ (β is a numerical coefficient close to the unity) as is done by Tokmalayeva [4], the error of the series for the longitude and latitude in the application to our problem is approximately 0.001°.

From the above it may be concluded that in order to obtain the coordinates of the AMS studied by us with a precision to 0.01°, the expansions of Delaunay are more than sufficient. The very same estimate allows the conclusion that for equatorial satellite of the Moon in general, moving at a distance from the Moon of no more than 8 lunar radii, the Delaunay method allows us to obtain the selenocentrical coordinates with a precision to 0.01° only for orbits with an eccentricity not exceeding 0.275.

As to orbits, not lying within the lunar equatorial plane, the question as to the possibility of applying the Delaunay method for such inclinations must be examined separately.

3. INEQUALITIES INDUCED BY THE ACTION OF THE EARTH

For the solution of the problem MOON - AMS - EARTH one must have the Earth's orbit elements relative to the Moon.

Placing the Moon in the center of the system of coordinates (Fig.2) and considering the motion of the Earth from the Moon, we shall have the elements of the Earth's orbit relative to the Moon, which are the same as the Moon's orbit has relative to the Earth. Only the longitudes will differ by 180°.

We assume for the Earth the following system of elements (denoting by stroke the elements of the Earth, and without stroke those of the Moon):

$e' = 0.054900489$,
$i' = 5.14540°$ (constant of Moon's orbit inclination to the ecliptic)
$n' = 13.176397°$ (mean daily motion of the Moon for the epoch 1900.0),
$a' = 384400$ km.

The Moon's mass ratio to that of the Earth is $\frac{1}{81.53}$. The mean longitude of the Moon, the mean longitude of the perigee and the mean longitude of the node, corresponding to the required initial moment of time, are computed by formulas brought out in the USSR Yearbook.

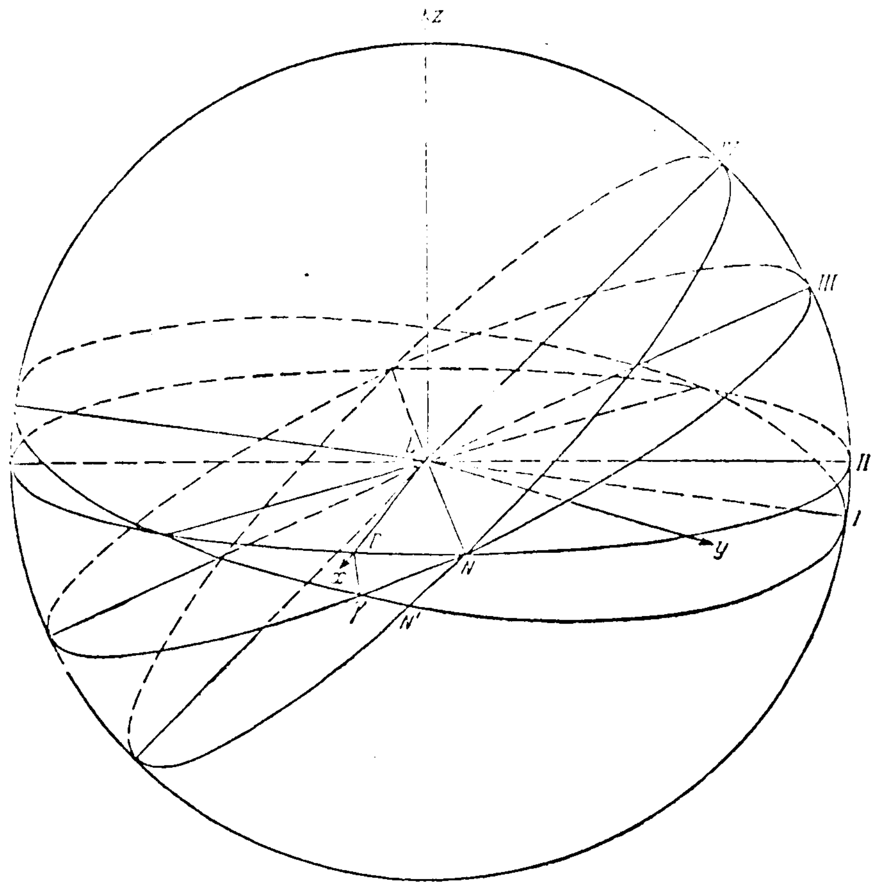

Fig.2.

I. Plane passing through the center of the Moon (L) parallelwise to the Earth's equatorial plane.

II. Earth's orbit plane relative to the Moon.

III. Plane passing through the center of the Moon parallelwise to the ecliptic plane.

IV. Orbit plane of the AMS (lunar equatorial plane).

Γ. Point of intersection of the plane II with the great circle passing through the point ♈ (point of the Spring equinox) and the pole of the plane II.

We select for orbits of lunar satellites the following elements: e=0.18, i=1.535° (constant inclination of the mean lunar equator to the ecliptic) are equal for all the three variants, and the major semiaxes are respectively of 2,4,8 lunar radii.

Table I gives the values of the major semiaxes in kilometers, the mean daily motions and the periods of satellite resolutions for each variant.

The motion of the Moon takes place in such a way that the planes of the lunar equator, the orbits of the Moon and the ecliptics mutually intersect along a single straight line, whereupon the ecliptic plane lies between the lunar equatorial planes and the plane of the Moon's orbit (Fig.2). This is why the angle of mutual inclination of satellite motion and Moon's orbit planes is equal to $i' + i = 6.68040°$.

TABLE I

Element	Var.I	Var.2	Var.3
a	3473.4 км	6946.8 км	13893.6 км
n	1687°7847	598°7524	211°8122
P	5ʰ11914	14ʰ43001	40ʰ79088

On the basis of the assumed data we obtain the values of the Delaunay parameters compiled in Table 2. For comparison we brought out in the same table the values of the Delaunay parameters for the Moon.

TABLE 2

	e	e'	γ	$\alpha = \frac{a}{a'}$	$m = \frac{n'}{n}$
Moon.	0.05490	0.01677	0.04489	0.00256	0.07480
AMS variant I.	0.18	0.0549	0.05826	0.009036	0.007807
AMS variant II. . . .	0.18	0.0549	0.05826	0.018072	0.022006
AMS variant III. . .	0.18	0.0549	0.05826	0.036144	0.062208

Limiting ourselves to the precision of 0.0I" when obtaining V, U and to 0.000I for the quantity $\frac{a}{r}$, we obtained on the basis of Delaunay expansions the following formulas for the computation of selenocentrical coordinates of AMS, valid for all the thre variants:

$$V = \varepsilon + nt + \left\{-3e'm + \frac{735}{16}e'm^3 + \frac{27}{2}\gamma^2 e'm - \frac{27}{8}e^2e'm - \frac{2925}{32}e^2e'm^2\right\}\sin l' - \frac{9}{4}e'^2 m \sin 2l' +$$

$$+ \left\{2e - \frac{1}{4}e^3\right\}\sin l + \left\{\frac{21}{4}ee'm + \frac{1233}{32}ee'm^2 + \frac{14913}{64}ee'm^3\right\}\sin(l - l') +$$

$$+ \left\{\frac{63}{16}ee'^2 m + \frac{5355}{128}ee'm^2\right\}\sin(l - 2l') + \left\{-\frac{21}{4}ee'm - \frac{717}{32}ee'm^2 -\right.$$

$$\left. - \frac{3069}{32}ee'm^3\right\}\sin(l + l') - \frac{63}{16}ee'^2 m \sin(l + 2l') + \left\{\frac{5}{4}e^2 - \frac{5}{4}\gamma^2 e^2 - \frac{11}{24}e^4\right\}\sin 2l +$$

$$+ \left\{\frac{105}{16}e^2e'm + \frac{6081}{128}e^2e'm^2 + \frac{134435}{512}e^2e'm^3\right\}\sin(2l - l') +$$

$$+ \left\{-\frac{105}{16}e^2e'm - \frac{3669}{128}e^2e'm^2\right\}\sin(2l + l') + \left\{\frac{13}{12}e^3 - \frac{43}{64}e^5\right\}\sin 3l +$$

$$+ \frac{273}{32}e^3e'm \sin(3l - l') - \frac{273}{32}e^3e'm \sin(3l + l') + \frac{103}{96}e^4 \sin 4l + \frac{1097}{960}e^5 \sin 5l +$$

$$+ \left\{-\gamma^2 - \frac{9}{4}\gamma^2 e^2 + \frac{675}{32}\gamma^2 e^2 m\right\}\sin 2F - 2\gamma^2 e \sin(2F + l) - \frac{13}{4}\gamma^2 e^2 \sin(2F + 2l) -$$

$$- \frac{59}{12}\gamma^2 e^3 \sin(2F + 3l) + \left\{-3\gamma^2 e + \frac{135}{8}\gamma^2 em + \frac{61}{8}\gamma^2 e^3\right\}\sin(2F - l) +$$

$$+ \left\{-\frac{3}{4}\gamma^2 m + \frac{75}{16}e^2 m + \frac{1101}{64}e^2m^2 + \frac{64271}{1024}e^2m^3 - \frac{45}{32}e^4 m + \frac{11}{8}m^2 + \frac{59}{12}m^3 +\right.$$

$$\left. + \frac{893}{72}m^4 + \frac{2855}{108}m^5\right\}\sin 2D + \left\{\frac{175}{16}e^2e'm + \frac{4541}{64}e^2e'm^2 + \frac{375979}{1024}e^2e'm^3 +\right.$$

$$+\frac{77}{16}e'm^2+\frac{479}{16}e'm^3\Big\}\sin(2D-l')+\Big\{\frac{195}{32}e^3m+\frac{17}{8}em^2+\frac{169}{24}em^3\Big\}\sin(2D+l)+$$

$$+\Big\{\frac{119}{16}ee'm^2+\frac{3131}{64}ee'm^3\Big\}\sin(2D+l-l')+\Big\{\frac{515}{64}e^4m+\frac{95}{32}e^2m^2\Big\}\sin(2D+2l)+$$

$$+\Big\{\frac{15}{4}em+\frac{263}{16}em^2+\frac{48217}{768}em^3+\frac{1880537}{9216}em^4+\frac{130463405}{221184}em^5+$$

$$+\frac{4389108607}{2654208}em^6-6\gamma^2em-\frac{75}{8}ee'^2m\Big\}\sin(2D-l)+$$

$$+\Big\{\frac{35}{4}ee'm+\frac{1801}{32}ee'm^2+\frac{31589}{128}ee'm^3\Big\}\sin(2D-l-l')+$$

$$+\Big\{\frac{255}{16}ee'^2m+\frac{17179}{128}ee'^2m^2\Big\}\sin(2D-l-2l')+$$

$$+\Big\{-\frac{15}{4}ee'm-\frac{173}{32}ee'm^2+\frac{50125}{384}ee'm^3\Big\}\sin(2D-l+l')-\frac{45}{16}ee'^2m\sin(2D-l+2l')+$$

$$+\Big\{\frac{45}{16}e^2m+\frac{53}{4}e^2m^2+\frac{263089}{3072}e^2m^3\Big\}\sin(2D-2l)+\Big\{\frac{105}{16}e^2e'm+$$

$$+\frac{577}{16}e^2e'm^2+\frac{249505}{1024}e^2e'm^3\Big\}\sin(2D-2l-l')-\frac{45}{16}e^2e'm\sin(2D-2l+l')+$$

$$+\Big\{\frac{105}{32}e^3m+\frac{6011}{384}e^3m^2+\frac{1647415}{18432}e^3m^3\Big\}\sin(2D-3l)+\frac{245}{32}e^3e'm\sin(2D-3l-l')+$$

$$+\frac{35}{8}e^4m\sin(2D-4l)-\frac{15}{4}\gamma^2em\sin(2D+2F-l)+\frac{9}{4}\gamma^2m\sin(2D-2F)-$$

$$-\frac{33}{8}\gamma^2em\sin(2D-2F+l)+\Big\{\frac{1125}{256}e^2m^2+\frac{18495}{512}e^2m^3\Big\}\sin(4D-2l)+$$

$$+\Big\{\frac{2625}{128}e^2e'm^2+\frac{212775}{1024}e^2e'm^3\Big\}\sin(4D-2l-l')+$$

$$+\alpha\Big\{-\frac{15}{8}m-\frac{93}{8}m^2-\frac{6887}{128}m^3-\frac{105}{16}e^2m-\frac{21429}{256}e^2m^2-\frac{736215}{1024}e^2m^3+\frac{165}{8}\gamma^2m\Big\}\sin D+$$

$$+\alpha\Big\{\frac{5}{2}e'-\frac{45}{4}e'm+\frac{6629}{96}e'm^2+\frac{15}{2}e^2e'+\frac{215}{2}e^2e'm+\frac{63803}{128}e^2e'm^2\Big\}\sin(D+l')+$$

$$+\alpha\Big\{-\frac{75}{32}em-\frac{117}{8}em^2-\frac{68215}{1024}em^3-\frac{15}{2}e^3m\Big\}\sin(D+l)+$$

$$+\alpha\Big\{\frac{25}{8}ee'-\frac{225}{16}ee'm+\frac{70103}{768}ee'm^2\Big\}\sin(D+l+l')-\alpha\Big\{\frac{195}{64}e^2m\Big\}\sin(D+2l)+$$

$$+\alpha\Big\{\frac{65}{16}e^2e'\Big\}\sin(D+2l+l')+\alpha\Big\{-\frac{165}{32}em-\frac{7317}{256}em^2-\frac{151307}{1024}em^3\Big\}\sin(D-l)+$$

$$+\alpha\Big\{\frac{25}{8}ee'-\frac{495}{16}ee'm+\frac{197429}{768}ee'm^2-\frac{1587479}{1536}ee'm^3\Big\}\sin(D-l+l')+$$

$$+\alpha\Big\{-\frac{435}{64}e^2m-\frac{14433}{256}e^2m^2-\frac{680863}{2048}e^2m^3\Big\}\sin(D-2l)+$$

$$+\alpha\Big\{\frac{105}{16}e^2e'+\frac{1045}{32}e^2e'm\Big\}\sin(D-2l+l')+$$

$$+\alpha\Big\{-\frac{875}{128}e^3m-\frac{2535}{256}em^2-\frac{81865}{1024}em^3\Big\}\sin(3D-l)+\alpha\Big\{\frac{375}{64}ee'm\Big\}\sin(3D-l+l')+$$

$$+\alpha\Big\{-\frac{175}{32}e^2m-\frac{5535}{128}e^2m^2\Big\}\sin(3D-2l), \qquad (26)$$

$$U=\{2\gamma-2\gamma e^2\}\sin F+\frac{3}{4}\gamma e'm\sin(F-l')-\frac{3}{4}\gamma e'm\sin(F+l')+\Big\{2\gamma e-\frac{5}{2}\gamma e^3\Big\}\sin(F+l)+$$

$$+6\gamma ee'm\sin(F+l-l')-6\gamma ee'm\sin(F+l+l')+\Big\{\frac{9}{4}\gamma e^2-\frac{27}{8}\gamma e^4\Big\}\sin(F+2l)+$$

$$+\frac{8}{3}\gamma e^3\sin(F+3l)+\Big\{-2\gamma e-5\gamma^3e+\frac{5}{4}\gamma e^3+\frac{189}{32}\gamma em^2\Big\}\sin(F-l)+$$

$$+\frac{9}{2}\gamma ee'm\sin(F-l-l')-\frac{9}{2}\gamma ee'm\sin(F-l+l')+\Big\{-\frac{3}{2}\gamma e^2+\frac{77}{48}\gamma e^4\Big\}\sin(F-2l)-$$

$$-\frac{17}{12}\gamma e^3\sin(F-3l)-4\gamma^3e\sin(3F-l)+$$

$$+\left\{\frac{135}{16}\gamma e^2 m+\frac{1929}{64}\gamma e^2 m^2+\frac{11}{8}\gamma m^2\right\}\sin(2D+F)+$$

$$+\left\{\frac{15}{4}\gamma em+\frac{211}{16}\gamma em^2+\frac{43721}{768}\gamma em^3\right\}\sin(2D+F-l)+$$

$$+\left\{\frac{35}{4}\gamma ee'm+\frac{423}{8}\gamma ee'm^2\right\}\sin(2D+F-l-l')-\frac{15}{4}\gamma ee'm\sin(2D+F-l+l')+$$

$$+\left\{\frac{3}{4}\gamma m+\frac{25}{16}\gamma m^2+\frac{27}{16}\gamma e^2 m\right\}\sin(2D-F)+\left\{\frac{7}{4}\gamma e'm+\frac{255}{32}\gamma e'm^2\right\}\sin(2D-F-l')-$$

$$-\frac{3}{4}\gamma e'm\sin(2D-F+l')+\frac{3}{4}\gamma em\sin(2D-F+l)+$$

$$+\left\{3\gamma em+\frac{105}{8}\gamma em^2+\frac{3681}{64}\gamma em^3\right\}\sin(2D-F-l)+$$

$$+\left\{7\gamma ee'm+\frac{171}{4}\gamma ee'm^2\right\}\sin(2D-F-l-l')-3\gamma ee'm\sin(2D-F-l+l')+$$

$$+\left\{\frac{147}{32}\gamma e^2 m+\frac{3257}{128}\gamma e^2 m^2\right\}\sin(2D-F-2l)+\left\{\frac{67}{8}\gamma e^3 m\right\}\sin(2D-F-3l)+$$

$$+\alpha\left\{-\frac{15}{8}\gamma m\right\}\sin(D+F)+\alpha\left\{\frac{5}{2}\gamma e'\right\}\sin(D+F+l')+$$

$$+\alpha\left\{-\frac{135}{32}\gamma em\right\}\sin(D+F+l)+\alpha\left\{\frac{45}{8}\gamma ee'\right\}\sin(D+F+l+l')+$$

$$+\alpha\left\{-\frac{15}{8}\gamma m\right\}\sin(D-F)+\alpha\left\{\frac{5}{2}\gamma e'\right\}\sin(D-F+l')+$$

$$+\alpha\left\{-\frac{195}{32}\gamma em\right\}\sin(D-F+l), \qquad (27)$$

$$\frac{a}{r}=1+\frac{1}{6}m^2-\frac{3}{2}e'm^2\cos l'+\left\{e-\frac{7}{12}em^2-\frac{285}{64}em^3-\frac{1}{8}e^3\right\}\cos l+$$

$$+\left\{\frac{21}{8}ee'm+\frac{1113}{64}ee'm^2+\frac{3269}{32}ee'm^3\right\}\cos(l-l')+\frac{63}{32}ee'^2 m\cos(l-2l')+$$

$$+\left\{-\frac{21}{8}ee'm-\frac{837}{64}ee'm^2-\frac{6811}{128}ee'm^3\right\}\cos(l+l')-\frac{63}{32}ee'^2 m\cos(l+2l')+$$

$$+\left\{e^2-\frac{5}{6}e^2m^2-\frac{1}{3}e^4\right\}\cos 2l+\left\{\frac{21}{4}e^2e'm+\frac{1161}{32}e^2e'm^2\right\}\cos(2l-l')+$$

$$+\left\{-\frac{21}{4}e^2e'm-\frac{789}{32}e^2e'm^2\right\}\cos(2l+l')+\left\{\frac{9}{8}e^3-\frac{81}{128}e^5\right\}\cos 3l+$$

$$+\frac{567}{64}e^3e'm\cos(3l-l')-\frac{567}{64}e^3e'm\cos(3l+l')+\frac{4}{3}e^4\cos 4l+\frac{625}{384}e^5\cos 5l+$$

$$+\left\{-5\gamma^2e^2+\frac{135}{8}\gamma^2e^2m\right\}\cos 2F-\frac{135}{16}\gamma^2e^3\cos(2F+l)+\left\{-\frac{5}{2}\gamma^2e+\frac{75}{16}\gamma^2e^3+\right.$$

$$\left.+\frac{135}{16}\gamma^2em\right\}\cos(2F-l)+\left\{\frac{15}{4}e^2m+\frac{189}{16}e^2m^2+\frac{10483}{256}e^2m^3-\frac{15}{2}\gamma^2e^2m+m^2+\right.$$

$$\left.+\frac{19}{6}m^3+\frac{131}{18}m^4\right\}\cos 2D+\left\{\frac{35}{4}e^2e'm+\frac{799}{16}e^2e'm^2+\frac{7}{2}e'm^2+\frac{157}{8}e'm^3\right\}\times$$

$$\times\cos(2D-l')+\left\{\frac{255}{16}e^2e'^2m+\frac{17}{2}e'^2m^2\right\}\cos(2D-2l')+$$

$$+\left\{-\frac{15}{4}e^2e'm-\frac{207}{16}e^2e'm^2-\frac{1}{2}e'm^2\right\}\cos(2D+l')+\left\{\frac{405}{64}e^3m+\frac{33}{16}em^2+\frac{101}{16}em^3\right\}\times$$

$$\times\cos(2D+l)+\left\{\frac{945}{64}e^3e'm+\frac{231}{32}ee'm^2+\frac{5727}{128}ee'm^3\right\}\cos(2D+l-l')+$$

$$+\left\{-\frac{405}{64}e^3e'm\right\}\cos(2D+l+l')+\left\{10e^4m+\frac{7}{2}e^2m^2+\frac{127}{12}e^2m^3\right\}\cos(2D+2l)+$$

$$+\frac{49}{4}e^2e'm^2\cos(2D+2l-l')+\frac{2125}{384}e^3m^2\cos(2D+3l)+$$

$$+\left\{\frac{15}{8}em+\frac{167}{32}em^2+\frac{29513}{1536}em^3-\frac{15}{4}\gamma^2em-\frac{75}{16}ee'^2m-\frac{463}{128}e^3m^2\right\}\cos(2D-l)+$$

$$+\left\{\frac{35}{8}ee'm+\frac{1269}{64}ee'm^2+\frac{41735}{768}ee'm^3\right\}\cos(2D-l-l')+$$
$$+\left\{\frac{255}{32}ee'^2m+\frac{12011}{256}ee'^2m^2\right\}\cos(2D-l-2l')+\left\{-\frac{15}{8}ee'm-\frac{97}{64}ee'm^2+\right.$$
$$\left.+\frac{51077}{768}ee'm^3\right\}\cos(2D-l+l')+\left\{-\frac{15}{4}e^2m^2-\frac{225}{16}e^2m^3\right\}\cos(2D-2l)-$$
$$-\frac{165}{8}e^2e'm^2\cos(2D-2l-l')+\left\{-\frac{105}{64}e^3m-\frac{7703}{768}e^3m^2\right\}\cos(2D-3l)-$$
$$-\frac{245}{64}e^3e'm\cos(2D-3l-l')-\frac{55}{16}e^4m\cos(2D-4l)-\frac{75}{8}\gamma^2e^2m\cos(2D+2F-2l)-$$
$$-\frac{21}{8}\gamma^2em\cos(2D-2F-l)+\frac{105}{8}e^2m^3\cos 4D+\left\{\frac{6075}{512}e^3m^2+\frac{495}{128}em^3\right\}\times$$
$$\times\cos(4D-l)+\frac{5775}{256}ee'm^3\cos(4D-l+l')+\left\{\frac{225}{64}e^2m^2+\frac{3195}{128}e^2m^3\right\}\times$$
$$\times\cos(4D-2l)+\frac{525}{32}e^2e'm^2\cos(4D-2l-l')+\alpha\left\{-\frac{15}{16}m-\frac{81}{16}m^2-\right.$$
$$\left.-\frac{5817}{256}m^3+\frac{165}{16}\gamma^2m-\frac{105}{32}e^2m\right\}\cos D+\alpha\left\{\frac{15}{16}e'm-\frac{977}{64}e'm^2\right\}\cos(D-l')+$$
$$+\alpha\left\{\frac{5}{4}e'+\frac{15}{4}e^2e'-\frac{45}{8}e'm+\frac{2211}{64}e'm^2\right\}\cos(D+l')+\alpha\left\{-\frac{15}{8}em-\frac{177}{16}em^2\right\}\times$$
$$\times\cos(D+l)+\alpha\left\{\frac{5}{2}ee'-\frac{45}{4}ee'm\right\}\cos(D+l+l')-\alpha\frac{405}{128}e^2m\cos(D+2l)+$$
$$+\alpha\frac{135}{32}e^2e'\cos(D+2l+l')+\alpha\frac{45}{16}em^2\cos(D-l)+\alpha\frac{435}{128}e^2m\cos(D-2l)-$$
$$-\alpha\frac{105}{32}e^2e'\cos(D-2l+l')+\alpha\frac{25}{64}m^2\cos 3D-\alpha\frac{475}{64}em^2\cos(3D-l)+$$
$$+\alpha\frac{75}{16}ee'm\cos(3D-l+l')-\alpha\frac{175}{64}e^2m\cos(3D-2l). \qquad (28)$$

In these formulas the point Γ (Fig.2) should be taken for the origin of the count of longitudes, and the arguments must be computed by formulas (25).

Compiled in Table 3 are the numerical values of the mean motion of the node and of the perigee of the AMS, obtained by Delaunay formulas of I872 [6]. The variations are given for 24 hours.

TABLE 3

	Var.1	Var.2	Var.3
$\frac{d\Omega}{dt}$	$-0^\circ.081969$	$-0^\circ.230621$	$-0^\circ.645881$
$\frac{d\pi}{dt}$	$+0^\circ.079782$	$+0^\circ.256701$	$+1^\circ.062658$

Tables 4, 5, 6 for the numerical values of coefficients of inequality, induced by the perturbing action of the Earth, have been compiled on the basis of formulas (26), (27) and (28).

One may judge by these tables of the character of perturbations produced by the Earth with regard to the motion of the lunar

TABLE 4

Coefficients of inequality induced by perturbation of the Earth (longitude)

No	No after Delaunay	Var.I	Var.II	Var.III	
		A_1	A_2	A_3	
1	2	− 0″07	− 0″22	− 0″60	$\sin l'$
2	3	0	− 1	− 2	$\sin 2l'$
3	7	+20.55	+20.55	+20.55	$\sin l$
4	8	+ 2	+ 8	+ 30	$\sin (l - l')$
5	9	0	0	1	$\sin (l - 2l')$
6	12	− 2	− 8	− 25	$\sin (l + l')$
7	13	0	0	− 1	$\sin (l + 2l')$
8	16	+ 2.28	+ 2.28	+ 2.28	$\sin 2l$
9	17	0	+ 1	+ 7	$\sin (2l - l')$
10	20	0	− 1	− 5	$\sin (2l + l')$
11	23	+ 0.35	+ 35	+ 35	$\sin 3l$
12	24	0	0	+ 1	$\sin (3l - l')$
13	26	0	0	− 1	$\sin (3l + l')$
14	28	+ 6	+ 6	+ 6	$\sin 4l$
15	33	+ 1	+ 1	+ 1	$\sin 5l$
16	37	− 20	− 20	− 20	$\sin 2F$
17	44	− 7	− 7	− 7	$\sin (2F + l)$
18	49	− 2	− 2	− 2	$\sin (2F + 2l)$
19	54	0	− 1	− 1	$\sin (2F + 3l)$
20	58	− 10	− 9	− 6	$\sin (2F - l)$
21	89	+ 7	+ 25	+ 1.05	$\sin 2D$
22	90	0	3	19	$\sin (2D - l')$
23	98	+ 2	+ 5	+ 23	$\sin (2D + l)$
24	99	0	0	3	$\sin (2D + l - l')$
25	105	0	+ 1	+ 5	$\sin (2D + 2l)$
26	118	+ 31	+ 93	+ 3.23	$\sin (2D - l)$
27	119	+ 4	+ 12	+ 47	$\sin (2D - l - l')$
28	120	0	+ 1	+ 5	$\sin (2D - l - 2l')$
29	123	− 2	− 5	− 12	$\sin (2D - l + l')$
30	124	0	0	− 1	$\sin (2D - l + 2l')$
31	127	+ 4	+ 13	+ 46	$\sin (2D - 2l)$
32	128	0	+ 1	+ 6	$\sin (2D - 2l - l')$
33	131	0	− 1	− 2	$\sin (2D - 2l + l')$
34	134	0	+ 2	+ 10	$\sin (2D - 3l)$
35	135	0	0	+ 1	$\sin (2D - 3l - l')$
36	139	0	+ 1	+ 2	$\sin (2D - 4l)$
37	161	0	0	− 1	$\sin (2D + 2F - l)$
38	183	0	+ 1	+ 3	$\sin (2D - 2F)$
39	190	0	0	+ 1	$\sin (2D - 2F + l)$
40	258	0	+ 1	+ 7	$\sin (4D - 2l)$
41	259	0	0	+ 1	$\sin (4D - 2l - l')$
42	342	0	− 5	− 41	$\sin D$
43	343	0	0	0	$\sin (D - l')$
44	346	+ 7	+ 16	+ 45	$\sin (D + l')$
45	349	0	− 1	− 9	$\sin (D + l)$
46	352	+ 2	+ 3	+ 5	$\sin (D + l + l')$
47	354	0	0	− 1	$\sin (D + 2l)$
48	357	0	+ 1	+ 1	$\sin (D + l2l + l')$
49	364	0	− 2	− 17	$\sin (D - l)$
50	367	+ 2	+ 3	+ 4	$\sin (D - 2 + l')$
51	369	0	− 1	− 5	$\sin (D - l)$
52	372	0	+ 1	+ 3	$\sin (D - 2$
53	427	0	0	− 3	$\sin (3D - l)$
54	430	0	0	+ 1	$\sin (3D - l + l'$
55	432	0	0	− 3	$\sin (3D - 2l)$

satellite as a function of its distance from the Moon.

Tables 4, 5 and 6 allow us also to attain the selenocentrical coordinates of the Moon for any moment of time.

According to the theory, we took in the equations of motion in the function R the multiplier $k^2m'=a'^3n'^2$ instead of the multiplier $k^2m'=\frac{a'^3n'^2}{\left(1+\frac{m}{m'}\right)}=a'^3n'^2\left[1-\frac{m}{m'}+2\left(\frac{m}{m'}\right)^2-\dots\right]$

If in the theory of motion of the Moon this imprecision is sufficiently accounted by multiplying the expansion factors by $\left(1-\frac{m}{m'}\right)$, in the given problem this imprecision must be taken into account by multiplying the coefficients given in Tables 4, 5, 6 by the multiplier $\left[1-\frac{m}{m'}+2\left(\frac{m}{m'}\right)^2\right]$, since $\left(\frac{m}{m'}\right)^2\approx 0.0002$ and this quantity cannot be neglected.

Coefficients of ineqaulity induced by perturbation of the Earth (latitude)

No	No after Delaun.	Var.I B_1	Var.2 B_2	Var.3 B_3	
1	1	+6°46	+6°46	+6°46	$\sin F$
2	2	0	0	+ 1	$\sin (F-l')$
3	6	0	0	− 1	$\sin (F+l')$
4	10	+1.15	+1.15	+1.15	$\sin (F+l)$
5	11	0	0	+ 1	$\sin (F+l-l')$
6	14	0	0	− 1	$\sin (F+l+l')$
7	17	+ 23	+ 23	+ 23	$\sin (F+2l)$
8	22	+ 5	+ 5	+ 5	$\sin (F+3l)$
9	31	−1.19	−1.19	−1.18	$\sin (F-l)$
10	32	0	0	+ 1	$\sin (F-l-l')$
11	35	0	0	− 1	$\sin (F-l+l')$
12	38	− 16	− 14	− 12	$\sin (F-2l)$
13	43	− 3	− 3	− 2	$\sin (F-3l)$
14	67	0	− 1	− 1	$\sin (3F-l)$
15	83	0	+ 2	+ 9	$\sin (2D+F)$
16	103	+ 2	+ 5	+ 19	$\sin (2D+F-l)$
17	104	0	+ 1	+ 3	$\sin (2D+F-l-l')$
18	107	0	0	− 1	$\sin (2D+F-l+l')$
19	143	+ 2	+ 6	+ 19	$\sin (2D-F)$
20	144	0	+ 1	+ 3	$\sin (2D-F-l')$
21	148	0	0	− 1	$\sin (2D-F+l')$
22	152	0	+ 1	+ 3	$\sin (2D-F+l)$
23	173	+ 1	+ 4	+ 15	$\sin (2D-F-l)$
24	174	0	+ 1	+ 2	$\sin (2D-F-l-l')$
25	177	0	0	− 1	$\sin (2D-F-l+l')$
26	180	0	+ 1	+ 4	$\sin (2D-F-2l)$
27	185	0	0	+ 1	$\sin (2D-F-3l)$
28	317	0	0	− 1	$\sin (D+F)$
29	320	0	+ 1	+ 2	$\sin (D+F+l')$
30	322	0	0	− 1	$\sin (D+F+l)$
31	325	0	0	− 1	$\sin (D+F+l+l')$
32	349	0	0	− 1	$\sin (D-F)$
33	352	0	+ 1	+ 2	$\sin (D-F+l')$
34	354	0	0	− 1	$\sin (D-F+l)$

TABLE 5

TABLE 6

Coefficients of inequality induced by perturbation of the Earth $\left(\frac{a}{r}\right)$

No	No after Delaun	Var.1 C_1	Var.2 C_2	Var.3 C_3	
1	1	+1.0000	+1.0001	+1.0006	
2	4	+0.1793	+0.1792	+0.1787	$\cos l$
3	5	+ 2	+ 7	+ 25	$\cos (l - l')$
4	7	− 2	− 7	− 22	$\cos (l + l')$
5	9	+ 321	+ 321	+ 320	$\cos 2l$
6	14	+ 65	+ 65	+ 65	$\cos 3l$
7	17	+ 14	+ 14	+ 14	$\cos 4l$
8	18	+ 3	+ 3	+ 3	$\cos 5l$
9	19	− 5	− 5	− 4	$\cos 2F$
10	22	− 2	− 2	− 2	$\cos (2F + l)$
11	23	− 15	− 14	− 12	$\cos (2F - l)$
12	27	+ 9	+ 34	+ 140	$\cos 2D$
13	28	+ 1	+ 4	+ 23	$\cos (2D - l')$
14	29	0	0	+ 2	$\cos (2D - 2l')$
15	30	0	− 1	− 6	$\cos (2D + l')$
16	32	+ 3	+ 10	+ 40	$\cos (2D + l)$
17	33	0	+ 1	+ 7	$\cos (2D + l - l')$
18	35	0	0	− 1	$\cos (2D + l + l')$
19	36	0	2	+ 12	$\cos (2D + 2l)$
20	37	0	0	+ 1	$\cos (2D + 2l - l')$
21	39	0	0	+ 1	$\cos (2D + 3l)$
22	40	+ 26	+ 77	+ 255	$\cos (2D - l)$
23	41	+ 3	+ 11	+ 36	$\cos (2D - l - l')$
24	42	+ 0	+ 1	+ 4	$\cos (2D - l - 2l)'$
25	43	− 1	− 4	− 11	$\cos (2D - l + l')$
26	45	0	− 1	− 6	$\cos (2D - 2l)$
27	46	0	0	− 1	$\cos (2D - 2l - l')$
28	48	0	− 2	− 8	$\cos (2D - 3l)$
29	49	0	0	− 1	$\cos (2D - 3l - l')$
30	51	0	− 1	− 2	$\cos (2D - 4l)$
31	53	0	0	− 1	$\cos (2D + 2F - 2l)$
32	61	0	0	− 1	$\cos (2D - 2F - l)$
33	65	0	0	+ 1	$\cos 4D$
34	69	0	0	+ 5	$\cos (4D - l)$
35	70	0	0	+ 1	$\cos (4D - l - l')$
36	72	0	+ 1	+ 6	$\cos (4D - 2l)$
37	73	0	0	+ 1	$\cos (4D - 2l - l')$
38	78	0	− 4	− 27	$\cos D$
39	79	0	0	+ 2	$\cos (D - l')$
40	81	+ 6	+ 14	+ 37	$\cos (D + l')$
41	83	0	− 1	− 11	$\cos (D + l)$
42	85	+ 2	+ 4	+ 7	$\cos (D + l + l')$
43	86	0	0	− 2	$\cos (D + 2l)$
44	87	0	+ 1	+ 3	$\cos (D + 2l + l')$
45	88	0	0	+ 1	$\cos (D - l)$
46	89	0	0	+ 2	$\cos (D - 2l)$
47	90	0	− 1	− 2	$\cos (D - 2l + l')$
48	93	0	0	+ 1	$\cos 3D$
49	97	0	0	− 2	$\cos (3D - l)$
50	98	0	0	+ 1	$\cos (3D - l + l')$
51	99	0	0	− 2	$\cos (3D - 2l)$

4. INEQUALITIES INDUCED BY THE ACTION OF THE SUN

This section gives the results of application of the Delaunay method in the problem MOON - AMS - SUN. This problem is stated and resolved entirely analogously to the preceding one, the Sun being taken instead of the Earth. Considered also is the problem of the motion of equatorial satellites in three variants.

In this problem we consider that the Sun moves relative to the Moon along an ellipse with elements given by Neucom in the theory of motion of the Earth, that is, the distance from the Moon to the Earth is neglected.

This error influences insignificantly the value of Delaunay parameters, and since in the given problem the latter are very small, the admitted error can be neglected.

On the basis of the above the values of elements determining the Delaunay parameters are compiled in Table 7.

TABLE 7

	ИСА (AMS)			(SUN) Солнце	
	I вариант	II вариант	III вариант		
e	0.18	0.18	0.18	e'	0.01675104
n	1687°7847	598°7524	211°8122	n'	0°98560911
a (а. е.)	$0.232334.10^{-4}$	$0.464668.10^{-4}$	$0.929336.10^{-4}$	a' (а. е.) . . .	1.00000023

In the given case the mutual inclination angle is equal to I.535° (constant inclination of the mean lunar equator to the ecliptic)

The ratio of the mass of the Moon to that of the Sun is $\frac{1}{81.53 \cdot 333430}$. In the quantity $\left(1 - \frac{m}{m'}\right)$ this ratio may obviously be neglected. The values of the Delaunay parameters are shown in the Table 8.

TABLE 8

	e	e'	γ	$\alpha = \frac{a}{a'}$	$m = \frac{n'}{n}$
Variant I..........	0.18	0.01675	0.01339	$0.232334 \cdot 10^{-4}$	$0.583966 \cdot 10^{-3}$
Variant II.........	0.18	0.01675	0.01339	$0.464668 \cdot 10^{-4}$	$0.164610 \cdot 10^{-2}$
Variant III........	0.18	0.01675	0.01339	$0.929336 \cdot 10^{-4}$	$0.465322 \cdot 10^{-2}$

It may be seen from Table 8 that the inequalites induced by the perturbing action of the Sun must be small, for in the given problem parameters a and m are many times smaller than in the theory of motion of the Moon, while parameters a and e' are of same order. Therefore, applied to the given problem, the Delaunay theory may provide a great precision.

But if we limit ourselves to the precision to 0.01" in coordinates V and U and to 0.0001 in $\frac{a}{r}$, as in the case of accounting for the perturbations due to the Earth, the formulas for the computation of these coordinates will be substantially simpler than in the preceding problem. They constitute only a small part of terms of formulas (26), (27), (28).

$$
\begin{aligned}
V = {} & \varepsilon + nt - 3e'm \sin l' + \left\{2e - \frac{1}{4}e^3\right\}\sin l + \left\{\frac{5}{4}e^2 - \frac{5}{4}\gamma^2 e^2 - \frac{11}{24}e^4\right\}\sin 2l + \\
& + \left\{\frac{13}{12}e^3 - \frac{43}{64}e^5\right\}\sin 3l + \left\{\frac{103}{96}e^4\right\}\sin 4l + \frac{1097}{960}e^5 \sin 5l + \\
& + \left\{-\gamma^2 - \frac{9}{4}\gamma^2 e^2\right\}\sin 2F - 2\gamma^2 e \sin(2F + l) - \frac{13}{4}\gamma^2 e^2 \sin(2F + 2l) - \\
& - 3\gamma^2 e \sin(2F - l) + \frac{75}{16}e^2 m \sin 2D + \frac{15}{4} em \sin(2D - l) + \\
& + \frac{35}{4} ee'm \sin(2D - l - l') + \frac{45}{16}e^2 m \sin(2D - 2l) + \frac{105}{32}e^3 m \sin(2D - 3l), \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
U = {} & (2\gamma - 2\gamma e^2)\sin F + \left\{2\gamma e - \frac{5}{2}\gamma e^3\right\}\sin(F + l) + \left\{\frac{9}{4}\gamma e^2 - \frac{27}{8}\gamma e^4\right\}\sin(F + 2l) + \\
& + \frac{8}{3}\gamma e^3 \sin(F + 3l) + \frac{625}{192}\gamma e^4 \sin(F + 4l) + \left\{-2\gamma e - 5\gamma^3 l + \frac{5}{4}\gamma e^3\right\}\sin(F - l) + \\
& + \left\{-\frac{3}{2}\gamma e^2 + \frac{77}{48}\gamma e^4\right\}\sin(F - 2l) - \frac{17}{12}\gamma e^3 \sin(F - 3l) + \\
& + \frac{15}{4}\gamma em \sin(2D + F - l) + \frac{3}{4}\gamma m \sin(2D - F) + 3\gamma em \sin(2D - F - l), \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\frac{a}{r} = {} & 1 + \left\{e - \frac{1}{8}e^3\right\}\cos l + \left\{e^2 - \frac{1}{3}e^4\right\}\cos 2l + \left\{\frac{9}{8}e^3 - \frac{81}{128}e^5\right\}\cos 3l + \frac{4}{3}e^4 \cos 4l + \\
& + \frac{625}{384}e^5 \cos 5l - 5\gamma^2 e^2 \cos 2F - \frac{135}{16}\gamma^2 e^3 \cos(2F + l) + \left\{-\frac{5}{2}\gamma^2 e + \frac{75}{16}\gamma^2 e^3\right\} \times \\
& \times \cos(2F - l) + \frac{15}{4}e^2 m \cos 2D + \frac{405}{64}e^3 m \cos(2D + l) + \frac{15}{8} em \cos(2D - l). \qquad (31)
\end{aligned}
$$

The daily variations of the mean motions of the node and of the perigee have the values shown in Table 9

TABLE 9

	Var.1	Var.2	Var.3
$\frac{d\Omega}{dt}$	−0°000457	−0°001288	−0°003641
$\frac{d\pi}{dt}$	+0.000416	+0.001185	+0.003440

The numerical values of the coefficients of formulas (29), (30), (31) are compiled in Tables 10, 11, 12.

It may be seen from these tables that the Sun's perturbations are small indeed.

TABLE 10

Coefficients of Solar inequalities (longitude)

No	No after Delaun.	Var.1 A_1	Var.2 A_1	Var.3 A_1	
1	2	0	0	− 0°01	sin l'
2	7	+20°55	+20°55	+20.55	sin l
3	16	+ 2.28	+ 2.28	+ 2.28	sin $2l$
4	23	+ 35	+ 35	+ 35	sin $3l$
5	28	+ 6	+ 6	+ 6	sin $4l$
6	33	+ 1	+ 1	+ 1	sin $5l$
7	37	− 20	− 20	− 20	sin $2F$
8	44	− 7	− 7	− 7	sin $(2F+l)$
9	49	− 2	− 2	− 2	sin $(2F+2l)$
10	54	− 1	− 1	− 1	sin $(2F+3l)$
11	58	− 11	− 11	− 11	sin $(2F-l)$
12	89	+ 1	+ 1	+ 4	sin $2D$
13	118	+ 2	+ 6	+ 18	sin $(2D-l)$
14	119	0	0	+ 1	sin $(2D-l-l')$
15	127	0	+ 1	+ 2	sin $(2D-2l)$
16	134	0	0	+ 1	sin $(2D-2l+l')$

Coefficients of Solar inequalities (latitude)

No	No after Delaun	Var.1 B_1	Var.2 B_1	Var.3 B_3	
1	1	+6°46	+6°46	+6°46	sin F
2	10	+1.15	+1.15	+1.15	sin $(F+l)$
3	17	+ 23	+ 23	+ 23	sin $(F+2l)$
4	22	+ 5	+ 5	+ 5	sin $(F+3l)$
5	27	+ 1	+ 1	+ 1	sin $(F+4l)$
6	31	−1.19	−1.19	−1.19	sin $(F-l)$
7	38	− 15	− 15	− 15	sin $(F-2l)$
8	43	− 3	− 3	− 3	sin $(F-3l)$
9	103	+ 1	+ 1	+ 1	sin $(2D+F-l)$
10	143	+ 1	+ 1	+ 1	sin $(2D-F)$
11	173	+ 1	+ 1	+ 1	sin $(2D-F-l)$

TABLE 11

TABLE 12

Coefficients of Solar inequality $\left(\frac{a}{r}\right)$

No	No after Delaun	Var.1	Var.2	Var.3	
		C_1	C_2	C_3	
1	1	+1.0000	+1.0000	+1.0000	
2	4	+0.1807	+0.1807	+0.1807	$\cos l$
3	9	+ 321	+ 321	+ 321	$\cos 2l$
4	14	+ 65	+ 65	+ 65	$\cos 3l$
5	17	+ 14	+ 14	+ 14	$\cos 4l$
6	18	+ 3	+ 3	+ 3	$\cos 5l$
7	19	− 5	− 5	− 5	$\cos 2F$
8	22	− 2	− 2	− 2	$\cos (2F+l)$
9	23	− 14	− 14	− 14	$\cos (2F-l)$
10	27	+ 1	+ 2	+ 6	$\cos 2D$
11	32	0	+ 1	+ 2	$\cos (2D+l)$
12	40	+ 2	+ 6	+ 16	$\cos (2D-l)$

* * * THE END * * *

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