# TRANSIENT SOLIDIFICATION OF A FLOWING LIQUID ON A COLD PLATE INCLUDING HEAT CAPACITIES OF FROZEN LAYER AND PLATE

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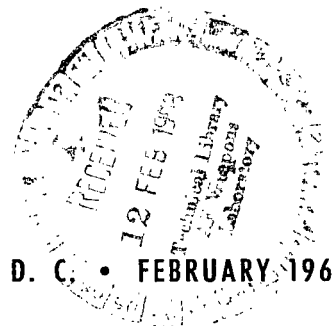
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## SUMMARY

An analysis is made of solidification within a flowing medium where a convective boundary condition is imposed on the moving interface of the solidifying layer. The configuration is one in which a frozen layer forms in a warm liquid as it is chilled while flowing over a plane wall that is convectively cooled on the opposite side. The analysis includes the heat capacities of both the wall and frozen layer. The solution for frozen layer thickness variation with time is given as a closed-form algebraic equation and in graphical form so that results can be readily evaluated for use in practical applications. Some illustrative examples are given for flowing water in contact with Inconel plates of various thicknesses to demonstrate the effect of the heat capacities and the liquid and coolant temperatures on the growth of the ice layer and the temperature distributions in the ice and wall.

## INTRODUCTION

This report is concerned with the analysis of solidification when a convective heat transfer condition is imposed at a moving interface of a frozen layer. This type of boundary condition is encountered in important applications such as the solidifying of metal castings in molds, freezing of rivers, and solidification within liquid flow heat exchangers that use a cryogen as the coolant. As discussed in references 1 and 2, relatively little analytical work and even less experimental work has been done for conditions where solidification is occurring within a flowing liquid phase.

The specific configuration considered here is the solidification of a warm liquid as it flows over one side of a plane wall that is convectively cooled on the opposite side. Initially there is no coolant flowing, and the wall and liquid are at a constant temperature that is above the fusion temperature of the liquid. Then a coolant is introduced whose temperature is below the freezing temperature of the liquid. As the wall cools

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below the fusion temperature, the flowing liquid in contact with the plate begins to solidify.

During the transient growth period, the heat supplied to the liquid - frozen layer interface is made up of the convection from the warm liquid and the latent heat of fusion generated by the freezing process. At steady state only the convective heat flux is supplied to the interface. The thickness of the steady-state layer is determined by the heat balance that this convective flux is exactly equal to the heat conducted through the frozen layer and wall and convected to the coolant.

In references 1 and 2, we investigated the transient solidification of a warm liquid on a thin wall. The heat capacity of the frozen layer was accounted for, but the wall heat capacity was considered to be negligible. The analytical solution was adequate for applications where a thick frozen layer forms on a thin wall. For thick walls, however, the wall heat capacity may be large compared with that of the frozen layer and may significantly retard the frozen layer growth. It is the purpose herein to develop a method that can be easily used to predict frozen layer growth when both the heat capacities of the frozen layer and the wall are accounted for.

To obtain the frozen layer growth, the transient heat conduction equation was integrated within the frozen layer and within the wall. This yielded coupled integral equations for the temperature distributions in the two media, and an integral equation for the frozen layer growth; this set of equations was solved iteratively. The frozen layer growth with time was obtained as a closed-form analytical equation that is easily evaluated for practical applications. Part of the growth relation is presented in graphical form so that hand calculations can be readily made.

To illustrate the influence of the wall and frozen layer heat capacities on the solidification rate, several examples are given for freezing of water. The examples illustrate the effect of varying the wall thickness, the coolant temperature, and the water temperature. The transient temperature distributions in the ice layer and wall are also shown for a few example cases.

## SYMBOLS

a     thickness of cooled wall

a'     dimensionless wall thickness, $a/X_s$

$c_p$     specific heat

F     frozen layer capacity term in growth equation, see eq. (26)

G     integral terms defined by eq. (17b)

h     convective heat-transfer coefficient

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| | |
|---|---|
| I | integral terms defined by eq. (16c) |
| $I_w$ | integral terms defined by eq. (16d) |
| J | integral terms defined by eq. (16e) |
| $J_w$ | integral terms defined by eq. (16f) |
| k | thermal conductivity of solidified material |
| $k_w$ | thermal conductivity of wall material |
| L | latent heat of fusion |
| q | heat flux in x direction |
| R | dimensionless parameter, $(X_s/k)/\left[(1/h_c) + (a/k_w)\right]$ |
| S | dimensionless parameter, $c_p(T_f - T_c)/L$ |
| T | temperature |
| $T'$ | dimensionless temperature, $(T - T_f)/(T_c - T_f)$ |
| $T'_w$ | dimensionless temperature, $(T_w - T_f)/(T_c - T_f)$ |
| W | wall capacity term in growth equation, see eq. (26) |
| X | thickness of frozen layer |
| $X'$ | dimensionless thickness of frozen layer, $X/X_s$ |
| $X_s$ | thickness of frozen layer at steady state |
| x | position coordinate measured from frozen layer-wall interface |
| $x'$ | dimensionless coordinate, $x/X_s$ |
| $\alpha$ | thermal diffusivity, $k/\rho c_p$ |
| $\beta$ | dimensionless coordinate, $x/a$ |
| $\gamma$ | dimensionless parameter, $1 + k_w/h_c a$ |
| $\xi$ | dimensionless coordinate, $x/X$ |
| $\tau$ | time from start of solidification |
| $\bar{\tau}$ | time during wall temperature transient preceeding solidification |
| $\tau'$ | dimensionless time, $\tau h_l (T_l - T_f)/\rho L X_s$ |
| $\Phi$ | function of $\gamma$ defined in eq. (27) |

Subscripts:

| | |
|---|---|
| c | coolant |
| f | at freezing temperature |

$l$     liquid phase of solidifying substance

s     steady state

w     wall

1     wall boundary in contact with coolant

2     wall boundary in contact with solidifying material

I, II     successive iterative approximations

# ANALYSIS

The model used for this study is the one-dimensional configuration shown in figure 1. The x coordinate has its origin at the interface of the frozen layer and wall, and the wall extends to x = -a. A warm liquid at a fixed temperature $T_l$ flows over one side of the wall, and it is assumed that the convective heat-transfer coefficient $h_l$ is a constant. A transient solidification process can then be initiated by introducing a flowing coolant on the other side of the wall. It is assumed that the coolant is at a fixed temperature $T_c$ and provides a constant heat-transfer coefficient $h_c$. After introduction of the coolant, the wall cools until the freezing temperature is reached on the surface of the wall (x = 0) exposed to the warm liquid. At this instant ($\tau$ = 0, X($\tau$) = 0) solidification is assumed to begin. The liquid - frozen layer interface is assumed to always be at the equilibrium fusion temperature $T_f$.

During the solidification process latent heat of fusion is being released at the interface of the solidified layer and the flowing warm liquid. The latent heat, along with the convective heat being transferred from the warm liquid boundary layer to the interface, is conducted through the frozen layer and wall and is then transferred to the coolant. The coolant also removes the additional amount of heat necessary to subcool the frozen layer and the wall as their temperatures decrease. The solid layer continues to grow until it achieves a steady-state thickness $X_s$. Constant properties are assumed throughout the analysis.

## Steady-State Thickness of Frozen Layer

In many solidification problems the frozen layer never approaches a steady-state thickness; rather, it continues to grow indefinitely with time. This occurs in a lake or river, for example, when the entire body of liquid is already at the freezing temperature and is exposed to a heat sink, the cold atmosphere in this instance. The only heat source in such a case is latent heat of fusion which is continually extracted so long as

the heat sink is present, thereby causing the frozen layer to grow continuously.

When, however, the liquid flowing over the frozen layer is at a temperature above the freezing point, the warm liquid continually supplies heat to the frozen layer - liquid interface by convection. Under this condition the frozen layer eventually achieves a steady-state thickness. This thickness, which will be used as a reference length in the later analysis of growing layers, is derived here.

It is assumed that the liquid-solid interface at the boundary of the frozen layer is always at the freezing temperature $T_f$. If the heat flow is taken as positive in the positive x direction, the convected flux from the liquid to the liquid - frozen layer interface is at all times

$$-q = h_\ell (T_\ell - T_f) \equiv constant$$

At steady state a heat balance at the liquid - frozen layer interface gives the relation

$$-q_s = h_\ell (T_\ell - T_f) = \frac{T_f - T_c}{\dfrac{X_s}{k} + \dfrac{a}{k_w} + \dfrac{1}{h_c}}$$

This is solved for the steady-state thickness

$$X_s = \frac{k}{h_\ell} \frac{T_f - T_c}{T_\ell - T_f} - k \left( \frac{a}{k_w} + \frac{1}{h_c} \right) \tag{1}$$

For $X_s = 0$, equation (1) gives the relation between variables required to just avoid freezing. For example, solving for $T_\ell$ gives

$$T_\ell \Big|_{X_s=0} = T_f + \frac{T_f - T_c}{h_\ell \left( \dfrac{a}{k_w} + \dfrac{1}{h_c} \right)} \tag{2}$$

To prevent freezing, the liquid temperature $T_\ell$ must be equal to or greater than the value given by equation (2).

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# Description of Method of Analysis for Frozen Layer

# Growth and Temperature Distributions

Before proceeding with the details of the analysis, it is instructive to outline the general features of the method. The heat flows within the frozen layer and the wall are governed by the one-dimensional transient heat conduction equations:

$$k \frac{\partial^2 T}{\partial x^2} = \rho c_p \frac{\partial T}{\partial \tau} \tag{3a}$$

for the frozen layer and

$$k_w \frac{\partial^2 T_w}{\partial x^2} = \rho_w c_{p_w} \frac{\partial T_w}{\partial \tau} \tag{3b}$$

for the wall. Equations (3a) and (3b) are each integrated twice with respect to the space coordinate x. The first integration in each medium is between one of the boundaries and an arbitrary position x. This results in expressions for the local heat fluxes $k(\partial T/\partial x)$ and $k_w(\partial T_w/\partial x)$ in terms of the heat fluxes at the boundaries which are generally unknown. The second integration, again between one of the boundaries (not necessarily the same boundary used in the first integration) and an arbitrary position x, yields the local temperatures $T(x, \tau)$ and $T_w(x, \tau)$ in terms of the temperature at one boundary of each medium and the heat flux at one boundary.

In this problem the only known boundary temperature is at the liquid - frozen layer interface. The instantaneous temperatures and heat fluxes are unknown at the boundaries between the solidified layer and wall and between the wall and coolant. When the unknown boundary heat fluxes and temperatures are eliminated by some rather lengthy algebraic manipulations, there results two rather complicated coupled integral equations, one each for $T(x, \tau)$ and $T_w(x, \tau)$.

When the equation for $T(x, \tau)$ is evaluated at the liquid - frozen layer interface, which is at the known solidification temperature, an equation is obtained for the rate of frozen layer growth. This equation is integrated to obtain the layer growth, but the integrated form contains integrals of $T$ and $T_w$. The frozen layer growth equation and the two equations for $T$ and $T_w$ form a set of three coupled integral equations. They are solved by an iterative method leading to closed-form approximate analytical solutions.

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## Temperature Distribution Equations

Equation (3a) is integrated from any position within the frozen layer $x$ to the solid-liquid interface $X$:

$$k \left. \frac{\partial T}{\partial x} \right|_X - k \left. \frac{\partial T}{\partial x} \right|_x = \rho c_p \int_x^X \frac{\partial T}{\partial \tau} \, dx \qquad 0 \le x \le X \qquad (4)$$

At the liquid - frozen layer interface the heat conducted into the solidified layer is equal to that supplied by the latent heat of fusion and the convection from the flowing liquid:

$$k \left. \frac{\partial T}{\partial x} \right|_X = \rho L \frac{dX}{d\tau} + h_l (T_l - T_f) \qquad (5)$$

Equation (5) is substituted into equation (4) to give

$$k \left. \frac{\partial T}{\partial x} \right|_x = +\rho L \frac{dX}{d\tau} + h_l (T_l - T_f) - \rho c_p \int_x^X \frac{\partial T}{\partial \tau} \, dx \qquad (6)$$

The term on the left side is the heat flow in the negative $x$ direction crossing any position $x$ at any time $\tau$. The last term on the right is the heat removed to subcool the portion of the solidified layer between $x$ and $X$. The integration of equation (6) from the wall $x = 0$ to any position $x$ results in an expression for the instantaneous temperature distribution in the frozen layer:

$$T(x, \tau) - T_2 = \frac{\rho L}{k} \frac{dX}{d\tau} x + \frac{h_l}{k} (T_l - T_f)x - \frac{\rho c_p}{k} \int_0^x \left( \int_x^X \frac{\partial T}{\partial \tau} \, dx \right) dx \qquad 0 \le x \le X$$

$$(7)$$

where $T_2 = T_2(\tau)$.

In a similar fashion equation (3b) is integrated to obtain the temperature distribution in the wall. Integrating from the boundary in contact with the coolant $x = -a$ to any $x$ location within the wall gives

$$k_w \frac{\partial T_w}{\partial x}\bigg|_x - k_w \frac{\partial T_w}{\partial x}\bigg|_{-a} = \rho_w c_{p_w} \int_{-a}^{x} \frac{\partial T_w}{\partial \tau} dx \qquad -a \leq x \leq 0 \tag{8}$$

At $x = -a$ the boundary condition is

$$k_w \frac{\partial T_w}{\partial x}\bigg|_{-a} = h_c(T_1 - T_c) \tag{9}$$

where $T_1 = T_1(\tau)$. Substituting equation (9) into (8) gives

$$k_w \frac{\partial T_w}{\partial x}\bigg|_x - h_c(T_1 - T_c) = \rho_w c_{p_w} \int_{-a}^{x} \frac{\partial T_w}{\partial \tau} dx \tag{10}$$

To obtain the temperature distribution in the wall, equation (10) is integrated again from $x = -a$ to $x$ (the use of $x = 0$ to $x$ as an alternate choice for the integration limits results ultimately in the same final expression for $T_w(x, \tau)$):

$$T_w(x, \tau) - T_1 = \frac{h_c}{k_w}(T_1 - T_c)(x + a) + \frac{\rho_w c_{p_w}}{k_w} \int_{-a}^{x} \left( \int_{-a}^{x} \frac{\partial T_w}{\partial \tau} dx \right) dx \qquad -a \leq x \leq 0 \tag{11}$$

Equations (7) and (11) provide the temperature distributions in the frozen layer and wall but are not yet in usable form because they contain $T_1$ and $T_2$ which are unknown functions of time. These temperatures can be eliminated by applying the conditions of continuity of temperature and heat flux that must hold at the boundary between the wall and frozen layer. Continuity of heat flux provides the relation

$$k \frac{\partial T}{\partial x}\bigg|_{x=0} = k_w \frac{\partial T_w}{\partial x}\bigg|_{x=0}$$

and by use of equations (6) and (10) evaluated at $x = 0$ this becomes

$$\rho L \frac{dX}{d\tau} + h_l (T_l - T_f) - \rho c_p \int_0^X \frac{\partial T}{\partial \tau} dx = h_c (T_1 - T_c) + \rho_w c_{p_w} \int_{-a}^0 \frac{\partial T_w}{\partial \tau} dx$$

(12)

Continuity of temperature requires that at $x = 0$, $T_w$ from equation (11) must equal $T_2$; therefore,

$$T_2 - T_1 = \frac{h_c}{k_w} (T_1 - T_c)a + \frac{\rho_w c_{p_w}}{k_w} \int_{-a}^0 \left( \int_{-a}^x \frac{\partial T_w}{\partial \tau} dx \right) dx$$

(13)

Equation (12) yields an expression for $T_1$ and this is substituted into equation (13) to obtain a relation for $T_2$. The $T_1$ and $T_2$ are then substituted into equations (7) and (11) to yield the temperature distributions

$$T(x, \tau) = T_c + \left( 1 + \frac{h_c a}{k_w} \right) \left[ \frac{\rho L}{h_c} \frac{dX}{d\tau} + \frac{h_l}{h_c} (T_l - T_f) - \frac{\rho_w c_{p_w}}{h_c} \int_{-a}^0 \frac{\partial T_w}{\partial \tau} dx \right.$$

$$\left. - \frac{\rho c_p}{h_c} \int_0^X \frac{\partial T}{\partial \tau} dx \right] + \frac{\rho L}{k} \frac{dX}{d\tau} x + \frac{h_l}{k} (T_l - T_f)x + \frac{\rho_w c_{p_w}}{k_w} \int_{-a}^0 \left( \int_{-a}^x \frac{\partial T_w}{\partial \tau} dx \right) dx$$

$$- \frac{\rho c_p}{k} \int_0^X \left( \int_x^X \frac{\partial T}{\partial \tau} dx \right) dx$$

(14a)

$$T_w(x, \tau) = T_c + \left[ 1 + \frac{h_c}{k_w} (x + a) \right] \left[ \frac{\rho L}{h_c} \frac{dX}{d\tau} + \frac{h_l}{h_c} (T_l - T_f) - \frac{\rho_w c_{p_w}}{h_c} \int_{-a}^0 \frac{\partial T_w}{\partial \tau} dx \right.$$

$$\left. - \frac{\rho c_p}{h_c} \int_0^X \frac{\partial T}{\partial \tau} dx \right] + \frac{\rho_w c_{p_w}}{k_w} \int_{-a}^x \left( \int_{-a}^x \frac{\partial T_w}{\partial \tau} dx \right) dx$$

(14b)

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The temperature distributions in equation (14) can then be placed in dimensionless form by letting $x' = x/X_s$, $\tau' = h_l(T_l - T_f)\tau/\rho L X_s$, $X' = X/X_s$, $T' = (T - T_f)/(T_c - T_f)$, $T_w' = (T_w - T_f)/(T_c - T_f)$, $R = (X_s/k)/[(1/h_c) + (a/k_w)]$, $S = c_p(T_f - T_c)/L$ and using equation (1) in the form

$$\frac{X_s h_l}{k} \frac{(T_l - T_f)}{(T_f - T_c)} = \frac{R}{1 + R}$$

This gives

$$T' = 1 - \frac{1 + Rx'}{1 + R}\left(1 + \frac{dX'}{d\tau'}\right) - \frac{RS}{1 + R}\left(\frac{1}{R}\int_0^{X'} \frac{\partial T'}{\partial \tau'}\, dx' + \int_0^{x'}\int_{x'}^{X'} \frac{\partial T'}{\partial \tau'}\, dx'\, dx'\right)$$

$$- \frac{\alpha}{\alpha_w}\frac{RS}{1 + R}\left(\frac{1}{R}\frac{k_w}{k}\int_{-a'}^{0} \frac{\partial T_w'}{\partial \tau'}\, dx' - \int_{-a'}^{0}\int_{-a'}^{x'} \frac{\partial T_w'}{\partial \tau'}\, dx'\, dx'\right) \quad (15a)$$

$$T_w' = 1 - \left(\frac{1 + R\frac{k}{k_w}x'}{1 + R}\right)\left(1 + \frac{dX'}{d\tau'} + S\int_0^{X'} \frac{\partial T'}{\partial \tau'}\, dx'\right)$$

$$+ \frac{RS}{1 + R}\frac{\alpha}{\alpha_w}\left[-\left(\frac{1}{R}\frac{k_w}{k} + x'\right)\int_{-a'}^{0} \frac{\partial T_w'}{\partial \tau'}\, dx' + \int_{-a'}^{x'}\int_{-a'}^{x'} \frac{\partial T_w'}{\partial \tau'}\, dx'\, dx'\right] \quad (15b)$$

A few transformations on equations (15) are now made. The time derivatives are taken out of the integrals by using the rule for differentiating under an integral. The double integrals are changed into single integrals; the method for accomplishing this is outlined in reference 1. Then the substitution $(\partial/\partial\tau') = (dX'/d\tau')[(\partial/\partial X')]$ is made. This yields the temperature distributions in the forms

$$T'(x', X') = 1 - \left(\frac{1 + Rx'}{1 + R}\right)\left(\frac{dX'}{d\tau'} + 1\right) - \frac{RS}{1 + R}\frac{dX'}{d\tau'}\left[\frac{\partial I(x', X')}{\partial X'} + \frac{\alpha}{\alpha_w}\frac{dI_w(X')}{\partial X'}\right] \quad (16a)$$

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$$T'_w(x', X') = 1 - \left(\frac{1 + \frac{k}{k_w} R x'}{1 + R}\right)\left(\frac{dX'}{d\tau'} + 1\right) - \frac{RS}{1+R} \frac{dX'}{d\tau'} \left[\frac{\partial J(x', X')}{\partial X'} + \frac{\alpha}{\alpha_w} \frac{\partial J_w(x', X')}{\partial X'}\right]$$

(16b)

where

$$I(x', X') = \frac{1}{R} \int_0^{X'} T' \, dx' + x' \int_{x'}^{X'} T' \, dx' + \int_0^{x'} x' T' \, dx'$$ (16c)

$$I_w(X') = \frac{k_w}{k} \frac{1}{R} \int_{-a'}^0 T'_w \, dx' + \int_{-a'}^0 x' T'_w \, dx'$$ (16d)

$$J(x', X') = \frac{1}{R} \int_0^{X'} T' \, dx' + \frac{k}{k_w} x' \int_0^{X'} T' \, dx'$$ (16e)

$$J_w(x', X') = \frac{1}{R} \frac{k_w}{k} \int_{-a'}^0 T'_w \, dx' + \int_{-a'}^{x'} T'_w x' \, dx' + x' \int_{x'}^0 T'_w \, dx'$$ (16f)

In the temperature distribution equations (eqs. (16a) and (16b)) there still remains the unknown quantity $dX'/d\tau'$. This quantity is also needed so that it can be integrated to obtain the frozen layer growth with time. The expression for $dX'/d\tau'$ can be found by imposing the physical boundary condition that at the liquid - frozen layer interface $T = T_f$ at $x = X$. This gives in equation (16a) $T' = 0$ at $x' = X'$, and equation (16a) can then be rearranged to yield

$$\frac{dX'}{d\tau'} = \frac{R(1 - X')}{1 + RX' + RS\left(\frac{dG}{dX'} + \frac{\alpha}{\alpha_w} \frac{dI_w}{dX'}\right)}$$ (17a)

where

$$G(X') = I(x' = X', X') = \frac{1}{R} \int_0^{X'} T' \, dx' + \int_0^{X'} x' T' \, dx'$$ (17b)

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Equation (17a) is substituted into equations (16a) and (16b) to eliminate $dX'/d\tau'$ and to obtain the final forms of the temperature distributions:

$$T' = \frac{R(1 - x')}{1 + R} - \left[\frac{1 + Rx' + RS\left(\frac{\partial I}{\partial X'} + \frac{\alpha}{\alpha_w}\frac{dI_w}{dX'}\right)}{1 + R}\right]\left[\frac{R(1 - X')}{1 + RX' + RS\left(\frac{dG}{dX'} + \frac{\alpha}{\alpha_w}\frac{dI_w}{dX'}\right)}\right]$$

(18a)

$$T'_w = \frac{R\left(1 - \frac{k}{k_w}x'\right)}{1 + R} - \left[\frac{1 + \frac{k}{k_w}Rx' + RS\left(\frac{\partial J}{\partial X'} + \frac{\alpha}{\alpha_w}\frac{\partial J_w}{\partial X'}\right)}{1 + R}\right]\left[\frac{R(1 - X')}{1 + RX' + RS\left(\frac{dG}{dX'} + \frac{\alpha}{\alpha_w}\frac{dI_w}{dX'}\right)}\right]$$

(18b)

## Frozen Layer Growth

Now return to equation (17a) and integrate to obtain the time variation of the frozen layer thickness. Separating the variables and integrating with the initial condition $X' = 0$ at $\tau' = 0$ yields

$$\tau' = \int_0^{X'} \frac{1 + RX'}{R(1 - X')}dX' + S\int_0^{X'}\frac{1}{1 - X'}\frac{dG}{dX'}dX' + S\frac{\alpha}{\alpha_w}\int_0^{X'}\frac{1}{1 - X'}\frac{dI_w}{dX'}dX'$$

(19)

Integrating the first term of equation (19) directly and the last two terms by parts gives

$$\tau' = \left[ -X' - \left( \frac{1 + R}{R} \right) \ln (1 - X') \right] + S \left[ \frac{G(X')}{1 - X'} - \int_0^{X'} \frac{G(X')}{(1 - X')^2} \, dX' \right]$$

$$+ S \frac{\alpha}{\alpha_w} \left[ \frac{I_w(X')}{1 - X'} - I_w(0) - \int_0^{X'} \frac{I_w(X')}{(1 - X')^2} \, dX' \right] \quad (20)$$

Equation (20) is the general expression that relates dimensionless frozen layer thickness and time. The quantities $G(X')$ and $I_w(X')$ contain the temperature distributions $T'(x', X')$ and $T'_w(x', X')$ in the frozen layer and wall, respectively; these distributions are provided by equations (18a) and (18b). The method of solution for the coupled temperature and layer growth relations will be given in the next section. The term $I_w(0)$ in equation (20) accounts for the temperature distribution in the wall at the instant that solidification begins. It should be noted that the influence of the wall heat capacity appears as the additive terms

$$\frac{\alpha}{\alpha_w} \frac{dI_w}{dX'}$$

and

$$\frac{\alpha}{\alpha_w} \frac{\partial J_w}{\partial X'}$$

in equations (18a) and (18b), and

$$\frac{\alpha}{\alpha_w} \left[ \frac{I_w}{1 - X'} - I_w(0) - \int_0^{X'} \frac{I_w}{(1 - X')^2} \, dX' \right]$$

in equation (20).

## Approximate Solution by Analytical Iterations

The temperature distributions in equation (18) depend on the instantaneous frozen layer thickness, and the layer growth time as given by equation (20) depends on integrals

of the temperature distributions. An approximate solution to this coupled set of equations is obtained by an iterative technique.

The first approximations for the frozen layer and wall temperature distributions $T'$ and $T'_w$ and the growth time $\tau'(X')$ are found by neglecting the effect of heat capacity in both media. When $c_p$ and $c_{p_w}$ are set equal to zero, the parameters $S$ and $1/\alpha_w$ become zero. Then from equations (18a), (18b), and (20) the first approximations are

$$T'_I = \frac{R(X' - x')}{1 + RX'} \tag{21a}$$

$$T'_{w_I} = \frac{R\left(X' - \dfrac{k}{k_w}\,x'\right)}{1 + RX'} \tag{21b}$$

and

$$\tau'_I = -X' - \frac{1 + R}{R}\ln(1 - X') \tag{22}$$

Improved approximations for $T'$, $T'_w$, and $\tau'(X')$ can now be obtained by substituting equations (21a) and (21b) into the integral quantities $G$, $I$, $I_w$, $J$, $J_w$ in equations (18a), (18b), and (20). When the integrations are carried out, the final equations for the second approximations take the form

$$T'_{II}(\xi, X') = \frac{R(1 - X'\xi)}{1 + R} - \frac{1 + RX'\xi + RS\left(\dfrac{\partial I_I}{\partial X'} + \dfrac{\alpha}{\alpha_w}\dfrac{dI_{w_I}}{dX'}\right)}{1 + R}\; \frac{R(1 - X')}{1 + RX' + RS\left(\dfrac{dG_I}{dX'} + \dfrac{\alpha}{\alpha_w}\dfrac{dI_{w_I}}{dX'}\right)} \tag{23a}$$

where

$$\frac{\partial I_I}{\partial X'} = \frac{R^2}{(1 + RX')^2}\left[\frac{1}{R}\left(\frac{X'}{R} + \frac{X'^2}{2}\right) + \left(\frac{X'^2\xi}{R} + \frac{X'^3\xi}{2}\right) - \left(\frac{X'^2\xi^2}{2R} + \frac{X'^3\xi^3}{6}\right)\right] \tag{23b}$$

$$\frac{dG_I}{dX'} = \frac{R}{(1 + RX')^2}\left(\frac{X'}{R} + X'^2 + \frac{RX'^3}{3}\right) \tag{23c}$$

$$\frac{dI_{w_I}}{dX'} = \frac{1}{(1 + RX')^2}\left(a'\frac{k_w}{k} - Ra'^2 + \frac{R^2}{3}\frac{k}{k_w}a'^3\right) \tag{23d}$$

$$\xi = \frac{x}{X} \quad \text{and} \quad 0 \leq \xi \leq 1$$

$$T'_{w,\Pi}(\beta, X') = \frac{R\left(1 - \frac{k}{k_w}a'\beta\right)}{1 + R} - \frac{1 + \frac{k}{k_w}Ra'\beta + RS\left(\frac{\partial J_I}{\partial X'} + \frac{\alpha}{\alpha_w}\frac{\partial J_{w_I}}{\partial X'}\right)}{1 + R}$$

$$\times \frac{R(1 - X')}{1 + RX' + RS\left(\frac{dG_I}{dX'} + \frac{\alpha}{\alpha_w}\frac{dI_{w_I}}{dX'}\right)} \tag{24a}$$

where $dG_I/dX'$ and $dI_{w_I}/dX'$ are given by equations (23c) and (23d) and

$$\frac{\partial J_I}{\partial X'} = \left(\frac{1}{R} + \frac{k}{k_w}a'\beta\right)\frac{RX'}{2}\frac{(2 + RX')}{(1 + RX')^2} \tag{24b}$$

$$\frac{\partial J_{w_I}}{\partial X'} = \frac{R}{(1 + RX')^2}\left(\frac{a'}{R}\frac{k_w}{k} - a'^2 + \frac{R}{3}\frac{k}{k_w}a'^3 - \frac{a'^2\beta^2}{2} - \frac{R}{6}\frac{k}{k_w}a'^3\beta^3\right) \tag{24c}$$

$$\beta = \frac{x}{a} \quad \text{and} \quad -1 \leq \beta \leq 0$$

and finally

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$$\tau'_{II} = \tau'_I - S\left\{\frac{RX'}{3(1 + RX')}\left[X' + \frac{2 + R}{R(1 + R)}\right] + \frac{3 + 3R + R^2}{3(1 + R)^2}\ln(1 - X') + \frac{\ln(1 + RX')}{3R(1 + R)^2}\right\}$$

$$+ S\frac{\alpha}{\alpha_w}\left(a'\frac{k_w}{k} - Ra'^2 + \frac{R^2 a'^3}{3}\frac{k}{k_w}\right)\left[\frac{X'R}{(1 + R)(1 + RX')} - \frac{1}{(1 + R)^2}\ln\frac{1 - X'}{1 + RX'}\right] \quad (25)$$

The solutions given by equations (23), (24), and (25) were the highest order approximations that were carried out for the present problem. It might be possible to obtain the third order approximations, but the results would be so complex that they would not be useful for practical problems. A numerical approach would probably be simpler than carrying out the third approximations. The solution equation (25), however, can be readily evaluated and reveals the influence of the several independent parameters. In reference 1 where $c_{p_w}$ was neglected, it was demonstrated that the second approximate solutions differed by only a few percent from the third approximations. It seems reasonable to assume that the same behavior holds true for the solutions presented in this report.

## Graphs for Predicting Solidified Layer Growth

Equation (25) for predicting frozen layer growth can be written as

$$\tau'_{II} = \tau'_I + SF + S\frac{\alpha}{\alpha_w}\left(a'\frac{k_w}{k} - Ra'^2 + \frac{R^2 a'^3}{3}\frac{k}{k_w}\right)W \quad (26a)$$

where $\tau'_I$, F, and W are functions of X' and R only:

$$\tau'_I = -X' - \frac{1 + R}{R}\ln(1 - X') \quad (26b)$$

$$F = -\left\{\frac{RX'}{3(1 + RX')}\left[X' + \frac{2 + R}{R(1 + R)}\right] + \frac{3 + 3R + R^2}{3(1 + R)^2}\ln(1 - X') + \frac{\ln(1 + RX')}{3R(1 + R)^2}\right\} \quad (26c)$$

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$$W = \frac{X'R}{(1+R)(1+RX')} - \frac{1}{(1+R)^2} \ln \frac{1-X'}{1+RX'} \qquad (26d)$$

When the specific heat of the wall is zero, $1/\alpha_w = 0$, the contribution arising from wall heat capacity will be zero; that is, the term containing $W$ vanishes. Similarly, SF is the contribution from the heat capacity of the frozen layer. The term $\tau_I'$ is the solution for no heat capacity in either wall or frozen layer.

For convenience in making quick desk calculations, the quantities $\tau_I$, F, and W are given in figures 2(a), (b), and (c) as functions of R for various X' values. For a given set of imposed conditions ($T_c$, $T_l$, $h_c$, $h_l$, a, etc.) the R value is first computed. Then the dimensionless time $\tau'$ to form a particular dimensionless thickness X' can be obtained by reading the quantities from the graphs and combining them according to equation (26a). This gives a relation between X' and $\tau'$ from which $X = X(\tau)$ can be found in dimensional form by using the definitions for X' and $\tau'$.

# DISCUSSION

As discussed in conjunction with the application of equation (26a), the three terms on the right side of the equation represent, respectively, the dimensionless growth time $\tau'$ when all heat capacities are neglected, the contribution of the frozen layer capacity, and the contribution of the wall heat capacity. There are several parameters involved such as S, R, $\alpha/\alpha_w$, and a'. A general parametric study of the effect of these independent parameters does not seem worthwhile, especially since equation (26a) can be easily evaluated for any design application.

Before giving some illustrative examples, an examination of the last term on the right in equation (26a) (wall heat capacity term) will provide some additional information. The parameter R can be written in an alternate form:

$$R = \frac{\dfrac{X_s}{k}}{\dfrac{a}{k_w} \gamma}$$

where

$$\gamma \equiv 1 + \frac{k_w}{h_c a}$$

For large $h_c a$ and low $k_w$, $\gamma$ approaches 1 as a lower limit. On the other hand, for large $k_w$ and small $h_c a$, $\gamma$ can become very large. With this range of $\gamma$ in mind, the factor in the last term of equation (26a) can be examined:

$$a' \frac{k_w}{k} - Ra'^2 + \frac{R^2 a'^3}{3} \frac{k}{k_w} = a' \frac{k_w}{k}\left(1 - \frac{1}{\gamma} + \frac{1}{3\gamma^2}\right) \equiv a' \frac{k_w}{k} \Phi(\gamma) \qquad (27)$$

As $\gamma \to 1$, $\Phi \to \frac{1}{3}$ and when $\gamma \to \infty$, $\Phi \to 1$. Hence, the $\Phi$ is confined within rather narrow limits of magnitude and is positive. Now the quantity $W$ is examined to determine its sign. The term $(X'R)/(1+R)(1+RX')$ is positive. The argument $(1-X')/(1+RX') < 1$; therefore, $\ln(1-X')/(1+RX') < 0$, and $W$ is positive for all $X'$ and $R$. It is then evident that the contribution to $\tau'$ by the wall heat capacity is positive; that is, the growth time $\tau$ for a given thickness $X$ is increased because of the extra heat extraction needed to cool the wall. Since the factor $F$ is positive, the same effect is true for the frozen layer capacity. This is what would be expected from intuitive reasoning.

To demonstrate the type of results obtained from the analysis a few illustrative examples are now given. To choose some reasonable values for the parameters, one of the experimental tests from reference 2 is used as a starting comparison. In those tests, warm water flowed over an Inconel plate that was cooled on the opposite side by flowing chilled alcohol. For the data shown in figure 3 the Inconel plate was 3/16 inch (0.476 cm) thick and a steady-state ice layer about 0.4 inch (1 cm) thick was formed. The data is a little above the analytical prediction, but the agreement is satisfactory for engineering purposes. Unfortunately the data does not provide a good check on all aspects of the theory because the heat capacity terms are relatively small for the thicknesses tested. Additional data would be desirable having larger frozen layer and plate thicknesses.

With the experimental test in figure 3 as a starting point, the predicted influence on the ice growth of changing some of the conditions is shown in figure 4. Figure 5 illustrates the corresponding temperature variations in the wall and frozen layer.

Figure 4(a) differs from figure 3 only by having the wall thickness increased from 3/16 to 1 inch (0.476 to 2.54 cm). The resulting steady-state ice layer is relatively thin. The growth curves reveal that the ice capacity has a negligible effect while the wall capacity significantly slows the frozen layer formation. In figure 4(b) the coolant temperature is decreased from -40.5° to -400° F (232.9° to 33.2° K) with the remaining conditions kept the same as in figure 4(a). Lowering the coolant temperature causes the steady-state ice layer to become quite thick. As shown by the growth curves, the ice capacity now has a significant effect throughout the layer growth; the wall capacity is

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only important during the early portion of the transient (the temperature distributions of fig. 5(b) show it is during this period that the wall undergoes most of its cooling). In figure 4(c) the conditions remain the same as for figure 4(b) except that the water temperature is increased from 53. 5° to 150° F (285. 1° to 338. 7° K). This causes the ice layer to be relatively thin and provide a minor heat capacity effect; the effect of the wall capacity is large since the wall is substantially cooled by the low temperature coolant.

The solid lines in figure 5 show the transient temperature distributions in the wall and ice as computed from equations (23) and (24) for the conditions in figure 4. The curves in the ice layer terminate at the various x values equal to the instantaneous ice thicknesses at the times shown.

The dashed curves in figure 5 are solutions to the one-dimensional transient heat conduction equation for the wall before the frozen layer begins to form. The solutions were carried out for the case where the wall was subjected to the water flow on one side and was initially at a uniform temperature equal to the water temperature. Then at time $\bar{\tau} = 0$ the convective cooling condition was applied to the other side of the wall. The dashed curve for the largest $\bar{\tau}$ is the temperature distribution at the instant when the surface of the plate in contact with the water reaches 32° F (273. 2° K), at which time freezing is assumed to begin. At this $\bar{\tau}$ the dashed curve does not agree precisely with the solid curve for $\tau = 0$ (as calculated from eq. (24)). This brings us to some comments about the initial condition imposed on the freezing process.

An instant before a frozen layer forms, $\tau = 0^-$, the wall surface at $x = 0$ is at the freezing temperature of the liquid and the heat flux into the surface is that supplied by convection:

$$T_w(x = 0, \tau = 0^-) = T_f$$

$$k_w \left. \frac{\partial T_w}{\partial x} \right|_{\substack{x=0 \\ \tau=0^-}} = h_l (T_l - T_f)$$

An instant later, $\tau = 0^+$, the frozen layer has just started to form and the heat balance at the surface becomes

$$k_w \left. \frac{\partial T_w}{\partial x} \right|_{\substack{x=0 \\ \tau=0^+}} = k \left. \frac{\partial T}{\partial x} \right|_{\substack{x=X \to 0 \\ \tau=0^+}} = h_l (T_l - T_f) + \rho L \left. \frac{dX}{d\tau} \right|_{\substack{X \to 0 \\ \tau=0^+}}$$

A comparison of these equations for $k_w (\partial T_w / \partial x) \big|_{x=0}$ at $\tau = 0^-$ and $\tau = 0^+$ reveals

that the heat flux into the surface at $x = 0$ undergoes a step increase in heat flux by an amount $\rho L (dX/d\tau)\big|_{X=0}$ when freezing begins. The effect of this additional heat flux is to retard the cooling of the wall in the vicinity of $x = 0$. In the remainder of the wall this effect has not been felt at small $\tau$ and the temperatures continue to decrease in the same manner as before the freezing started. As a result, the temperature profiles change in shape from those shown by the dashed curves to those shown by solid curves.

The present solution being approximate, since only the second approximation was carried out, does not reveal in precise detail the transient adjustment that occurs at small $\tau$. As shown by equation (20) the initial wall temperature distribution enters the successive approximations in an integrated form by means of the $I_w(0)$ term. The reasonably good agreement of the dashed and solid curves when the surface at $x = 0$ reaches the freezing temperature shows that the freezing solution given here corresponds well with the following initial condition at the beginning of the entire transient process. When $\overline{\tau} < 0$ there is no coolant flowing and the wall and flowing warm liquid are all isothermal at the temperature of the liquid that is flowing over one side of the wall. Then at $\overline{\tau} = 0$ the coolant is introduced on the other side of the wall.

# CONCLUDING REMARKS

In this report a solidification analysis is presented for conditions that arise in some important engineering applications such as casting of metals in molds, heat exchangers using cryogenics, and freezing of ice sheets on rivers. The problem studied is the transient solidification of a flowing warm liquid in contact with a plane wall that is suddenly cooled convectively from the opposite side. The purpose of the analysis was to derive a method whereby the instantaneous frozen layer thickness could be predicted at any time, and which would account for the heat capacities in both the frozen layer and wall. Although this is a complex problem, a closed-form analytical solution (eq. (26)) was found that can be easily evaluated for engineering use. Portions of the equation are presented in graphical form to further facilitate its use.

To demonstrate the type of results that are obtainable from this equation some examples are given for ice forming on a stainless-steel plate. The examples illustrate that under some conditions the heat capacities of the frozen layer and wall appreciably retard the frozen layer growth.

Lewis Research Center,
    National Aeronautics and Space Administration,
        Cleveland, Ohio, October 18, 1967,
            129-01-11-06-22.

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2. Savino, Joseph M.; and Siegel, Robert: Experimental and Analytical Study of the Transient Solidification of a Warm Liquid Flowing over a Chilled Flat Plate. NASA TN D-4015, 1967.
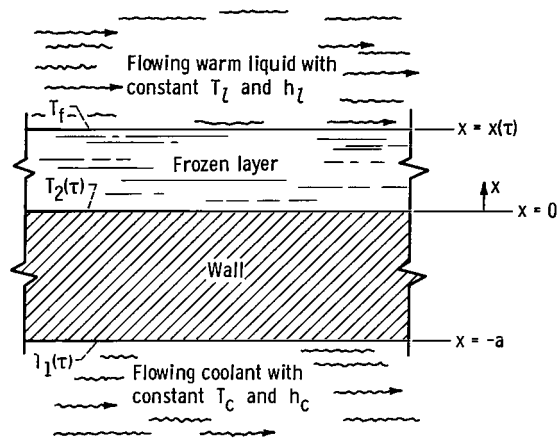
Figure 1. - One-dimensional model for transient solidification of a flowing liquid on a convectively cooled wall.
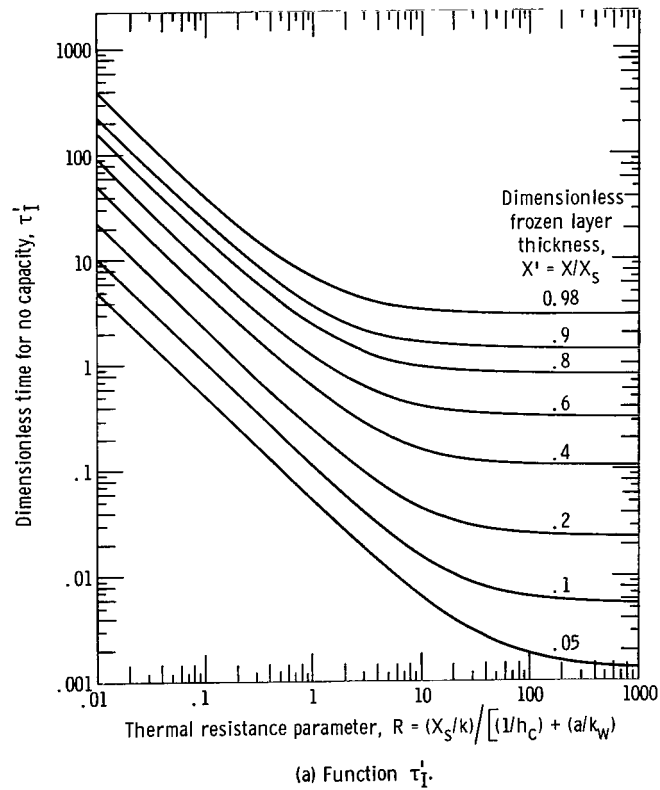


(a) Function $\tau'_I$.

Figure 2. - Graphical presentation of functions of $X'$ and $R$ contained in equation (26) for frozen layer growth.
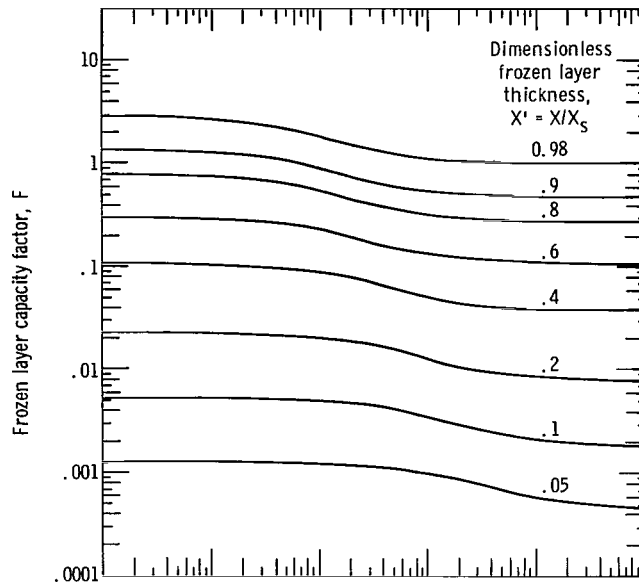
(b) Relation between the frozen layer capacity factor $F$ and the thermal resistance parameter $R$.
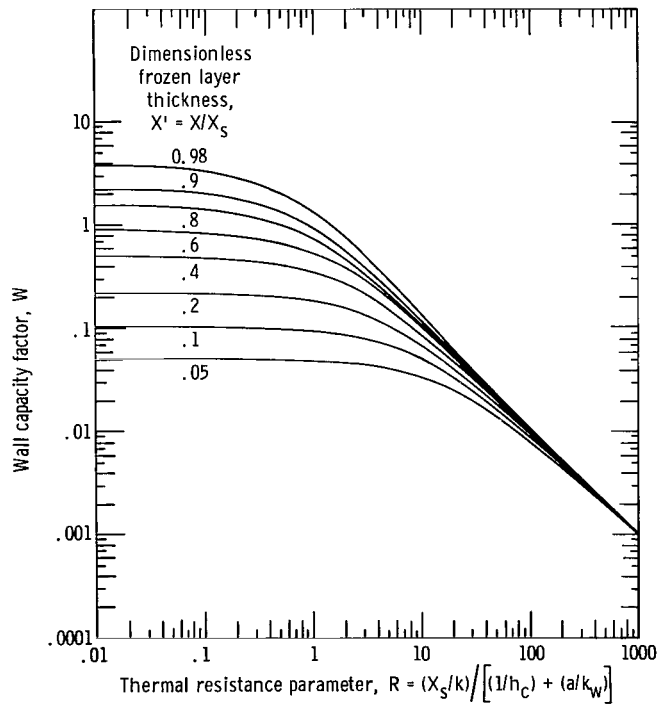


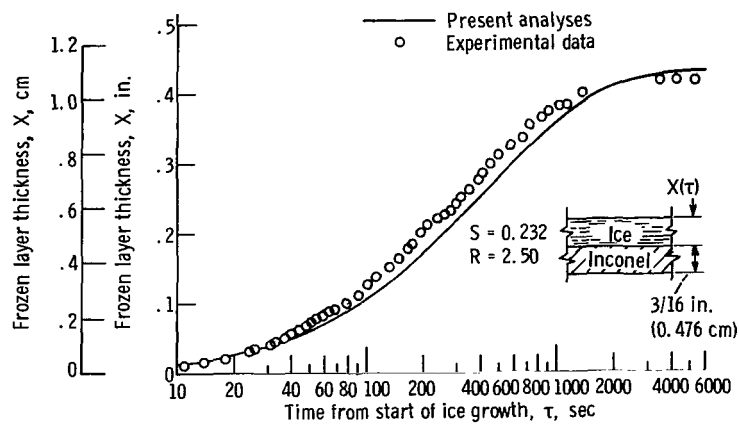(c) Relation between the wall heat capacity factor $W$ and the thermal resistance parameter $R$.

Figure 2. – Concluded.

Figure 3. - Comparison of analysis with experimental data for ice forming on an Inconel plate 3/16 inch (0.476 cm) thick. Water temperature, $T_\ell$ = 53.5° F (285.1° K); coolant temperature, $T_c$ = -40.5° F (232.9° K); heat-transfer coefficients, $h_\ell$ = 92 Btu/(hr)(ft$^2$)(°F) (522 W/(m$^2$)(°K)), and $h_c$ = 118 Btu/(hr)(ft$^2$)(°F) (670 W/(m$^2$)(°K)).

(a) Water temperature, 53.5° F (285.1° K); coolant temperature, -40.5° F (232.9° K).

(b) Water temperature, 53.5° F (285.1° K); coolant temperature, -400° F (33.2° K).

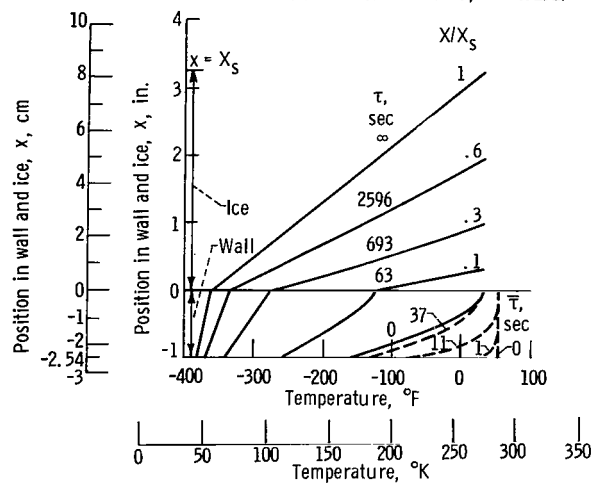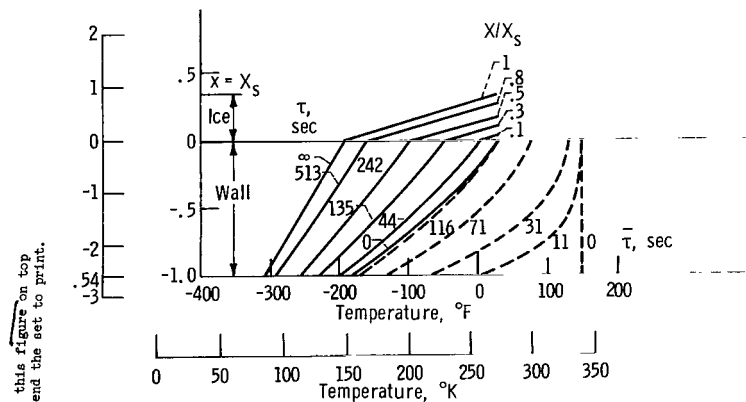(c) Water temperature, 150° F (338.7° K); coolant temperature, -400° F (33.2° K).
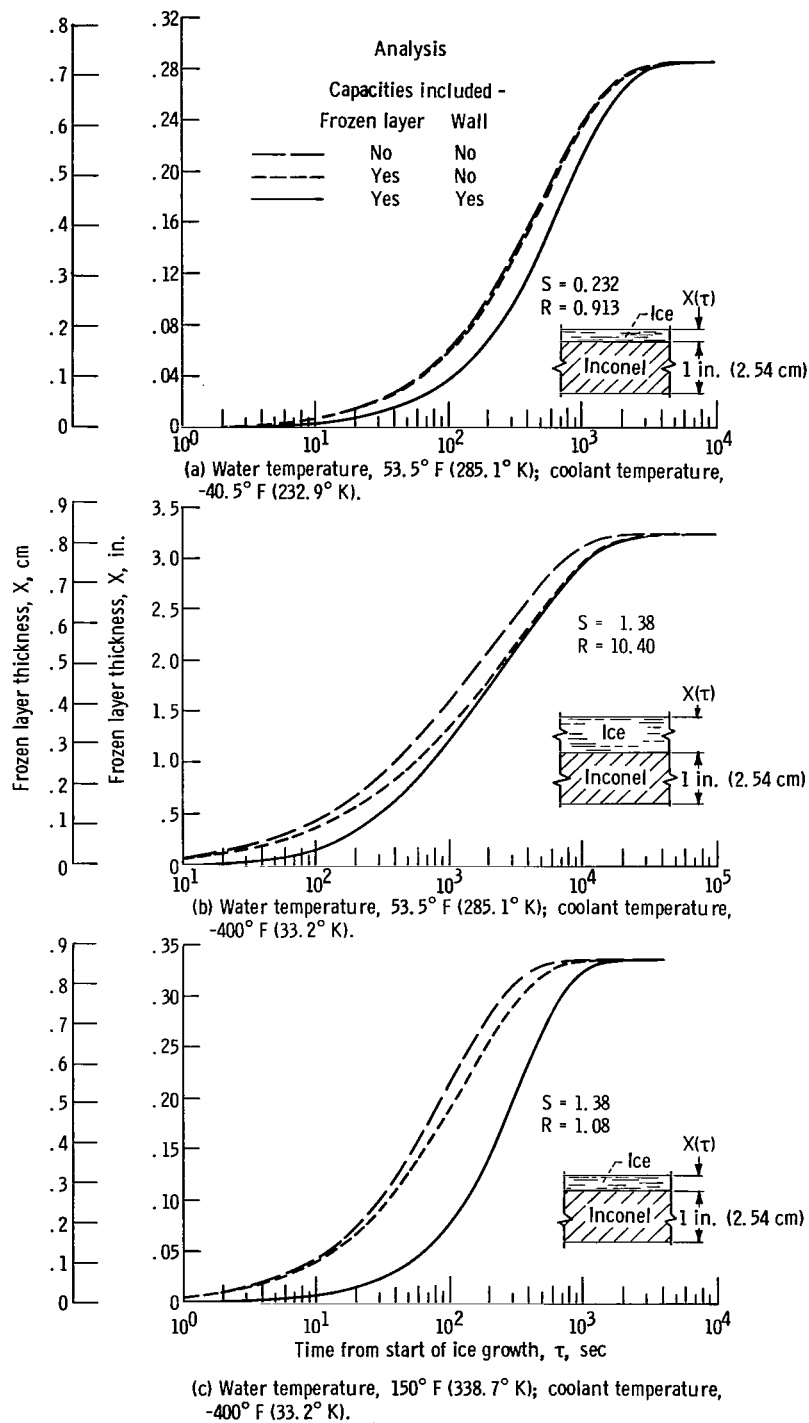
Figure 4. – Effect of heat capacity in prediction of ice layer growth on a thick plate. Heat-transfer coefficients:
$h_\ell$ = 92 Btu/(hr)(ft$^2$)(°F) (522 W/(m$^2$)(°K));
$h_c$ = 118 Btu/(hr)(ft$^2$)(°F) (670 W/(m$^2$)(°K)).
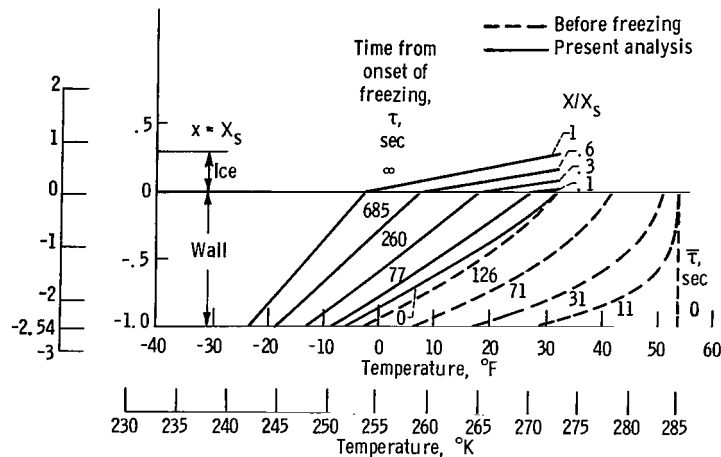
(a) Water temperature, 53.5° F (285.1° K); coolant temperature, -40.5° F (232.9° K). R = 0.913, S = 0.232.



(b) Water temperature, 53.5° F (285.1° K); coolant temperature, -400° F (33.2° K). R = 10.40, S = 1.38.



(c) Water temperature, 150° F (338.7° K); coolant temperature, -400° F (33.2° K). R = 1.08, S = 1.38.

Figure 5. - Transient temperature distributions in wall and frozen layer. Heat-transfer coefficients: $h_\ell$ = 92 Btu/(hr)(ft$^2$)(°F) (522 W/(m$^2$)(°K)); $h_c$ = 118 Btu/(hr)(ft$^2$)(°F) (670 W/(m$^2$)(°K)).

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