Technical Report No. 111

ON SOME MIXED BOUNDARY VALUE PROBLEMS FOR SPHERICALLY ISOTROPIC ELASTIC MEDIA

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April 1967

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In this paper, some mixed boundary value problems for spherically isotropic elastic medium are discussed. General solutions to these problems are obtained in terms of infinite series. Mixed boundary values on the surface give rise to triple series equations. They are solved by reducing to a single Fredholm integral equation.

INTRODUCTION

The boundary value problems for aelotropic media have been considered by many investigators in the past[1,2]*. Interests have been focused on spherically isotropic materials. For such materials the physical properties are transversely isotropic about the radius vector. Recently Chen[3] solved some problems concerning spherically isotropic elastic medium bounded by two concentric spherical surfaces. His considerations were particularly centered on several axi-symmetric problems such as a concentrated force in an infinite medium, the stress concentration due to spherical cavity and a steadily rotating shell. In his paper only one type of boundary value problems wherein stress components are prescribed on the boundary has been dealt with. The results are based essentially on the solutions similar to those obtained by Hu[4] who has shown that the general solutions to the equilibrium equations may be expressed in terms of spherical harmonics.

In this report an attempt is made to investigate general mixed boundary value problems for a spherically isotropic elastic medium bounded by spherical surfaces. For these types of problems a set of triple infinite series equations is generally involved and may be reduced to a double series equations as special cases. A considerable amount of effort has been given in solving these equations by Collins[5] and others[6]. Results have also been applied to the study of boundary value problems. The purpose of this investigation is to show

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*Numbers in the square brackets refer to references at the end.

that certain equilibrium problems with prescribed boundary displacements and stresses give rise to the triple series equations which can be solved as a single Fredholm integral equation.

EQUILIBRIUM PROBLEM

The fundamental system of field equations in elasticity consists of the linearized strain-displacement relations, stress-strain relations and equilibrium equations together with the proper boundary conditions. Using spherical coordinate system (r, θ, ϕ) the stress-strain relations for a spherically isotropic and homogeneous medium, symmetric about the radius vector, may be given as follows:

$$\begin{aligned}
\sigma_{\theta\theta} &= C_{11}\, e_{\theta\theta} + C_{12}\, e_{\phi\phi} + C_{13}\, e_{rr} \\
\sigma_{\phi\phi} &= C_{12}\, e_{\theta\theta} + C_{11}\, e_{\phi\phi} + C_{13}\, e_{rr} \\
\sigma_{rr} &= C_{13}\, e_{\theta\theta} + C_{13}\, e_{\phi\phi} + C_{33}\, e_{rr} \\
\sigma_{r\theta} &= C_{44}\, e_{r\theta} \\
\sigma_{r\phi} &= C_{44}\, e_{r\phi} \\
\sigma_{\theta\phi} &= \tfrac{1}{2}(C_{11} - C_{12})\, e_{\theta\phi}
\end{aligned} \tag{1}$$

where the e quantities represent the strains in spherical coordinates and the elasticity coefficients are shown as the various C quantities. The strain-displacement and equilibrium equations in spherical coordinates may be found in [2]. We express the displacements (u_r, u_θ, u_ϕ) along the (r, θ, ϕ) directions in terms of the potential functions $\Phi_n(r, \theta, \phi)$ and $\Psi_n(r, \theta, \phi)$ as

$$\begin{aligned}
u_\theta &= \sum_{n=0}^{\infty} \frac{1}{r} \frac{\partial \Phi_n}{\partial \theta} + \sum_{n=1}^{\infty} \frac{1}{r \sin\theta} \frac{\partial \Psi_n}{\partial \phi} \\
u_\phi &= \sum_{n=0}^{\infty} \frac{1}{r \sin\theta} \frac{\partial \Phi_n}{\partial \phi} - \sum_{n=1}^{\infty} \frac{1}{r} \frac{\partial \Psi_n}{\partial \theta} \\
u_r &= \sum_{n=0}^{\infty} k_n \frac{\partial \Phi_n}{\partial r}
\end{aligned} \tag{2}$$

and assume that

$$\Phi_n = r^{\nu_n + \frac{1}{2}} \; Y_n(\theta, \phi)$$

$$\Psi_n = r^{\lambda_n + \frac{1}{2}} \; Y_n(\theta, \phi) \tag{3}$$

where $Y_n(\theta, \phi)$ is a surface harmonic of order n. As shown in [3] the equilibrium equations in the absence of body forces are then satisfied by Φ_n and Ψ_n if

$$\nu_n^4 - 2a_n \nu_n^2 + b_n = 0 \tag{4}$$

where

$$\begin{aligned} a_n = \frac{1}{2C_{33}C_{44}} \Big\{ & n(n+1) \left[C_{44}^2 + C_{11}C_{33} - (C_{13}+C_{44})^2 \right] \\ & + \tfrac{1}{2} C_{33}C_{44} + C_{33}(2C_{44} + C_{12} - C_{11}) \\ & + 2C_{44}(C_{11} + C_{12} - C_{13}) \Big\} \\ b_n = \frac{1}{C_{33}C_{44}} \Big\{ & \left[C_{44}\, n(n+1) + 2(C_{11}+C_{12}-C_{13}) + \tfrac{1}{4} C_{33} \right] \times \\ & \left[C_{11}\, n(n+1) + 2C_{44} + C_{12} - C_{11} + \tfrac{1}{4} C_{33} \right] \\ & - n(n+1) \left[2C_{44} + C_{12} + C_{11} - \tfrac{1}{2}(C_{13}+C_{44})^2 \right] \Big\} \end{aligned} \tag{5}$$

Equation (4) has four roots ν_{ni} $(i = 1, 2, 3, 4)$. For convenience, they are assumed to be distinct and real. Similarly λ_n has two values λ_{ni} $(i = 1, 2)$ which satisfies

$$\lambda_n^2 - \left[\frac{C_{11} - C_{12}}{2C_{44}} (n-1)(n+2) + \frac{9}{4} \right] = 0 \tag{6}$$

One important result from the above equations is that for large n equation (4) becomes

$$C_{44} C_{33} \nu_n^4 + \left[C_{13}(2C_{44} + C_{13}) - C_{11}C_{33} \right] n^2 \nu_n^2 + C_{11}C_{44} n^4 = 0 \tag{7}$$

and (6) becomes

$$\lambda_n^2 - \frac{C_{11} - C_{12}}{2\, C_{44}} n^2 = 0 \tag{8}$$

Therefore from (7) and (8) we can expect that ν_n and λ_n are proportional to n .

Without losing generality, let us specialize to the problems with prescribed boundary conditions symmetrical with respect to the z -axis. As a consequence the surface harmonic $Y_n\,(\theta, \phi)$ in (3) becomes a Legendre polynomial $P_n(\cos\theta)$. Using the notation

$$\xi = \cos\theta \qquad ; \qquad P_n'(\xi) = \frac{dP_n(\xi)}{d\xi} \tag{9}$$

the displacement components can be expressed as:

$$u_\theta = -\sum_{n=1}^{\infty} \sum_{i=1}^{4} A_{ni}\, r^{\nu_{ni} - \frac{1}{2}} \sqrt{1 - \xi^2}\; P_n'(\xi)$$

$$u_\phi = \sum_{n=1}^{\infty} \sum_{i=1}^{4} B_{ni}\, r^{\lambda_{ni} - \frac{1}{2}} \sqrt{1 - \xi^2}\; P_n'(\xi)$$

$$u_r = \sum_{n=0}^{\infty} \sum_{i=1}^{4} K_{ni}\, A_{ni}\, r^{\nu_{ni} - \frac{1}{2}}\; P_n(\xi) \tag{10}$$

where

$$K_{ni} = \frac{C_{11}\, n(n+1) + 2\, C_{44} + C_{12} - C_{11} - C_{44}\,(\nu_{ni}^2 - \frac{1}{4})}{(C_{13} + C_{44})(\nu_{ni} - \frac{1}{2}) + 2\, C_{44} + C_{11} + C_{12}}$$

$$(i = 1, 2, 3, 4) \tag{11}$$

The stress components are expressible as follows:

$$\sigma_{rr} = \sum_{n=0}^{\infty} \sum_{i=1}^{4} A_{ni} \left\{ c_{33} K_{ni} (\nu_{ni} - \tfrac{1}{2}) + c_{13} \left[2 K_{ni} - n(n+1) \right] \right\} \times r^{\nu_{ni} - \frac{3}{2}} P_n(\xi)$$

$$\sigma_{\theta\theta} + \sigma_{\phi\phi} = \sum_{n=0}^{\infty} \sum_{i=1}^{4} A_{ni} \left\{ (c_{11} + c_{12}) \left[2 K_{ni} - n(n+1) \right] + 2 c_{13} (\nu_{ni} - \tfrac{1}{2}) K_{ni} \right\} r^{\nu_{ni} - \frac{3}{2}} P_n(\xi)$$

$$\sigma_{\theta\theta} - \sigma_{\phi\phi} = - \sum_{n=0}^{\infty} \sum_{i=1}^{4} A_{ni} (c_{11} - c_{12}) r^{\nu_{ni} - \frac{3}{2}} \times \left[n(n+1) P_n(\xi) - 2 \sqrt{1-\xi^2} P_n'(\xi) \right]$$

$$\sigma_{r\theta} = - \sum_{n=1}^{\infty} \sum_{i=1}^{4} c_{44} A_{ni} \left[K_{ni} + \nu_{ni} - \tfrac{3}{2} \right] r^{\nu_{ni} - \frac{3}{2}} \sqrt{1-\xi^2} P_n'(\xi)$$

$$\sigma_{r\phi} = \sum_{n=1}^{\infty} \sum_{i=1}^{4} c_{44} (\lambda_{ni} - \tfrac{3}{2}) B_{ni} r^{\lambda_{ni} - \frac{3}{2}} \sqrt{1-\xi^2} P_n'(\xi)$$

$$\sigma_{\theta\phi} = \sum_{n=1}^{\infty} \sum_{i=1}^{4} \frac{c_{11} - c_{12}}{2} B_{ni} r^{\lambda_{ni} - \frac{3}{2}} \left[2 \xi P_n'(\xi) + n(n+1) P_n(\xi) \right] \tag{12}$$

where A_{ni} and B_{ni} are constants to be determined from the boundary conditions. Equations (10) and (12) are general expressions, good for any domain.

Now let us consider a general boundary value problem for a solid sphere ($0 \leq r \leq R$) with the prescribed boundary conditions on $r = R$ surface as:

$$u_r(R, \theta, \phi) = \bar{g}(\theta) \qquad (0 \leq \theta < \theta_1 \;,\; \theta_2 < \theta \leq \pi)$$

$$\sigma_{rr}(R, \theta, \phi) = \bar{f}(\theta) \qquad (\theta_1 < \theta < \theta_2)$$

$$\sigma_{r\theta}(R, \theta, \phi) = 0 \qquad (0 \leq \theta \leq \pi) \tag{13}$$

For simplicity, let us assume that $\sigma_{r\phi} = \sigma_{\theta\phi} = 0$ $(0 \leq \theta \leq \pi)$ and $\bar{g}(\theta)$ and $\bar{f}(\theta)$ are sufficiently smooth functions of θ. Then since the solution is required to be bounded everywhere in the region $0 \leq r \leq R$, we choose those A_{ni} and B_{ni} corresponding to terms which involve negative exponents of r, to be zero. We assume also that these terms are given by $i = 3,4$ in ν_{ni} and $i = 2$ in λ_{ni} then set $A_{n3} = A_{n4} = B_{n2} = 0$. Substituting (10) and (12) into (13) we obtain

$$B_{n1} = B_{n2} = A_{n3} = A_{n4} = 0$$

and

$$A_{n2} = -\frac{K_{n1} + \nu_{n1} - \frac{3}{2}}{K_{n2} + \nu_{n2} - \frac{3}{2}} A_{n1} \tag{14}$$

Finally we obtain with some algebraic manipulations,

$$\sum_{n=0}^{\infty} (n+\tfrac{1}{2}) A_n P_n(\xi) = g(\xi) \qquad \begin{matrix}(1 \geq \xi > \alpha) \\ (\beta > \xi \geq -1)\end{matrix}$$
$$\sum_{n=0}^{\infty} (1+H_n) A_n P_n(\xi) = f(\xi) \qquad (\alpha > \xi \geq \beta) \tag{15}$$

where

$$\cos\theta_1 = \alpha, \qquad \cos\theta_2 = \beta$$

$$A_n = -\frac{2A_{n1}}{2n+1}\left[K_{n1} R^{\nu_{n1}-\frac{1}{2}} - K_{n2} R^{\nu_{n2}-\frac{1}{2}} \frac{K_{n1}+\nu_{n1}-\frac{3}{2}}{K_{n2}+\nu_{n2}-\frac{3}{2}}\right]$$

$$1+H_n = -\frac{2n+1}{2}\Big\{R^{\nu_{n1}-\frac{3}{2}}\left[C_{33} K_{n1}(\nu_{n1}-\tfrac{1}{2}) + C_{13}\left[2K_{n1} - n(n+1)\right]\right]$$
$$- R^{\nu_{n2}-\frac{3}{2}}\left[C_{33} K_{n2}(\nu_{n2}-\tfrac{1}{2}) + C_{13}\left[2K_{n2} - n(n+1)\right]\frac{K_{n1}+\nu_{n1}-\frac{3}{2}}{K_{n2}+\nu_{n2}-\frac{3}{2}}\right]\Big\}$$
$$\times\left[K_{n1} R^{\nu_{n1}-\frac{1}{2}} - K_{n2} R^{\nu_{n2}-\frac{1}{2}} \frac{K_{n1}+\nu_{n1}-\frac{3}{2}}{K_{n2}+\nu_{n2}-\frac{3}{2}}\right]^{-1} \tag{16}$$

and

$$f(\xi) = \bar{f}(\theta) \lim_{n\to\infty} (1 + H_n) \quad , \quad g(\xi) = \bar{g}(\theta)$$

Here it can be shown easily through the use of (7), (8) and (11) that for large n, $H_n \to O(\frac{1}{n})$. The system of equations (15) constitutes triple series equations and have been investigated in details by Collins[5]. Following [5] the solution to (15) can be obtained and shown to exist if $H_n \to O(\frac{1}{n})$ for large n. Unfortunately the final form of the solution is quite complicated and evaluation may be very much involved. However, following the approach of [6] the solution to (15) can be obtained conveniently in the form of a single Fredholm integral equation and thus simplify the solution. To do this, let us define a function $\mu(\xi)$ such that

$$\mu(\xi) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) A_n P_n(\xi) \qquad (\alpha > \xi \geq \beta) \tag{17}$$

Multiply both sides of (17) by $P_m(\xi)$ and integrate over $(-1, 1)$ with the help of the orthogonality conditions of Legendre polynomials we get

$$A_n = \int_{\alpha}^{1} g(\xi) P_n(\xi)\, d\xi + \int_{\beta}^{\alpha} \mu(\xi) P_n(\xi)\, d\xi + \int_{-1}^{\beta} g(\xi) P_n(\xi)\, d\xi \tag{18}$$

The use of Mehler - Dirichlet formula

$$P_n(\xi) = \frac{2}{\pi} \int_0^{\theta} \frac{\cos[(n+\frac{1}{2})x]\, dx}{\sqrt{2\cos x - 2\cos\theta}}$$

results in a useful relation as follows:

$$\cos[(n+\tfrac{1}{2})\cos^{-1}\gamma] = -\sqrt{\frac{1-\gamma^2}{2}}\, \frac{d}{d\gamma} \int_{\gamma}^{1} \frac{P_n(x)\, dx}{\sqrt{x-\gamma}} \tag{19}$$

Now multiply both sides of the second equation of (15) by $(x-\gamma)^{-1/2}$ and integrate over $(\gamma, 1)$. Then through the use of (19) we obtain

$$\sum_{n=0}^{\infty} (1+H_n) A_n \cos\left[(n+\tfrac{1}{2})\cos^{-1}\gamma\right] = -\sqrt{\frac{1-\gamma^2}{2}}\,\frac{d}{d\gamma}\int_{\gamma}^{1} \frac{f(\xi)\,d\xi}{\sqrt{\xi-\gamma}} \tag{20}$$

$$(\alpha > \gamma > \beta)$$

Substituting for A_n the values shown in (18) and using a well known relation

$$\sum_{n=0}^{\infty} P_n(\xi)\cos\left[(n+\tfrac{1}{2})\cos^{-1}\gamma\right] = \begin{cases} \dfrac{1}{\sqrt{2(\gamma-\xi)}} & (\gamma > \xi) \\ 0 & (\gamma < \xi) \end{cases} \tag{21}$$

we can reduce (20) to the following form:

$$\int_{-1}^{\beta} \frac{g(\xi)\,d\xi}{\sqrt{2(\gamma-\xi)}} + \int_{\beta}^{\gamma} \frac{\mu(\xi)\,d\xi}{\sqrt{2(\gamma-\xi)}} + \sum_{n=0}^{\infty} H_n \cos\left[(n+\tfrac{1}{2})\cos^{-1}\gamma\right] \times$$

$$\left\{\int_{\beta}^{\alpha} \mu(\xi) P_n(\xi)\,d\xi + \left(\int_{\alpha}^{1} + \int_{-1}^{\beta}\right) g(\xi) P_n(\xi)\,d\xi\right\} = -\sqrt{\frac{1-\gamma^2}{2}}\,\frac{d}{d\gamma}\int_{\gamma}^{1} \frac{f(\xi)\,d\xi}{\sqrt{\xi-\gamma}} \tag{22}$$

This is an Abel type singular integral equation and can be inverted to determine $\mu(\xi)$ to give

$$\mu(\xi) = -\frac{1}{\pi}\,\frac{d}{d\xi}\int_{\beta}^{\xi} \frac{d\gamma}{\sqrt{\xi-\gamma}} \left\{\int_{\beta}^{1} S(\gamma,x)\mu(x)\,dx\right.$$

$$+ \int_{-1}^{\beta} \frac{g(x)\,dx}{\sqrt{\gamma-x}} + \sqrt{1-\gamma^2}\,\frac{d}{d\gamma}\int_{\gamma}^{1} \frac{f(x)\,dx}{\sqrt{x-\gamma}}$$

$$\left. + \left(\int_{\alpha}^{1} + \int_{-1}^{\beta}\right) S(\gamma,x)\, g(x)\,dx\right\} \tag{23}$$

where

$$S(\gamma,\xi) = 2 \sum_{n=0}^{\infty} H_n P_n(\xi) \cos\left[(n+\tfrac{1}{2})\cos^{-1}\gamma\right] \quad (24)$$

If we denote

$$G(\gamma) = \int_{-1}^{\beta} \frac{g(x)\,dx}{\sqrt{\gamma - x}} + \sqrt{1-\gamma^2}\,\frac{d}{d\gamma}\int_{\gamma}^{1} \frac{f(x)\,dx}{\sqrt{x-\gamma}}$$
$$+ \left(\int_{\alpha}^{1} + \int_{-1}^{\beta} \right) S(\gamma, x)\, g(x)\, dx \quad (25)$$

Then (23) becomes

$$\mu(\xi) = -\frac{1}{\pi}\,\frac{d}{d\xi}\int_{\beta}^{\xi} \frac{d\gamma}{\sqrt{\xi-\gamma}}\left[G(\gamma) + \int_{\beta}^{\alpha} S(\gamma,x)\,\mu(x)\,dx\right] \quad (26)$$

This is a Fredholm integral equation of the second kind and can be solved easily. It can be readily shown that the kernel $S(\gamma,\xi)$ is regular if $H_n \to O(\frac{1}{n})$ and the equation has a unique solution. Knowing $\mu(\xi)$ it is a matter of simple computation of A_n and all the required quantities.

The equation (23) or (26) can further be simplified if some more information about the values of θ_1 and θ_2 is known. For instance, if $\theta_1 + \theta_2 = \pi$ then $\alpha = -\beta$ and

$$G(\gamma) = \int_{-1}^{\beta} \frac{g(\xi)\,d\xi}{\sqrt{\gamma-\xi}} + \sqrt{1-\gamma^2}\,\frac{d}{d\gamma}\int_{\gamma}^{1}\frac{f(\xi)\,d\xi}{\sqrt{\xi-\gamma}}$$
$$+ \int_{-1}^{\beta}\left[S(\gamma,\xi)\,g(\xi) + S(\gamma,-\xi)\,g(-\xi)\right]d\xi$$

$$\mu(\xi) = -\frac{1}{\pi}\frac{d}{d\xi}\int_{\beta}^{\xi}\frac{d\gamma}{\sqrt{\xi-\gamma}}\left[G(\gamma) - \int_{-\beta}^{\beta} S(\gamma,x)\mu(x)\,dx\right]$$

$$A_n = \int_{-1}^{\beta}\left[g(\xi)+(-1)^n g(-\xi)\right] P_n(\xi)\,d\xi - (-1)^n\int_{-\beta}^{\beta}\mu(\xi)P_n(\xi)\,d\xi \tag{27}$$

also

$$S(\gamma,-\xi) = 2\sum_{n=0}^{\infty}(-1)^n P_n(\xi)\cos\left[(n+\tfrac{1}{2})\cos^{-1}\gamma\right]$$

In addition, if $g(\xi)$ is an even function i.e. $g(\xi) = g(-\xi)$ then the last term in $G(\gamma)$ will have only even terms and thus (27) reduces to the following set of equations:

$$G(\gamma) = \int_{-1}^{\beta} g(\xi)\left[\frac{1}{\sqrt{\gamma-\xi}} + S_1(\gamma,\xi)\right]d\xi + \sqrt{1-\gamma^2}\,\frac{d}{d\gamma}\int_{\gamma}^{1}\frac{f(\xi)\,d\xi}{\sqrt{\xi-\gamma}}$$

$$\mu(\xi) = -\frac{1}{\pi}\frac{d}{d\xi}\int_{\beta}^{\xi}\frac{dx}{\sqrt{\xi-x}}\left[G(x) - \int_{0}^{\beta} S_1(x,\gamma)\mu(\gamma)\,d\gamma\right]$$

$$A_{2n} = 2\int_{-1}^{\beta} g(\xi)\,P_{2n}(\xi)\,d\xi - 2\int_{0}^{\beta}\mu(\xi)\,P_{2n}(\xi)\,d\xi \tag{28}$$

and $$A_{2n+1} = 0$$

where $$S_1(\gamma,\xi) = S(\gamma,\xi) + S(\gamma,-\xi)$$

If $g(\xi)$ is an odd function, $g(\xi) = -g(-\xi)$, then we will get equations similar to (28) for A_{2n+1}. It can be seen that $A_{2n}=0$ and $S_1(\gamma,\xi)$ is to be replaced by $S_2(\gamma,\xi)$ defined as

$$S_2(\gamma,\xi) = S(\gamma,\xi) - S(\gamma,-\xi) \tag{29}$$

Furthermore, let us now consider $\theta_1=0$. Then $\alpha=1$ and the system of triple series equations reduces to a dual series system. The solution to these equations can be obtained by substituting $\alpha=1$ in (25) and (26) as a special case.

If, instead of (13), the boundary conditions are prescribed on $r=R$ for the shear components as the following:

$$u_\theta(R,\theta) = \bar{q}(\theta) \qquad (0\le\theta<\theta_1\,;\ \theta_2<\theta\le\pi)$$

$$\sigma_{r\theta}(R,\theta) = \overline{m}(\theta) \qquad (\theta_1<\theta<\theta_2)$$

and

$$\sigma_{rr}(R,\theta) = 0 \qquad (0\le\theta\le\pi) \tag{30}$$

where $\bar{q}(\theta)$ and $\overline{m}(\theta)$ are sufficiently smooth functions of θ.

For simplicity, we consider also that

$$\sigma_{r\phi} = \sigma_{\theta\phi} = 0 \qquad (0\le\theta\le\pi)$$

Substituting (10) and (12) into (30) we obtain,

$$A_{n4} = \frac{c_{13}\left[2K_{n3}-n(n+1)\right]+c_{33}K_{n3}\left(\nu_{n3}-\frac{1}{2}\right)}{c_{13}\left[2K_{n4}-n(n+1)\right]+c_{33}K_{n4}\left(\nu_{n4}-\frac{1}{2}\right)}A_{n3}$$

$$B_{n1} = B_{n2} = 0$$

be seen that $A_{2n} = 0$ and $S_1(\gamma, \xi)$ is to be replaced by $S_2(\gamma, \xi)$ defined as

$$S_2(\gamma, \xi) = S(\gamma, \xi) - S(\gamma, -\xi) \tag{29}$$

Furthermore, let us now consider $\theta_1 = 0$. Then $\alpha = 1$ and the system of triple series equations reduces to a dual series system. The solution to these equations can be obtained by substituting $\alpha = 1$ in (25) and (26) as a special case.

If, instead of (13), the boundary conditions are prescribed on $r = R$ for the shear components as the following:

$$u_\theta(R, \theta) = \bar{q}(\theta) \qquad (0 \leq \theta < \theta_1 ; \ \theta_2 < \theta \leq \pi)$$

$$\sigma_{r\theta}(R, \theta) = \bar{m}(\theta) \qquad (\theta_1 < \theta < \theta_2)$$

and

$$\sigma_{rr}(R, \theta) = 0 \qquad (0 \leq \theta \leq \pi) \tag{30}$$

where $\bar{q}(\theta)$ and $\bar{m}(\theta)$ are sufficiently smooth functions of θ.

For simplicity, we consider also that

$$\sigma_{r\phi} = \sigma_{\theta\phi} = 0 \qquad (0 \leq \theta \leq \pi)$$

Substituting (10) and (12) into (30) we obtain,

$$A_{n4} = \frac{c_{13}\left[2K_{n3} - n(n+1)\right] + c_{33} K_{n3}(\nu_{n3} - \frac{1}{2})}{c_{13}\left[2K_{n4} - n(n+1)\right] + c_{33} K_{n4}(\nu_{n4} - \frac{1}{2})} A_{n3}$$

$$B_{n1} = B_{n2} = 0$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{2} R^{\nu_{ni}-\frac{1}{2}} A_{ni} P_n'(\xi) = \frac{\bar{q}(\theta)}{\sqrt{1-\xi^2}} \qquad \begin{matrix}(0 \le \theta < \theta_1) \\ (\theta_2 < \theta \le \pi)\end{matrix}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{2} c_{44}\left[K_{ni} + \nu_{ni} - \tfrac{3}{2}\right] R^{\nu_{ni}-\frac{3}{2}} P_n'(\xi) = \frac{\overline{m}(\theta)}{\sqrt{1-\xi^2}} \qquad (31)$$

$$(\theta_1 < \theta < \theta_2)$$

Integrating both sides of the last two equations with respect to ξ we can write after some algebraic manipulations

$$\sum_{n=1}^{\infty} (n+\tfrac{1}{2}) \bar{A}_n P_n(\xi) = q(\xi) \qquad \begin{matrix}(1 \ge \xi > \alpha) \\ (\beta > \xi \ge -1)\end{matrix}$$

$$\sum_{n=1}^{\infty} (1+\overline{H}_n) \bar{A}_n P_n(\xi) = m(\xi) \qquad (\alpha > \xi > \beta)$$

$$(32)$$

In the above equations,

$$q(x) = \int_1^x \frac{\bar{q}(\theta)\, d\xi}{\sqrt{1-\xi^2}} + q_0$$

$$m(x) = \Omega \int_\beta^x \frac{\overline{m}(\theta)\, d\xi}{\sqrt{1-\xi^2}} + m_0 \,, \quad (\xi = \cos\theta) \qquad (33)$$

where $\Omega = \lim_{n\to\infty}(1+\overline{H}_n)$. q_0 and m_0 are arbitrary constants and must be chosen consistently. One of the constants can be chosen to be zero since otherwise it contributes only to the rigid displacement and does not influence the stress system. The other constant can be chosen such that the displacement u_θ on the surface is continuous. Here $\overline{H}_n$ is $O(\frac{1}{n})$ for large n and is defined as

$$1+\overline{H}_n = \frac{2n+1}{2}\left[R^{\nu_{n3}-\frac{3}{2}} c_{44}(K_{n3}+\nu_{n3}-\tfrac{3}{2})A_{n3} + c_{44}R^{\nu_{n4}-\frac{3}{2}}(K_{n4}+\nu_{n4}-\tfrac{3}{2})A_{n4}\right] \times \left[R^{\nu_{n3}-\frac{1}{2}}A_{n3} + R^{\nu_{n4}-\frac{1}{2}}A_{n4}\right]^{-1}$$

and

$$\overline{A}_n = \frac{2}{2n+1}\left[R^{\nu_{n3}-\frac{1}{2}}A_{n3} + R^{\nu_{n4}-\frac{1}{2}}A_{n4}\right] \qquad (34)$$

Equations in (32) are similar to (15) and can be solved easily as before.

A simple but important problem when only σ_{rr} and $\sigma_{r\theta}$ are prescribed on the surface can be considered again as a special case of (13) and (30). The solution in this case is directly obtained if the prescribed stresses are expanded in terms of Legendre polynomials. In this way, we obtain a system of simple algebraic equations for the unknown constants A_{ni} which can be readily solved.

So far we have discussed the mixed boundary value problems for a solid sphere ($0 \leq r \leq R$). In the case of infinite medium with spherical cavity ($R \leq r < \infty$) the same procedure can be followed with the only difference that we choose A_{n1}, A_{n2}, rather than A_{n3}, A_{n4}, and B_{ni} to be zero. This corresponds to assuming the solutions Φ_n in (3) such that Φ_n goes to zero as $r \to \infty$. The problem ultimately reduces to solving a triple series equations similar to (15) or (32). Again the solution may be obtained in terms of Fredholm integral equation of the second kind without any difficulty.

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ACKNOWLEDGMENT

It is a great pleasure to thank Professor C. C. Hsiao for his help and interest during this investigation.