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THE HIGHER ORDER THEORY OF SUPERCAVITATING CASCADES WITH CONSTANT PRESSURE CAMBERED BLADES

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NOTATION

a_o, a_1, a_3	Points in the ζ plane corresponding to the end of foil, and the end points of upper and lower cavity respectively
c	Chord length of foil
C_L	Lift coefficient defined by [18]
C_D	Drag coefficient defined by [16]
d	Distance between leading edges of adjacent foils in the cascade
k	Strength of a singularity at the leading edge
ℓ	Cavity length
$P_{-\infty}, P_c$	Pressure at $x = -\infty$ and on the cavity respectively
Q	A quantity defined in [5]
$q, q_I, q_{II}, q_1, q_2, q_a, q_b$	Speed defined in [19], and [21]
u, v	x,y components of perturbation velocity respectively
U	Speed at $x = -\infty$
$W = \varphi + i\psi$	Complex potential
$z = x + yi$	Coordinate in the physical plane
γ	Stagger angle defined in Figure 1
$\theta, \theta_I, \theta_{II}, \theta_1, \theta_2, \theta_a, \theta_b$	Flow angle defined in [20] and [22]
c	Cavitation number defined by [1]
ρ	Density of water

$\lambda_o, \lambda_1, \lambda_2, \lambda_3$	Constants defined in [8]
$\zeta = \xi + \eta i$	Coordinates in the transformed plane

Superscript

(n)	Quantity correct up to nth order

INTRODUCTION

When the hub to diameter ratio of axial flow pumps is large, the hydrodynamics of such machines may be approximated by that of two-dimensional cascades. The recent development of high rotational speed rocket pumps necessitates the study of supercavitating cascades.

Both linear [Acosta (1960), Cohen and Sutherland (1958)] and nonlinear [Betz and Petersohn (1931)] theories for flat plate cascades with infinitely long cavities have been derived. For either large solidities, large stagger angles, or large angles of attack, an important discrepancy is known to exist between the linear and the nonlinear theories.

For cambered foils, the exact nonlinear theory is quite complicated to derive since we need the solution of a difficult nonlinear integral equation. The corresponding case of flat plates is simpler and has been derived [Sedov (1965), Jacobsen (1964)]. A linearized cascade theory for circular arc foils with infinite cavities has been given by Acosta (1960).

Recently, a linear cascade theory for constant pressure camber foils with finite cavities has been solved (Yim 1967). One of the advantages for the constant pressure camber foils lies not only in simplifications introduced in application of the linear theory but also in corresponding simplications in the appropriate nonlinear theory for the cascade characteristics. In this report, the solution is found for the nonlinear problem for supercavitating, constant pressure cambered cascades with finite cavities; the second order solution for the same problem is found to be simple, so that a second order correction can be made only from the use of results obtained in the first order theory. For the mathematical model of the collapse of the cavity, the double spiral vortex model (Tulin 1964) is adopted as in the case of the linear theory (Yim 1967).

NONLINEAR PROBLEM

We consider a potential flow through a two-dimensional cascade in the $z = x + iy$ plane as shown in Figure 1a with the stagger angle γ and the solidity c/d. A foil is represented by OA_4A_0 which is unknown a priori. On OA_4 the pressure is uniform or the speed q is constant, from the Bernoulli equation. The length of A_4A_0 is small and the pressure there changes continuously from that of OA_4 to that of the cavity where the cavitation number σ is defined as

$$\sigma = \frac{P_{-\infty} - P_c}{\frac{1}{2}\rho U^2} \qquad [1]$$

$P_{-\infty}$: the pressure at $x = -\infty$

U : the speed at $x = -\infty$

P_c : the pressure at the cavity.

From the Bernoulli equation, on the cavity

$$\sigma = \frac{q_c^{\,2}}{U^2} - 1$$

or

$$\frac{q_c}{U} = \sqrt{\sigma + 1} \qquad [2]$$

If we use the double spiral vortex model (Tulin 1964), the pressure on the wake is constant, or the speed is equal to a constant q_∞ (the speed at $x = \infty$) which will be determined from the conditions, at $x = -\infty$ and of mass continuity between the inlet and the outlet of cascades.

We consider the complex potential

$$\frac{W}{U} = \frac{\varphi}{U} + \frac{i\psi}{U} \qquad [3]$$

which has a dimension of length.

Then the complex velocity is

$$\frac{w}{U} = \frac{d\,\frac{W}{U}}{dz} = \frac{u}{U} - \frac{iv}{U} = \frac{q}{U}\,e^{-i\theta} \qquad [4]$$

The cascade in the W/U plane corresponding to that in the physical (z) plane is shown in Figure 1b, with the same notations for points corresponding to those in the physical plane.

The stagger angle of the cascade in the W/U plane is exactly the same as that in the z plane for the following reasons. In Figure 1a the flow at $x = -\infty$ is uniform. Suppose $O'D_\infty$ and OD_∞ in Figure 1b are streamlines from the leading edges of foils. Thus if $\psi(D_\infty) = 0$, then

$$\psi(D_\infty') = Ud \cos \gamma$$

If

$$\varphi(D_\infty) = \varphi(E_\infty) = 0$$

where the x coordinates of D_∞ and E_∞ are the same. Then

$$\varphi(D_\infty') = U \cdot \overline{E_\infty D_\infty}' = Ud \cdot \sin \gamma$$

Because of the periodicity of the flow

$$\varphi(O) - \varphi(D_\infty) = \varphi(O') - \varphi(D_\infty')$$

Therefore

$$\varphi(O) = \varphi(O') - Ud \sin \gamma$$

This proves that the stagger angle $\angle Y\ O\ O' = \gamma$.

Following the streamline theory by Kirchoff and Helmholtz we consider

$$\frac{Q}{U} = \log \frac{W}{U} = \log \frac{q}{U} - i\theta \qquad [5]$$

which is an analytic function of W/U in the W/U plane except at the singularities. From now on we use for convenience W, φ, ψ, Q, q, u and v for the same physical quantities divided by U. For the boundary conditions all the values of log q are given on $\psi = \pm 0$, $\varphi > 0$ and $\varphi = \pm\infty$ corresponding to those in the physical (z) plane.

To solve this problem, we use a conformal transformation

$$W = e^{i\gamma} \log\left(1 - \zeta\, e^{i\left(\frac{\pi}{2} - \gamma\right)}\right) + e^{-i\gamma} \log\left(1 - \zeta\, e^{-i\left(\frac{\pi}{2} - \gamma\right)}\right) \qquad [6]$$

$$\frac{d\varphi}{d\xi} = \frac{2\xi \cos\gamma}{1 - 2\xi \sin\gamma + \xi^2} \qquad [7]$$

which is in general used for cascade theory [Acosta (1960), Cohen and Sutherland (1958), Yim (1967)]. The transformation [6] transforms all the foil-cavity-wake in the W plane onto the real axis of ζ plane with branch points at $\zeta = e^{i(\pi/2-\gamma)}$ and $\zeta = \infty\, e^{i(\pi/2-\gamma)}$ as shown in Figure 1c with the same notation as in the W plane for the corresponding points. The boundary conditions in the ζ plane are as follows

$$\log q = \lambda_1 \equiv \log \lambda_o \quad \text{on} \quad 0 > \xi > -a_4$$

$$\log q = \lambda_2 + \frac{\lambda_1 - \lambda_2}{a_o - a_4} (\xi + a_o) \qquad [8]$$

$$\text{on} \quad -a_4 > \xi > -a_o$$

$$\log q = \lambda_2 \equiv \frac{1}{2} \log (1 + \sigma)$$

$$\text{on} \quad -a_o > \xi > -a_3$$

$$\text{and on} \quad 0 < \xi < a_1$$

$$\log q = \lambda_3 \equiv \log q_\infty \text{ on } \quad a_1 < \xi$$

$$\text{and on } -a_3 > \xi$$

$$\log q = 0 \text{ and } \theta = 0 \quad \text{at} \quad \zeta = e^{i\left(\frac{\pi}{2} - \gamma\right)}$$

Thus, all the real part of an analytic function Q is given on the φ axis in the W plane, and we have

$$Q - \lambda_3 + i\theta_\infty = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{Re(Q - \lambda_3)}{t - \zeta} dt$$

$$= \frac{1}{\pi i} \left[\log \left\{ \frac{(\zeta + a_o)^{\overline{\lambda_2}} \zeta^{\overline{\lambda_1}} (a_1 - \zeta)^{\overline{\lambda_2}}}{(\zeta + a_3)^{\overline{\lambda_2}} (\zeta + a_4)^{\overline{\lambda_1}} \zeta^{\overline{\lambda_2}}} \right\} \right.$$

$$+ \left\{ \overline{\lambda_2} + \frac{\overline{\lambda_1} - \overline{\lambda_2}}{a_o - a_4} (\zeta + a_o) \right\} \log \frac{a_4 + \zeta}{a_o + \zeta}$$

$$\left. + \overline{\lambda_1} - \overline{\lambda_2} + \frac{k}{\zeta} \right] \qquad [9]$$

where

$\overline{\lambda_1} = \lambda_1 - \lambda_3$

$\overline{\lambda_2} = \lambda_2 - \lambda_3$

θ_∞ is the exit flow angle at $x = \infty$, and

k is an arbitrary constant related to the leading edge condition.

Before applying the condition at $\zeta = e^{i(\pi/2-\gamma)}$ we consider the continuity condition between the inlet and the outlet of the cascade. If we consider the uniformity of w along $z = \pm\infty + r\, e^{i(\pi/2-\gamma)}$ in the physical plane it can be easily shown from the mass continuity

$$\cos\gamma = q_\infty \cos(\gamma + \theta_\infty)$$

or

$$q_\infty = \frac{1}{\cos\theta_\infty - \sin\theta_\infty \tan\gamma} \qquad [10]$$

Now to apply the condition at $\zeta = e^{i(\pi/2-\gamma)}$ in [8], we substitute $\zeta = e^{i(\pi/2-\gamma)}$ in the right hand side of [9], put

$$Q\left(e^{i\left(\frac{\pi}{2} - \gamma\right)}\right) = 0 \qquad [11]$$

and we compare the real and the imaginary part respectively. Thus we obtain two simultaneous equations, say [11-a] and [11-b]. If we apply the condition of uniformity at $x = \infty$ or $|\zeta| = \infty$,

$$\lim_{|\zeta| \to \infty} \left|\frac{dQ}{d\zeta}\right| = 0 \qquad [12]$$

Thus we have four simultaneous equations for σ, q_∞, θ_∞ and a_3 with a given set of γ, a_o and a_1 which are related to a solidity and the length of cavity. a_4 is arbitrary.

For a relation between the W plane and the z plane we use

$$w(\zeta) = e^{Q(\zeta)} = \frac{dW(\zeta)}{d\zeta}\frac{d\zeta}{dz}$$

or

$$dz(\zeta) = \frac{\frac{dW}{d\zeta}\, d\zeta}{w(\zeta)} \qquad [13]$$

Thus by an integration of [13] along ξ axis we obtain

$$x(\xi, 0) = \int_0^{\xi} \frac{\operatorname{Re} w(\xi, 0) 2\xi \cos\gamma}{q^2\{1 - 2\xi \sin\gamma + \xi^2\}}\, d\xi \qquad [14]$$

From here we obtain the points on the x axis corresponding to a_i on ξ axis.

For the foil and cavity shapes, we have

$$y(\xi, 0) = \int_0^{\xi} \frac{\operatorname{Im} w(\xi, 0) 2\xi \cos\gamma}{q^2\{1 - 2\xi \sin\gamma + \xi^2\}} d\xi \qquad [15]$$

with the coordination with [14], having ξ as a parameter.

When we assume k = 0, or the case of shock free entry the drag coefficient can be obtained from

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 c} = -\frac{1}{c} \int_0^{-a_o} \left[\frac{P - P_c}{\frac{\rho}{2} U^2}\right] \frac{dy(\xi, 0)}{d\xi} d\xi$$

$$= -\frac{1}{c} (\sigma + 1 - \lambda_o^2) \int_0^{-a_o} \frac{\operatorname{Im} w(\xi, 0) 2\xi \cos\gamma}{q^2\{1 - 2\xi \sin\gamma + \xi^2\}} d\xi$$

$$= -\frac{1}{c} (\sigma + 1 - \lambda_o^2)\, y(-a_o, 0) \qquad [16]$$

where c is the chord length

$$c = \sqrt{x^2(-a_o, 0) + y^2(-a_o, 0)}$$

The solidity is

$$\frac{c}{d} = \frac{1}{2\pi}\sqrt{x^2(-a_o, 0) + y^2(-a_o, 0)} \qquad [17]$$

The lift coefficient is

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 c} = \frac{1}{c}(\sigma + 1 - \lambda_o^2)\, x(-a_o, 0) \qquad [18]$$

When we consider the total force F_n normal to the chord c the normal force coefficient will be

$$C_N = \frac{F_n}{\frac{1}{2}\rho U^2 c} = \frac{1}{c}(\sigma + 1 - \lambda_o^2)$$

Thus, C_N will be known with a given cavitation number σ, the pressure on the foil, and the solidity.

The contribution from the leading edge singularity can be obtained through the Blasius theorem (see Yim (1967)).

REMARKS FOR THE NONLINEAR PROBLEM

For the numerical computation, we must calculate the drag coefficient of the cascade with a given set of parameters: a stagger angle γ, a speed on the foil λ_o, a cavitation number σ and the solidity c/d. However, first we may give γ, λ_o (or C_N), a_o and a_1, and solve the four simultaneous equations, [10], [11-a], [11-b] and [12] for σ, q_∞, θ_∞ and a_3. Now if we insert these values into [14] and [15] we will have a solidity from [17], the drag coefficient from [16], and the length of cavity whose upper part may be different in length from the lower part. If we want these to be equal, we may have to solve a set of simultaneous equations which include an integral equation

$$x(-a_3, 0) = x(a_1, 0)$$

associated with [14]. The integration of [14] and [15] may require a numerical scheme.

It is easy to see that θ has logarithmic singularities at the end of the cavity whose strengths (coefficients) are different depending on whether we approach from the cavity or from the wake. If $a_4 = a_o$ this same situation will take place at the trailing edge of the foil. The speed has only a discontinuity there. Thus if $a_4 = a_o$ the Kutta condition is not satisfied at the trailing edge. However, it does not seem to affect the flow too much.

A difficulty of a general nonlinear theory for a curved foil with a finite cavity lies in the fact that the boundary condition for the foil in the W plane is given as a function of z rather than of W, which leads only to an integral equation. Thus it is not a surprise that the constant pressure cambered foil has as easy a nonlinear solution as that of the constant slope flat plate foil. The former case gives the real part of Q, the latter case gives the imaginary part of Q, both constants on the foil. Both problems can be treated as a simple Rieman Hilbert problem (see Muskhelishvili, 1953) in the W plane.

FIRST AND SECOND ORDER PROBLEM

When we expand Q in small parameters, the angle of attack α and the cavitation number as in the method adopted by Tulin (1964), and Q, q, u and v are nondimensionalized with respect to the speed U at $x = -\infty$, and φ and ψ are the quantities already divided by U,

$$\log q = [\alpha q_1 + \alpha^2 q_2 + ----]$$

$$+ [\sigma q_a + \sigma^2 q_b + ----]$$

$$\theta = [\alpha\theta_1 + \alpha^2\theta_2 + ----]$$

$$+ [\sigma\theta_a + \sigma^2\theta_b + ----]$$

$$\text{or} \quad \log q = \alpha q_I + \alpha^2 q_{II} + ---- \qquad [19]$$

$$\theta = \alpha\theta_I + \alpha^2\theta_{II} + ---- \qquad [20]$$

where

$$q_I = q_1 + (\sigma/\alpha)q_a$$

$$q_{II} = q_2 + (\sigma/\alpha)^2 q_b \qquad [21]$$

$$\theta_I = \theta_1 + (\sigma/\alpha)\theta_a$$
$$\theta_{II} = \theta_2 + (\sigma/\alpha)^2\theta_b \qquad [22]$$

$$u = 1 + \alpha q_I + \alpha^2(q_{II} + q_I^2/2 - \theta_I^2/2) + 0(\alpha)^3 \qquad [23]$$

$$v = \alpha\theta_I + \alpha^2(\theta_{II} + \theta_I q_I) + 0(\alpha)^3 \qquad [24]$$

$$\varphi = x + \alpha\int_0^x q_I \, dx + 0(\alpha^2) \qquad [25]$$

$$\psi = y - \alpha\int_{-\infty}^x \theta_I(y,x) \, dx + 0(\alpha^2) \qquad [26]$$

$$q = 1 + \alpha q_I + \alpha^2(q_{II} + q_I^2/2) + 0(\alpha^3) \qquad [27]$$

$$\frac{P_{-\infty} - P}{\frac{1}{2}\rho U^2} = q^2 - 1 = \alpha 2q_I + \alpha^2(2q_{II} + 2q_I^2) + 0(\alpha^3) \qquad [28]$$

Thus, in the first order of our problem φ, ψ, log q and θ can be replaced by x, y, $u-1 = u_1$, and v respectively. The linear boundary conditions and the solution is exactly the same whether the problem is originally started from the logarithm of the complex velocity Q on the complex potential (W) plane as in the nonlinear formulation or from the complex velocity w on the physical (z) plane. However, the second order problem formulated in the W plane and that in the z plane are quite different in their form. The developments such as [19] - [28] prefer the formulation of the second order problem in the W plane first as shown in the nonlinear problem. Then q_I, q_{II}, θ_I, θ_{II} .... will be given as functions of $W = \varphi + i\psi$, which are in general neat and easy to obtain if the boundary conditions are given as functions of W too. For the conversion of the independent variables φ, ψ to $z = x + iy$, especially on $\psi = 0$, we use the integration of Equation [13] on $\psi = 0$, and substitute [23], [24] and [28],

$$\left.\begin{aligned} x &= \int_0^{\varphi} \frac{u d\varphi}{q^2} = \varphi - \int_0^{\varphi} [\alpha q_I + \alpha^2 (q_{II} + q_I^2 + \theta_I^2/2)] d\varphi \\ y &= \int_0^{\varphi} \frac{v d\varphi}{q^2} = \int_0^{\varphi} [\alpha\theta_I + \alpha^2 (\theta_I + \alpha^2 (\theta_{II} - \theta_I q_I)] d\varphi \end{aligned}\right\} \quad [29]$$

The boundary conditions for the case of a constant pressure camber foil are as follows: On the foil, from [28]

$$\left.\begin{aligned} \alpha q_I &= \frac{\lambda}{2} \equiv \frac{\lambda_o^2 - 1}{2}, \\ \alpha 2q_I + \alpha^2 (2q_{II} + 2q_I^2) &= \frac{\lambda}{2}, \\ \text{or} \quad \alpha^2 q_{II} &= \frac{-\lambda^2}{4} \end{aligned}\right\} \quad [30a]$$

Similarly on the cavity

$$\alpha q_I = \frac{\sigma}{2} \quad , \quad \alpha^2 q_{II} = -\frac{\sigma^2}{4}$$

$$\text{at} \quad x = -\infty \quad , \quad q_I = 0 \quad , \quad q_{II} = 0 \qquad [30b]$$

The value of q_I and q_{II} at $x = \infty$ are not zero in cascades depending on the mass continuity, and the condition at $x = -\infty$. On the wake, the pressure is assumed to be the same as that at $x = \infty$ (Tulin 1964) or

$$\alpha q_I + \alpha^2 (q_{II} + q_I^2/2) = \alpha q_{\infty 1} + \alpha^2 q_{\infty 2}$$

$$\text{or} \quad q_I = q_{\infty 1} \quad , \quad q_{II} = q_{\infty 2} - q_{\infty 1}^2/2 \qquad [30c]$$

where q_1 and q_2 can be determined later.

From [23], [26] and [10]

$$\alpha q_I + \alpha^2 (q_{II} + q_I^2/2 - \theta_I^2/2) = \{\alpha\theta_I + \alpha^2 (\theta_{II} + \theta_I q_I)\} \tan\gamma$$

$$\theta_I = q_{\infty 1} \cot\gamma \qquad [31]$$

$$\theta_{II} = -q_{\infty 1}^2 \left(1 + \frac{\cot^2\gamma}{2}\right)\cot\gamma + q_{\infty 2} \cot\gamma \qquad [32]$$

The first order solution for this problem is given by Yim (1967). We write down the solution here

$$w^{(1)}(\zeta) = u^{(1)} - iv^{(1)}$$

$$= \frac{1}{\pi i}\left[\left(\frac{\sigma}{2} - q_{\infty 1}\right) \log \frac{(a_o + \zeta)(\zeta - a_1)}{(a_3 + \zeta)\zeta} + \left(\frac{\lambda}{2} - q_{\infty 1}\right) \log \frac{\zeta}{a_o + \zeta} + \frac{k}{\zeta}\right] + 1 + q_{\infty 1}$$

$$- i\, q_{\infty 1} \cot \gamma \qquad [33]$$

where

$$1 + q_{\infty 1} = q(x = \infty) \cos \theta + O(\alpha^2) \qquad [34]$$

and the superscript (1) denotes the first order quantities.

$$w\left(e^{i\left(\frac{\pi}{2} - \gamma\right)}\right) = 1 \qquad [35]$$

$$C_D^{(1)} = \frac{(\sigma - \lambda)}{c^{(1)}} \int_0^{-a_o} v \frac{dx}{d\xi}\, d\xi$$

$$= 2 \cos \gamma \frac{(\sigma - \lambda)}{\pi c^{(1)}} \int_0^{-a_o} \left\{\left(\frac{\sigma}{2} - q_{\infty 1}\right) \log \left|\frac{(\xi - a_1)(a_o + \xi)}{(a_3 + \xi)\,\xi}\right| + \frac{k}{\xi}\right.$$

$$\left. + \left(\frac{\lambda}{2} - q_{\infty 1}\right) \log \left|\frac{\xi}{a_o + \xi}\right| + c^{(1)}\pi\, q_{\infty 1} \cot \gamma\right\} \frac{\xi\, d\xi}{1 - 2\xi \sin \gamma + \xi^2}$$

[36]

$$C_L^{(1)} = \sigma - \lambda \qquad [37]$$

The second order solution can be obtained in a similar way as for the first order solution. From the boundary conditions on $\psi = 0$, [30a], [30b], [30c], [31] and [32], we obtain

$$\begin{aligned} Q^{(2)}(\zeta) &= \alpha q_I + \alpha^2 q_{II} + i(\alpha\theta_I + \alpha^2\theta_{II}) \\ &= \frac{1}{\pi i}\left[\left(\frac{\sigma}{2} - \frac{\sigma^2}{4} - q_\infty\right)\log\frac{(a_o + \zeta)(\zeta - a_1)}{(a_3 + \zeta)\zeta}\right. \\ &\left. + \left(\frac{\lambda}{2} - \frac{\lambda^2}{4} - q_\infty\right)\log\frac{\zeta}{a_o + \zeta} + \frac{k}{\zeta}\right] + q_\infty \\ &- i\left\{q_\infty - \frac{\alpha^2 q_{\infty 1}{}^2}{2}(1 + \cot^2\gamma)\right\}\cot\gamma \end{aligned} \qquad [38]$$

where $q_\infty = \alpha q_1 + \alpha^2 q_2 - \alpha^2 q_1{}^2/2$.

The second order lift coefficient is

$$C_L^{(2)} = \sigma - \lambda + O(\alpha^3) = C_L^{(1)} \qquad [39]$$

A superscript (2) represents the quantity correct up to the second order.

Here we cannot deal with simplified parameters as in the case of the first order solution where all the physical quantities are divided by C_L, but C_L should be given in addition to a_o, and a_1 which are related to the solidity and the cavity length through [6] and [29], namely

$$c^{(2)} = x(\xi = -a_o, \eta = 0) = \varphi(-a_o, 0)(1 - \lambda/2)$$

Hence, the solidity is

$$c^{(2)}/(2\pi) = \varphi(-a_o, 0)\, \frac{1 - \lambda/2}{2\pi} = \frac{c^{(1)}}{2\pi}\,(1 - \lambda/2) \qquad [40]$$

The cavity length is

$$\ell^{(2)} = x(\xi = a_1, \eta = 0) = \varphi(a_1, 0)\left(1 - \frac{\sigma}{2}\right)$$

$$= \varphi(-a_3, 0)\left(1 - \frac{\sigma}{2}\right) + \left(\frac{\sigma}{2} - \frac{\lambda}{2}\right)\varphi(-a_o, 0) \qquad [41]$$

or from [39]

$$\varphi(-a_3, 0) = \varphi(a_1, 0) + \frac{C_L}{2}\varphi(-a_o, 0) + O(\alpha^2) \qquad [42]$$

from which a_3 will be determined.

$$\ell^{(2)}/c^{(2)} = \frac{\ell^{(1)}}{c^{(1)}}\left(\frac{1 - \sigma/2}{1 - \lambda/2}\right) \qquad [43]$$

at $x = -\infty$. Equations [11] and [38] may be combined to give

$$Q^{(2)}\left(e^{i\left(\frac{\pi}{2}-\gamma\right)}\right)=0 \qquad [44]$$

Thus we have two simultaneous equations; the real and the imaginary parts of [42], from which we can obtain q_∞ and $\sigma - \frac{\sigma^2}{2}$ for a given set of γ, C_L, a_o, and a_1 with a_3 from [42].

In the results of the first order, we notice that σ and q_∞ are much smaller than C_L for $\ell/c > 1.25$, especially in the cases of large stagger angles or high solidities where the nonlinear effect is large. In addition, the influence due to the difference in a_3 between the theories of the first order and the second order does not seem to affect physical quantities such as lift and drag coefficients too much, since even complete neglect of this fact in the theory of an isolated supercavitating foil leads to good agreement in the lift coefficients not only with the case of the other model but also with the experiments (Hsu 1966).

From [16] and [24] the drag coefficient correct up to the second order is

$$C_D^{(2)} = \frac{(\sigma-\lambda)}{c^{(2)}}\int_0^{-a_o} \frac{[\alpha\theta_I(\xi,0) + \alpha^2\{\theta_{II}(\xi,0) + \theta_I(\xi,0)q_I(\xi,0)\}]}{1+2\alpha q_I}$$

$$\times \frac{d\varphi(\xi,0)}{d\xi}\,d\xi \qquad [45]$$

This can be computed by a numerical integration as in the first order theory.

Thus, for a given set of data γ, C_L, a_o, and a_1 we can obtain the solidity, the cavity length, the parameter a_2 from Equations [40], [41], [42] and [6]. Then we find q_∞ and $\sigma - \sigma^2/2$ from [44]. Using these results we can obtain $C_D^{(2)}$ from [45]. The process of calculation is exactly the same as in the case of the first order theory (Yim 1967). Thus we can write

$$Q^{(2)}(\zeta,\sigma,C_L) = Q^{(1)}\left(\zeta,\sigma - \frac{\sigma^2}{2}, C_L - \frac{\sigma^2}{2} + \frac{\lambda^2}{2}\right) \qquad [46]$$

Hence from [40] and [45]

$$\frac{C_D^{(2)}}{C_L}(\sigma,C_L) = \left(1 + \frac{\lambda}{2}\right)\frac{C_D^{(1)}}{C_L}\left(\sigma - \frac{\sigma^2}{2}, C_L - \frac{\sigma^2}{2} - \frac{\lambda^2}{2}\right)$$

$$- \frac{\lambda}{2}\frac{C_D^{(1)}}{C_L}(\sigma,C_L) \qquad [47]$$

The first order drag is represented (Yim 1967) in the form of

$$\frac{C_D^{(1)}}{C_L^2} = \frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma}{C_L}, \frac{c}{d}, \gamma\right) \qquad [48]$$

and many curves of $C_D^{(1)}/C_L^2$ versus σ/C_L are given for different solidities c/d and stagger angles γ. Since

$$\frac{C_D^{(1)}}{C_L\left(C_L - \frac{\sigma^2}{2} + \frac{\lambda^2}{2}\right)}\left(\sigma - \frac{\sigma^2}{2}\, ,\, C_L - \frac{\sigma^2}{2}\right) = \frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma - \frac{\sigma^2}{2}}{C_L - \frac{\sigma^2}{2} + \frac{\lambda^2}{2}}\, ,\, \frac{c}{d}\, ,\, \gamma\right)$$

Equation [45] can be rewritten as

$$\begin{aligned}\frac{C_D^{(2)}}{C_L^2}\left(\sigma, C_L, \frac{c}{d}, \gamma\right) &= \left(1 + \frac{\lambda}{2}\right)\frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma}{C_L} - \frac{\sigma^2}{2C_L}, \frac{c}{d}, \gamma\right) - \frac{\lambda}{2}\,\frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma}{C_L}\, ,\, \frac{c}{d}\, ,\, \gamma\right)\\ &\quad - \left(\frac{\sigma^2}{2C_L} - \frac{\lambda^2}{2C_L}\right)\frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma}{C_L}\, ,\, \frac{c}{d}\, ,\, \gamma\right) + O(\alpha^2)\\ &= \frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma}{C_L} - \frac{\sigma^2}{2C_L}\, ,\, \frac{c}{d}\, ,\, \gamma\right) + \left(\frac{C_L}{2} - \sigma\right)\frac{C_D^{(1)}}{C_L^2}\left(\frac{\sigma}{C_L}, \frac{c}{d}, \gamma\right) + O(\alpha)\end{aligned} \qquad [49]$$

Once we have the result of the first order theory, Equation [49] furnishes an easy estimation of a second order correction. As an example, the case of $\gamma = 60^{\circ}$ is shown in Figure 2.

DISCUSSION OF RESULTS AND THE APPLICATION IN THE SECOND ORDER CORRECTION OF FLOW FOR A SUPERCAVITATING CASCADE OF FLAT PLATE FOILS

The solutions for the flow due to a supercavitating cascade of flat plate foils are known both for the nonlinear theory (Betz and Peterson (1932)) in the case of infinite cavities and for linear theory (Cohen and Sutherland (1958)). For the linear theory, the solution can be written

$$Q^{(1)} = Q(\gamma, c/d, \alpha, \sigma) \qquad [50]$$

where α represents the angle of attack. As explained in the previous section, if we neglect $q_{\infty 1}{}^2$ term in the condition $x = \infty$, Equation [32], the second order solution would be

$$Q^{(2)} = Q^{(1)}(\gamma, c/d, \alpha, \sigma - \sigma^2/2) \qquad [51]$$

The lift coefficient can be written

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 C} = \mathrm{Re}\, \frac{1}{c} \int_0^c \left[\frac{P - P_c}{\frac{\rho}{2} U^2} \right] \frac{dz}{dW} \; dW$$

From [1], [2], [23] and [28]

$$C_L^{(2)} = \frac{1}{c^{(2)}}\left\{\sigma - 2\int_0^{\varphi(x=c)} \frac{(\alpha q_I + \alpha^2 q_{II} + \alpha^2 q_I^2)}{1 + 2\alpha q_I}(1+\alpha q_I)\, dW\right.$$

$$= \frac{1}{c^{(2)}} \quad \sigma - 2\int_0^{\varphi(x=c)} [\alpha q_I + \alpha^2 q_{II}] \quad d\varphi \Bigg\}$$

$$= \frac{1}{c^{(1)}}\left\{\sigma - 2\ C_L(\gamma,\ c/d,\ \alpha,\ \sigma - \sigma^2/2)\right\}\left(1 + \frac{\sigma - C_L^{(1)}}{2}\right)$$

$$= C_L^{(1)}(\sigma,\ c/d,\ \alpha,\ \sigma - \sigma^2/2) + C_L^{(1)}\left(\frac{\sigma - C_L^{(1)}}{2}\right) + \frac{\sigma^2}{2}$$

$$+ O(\alpha)^3 \qquad [52]$$

where the simplicity arises because of the cancellation of $\alpha^2 q_I^2$. In fact, in the case of the isolated supercavitating hydrofoil, Tulin (1964) developed a second order theory like this, and Hsu (1966) verified that it was a good approximation for a non-linear theory and the experimental results. Taking the slope from figures in the paper by Cohen and Sutherland (1958) and using the results of the paper by Acosta (1960), a second order correction utilizing Equation [52] is made in Figure 3. This seems to justify the negligence of the $q_{\infty 1}^2$ term for our practical second order correction. For the case of small σ the second term in [52] mainly contributes to the second order as in the case of an isolated foil (Hsu 1966). When σ is fairly large and either the solidity or the stagger angle is large, the large slope of C_L curve versus σ, or the first term contributes a great deal to the second order correction.

In general the first order problem is much simpler and serves to understand the physical behavior and the dependence on parameters. In fact, the two-dimensional approximation, the neglect of viscosity or the approximated mathematical model may already involve too severe approximations. However, it should be admitted that not only is it mathematically interesting, but that also our physical understanding becomes clearer when we possess the nonlinear or the second order solution. Besides, if the second order correction is so simple as [49] or [52], there is no reason why we should not utilize it regularly in the design of supercavitating stages.

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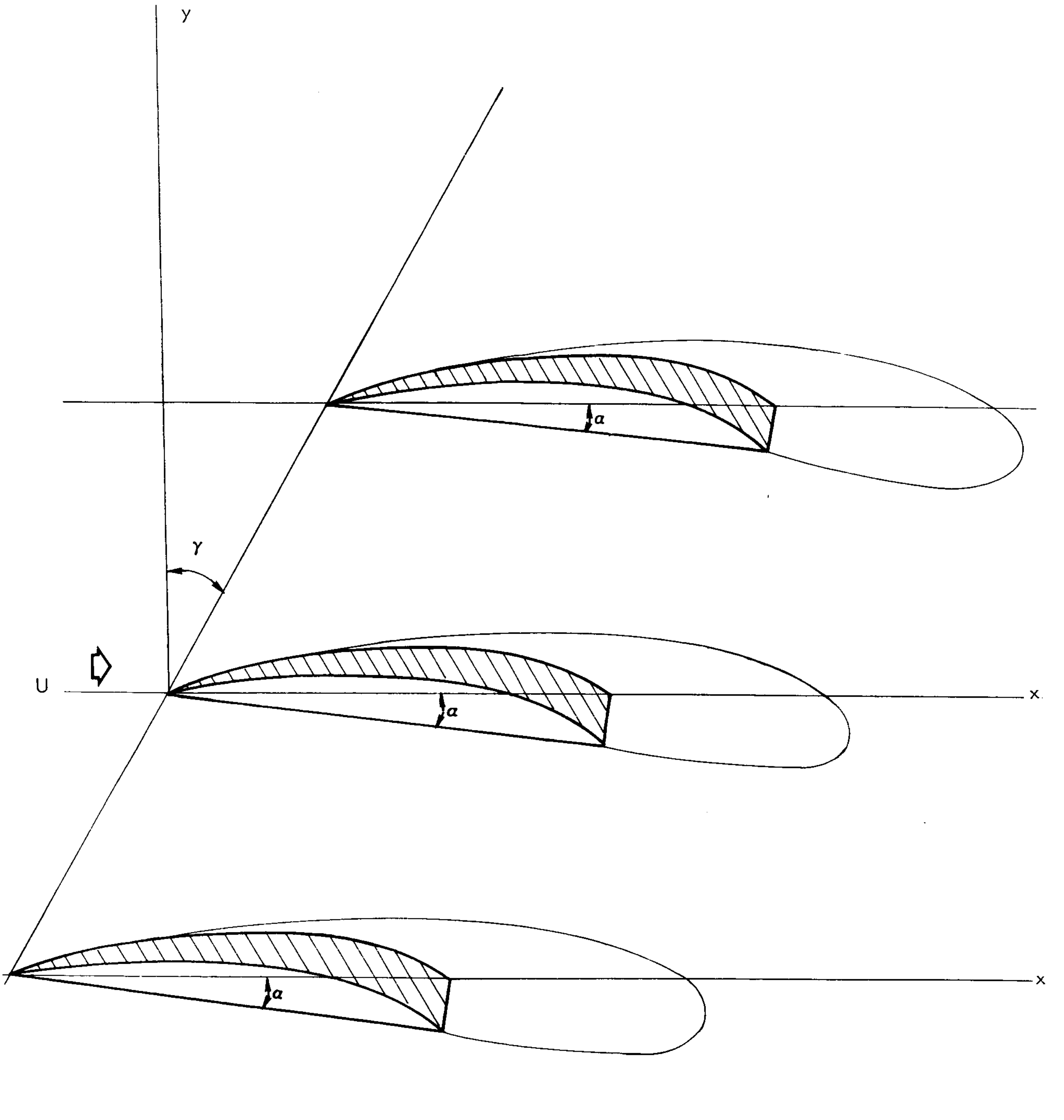

FIGURE 1a - Z PLANE (NON LINEAR)

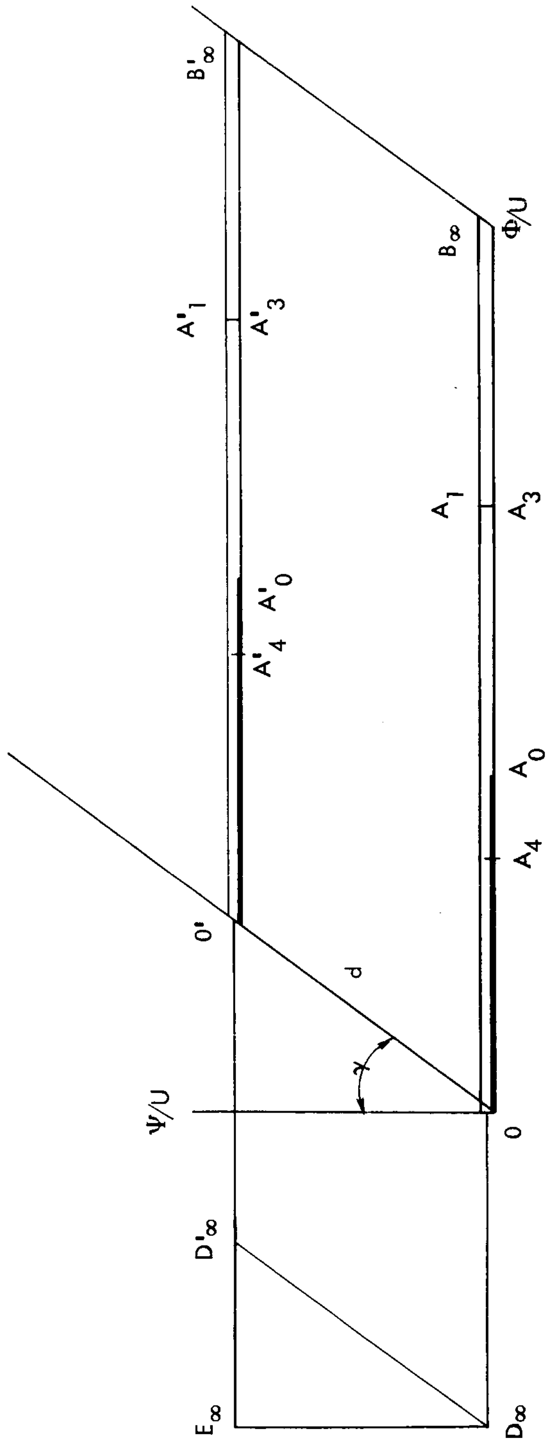

FIGURE 1 b - W/U PLANE

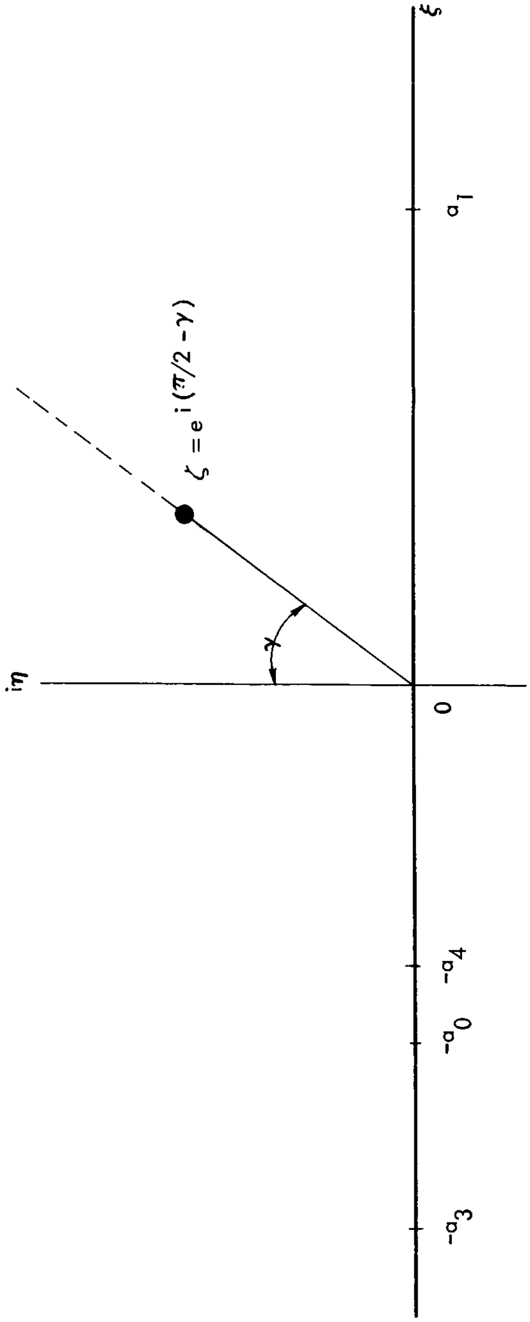

FIGURE 1 c - ζ PLANE

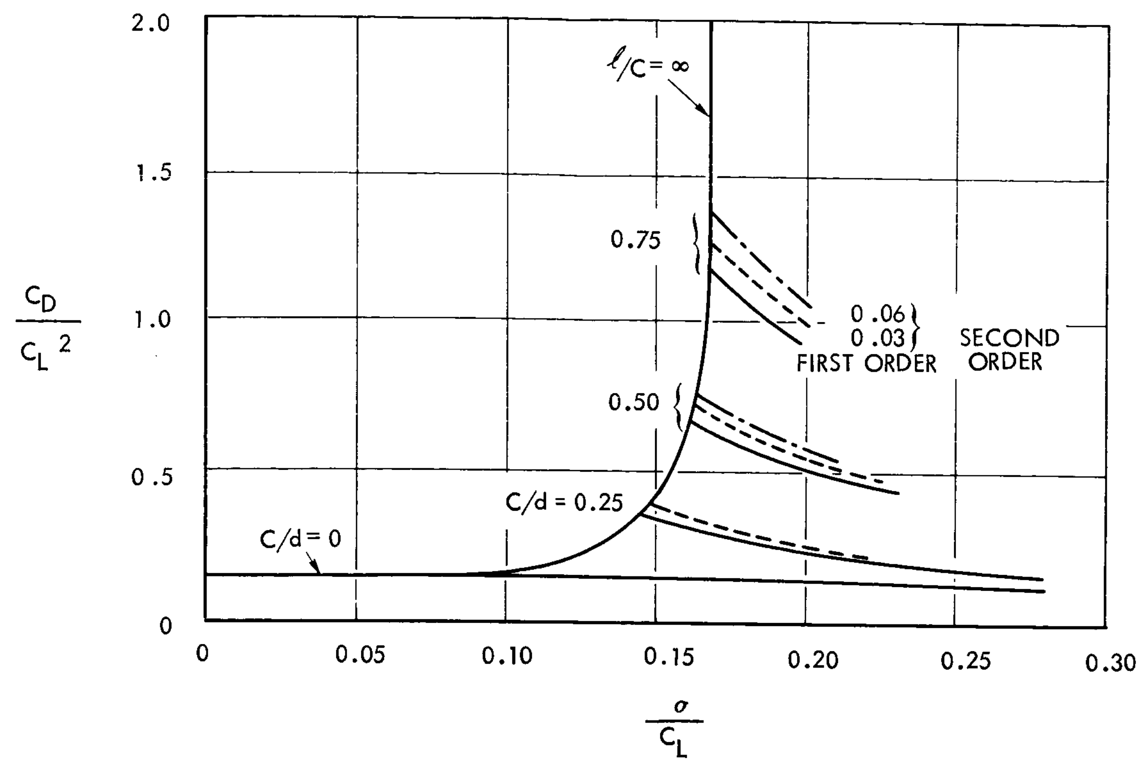

FIGURE 2 - RELATIONS BETWEEN DRAG COEFFICIENTS, LIFT COEFFICIENTS, AND CAVITATION NUMBERS OF SUPERCAVITATING CASCADES WITH A STAGGER ANGLE OF 60°

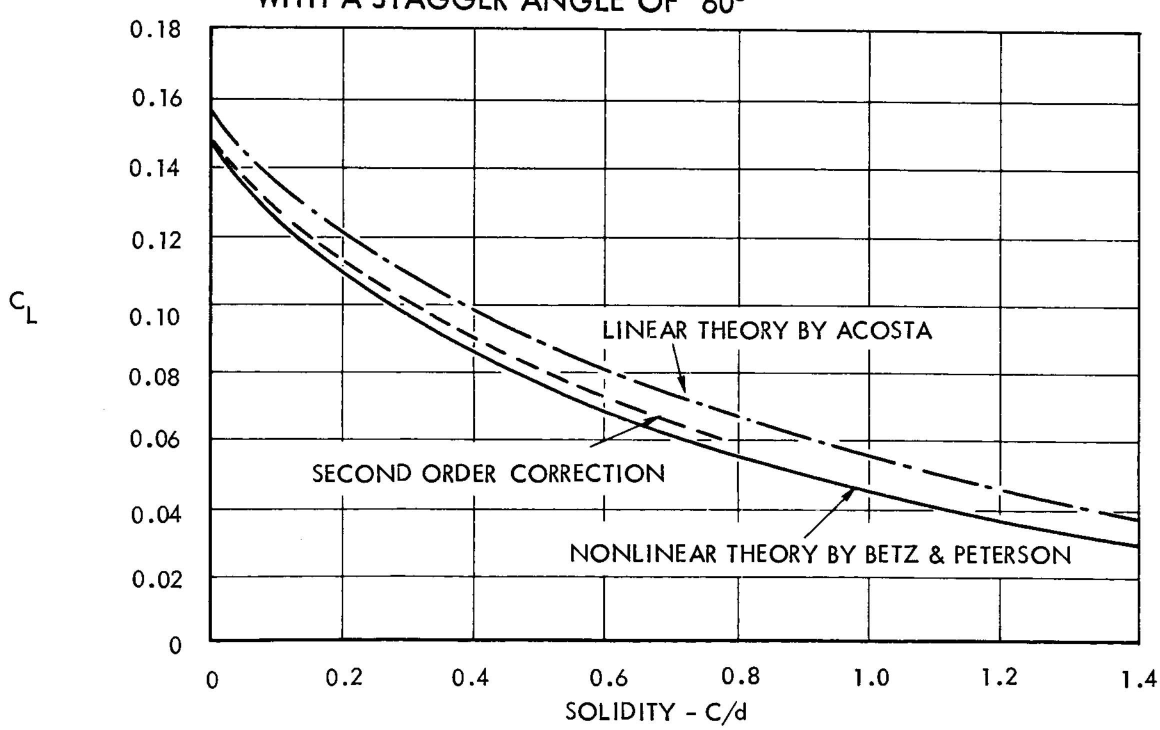

FIGURE 3 - LIFT COEFFICIENT VS SOLIDITY FOR A FULLY CAVITATING FLAT PLATE CASCADE WITH $\alpha = 6°$, $\gamma = 60°$