MATHEMATICAL EXPRESSION
FOR
DROP SIZE DISTRIBUTION IN SPRAYS

by

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SUMMARY

Several mathematical expressions for the distribution of drop sizes in liquid sprays have been proposed by different researchers. These various distributions can be classified into two main types, the logarithmic normal distributions and the Chi-square distributions. In this paper, the theoretical soundness of applying these two main types of distribution functions to drop size distributions is discussed and several figures that are useful when applying them to spray data are included. It is shown how some empirical drop size distributions can be derived from the general Chi-square distribution function. This suggests that it may be possible to relate the parameters in the Chi-square distribution equation to the fundamental mechanisms of spray formation. It is concluded that the Chi-square distribution fits the available spray data well, can be graphed fairly simply, and furthermore if the parameters $\beta$ and $\phi$ are defined, several expressions of distribution (volume, number, cumulative, etc.) can be found and several types of mean diameters can be calculated.
### LIST OF SYMBOLS

- **A, A'**: Distribution constants (Table 4)
- **B, B'**: Distribution constants (Table 4)
- **b**: Distribution constant
- **C, C'**: Distribution constants (Table 4)
- **D, D'**: Distribution constants (Table 4)
- **f_n(x)**: Frequency distribution function of number of droplets
- **f_v(x)**: Frequency distribution function of volume of droplets
- **I(x, t)**: Ratio of complete and incomplete Gamma function,
  \[ I(x, t) = \frac{\Gamma_x(t)}{\Gamma(t)} \]
- **k'**: Distribution constant, Eq. 53
- **n**: Cumulative number fraction of droplets
- **N**: Total number of droplets in a spray
- **p**: Injection pressure
- **P_0**: Opening pressure for injection nozzle
- **p**: Droplet characteristic symbol (Table 1)
- **q**: Droplet characteristic symbol (Table 1)
- **r**: Distribution constant, Eq. 53
- **t**: Standardized normal variable (for normal distribution function)
- **V**: Total volume of droplets in a spray
- **v**: Cumulative volume of droplets
- **x**: Droplet diameter
- **x^+**: Droplet size arbitrarily chosen and used to shift the coordinate scale from zero droplet size to the value x
\( x_{\text{max}} \) maximum droplet diameter
\( x_{32} \) Sauter's mean diameter
\( x_{\text{nm}} \) number median diameter
\( x_{\text{vm}}, x_{m} \) volume median diameter, \( x_{\text{vm}} \) used up to Eq. 13, \( x_{m} \) used thereafter, see foot note, p. 7
\( \alpha \) distribution constant, Eq. 51
\( \beta \) chi-square distribution constant (Table 4)
\( \Gamma(x) \) Gamma function
\( \Gamma_{x}(x) \) incomplete Gamma function
\( \delta \) \( 1/\sqrt{2\sigma} \), Eq. 18
\( \mu \) viscosity of fuel
\( \rho \) density of fuel
\( \sigma \) standard deviation
\( \phi \) degree of freedom
\( \chi^{2} \) chi-square distribution
\( \chi_{m}^{2} \) median \( \chi^{2} \) value
superscripts
* indicate normalization
subscripts
\( i \) \( i \)-th section of droplet diameter, Eq. 14
\( n \) number of droplets
\( v \) volume of droplets
LIST OF FIGURES AND TABLES

Fig. 1  Variation of logarithmic-normal volume distribution with logarithmic dimensionless diameter
Fig. 2  Variation of logarithmic-normal volume distribution with dimensionless diameter \( \frac{x}{x_m} \)
Fig. 3  Cumulative volume on the probability scale vs. \( \log R \)
Fig. 4  Upper-limit analysis of Ingebo's data
Fig. 5  Explanatory figure of chi-square distribution
Fig. 6  Chi-square distribution with various \( \phi \)
Fig. 7  Variation of chi-square distribution with dimensionless diameter \( \frac{x}{x_3} \)
Fig. 8  Variation of chi-square distribution with dimensionless diameter \( \frac{x}{x_m} \)
Fig. 9  Chi-square volume distribution curve for various \( \beta \)
Fig. 10 Cumulative volume chi-square distribution
Fig. 11 Cumulative volume on the Chi-square scale vs. dimensionless diameter
Fig. 12 A comparison of a logarithmic-normal distribution and a chi-square distribution
Fig. 13 Droplet size distribution of Ingebo's data
Fig. 14 Logarithmic-normal analysis of Ingebo's data
Fig. 15 Upper-limit analysis of Ingebo's data
Fig. 16 Chi-square distribution analysis of Ingebo's data
Fig. 17 Chi-square distribution analysis of Houghton's data
Fig. 18 Chi-square distribution analysis of Rice's data
Fig. 19 Drop size distribution for throttle nozzle
Fig. 20 Drop size distribution for pintle nozzle
Fig. 21 Drop size distribution for hole nozzle
Fig. 22 The effect of viscosity of fuel on the size distribution

Table 1  Mean and median diameters
Table 2  Relation of the median diameter to Sauter's mean diameter for several values of \( \phi \)
Table 3  Frequency, cumulative and mean diameter expressions for log-normal distribution and chi-square distribution
Table 4  Dimensionless expression of chi-square distribution
The value of dimensionless distribution constants with various $\phi$ ($\beta = 1$)

Experimental results of Ingebo

Atomization in a small air-atomizing nozzle, data of Houghton

Droplet data of swirl atomizer obtained by Rice
INTRODUCTION

Various mathematical expressions for the distribution of drop sizes in liquid sprays have been proposed by different researchers. For example, the Rosin-Rammler distribution, the Nukiyama-Tanasawa distribution, the logarithmic normal distribution, the upper-limit logarithmic normal distribution and the square root normal distribution are widely used for this purpose. Unfortunately, we cannot determine which expression is theoretically best because we have no accurate method of measuring the size distribution of drops and there is little knowledge of the mechanism of droplet breakup.

Under these circumstances, a suitable expression should (1) fit the data adequately, (2) permit easy calculation of mean sizes and other mathematical parameters of interest and (3) give an insight into the fundamental mechanism of droplet production.

This paper will discuss how well the various distributions satisfy the above requirements.

THE CONCEPT OF THE DISTRIBUTIONS

There are various experimental methods of measuring the size of liquid drops sprayed from a nozzle. By these experimental methods we typically characterise a spray either by number versus diameter or by volume versus diameter (for some special cases, there is also a method for characterizing by surface area).

Let us first define the terms we shall use in describing the number and volume of drops in a spray. Speaking first of the number of droplets in a spray we define:
1. The total number \( N \) of droplets in a spray, is mathematically defined as

\[
N = \int_{0}^{\infty} f_n(x) \, dx
\]  

(1)

where \( f_n(x) = \frac{dn}{dx} \) and is a mathematical expression for the number of drops of a size \( x \) in a given size range \( dx \).

2. The cumulative number \( n \) of droplets less than a given size range \( x \), is mathematically defined as

\[
n = \int_{0}^{x} f_n(x) \, dx
\]  

(2)

Likewise the number of droplets of a size larger than \( x \) will be given by

\[
N - n = \int_{x}^{\infty} f_n(x) \, dx
\]  

(3)

3. The incremental number \( \Delta n \) of droplets in a size range between

\[
x - \frac{\Delta x}{2} \quad \text{and} \quad x + \frac{\Delta x}{2}
\]

4. In many cases we shall wish to normalize these terms and shall use the \( * \) superscript to indicate normalization, i.e.,

\[
N^* = \int_{0}^{\infty} f_n^*(x) \, dx = 1
\]  

(4)

The cumulative number fraction is

\[
n^* = \int_{0}^{x} f_n^*(x) \, dx
\]  

or

\[
N^* - n^* = \int_{x}^{\infty} f_n^*(x) \, dx
\]  

(5) (6)

We will follow the same symbolism for the volume \( V \) of \( N \) droplets in a spray using \( f_v = dv/dx \) to indicate the mathematical expression for the volume of the drops in a size range \( dx \)
The normalized expression is

\[ V^* = \int_0^\infty f_v^*(x) \, dx \]  \hspace{1cm} (10)

The cumulative volume fraction is

\[ v^* = \int_0^x f_v^*(x) \, dx \] \hspace{1cm} (11)

or

\[ V^* - v^* = \int_x^\infty f_v^*(x) \, dx \] \hspace{1cm} (12)

where the subscript \( v \) indicates a volume function as opposed to the number function \( f_n \). Note that \( f_n \) and \( f_v \) are related by

\[ f_v = \frac{\pi}{6} x^3 f_n \] \hspace{1cm} (13)

In some cases the cumulative volume fraction \( v^* \) is expressed as a percentage and called the percent passing volume quantity. The term percent passing volume quantity comes from the size of solid particles that would be passed through a sieve which has opening of size \( x \). By analogy the quantity \( V^* - v^* \) is sometimes called the residual volume quantity.
THE CONCEPT OF MEAN DIAMETERS

In many spray problems it is desirable to work only with average diameters instead of the complete drop size distribution.

There are at least six types of mean diameters which may be used to represent a given spray distribution. This is because a spray has four characteristics, that is, the number of droplets, and the diameter, volume and surface area of each droplet.

Using these characteristics of a spray, the general equations for the various types of mean diameters can be written

\[ \bar{x}_{qp} = \left( \frac{\int x^p f(x) dx}{\int x^q f(x) dx} \right)^{1\over q-p} \left( \frac{\int x^q f(x) dx}{n} \right)^{1\over q-p} \]

(14)

where q and p may take on values corresponding to the characteristic investigated.

Some of the more important mean diameters are listed in Table 1 with the corresponding values for p and q.

<table>
<thead>
<tr>
<th>q</th>
<th>p</th>
<th>Name of Mean Diameter</th>
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<td>1</td>
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<td>1</td>
<td>Volume - Diameter</td>
</tr>
<tr>
<td>32</td>
<td>2</td>
<td>Volume - Surface Sauter's</td>
</tr>
</tbody>
</table>

| \( \bar{x}_{nm} \) | Number Median Diameter |
| \( \bar{x}_{vm} \) | Volume Median Diameter |
In addition to mean diameters, median diameters have also been defined. They are convenient average sizes because they can be evaluated immediately at the 50 percent mark of the cumulative distribution curves.

Two median diameters are commonly used. The first, \( \bar{x}_{\text{nm}} \), is the number median diameter, i.e., the diameter of a drop such that half the total number of drops have diameters greater and half have diameters less than \( \bar{x}_{\text{nm}} \), or mathematically using Eqs. 5 and 6

\[
\frac{1}{2} = \int_0^{\bar{x}_{\text{nm}}} f(x)dx = \int_{\frac{1}{2}}^{\infty} f(x)dx = \frac{1}{2} \int_0^{\infty} f(x)dx.
\]

The second, \( \bar{x}_{\text{vm}} \), is the volume (or weight) median diameter, i.e., the diameter of a drop such that half the total volume is composed of drops with diameters less than \( \bar{x}_{\text{vm}} \) and half is composed of drops with diameters greater than \( \bar{x}_{\text{vm}} \), defined mathematically by

\[
\frac{1}{2} = \int_0^{\bar{x}_{\text{vm}}} x^3 f(x)dx = \int_{\frac{1}{2}}^{\infty} x^3 f(x)dx = \frac{1}{2} \int_0^{\infty} x^3 f(x)dx.
\]

**Logarithmic-probability distribution**

Most distribution functions have been defined inductively from many experimental results. To be useful, a distribution function should be well known mathematically. A normal probability distribution fits this criteria.

The normal distribution function can be adapted to give a volume distribution for sprays

\[
f(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left[\frac{-(x - \bar{x}^\dagger)^2}{2\sigma^2}\right]
\]

where \( \sigma \) is the standard deviation and \( \bar{x}^\dagger \) is called the characterizing parameter; \( f(x) \) is maximum at \( x = \bar{x}^\dagger \).
It is found that this function gives a very poor fit to many experimental droplet distributions since they are ordinarily distributions somewhat skewed from the normal distribution. Consequently, it was suggested that the distribution be skewed from the normal function by using the logarithm of x as the variable. The first to apply the logarithmic-normal distribution to a problem of drop size distribution was Galton.

This typical logarithmic normal representation of the volume size distribution is

$$\frac{dv^*}{dy} = \frac{\delta}{\sqrt{\pi}} e^{-\delta^2 y^2} = f_v^*(x)$$

(18)

where

$$y = \ln\left(\frac{x}{x_0}\right)$$

$$\delta = \frac{1}{\sqrt{2} \sigma}$$

The cumulative volume fraction is determined by integrating equation (18)

$$v^* = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-z^2} dz$$

(19)

where

$$z = \delta y$$

$$v^* = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-z^2} dz + \frac{1}{\sqrt{\pi}} \int_{0}^{z} e^{-z^2} dz$$

$$= \frac{1}{2} + \text{erf}(z)$$

(20)

where erf(z) is the error function, or probability integral, of z.

*Numbers refer to references at the end of the paper.
The logarithmic normal number distribution can be shown to be

$$\frac{dn^*}{dy} = \frac{\delta}{\sqrt{n}} e^{-(\delta y + \frac{3}{2\delta})^2}. \quad (21)$$

For the $x\dagger$ in the log-normal distributions, Theodore Hatch and Sarah P. Chaté\textsuperscript{2} have used the geometric mean diameter while B. Epstein\textsuperscript{3} and F. Kottler\textsuperscript{4} have used volume median diameter $\bar{x}_m$.\,

Theoretically, however, we should only use the mean diameter for $x\dagger$. From equations (14) and (18) plus some algebra it can be shown that the general expression of mean diameter for this distribution is

$$\bar{x}_{qm} = x\dagger \exp \left[ \frac{p + q - 6}{4\delta^2} \right]. \quad (22)$$

In the following discussion, we shall use the volume median diameter $\bar{x}_m$ for $x\dagger$.

Fig. 1 shows the variation of volume distribution $dv/dy$ with dimensionless diameter $(x/\bar{x}_m)$ for the various values of $\delta$. From this figure, it is apparent that the logarithmic-volume distribution has a maximum point at the median diameter $\bar{x}_m$.

As it is simpler to represent volume distribution versus dimensionless diameter $(x/\bar{x}_m)$ rather than volume distribution versus the logarithm of the dimensionless diameter, we replace $dy = 1/(x/\bar{x}_m) \cdot d(x/\bar{x}_m)$ in equation (18) to obtain

$$\frac{dV^*}{d(x/\bar{x}_m)} = \frac{\delta}{\sqrt{n}} \frac{\bar{x}_m}{x} e^{-\delta^2 (\ln(x/\bar{x}_m))^2} \quad (23)$$

\textsuperscript{†}In order to write simpler expressions we hereafter write $\bar{x}_m$ to mean the same as was previously written $\bar{x}_{vm}$. 
The variations of the volume distribution $\frac{dv}{d(x/x_m)}$ with dimensionless diameter $(x/x_m)$ are shown in Figure 2 for different values of $\delta$.

Figure 3 is an example of a plot of cumulative volume on the probability scale versus $\log(x/x_m)$. If the data follow the logarithmic-normal distribution, they will form straight lines. The value of $\delta$ represents the degree of uniformity of the drop size in the spray, as illustrated by Figures 1, 2, and 3. The value of $\delta$ can be calculated from the slope of the lines in Figure 3, that is

$$\delta = \frac{0.394}{\log_{10}(x_{90}/x_m)} = \frac{0.394}{\log_{10}(x_m/x_{10})}$$

(24)

where $x_{90}$ and $x_{10}$ are the diameter for a cumulative percentage of 90 and 10 respectively.

Although the plot of an exact logarithmic-normal distribution will be a straight line on log-probability coordinates, typically the experimental data deviate from a straight line at large values of $x$. In an attempt to develop a different function that would not have this problem, Mugel and Evans suggest an upper limit log-normal distribution which used $y = \ln(ax/(x_{max} - x))$ instead of $y = \ln(x/x^*)$ in equation (18), where $x_{max}$ is the maximum diameter droplet in the spray.

Figure 4 illustrates the application of the upper-limit equation to Ingebo's data. Here, $x_{max}$ is determined as follows. Plot the data as $P$ against $x$, drawing a smooth curve to fit these points. From these curves read the 10th, 50th, and 90th percentiles, $x_{10}$, $x_{50} = x_m$, and $x_{90}$. Then calculate $x_{max}$ from the formula

$$x_{max} = x_{50} \cdot \frac{x_{50}(x_{90} + x_{10}) - 2 \cdot x_{90} \cdot x_{10}}{x_{50}^2 - x_{90} \cdot x_{10}}$$

(25)
The parameter $\alpha$ is readily determined from the line representing the distribution. Since $y = 0$ at the 50th percentile, (where $x_{50} = \bar{x}_m$, the volume median diameter), we have here

$$a = \frac{x_{\text{max}} - \bar{x}_m}{\bar{x}_m}$$

(26)

The parameter $\delta$ is determined by the slope of the line, hence by any two points on it. Let us designate the coordinate on the log scale by $u$;

$$u = \frac{x}{x_{\text{max}} - x}$$

Then if we read the values $u_{90}$ and $u_{50}$ at the 90th and 50th percentiles, we find

$$\delta = \frac{0.394}{\log_{10}(u_{90}/u_{50})}$$

(27)

**CHI-SQUARE DISTRIBUTION**

Another distribution function which can be derived from the normal distribution function is the Chi-square ($\chi^2$) distribution. In general, if $t_1, t_2, t_3, \ldots, t_\phi$ are standardized normal variables and we define

$$\chi^2 = t_1^2 + t_2^2 + t_3^2 + \ldots + t_\phi^2 = \sum_{i=1}^{\phi} t_i^2$$

(28)

then $\chi^2$ has the distribution

$$f(\chi^2) = \frac{1}{2^{\phi/2} \Gamma(\phi/2)} (\chi^2)^{\phi/2 - 1} e^{-\chi^2/2} d(\chi^2)$$

(29)

where $\Gamma$ is the Gamma function in the above equation.
which is known as the $\chi^2$ distribution with $\phi$ degrees of freedom.

As an illustration of the meaning of the $\chi^2$ distribution consider a target at which we have fired a very large number of arrows. If we call the non-dimensional rectangular coordinates of a point on the target $(t_1, t_2)$ as in Figure 5-a we can expect that the probability of a hit depends upon the radius, $\chi$, from the target and that the frequency versus distance curves will follow a normal distribution for both $t_1$ and $t_2$ as in Fig. 5-b. Note that if the probability of a hit is the same on the $t_1$ and the $t_2$ axes, the curves in Fig. 5-b will fall on top of each other. If the probability of a hit is different between the two axes, the curves for $t_1$ and $t_2$ will not coincide as illustrated in Fig. 5-b. Now if $\chi^2 = t_1^2 + t_2^2$ as is shown in Fig. 5-a, the frequency versus $\chi^2$ curves can be expected to follow a Chi-square distribution function with degrees of freedom $\phi = 2$ (Eq. 29) and have a shape similar to Fig. 5-c.

Another example of a form of $\chi^2$ distribution function is the familiar Maxwell distribution of molecular velocities with degrees of freedom $\phi = 3$.

The $\chi^2$ distribution function has been applied to some problems of the distributions of droplets in sprays. In this case, $\chi$ is related to the droplet diameter $x$ by

$$\chi^2 = 2bx^\beta,$$

where $\phi$, $b$, and $\beta$ are the distribution constants. Thus for the volume distribution the following equation is presented using Eq. 27

$$\frac{dv^*}{dx^\phi} = \frac{1}{2^{\phi/2} \Gamma(\phi/2)} (x^\phi) \left(\frac{\phi}{2} - 1\right) - x^\phi \text{e}$$

Since $\chi^2 = 2bx^\beta$, equation (30) becomes

$$\frac{dv^*}{dx} = \frac{\beta b^\frac{\phi}{2} x^{\frac{\phi}{2} - 1} - bx^\beta}{\Gamma(\phi/2)} \text{e} = f_v(x)$$
When $\beta = 1$, equation (31) becomes

$$
\frac{dv^*}{dx} = \frac{\phi}{2} - 1 - bx
$$

Using $\beta = 1$ has provided a good fit to much experimental data. Equation (32) is a simple and a useful equation for drop size distributions.

The $\chi^2$ distribution is always highly skewed for small values of $\phi$, becomes more symmetric when $\phi$ increases, and approaches a normal distribution in shape for large $\phi$ as shown in Figure 6.

The number distribution corresponding to the $\chi^2$ equation can be obtained from Eq. (31) by using the relationship between number and volume as given by Eq. (13) to give

$$
\frac{dn^*}{dx} = \frac{b}{\Gamma(\phi/2 - 3)} \frac{\phi}{2 - 3} x^{\phi/2 - 4} e^{-bx^\phi}
$$

When $\beta = 1$, Eq. (33) becomes

$$
\frac{dn^*}{dx} = \frac{b}{\Gamma(\phi/2 - 3)} \frac{\phi}{2} x^{\phi/2 - 4} e^{-bx}
$$

The cumulative volume fraction $v^*$ and the cumulative number fraction $n^*$ are determined by integrating Eqs. (30) and (33), respectively to give

$$
v^* = \int_0^{\chi^2} \frac{1}{2^{\phi/2} \Gamma(\phi/2)} e^{-(\phi/2 - 1) x^2} \frac{\chi^2}{2} \right) d(\chi^2)
$$

$$
v^* = \frac{\Gamma(\phi/2)}{2bx^\phi (\phi/2) \Gamma(\phi/2)}
$$
and
\[ n^* = \frac{r}{2b} \beta \left( \frac{\phi}{2} - \frac{3}{\beta} \right) \frac{\Gamma \left( \frac{\phi}{2} - \frac{3}{\beta} \right)}{\Gamma \left( \frac{\phi}{2} - \frac{3}{\beta} \right)} \]

(37)

where \( \Gamma_x(t) \) is the incomplete gamma function, as defined by
\[ \Gamma_x(t) = \int_0^x z^{\ell-1} e^{-z} \, dx \quad 0 < x < \infty \]

Extensive tables of the ratio \( I(x, \ell) = \Gamma_x(t)/\Gamma(\ell) \) are given in "Tables of the Incomplete Gamma Function." \(^8\)

Then we can easily obtain the values of \( n^* \) and \( v^* \) from both the Tables of Chi-Square Distributions and from Tables of the Incomplete Gamma Function.

If we wish the commonly used Sauter's mean diameter \( \bar{x}_{32} \), the value of \( b \) in Eqs. (31), (33) and related equations can be shown to be

\[ b = \left[ \frac{\Gamma(\ell/2)}{\Gamma(\phi/2) - 1/\beta} \right]^{\beta} \left[ \frac{1}{\bar{x}_{32}} \right]^{\beta} \]

(38)

On the other hand, if we wish the volume median diameter \( \bar{x}_m \), the value of \( b \) becomes

\[ b = \left[ \frac{\bar{x}_m}{2} \right]^{\phi} \left[ \frac{1}{\bar{x}_m} \right]^{\beta} \]

(39)

where \( \bar{x}_m \) is median \( \chi^2 \) value which is discussed later.

If we introduce the dimensionless diameter as \( x/\bar{x}_{32} \), then Eq. (31) may be written

\[ \frac{dv^*}{d(\bar{x}/\bar{x}_{32})} = \beta \left[ \frac{\Gamma(\phi/2)}{\Gamma(\phi/2) - 1/\beta} \right]^{\beta/2} \left[ \frac{x}{\bar{x}_{32}} \right]^{\phi/2} \exp \left[ - \frac{\Gamma(\phi/2)(x/\bar{x}_{32})}{\Gamma(\phi/2) - 1/\beta} \right]^{\beta} \]

(40)
Since in many cases $\beta = 1$, $\phi$ is usually the only distribution constant. Thus, Fig. 7 shows the variation of $dv/d(x_{32}^\phi)$ with dimensionless diameter ($x/\bar{x}_{32}$) as given by Eq. 40 for various values of $\phi$. The curves on this figure give a maximum point at $x/\bar{x}_{32} = 1$, that is, at Sauter's mean diameter. For this reason the form of the $\chi^2$ distribution is very convenient.

Next, we will develop a dimensionless expression using the volume mean diameter $\bar{x}_m$. To obtain the median diameter, we define the mean value of $\chi^2$ for each $\phi$ in the $\chi^2$ distribution by the following equation

$$\int_0^\infty \frac{1}{2\phi/2 \Gamma(\phi/2)} (\chi^2)^{\phi/2 - 1} e^{-\chi^2/2} d(\chi^2) = \frac{1}{2}$$

(41)

Values of $\chi^2_m$ are listed in $\chi^2$ distribution tables.

Thus, the relation of the median diameter to Sauter's mean diameter may be obtained from Eqs. (38) and (39)

$$\frac{\bar{x}_m}{\bar{x}_{32}} = \left[\frac{\chi^2_m}{\chi^2_{32}}\right]^{1/\beta} \frac{\Gamma(\phi/2 - 1/\beta)}{\Gamma(\phi/2)}$$

(42)

Table 2 shows the above relation for several values of $\phi$ when $\beta = 1$.

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<th>$\chi^2_m/2$</th>
<th>$\bar{x}<em>m/\bar{x}</em>{32}$</th>
<th>$\phi$</th>
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<td>13</td>
<td>6.16990</td>
<td>1.12180</td>
<td>23</td>
<td>11.16845</td>
<td>1.06366</td>
</tr>
</tbody>
</table>
Eq. (31) may be written with \((x/\bar{x}_m)\) by using Eq. 39 for \(b\),

\[
dv^* = \frac{\beta}{\Gamma(\phi/2)} \left[ \frac{x^2}{\bar{x}_m^2} \right]^{\phi/2} \left[ \frac{x}{\bar{x}_m} \right]^{\phi/2 - 1} \exp \left[ -\frac{x^2}{2\bar{x}_m} \right] \exp \left[ -\frac{x}{\bar{x}_m} \right] d(x/\bar{x}_m) \tag{43}
\]

If \(\phi\) is defined, \(x'_m\) is obtained from Table 2.

Fig. 8 shows the variation of \(dv/d(x/\bar{x}_m)\) with dimensionless diameter \((x/\bar{x}_m)\) for various values of \(\phi\).

The value of \(\phi\) is related to the degree of uniformity of the drop sizes in a spray, and experimental data shows \(\phi\) is a constant for a given nozzle over a wide range of operating conditions.

When \(\phi = 12\), volume distribution curves for various \(\beta\) are shown in Fig. 9. The values of \(\beta\) are also associated with the degree of uniformity of drop size and it is quite sensitive to variations in drop size distribution.

Fig. 10 shows the cumulative volume distribution curves for various values of \(\phi\). A plot of cumulative volume distribution on a Chi-square scale against a linear \((x/\bar{x}_m)^{6}\) scale should yield a straight line for \(\beta = 1\) as shown in Fig. 11.

Analysis of experimental data on a liquid spray for number versus diameter distribution is easily made by converting equation (33) into the form

\[
\log_{10} \left[ \frac{1}{x^{2} \cdot e^{-4}} \cdot \frac{dn}{dx} \right] = \log_{10} \left[ \frac{\beta \cdot b}{\Gamma(\phi/2 - 3/\beta)} \right] - \frac{b}{2.3} x^{\beta} \tag{44}
\]

A plot of \(\log \left[ \frac{1}{x^{2} \cdot e^{-4}} \frac{dn}{dx} \right] \) against \(x^{\beta}\) should yield a straight line with slope equal to \(b/2.3\). Here, the values of \(\phi\) and \(\beta\) are determined by trial and error to give the best alignment of data points. In actual practice, construction of straight line plots from original data is quite simple, as will be illustrated later.
If we have data on the volume distribution fraction, \( f_v \), we can put Eq. (31) in the form

\[
\log_{10} \left[ \frac{1}{x^{\beta}} \cdot \frac{dv}{dx} \right] = \log_{10} \left[ \frac{\beta \, b^{\phi/2}}{\Gamma(\phi/2)} \right] - \frac{b}{2.3} \times x^\beta
\]  

(45)

which expresses \( \log \left[ \frac{1}{x^{\beta}} \cdot \frac{dv}{dx} \right] \) as a linear function of \( x^\beta \).

Finally, the general equation for the mean diameter \( \bar{x}_{QP} \) for the \( x^\phi \) distribution function is obtained by substituting Eq. 33 into Eq. 14 to give

\[
\frac{\bar{x}_{QP}}{x} = b \cdot \frac{\Gamma(\phi/2 + (q-3)/\beta)}{\Gamma(\phi/2 + (p-3)/\beta)}
\]  

(46)

When \( \beta = 1 \), this simplifies to

\[
\frac{\bar{x}_{QP}}{x} = 1 \cdot \frac{\Gamma(\phi/2 + q - 3)}{\Gamma(\phi/2 + p - 3)}
\]  

(47)

If \( \phi = 2 \) in a \( x^2 \) distribution equation, Eq. 31, it becomes the Rosin-Rammler equation which was presented by Rosin and Rammler for application to powdered materials. That is

\[
\frac{dv_\phi}{dx} = b \cdot x^{\beta-1} \cdot e^{-bx^\beta}
\]  

(48)

\[
\nu^* = \int_0^x \frac{dv^*}{dx} \, dx = e^{-bx^\beta}
\]  

(49)

\[

\nu^* - \nu^* = 1 - e^{-bx^\beta}
\]  

(50)

where \( \beta \) and \( b \) are distribution constants. Thus the Rosin-Rammler equation can be obtained from the \( x^\phi \) distribution function.
When \( \frac{\alpha}{2} \beta = \alpha + 4 \), the \( \chi^2 \) distribution equation becomes the Nukiyama-Tanasawa equation which was empirically obtained from extensive experimental data on drop sizes in sprays formed by air atomization by S. Nukiyama and Y. Tanasawa.\(^{11}\) That is, from Eqs. (31) and (33)

\[
\frac{dy^*}{dx} = \frac{\alpha+4}{\beta} \frac{\alpha+3}{(\alpha+4)\beta} x^{\alpha} e^{-bx^\beta} e
\]

\[
\frac{dn^*}{dx} = \frac{\alpha+1}{\beta} \frac{\beta}{\Gamma(\frac{\alpha+1}{\beta})} x^{\alpha} e^{-bx^\beta} e
\]

where \( \alpha, \beta, \) and \( b \) are distribution constants. Thus, the Nukiyama-Tanasawa equation can also be obtained from the \( \chi^2 \) distribution function.

Further, when \( \frac{\alpha}{2} + 4 \approx r \), and \( \beta = -1 \), the \( \chi^2 \) distribution equation Eq. (33) becomes Griffith comminution equation which was derived by L. Griffith\(^{12}\) by applying the theory of probability to the molecular surface energy in an elementary comminuted system, that is

\[
\frac{dn^*}{dx} = k' x^{-r} e^{-b/x}
\]

where the values of \( k' \), \( r \), and \( b \) can be determined from experimental data. Note that

\[
k' = \frac{-b^{r-1}}{\Gamma(r-1)}
\]

so that \( k' \), \( r \), and \( b \) are related and there are really only two independent variables.

**DISCUSSION AND COMPARATIVE FIT OF DISTRIBUTION EQUATIONS**

The log-normal distribution and the \( \chi^2 \) distribution can be compared directly in terms of their (a) mathematical characteristics (b) numerical distribution (c) volume distribution and (d) cumulative volume distribution.
Frequency, cumulative and mean diameter expressions for the log-normal distribution and chi-square distribution respectively are listed in Table 4. Dimensionless expressions of chi-square distribution are also shown in Table 4. The value of the dimensionless distribution constants for various values of $\phi$ and $\beta=1$ are shown in Table 5.

As discussed previously, both distributions can be obtained from normal distributions. The log-normal distribution is skewed into the normal shape by using a logarithmic abscissa, while the $\chi^2$ distribution is formed from several normal distributions. The fact that the $\chi^2$ distribution may be given the form of empirical drop size distribution equations suggests that it may be possible to relate the parameters in the $\chi^2$ distribution equation to the fundamental mechanisms of spray formation.

The log-normal distribution has two parameters $\delta$ and $\bar{x}_m$, and dimensionless expressions can be made easily with $\bar{x}_m$. The $\chi^2$ distribution has three parameters $\phi$, $\beta$, and $b$ (usually, for sprays $\beta=1$). While the $\chi^2$ distribution can also be made nondimensional by using $\bar{x}_m$, it has the advantage of also being conveniently normalized using Sauter's mean diameter ($\bar{x}_{32}$). This is very convenient because Sauter's mean diameter has a useful physical meaning and is an important value for combustion and chemical reaction calculations. The nondimensional volume expression using $\bar{x}_{32}$ always has its maximum point at $x = \bar{x}_{32}$.

In Fig. 12, a log-normal curve (Fig. 2) is compared with a $\chi^2$ curve (Fig. 8) of the same maximum height. Because of its broader peak, its gentler slopes near the peak, and its lower tail, the distribution frequently provides a better fit to experimental data than the log-normal distribution.

The upper-limit log-normal distribution provides a good fit to some experimental data. However, this distribution is difficult to apply because of the complications involved in determining $\delta$ and $x_{\text{max}}$ for each set of experimental data. Because of these difficulties, the upper limit log-normal distribution is not a commonly used distribution.
TABLE 3

Logarithmic-Normal Distribution

\[ \frac{dv^*}{d(\ln\left(\frac{x}{x^*}\right))} = \frac{\delta}{\sqrt{\pi}} e^{-(\delta \ln\left(\frac{x}{x^*}\right))^2} \]

Frequency Distribution

\[ \frac{dn^*}{d(\ln\left(\frac{x}{x^*}\right))} = \frac{\delta}{\sqrt{\pi}} e^{-(\delta \ln\left(\frac{x}{x^*}\right) + \frac{3}{2\delta})^2} \]

Cumulative Distribution

\[ v^* = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-z^2} dz \]

\[ z = \ln\left(\frac{x}{x^*}\right) \delta \]

(error function)

Mean Diameter

\[ x_{qp} = x^* e^{\frac{p+q-6}{4\delta^2}} \]

Chi-Square Distribution

\[ \frac{dv}{dx} = \frac{\beta b}{\Gamma\left(\frac{\phi}{2}\right)} x^{\frac{\phi}{2}-1} e^{-bx^\frac{\phi}{2}} \]

\[ \frac{dn}{dx} = \frac{\beta b}{\Gamma\left(\frac{\phi}{2} - \frac{3}{\beta}\right)} x^{\frac{\phi}{2}-4} e^{-bx^\frac{\phi}{2}} \]

\[ v^* = \frac{1}{\Gamma\left(\frac{\phi}{2}\right)} \int_{0}^{\frac{x^2}{2\beta}} \frac{1}{\left(\chi^2\right)^{\frac{\phi}{2}-1}} e^{\frac{\phi}{2} \chi^2} d(\chi^2) \]

(Chi-Square function)
Table 4

Dimensionless Expression of Chi-Square Distribution

\[
\begin{align*}
\frac{dn}{n} &= A \left( \frac{x}{x_{32}} \right) \frac{\phi}{2} \beta - 4 \exp \left( -B \left( \frac{x}{x_{32}} \right) \beta \right) d \left( \frac{x}{x_{32}} \right) \\
\frac{dv}{v} &= A' \left( \frac{x}{x_{32}} \right) \frac{\phi}{2} \beta - 1 \exp \left( -B' \left( \frac{x}{x_{32}} \right) \beta \right) d \left( \frac{x}{x_{32}} \right) \\
\frac{dn}{n} &= C \left( \frac{x}{x_m} \right) \frac{\phi}{2} \beta - 4 \exp \left( -D \left( \frac{x}{x_m} \right) \beta \right) d \left( \frac{x}{x_m} \right) \\
\frac{dv}{v} &= C' \left( \frac{x}{x_m} \right) \frac{\phi}{2} \beta - 1 \exp \left( -D' \left( \frac{x}{x_m} \right) \beta \right) d \left( \frac{x}{x_m} \right)
\end{align*}
\]

where

\[
A = \left[ \frac{\Gamma \left( \frac{\phi}{2} \right)}{\Gamma \left( \frac{\phi}{2} - \frac{1}{\beta} \right)} \right]^{\frac{\phi}{2} \beta - 3} \frac{\beta}{\Gamma \left( \frac{\phi}{2} - \frac{3}{\beta} \right)}
\]

\[
B = \left[ \frac{\Gamma \left( \frac{\phi}{2} \right)}{\Gamma \left( \frac{\phi}{2} - \frac{1}{\beta} \right)} \right]^{\beta}
\]

\[
A' = \beta \left[ \frac{\Gamma \left( \frac{\phi}{2} \right)}{\Gamma \left( \frac{\phi}{2} - \frac{1}{\beta} \right)} \right]^{\frac{\phi}{2} \beta - 1} \frac{\phi}{2} \beta
\]

\[
B' = \left[ \frac{\Gamma \left( \frac{\phi}{2} \right)}{\Gamma \left( \frac{\phi}{2} - \frac{1}{\beta} \right)} \right]^{\beta}
\]

\[
C = \frac{\beta}{\Gamma \left( \frac{\phi}{2} - \frac{3}{\beta} \right)} \left( \frac{x_m}{2} \right)^{\frac{\phi}{2} - \frac{3}{\beta}}
\]

\[
D = \frac{x_m^2}{2}
\]

\[
C' = \frac{\beta}{\Gamma \left( \frac{\phi}{2} \right)} \left( \frac{x_m}{2} \right)^{\frac{\phi}{2}}
\]

\[
D' = \frac{x_m^2}{2}
\]
TABLE 5

The Value of Dimensionless Distribution Constants shown in Table 4 with Various $\phi$ ($\beta = 1$)

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>A</th>
<th>B</th>
<th>A'</th>
<th>B'</th>
<th>C</th>
<th>D</th>
<th>C'</th>
<th>D'</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.8921</td>
<td>2.5</td>
<td>7.4338</td>
<td>2.5</td>
<td>1.0050</td>
<td>3.173</td>
<td>17.1208</td>
<td>3.173</td>
</tr>
<tr>
<td>8</td>
<td>3.0000</td>
<td>3.0</td>
<td>13.5000</td>
<td>3.0</td>
<td>3.6721</td>
<td>3.672</td>
<td>30.3032</td>
<td>3.672</td>
</tr>
<tr>
<td>9</td>
<td>7.3885</td>
<td>3.5</td>
<td>24.1360</td>
<td>3.5</td>
<td>9.6134</td>
<td>4.171</td>
<td>53.1658</td>
<td>4.171</td>
</tr>
<tr>
<td>10</td>
<td>16.0000</td>
<td>4.0</td>
<td>42.6667</td>
<td>4.0</td>
<td>21.8174</td>
<td>4.671</td>
<td>92.6393</td>
<td>4.671</td>
</tr>
<tr>
<td>11</td>
<td>32.3143</td>
<td>4.5</td>
<td>74.7845</td>
<td>4.5</td>
<td>45.7293</td>
<td>5.171</td>
<td>160.5360</td>
<td>5.171</td>
</tr>
<tr>
<td>12</td>
<td>62.5000</td>
<td>5.0</td>
<td>130.2086</td>
<td>5.0</td>
<td>91.1493</td>
<td>5.670</td>
<td>276.9398</td>
<td>5.670</td>
</tr>
<tr>
<td>13</td>
<td>117.4071</td>
<td>5.5</td>
<td>225.4964</td>
<td>5.5</td>
<td>175.5492</td>
<td>6.170</td>
<td>475.9812</td>
<td>6.170</td>
</tr>
<tr>
<td>14</td>
<td>216.0000</td>
<td>6.0</td>
<td>388.7988</td>
<td>6.0</td>
<td>329.8080</td>
<td>6.670</td>
<td>815.4350</td>
<td>6.670</td>
</tr>
<tr>
<td>15</td>
<td>391.2615</td>
<td>6.5</td>
<td>667.9124</td>
<td>6.5</td>
<td>608.1952</td>
<td>7.170</td>
<td>1393.1954</td>
<td>7.170</td>
</tr>
<tr>
<td>16</td>
<td>700.2918</td>
<td>7.0</td>
<td>1143.8076</td>
<td>7.0</td>
<td>1105.4847</td>
<td>7.670</td>
<td>2374.6056</td>
<td>7.670</td>
</tr>
</tbody>
</table>
The $\chi^2$ distribution can conveniently be expressed for number frequency distribution, volume frequency distribution and cumulative distribution, but the log-normal distribution can be conveniently expressed only for cumulative volume distributions.

For example, Table 6 presents Ingebo's data. He measured number versus diameter of droplet for impinging jets in a rocket combustor. The data of the third column of Table 6, $n$, are measured directly. From this, the data of the fourth column of Table 6, $v$, are calculated using the data of droplet number, i.e., third column. Fig. 13 shows these results for Run 4. From curve A it is apparent that the number frequency curve is a fairly smooth curve but the volume-frequency distribution, curve B, is irregular, i.e., the data are scattered. Because of the scattering, it is very difficult to analyze correctly for volume distribution data. Curve C of Fig. 13 is residual volume distribution data obtained from the volume fraction data. This is a fairly smooth curve, but because it was obtained from too small a sample, each data point has a large uncertainty.

The log-normal and the upper limit log-normal distributions are suited only to cumulative volume data while the $\chi^2$ distribution can be applied directly to number frequency data as well as to cumulative volume data. For this reason the $\chi^2$ distribution is the most generally applicable.

Figs. 14, 15, and 16 show Ingebo's data plotted for log-normal, upper limit log-normal and $\chi^2$ distributions. Since the plot for the log-normal distribution is not a straight line, these data do not follow the log-normal distribution. While the $\chi^2$ distribution shows some scatter, it does have the previously mentioned advantage of being able to work with volume or number distributions.

Data for one run with a small air atomizing nozzle are given by Houghton as shown in Table 7. As shown in column 5, 99% of the number of droplets fall in the range of $2 \sim 20 \mu$ in diameter.
## TABLE 6

**EXPERIMENTAL RESULTS OF INGEBÖ (13)**

<table>
<thead>
<tr>
<th>(x_\mu)</th>
<th>(\Delta x_\mu)</th>
<th>(\Delta n)</th>
<th>(\Delta v)</th>
<th>(v^<em>-v^</em>)</th>
<th>(\frac{\Delta n}{\Delta x} \cdot \frac{1}{N x^2})</th>
<th>(\frac{x}{x_{\text{max}}-x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.75</td>
<td>12.5</td>
<td>60</td>
<td>0.0002</td>
<td>1.0000</td>
<td>1.2675 \times 10^{-5}</td>
<td>0.0617</td>
</tr>
<tr>
<td>26.25</td>
<td>12.5</td>
<td>717</td>
<td>0.0285</td>
<td>.9997</td>
<td>4.1537 \times 10^{-5}</td>
<td>0.1223</td>
</tr>
<tr>
<td>38.75</td>
<td>12.5</td>
<td>319</td>
<td>0.0603</td>
<td>.9712</td>
<td>8.4830 \times 10^{-6}</td>
<td>0.1957</td>
</tr>
<tr>
<td>51.25</td>
<td>12.5</td>
<td>291</td>
<td>0.0441</td>
<td>.9109</td>
<td>4.4227 \times 10^{-6}</td>
<td>0.2763</td>
</tr>
<tr>
<td>63.75</td>
<td>12.5</td>
<td>127</td>
<td>0.0515</td>
<td>.8668</td>
<td>1.2475 \times 10^{-6}</td>
<td>0.3686</td>
</tr>
<tr>
<td>76.25</td>
<td>12.5</td>
<td>76</td>
<td>0.0305</td>
<td>.8154</td>
<td>5.2196 \times 10^{-7}</td>
<td>0.4752</td>
</tr>
<tr>
<td>88.75</td>
<td>12.5</td>
<td>104</td>
<td>0.0735</td>
<td>.7849</td>
<td>5.2710 \times 10^{-7}</td>
<td>0.6000</td>
</tr>
<tr>
<td>101.25</td>
<td>12.5</td>
<td>67</td>
<td>0.0589</td>
<td>.7114</td>
<td>2.6090 \times 10^{-7}</td>
<td>0.7475</td>
</tr>
<tr>
<td>113.75</td>
<td>12.5</td>
<td>52</td>
<td>0.0735</td>
<td>.6525</td>
<td>1.6043 \times 10^{-7}</td>
<td>0.9251</td>
</tr>
<tr>
<td>126.25</td>
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<td>70</td>
<td>0.1130</td>
<td>.5790</td>
<td>1.7532 \times 10^{-7}</td>
<td>1.1429</td>
</tr>
<tr>
<td>138.75</td>
<td>12.5</td>
<td>38</td>
<td>0.0860</td>
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<tr>
<td>151.25</td>
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<td>0.0523</td>
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<td>226.25</td>
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<td>3</td>
<td>0.0257</td>
<td>.0257</td>
<td>2.3433 \times 10^{-9}</td>
<td>21.6300</td>
</tr>
</tbody>
</table>

\[ \begin{align*}
    x_{\text{max}} &= 236.71 \mu \\
    \bar{x}_m &= 137 \mu \\
    x_{10} &= 55 \mu \\
    x_{99} &= 204 \mu
\end{align*} \]
TABLE 6 cont.

(2) Run 11

<table>
<thead>
<tr>
<th>( x \mu )</th>
<th>( \Delta n )</th>
<th>( \Delta v )</th>
<th>( V^* - V^* )</th>
<th>( \frac{\Delta n}{\Delta x} \left( \frac{1}{N_{x^2}} \right) )</th>
<th>( x )</th>
</tr>
</thead>
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<tr>
<td>13.75</td>
<td>1070</td>
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<td>1.0000</td>
<td>1.5518\times10^{-4}</td>
<td>0.09776</td>
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<tr>
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<td>681</td>
<td>0.0280</td>
<td>0.9905</td>
<td>2.6864\times10^{-5}</td>
<td>0.20484</td>
</tr>
<tr>
<td>38.75</td>
<td>345</td>
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<td>0.33506</td>
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<td>4.48490</td>
</tr>
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<td>0.0311</td>
<td>0.0311</td>
<td>4.7537\times10^{-9}</td>
<td>48.0159</td>
</tr>
</tbody>
</table>

\[ x_{\text{max}} = 154.4 \mu \]
\[ \bar{x}_{90} = 132 \mu \]
\[ \bar{x}_{10} = 52 \mu \]
\[ \bar{x}_{m} = 95 \mu \]

\[ dn^* = 2.1 \times 10^{-4} x^2 e^{-0.0749x} \]
TABLE 6 cont.

(3) Run 23

<table>
<thead>
<tr>
<th>$x_\mu$</th>
<th>$\Delta n$</th>
<th>$\Delta v$</th>
<th>$v^<em>-v^</em>$</th>
<th>$\frac{\Delta n}{\Delta x} \frac{1}{N x^2}$</th>
<th>$\frac{x}{x_{\text{max}}-x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.75</td>
<td>78</td>
<td>0.0187</td>
<td>1.0000</td>
<td>$1.0316 \times 10^{-4}$</td>
<td>0.11331</td>
</tr>
<tr>
<td>26.25</td>
<td>100</td>
<td>0.0656</td>
<td>0.9813</td>
<td>$3.6281 \times 10^{-5}$</td>
<td>0.24116</td>
</tr>
<tr>
<td>38.75</td>
<td>61</td>
<td>0.1360</td>
<td>0.9157</td>
<td>$1.0156 \times 10^{-5}$</td>
<td>0.40218</td>
</tr>
<tr>
<td>51.25</td>
<td>39</td>
<td>0.1625</td>
<td>0.7797</td>
<td>$3.7122 \times 10^{-6}$</td>
<td>0.61121</td>
</tr>
<tr>
<td>63.75</td>
<td>18</td>
<td>0.1573</td>
<td>0.6172</td>
<td>$1.1072 \times 10^{-6}$</td>
<td>0.8935</td>
</tr>
<tr>
<td>76.25</td>
<td>12</td>
<td>0.1710</td>
<td>0.4599</td>
<td>$5.1600 \times 10^{-7}$</td>
<td>1.29567</td>
</tr>
<tr>
<td>88.75</td>
<td>8</td>
<td>0.1628</td>
<td>0.2889</td>
<td>$2.5594 \times 10^{-7}$</td>
<td>1.91478</td>
</tr>
<tr>
<td>101.25</td>
<td>3</td>
<td>0.0837</td>
<td>0.1261</td>
<td>$7.3156 \times 10^{-8}$</td>
<td>2.99113</td>
</tr>
<tr>
<td>113.75</td>
<td>1</td>
<td>0.0425</td>
<td>0.0423</td>
<td>$1.9322 \times 10^{-8}$</td>
<td>5.32787</td>
</tr>
</tbody>
</table>

$x_{\text{max}} = 135.1 \mu$

$\bar{x}_m = 74 \mu$

$\bar{x}_{10} = 40 \mu$

$\bar{x}_{90} = 105 \mu$

$dn^* = 4.2 \times 10^{-4} x^2 e^{-0.0944x} \, dx$
TABLE 7
ATOMIZATION IN A SMALL AIR-ATOMIZING NOZZLE

Data of Houghton (14)

<table>
<thead>
<tr>
<th>$x_\mu$</th>
<th>$\Delta x_\mu$</th>
<th>$\Delta n$</th>
<th>$\Delta n/N\Delta x$</th>
<th>$n^*$</th>
<th>$\Delta n/xN\Delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.5</td>
<td>390000</td>
<td>0.16275939</td>
<td>57.43</td>
<td>0.081379695</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>340000</td>
<td>0.07094640</td>
<td>82.47</td>
<td>0.014189280</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>165000</td>
<td>0.03442987</td>
<td>95.35</td>
<td>0.003442987</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>40200</td>
<td>0.00838837</td>
<td>98.31</td>
<td>0.000559225</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>11680</td>
<td>0.00243722</td>
<td>99.18</td>
<td>0.000121861</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>4970</td>
<td>0.00103707</td>
<td>99.54</td>
<td>0.000041483</td>
</tr>
<tr>
<td>30</td>
<td>5</td>
<td>2160</td>
<td>0.00045072</td>
<td>99.70</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>5</td>
<td>1730</td>
<td>0.00036099</td>
<td>99.83</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>5</td>
<td>1080</td>
<td>0.00022536</td>
<td>99.91</td>
<td></td>
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<tr>
<td>45</td>
<td>5</td>
<td>650</td>
<td>0.00013563</td>
<td>99.96</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>7.5</td>
<td>430</td>
<td>0.00005982</td>
<td>99.98</td>
<td></td>
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<tr>
<td>60</td>
<td>10</td>
<td>350</td>
<td>0.00003652</td>
<td>99.99</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>10</td>
<td>220</td>
<td>0.00002295</td>
<td>100.00</td>
<td></td>
</tr>
</tbody>
</table>

$$dn = 0.062 \times e^{-0.25x} dx$$
while only 1% fall in the range of 25 μ - 70 μ in diameter.

The data of 99% of droplets are plotted as shown in Fig. 17, with the result that the data are found to agree in range of 2 - 25 μ quite well with a χ² distribution for dn/dx when φ = 10, and b = 0.25. The the distribution equation is expressed by

$$\frac{dn^*}{dx} = 0.062 x e^{-0.25x}$$

The drop size distribution produced by a particular swirl type nozzle has been measured by Edward Rice. He took droplet pictures with 25 times linear magnification using a fluorescent technique. The number and size of droplets on the photographs were determined with a flying spot automatic counter.

Data for ethanol is given in Table 8. The data were analyzed using the χ² distribution by the authors. As expected, the data follow the same dimensionless equations even if the injection pressure is changed, as shown in Fig. 18, i.e.,

$$dn^* = 0.8921 \left( \frac{x}{x_{32}} \right)^{-0.5} e^{-2.5x} d(\frac{x}{x_{32}})$$

Thus for this nozzle the property combination the distribution is determined only by the value of Sauter's mean diameter (x_{32}), and x_{32} is found to be a function of the injection pressure. This suggests that for a given nozzle and fluid property combination the distribution equation is determined by the value of x_{32} and that x_{32} might be expected, in general, to be a function of the injection pressure.

Hiroyasu and Tanasawa, using a new type of drop size analyzer based on sedimentation, measured the drop size distributions produced by various types of Diesel injection nozzles.

The droplets sprayed from a nozzle were allowed to fall down a settling tower onto and automatic recording balance to give weight versus time data. Then diameter versus falling time data were used with the weight versus time data to calculate the cumulative weight versus diameter distribution. Then this data was analyzed using the cumulative volume χ² distribution expressions.
TABLE 8
DROPLET DATA OF SWIRL ATOMIZER
OBTAINED BY EDWARD RICE (15)  

<table>
<thead>
<tr>
<th>$x_\mu$</th>
<th>$\Delta x_\mu$</th>
<th>$\Delta n$</th>
<th>$\Delta n/\Delta x_\mu$</th>
<th>$x^{0.5} \Delta n/\Delta x_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.07</td>
<td>4.14</td>
<td>5260</td>
<td>$9.0765 \times 10^{-2}$</td>
<td>$3.150 \times 10^{-1}$</td>
</tr>
<tr>
<td>17.07</td>
<td>5.86</td>
<td>2278</td>
<td>$2.7771 \times 10^{-2}$</td>
<td>$1.147 \times 10^{-1}$</td>
</tr>
<tr>
<td>24.14</td>
<td>8.28</td>
<td>3022</td>
<td>$2.6073 \times 10^{-2}$</td>
<td>$1.278 \times 10^{-1}$</td>
</tr>
<tr>
<td>34.14</td>
<td>11.72</td>
<td>1270</td>
<td>$7.7412 \times 10^{-3}$</td>
<td>$4.521 \times 10^{-2}$</td>
</tr>
<tr>
<td>48.29</td>
<td>16.57</td>
<td>1166</td>
<td>$5.0270 \times 10^{-3}$</td>
<td>$3.489 \times 10^{-2}$</td>
</tr>
<tr>
<td>68.29</td>
<td>23.43</td>
<td>647</td>
<td>$1.5727 \times 10^{-3}$</td>
<td>$1.630 \times 10^{-2}$</td>
</tr>
<tr>
<td>96.57</td>
<td>33.14</td>
<td>222</td>
<td>$4.7856 \times 10^{-4}$</td>
<td>$4.695 \times 10^{-3}$</td>
</tr>
<tr>
<td>136.57</td>
<td>46.86</td>
<td>114</td>
<td>$1.7379 \times 10^{-4}$</td>
<td>$2.030 \times 10^{-3}$</td>
</tr>
<tr>
<td>193.14</td>
<td>66.27</td>
<td>19</td>
<td>$2.0482 \times 10^{-5}$</td>
<td>$2.849 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

$dn^* = 0.37 x 10^{-0.5} - 0.43 x 10^{-0.43}$  

P = 70 psi

<table>
<thead>
<tr>
<th>$x_\mu$</th>
<th>$\Delta x_\mu$</th>
<th>$\Delta n$</th>
<th>$\Delta n/\Delta x_\mu$</th>
<th>$x^{0.5} \Delta n/\Delta x_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.07</td>
<td>4.14</td>
<td>6206</td>
<td>$1.1976 \times 10^{-1}$</td>
<td>$4.156 \times 10^{-1}$</td>
</tr>
<tr>
<td>17.07</td>
<td>5.86</td>
<td>2258</td>
<td>$3.0784 \times 10^{-2}$</td>
<td>$1.271 \times 10^{-1}$</td>
</tr>
<tr>
<td>24.14</td>
<td>8.28</td>
<td>2667</td>
<td>$2.5733 \times 10^{-2}$</td>
<td>$1.261 \times 10^{-1}$</td>
</tr>
<tr>
<td>34.14</td>
<td>11.72</td>
<td>730</td>
<td>$4.9762 \times 10^{-3}$</td>
<td>$2.906 \times 10^{-2}$</td>
</tr>
<tr>
<td>48.29</td>
<td>16.57</td>
<td>509</td>
<td>$2.4541 \times 10^{-3}$</td>
<td>$1.212 \times 10^{-2}$</td>
</tr>
<tr>
<td>68.29</td>
<td>23.43</td>
<td>127</td>
<td>$4.3304 \times 10^{-4}$</td>
<td>$3.577 \times 10^{-3}$</td>
</tr>
<tr>
<td>96.57</td>
<td>33.14</td>
<td>19</td>
<td>$4.5804 \times 10^{-5}$</td>
<td>$4.493 \times 10^{-4}$</td>
</tr>
<tr>
<td>136.57</td>
<td>46.86</td>
<td>1</td>
<td>$1.7049 \times 10^{-6}$</td>
<td>$1.991 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

$dn^* = 0.48 x 10^{-0.5} - 0.72 x 10^{-0.72}$  

P = 70 psi
TABLE 8
(continued)

\[ P = 100 \text{ psi} \]

\[
\begin{array}{cccccc}
\mu & \Delta x & \Delta n & \Delta n/N\Delta x & x^{0.5} & \Delta n/N\Delta x \\
12.07 & 4.14 & 7752 & 1.386 \times 10^{-1} & 4.809 \times 10^{-1} \\
17.07 & 5.86 & 2600 & 3.285 \times 10^{-2} & 1.357 \times 10^{-1} \\
24.14 & 8.28 & 2319 & 2.074 \times 10^{-2} & 1.016 \times 10^{-1} \\
34.14 & 11.72 & 534 & 3.373 \times 10^{-3} & 1.970 \times 10^{-2} \\
48.29 & 16.57 & 263 & 1.175 \times 10^{-3} & 8.155 \times 10^{-3} \\
68.29 & 23.43 & 37 & 1.169 \times 10^{-4} & 9.656 \times 10^{-4} \\
96.57 & 33.14 & 2 & 4.468 \times 10^{-6} & 4.383 \times 10^{-5} \\
\end{array}
\]

\[ dn^* = 0.59 \times e^{-1.09x} \text{ dx} \]
Fig. 19 shows the effect of rotational speed of the pump on the size distribution of drops produced by a throttle nozzle. From these curves we can obtain the distribution parameters \( \phi \) and \( \bar{x}_m \). Having \( \phi \), the value of \( b \) can be obtained using Eq. 39 and Table 2. The results of this experiment showed that even if operating conditions are changed, the distribution parameter \( \phi \) does not change. It was also found that the median diameter \( \bar{x}_m \) of sprays decreases with increase in rotational speed, i.e., an increase of injection pressure. The expression obtained for the drop size distribution of sprays produced by a throttle nozzle is

\[
dv^* = 30.30 \left( \frac{x}{\bar{x}_m} \right)^{3.67} \frac{1}{\bar{x}_m} \left( \frac{x}{\bar{x}_m} \right) d(x/\bar{x}_m)
\]

Similarly, the results obtained for the pintle nozzle are shown in Fig. 20, and the drop size distribution of sprays is expressed by

\[
dv^* = 53.17 \left( \frac{x}{\bar{x}_m} \right)^{3.5} \frac{1}{\bar{x}_m} \left( \frac{x}{\bar{x}_m} \right) d(x/\bar{x}_m)
\]

Fig. 22 shows the effect of viscosity of fuel on the size distribution of drops. These results show that the effect of viscosity of fuel is larger than it is usually expected to be. When the viscosity of the liquid becomes larger than 10 cp, the values of \( \phi \) decrease with increase in viscosity as shown in Fig. 22 and \( \phi \) takes values of \( \phi = 8 \) for \( \mu = 10 \) cp to \( \phi = 4.4 \) for \( \mu = 200 \) cp.

As has been illustrated in the above examples, experimental data show that the size distributions produced by a number of spray producing devices follow the \( \chi^2 \) distribution reasonably well. This distribution can be graphed fairly simply, and furthermore, if the parameters \( \beta \) and \( \phi \) are defined, several expressions of distribution (volume, number, cumulative, etc.) can be found and several mean diameters can be calculated.
ACKNOWLEDGMENT

The author wishes to thank Mr. Richard Sowls for his valuable assistance on this paper. The author would also like to thank Prof. P.S. Myers and Prof. O.A. Uyehara for their many helpful suggestions and encouragement. The review of the manuscript by Paul Wieber was very helpful.
BIBLIOGRAPHY


Fig. 1

Logarithmic-Normal Distribution

\[ y = \frac{\delta}{2} \left( e^{-\delta^2 y^2} - 2^n(y) \right) \]

\[ \frac{dv^*}{dy} = \frac{\delta}{2} \left( e^{-\delta^2 y^2} - 2^n(y) \right) \]

\[ x/x_m \]

\[ y = \ln(x/x_m) \]
\[
dv^*/d\left(\frac{x}{\bar{x}m}\right)
\]

Fig. 2

Logarithmic-Normal Distribution

\[
dv^* = \frac{\delta}{(\bar{x}m)} \cdot x^{-1} \cdot \left(\frac{\ln(x)}{m}\right)^2
\]
\[
\frac{dv^*}{dy} = \frac{\delta}{\sqrt{\pi}} e^{-\delta^2 y^2}
\]

\[
y = \ln \left( \frac{x}{x_m} \right)
\]

\[
\delta = 0.394 \frac{x_{90}}{\log \left( \frac{x_{50}}{x_m} \right)}
\]

Fig. 3
INEGBO'S DATA
RUN 4

**CUMULATIVE VOLUME V**

\[ u = \frac{x}{x_{\text{max}} - x} \]

\[ X_{10} = 55.0\mu \]
\[ X_{90} = 204.0\mu \]
\[ X_{50} = 135.0\mu \]
\[ X_{\text{max}} = 236.71\mu \]
\[ \delta = 0.63 \]

Fig. 4
\[
\frac{2}{\chi} \left[ 1 - \frac{2}{\phi} \right] = \frac{(z\chi)_p}{\phi^{\nu p}}
\]
CUMULATIVE VOLUME v*

Fig. 10

x/\bar{x}^m

CHI-SQUARE DISTRIBUTION

\phi = 14
\phi = 12
\phi = 10
\phi = 8
\phi = 6
\phi = 4
FIG. 12
DROP DIAMETER $x/x_m$

$\frac{dv}{d(r)} |_{x}$

$P_{in}(x_x)$

LOG-NORMAL DISTRIBUTION

$\phi = 12, P = 1 (Eq. 43)$

CHI-SQUARE DISTRIBUTION

$8 = 1.6 (Eq. 23)$
Fig. 15

\[ u = \frac{x}{x_{\text{max}} - x} \]
INGEBO'S DATA

Run | Air Vel. | Equation
--- | --- | ---
4 | 100 ft/sec | $dn = 5.3 \times 10^{-5} x^2 e^{-0.0473x} dx$
11 | 180 | $dn = 2.1 \times 10^{-4} x^2 e^{-0.0749x} dx$
23 | 300 | $dn = 4.2 \times 10^{-4} x^2 e^{-0.0944x} dx$

Fig. 16
HOUGHTON'S DATA

\[ \frac{dn^*}{dx} = 0.062 \times e^{-0.25x} \]
RICE'S DATA
- P = 40 psi $\bar{x}_{32} = 58.1 \mu$
- P = 70 psi $\bar{x}_{32} = 34.5 \mu$
- P = 100 psi $\bar{x}_{32} = 22.9 \mu$

$$dn^* = 0.892 \left( \frac{x}{\bar{x}_{32}} \right)^{-0.5} e^{-2.5x} \cdot d \left( \frac{x}{\bar{x}_{32}} \right)$$

Fig. 18
Figure 20

DROPLET DIAMETER

CUMULATIVE VOLUME $V^*$

PIN TLE NOZZLE ($P_0 = 100 \, \text{kPa}$)
Fig. 22

DROPLET DIAMETER

CUMULATIVE VOLUME V* - V*

THROTTELED NOZZLE (P0=100 Kgf/cm²)

\[ \phi = 4.2 \]

\[ \phi = 5.4 \]

\[ \phi = 4.8 \]

\[ \phi = 4.6 \]

\[ \phi = 5.0 \]

\[ \phi = 21 \text{ mm} \]

\[ \phi = 500 \text{ rpm} \]

\[ \phi = 225.4 \text{ cp} \]

\[ \phi = 8 \]
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