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SOME APPLICATIONS OF ALGEBRAIC METHODS

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N O T I C E

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CONTENTS

INTRODUCTION

Classical techniques for analysis and design of dynamic systems are largely restricted to cases in which only one parameter of the system is adjustable. As a consequence complex systems cannot be treated adequately with classical techniques. Algebraic methods, as developed in NASA CR-617*. are capable of treating systems in which two parameters are adjustable, and thus permit analysis and synthesis of systems which are too complex for treatment with classical methods.

The treatment of algebraic methods presented in CR-617 develops the fundamental theoretical basis for the coefficient plane and parameter plane methods. It also applies these methods to basic problems such as stability analysis, cascade compensation of systems, and related topics. The applications indicated in CR-617 are rather elementary, i.e., the problems considered illustrated the procedures to be used but were not very complex problems. This report is based on the findings of CR-617, and extends the applications of the algebraic methods to problems of a more complex nature.

When cascade compensation is used in a feedback control system, more than one filter section may be required to achieve desired performance. Frequency response methods involving trial and error are often used, but parameter plane methods permit analysis and design without trial and error if it is permissible

*Algebraic Methods for Dynamic Systems by G. J. Thaler, D. D. Siljak and R. C. Dorf, Nasa Contractor Report NASA CR-617, Nov., 1966.

to use two identical filter sections. This problem is treated in Chapter I of this report. The applicable parameter plane equations are derived and a digital computer program based on these equations is presented. The program is used to study the effects of compensation on several systems.

Chapters II and III are concerned with nonlinear systems. Conventional methods such as frequency domain analysis of systems with the Describing function have proven useful when the system contains only one nonlinearity (or several nonlinearities conveniently located so that they can be incorporated in one describing function). These techniques can define stability and estimate relative stability for fairly complex systems as long as the conditions of nonlinearity are not too complex. Such cases are easily treated using algebraic methods, the effect of the nonlinearity being represented as a movement of the operating point on the parameter plane, which in turn represents a variation of the characteristic roots as a function of signal amplitude. The algebraic methods are capable of extending such analysis to systems containing two distinct nonlinear components, and can be used to predict the transient response of the system rather accurately. Techniques for such problems are developed in Chapter II.

Chapter III is concerned with a much more difficult nonlinear problem, that of asymmetrical nonlinear oscillations. These are oscillations consisting of a limit cycle superimposed on another signal. The problems studied on the parameter plane involve steady-state operating conditions (rather than transient conditions), and permit analysis of the existence of oscillations as well as their dependence on parameter values and input signal values. Extension to linearization with either signals is included, as well as some design considerations.

It is felt that the results presented here indicate the capabilities of the algebraic methods in dealing with complex linear and nonlinear problems. It is also felt that the results presented here will be directly applicable to a number of practical problems, and will point out avenues of approach to still additional problems.

I

SOLUTION OF EQUATIONS WITH COEFFICIENTS THAT ARE QUADRATIC IN α and β

1.1 INTRODUCTION

It has been shown that the characteristic equation can be solved for $\alpha = \alpha(\zeta, \omega_n)$ and $\beta = \beta(\zeta, \omega_n)$ when the coefficients of the characteristic equation are of the forms:

$$
\begin{aligned}
&\text{a)} \quad a_k = b_k\alpha + c_k\beta + d_k \\
&\text{b)} \quad a_k = b_k\alpha + c_k\beta + h_k\alpha\beta + d_k \\
&\text{c)} \quad a_k = b_{k2}\alpha^2 + b_{k1}\alpha + h_k\alpha\beta + c_{k1}\beta + c_{k2}\beta^2 + d_k \\
&\text{d)} \quad a_k = b_{kn}\alpha^n + b_{k(n-1)}\alpha^{n-1} + \ldots h_{k(n-1)}\alpha^{n-1}\beta + \\
&\qquad\qquad c_{k(n-1)}\beta^{n-1} + c_{kn}\beta^n + d_k
\end{aligned}
\tag{1}
$$

In addition practical solutions have been obtained for the first two of these coefficient forms, i.e., computer programs have been written for them and successfully applied. The development to be presented here is a particular solution for case 1-1c, particularly in the sense that a computer program has been obtained which solves the equations of a third order system for which the coefficients are quadratic in α and β, but which do not contain <u>all</u> of the α and β combinations indicated. At the same time the solution is a general solution in the sense that the program ca be modified to solve the equations of an n^{th} order system, an can also be modified to accept all of the α and β forms indicated in

$$a_k = b_{k2}\alpha^2 + b_{k1}\alpha + h_k\alpha\beta + c_{k1}\beta + c_{k2}\beta^2 + d_k$$

The modifications to be made in the program are discussed, but the necessary programming has not been done.

1.2 THE PROBLEM: Cascade Compensation with two identical filter sections.

In the design of feedback control systems it is common to use compensators which are filters placed in cascade with the main transmission path. Frequently two sections of filter are needed, and if identical sections are used with an isolation amplifier so that their transfer functions can be multiplied, then manipulation of the transfer function equation provides a characteristic equation in which the coefficients are quadratic in z and p, the zero and pole of the compensators. For example let:

$$G = \frac{K}{s^3+Xs^2+Ys} \tag{1-2}$$

$$G_c = \left(\frac{s+z}{s+p}\right)^2 = \frac{s^2+2zs+z^2}{s^2+2ps+p^2} \tag{1-3}$$

$$1+G_cG = 0 = 1 + \frac{K(s^2+2zs+z^2)}{(s^3+Xs^2+Ys)(s^2+2ps+p^2)} \tag{1-4}$$

from which the characteristic equation is

$$s^5 + (X+2p)s^4 + (p^2+2Xp+Y)s^3 + (Xp^2+2Yp + K)s^2 + \\ + (Yp^2 + 2Kz)s + Kz^2 = 0 \tag{1-5}$$

Letting $p \overset{\Delta}{=} \alpha$ and $z \overset{\Delta}{=} \beta$ it is noted that all of the forms specified in the quadratic case definition of a_k do appear in at least some of the coefficients except that there is no $\alpha\beta$ product term.

The formulation just given does not conform to normal control system practice, however, in that an important restriction on the design of the compensator is the usual requirement that steady state accuracy must be maintained by keeping the error coefficient unchanged. To do this the physical adjustment is to alter the gain of the amplifier, but in the mathematical analysis it is more convenient to include this restriction in the transfer function of the compensator by defining (for this case)

$$G_c = \left(\frac{p}{z}\right)^2 \left(\frac{s+z}{s+p}\right)^2 \tag{1-6}$$

This alters the algebraic form of the characteristic equation which becomes:

$$0 = 1 + \frac{K\left(\frac{p}{z}\right)^2 (s^2+2zs+z^2)}{(s^3+Xs^2+Ys)(s^2+2ps+p^2)}$$

$$= (s^3+Xs^2+Ys)(s^2+2ps+p^2)+K\,\frac{p^2}{z^2}(s^2+2zs+z^2)$$

$$= s^5+(X+2p)s^4 + (p^2+2Xp+Y)s^3+ \left[Xp^2+2Yp+K\left(\frac{p}{z}\right)^2\right]s^2$$

$$+\left[Yp^2+2Kp\left(\frac{p}{z}\right)\right]s + Kp^2 \tag{1-7}$$

Choosing $p \triangleq \beta$ and $\frac{p}{z} \triangleq \alpha$ this becomes

$$0 = s^5 + (X+2\beta)s^4 + (\beta^2+2X\beta+Y)s^3 + (X\beta^2+2Y\beta+K\alpha^2)s^2$$

$$+ (Y\beta^2+2K\beta\alpha)s + K\beta^2 \tag{1-8}$$

In equation 1-8 the coefficients are quadratic in α and β, but there is no term of the form $b_{kl}\alpha$, and the program as written does not make provision for such a term, though modification of the program to include it is not difficult. The problem to be studied, then is that of a third order system compensated with two cascaded identical sections of filter, and with the added requirement that the error coefficient be maintained constant at a predetermined value.

1.3 DERIVATION OF THE GENERAL THIRD ORDER SYSTEM RELATIONSHIPS

The general third order system is defined by the transfer function

$$G(s) = \frac{K}{(s+A)(s+B)(s+C)} \tag{1-9}$$

which is a Type Zero system, but which can be changed to Type 1, 2, or 3 by setting one or more of the poles to zero. The compensator transfer function, including the gain multiplier which maintains the error coefficient is

$$G_c = \left(\frac{p}{z}\right)^2 \left(\frac{s+z}{s+p}\right)^2 = \frac{p^2(s^2+2zs+z^2)}{z^2(s^2+2ps+p^2)} \tag{1-10}$$

From 1-9 and 1-10 the characteristic equation is

$$\left[s^3 + (A+B+C)s^2 + (AB+BC+AC)s + ABC\right](s^2+2ps+p^2) + $$

$$+ K\frac{p^2}{z^2}(s^2+2zs+z^2) = 0 \tag{1-11}$$

This expands to

$${}^{5} + (A+B+C+2p)s^4 + \left[AB+BC+AC+2p(A+B+C)+p^2\right]s^3$$

$$+ \left[ABC+2p(AB+BC+CA) + p^2(A+B+C) + K\frac{p^2}{z^2}\right]s^2$$

$$+ \left[2pABC+p^2(AB+BC+AC) + 2Kp\left(\frac{p}{z}\right)\right]s+p^2(ABC+2K) = 0$$

Let $p \triangleq \beta \qquad \frac{p}{z} \triangleq \alpha$

$A+B+C \quad \triangleq \sum r_i$ = sum of roots (poles)

$AB+BC+AC \triangleq \sum \prod_2 r_i$ = sum of root products taken 2 at a time

$\sum \prod_n r_i \triangleq$ sum of root products taken n at a time

$ABC \quad \triangleq \quad \prod r_i \triangleq$ products of the roots

Then equation 1-12 becomes:

$$s^5 + (\sum r_i + 2\beta)s^4 + (\sum \prod_2 r_i + 2\sum r_i\beta+\beta^2)s^3 +$$

$$(\prod r_i+2\sum \prod_2 r_i\beta + \sum r_i\beta^2+ K\alpha^2)s^2 + \tag{1-13}$$

$$(2\prod r_i\beta + \sum \prod_2 r_i\beta^2 + 2K\alpha\beta)s + (\prod r_i+2K)\beta^2 = 0$$

Collecting like terms in α and β:

$$\alpha^2(Ks^2) + \alpha\beta(2Ks) + \beta^2(s^3+ \sum r_i s^2 + \sum \prod_2 r_i s + \prod r_i+K) +$$

$$+ \beta(2s^4 + 2\sum r_i s^3 + 2\sum \prod_2 r_i s^2 + 2\prod r_i s)$$

$$+ (s^5 + \sum r_i s^4 + \sum \prod_2 r_i s^3 + \prod r_i s^2) = 0 \tag{1-14}$$

Using the basic parameter plane relationships:

$$\sum_{k=0}^{n} (-1)^k a_k\omega^k \overline{U}_{k-1}(\zeta) = 0 \tag{1-15}$$

$$\sum_{k=0}^{n} (-1)^k a_k\omega^k \overline{U}_k(\zeta) = 0 \tag{1-16}$$

and defining:

$$B_{21} = K\omega^2U_1(\zeta) \tag{1-17}$$

$$B_{22} = K\omega^2U_2(\zeta) \tag{1-18}$$

$$D_1 = -2K\omega U_o(\zeta) \tag{1-19}$$

$$D_2 = -2K\omega U_1(\zeta) \tag{1-20}$$

$$E_{11} = 2\omega^4U_3(\zeta) -2\sum r_i\omega^3U_2(\zeta) + 2\sum \prod_2 r_i\omega^2U_1(\zeta)$$

$$- 2\prod r_i\omega U_o(\zeta) \tag{1-21}$$

$$E_{12} = 2\omega^4U_4(\zeta) -2\sum r_i\omega^3U_3(\zeta) + 2\sum \prod_2 r_i\omega^2U_2(\zeta)$$

$$-2\prod r_i\omega U_1(\zeta) \tag{1-22}$$

$$F_{21} = -\omega^3 U_2(\zeta) + \sum r_i \omega^2 U_1(\zeta) - \sum \prod_2 r_i \omega U_o(\zeta)$$
$$+ (\Pi r_i + K) U_{-1}(\zeta) \qquad (1\text{-}23)$$

$$F_{22} = -\omega^3 U_3(\zeta) + \sum r_i \omega^2 U_2(\zeta) - \sum \prod_2 r_i \omega U_1(\zeta)$$
$$+ (\Pi r_i + K) U_o(\zeta) \qquad (1\text{-}24)$$

$$G_1 = -\omega^5 U_4(\zeta) + \sum r_i \omega^4 U_3(\zeta) - \sum \prod_2 r_i \omega^3 U_2(\zeta)$$
$$+ \Pi r_i \omega^2 U_1(\zeta) \qquad (1\text{-}25)$$

$$G_2 = -\omega^5 U_5(\zeta) + \sum r_i \omega^4 U_4(\zeta) - \sum \prod_2 r_i \omega^3 U_3(\zeta)$$
$$+ \Pi r_i \omega^2 U_2(\zeta) \qquad (1\text{-}26)$$

$$P_1 = \beta D_1 \qquad (1\text{-}27)$$

$$P_2 = \beta D_2 \qquad (1\text{-}28)$$

$$Q_1 = \beta E_{11} + \beta^2 F_{21} + G_1 \qquad (1\text{-}29)$$

$$Q_2 = \beta E_{12} + \beta^2 F_{22} + G_2 \qquad (1\text{-}30)$$

This results in

$$\alpha^2 B_{21} + \alpha P_1 + Q_1 = 0 \qquad (1\text{-}31)$$

$$\alpha^2 B_{22} + \alpha P_2 + Q_2 = 0 \qquad (1\text{-}32)$$

which are two non-linear algebraic equations completely generalized in terms of the uncompensated system poles and root locus gain, ζ, ω and the first kind of Chebyshev Functions. These must be solved simultaneously for the correct values of α and β. To do this, the method with the best chance of success appears to be Sylvester's Method in which we form a set of four equations by taking the original Equations (1-31) and (1-32) and forming two more by a multiplication with α giving:

$$\alpha^2 B_{21} + \alpha P_1 + Q_1 = 0 \qquad (1\text{-}33)$$

$$\alpha^2 B_{22} + \alpha P_2 + Q_2 = 0 \qquad (1\text{-}34)$$

$$\alpha^3 B_{21} + \alpha^2 P_1 + \alpha Q_1 = 0 \tag{1-35}$$

$$\alpha^3 B_{22} + \alpha^2 P_2 + \alpha Q_2 = 0 \tag{1-36}$$

Now placing these equations in matrix form:

$$\begin{bmatrix} 0 & B_{21} & P_1 & Q_1 \\ 0 & B_{22} & P_2 & Q_2 \\ B_{21} & P_1 & Q_1 & 0 \\ B_{22} & P_2 & Q_2 & 0 \end{bmatrix} \begin{bmatrix} \alpha^3 \\ \alpha^2 \\ \alpha \\ 1 \end{bmatrix} = 0 \tag{1-37}$$

If the α's are not zero then:

$$\begin{vmatrix} 0 & B_{21} & P_1 & Q_1 \\ 0 & B_{22} & P_2 & Q_2 \\ B_{21} & P_1 & Q_1 & 0 \\ B_{22} & P_2 & Q_2 & 0 \end{vmatrix} = 0 \tag{1-38}$$

Expanding this determinant

$$-B_{21}^2 Q_2^2 + B_{21}B_{22}Q_1Q_2 + P_1P_2B_{21}Q_2 - P_1^2Q_2B_{22} + Q_1Q_2B_{21}B_{22}$$
$$-Q_1^2B_{22}^2 - Q_1P_2^2B_{21} + Q_1P_1P_2B_{22} = 0 \tag{1-39}$$

Substituting equations (1-27) through (1-30) in equation (1-39) provides a fourth order equation in β:

$$\beta^4(-F_{22}^2B_{21}^2 + 2F_{21}F_{22}B_{21}B_{22} + D_1D_2F_{22}B_{21} - F_{22}D_1^2B_{22} -$$
$$F_{21}^2B_{22}^2 - F_{21}D_2^2B_{21} + D_1D_2F_{21}B_{22}) +$$
$$\beta^3(-2E_{12}F_{22}B_{21}^2 + 2E_{11}F_{22}B_{21}B_{22} + 2F_{21}E_{12}B_{21}B_{22} +$$
$$D_1D_2E_{12}B_{21} - D_1^2E_{12}B_{22} - 2E_{11}F_{21}B_{22}^2 -$$
$$D_2^2E_{11}B_{21} + D_1D_2E_{11}B_{22}) +$$
$$\beta^2(-E_{12}^2B_{21}^2 - 2F_{22}G_2B_{21}^2 + 2E_{11}E_{12}B_{21}B_{22} + 2F_{21}G_2B_{21}B_{22} +$$

$$2G_1F_{22}B_{21}B_{22} + D_1D_2G_2B_{21} - D_1^2G_2B_{22} - E_{11}^2B_{22}^2 -$$

$$2F_{21}G_1B_{22}^2 - D_2^2G_1B_{21} + D_1D_2G_1B_{22}) +$$

$$\beta(-2E_{12}G_2B_{21}^2 + 2E_{11}G_2B_{21}B_{22} + 2G_1E_{12}B_{21}B_{22} - 2E_{11}G_1B_{22}^2) +$$

$$(-G_2^2B_{21}^2 + 2G_1G_2B_{21}B_{22} - G_1^2B_{22}^2) = 0 \qquad (1\text{-}40)$$

from which the coefficients may be determined by a substitution of (1-17) through (1-26) and the values of the first kind of Chebyshev functions in terms of ζ and ω. Since the solution of a fourth order equation is at best difficult, it is at this point a digital computer becomes a necessity.

The major problem is not the actual solution of the quartic itself, but rather the proper choice of one of the four solutions. There are two marked characteristics, however, which help in the selection. These are:

a) Complex answers to the quartic have no physical significance and may therefore be discarded as erroneous.

b) The definition of α requires that α and β be of the same sign so that p and z will be of identical sign.

Using this information and that available from the Ross-Warren[2] method as to compensator pole and zero location, it is found that the solution to the β quartic is the largest, positive, rea value.

Now entering equation (1-27) with this value, and evaluating the other coefficients

$$\alpha = [-Q_1/B_{21}]^{\frac{1}{2}} \qquad (1\text{-}41)$$

for in the third order case P_1 is always identically zero.

Thus, with the programming of the appropriate equations, the digital computer could give all of the values and plot the constant zeta and constant omega loci on the Parameter Plane for any desired values.

1.4 SOME APPLICATIONS OF THE PROGRAM

Several third order systems were investigated by the application of the generalized equations and the Parameter Plane curves, Figures 1-1 through 1-8 were plotted. Of these, the K/s^3 family appears the most interesting. Further investigation of three of the curves in this family, Figures 1-1, 1-2 and 1-3 shows that there is a relationship between K, the root locus gain, α and β.

These relations are:

a) Choose a point on the $1/s^3$ α-β plane.

b) Zeta reads directly.

c) Determine the actual omega at that point by multiplying the value read by the cube root of the uncompensated system gain.

d) Read the value of α directly from the point chosen.

e) Read the value of β from the point chosen.

f) Obtain the true value of β by multiplying this value by the cube root of the uncompensated system gain.

By this method, the values of α and β may be determined for all $\frac{K}{s^3}$ systems from one universal curve.

1.5 BANDWIDTH CURVES ON THE α-β Plane

In many instances, there is also a bandwidth criterion imposed on the engineer as well as an optimal operating point for the plant under consideration. With this in mind, equations for the plotting of constant bandwidth curves on the $\alpha-\beta$ plane are developed. For the purpose of this development a constant bandwidth curve will be defined as:

> A constant bandwidth curve for $G(j\omega_b) = M$ is a curve drawn upon the parameter plane which specifies the relation between the parameters necessary if the transfer function $G(s)$, which is a function of the parameters, is to have magnitude M at the real frequency ω_b.

Once these curves are obtained they may be superimposed on the parameter plane thus indicating what values of the parameters are necessary in order to meet the specifications.

Taking the rational transfer function and defining it:

$$G(s) = \frac{P(s)}{Q(s)} = \frac{p_m s^m + p_{m-1} s^{m-1} + \ldots p_1 s + p_o}{q_n s^n + q_{n-1} s^{n-1} + \ldots q_1 s + q_o} \qquad (1\text{-}42)$$

where the p_m's and q_n's are of the form:

$$p_u = g_u \alpha^2 + h_u \alpha + i_u \alpha\beta + j_u \beta + k_u \beta^2 + l_u$$

$$u = 0,1,2,\ldots,m \qquad (1\text{-}43)$$

$$q_v = a_v \alpha^2 + b_v \alpha + c_v \alpha\beta + d_v \beta + e_v \beta^2 + f_v$$

$$v = 0,1,2,\ldots,n \qquad (1\text{-}44)$$

Therefore

$$G(s) = \frac{\sum_{u=0}^{m} p_u s^u}{\sum_{v=0}^{n} q_v s^v} \qquad (1\text{-}45)$$

Employing Equation 1-45 in the parameterized form the generalized compensated third order transfer function is:

$$G(s) = \frac{P(s)}{Q(s)} \tag{1-46}$$

where:

$$P(s) = \alpha^2 Ks^2 + 2\alpha\beta Ks + \beta^2 K \tag{1-47}$$

and:

$$\begin{aligned} Q(s) = {} & \beta^2\left[s^3 + \sum \prod_1 r_i s^2 + \sum \prod_2 r_i s + \sum \prod_3 r_i\right] + \\ & \beta\left[2s^4 + 2\sum \prod_1 r_i s^3 + 2\sum \prod_2 r_i s^2 + 2\sum \prod_3 r_i s\right] + \\ & \left[s^5 + \sum \prod_1 r_i s^4 + \sum \prod_2 r_i s^3 + \sum \prod_3 r_i s^2\right] \end{aligned} \tag{1-48}$$

Making the definitions:

$$A_r = \sum_{\substack{v=0 \\ \text{even}}}^{n} (-1)^{\frac{1}{2}v} \omega_b^v a_v; \text{ etc. for } B_r, C_r, D_r, E_r, F_r \tag{1-49}$$

$$A_i = \sum_{\substack{v=0 \\ \text{odd}}}^{n} (-1)^{\frac{1}{2}(v-1)} \omega_b^v a_v; \text{ etc. for } B_i, C_i, D_i, E_i, F_i \tag{1-50}$$

$$G_r = \sum_{\substack{u=0 \\ \text{even}}}^{m} (-1)^{\frac{1}{2}u} \omega_b^u g_u; \text{ etc. for } H_r, I_r, J_r, K_r, L_r \tag{1-51}$$

$$G_i = \sum_{\substack{u=0 \\ \text{odd}}}^{m} (-1)^{\frac{1}{2}(u-1)} \omega_b^u g_u; \text{ etc. for } H_i, I_i, J_i, K_i, L_i \tag{1-52}$$

and substituting in Equation 1-46,

$$G(j\omega_b) = \frac{(\alpha^2 G_r + K_r) + j(\alpha\beta I_i)}{(\beta^2 D_r + \beta E_r + F_r) + j(\beta^2 D_i + \beta E_i + F_i)} \tag{1-53}$$

Setting the magnitude of $G(j\omega_b) = M$:

$$M^2 = |G(j\omega_h)^2| = \frac{(\alpha^2 G_r + K_r)^2 + (\alpha\beta I_i)^2}{(\beta^2 D_r + \beta E_r + F_r)^2 + (\beta^2 D_i + \beta E_i + F_i)^2} \tag{1-54}$$

Manipulating Equation (1-54) algebraically

$$\phi(\alpha,\beta) - M^2\theta(\alpha,\beta) = 0 \tag{1-55}$$

where

$$\phi(\alpha,\beta) = \alpha^4 G_r^2 + 2\alpha^2 K_r G_r + K_r^2 + \alpha^2\beta^2 I_i^2 \tag{1-56}$$

$$\begin{aligned} \theta(\alpha,\beta) = {} & \beta^4 D_r^2 + 2\beta^3 D_r E_r + 2\beta^2 D_r F_r + \beta^2 E_r^2 \\ & 2\beta E_r F_r + F_r^2 + \beta^4 D_i^2 + \beta^2 E_i^2 + F_i^2 \\ & 2\beta^3 E_i D_i + 2\beta^2 D_i F_i + B^2 E_i^2 + F_i^2 \\ & 2\beta^3 E_i D_i + 2\beta^2 D_i F_i + 2\beta E_i F_i \end{aligned} \tag{1-57}$$

Substituting Equations (1-56) and (1-57) in Equation (1-55) and defining:

$$P_1 = D_r^2 + D_i^2 \tag{1-58}$$

$$Q_1 = 2D_r E_r + 2E_i D_i \tag{1-59}$$

$$R_1 = 2D_r F_r + E_r^2 + E_1^2 + 2D_i F_i \tag{1-60}$$

$$R_2 = \alpha^2 I_i^2 \tag{1-61}$$

$$V_1 = 2E_r F_r + 2E_i F_i \tag{1-62}$$

$$W_1 = F_r^2 + F_i^2 \tag{1-63}$$

$$W_2 = \alpha^4 G_r^2 + 2\alpha^2 K_r G_r + K_r^2 \tag{1-64}$$

It follows that

$$M^2P_1\beta^4 + M^2Q_1\beta^3 + (M^2R_1 - R_2)\beta^2 + M^2V_1\beta +$$

$$(M^2W_1 - W_2) = \tag{1-65}$$

Since the Parameter Plane for compensation purposes has already been determined it is now a matter of taking the computed α values and substituting them along with a constant value of omega and M into Equation (1-65) and then solving the β quartic. This has as its solution the largest, real and positive value of the four roots as before.

1.5 EXTENSIONS TO HIGHER ORDER SYSTEMS

Although the work presented to this point has been limited to third order systems and the program written for this specific case, investigation shows that generalized equations may be written which will allow the extension of the program to higher ordered systems. It can be shown for a given n^{th} order system with no zeros to be compensated with two identical sections of cascade compensation, that the characteristic equation of the system may be generally written as:

$$s^{n+2} + 2ps^{n+1} + p^2s^n + (z^2s^2 + 2pzs + p^2)K +$$

$$2p\sum_{\substack{k=n\\j=1}}^{\substack{j=n\\k=1}}\left(\sum\prod_j r_i\right)s^k + p^2\sum_{\substack{k=n-1\\j=1}}^{\substack{j=n\\k=0}}\left(\sum\prod_j r_i\right)s^k +$$

$$\sum_{\substack{k=n+1\\j=1}}^{\substack{k=2\\k=n}}\left(\sum\prod_j r_i\right)s^k = 0 \tag{1-66}$$

where for n=4 the equation would be written:

$$s^6 + 2ps^5 + p^2s^4 + (z^2s^2 + 2pzs + p^2)K +$$

$$2p(\sum_1 \Pi\, r_i s^4 + \sum_2 \Pi\, r_i s^3 + \sum_3 \Pi\, r_i s^2 + \sum_4 \Pi\, r_i s)$$

$$p^2(\sum_1 \Pi\, r_i s^3 + \sum_2 \Pi\, r_i s^2 + \sum_3 \Pi\, r_i s + \sum_4 \Pi\, r_i) +$$

$$(\sum_1 \Pi\, r_i s^5 + \sum_2 \Pi\, r_i s^4 + \sum_3 \Pi\, r_i s^3 + \sum_4 \Pi\, r_i s^2) = 0 \tag{1-67}$$

It may be further shown that the parameters defined by Equations (1-17) through (1-26) may be written:

$$B_{21} = K\omega^2 U_1(\xi) \tag{1-68}$$

$$B_{22} = K\omega^2 U_2(\xi) \tag{1-69}$$

$$D_1 = -2K\omega U_o(\xi) \tag{1-70}$$

$$D_2 = -2K\omega U_1(\xi) \tag{1-71}$$

$$E_{11} = 2(-1)^{n+1}\omega^{n+1}U_n(\xi) + 2\sum_{\substack{k=n\\j=1}}^{\substack{j=n\\k=1}} \left[(-1)^k\omega^k U_{k-1}(\xi)(\sum_j \Pi\, r_i)\right] \tag{1-72}$$

$$E_{12} = 2(-1)^{n+1}\omega^{n+1}U_{n+1}(\xi) + 2\sum_{\substack{k=n\\j=1}}^{\substack{j=n\\k=1}} \left[(-1)^k\omega^k U_k(\xi)(\sum_j \Pi\, r_i)\right] \tag{1-73}$$

$$F_{21} = (-1)^n\omega^n U_{n-1}(\xi) + \sum_{\substack{k=n-1\\j=1}}^{\substack{j=n\\k=0}} \left[(-1)^k\omega^k U_{k-1}(\xi)(\sum_j \Pi\, r_i)\right] + KU_{-1}(\xi) \tag{1-74}$$

$$F_{22} = (-1)^n \omega^n U_n(\xi) + \sum_{\substack{k=n-1 \\ j=1}}^{\substack{j=n \\ k=0}} \left[(-1)^k \omega^k U_k(\xi) \left(\sum_j \prod r_i\right) \right] + KU_o(\xi) \tag{1-75}$$

$$G_1 = (-1)^{n+2} \omega^{n+2} U_{n+1}(\xi) + \sum_{\substack{k=n+1 \\ j=1}}^{\substack{j=n \\ k=2}} \left[(-1)^k \omega^k U_{k-1}(\xi) \left(\sum_j \prod r_i\right) \right] \tag{1-76}$$

$$G_2 = (-1)^{n+2} \omega^{n+2} U_{n+2}(\xi) + \sum_{\substack{k=n+1 \\ j-1}}^{\substack{j=n \\ k=2}} \left[(-1)^k \omega^k U_k(\xi) \left(\sum_j \prod r_i\right) \right] \tag{1-77}$$

These then are the recursive equations required for the complete generalization to a n^{th} order system. By employing the above equations and replacing in PROGRAM PROJECT cards 100 through 150 and 300 through 540 with the appropriate programming, the program may be used for any given n^{th} order system.

In like manner by generally defining:

$$P(s) = \alpha^2 Ks^2 + \alpha\beta Ks + \beta^2 K \tag{1-78}$$

and:

$$Q(s) = s^{n+2} + 2\beta s^{n+1} + \beta^2 s^n + 2 \sum_{\substack{k=n \\ j=1}}^{\substack{j=n \\ k=1}} \left(\sum_j \prod r_i\right) s^k +$$

$$\beta^2 \sum_{\substack{k=n-1 \\ j=1}}^{\substack{j=n \\ k=0}} \left(\sum_j \prod r_i\right) s^k + \sum_{\substack{k=n+1 \\ j=1}}^{\substack{j=n \\ k=2}} \left(\sum_j \prod r_i\right) s^k \tag{1-79}$$

and using Equations (1-49) through (1-52) we may replace in the program cards 2860 and 2880 through 2920, thus adapting this part of the program to a general n^{th} order application

1.6 COMMENTS

Throughout this development of the Parameter Plane quadratic extension, the c_k's in the generalized coefficient form:

$$\sum_{k=0}^{n} (b_k\alpha^2 + c_k\alpha + d_k\alpha\beta + e_k\beta + f_k\beta^2 + g_k) = 0 \tag{1-80}$$

have been identically zero. This at first appearance might seem to detract from the generalization. The inclusion of this parameter does not however introduce any great difficulty in the solution. The change in the development would be to the value of P_1 and P_2 which would become:

$$P_1 = C_1 + \beta D_1 \tag{1-81}$$

$$P_2 = C_2 + \beta D_2 \tag{1-82}$$

and the final solution for α which would change to:

$$\alpha = -\frac{P_1}{2B_{21}} \pm \left[\frac{P_1^2 - 2B_{21}Q_1}{4B^2{}_{21}}\right]^{\frac{1}{2}} \tag{1-83}$$

For this case, new selection rules for acceptable values of α would be used, and would be much like those presented for β.

Though the extension of the Parameter Plane to include the $\alpha - \beta$ quadratic case makes this tool even more useful, further work is still to be done in this field. Not only must the equations for the solutions of the Parameter Plane curves for such cases as:

$$a_k = b_k\alpha^2\beta^2 + c_k\alpha^2\beta + d_k\alpha\beta^2 + e_k\alpha^2 + f_k\beta^2 + g_k\alpha\beta + h_k\alpha + q_k\beta + t_k \quad [5] \tag{1-84}$$

and higher ordered combinations of the parameters be developed, but more efficient programming techniques must be developed. In the use of PROGRAM PROJECT, for instance, as the location of the system poles on the σ axis of the S-plane move to the left, the computational time becomes excessive due to present programming technique and computer speed.

Another major problem in further extensions of these techniques, and indeed even other applications of the curves from the proceeding development, will be interpretation. In this case, the initial substitution of variables immediately allowed interpretation of the curves sight unseen. Here then, will be most likely the one single drawback to further extension, for as the parameters α and β are used as representations of other variables in control systems, each application will have its own unique interpretation.

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APPENDIX I

PROGRAM PROJECT is designed to solve the α quadratic and β quartic. The program is divided into two main sections, the first for the computation of the α-β points and the second for the bandwidth points.

The first section computes an 80 by 80 matrix of the α and β points corresponding to set values of ζ and ω. The computational part is followed by two distinct graphing sections, one for lag and the other for lead compensation.

The lag graphing section is set up so that during the plotting of the curves each value of α is tested to determine if its value is $10^{-7} \leq \alpha \leq 1.0001$. If no points are found within this range then a print out is made:

NO LAG COMPENSATION POSSIBLE

For the lead section graphs, α is again tested by the criterion $1.0001 \leq \alpha \leq$ (X-graph scale)(X graph width). Again if there are no values of α within this region the statement:

NO LEAD COMPENSATION POSSIBLE

is printed. In this case however a study of the printed values of α must be made to insure that the points are indeed non-existant or rather just lie outside the range of the graph.

The second main section of the program computes the value of β for a given value of α is determined by the X graph scale. Here the plotting routine is set up so as to <u>not</u> plot zero points and to stop the curve when either the α or β value exceeds the range of the graph.

```
      PROGRAM PROJECT                                                   000000
C  THIS PROGRAM COMPUTES THE VALUES OF BETA(POLE LOCATION) AND ALFA(POLE
C    -ZERO RATIO) BASED ON PARAMETER PLANE TECHNIQUES.  THE COMPUTED
C     ALFA AND BETA VALUES ARE THOSE REQUIRED TO PLACE THE ROOTS OF ANY
C     THIRD ORDER SYSTEM, TYPE 0,1,2 OR 3, AT A DESIRED ZETA AND OMEGA
C    LOCATION WHILE MAINTAINING A CONSTANT VELOCITY COEFFICIENT.  AFTER
C     COMPUTING THE VALUES IT WILL PLOT THE PARAMETER PLANE CONSTANT ZETA
C    CURVES FROM 0.0 TO 0.9 AND THE CONSTANT OMEGA CURVES FOR EVEY ONE
C     TENTH OF THE VALUE OF THE MAXIMUM VALUE OF OMEGA USED.  A 9 BY 15
C     INCH GRAPH IS OUTPUT BY THE ROUTINE.  THIS IS DONE
C     ON TWO SEPARATE GRAPHS, ONE FOR POSSIBLE LAG COMPENSATION AND ONE
C    FOR LEAD COMPENSATION.  THE PROGRAM THEN HAS THE ADDITIONAL FEATURE
C     OF COMPUTING AND PLOTTING THE CONSTANT BANDWIDTH CURVES.
C  THE FOLLOWING FEATURES ARE AVAILABLE WITH THE PROPER USE OF THE DATA
C     CARDS.
C    1. THE ALFA-BETA COMPUTATIONS MAY OR MAY NOT BE DONE.
C    2. LAG COMPENSATION MAY OR MAY NOT BE PLOTTED.
C    3. LEAD COMPENSATION MAY OR MAY NOT BE PLOTTED.
C    4. BANDWIDTH COMPUTATIONS MAY OR MAY NOT BE COMPLETED.
C    5. BANDWIDTH CURVES MAY OR MAY NOT BE PLOTTED. (AVAILABLE ONLY IF
C         THE BANDWIDTH COMPUTATIONS HAVE BEEN MADE)
C  THE FOLLOWING DATA CARDS ARE REQUIRED.
C***CARD ONE - A,B,C,G - TEN COLUMNS PER NUMBER IN FLOATING POINT.
C      THESE ARE THE LOCATIONS OF THE UNCOMPENSATED POLES AND THE
C    UNCOMPENSATED ROOT LOCUS GAIN.
C***CARD TWO - WFIN - TEN COLUMNS IN FLOATING POINT.
C    THIS IS THE MAXIMUM VALUE OF OMEGA TO BE USED.
C***CARD THREE - IABCMP - COLUMN ONE IN FIXED POINT
C     0 - THE ALFA-BETA COMPUTATIONS WILL BE DONE
C     1 - THE ALFA-BETA COMPUTATIONS WILL NOT BE DONE
C                            *****
C   IF IABCMP=1 CARDS FOUR THROUGH THIRTEEN ARE OMITTED
C                            *****
C***CARD FOUR - ILGPLT - COLUMN ONE IN FIXED POINT
C     0 - THE LAG ZONE CURVES WILL BE PLOTTED
```

```
C      1 - THE LAG ZONE CURVES WILL NOT BE PLOTTED
C                     ******
C    IF ILGPLT=1 THE NEXT FOUR CARDS ARE OMITTED
C                     ******
C***CARD FIVE -IT(1)-IT(6) - COLUMNS 1-48 IN ALFANUMERIC CHARACTERS
C     THIS IS THE FIRST LINE OF THE LAG GRAPH TITLE
C***CARD SIX - IT(7)-IT(12) - COLUMNS 1-48 IN ALFANUMERIC CHARACTERS
C     THIS IS THE SECOND LINE OF THE LAG GRAPH TITLE.
C***CARD SEVEN - LBL(11)-LBL(20) - FOUR COLUMNS PER LABEL (TEN LABELS IN
C      CONSECUTIVE COLUMNS) IN ALFANUMERIC CHARACTERS.
C     THESE ARE THE LABELS TO BE PUT ON THE CONSTANT OMEGA CURVES.   TO
C      DETERMINE WHICH VALUES WILL BE PLOTTED, DIVIDE WFIN BY 10 .  THIS
C      VALUE AND INTEGER MULTIPLES OF IT TO 10 WILL BE PLOTTED.
C***CARD EIGHT - XLGZ,YLGZ - TEN COLUMNS PER NUMBER IN EXPONENTIAL OR
C      FLOATING POINT.
C     THESE ARE THE X AND Y SCALES FOR THE LAG GRAPH.  ONLY ONE SIGNI-
C     FICANT NUMBER IS TO BE USED.
C***CARD NINE - ILDPLT - COLUMN ONE IN FIXED POINT
C      0 - THE LEAD CURVES WILL BE PLOTTED
C      1 - THE LEAD CURVES WILL NOT BE PLOTTED
C                     ******
C    IF ILDPLT=1 THE NEXT FOUR CARDS ARE OMITTED
C                     ******
C***CARD TEN - THE SAME AS CARD FIVE EXCEPT FOR THE LEAD GRAPH
C***CARD ELEVEN - THE SAME AS CARD SIX EXCEPT FOR THE LEAD GRAPH
C***CARD TWELVE - THE SAME AS CARD EIGHT EXCEPT FOR THE LEAD GRAPH
C***CARD THIRTEEN - A***DUPLICATE*** OF CARD SEVEN
C***CARD FOURTEEN - IBWCMP - COLUMN ONE FIXED POINT
C      0 - BANDWIDTH COMPUTATIONS AND GRAPHING WILL NOT BE DONE.
C      1 - BANDWIDTH COMPUTATIONS WILL BE DONE
C                     ******
C    IF IBWCMP=0 THE REMAINING CARDS ARE OMITTE
C                     ******
C***CARD FIFTEEN - BWX,BWY - THE SAME AS CARD EIGHT EXCEPT FOR THE
C     BANDWIDTH CURVES.
C      BWY IS ALSO USED TO DETERMINE WHICH VALUES OF ALFA WILL BE USED IN
```

```
     THE BANDWIDTH COMPUTATIONS.
***CARD SIXTEEN - WEND - TEN COLUMNS IN FLOATING POINT
     THIS IS THE MAXIMUM VALUE OF OMEGA FOR WHICH THE BANDWIDTH
     COMPUTATIONS WILL BE DONE
***CARD SEVENTEEN - IBWPLT - COLUMN ONE IN FIXED POINT
     0 - THE BANDWIDTH CURVES WILL BE PLOTTED
     1 - THE BANDWIDTH CURVES WILL NOT BE PLOTTED
                    ******
   IF IBWPLT=1 THE REMAINING CARDS ARE OMITTED
                    ******
***CARD EIGHTEEN - THE SAME AS CARD FIVE EXCEPT FOR THE BANDWIDTH CURVES
***CARD NINETEEN - THE SAME AS CARD SIX EXCEPT FOR THE BANDWIDTH CURVES
***CARD TWENTY - BANDWIDTH CURVE LABELS
     TO DETERMINE WHICH CURVES WILL BE PLOTTED, DIVIDE  WEND  BY 10  .
     THE PROGRAM PLOTS THIS CURVE AND INTEGER MULTIPLES OF IT UP TO 10


   IT IS RECOMMENDED THAT FOR THE INITIAL RUN THE FOLLOWING DATA CARDS
     BE USED.
        CARDS 1,2,3(IABCMP=0),4(ILGPLT=1),9(ILDPLT=1),14(IBWCMP=0)

   THESE DATA CARDS WILL ALLOW ONLY THE ALFA-BETA COMPUTATIONS TO BE
     COMPLETED.  A PRINT OUT OF THE VALUES WILL BE OUTPUT WHICH WILL ALLOW
     YOU TO CHOOSE THE PROPER CURVES AND SCALES.  CAREFUL SELECTION
     OF CURVE SCALES IS IMPORTANT, FOR THE PROGRAM WILL NOT ALLOW POINTS
     OUTSIDE THE AXIX LIMITS TO BE PLOTTED.

      DIMENSION AFIN(80,80),BFIN(80,80),XAZ(80),YBZ(80),XAW(80),
     1 YBW(80),IT(12),LBL(20),BCOFI(5),ROOTR(4),ROOTI(4),ACOFI(3),
     2 U(10),AROOTI(4),ACOFR(3),BCOFR(5),WLAB(80),ZLAB(80),AROOTR(4)
      COMMON BCOFR,BCOFI,ROOTR,ROOTI,BFINAL,IFLAG,AFIN,BFIN
 9999 PRINT 140
  140 FORMAT (1H1)
      DO 60 JK=1,6400
      AFIN(JK) =0.0
   60 BFIN(JK) = 0.0
```

```
      READ 1,A,B,C,G
    1 FORMAT(4F10.0)
      PROD = A*B*C
      SUM = A+B+C
      SMPRD = A*B + A*C + B*C
      PRDGN = PROD + G
      ZETA = 0.0
      READ 2,WFIN
    2 FORMAT (F10.0)
      READ 9, IABCMP
    9 FORMAT (I1)
      IF(IABCMP-1)23,24,24
   23 STP = WFIN/80.
      DO 12 L = 1,80
      LJ = 80*(L-1)
      W = STP
   30 U(1)=-1.
      U(2)=0.
      U(3)=1.
      DO 10 N=2,6
      U(N+2)=2.*ZETA*U(N+1)-U(N)
      DO 11 J=1,80
      LJ = LJ + 1
      W2=W*W
      W3=W2*W
      W4=W2*W2
      W5=W2*W3
      CONN = G*W2
      CON = -2.*G*W
      CON1 = 2.*W4
      CON2 = -2.*SUM*W3
      CON3 = 2.*SMPRD*W2
      CON4 = -2.*PROD*W
      CON5 = SUM*W2
      CON6 = -SMPRD*W
      CON7 = SUM*W4
```

42

```
      CON8 = -SMPRD*W3
      CON9 = PROD*W2
      B21 = CONN*U(3)
      B22 = CONN*U(4)
      D1 = CON*U(2)
      D2 = CON*U(3)
      E11 = CON1*U(5) + CON2*U(4) + CON3*U(3) + CON4*U(2)
      E12 = CON1*U(6) + CON2*U(5) + CON3*U(4) + CON4*U(3)
      F21 = -W3*U(4) + CON5*U(3) + CON6*U(2) + PRDGN*U(1)
      F22 = -W3*U(5) + CON5*U(4) + CON6*U(3) + PRDGN*U(2)
      G1 = -W5*U(6) + CON7*U(5) + CON8*U(4) + CON9*U(3)
      G2 = -W5*U(7) + CON7*U(6) + CON8*U(5) + CON9*U(4)
      COF1 = B21*F22*(2.*F21*B22-F22*B21)-F21*(F21*B22*B22+D2*D2*B21)
      COF2 = E11*(2.*B22*(F22*B21-F21*B22)-D2*D2*B21)+2.*E12*B21*(F21*B2
     12-F22*B21)
      COF3 = B21*(-B21*(E12*E12+2.*F22*G2)-D2*D2*G1+2.*B22*(E11*E12+F21*
     1 G2+G1*F22))-B22*B22*(E11*E11+2.*F21*G1)
      COF4=2.*G2*B21*(E11*B22-E12*B21)+2.*B22*G1*(E12*B21-E11*B22)
      COF5= -(G2*B21-G1*B22)*(G2*B21-G1*B22)
      DO 50 I =1,5
   50 BCOFI(I) = 0.0
      BCOFR(1) = 1.0
      BCOFR(2) = COF2/COF1
      BCOFR(3) = COF3/COF1
      BCOFR(4) = COF4/COF1
      BCOFR(5) = COF5/COF1
      CALL ABETART
      IFLAG = 0
      CALL SORT
      IF (IFLAG-1)300,11,11
  300 BFIN(LJ) = BFINAL
      Q1 = BFIN(LJ)*(E11+BFIN(LJ)*F21)+G1
      ACOFR(1)=1.0
      ACOFR(2)=0.0
      ACOFR(3) = Q1/B21
      ALFASQ = ABSF(ACOFR(3))
```

```
      AFIN(LJ) = SQRTF(ALFASQ)                                          000790
   11 W = W+STP                                                         000800
   12 ZETA = ZETA + .0125                                               000810
      LBL(1) = 4HZ=.0                                                   000820
      LBL(2) = 4HZ=.1                                                   000830
      LBL(3) = 4HZ=.2                                                   000840
      LBL(4) = 4HZ=.3                                                   000850
      LBL(5) = 4HZ=.4                                                   000860
      LBL(6) = 4HZ=.5                                                   000870
      LBL(7) = 4HZ=.6                                                   000880
      LBL(8) = 4HZ=.7                                                   000890
      LBL(9) = 4HZ=.8                                                   000900
      LBL(10) = 4HZ=.9                                                  000910
      READ 7, ILGPLT                                                    000920
    7 FORMAT (I1)                                                       000930
      IF(ILGPLT-1)8,67,67                                               000940
    8 READ 3, (IT(I),I=1,12)                                            000950
    3 FORMAT (6A8)                                                      000960
      READ 6, (LBL(I),I=11,20)                                          000970
    6 FORMAT (10A4)                                                     000980
      READ 4, XLGZ,YLGZ                                                 000990
    4 FORMAT (2E10.0)                                                   001000
      XLGLM = 9.*XLGZ                                                   001010
      YLGLM = 15.*YLGZ                                                  001020
      MODE = 1                                                          001030
      IL = 0                                                            001040
      DO 62 K=1,80,8                                                    001050
      LL = 1                                                            001060
      KJ = (K-1)*80                                                     001070
      DO 61 J=1,80                                                      001080
      KJ = KJ+1                                                         001090
      IF(AFIN(KJ)-.0000001)61,6110,6110                                 001095
 6110 IF(AFIN(KJ)-1.0001)6113,61,61                                     001100
 6113 IF(AFIN(KJ) - XLGLM)6114,61,61                                    001110
           CARDS 1120 - 1130 ARE MISSING
 6114 XAZ(LL) = AFIN(KJ)                                                001140
```

```
      IF(BFIN(KJ) - YLGLM)6112,61,61                                    001150
           CARDS 1160 - 1170 ARE MISSING
 6112 YBZ(LL) = BFIN(KJ)                                                001180
      LL = LL + 1                                                       001190
   61 CONTINUE                                                          001200
      JJ = LL - 1                                                       001210
      IL = IL + 1                                                       001220
      IF(JJ-1)62,62,6116                                                001230
 6116 LAL = LBL(IL)                                                     001240
      CALL DRAW(JJ,XAZ,YBZ,MODE,0,LAL,IT,XLGZ,YLGZ,0,0,0,0,9,15,0,LAST) 001250
 6111 MODE = 2                                                          001260
   62 CONTINUE                                                          001270
      IF(MODE-1)65,65,6120                                              001280
 6120 DO 66 K=8,80,8                                                    001290
      LL = 1                                                            001300
      DO 63 J=1,80                                                      001310
      JK = (J-1)*80 + K                                                 001320
      IF(AFIN(JK)-.0000001)63,6127,6127                                 001325
 6127 IF(AFIN(JK)-1.0001)6123,63,63                                     001330
 6123 IF(AFIN(JK) - XLGLM)6124,63,63                                    001340
C          CARDS 1350 - 1360 ARE MISSING
 6124 XAW(LL) = AFIN(JK)                                                001370
      IF(BFIN(JK) - YLGLM)6122,63,63                                    001380
C             CARDS 1390 - 1400 ARE MISSING
 6122 YBW(LL) = BFIN(JK)                                                001410
      LL = LL + 1                                                       001420
   63 CONTINUE                                                          001430
      JJ = LL - 1                                                       001440
      IL = IL + 1                                                       001450
      IF(JJ-1)6121,6121,612                                             001460
 6121 IF(K-80)66,6125,6125                                              001470
 6125 MODE = 3                                                          001480
      LAL = 4H                                                          001490
      JJ = 2                                                            001500
      XAW(1) = XLGLM                                                    001510
      XAW(2) = XLGLM                                                    001520
```

```
      YBW(1) = 0.0                                                      001530
      YBW(2) = YLGZ                                                     001540
      GO TO 2000                                                        001550
 6126 LAL = LBL(IL)                                                     001560
 2000 CALL DRAW(JJ,XAW,YBW,MODE,0,LAL,IT,XLGZ,YLGZ,0,0,0,0,9,15,0,LAST) 001570
      MODE = 2                                                          001580
      IF(K-7[illegible])66,64,64                                        001590
   64 MODE = 3                                                          001600
   66 CONTINUE                                                          001610
      GO TO 67                                                          001620
   65 PRINT 130                                                         001630
  130 FORMAT (1X,33H NO LAG COMPENSATION IS POSSIBLE  ,///)             001640
   67 READ 20, ILDPLT                                                   001650
   20 FORMAT (I1)                                                       001660
      IF(ILDPLT-1)68,1000,1000                                          001670
   68 READ 5, (IT(I),I=1,12)                                            001680
    5 FORMAT (6A8)                                                      001690
      READ 21, XLDZ,YLDZ                                                001700
   21 FORMAT (2E10.0)                                                   001710
      READ 22, (LBL(I),I=11,20                                          001713
   22 FORMAT (10A4)                                                     001716
      XLDLM = 9.*XLDZ                                                   001720
      YLDLM = 15.*YLDZ                                                  001730
      IL = 0                                                            001740
      MODE = 1                                                          001750
      DO 72 K = 1,80,8                                                  001760
      KJ = (K-1)*80                                                     001770
      KK = 1                                                            001780
      DO 71 J = 1,80                                                    001790
      KJ = KJ + 1                                                       001800
      IF(AFIN(KJ)-1.0001)71,7111,7111                                   001810
 7111 IF(AFIN(KJ) - XLDLM)7117,71,71                                    001820
C       CARDS 1830 - 1840 ARE MISSING
 7117 XAZ(KK) = AFIN(KJ)                                                001850
       F(BFIN(KJ) - YLDLM)7118,71,71                                    001860
       ARDS 1870 - 1880 ARE MISSING
```

```
7118 YBZ(KK) = BFIN(KJ)
     KK = KK + 1
  71 CONTINUE
     MM = KK-1
     IL = IL + 1
     IF(MM-1)72,72,7119
7119 LAL = LBL(IL)
     CALL DRAW(MM,XAZ,YBZ,MODE,0,LAL,IT,XLDZ,YLDZ,0,0,0,0,9,15,0,LAST)
7110 MODE = 2
  72 CONTINUE
     IF(MODE-1)70,70,78
  78 DO 76 K=8,80,8
     KK = 1
     DO 73 J = 1,80
     JK = (J-1)*80 +
     IF(AFIN(JK)-1.0001)73,7121,7121
7121 IF(AFIN(JK) - XLDLM)7127,73,73
      CARDS 2060 - 2070 ARE MISSING
7127 XAW(KK) = AFIN(JK)
     IF(BFIN(JK) - YLDLM)7128,73,73
      CARDS 2100 - 2110 ARE MISSING
7128 YBW(KK) = BFIN(JK)
     KK = KK + 1
  73 CONTINUE
     MM = KK-1
     IL = IL + 1
     IF(MM-1)7120,7120,7129
7120 IF(K-80)76,7122,7122
7122 MODE = 3
     LAL = 4H
     MM = 2
     XAW(1) = XLDLM
     XAW(2) = XLDLM
     YBW(1) = 0.0
     YBW(2) = YLDZ
     GO TO 2001
```

```
7129 LAL = LBL(IL)
2001 CALL DRAW(MM,XAW,YBW,MODE,0,LAL,IT,XLDZ,YLDZ,0,0,0,0,9,15,0,LAST)
     MODE = 2
     IF(K-72)76,75,75
  75 MODE = 3
  76 CONTINUE
     GO TO 1000
  70 PRINT 131
 131 FORMAT (1X,34H NO LEAD COMPENSATION IS POSSIBLE  ,///)
1000 CONTINUE
     ZLAB(1) = 0.0
     DO 81 I=2,10
  81 ZLAB(I) = ZLAB(I-1) +
     WLAB(8) = 8.*STP
     DO 82 N=16,80,8
  82 WLAB(N) = WLAB(N-8) + WLAB(8)
     PRINT 100
 100 FORMAT (1H1)
     PRINT 101
 101 FORMAT(2X,20H THE ALFA VALUES ARE  ,///
     PRINT 102, (ZLAB(I) I=1,10)
 102 FORMAT (1X,6H ZETA    ,10F11.6)
     PRINT 103, (WLAB(J),(AFIN(J,I),I=1,73,8)J=8,80,8)
 103 FORMAT (/,1X,F6.2,10E11.5)
     PRINT 111
 111 FORMAT (/////2X,20H THE BETA VALUES ARE
     PRINT 112, (ZLAB(I) I=1,10)
 112 FORMAT (1X,6H ZETA    ,10F11.6)
     PRINT 113, (WLAB(J),(BFIN(J,I),I=1,73,8)J=8,80,8
 113 FORMAT (/,1X,F6.2,10E11.5)
  24 PRINT 114
 114 FORMAT (1H1)
     READ 218, IBWCMP
 218 FORMAT (I1)
     IF(IBWCMP-1)1002,219,219
 219 READ 217, BWX,BWY
```

```
  217 FORMAT (2E10.0)
      READ 223, WEND
  223 FORMAT (F10.0)
      YBWLM = 15.*BWY
      XLM = 9.*BWX
      AM2=.5
      STEP = WEND/20.
      W = STEP
      ALFASP = XLM/20.
      XAW(1) = ALFASP
      DO 200 K=2,20
  200 XAW(K) = XAW(K-1) + XAW(1)
      DO 203 N=1,20
      ALFA = ALFASP
      DO 210 M=1,20
      XAZ(M) = 0.0
  210 YBZ(M) = 0.0
      DO 207 I=1,20
      YBW(N) = W
      W2=W*W
      W3=W*W2
      W4=W2*W2
      W5=W2*W3
      AGR = -W2*
      AKR = G
      ADR=2.*W4-2.*W2*SMPRD
      AII = 2.*G*W
      AER=-W2*SUM + PROD
      AFR=W4*SUM - W2*PROD
      ADI=-2.*W3*SUM + 2.*W*PROD
      AEI = -W3 + W*SMPRD
      AFI = W5 - W3*SMPRD
      P1 = ADR*ADR+ADI*ADI
      Q1 = 2.*ADR*AER + 2.*AEI*ADI
      R1 = 2.*ADR*AFR + AER*AER + AEI*AEI + 2.*ADI*AFI
      V1 = 2.*AER*AFR + 2.*AEI*AFI
```

30

```
      W1 = AFR*AFR + AFI*AFI
      A2 = ALFA*ALFA
      A4 = A2*A2
      R2 = A2*AII*AI
      WW2 = A4*AGR*AGR + 2.*A2*AKR*AGR + AKR*AKR
      DO 51 M = 1,5
   51 BCOFI(M) = 0.0
      BCOFR(1) = 1.0
      BCOFR(2) = Q1/P1
      BCOFR(3) = (AM2*R1-R2)/(AM2*P1)
      BCOFR(4) = V1/P1
      BCOFR(5) = (AM2*W1-WW2)/(AM2*P1)
      CALL ABETART
      IFLAG = 0
      CALL SORT
      IF(IFLAG-1)900,206,206
  206 BFIN(N,I) = 0.0
      GO TO 207
  900 BFIN(N,I) = BFINAL
  207 ALFA = ALFA + ALFASP
  203 W = W + STEP
      READ 216, IBWPLT
  216 FORMAT (I1)
      IF(IBWPLT-1)214,1001,1001
  214 MODE = 1
      READ 202,(IT(K),K=1,12)
  202 FORMAT (6A8)
      READ 201, (LBL(N),N=2,20,2)
  201 FORMAT (10A4)
      DO 211 N=2,20,2
      KK = 1
      IF(N-20)204,205,205
  205 MODE = 3
  204 CONTINUE
      DO 212 I=1,20
      IF(BFIN(N,I)   .000001)212,212,209
```

```
  209 IF(BFIN(N,I) - YBWLM)905,906,906                                    003330
  905 YBZ(KK) = BFIN(N,I)                                                 003340
      XAZ(KK) = XAW(I)                                                    003350
       CARD 3360 IS MISSSING
      KK = KK + 1                                                         003370
  212 CONTINUE                                                            003380
  906 JJ = KK - 1                                                         003390
      IF(JJ-1)221,221,220                                                 003400
  221 IF(N-20)211,225,225.                                                003410
  225 IF(MODE-1)224,224,215                                               003415
   15 MODE = 3                                                            003420
      LAL = 4H                                                            003430
      XAZ(1) = 0.0                                                        003440
      XAZ(2) = 0.0                                                        003450
      YBZ(1) = 0.0                                                        003460
      YBZ(2) = BWY                                                        003470
      JJ = 2                                                              003480
      GO TO 2002                                                          003490
  220 LAL = LBL(N)                                                        003500
 2002 CALL DRAW (JJ,XAZ,YBZ,MODE,0,LAL,IT,BWX,BWY,0,0,0,0,9,15,0,LAST)    003510
  222 IF(N-20)208,211,211                                                 003520
  208 MODE = 2                                                            003530
  211 CONTINUE                                                            003540
      GO TO 1001                                                          003541
  224 PRINT 226                                                           003542
  226 FORMAT (1H1,1X,76HTHERE ARE NO POSSIBLE BANDWIDTH CURVES FOR THE F  003543
     1INAL VALUE OF OMEGA STATED                                          003544
      GO TO 1002                                                          003545
 1001 PRINT 120                                                           003550
  120 FORMAT (1H1,20X,38H VALUES OF BETA FOR CONSTANT BANDWIDTH           003560
      PRINT 122, (YBW(K),K=2,20,2)                                        003570
  122 FORMAT (/,9X,10F11.6)                                               003580
      PRINT 121, (XAW(K),(BFIN(J,K),J=2,20,2),K=1,20)                     003590
  121 FORMAT (/,1X,F6.2,2X,10E11.6)                                       003600
 1002 CONTINUE                                                            003610
      GO TO 9999                                                          003615
```

```
      END

      SUBROUTINE ABETART
      DIMENSION A(5),YIMAG(5),U(4),V(4),H(50),B(50),C(50),D(50),E(50)
     1 ,CONV(50)
      DIMENSION AFIN(80,80),BFIN(80,80)
      COMMON A,YIMAG,U,V,DUMMY1,DUMMY2,AFIN,BFIN
      N = 4
       =10.0
      .=25
       ER=0
      .F(N) 54,54,52
  54  IER=1
  52  NP3=N+3
 100  B(2)=0.0
      B(1)=0.0
      C(2)=0.0
      C(1)=0.0
      D(2)=0.0
      E(2)=0.0
      H(2)=0.0
      DO 101 J=3,NP3
 101  H(J)=A(J-2)
      T=1.0
      SK=10.0**F
 150  IF(H(NP3)) 200,151,200
 15    )(NP3)=0.0
       '(NP3)=0.0
      CONV(NP3)=SK
      NP3=NP3-1
      IF(NP3)152,152,150
 152  IER=1
 200  IF(NP3-3)205,51,201
 205  IER=1
 201  PS=0.0
```

```
      QS=0.0
      PT=0.0
      QT=0.0
      S=0.0
      REV=1.0
      SK=10.0**F
      IF(NP3-4)206,202,203
  206 IER=1
  202 R=-H(4)/H(3)
      GO TO 500
  203 DO 207 J=3,NP3
      IF(H(J))204,207,204
  204 S=S+LOGF(ABSF(H(J)))
  207 CONTINUE
      FPN1=N+1
      S=EXPF(S/FPN1)
      DO 208 J=3,NP3
  208 H(J)=H(J)/S
  210 IF(ABSF(H(4)/H(3))-ABSF(H(NP3-1)/H(NP3)))250,252,252
  250 T=-T
      M=(NP3-4)/2 + 3
      DO 251 J=3,M
      S=H(J)
      JJ=NP3-J+3
      H(J)=H(JJ)
  251 H(JJ)=S
  252 IF(QS) 253,254,253
  253 P=PS
      Q=QS
      GO TO 300
  254 HH2=H(NP3-2)
      IF(HH2) 256,255,256
  255 Q=1.0
      P=-2.0
      GO TO 257
  256 Q=H(NP3)/HH2
```

```
      P=(H(NP3-1)-Q*H(NP3-3))/HH2
  257 IF(NP3-5)258,550,258
  258 R=0.0
  300 DO 490 I=1,L
  350 DO 351 J=3,NP3
      B(J)=H(J)-P*B(J-1)-Q*B(J-2)
  351 C(J)=B(J)-P*C(J-1)-Q*C(J-2)
      IF(H(NP3-1))352,400,352
  352 IF(B(NP3-1))353,400,353
  353 AVHB1=ABSF(H(NP3-1)/B(NP3-1))
  356 IF(AVHB1-SK)450,354,354
  354 B(NP3)=H(NP3)-Q*B(NP3-2)
  400 IF(B(NP3))401,550,401
  401 AVHB2=ABSF(H(NP3)/B(NP3))
  403 IF(SK-AVHB2)550,450,450
  450 DO 451 J=3,NP3
      D(J)=H(J)+R*D(J-1)
  451 E(J)=D(J)+R*E(J-1)
      IF(D(NP3))452,500,452
  452 AVHD3=ABSF(H(NP3)/D(NP3))
  460 IF(SK-AVHD3)500,453,453
  453 CC2=C(NP3-2)
      CC3=C(NP3-3)
      C(NP3-1)=-P*CC2-Q*CC3
      CC1=C(NP3-1)
      S=CC2*CC2-CC1*CC3
      IF(S)455,454,455
  454 P=P-2.0
      Q=Q*(Q+1.0)
      GO TO 456
  455 P=P+(B(NP3-1)*CC2-B(NP3)*CC3)/S
      Q=Q+(-B(NP3-1)*CC1+B(NP3)*CC2)/S
  456 IF(E(NP3-1))458,457,458
  457 R=R-1.0
      GO TO 490
  458 R=R-D(NP3)/E(NP3-1)
```

```
490 CONTINUE
    PS=PT
    QS=QT
    PT=P
    QT=Q
    IF(REV)491,492,492
491 SK=SK/10.0
492 REV=-REV
    GO TO 250
500 IF(T)501,502,502
501 R=1.0/R
502 NP=NP3-3
    U(NP)=R
    V(NP)=0.0
    CONV(NP)=SK.
    NP3=NP3-1
    DO 503 J=3,NP3
503 H(J)=D(J)
    IF(NP3-3)300,51,300
550 IF(T)551,552,552
551 P=P/Q
    Q=1.0/Q
552 PP2=P/2.0
    QMPSQ=Q-PP2*PP2
560 IF(QMPSQ)554,554,553
553 NP=NP3-3
    U(NP)=-PP2
    U(NP-1)=-PP2
    S=SQRTF(QMPSQ)
    V(NP)=S
    V(NP-1)=-S
    GO TO 561
554 S=SQRTF(-QMPSQ)
    NP=NP3-3
    IF(P)555,556,556
555 U(NP)=-PP2+S
```

```
      GO TO 557
  556 U(NP)=-PP2-S
  557 U(NP-1)=Q/U(NP)
      V(NP)=0.0
      V(NP-1)=0.0
  561 CONV(NP)=SK
      CONV(NP-1)=SK
      NP3=NP3-2
      DO 558 J=3,NP3
  558 H(J)=B(J)
      GO TO 200
   51 RETURN
      END


      SUBROUTINE SORT
      DIMENSION RR(4),RI(4),REAL(5),YIMAG(5)
      DIMENSION AFIN(80,80),BFIN(80,80)
      COMMON REAL,YIMAG,RR,RI,BF,IFLAG,AFIN,BFIN
      B = 0.0
      DO 800 I=1,4
      IF(ABSF(RI(I))-1.E-7)801,800,800
  801 B = MAX1F(B,RR(I))
  800 CONTINUE
      IF(B) 802,802,803
  802 PRINT 804
  804 FORMAT (34H THERE ARE NO POSITIVE REAL ROOTS
      IFLAG = 1
       RETURN
  803 BF = B
       RETURN
      END
      END
```

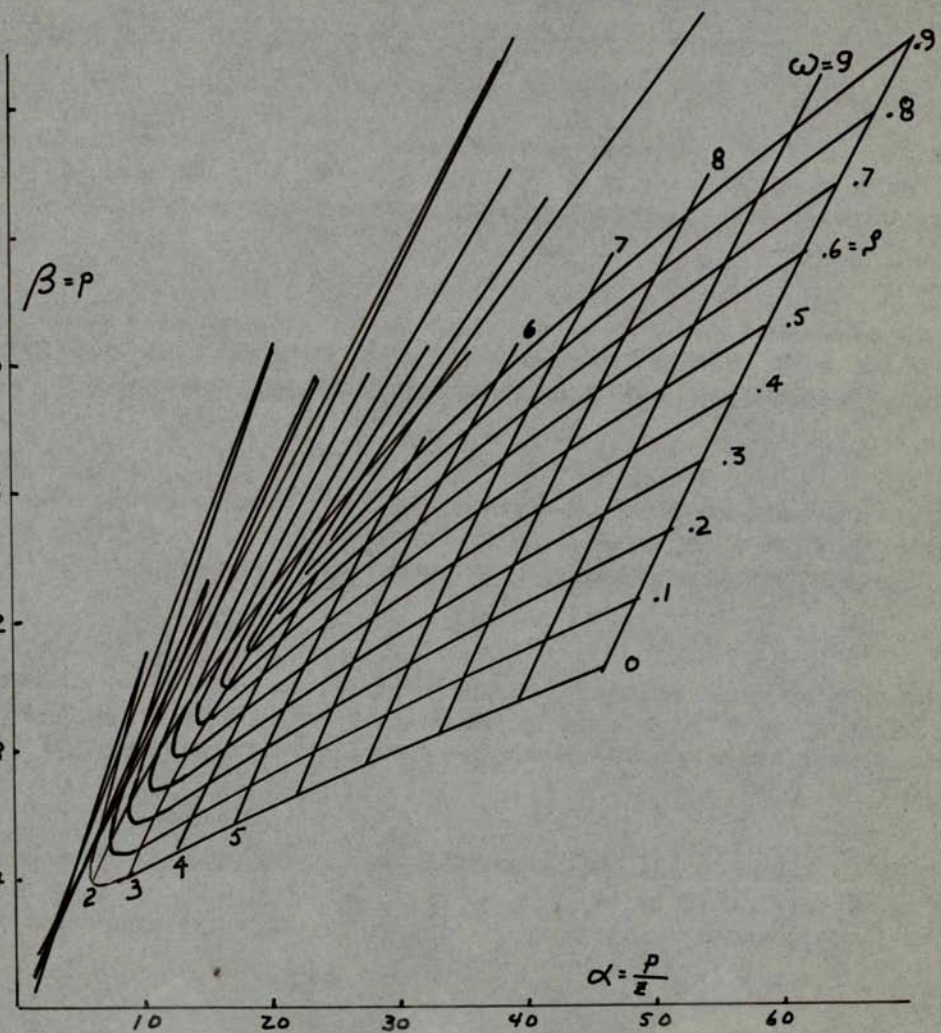

Fig. 1-1. Double Lead Compensation of Plant with $G(s) = \frac{1}{s^3}$

38

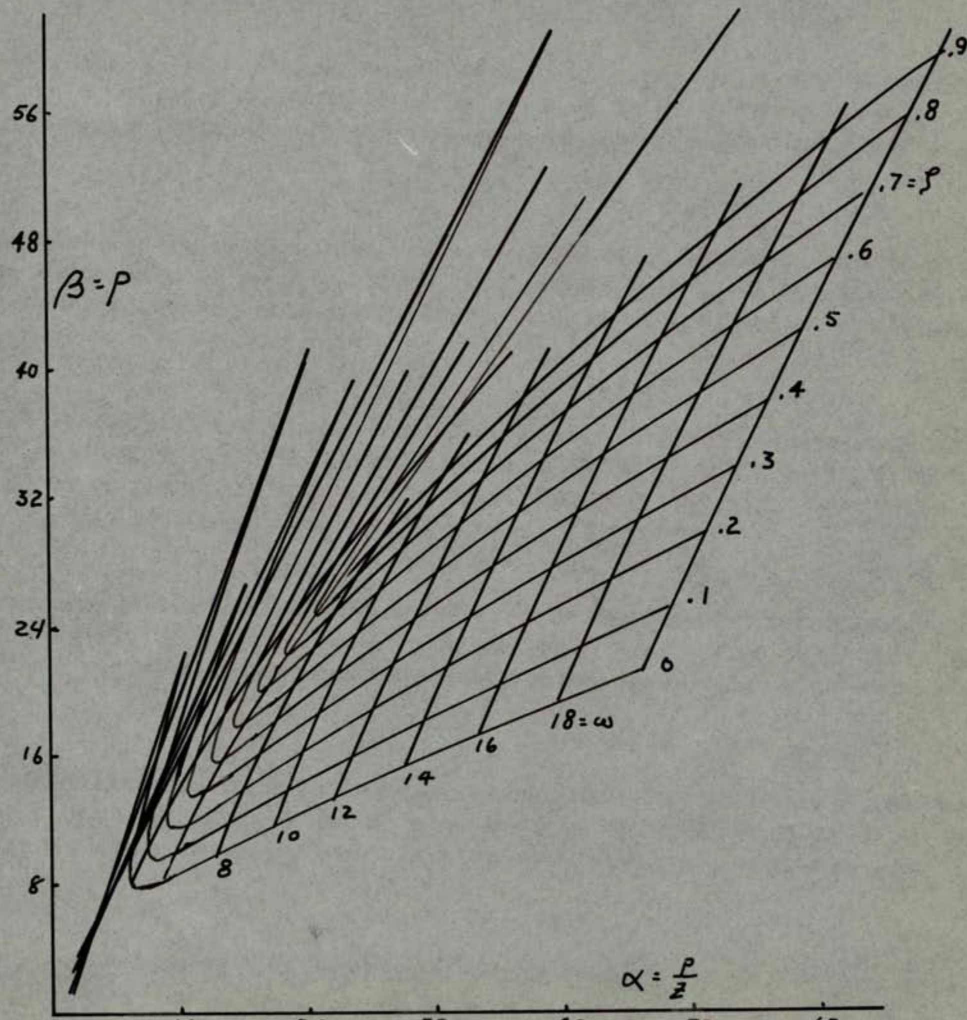

Fig. 1-2. Double Lead Compensation of Plant with $G(s) = \frac{8}{s^3}$

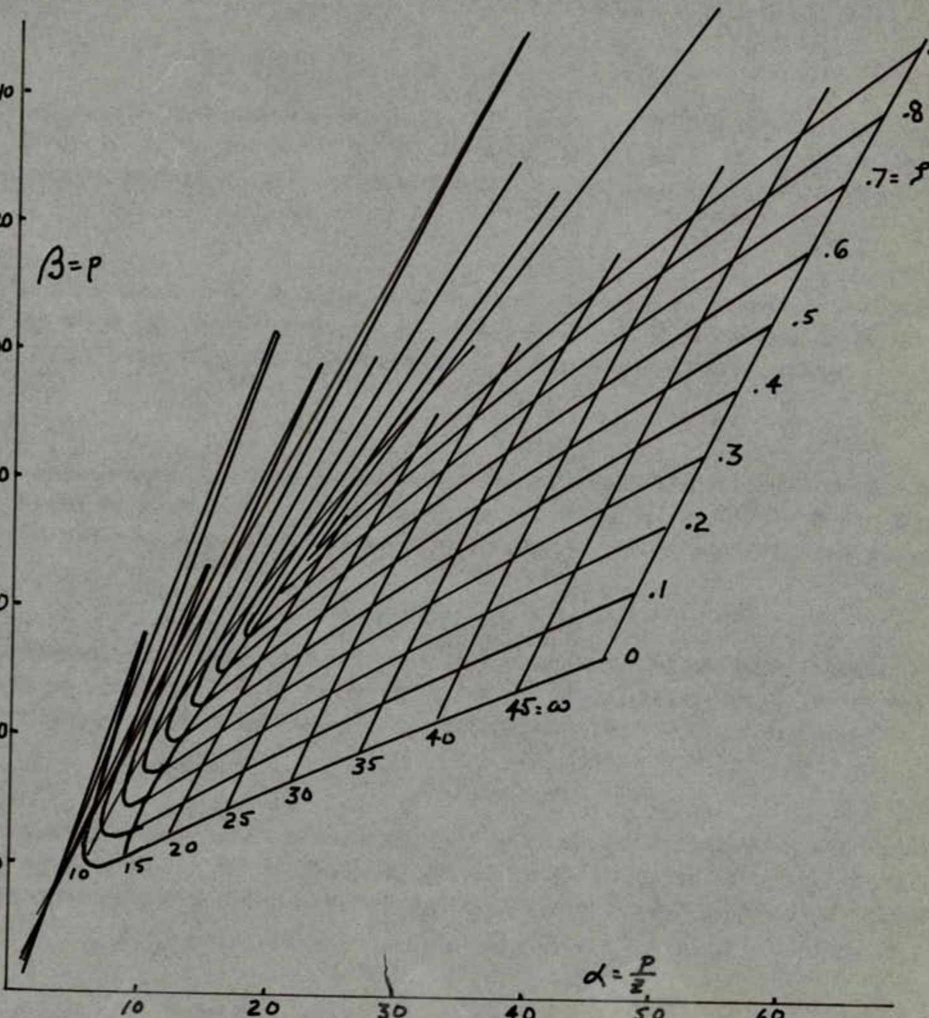

Fig. 1-3. Double Lead Compensation of Plant with $G(s) = \frac{125}{s^3}$

40

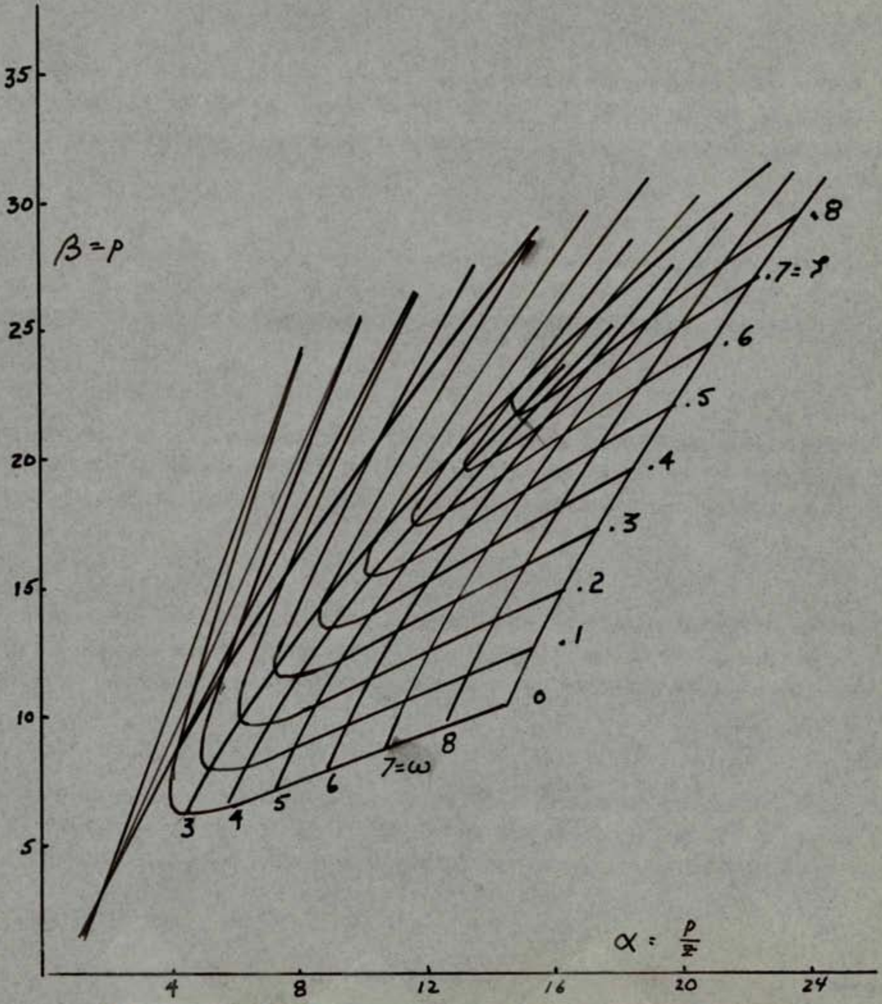

Fig. 1-4. Double Lead Compensation of Plant with $G(s) = \dfrac{10}{s^2(s+1)}$

41

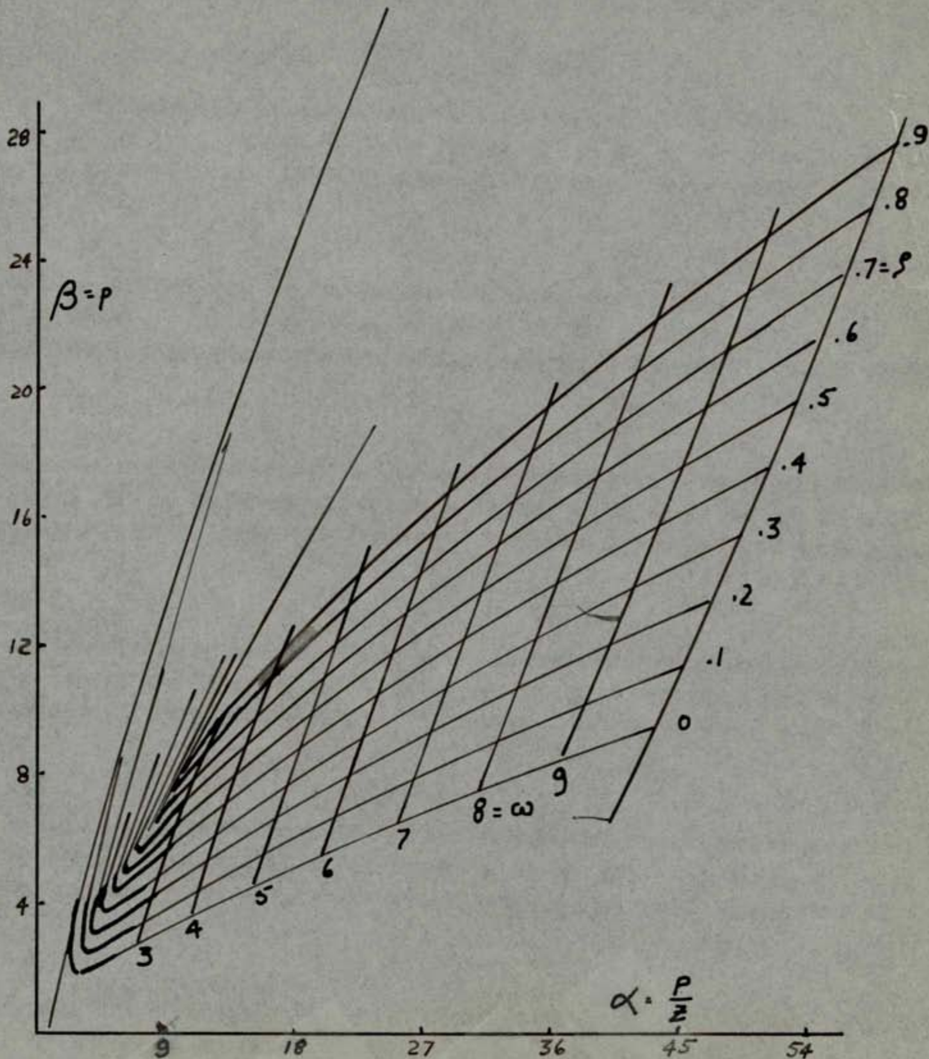

Fig. 1-5. Double Lead Compensation of Plant with $G(s) = \frac{1}{s^2(s+1)}$

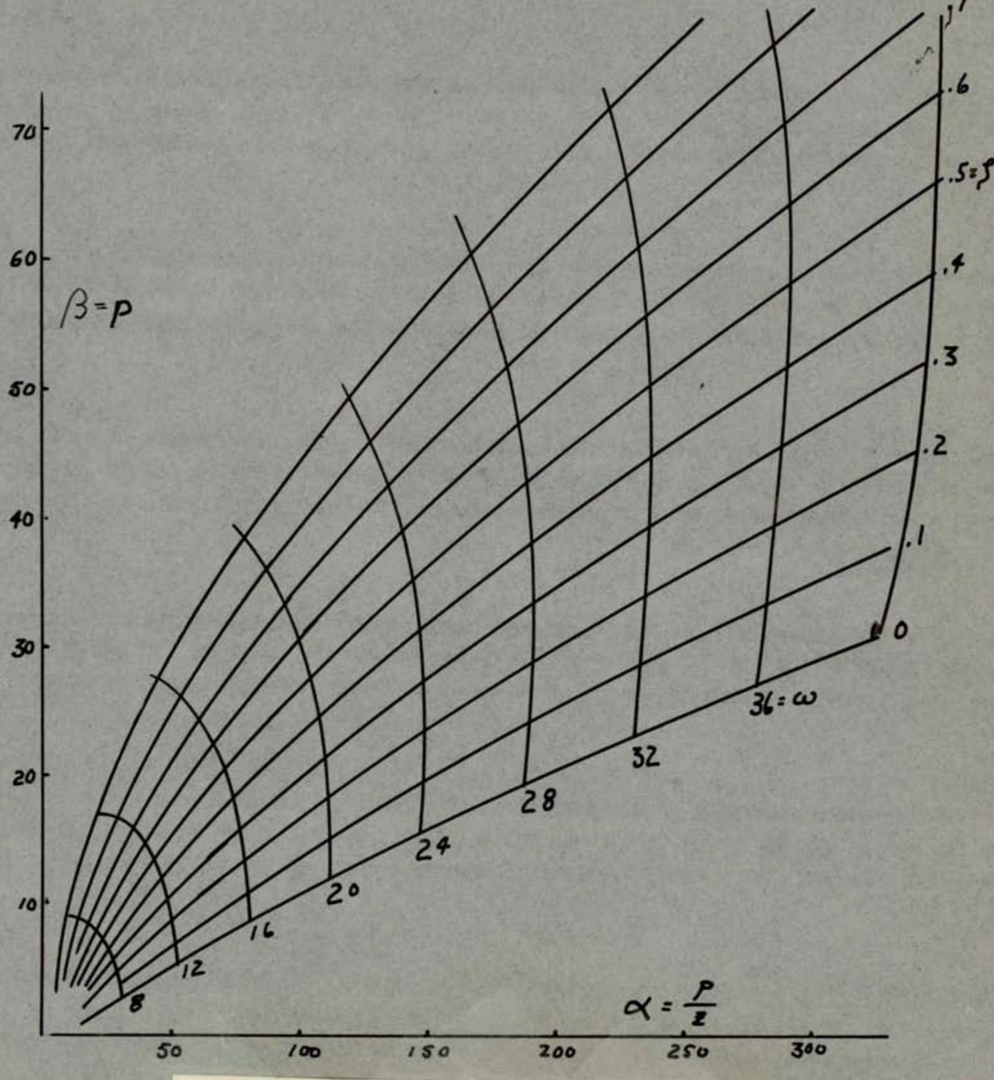

Fig. 1-6. Double Lead Compensation of Plant with $G(s) = \frac{1}{s^2(s+10)}$

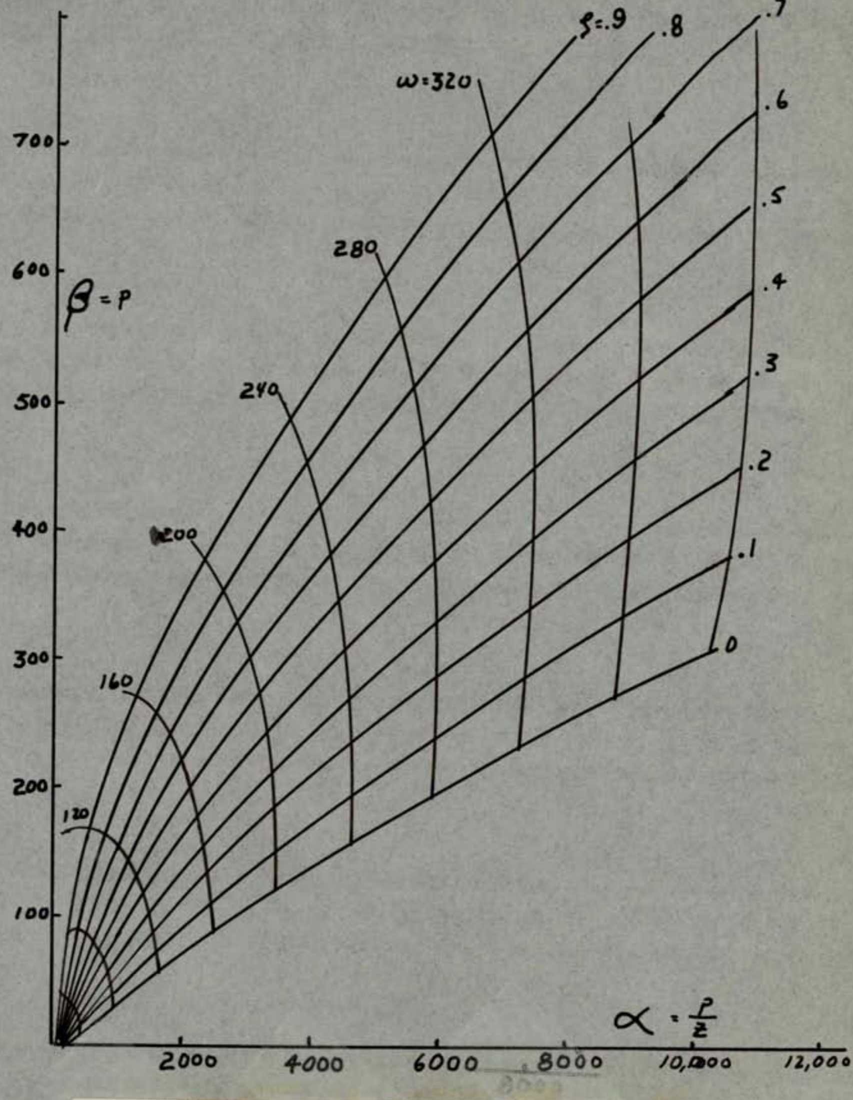

Fig. 1-7. Double Lead Compensation of Plant with $G(s) = \dfrac{1}{s^2(s+100)}$

44

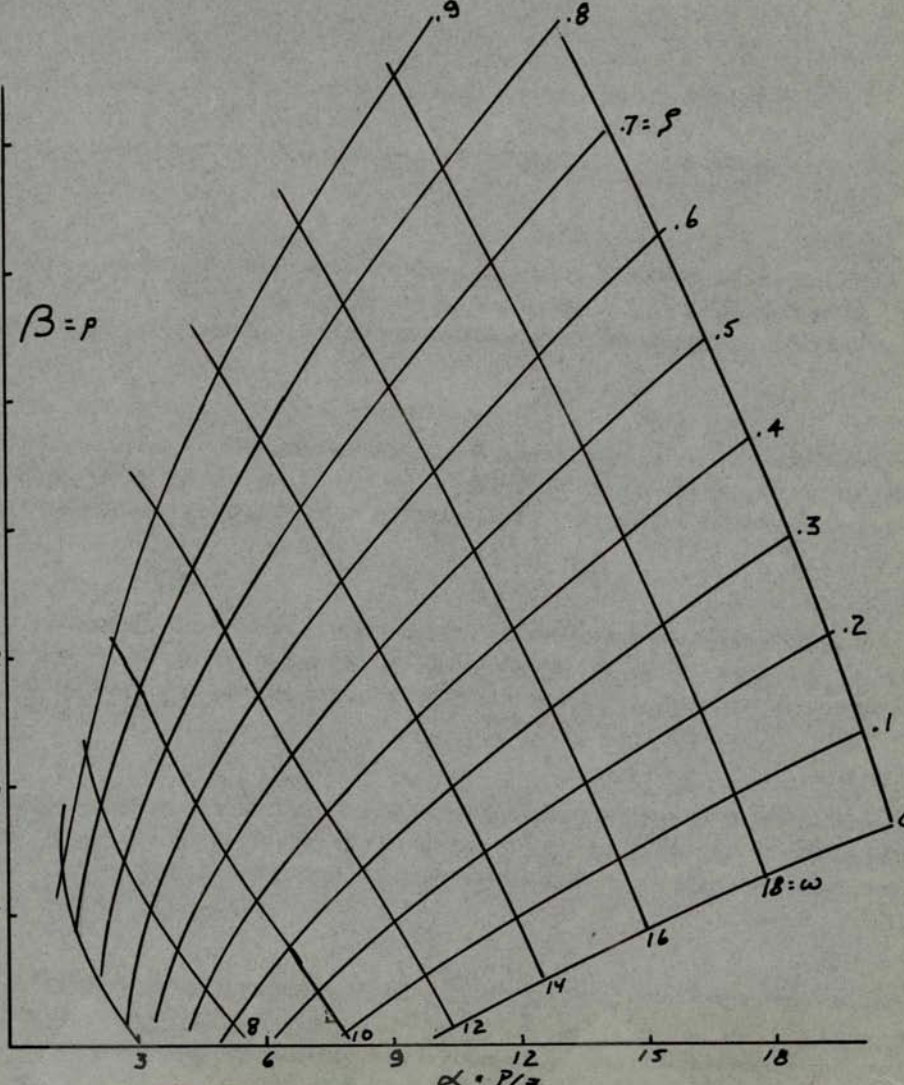

Fig. 1-8. Double Lead Compensation of Plant with with $G(s) = \dfrac{25}{(s+5)^2(s+10)}$

45

2

TRANSIENT RESPONSE OF NONLINEAR SYSTEMS

2.1 INTRODUCTION

When a system has one nonlinear element that is single valued and non-frequency dependent, analysis of the system is conveniently accomplished using the parameter plane methods. The nonlinear element is represented by a describing function, which is a function of signal amplitude only. The describing function is designated as one of the parameters, α or β. This designation removes the nonlinear parameter from the functions that determine the parameter plane curves* so that these may be plotted on the α-β plane. The M-point is located on the α-β plane in the usual way, but for the case of one nonlinear element one coordinate of the M-point is the numerical value of the describing function of the nonlinear parameter. For linear systems the M-point is stationary on the α-β plane, but for a nonlinear system the M-point moves because the numerical value of the describing function is a function of signal amplitude. For a system with one single valued nonlinearity, N, where N is designated as β, the locus followed by the M-point is a straight line parallel to the β-axis. This locus of M-point motion can be said to start at the value of β corresponding to very small (zero) signal amplitude into the nonlinear element. The displacement of the M-point along this locus is determined by the way in which β varies as a function of signal amplitude, and this is determined by using the

*Constant -ζ and constant -ω_n curves, or constant -σ and constant -ω curves.

the describing function of the nonlinear element.

Previous work has shown how to predict limit cycles using M-point locus on the parameter plane. If this locus crosses the stability boundary (ζ=0 curve or σ=0 curve) the intersection of these curves defines the frequency of the limit cycle. If an amplitude scale can be determined for the location of the M-point on the M-locus, then this scale is used to define the amplitude of the limit cycle.

The concept of a moving M-point on the parameter plane can be used to calculate the transient response of nonlinear systems. As the M-point moves along the M-locus, each point defines both signal amplitude and <u>all</u> roots of the characteristic equation. This information can be used to determine the amplitude vs time relationship which is the transient response. Computations are based on Siljak's extension of some basic work by Krylov and Bogoliubov, and details are given in the following paragraphs.

Assume* that the system is second order, and that the nonlinear element is represented by its describing function. Then for an initial signal amplitude A_o, the transient response is defined by

$$X(t) = A_o \, \epsilon^{\sigma t} \cos(\omega t + \phi) \tag{2-1}$$

where σ and ω are both functions of the signal amplitude.

*These assumptions restrict use of this method to systems in which a pair of complex roots dominate the transient response, and these systems must have low pass filter characteristics to justify use of a describing function.

$$\sigma \triangleq \sigma(A)$$
$$\omega \triangleq \omega(A) \tag{2-2}$$

The parameter plane curves are prepared, the M-locus is superimposed on them, and the describing function is used to associate an amplitude scale with the M-locus. Then the values of $\sigma(A)$ and $\omega(A)$ may be read from the parameter plane for any X.

The transient response of the system from any initial displacement, A_o, is determined in two steps, the first of which is to calculate the envelope of the transient. Assuming that ϕ (in eqn. 2-1) is zero, the envelope is defined by

$$X(t) = A_o\, \epsilon^{\sigma(A)t} \tag{2-3}$$

which may be approximated over a short time interval by a straight line tangent to the exponential curve. Thus at $t = 0$, $X=A_o$ and from the parameter plane $\sigma(A_o)=\sigma_o$ is evaluated. Then $X(t)=A_o\epsilon^{\sigma_o t}$ is approximated by a short straight line segment on the X vs t plane. This straight line is terminated at $t = t_1$ and at t_1 a new amplitude A_1 is read from the curve. Entering the M-locus on the parameter plane with A_1 values are obtained for σ_1 and ω_1. The envelope of the transient is extended from t_1 to t_2 with another straight line segment defined by $X = A_1\epsilon^{\sigma_1 t}$. This procedure is repeated until the envelope is defined over an acceptable time interval.

As a by-product of this procedure, ω has been determined quantitatively as a function of amplitude and also as a function of time. Using the definition

48

$$\Phi = \int_0^t \omega(A)\ dt \tag{2-4}$$

the phase can be determined at any t by graphical integration (i.e., evaluation of the area under the $\omega(A)$ vs t curve). If ϕ in eqn. 2-1 is zero, then $X(t) = 0$ for $\Phi = (2n-1)(\pi/2)$. Values of t corresponding to $\Phi = 90^\circ$, 270°, 450°, etc., are determined by graphical integration, are marked on the axis of the X vs t plane, and the transient response is drawn tangent to the envelope and intersecting the $X = 0$ axis at the indicated values of t.

The above procedures are readily applied to systems with one nonlinearity, and correlation with simulation results is excellent. Since such applications are elementary no illustrations are given here, and the study is extended to systems with two single valued nonlinear elements. In general no other methods exist for predicting the transient response of systems with two nonlinear elements, so the results obtained here represent a significant advance in the state of the art.

2.2 CLASSIFICATION OF SYSTEMS WITH TWO NONLINEARITIES

When a system contains two nonlinear elements, N_1 and N_2, that are single valued and are not frequency dependent, parameter plane representation may be used but both α and β become functions of N_1 and N_2. Computation of the parameter plane curves presents no difficulty, but determination of the M-locus may be difficult. As a result it is convenient to classify nonlinear systems according to the structural conditions which complicate the evaluation of the M-locus. The following classes are proposed:

CLASS 1. Identical signal excitation to both nonlinear elements.

In Fig. 2-1a, the signal X is the input to both nonlinear elements N_1 and N_2. For every value of X corresponding values of N_1 and N_2 are uniquely defined and are independent of frequency so evaluation of the M-locus is easy.

CLASS 2. The input signals to the two nonlinear elements are related by a linear differential equation.

In Fig. 2-1b the signal X is the input to N_2, but the input to N_1 is X $G_{-1}(s)$. Thus the input to N_2 is a function of amplitude only, but the input to N_1 is a function of both amplitude and frequency. For a given amplitude of the signal X, the describing function for N_2 provides one unique value, but for each amplitude of X the describing function for N_1 has an infinite number of possible values, one for each possible value of frequency. As a result the evaluation of the M-locus is considerably more difficult than for Class 1.

CLASS 3. The input signals to the two nonlinear elements are related by a nonlinear differential equation.

Fig. 2-1c illustrates this class of nonlinear systems. The signal X is the input to N_1, but the input to N_2 is $X\{N_1\}\{G_2(s)\}$ where the brackets are intended to represent some functional relationship rather than a multiplication. Evaluation of the M-locus can be very difficult for such systems.

2.3 EVALUATION OF THE M-LOCUS. THE DYNAMIC DESCRIBING FUNCTION.

When a system with one single valued nonlinear element is represented on the parameter plane the M-locus is clearly a

50

straight line parallel to one of the coordinate axes. Thus the M-locus itself is readily found but the amplitude scale associated with this locus must be evaluated. For systems with two nonlinearities (especially Class 2 or 3) the path of the M-point on the parameter plane cannot be predicted by inspection. It can be calculated, however, using the ordinary describing function to define the amplitude relationships.

To justify the choice of the describing function as a tool, consider the fact that parameter plane predictions of limit cycles are defined on the basis of a single point where the M-locus interests the stability boundary. This single point defines both the fundamental frequency of the oscillation and also the amplitude of this fundamental component. It is clear that the location of the M-point represents some sort of average value of amplitude, since the instantaneous value of amplitude varies cyclically during a limit cycle. The describing function of a nonlinear element effectively averages the response of the element to a sinusoidal input over one cycle of operation. Thus its use is clearly justified when system operation is periodic and lightly damped. While not so clearly justified for other operating conditions it has given surprisingly accurate results and therefore will be used until a better technique becomes available.

Using the describing functions of the two nonlinearities in a system, a family of decribing function curves are computed and plotted on the α-β parameter plane. When these curves are superimposed on the regular parameter plane curves, the M-locus can

51

be determined. The M-locus represents the curve along which the M-point moves when the system is in dynamic operation, and it consists of the locus of all points at which the describing function curves and the parameter plane curves have common frequency intersections. We choose to call this curve the "Dynamic Describing Function Locus". The procedure and also a justification is as follows:

a) Assume a constant amplitude, constant ω signal at X, the input to one nonlinear element. Using the describing function compute the equivalent gain of that element; also compute the signal amplitude at the input to the second nonlinear element, and the equivalent gain of this second element.

b) The two equivalent gains evaluated in (a) determine one point on a describing function curve on the α-β plane. Repetition using the same value of ω but different amplitudes at X determines a describing function curve for a constant ω signal.

c) Repetition of a) and b) for other values of ω provides a family of describing function curves, each curve being for a designated value of ω.

d) These curves are then superimposed on the usual* parameter plane curves. The constant -ω describing function

*Curves for constant -σ and constant ω are most convenient, but constant -ζ and constant ω_n curves can be used if it is noted that $\omega = \omega_n\sqrt{1-\zeta^2}$.

52

curves will intersect the constant $-\omega$ parameter plane curves, and those intersections for which the ω is the same. Define the Dynamic Describing Function locus.

The nonlinear system is described by one nonlinear differential equation. The procedures used here effectively partition this equation into two parts, a linear part represented by the parameter plane curves, and a nonlinear part represented by the describing function curves. Then parts are "coupled" by the parameters α and β which are the coordinates of both plots. If the system is in steady state periodic motion at a given frequency the nonlinear differential equation of the system must be satisfied, so the linear and nonlinear partitions must be satisfied at that frequency. This condition can exist only at the intersection of the common frequency curves. The points thus defined on the "Dynamic Describing Function Locus" are determined on the basis of steady state sinusoidal operation (unforced). Under transient conditions the M-point moves along some locus on the parameter plane, and we assume that the points on The Dynamic Describing Function locus apply to transient operation although they are determined by means of steady state sinusoidal concepts. Experimental results indicate that this is a good assumption.

2.4 CALCULATED AND EXPERIMENTAL RESULTS

In order to verify the correctness and the applicability of the dynamic describing function and the graphical transient response calculations, specific examples of each of the three general cases of Fig. 2-1 were investigated. The details of some of these examples, and the corresponding calculated results are presented here. Simulation of the systems provided experimental results which are also presented to permit comparison between theory and experiment.

System 1. Two nonlinear elements with identical excitation:

The block diagram is given in Fig. 2-2. The characteristic equation is

$$s^3 + 10s^2 + (10N_1 + 10N_2)s + 100N_1 = 0 \qquad (2\text{-}5)$$

and it is convenient to let $N_1 = \alpha$, $N_2 = \beta$. Fig. 2-3 gives the parameter plane plot (in σ- and ω- curves). Since the two nonlinear elements have identical excitation a single dynamic describing function curve is obtained which is independent of frequency. However, the dynamic describing function is dependent on the specific numerical characteristics of the nonlinearities, and Fig. 2-3 contains three dynamic describing function curves (dotted) for three different sets of characteristics in N_1 and N_2. These three curves were chosen to illustrate different root variations. For curve 1 a real root becomes dominant early in the transient, for curves 2 and 3 complex roots are dominant, the system being moderately damped for curve 2 but going to a stable limit cycle for curve 3.

Calculated and analog computer results are given on Figs. 2-4, 5,6. It is seen from Fig. 2-4 that the dominant real root condition cannot be handled accurately with the graphical computations. It is not known whether the discrepancy lies solely in the graphical

54

method which is based on complex roots, or whether the dynamic describing function also contributes to the errors. Research on this point is continuing. For the cases of Fig. 2-5 and 2-6 the calculated results compare well with the computer results.

System 2. Two nonlinear elements related to a common signal by a linear differential equation.

The block diagram is given in Fig. 2-7, and the parameter plane curves with dynamic describing function curve shown dotted are given on Fig. 2-8. Fig. 2-9 gives the describing function grid needed to obtain the dynamic describing function curve. To obtain the grid of Fig. 2-9 the point A_o on Fig. 2-7 was chosen as a reference point, and at each value of ω the amplitude of the (assumed) sinusoidal signal at A_o was varied to obtain the N_1 vs N_2 values for a constant ω curve on Fig. 2-9. The dynamic describing function curve on Fig. 2-8 is obtained by superimposing the parameter plane curves of Fig. 2-8 on the describing function net of Fig. 2-9 and locating intersections of constant ω curves of the same ω value.

Limit cycle predictions of the dynamic describing function curve on the parameter plane agree with analog computer simulation results. In addition Figs. 2-10, 11,12 compare predicted transient response with simulation results.

Additional checks were run using different values for the deadzone and saturation limits in the two nonlinearities, but the detailed data is not given here. In general the predicted and simulated results were in good agreement except when a real root became dominant during the transient response, in which case the frequency of the oscillatory component was usually predicted with reasonable accuracy, but amplitudes were not, nor was the total response time due to the influence of this real root.

The calculations and simulations were also repeated with the nonlinearities interchanged (i.e., in Fig. 2-7, N_1 becomes a saturated element and N_2 a dead zone element). Using the same techniques the results obtained were always in agreement with about the same degree of accuracy and with the shortcomings as previously noted.

System 3. Two nonlinear elements related to a common signal by nonlinear differential equation.

The classification described as System 3 can contain a wide variety of combinations of linear and nonlinear elements, of which the parameter plane method may be applicable to only a small subset. A specific system which belongs in this class is shown in Fig. 2-13. The characteristic equation of this system is

$$s^3 + 3s^7 + 2s + 40KN_1(N_a + jN_b) \qquad (2\text{-}6)$$

where $N_2 \triangleq N_a + jN_b$ for the hysteretic nonlinearity, and we define $\alpha = N_1N_a$; $\beta = N_1N_b$. The parameter plane equations are still applicable and the parameter plane curves can be computed. For the purposes of this study only $\zeta = 0$ curve was calculated, and only the limit cycle predictions were checked. The describing function net is required, and in this case relates the N_1N_a and N_1N_b pairs to the common signal at A on Fig. 2-13. The results of these computations are given on Fig. 2-14, which shows the $\zeta = 0$ curve from the parameter plane equations and the describing function net for the case where K = 0.15. Only one point is defined on the dynamic describing function curve, and this is marked on the $\zeta = 0$ curve at the point where the ω value on the $\zeta = 0$ curve is the same as the value of the constant ω describing function curve passing through that point. This defines the frequency and amplitude of the limit cycle, and the results agree with simulation results.

Note that a change in the value of K changes the differential equation of the system, thus requiring a new set of curves. Results were obtained with other values of K and again the predictions agreed with simulation results.

2.5 COMMENTS

The results obtained thus far indicate that the parameter plane is a useful tool in predicting the stability and response of nonlinear systems. The accuracy available is only fair, but is more than adequate for many engineering applications. The transient response predictions - in particular for systems containing two nonlinearities, - are better than are available with any other method.

The graphical presentation of the dynamic describing function curve on the parameter plane is potentially a valuable <u>design</u> tool. It indicates at a glance the range of variations of the roots, and thus permits prediction of a <u>desired</u> location of the describing function curve, which in turn implicitly defines the required characteristics of the nonlinear element. Further research is required in this area.

The technique becomes inaccurate when the transient response is influenced by more than two complex roots. Again more research is required to evaluate this situation.

It is too early to assess the true value of studying nonlinear systems on the parameter plane. Without question it does make possible many types of analyses that are not readily available otherwise. However, the limitations of the technique are not clearly defined, and it obviously is important to know under what conditions the methods are not applicable, or should be applied with care.

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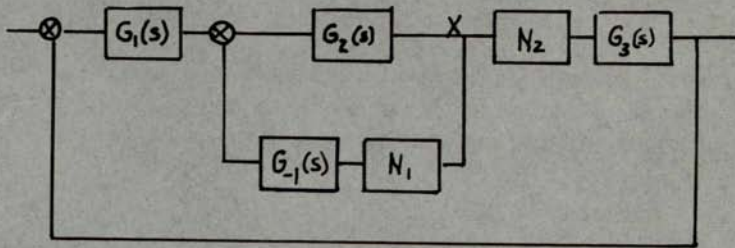
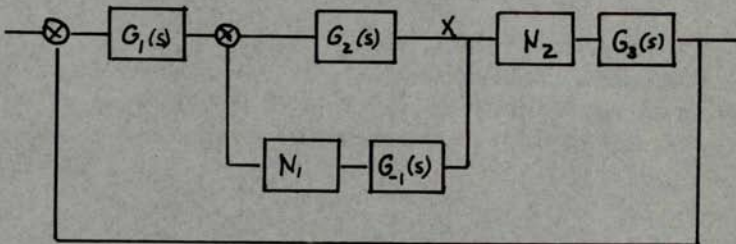
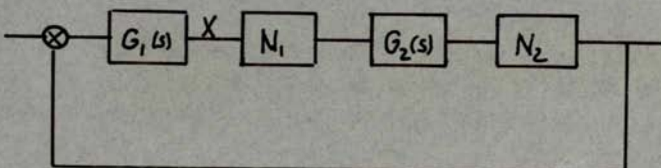
59



a. CASE 1.



b. CASE 2.



c. CASE 3.

Fig. 2-1. General Classification of Control Systems with Two Nonlinear Elements.

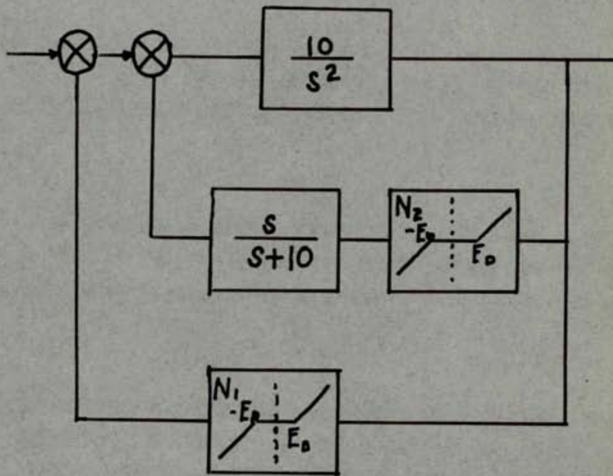


Figure 2-2. Block Diagram of Third Order System with Two Nonlinear Elements.

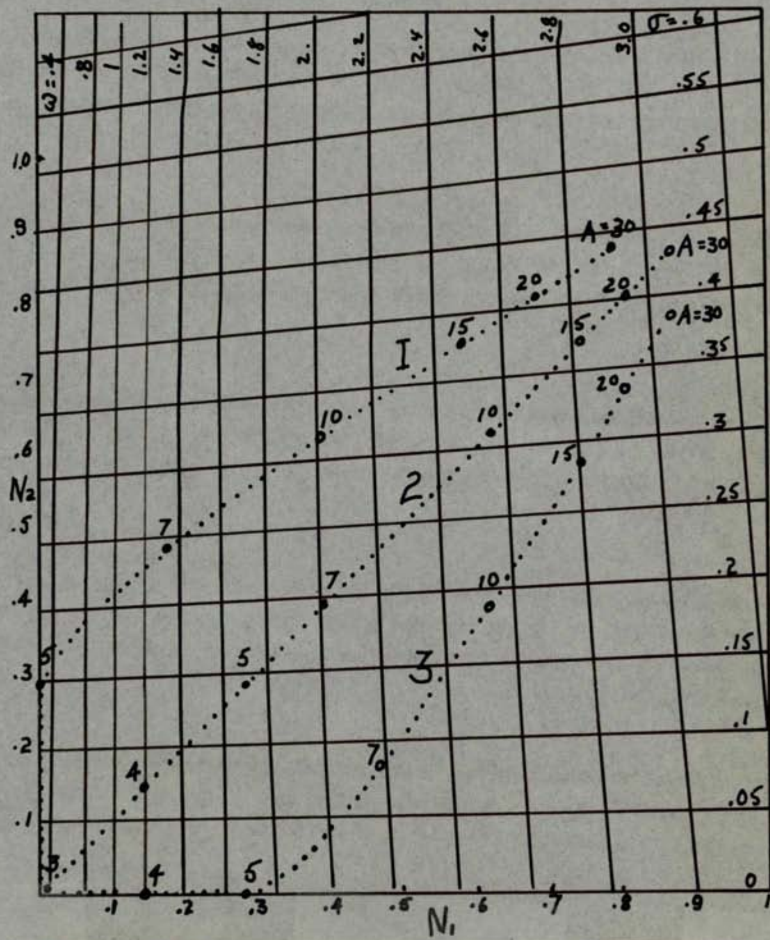


Fig. 2-3. Dynamic Describing Function Curve on Sigma-Omega Curves.

	N_1 - Dead Zone	N_2 - Dead Zone
1	$E_d = \pm 5$ Volts	$E_d = \pm 3$ Volts
2	$E_d = \pm 3$ Volts	$E_d = \pm 3$ Volts
3	$E_d = \pm 3$ Volts	$E_d = \pm 5$ Volts

62

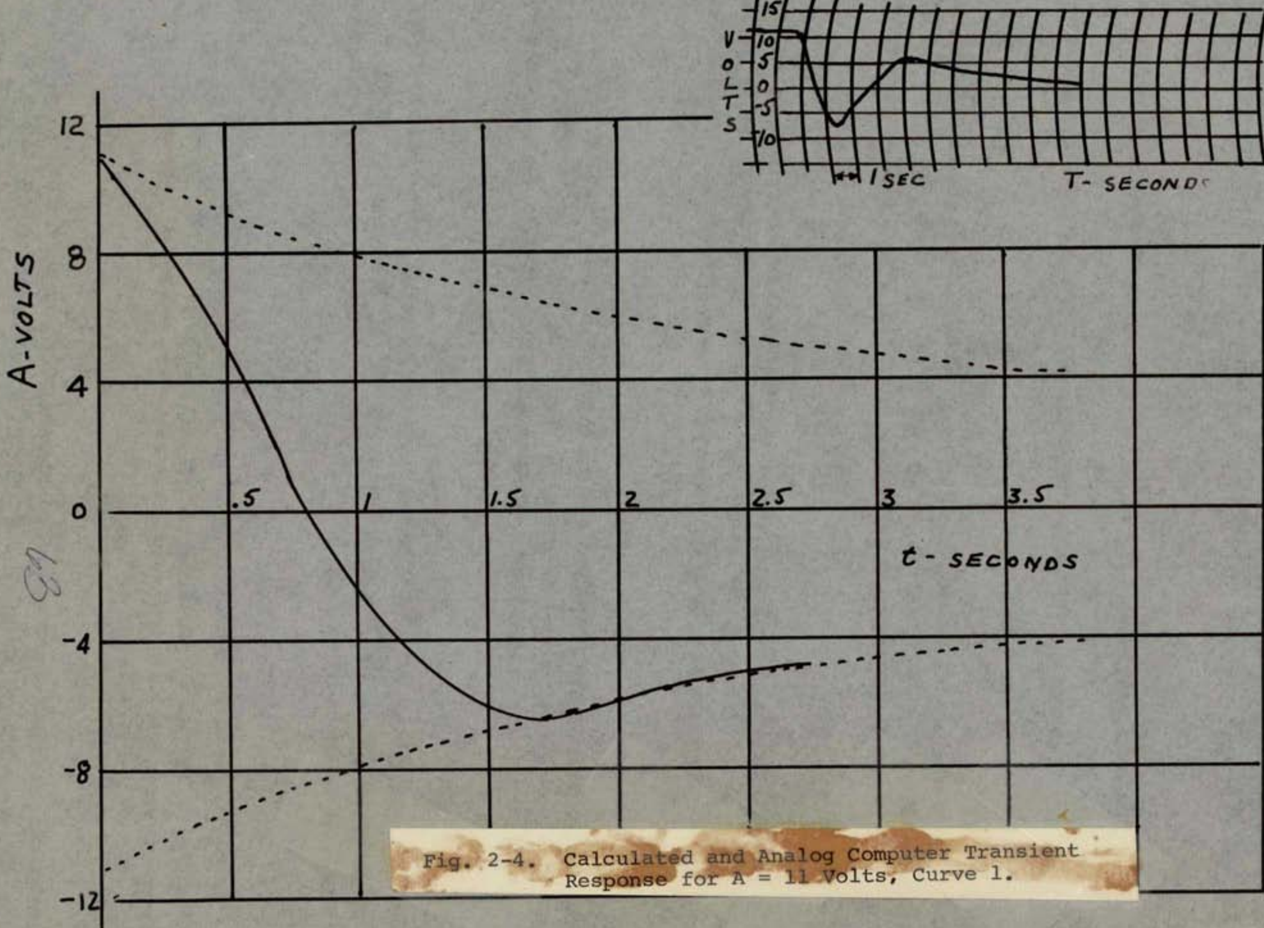


Fig. 2-4. Calculated and Analog Computer Transient Response for A = 11 Volts, Curve 1.

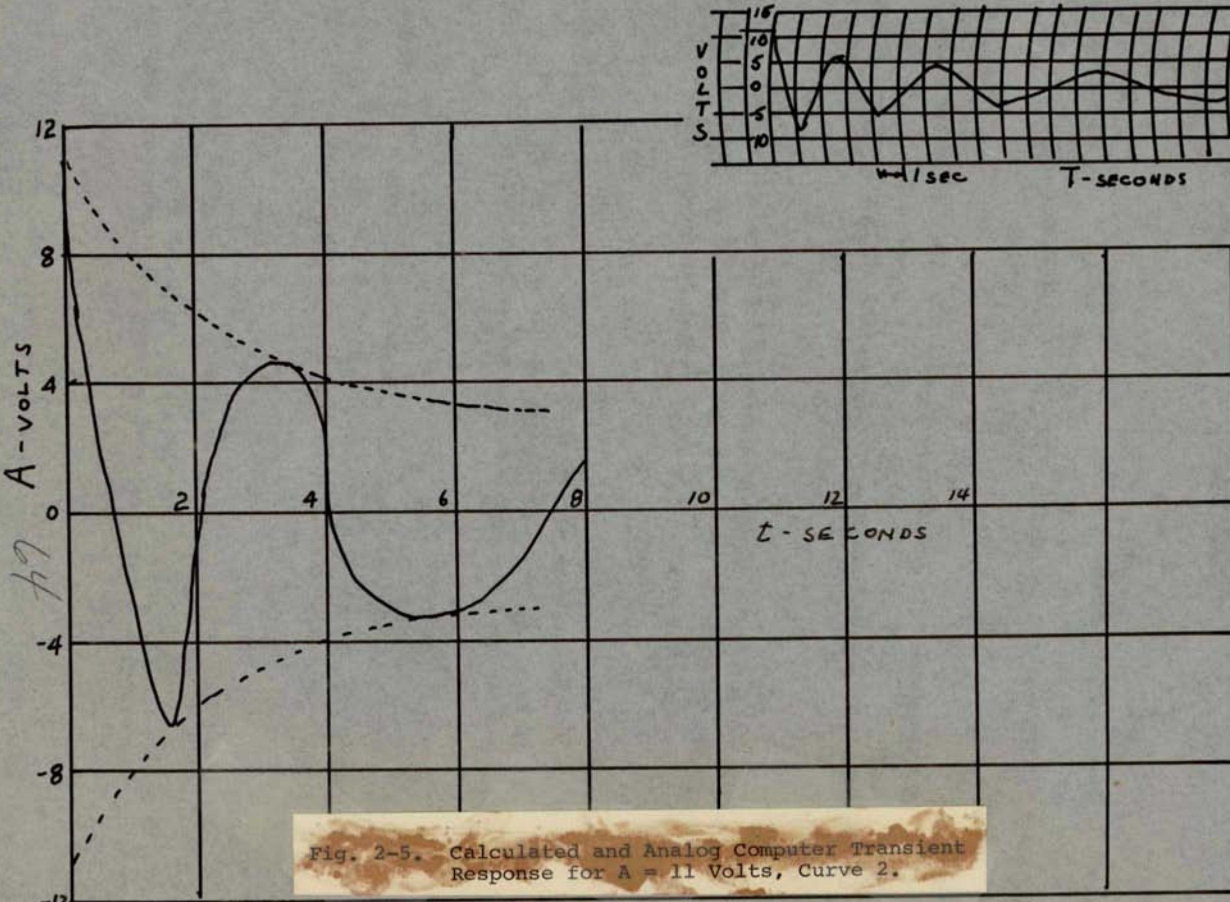


Fig. 2-5. Calculated and Analog Computer Transient Response for A = 11 Volts, Curve 2.

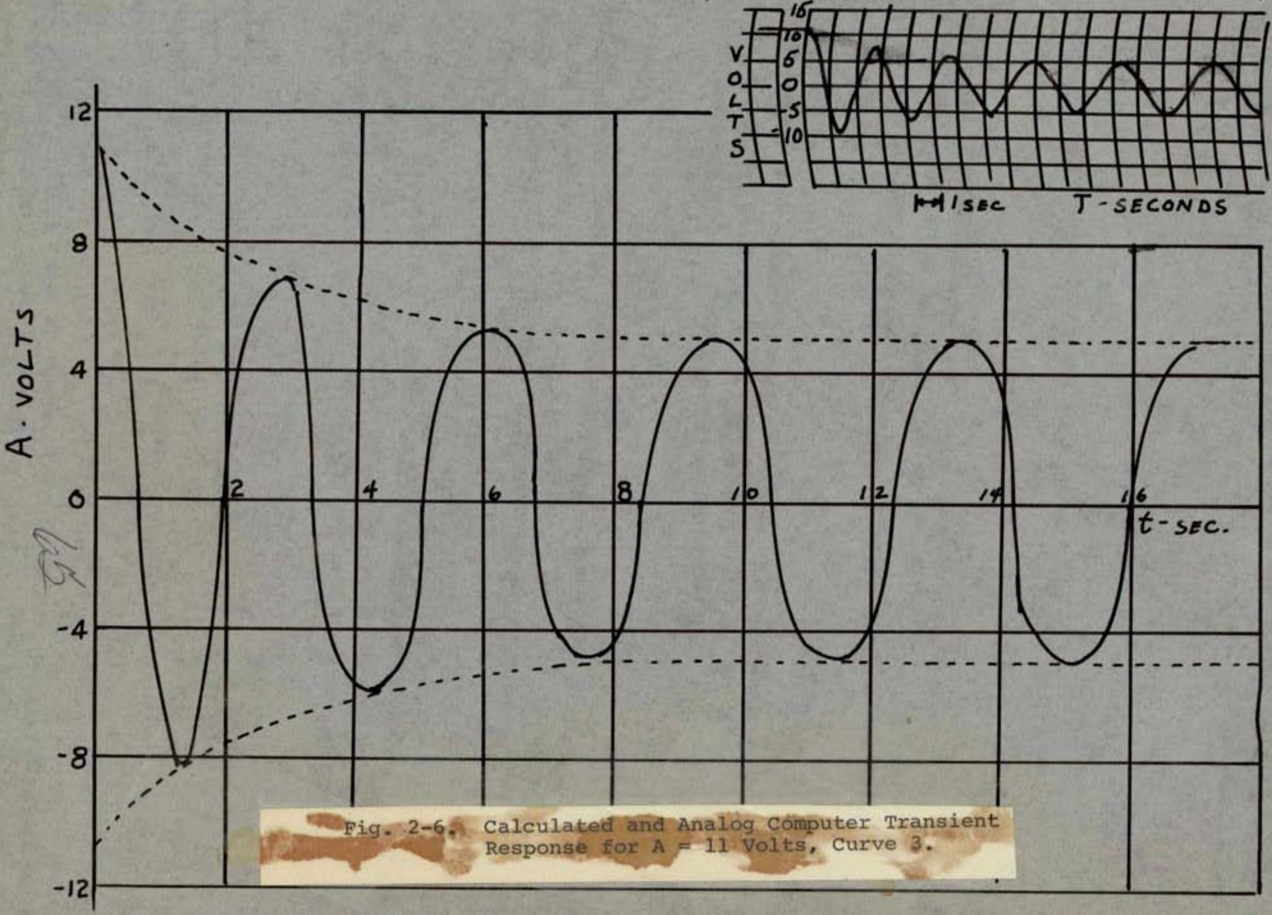


Fig. 2-6. Calculated and Analog Computer Transient Response for A = 11 Volts, Curve 3.

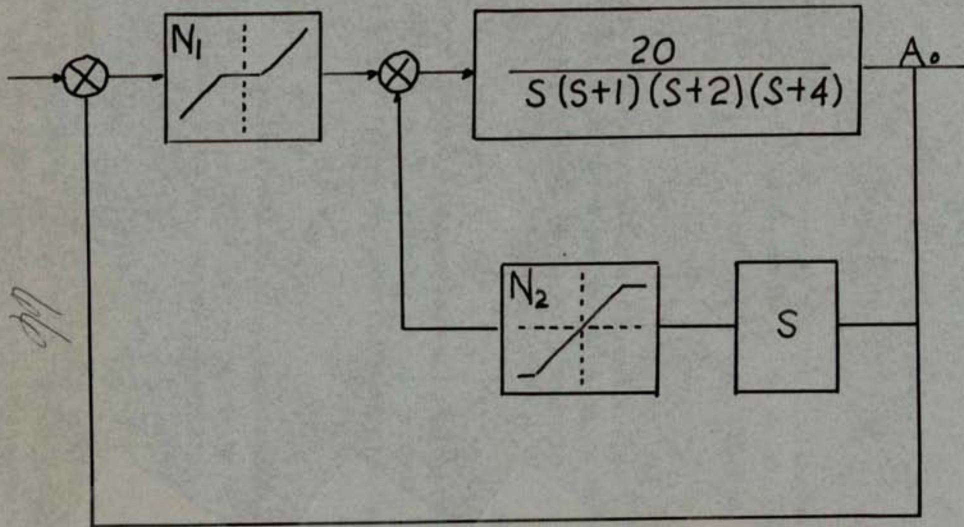


Fig. 2-7. Block Diagram of Fourth Order System with Two Nonlinear Elements.

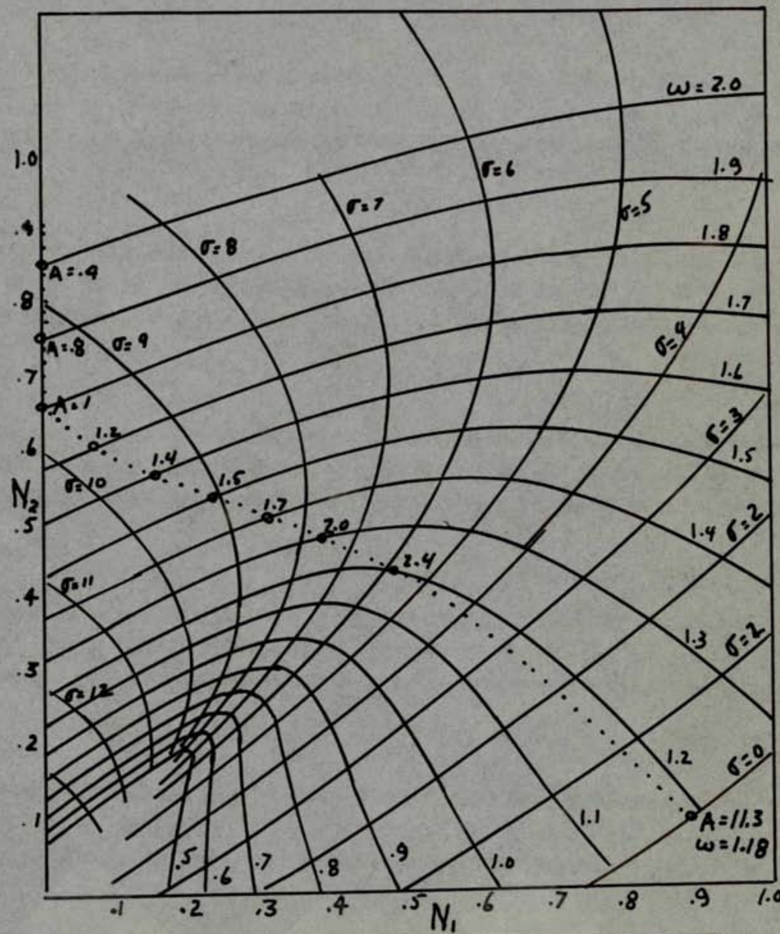


Fig. 2-8. Dynamic Describing Function Curve on Sigma-Omega Curves.
N_1 - Dead Zone $E_d = \pm 1$ Volt
N_2 - Saturation $E_{sat} = \pm 1$ Volt

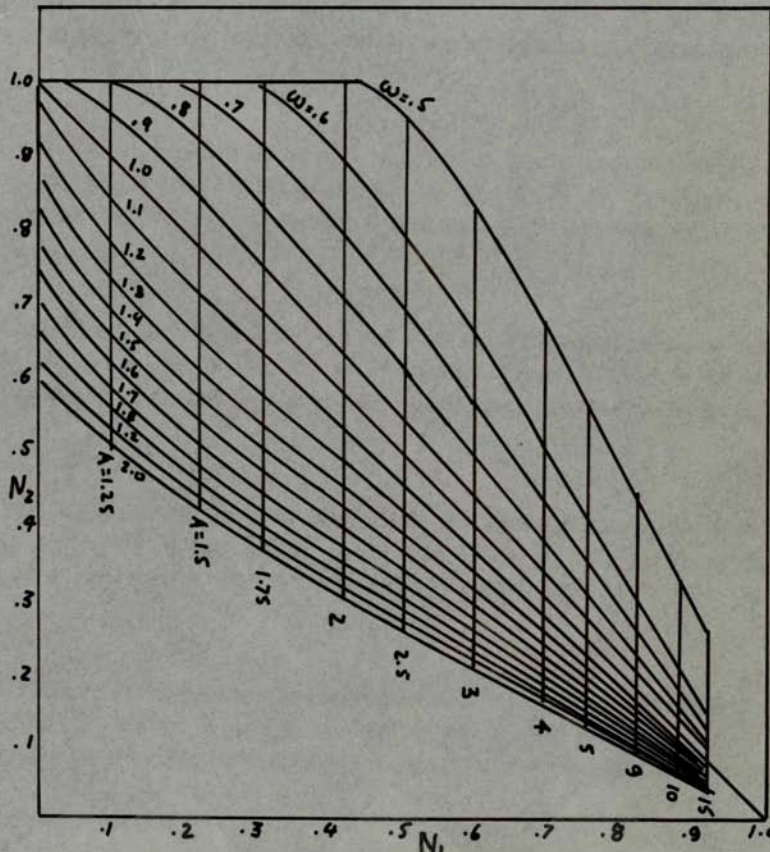


Fig. 2-9. ω-A Grid for Figures 4-2 and 4-3.
N_1 - Dead Zone $E_d = \pm 1$ Volt
N_2 - Saturation $E_{sat} = \pm 1$ Volt

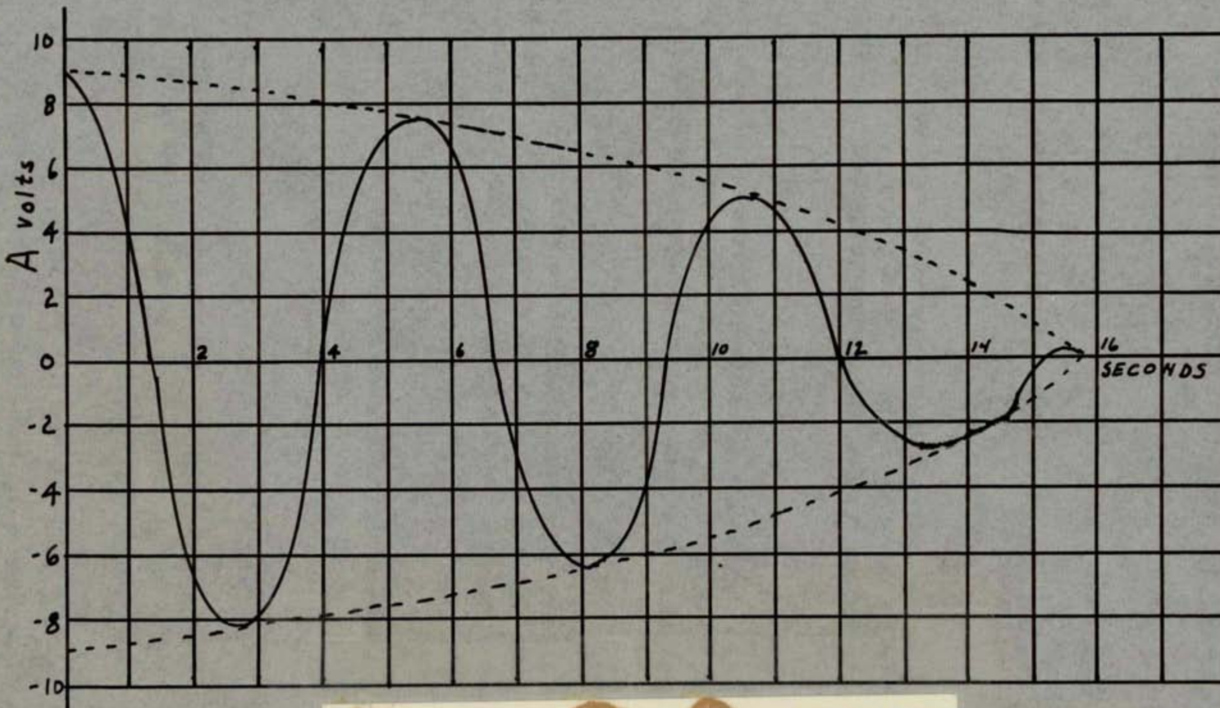


Figure 2-10a. Calculated Transient Response for A = 9 Volts.

69

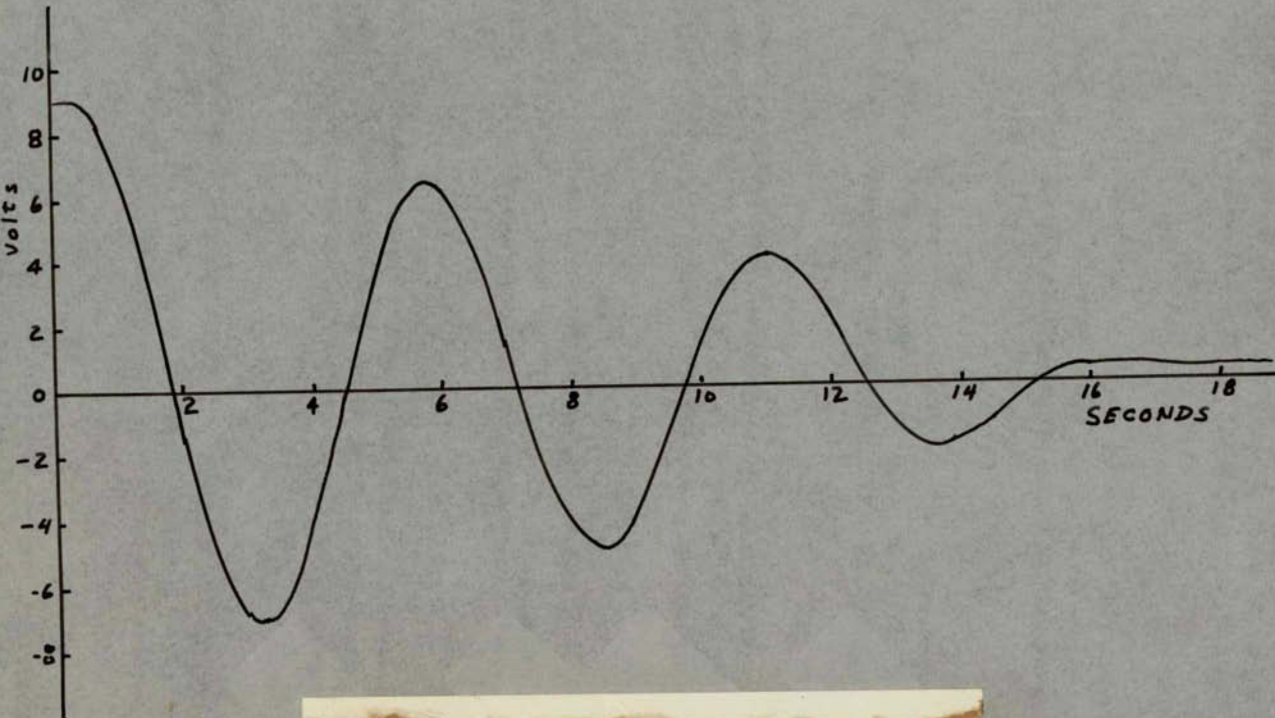


Fig. 2-10b. Analog Computer Transient Response for A = 9 Volts.

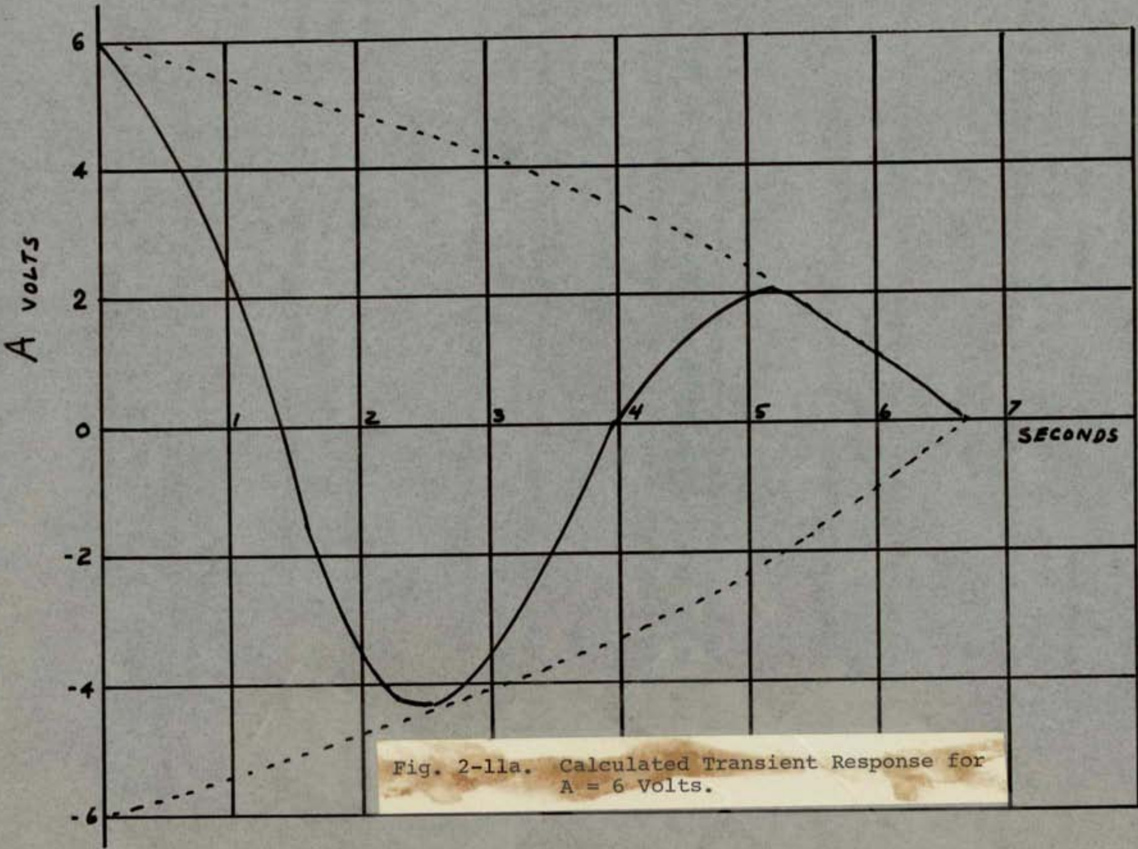


Fig. 2-11a. Calculated Transient Response for
A = 6 Volts.

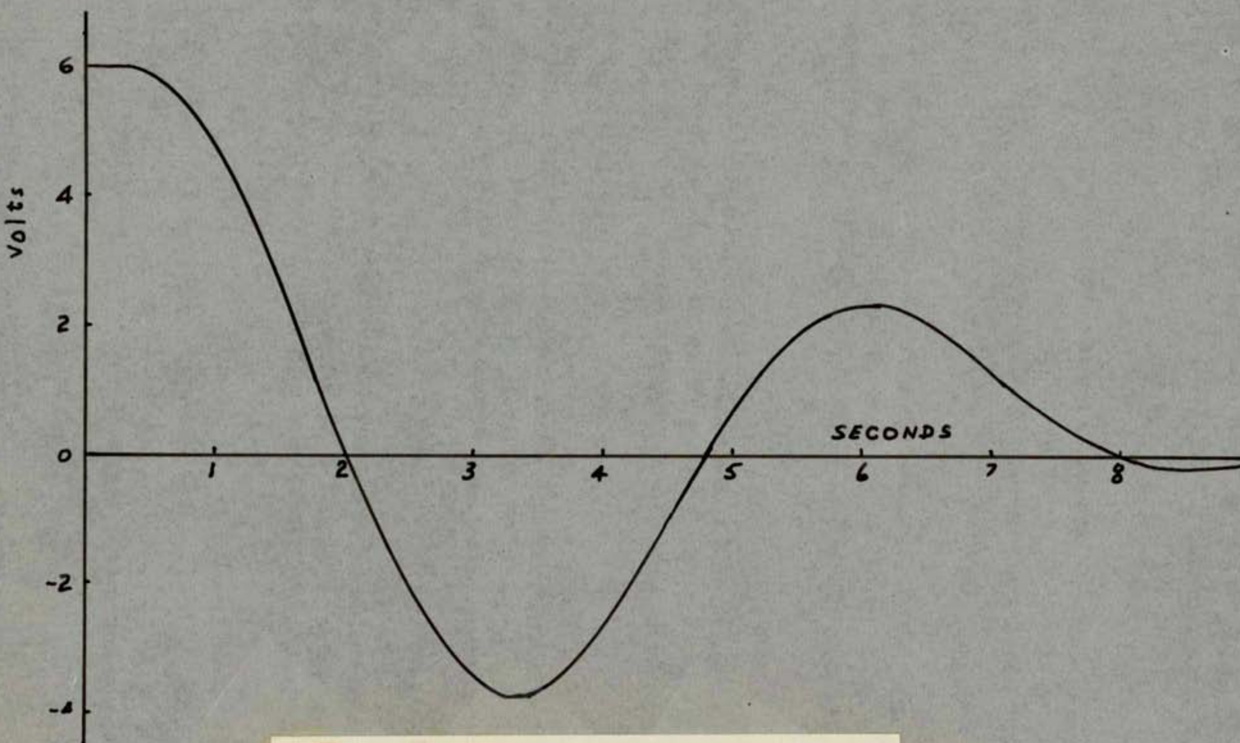


Fig. 2-11b. Analog Computer Transient Response for A = 6 Volts.

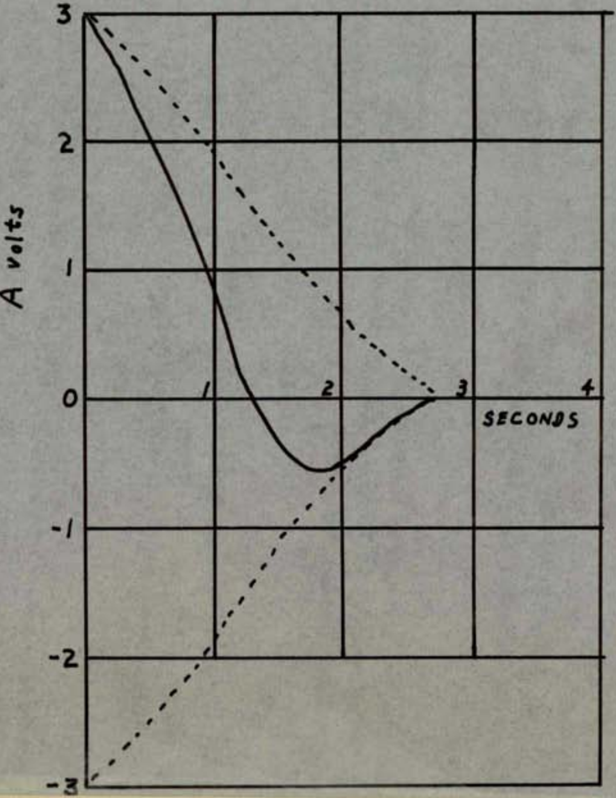


Fig. 2-12a. Calculated Transient Response for A = 3 volts.

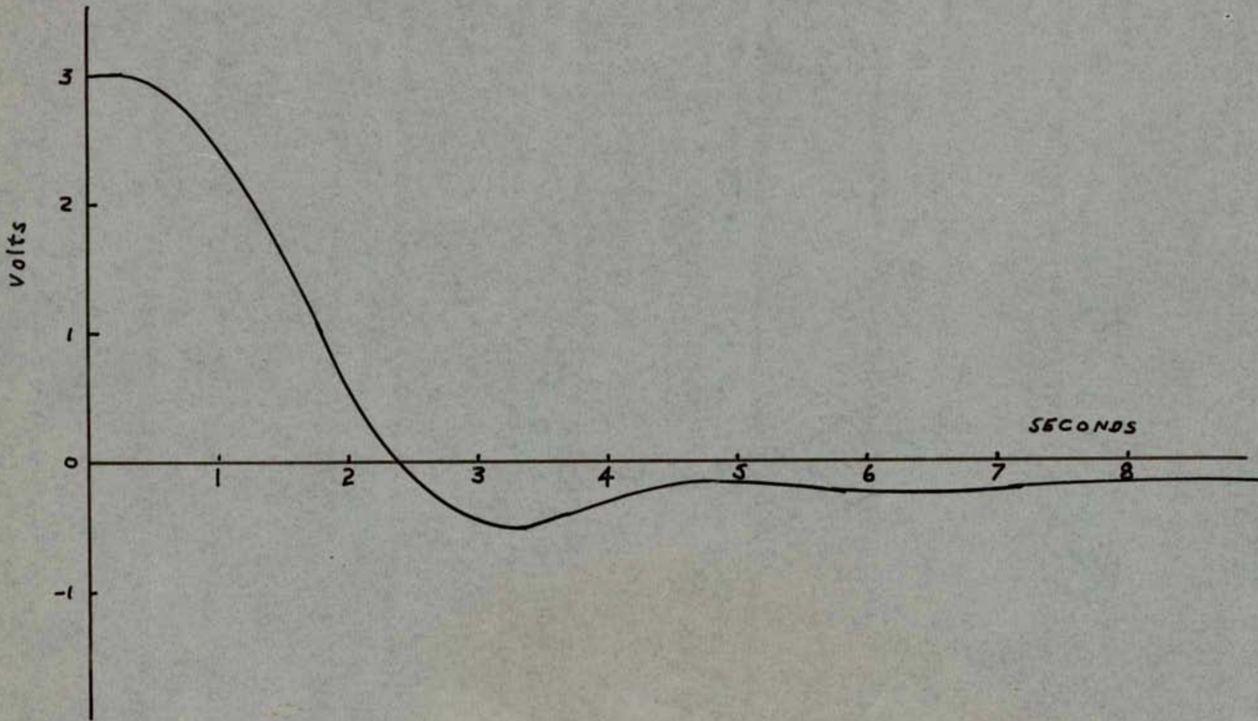


Fig. 2-12b. Analog Computer Transient Response for A = 3 Volts.

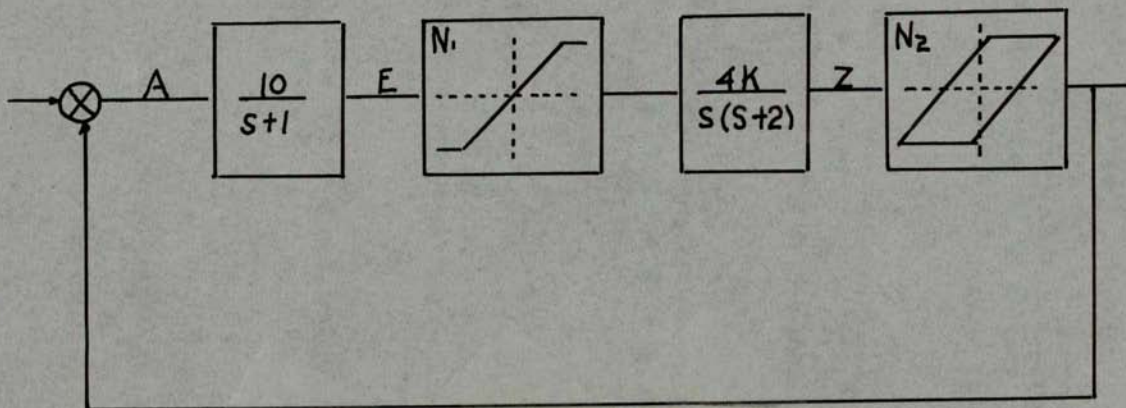


Fig. 2-13. Block Diagram of Test Example.

76

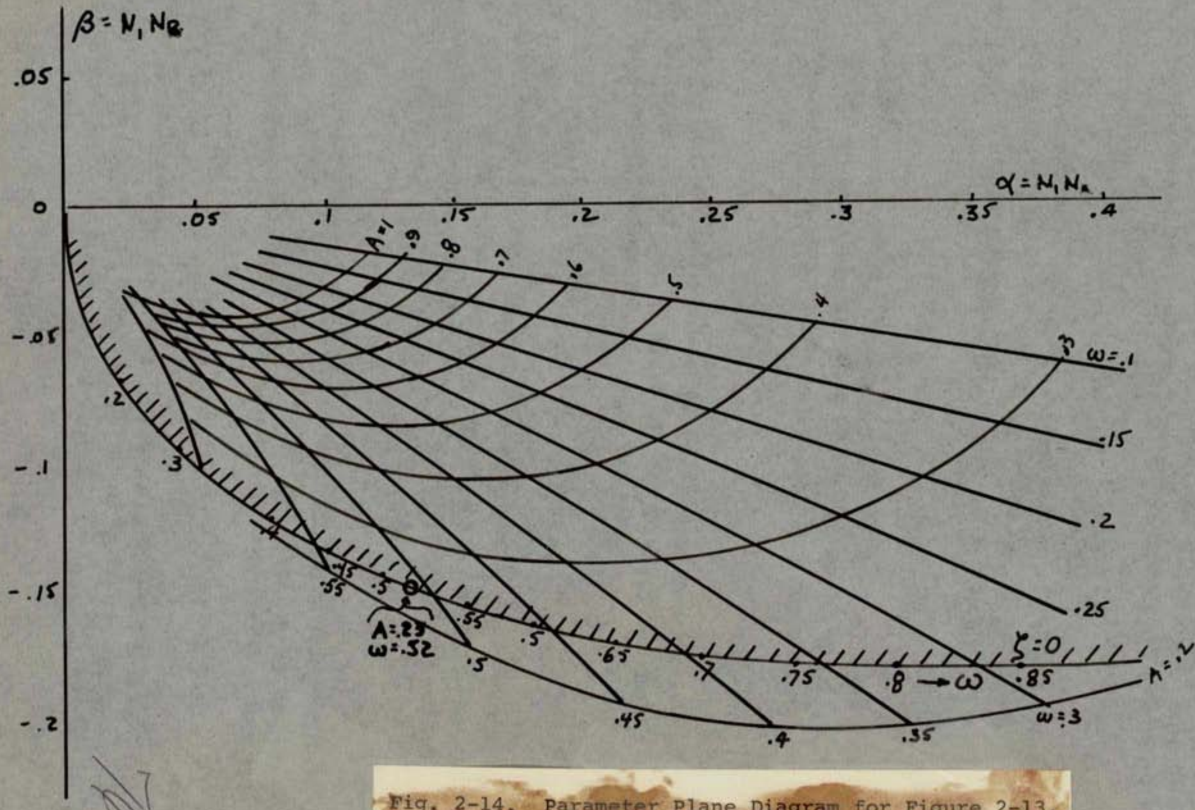


Fig. 2-14. Parameter Plane Diagram for Figure 2-13, K = .15.

CHAPTER III

ASYMMETRICAL NONLINEAR OSCILLATIONS

3.1 Introduction.

In certain classes of nonlinear systems, oscillations may consist of a limit cycle superimposed on a constant or slow-varying signal. These oscillations are referred to as asymmetrical oscillations since the center of the limit cycle is shifted according to the corresponding value of the constant or slow-varying signal. In general, asymmetrical oscillations may occur when the input-output characteristic of the nonlinearity in the system is not symmetrical about the origin, or when the system is subject to forcing signals. When the nonlinear characteristic is asymmetric, the output of the nonlinearity may contain a constant term even though the corresponding input is a single sinusoidal wave. If the nonlinear characteristic is symmetric, asymmetrical oscillations can arise whenever the system is subject to forcing input signals. Evidently these oscillations may take place at certain points of the system if both conditions are present. Before the analysis of asymmetrical oscillations in the parameter plane is presented, the previous work and results in considering these oscillations and related problems are reviewed.

It has been shown first by MacColl [3.1] that the introduction of an external sinusoidal signal at the input to an on-off servomechanism yields a system that behaves like a linear one for small inputs superimposed on the sinusoidal signal. This phenomena has been later investigated under various names, such as "dither effect", "signal stabilization", etc. Asymmetrical nonlinear oscillations has been found by a majority of authors as the most appropriate term for the mentioned phenomena.

In analyzing a carrier-controlled relay servo, Lozier [3.2] has introduced an idea to accomplish the linearization of the relay by a limit cycle existing in the system and without an external signal. This idea has been further developed by several authors [3.3-3.9] and a detailed treatment of the problem has been given by Popov and Palitov [3.8]. On the other hand, the external signal application has been developed by Loeb [3.9] and Oldenburger with his associates [3.10-3.12]. The latter introduced the name "signal stabilization" to indicate that the nonlinear system is stabilized in the state of sustained oscillations with sufficiently high frequency. The stabilization is actually a consequence of the linearizing effect discovered by MacColl. The concept of signal stabilization has been extended by Sridhar [3.13-3.14] to the case of a nonlinear system which has one single-valued nonlinearity in the loop, and the stabilizing signal is a stationary random process with a Gaussian distribution and obeys the ergodic hypothesis.

The above defined problem can be treated by dual-input describing functions as proposed by West [3.15]. This approach has been significantly simplified by Boyer [3.16] as outlined by Gibson [3.17]. A similar approach is used by Gelb and Van der Veld [3.18], and significant results have been obtained by Atherton and others [3.19-3.20] who made a comparison of the utilized concept with the Tsypkin method [3.21].

The study of asymmetrical nonlinear oscillations has been extensively performed in the analysis and design of a large class of plant adaptive control systems. This class of system is sometimes called the limit cycling adaptive systems because of the fact that the existing limit cycle is used as an identification signal. Some of the references on this subject are listed here [3.22-3.26]. A majority of the authors proposed an external sinusoidal signal for identification. More recently, Gelb and Van der Velde [3.18] have examined to a limited extent and in a quantitative manner the properties of self-oscillating adaptive systems which have several advantages over the external adaptation, such as simplicity, cost, reliability, etc. The following analysis of asymmetrical nonlinear oscillations in the parameter plane can be applied directly to self-oscillating adaptive systems.

In the following developments, the asymmetrical nonlinear oscillations are analyzed in the parameter plane [3.27]. The control systems with asymmetrical nonlinear characteristics are considered to determine stability and sustained oscillations. The same type of oscillations is investigated in nonlinear control systems subject to constant reference and perturbing input signals. The procedure is further extended to the analysis of systems with slow-varying input signals. In this case, it is shown how a nonlinear characteristic can be modified for the slow-varying signals. The presented analysis is performed with respect to both input signals and the values of adjustable system parameters. The analysis procedure is illustrated by examples in which multiloop feedback structures with several adjustable parameters are considered. In addition, various nonlinear characteristics are used in either the forward or the feedback

80

path. The obtained results are checked by computer simulations which indicate a sufficient accuracy of the presented procedure.

3.2 Basic Developments

Consider a nonlinear system described by the nonlinear differential equation

$$B(s)x + C(s)\ F(x,sx) = H(s)f, \qquad s \equiv \frac{d}{dt} \tag{3.1}$$

where B(s), C(s), and H(s) are polynomials in s and the degree of the polynomial B(s) is greater than the degree of the polynomials C(s) and H(s). The function F(x,sx) describes the nonlinearity. Function $f = f(t)$ is a forcing signal, which may be either a reference input or a perturbing signal, and it is assumed to be a constant or a slowly-varying function of time.

As a first approximation, the steady-state solution $x = x(t)$ of equation 3.1 which represents the input to the nonlinearity, is assumed to be

$$x = x^o + x^* \tag{3.2}$$

where $x^o = x^o(t)$ is either a slowly-varying function of time or is constant, and x^*, which is

$$x^* = A \sin \phi, \qquad \phi = \Omega t + \theta, \tag{3.3}$$

represents the periodic component of the solution x(t). Since θ in (3.3) merely corresponds to a shift in t, one can put $\theta = 0$ and use $x^* = A \sin \Omega t$.

The forcing function f(t) is considered as a slowly-varying function of time if it can be assumed approximately as constant over any cycle of the periodic component x^*; i.e.,

$$|f(t+T) - f(t)| << |f(t)| \tag{3.4}$$

91

where the period $T = 2\pi/\Omega$. In the frequency domain, equation 3.4 means that the frequency Ω of the periodic component x^* is much greater (practically ten times or more) than the highest frequency of the slowly-varying component x^o. In this case, no harmonic relation between the components x^o and x^* nonlinear system subject to forcing signals, such as jump-resonance, generation of subharmonics, etc., cannot take place. The forced nonlinear oscillations for which the condition (3.4) is not satisfied necessarily, are considered in other works.

Under the condition (3.4), the values of x^o, A, and Ω, which appear in the solution $x = x^o + A \sin \Omega t$, are slowly-varying quantities in time. This enables the extension of the conventional harmonic linearization in which the describing function is defined for the signal $x = x^o + x^*$ as an input to the nonlinear element. Thus, the nonlinear function $F(x, sx)$ is approximately expressed by the first terms of the Fourier series as

$$F(x,sx) = F^o + N_1 x^* + \frac{N_2}{\Omega} sx^* \tag{3.5}$$

where

$$F^o = \frac{1}{2\pi} \int_o^{2\pi} F(x^o + A \sin \phi, A\Omega \cos \phi)\, d\phi \tag{3.6a}$$

$$N_1 = \frac{1}{\pi A} \int_o^{2\pi} F(x^o + A \sin \phi, A\Omega \cos \phi) \sin \phi \, d\phi \tag{3.6b}$$

$$N_2 = \frac{1}{\pi A} \int_o^{2\pi} F(x^o + A \sin \phi, A\Omega \cos \phi) \cos \phi \, d\phi \tag{3.6c}$$

and $\phi = \Omega t$.

As can be seen from equations 3.5 and 3.6a, the component F^o of the output of the nonlinearity $F(x,sx)$ is not considered zero as was the case in the analysis of symmetrical nonlinear oscillations presented in the previous chapter. This results from the fact that either the nonlinear function $F(x,sx)$ is not symmetric or the system is subject to an external input signal, or that both facts are present in the system.

According to equations 3.6, all coefficients F^o, N_1, and N_2 are generally functions of x^o, A, and Ω, i.e.,

$$F^o+F^o(x^o,A,\Omega), \quad N_1 = N_1(x^o,A,\Omega), \quad N_2 = N_2(x^o,A,\Omega) \qquad (3.7)$$

For a majority of the nonlinear functions $F(x,sx)$ encountered in practical applications, the above functions (3.7) are obtained once and for all.

By applying the linearization of the function $F(x,sx)$ given in equation 3.5, the solution $x = x^o + x^*$ of (3.1) can be obtained by considering the following linearized differential equation

$$B(s)(x^o+x^*) + C(s)(F^o+N_1x^* + \frac{N_2}{\Omega} sx^*) = H(s)f \qquad (3.8)$$

instead of equation 3.1. If x^o, A, and Ω are slowly-varying functions of time as a consequence of the same property associated with the forcing function f, equation 3.8 can be rewritten as two simultaneous equations corresponding to the slowly-varying signal x^o and the periodic signal x^* as follows:

$$B(s)x^o + C(s)F^o = H(s)f \qquad (3.9a)$$

$$B(s)x^* + C(s)(N_1x^* + \frac{N_2}{\Omega} sx^*) = 0 \qquad (3.9b)$$

Equations 3.9, however, cannot be solved independently since they are related to each other by the nonlinear equations 3.7. This fact indicates that the applied linearization preserves the essential feature of nonlinear systems and that the superposition principle from linear analysis is not valid.

An analytical solution of equations 3.9 is difficult to obtain since F^o in (3.9a) is usually a trancendental function with respect to x^o. A graphical procedure is presented for solving equations 3.9 in the parameter plane. A necessary condition for equation 3.1 to have a solution x(t) close to 3.2 is that the characteristic equation

$$B(s) + C(s)(N_1 + \frac{N_2}{\Omega}s) = 0 \qquad (3.10)$$

corresponding to the linearized differential equation 3.9b, have a pure imaginary root $s = j\Omega$.

By using the parameter plane approach, equation 3.10 can be solved for α and β as

$$\begin{aligned} \alpha &= \alpha(\Omega) \\ \beta &= \beta(\Omega) \end{aligned} \qquad (3.11)$$

where α and β are N_1 and N_2 or some other system adjustable parameter. Equations 3.11 represent the $\Sigma = 0$ (or $\zeta = 0$) curve for which $s = j\Omega$. The $\Sigma = 0$ curve determines the stable region in the $\alpha\beta$ plane in the usual manner. After the stable region is found, the loci of points $M(\alpha, \beta)$ are plotted according to the variations of α and/or β representing N_1 and/or N_2. The M loci incorporates the additional variable x^o, and a family of the loci should be constructed for different values of x^o. Then the stability of the nonlinear system is determined by the relative location of the Σ curve and the M loci and the limit cycles are

84

found at their intersections. The stability of the limit cycles is determined in the usual manner. This part of the solution process will be best described by the examples that follow.

The presence of a limit cycle in the system can modify the nonlinear characteristic for the slowly-varying input signal. In order to determine the modified characteristic, the intersections of the $\Sigma = 0$ curve and the M loci are considered to evaluate the amplitude A and the frequency Ω of the limit cycle as functions of the slowly-varying component x^o; i.e.,

$$A = A(x^o), \quad \Omega = \Omega(x^o) \tag{3.12}$$

These functions, when substituted into the function $F^o(x^o, A, \Omega)$ yield the modified nonlinear characteristic for the slowly-varying signal

$$F^o = \psi(x^o) \tag{3.13}$$

The function $\psi(x^o)$ is continuous in a limited range of x^o, which indicates the smoothing effect due to the presence of the limit cycle.

Substitution of equation 3.13 into equation 3.9a gives

$$B(s)x^o + C(s)\ \psi(x^o) = H(s) f \tag{3.14}$$

Equation 3.14 is a nonlinear differential equation in x^o, which can be solved graphically for x^o after the function $\psi(x^o)$ is obtained. This, in turn, yields the related values of the functions $A(x^o)$ and $\Omega(x^o)$ of equations 3.12, and the solution $x = x^o + A \sin \Omega t$ is thereby determined.

The function $\psi(x^o)$ is a continuous function of x^o and it can

15

be assumed approximately linear for small variations of x^o. Then the stability problem related to equation 3.14 can be solved by known linear methods. If it is regarded as a nonlinear function of x^o, it can be linearized by harmonic linearization and the results of the previous chapter can be applied.

It should be noted here that the same parameter plane procedure can be used when the right side of equation 3.1 has more than one forcing function; i.e., the right-hand side is expressed by $\sum_{i=1}^{r} H_i(s) f_i$. The solution x, however, must be found by considering all existing inputs simultaneously since the superposition principle of linear analysis is not valid. Furthermore, if the polynomial H(s) of equation 3-1 can be factored in the form $sH_1(s)$, the procedure applied to the case in which the rate sf of the function f is considered as a slowly-varying signal; i.e., $|sf(t+T) - sf(t)|$.

The presented graphical procedure can be extended to nonlinear control systems with two nonlinear functions $F_1(s)$ and $F_2(x)$, whereby the following nonlinear differential equation is investigated:

$$B(s)\dot{x} + C(s)\ F_1(x) + D(s)\ F_2(x) = H(s)f. \tag{3.15}$$

In this case, a procedure similar to that given in Section can be extended to determine the solution $x = x^o + x^*$.

The general procedure outlined in this section is modified depending on the actual problem involved. These problems may be divided into three major groups: asymmetrical nonlinearities; 86

constant forcing signals; and slow-varying signals. In the following, each group is considered separately.

3.3 Asymmetrical Nonlinearities.

In an autonomous nonlinear system, which is described by the differential equation 3.1 and where $f \equiv 0$, the asymmetrical oscillations may occur whenever the function $F(x,sx)$ is not symmetrical to the origin. Then, under the conditions discussed in the previous section, the system may be described by equations 3.9 which has the form

$$B(o)\, x^o + C(o) F^o = 0 \tag{3.16a}$$

$$[B(s) + C(s)(N_1 + \frac{N_2}{\Omega}s)]x^* = 0 \tag{3.16b}$$

In equation 3.16a, which corresponds to equation 3.9a, there is no forcing slowly-varying function ($f \equiv 0$), and in the steady-state solution $x = x^o + x^*$, the x^o is constant and hence s is replaced by zero in $B(s)$ and $C(s)$.

In practical situations, $B(o)$ or $C(o)$ can be zero. Also, the nonlinearity in the system is often described by a single-valued function $F(x)$ and $N_2=0$. Thus, an adjustable parameter appearing in $B(s)$ or $C(s)$ can be chosen as one of the axes in the parameter $\alpha\beta$ plane, while the other axes is related to the describing function coefficient N_1. Some of these situations are discussed in the following examples.

Consider a feedback control system with the block diagram of Fig. 3.1 in which the transfer functions are

$$G_1(s)=K_1,\quad G_2(s)=\frac{K_2}{s(s+1)},\quad G_3=\frac{K_3}{s+2},\quad G_{-1}(s)=K_{-1}s. \tag{3.17}$$

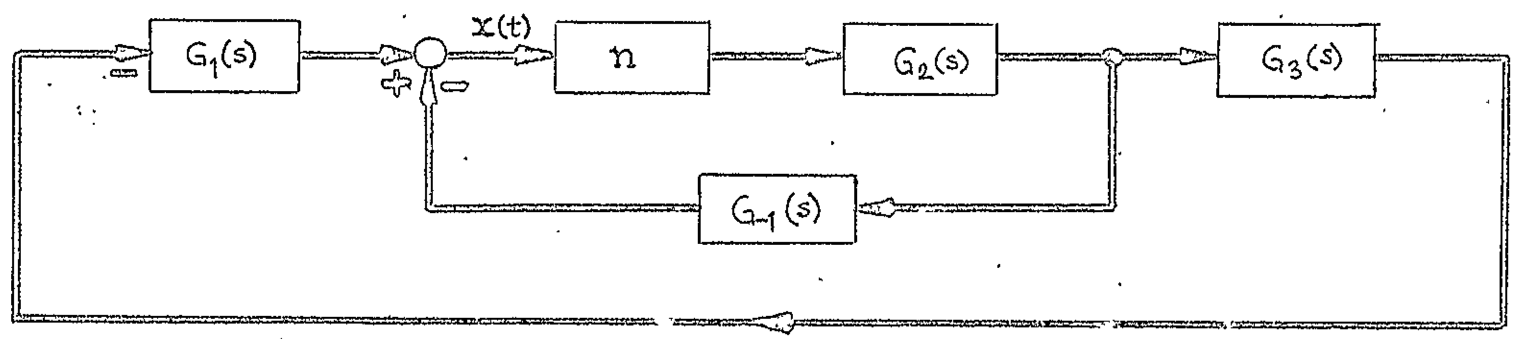


Fig. 3.1 – System block diagram

The nonlinearity n has the form shown in the upper left corner of Fig. 3.2.

Equations 3.16, for the system under investigation, have the form

$$F^{o} = 0 \tag{3.18a}$$

$$\{s(s+1)(s+2) + [K_2K_{-1}s(s+2) + K_1K_2K_3]N_1\}x^{*} = 0 \tag{3.18b}$$

where, according to the function F(x) of Fig. 3.2 and equations 3.6, one has

$$F^{o} = \frac{(1-m)c}{2} + \frac{(1+m)c}{\pi} \text{ arc sin } \frac{x^{o}}{A} \tag{3.19a}$$

$$N_1 = \frac{2(1-m)c}{A}\sqrt{1 - \left(\frac{x^{o}}{A}\right)^2} \tag{3.19b}$$

$$N_2 \equiv 0 \tag{3.19c}$$

and $x = x(t)$ is the input signal to the nonlinearity n as indicated in Fig. 3.1.

The characteristic equation of equation 3.18b is

$$s(s+1)(s+2) + [K_2K_{-1}s(s+2) + K_1K_2K_3]N_1 = 0 \tag{3.20}$$

By denoting $K_2K_{-1}N_1 = \alpha$ and $K_1K_2K_3N_1 = \beta$, the $\zeta = 0$ curve is obtained as

$$\alpha = \frac{1}{2}(\Omega^2 - 2) \tag{3.21}$$

$$\beta = \frac{1}{2}\Omega^2(\Omega^2 + 4)$$

and the stable region is determined in the $\alpha\beta$ plane in the usual fashion as shown in Fig. 3.2.

From equations 3.18a and 3.19a, one obtains

$$x^{o} = A \cos \frac{\pi}{1+m} \tag{3.22}$$

89

and N_1 of equation 6.19b becomes

$$N_1 = \frac{2(1+m)c}{A} \sin \frac{\pi}{1+m} \tag{3.23}$$

By using equation 3.23 and the expressions $\alpha = K_2K_{-1}N_1$, $\beta = K_1K_2K_3N_1$, three M loci (a), (b), and (c), are drawn in Fig. 3.2. They correspond to the parameter values $m = 0.5$, $c = 1$, $K_2 = 1$ and (a) K_1K_3, $K_{-1} = 0.125$; (b) $K_1K_3 = 8.39$, $K_{-1} = 0.28$; (c) $K_1K_3 = 26$, $K_{-1} = 1.75$. The stable asymmetrical oscillations are found at the point M_1 and M_2 where the M loci (a) and (b) intersect the $\zeta = 0$ curve. The amplitudes of the oscillations are approximately $A_1 = 0.85$ and $A_2 = 0.8$, which is read from the M loci (a) and (b) at the intersections M_1 and M_2. The corresponding frequencies $\Omega_1 = 1.5$ and $\Omega_2 = 1.6$ are indicated on the $\gamma = 0$ curve. The related values of x^o in the solution $x = x^o + a_1 \sin \Omega t$ is calculated for each point M_1 and M_2 using equation 3.22, namely, $x_1^o = -0.42$ and $x_2^o = -0.39$.

In Fig. 3.3, the solution $x_1 = 0.42 + 0.85 \sin 1.5t$ for the case (a) is shown as obtained by a digital computer simulation. The calculated results are sufficiently close to that obtained by the simulation. From Fig. 3.3, it can be seen that an initial condition $x_1(o) = 4.25$ is used and the variable $x_1(t)$ approached a stable limit cycle. That the limit cycle is stable and will be reached by $x_1(t)$ starting from $x_1(o) = 4.25$ can be concluded from the relative location of the $\zeta = 0$ curve and the M locus (a), as explained in the preceding chapter on the symmetrical oscillations. The additional component x^o of the solution $x(t)$ does not alter the stability analysis of the oscillations.

90

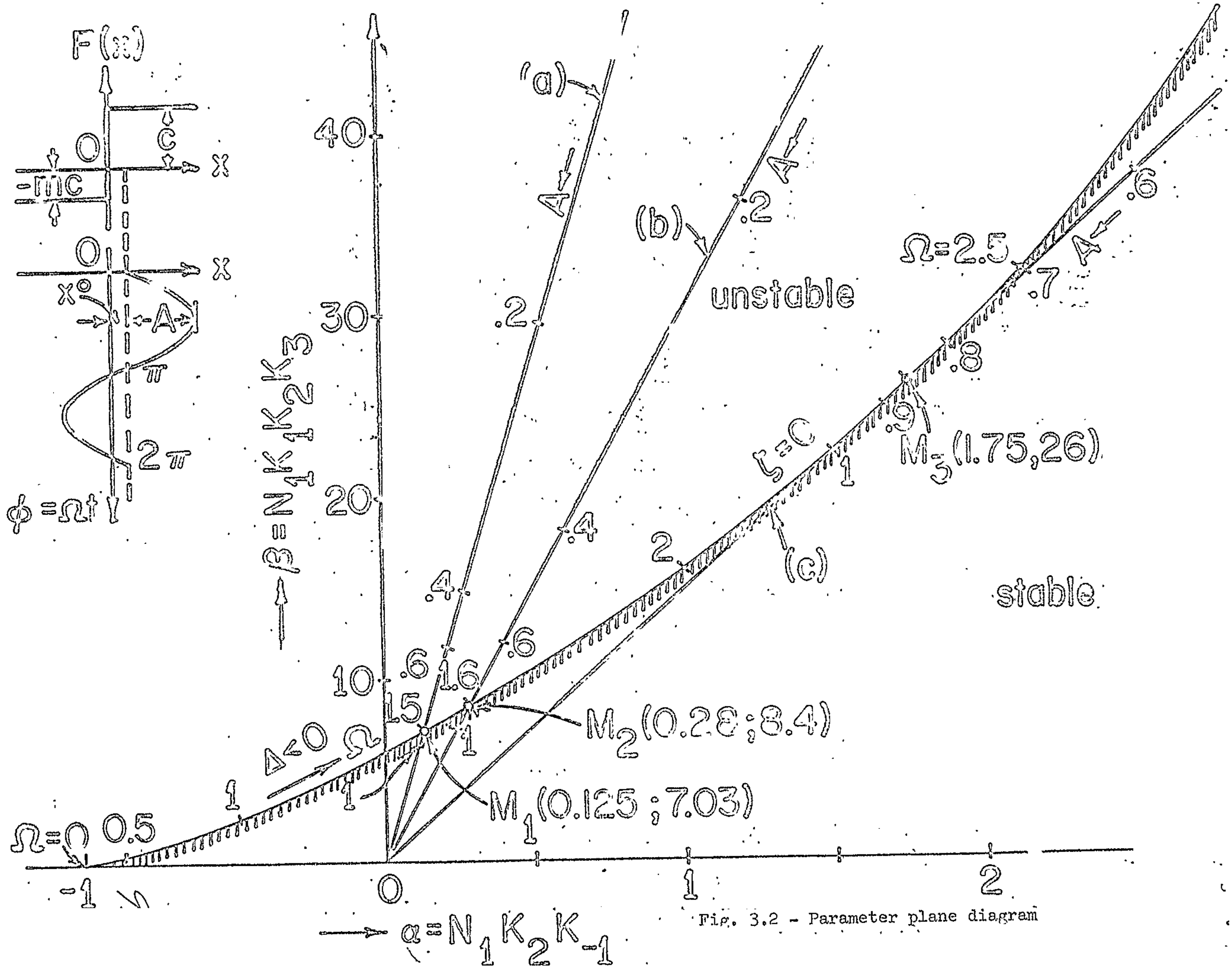
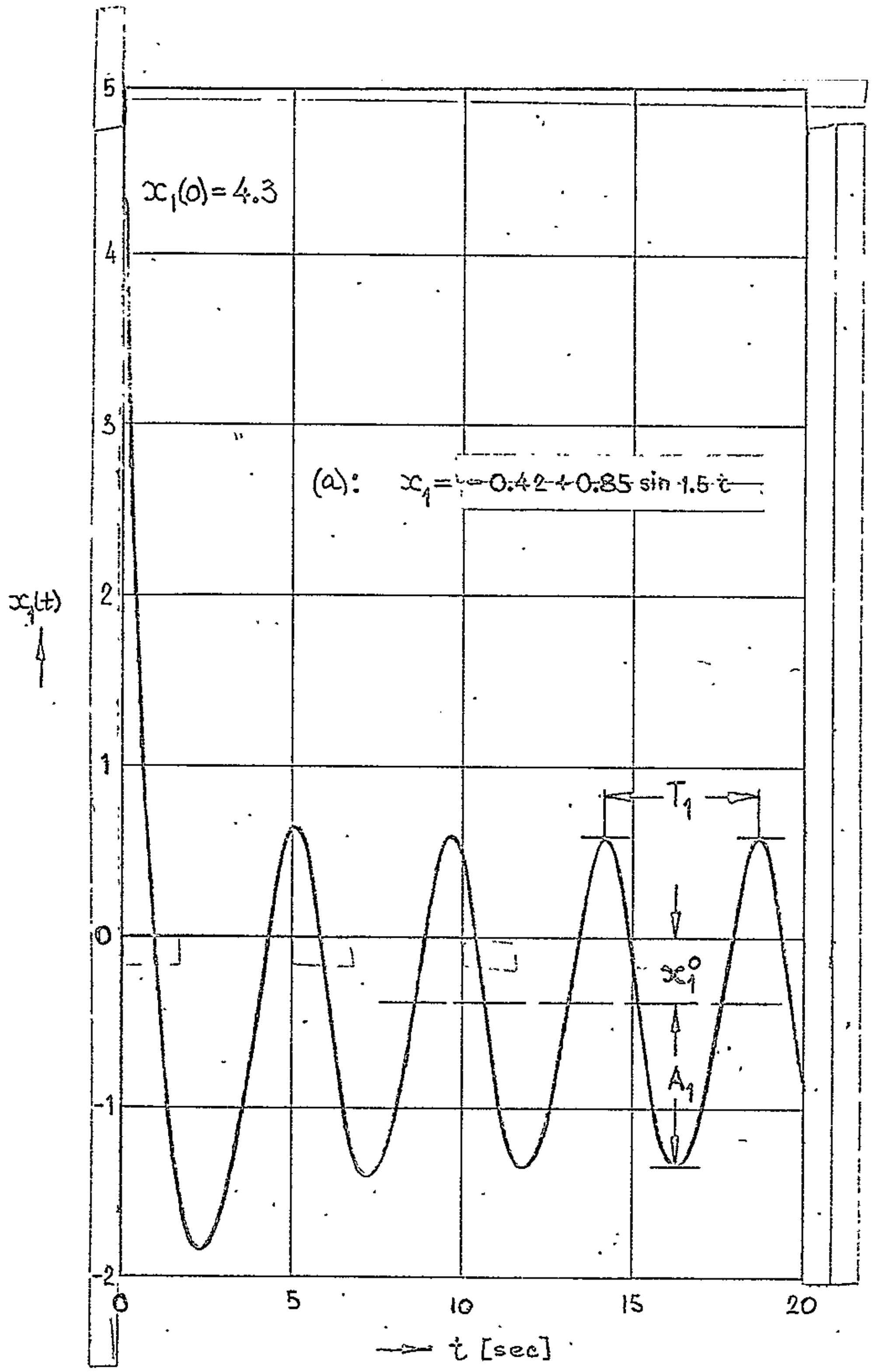


Fig. 3.2 - Parameter plane diagram



92

Fig. 3.3 - Digital computer solution in case (a)

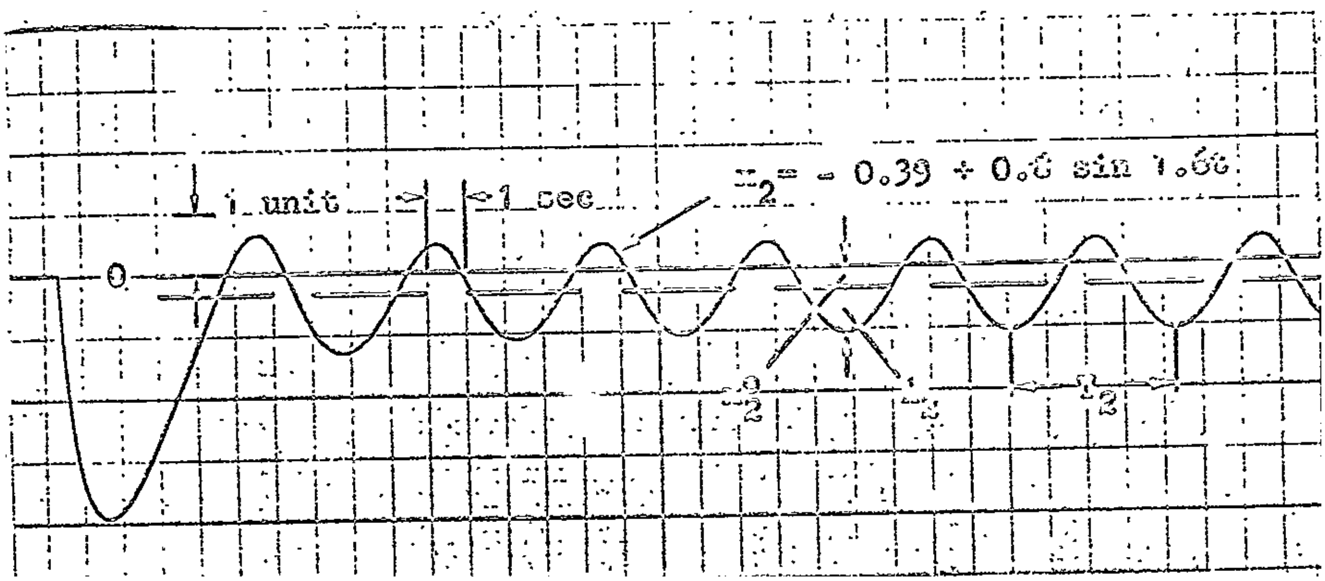


Fig. 3.4 Analog computer solution in case (b)

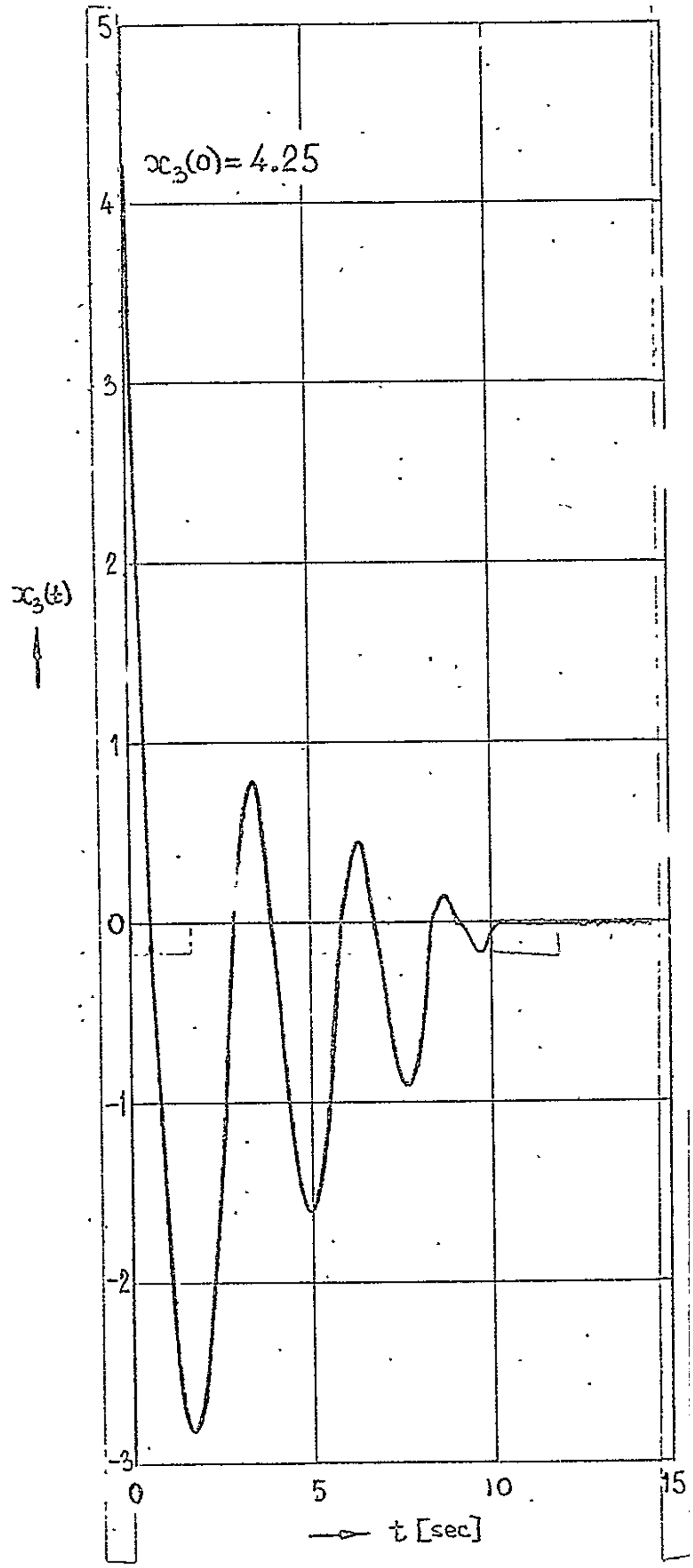


Fig. 3.5 - Digital computer solution in case (c)

An analog computer simulation of the case (b) gives the solution $x_2 = -0.39 + 0.8 \sin 1.6t$ as shown in Fig. 3.4. A sufficient accuracy is indicated. The initial condition $x_2(o) = 0$ and $x_2(t)$ reached a limit cycle. This could be concluded from Fig. 3.2 as previously noted.

It is of particular interest to consider the case (c) of Fig. 3.2. The M locus (c) is tangent to the = 0 curve and corresponds to the ratio $\alpha/\beta = K_1K_3/K_{-1} = 14.8$. If this ratio is higher than 14.8, then there is a limit cycle as shown by cases (a) and (b). On the other hand, if this ratio is less than 14.8, the entire M locus is situated in the stable region and the corresponding system is always stable. The tangent case (c): $m = 0.5$, $c = 1$, $K_2 = 1$, $K_1K_3 = 26$, $K_{-1} = 1.75$, is simulated on a digital computer and the obtained solution $x_3(t)$ is shown in Fig. 3.5, which indicates that the system is stable.

3.4 Constant Forcing Signals

When the forcing signal at certain points of a nonlinear system is constant, the solution $x = x^o + A \sin \Omega t$(if it exists) will have x^o, A, and Ω as constant values. To determine these values, note that the equations to solve in the presence of a constant forcing signal f^o have the form

$$B(o)x^o + C(o)F^o = H(o)f^o \tag{3.24a}$$

$$[B(s) + C(s)N_1 + \frac{N_2}{\Omega} s)]x^* = 0 \tag{3.24b}$$

In general B(o), C(o), and H(o) are constants different from zero, and the solution procedure is somewhat more complicated to perform than in the previous section where the right

95

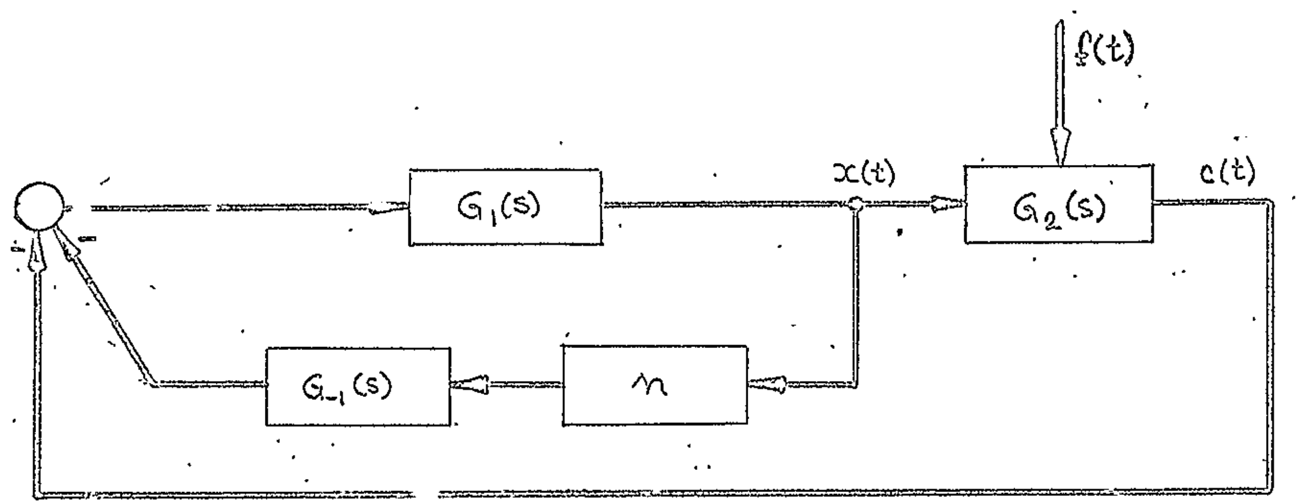


Fig. 3.6 - System block diagram

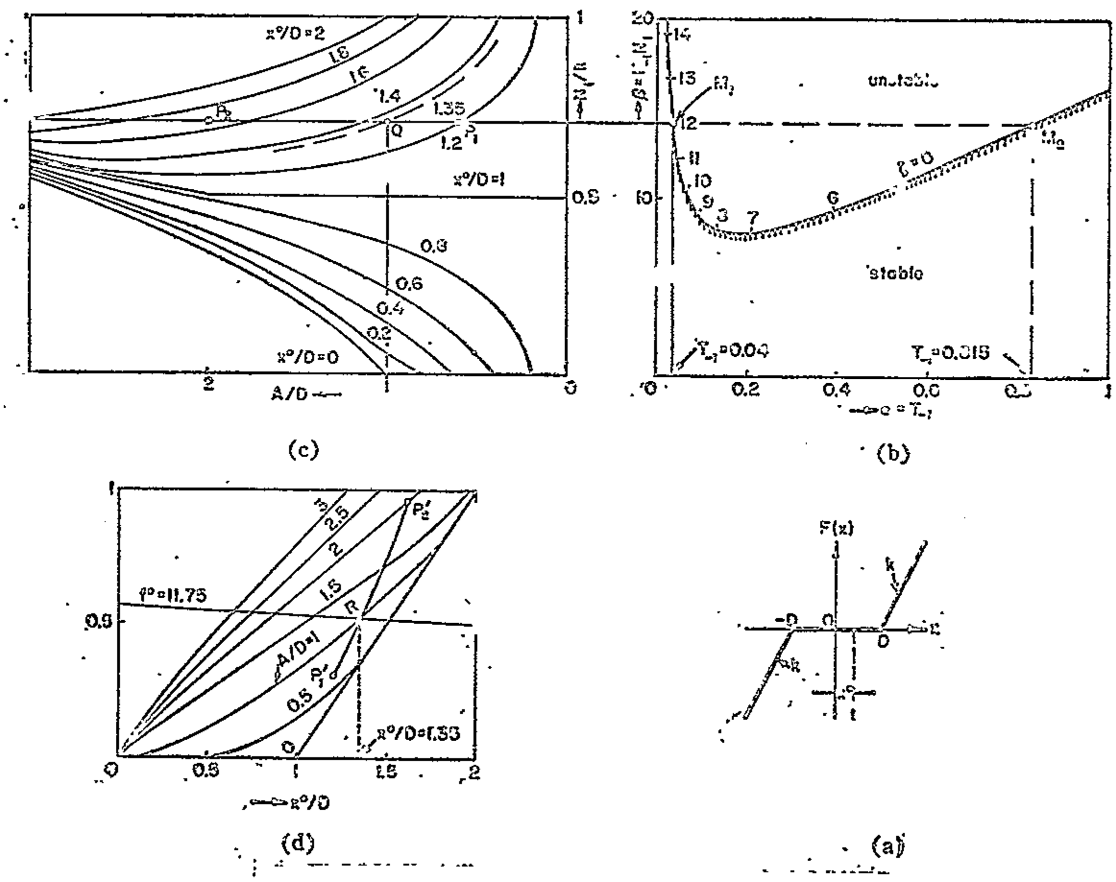


Fig. 3.7 - Parameter plane diagram

97

side of equation 3.24a was zero.

To illustrate the solution procedure, consider a nonlinear feedback system with the block diagram of Fig. 3.6 and the transfer functions

$$G_1(s) = \frac{2}{0.2s^2+0.8s+1}, \quad G_2(s) = \frac{0.5(s+1)}{0.2s+1}, \quad G_{-1} = \frac{K_{-1}}{T_{-1}s+1} \tag{3.25}$$

The nonlinearity n is given in Fig. 3.7a. The input to the system is a perturbation signal $f = f(t)$ which is related to the signal $x = x(t)$ and $c = c(t)$ of Fig. 3.6 as

$$(0.2s+1)c = 0.5(s+1)x-f \tag{3.26}$$

If the perturbation signal is $f(t) = f^o$ = const., equations 3.24 have the form

$$x^o + K_{-1}F^o = f^o \tag{3.27a}$$

$$(0.04s^4 + 0.36s^3 + 2s^2 + 2s)T_{-1} + (0.4s+2)K_{-1}N_1 + \\ + 0.04s^3 + 0.36s^2 + 2s + 2 = 0 \tag{3.27b}$$

where equation 3.27b represents the characteristic equation of the linearized equation 3.24b. By substituting $T_{-1} = \alpha$ and $K_{-1}N_1 = \beta$, the parameter plane diagram is plotted in Fig. 3-7b according to the parameter plane equations

$$\alpha = \frac{0.64\Omega^2 + 3.2}{0.016\Omega^4 - 0.08\Omega^2 - 4}$$

$$\beta = \frac{0.016\Omega^6 - 0.03\Omega^4 + 2.56\Omega^2 + 4}{0.016\Omega^4 - 0.08\Omega^2 - 4} \tag{3.28}$$

The variation of the M point due to the function $N_1 = N_1(x^o, A)$ given as

$$N_1 = k - \frac{k}{\pi}(\arcsin\frac{D-x^o}{A} + \arcsin\frac{D+x^o}{A} + \frac{D-x^o}{A}\sqrt{1-(\frac{D-x^o}{A})^2} + \frac{D+x^o}{A}\sqrt{1-(\frac{D+x^o}{A})^2}, \quad A \geq D + |x^o| \tag{3.29}$$

is plotted in Fig. 3.7c. (The expression (3.29), corresponds to the nonlinearity of Fig. 3.7a). In order to find a solution $x = x^o + x^*$ of equations 3.27, the parameter k is assumed equal to one, and the function $F^o(x^o, A)$ is plotted in Fig. 3.7d by using

$$F^o = \frac{kA}{\pi}\sqrt{1-(\frac{D-x^o}{A})^2} - \sqrt{1-(\frac{D+x^o}{A})^2} + kx^o + \frac{k}{\pi}[D(\arcsin\frac{D-x^o}{A} - \arcsin\frac{D+x^o}{A}) - x^o(\arcsin\frac{D-x^o}{A} + \arcsin\frac{D+x^o}{A})], \quad A \geq D + |x^o| \tag{3.30}$$

For $T_{-1} = 0.04$, the point $M_1(0.04; 14.3)$ corresponds to a solution $x = x^o + x^*$ which will have $\Omega = 12$ rad/sec as indicated on the curve $\zeta = 0$. If $K_{-1} = 20$, from M_1 it follows that $N_1 = \beta/K_{-1} = 0.715$. This value of N_1 determines the relationship between the values of x^o and A for a possible solution x. This relationship, expressed as a function $A = A(x^o)$, can be graphically obtained from the diagram $N_1 = N_1(x^o, A)$ by plotting the straight line P_1P_2 corresponding to the value $N_1 = 0.715$.

The function $A = A(x^o)$ represents the solution of equation 3.27b only. The pair of values (x^o, A) which enter into the actual solution of equation 3.27, is replotted on the diagram

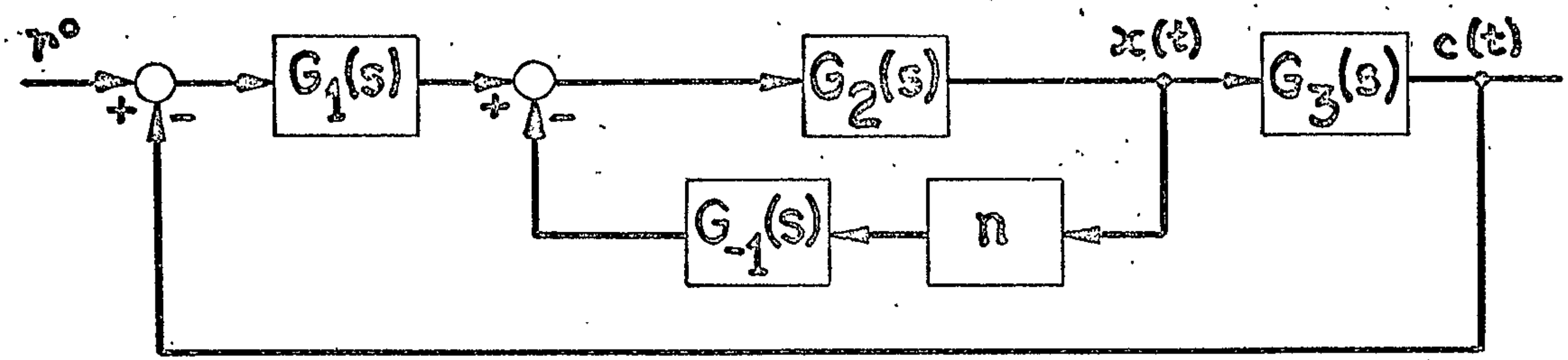


Fig. 3.8 - System block diagram

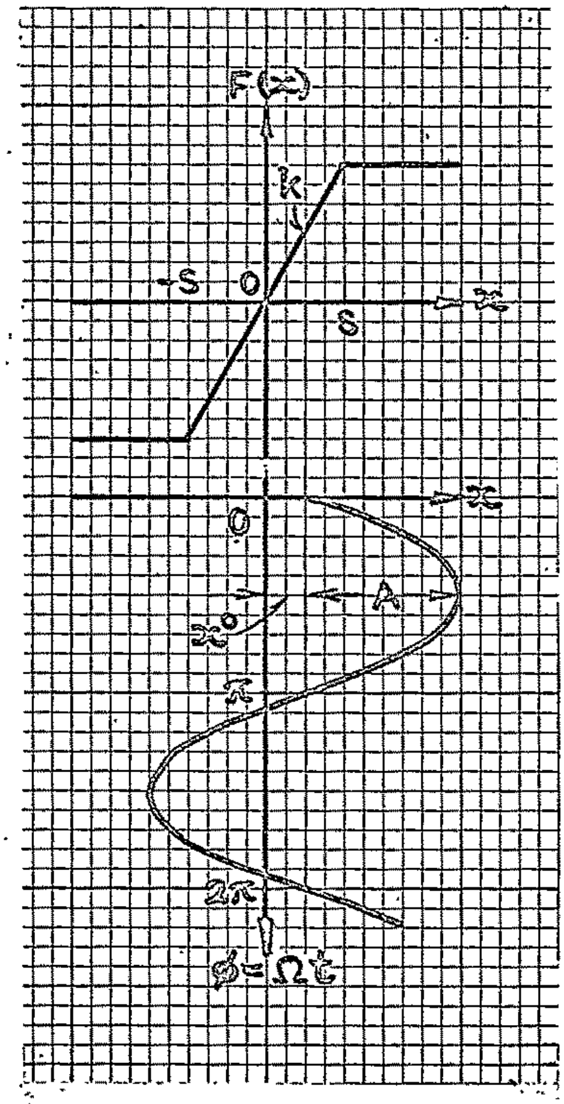


Fig. 3.9 - Nonlinear characteristic

$F^o = F^o(x^o, A)$ of Fig. 3.7d into the curve $P_1'P_2'$. Suppose that the constant perturbing signal has a value of $f^o = 11.75$; then equation 3.27a determines the straight line $f^o = 11.75$ plotted in the diagram $F^o = F^o(x^o, A)$ of Fig. 3.7d. The intersection R of that straight line and the curve $P_1'P_2'$ gives the pair (x^o, A) of the solution x(t) which satisfies equation 3.27 simultaneously. At this point R, the values are $x^o/D = 1.35$ and $A/D = 1$. The same values are obtained at the point Q on the diagram $N_1 = N_1(x^o,A)$ and the solution $x = x^o + A \sin \Omega t$ of equations 3.27 is found. If D = 1, it is $x = 1.35 + \sin 12t$. Note that the same solution is obtained if the point M_2 of Fig. 3.7b is considered save that the frequency Ω is lower (approximately $\Omega = 5.5$ rad/sec).

Simpler situations may occur if one of the values B(o) or C(o) is zero. To illustrate, consider the nonlinear system of Fig. 3.8. The transfer functions are

$$G_1(s) = K_1, \quad G_2(s) = \frac{K_2}{s(s+1)}, \quad G_3(s) = \frac{K_3}{s+2}, \quad G_{-1}(s) = K_{-1}s \tag{3.31}$$

and the nonlinearity n in the system is given by the function F(x) of Fig. 3.9. The input to the system is the reference constant input signal $r(t) = r^o$.

The nonlinear differential equation describing the above system is

$$[s(s+1)(s+2)+K_1K_2K_3]x+K_2K_{-1}s(s+2)F(x) = K_1K_2(s+2)r^o \tag{3.32}$$

which may be rewritten according to equations 3.24 as

$$K_1K_2K_3x^o = 2r^o \tag{3.33a}$$

102

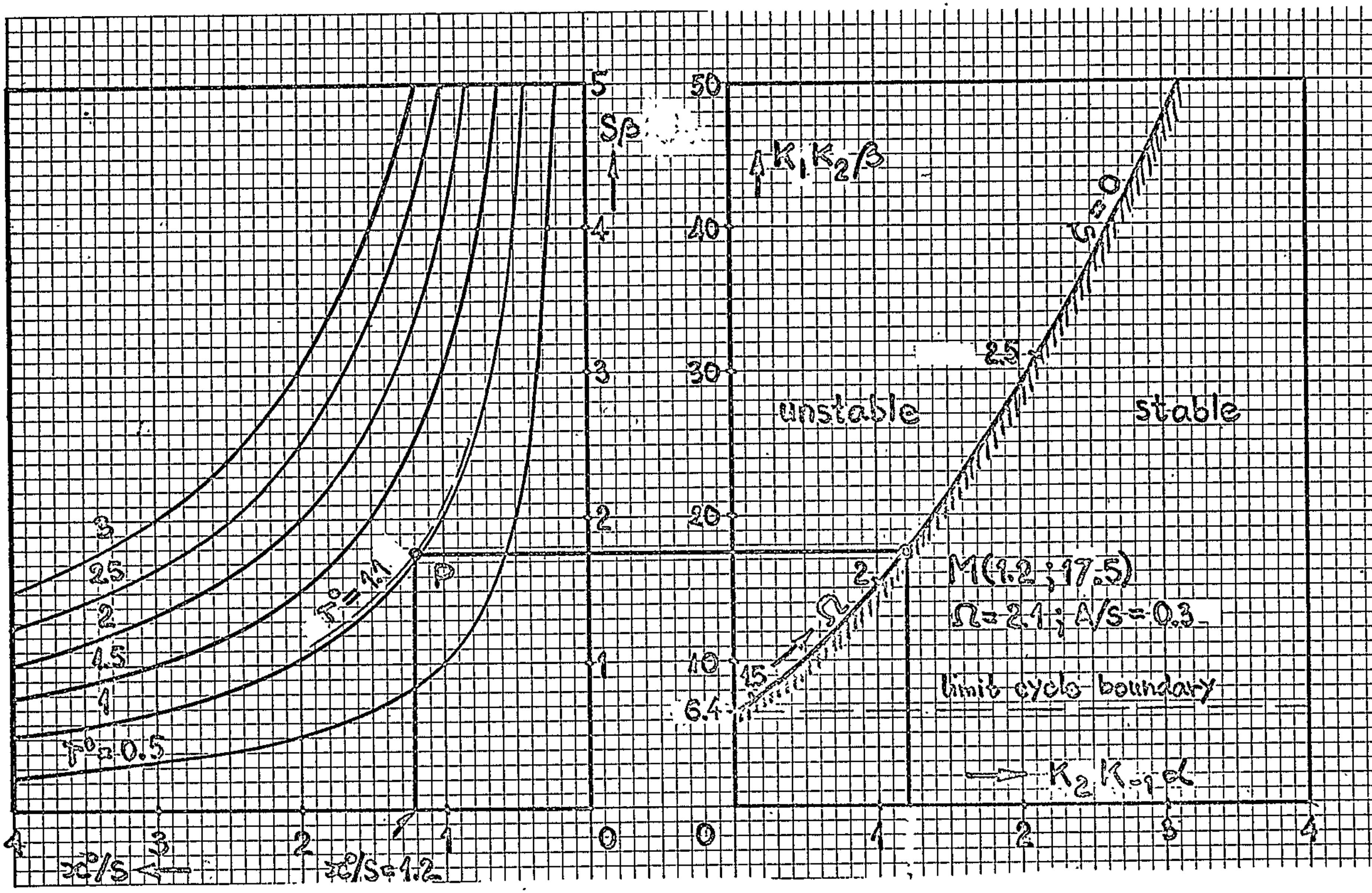


Fig. 3-10 - Parameter plane diagram

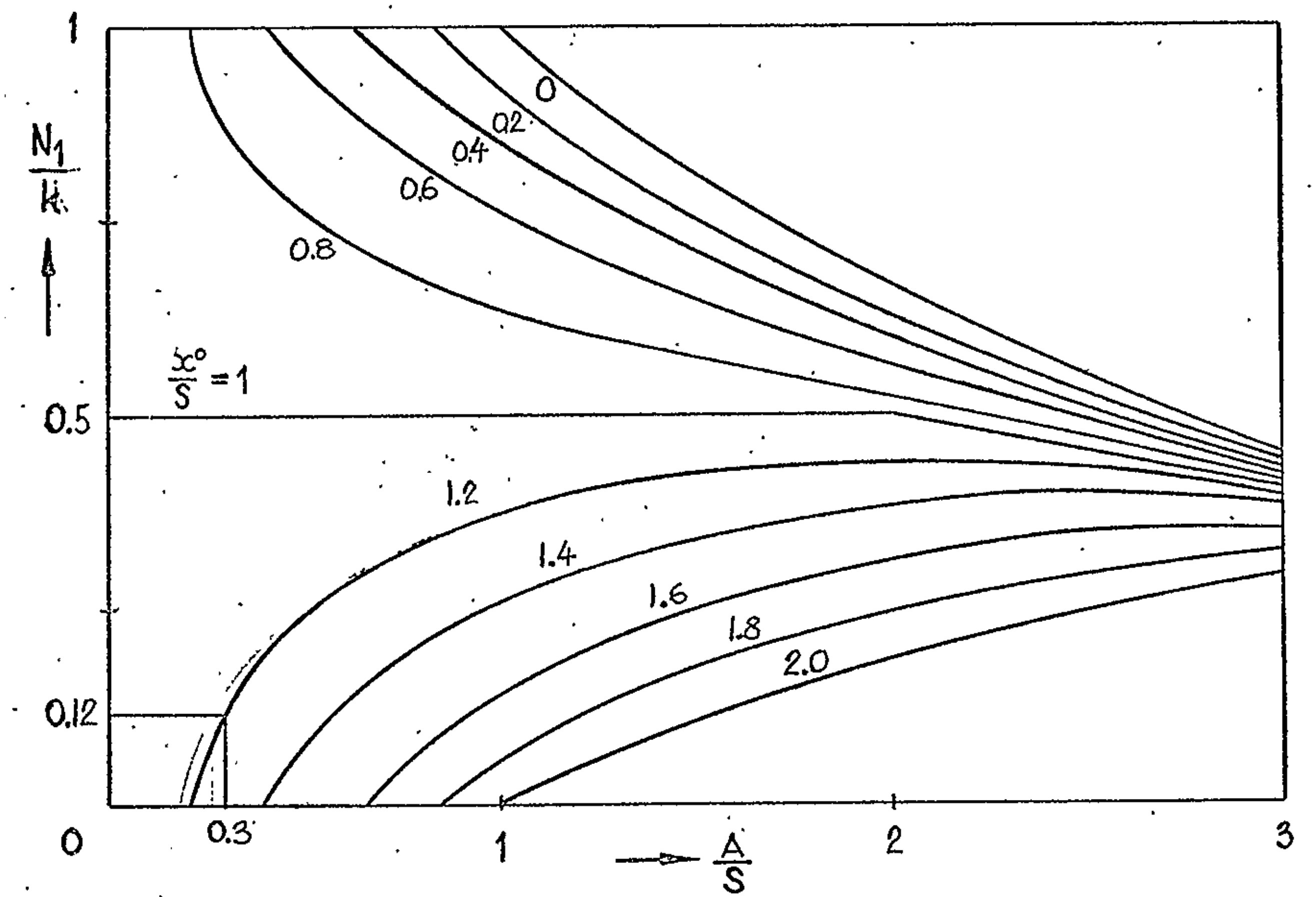


Fig. 3.11 - Function $N_1(A, x^o)$

$$[s(s+1)(s+2) + K_1K_2K_3 + K_2K_{-1}s(s+2)N_1]x^* = 0 \qquad (3.33b)$$

The characteristic equation of the equation 3.33b is evidently

$$s(s+1)(s+2) + K_1K_2K_3 + K_2K_{-1}s(s+2)N_1 = 0. \qquad (3.34)$$

By denoting

$$\alpha = N_1$$
$$\beta = K_3 \qquad (3.35)$$

the parameter plane diagram is plotted in Fig. 3.10 in the usual fashion. The function $N_1 = N_1(A, x^o)$, which appears as a variation of α in the point $M(a; \beta)$ is plotted in Fig. 3.11 by using general formula 3.6b.

From equation 3.33a, one can derive the following relationship between the input r^o, the constant term x^o, and the parameter $\beta = k_3$,

$$S\beta = \frac{2r^o}{x^o/S} \qquad (3.36)$$

where S is the parameter of the nonlinearity F(x) of Fig. 3.9. The function $S\beta$ given in (3.36) is plotted in Fig. 3.10.

Now, by using Fig. 3.10 and 3.11, it is possible to determine the sustained oscillations and their stability for various values of system parameters K_1, K_2, K_3, K_{-1}, S, k, and the input r^o. For example, if $K_1 = 1$, $K_2 = 10$, $K_3 = 1.75$, $K_{-1} = 1$, $S = 1$, $k = 1$, and $r^o = 1.1$, then the solution of equation 3.33 is determined by the values $x^o = 1.2$, $A = 0.3$, and $\Omega = 2.1$ rad/sec to be approximately

$$x = 1.2 + 0.3 \sin 2.1t \qquad (3.37)$$

For a given value of $\beta = K_3 = 1.75$, $r^o = 1.1$, and $S = 1$, the value of $x^o = 1.2$ is read from the left part of Fig. 3.10. Then the

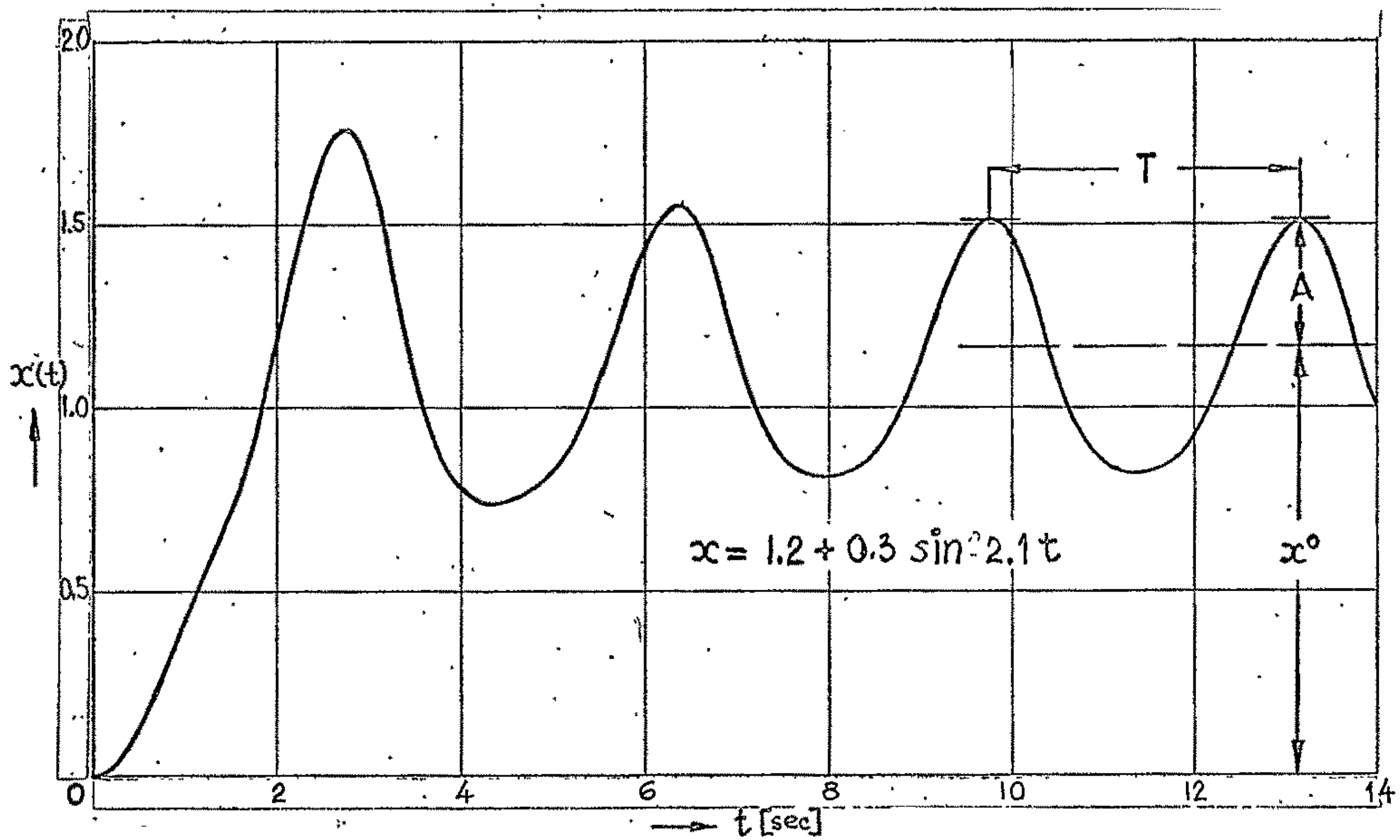


Fig. 3.12 - Computer solution

value of $K_1K_2\beta = 17.5$ determines the point $M(1.2; 17.5)$ on the $\zeta = 0$ curve where $\Omega = 2.1$ rad/sec. At this point, $K_2K_{-1}\alpha = 1.2$ which gives $N_1 = \alpha = 0.12$. Fig. 3.11 is used to evaluate the amplitude $A = 0.3$ from the curve $x^o/S = 1.2$. The value $A = 0.3$ is read directly from the diagram $N_1(A, x^o)$ of Fig. 3.11, since $K = S = 1$ are the parameters of the given nonlinearity in Fig. 3.9.

The solution (3.37) is stable since an increase in the amplitude A causes the point M to move into the stable region; while a decrease in the amplitude A places the point M inside the unstable region of the parameter plane (Fig. 3.10). It is of interest to note that if the produce $K_1K_2\beta$ where $\beta = K_3$ is such that it is less than 6.4, the system is always stable since there is no intersections of the variation of the M point and the $\zeta = 0$ curve.

The above solution (3.37) is checked by computer simulation to obtain the curve on Fig. 3.12. The accuracy of the calculated solution is sufficiently high and, for calculated values of x^o, A, and Ω, is approximately 10%. On the other hand, the computer solution indicates a distortion of the assumed solution $x = x^o + A \sin \Omega t$ which is due to the higher harmonics present in the actual solution.

3.5 Slowly-varying Signals

In this section, the problem of linearizing a nonlinear system by a high-frequency limit cycle is considered in more detail. The objective is to determine the conditions under which 107

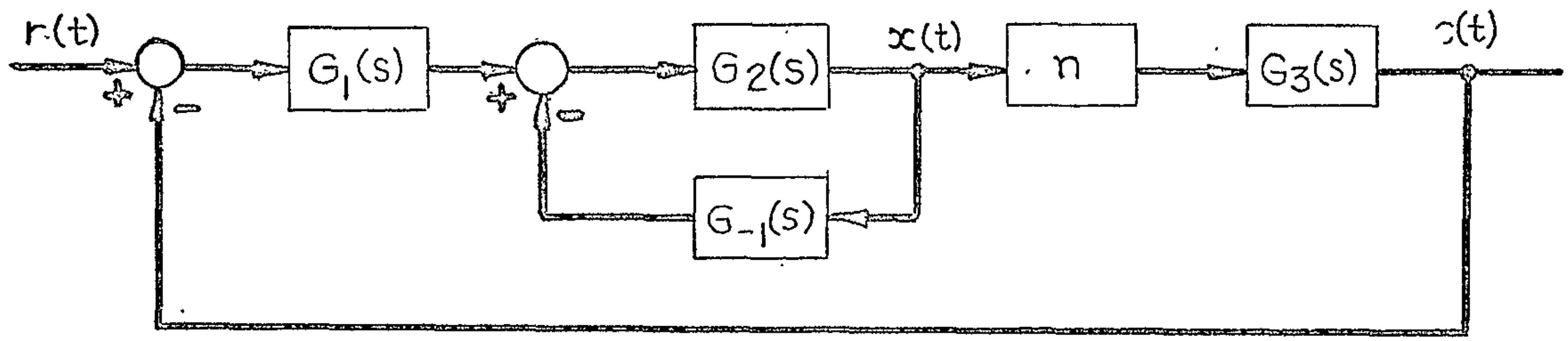


Fig. 3.13 - System block diagram

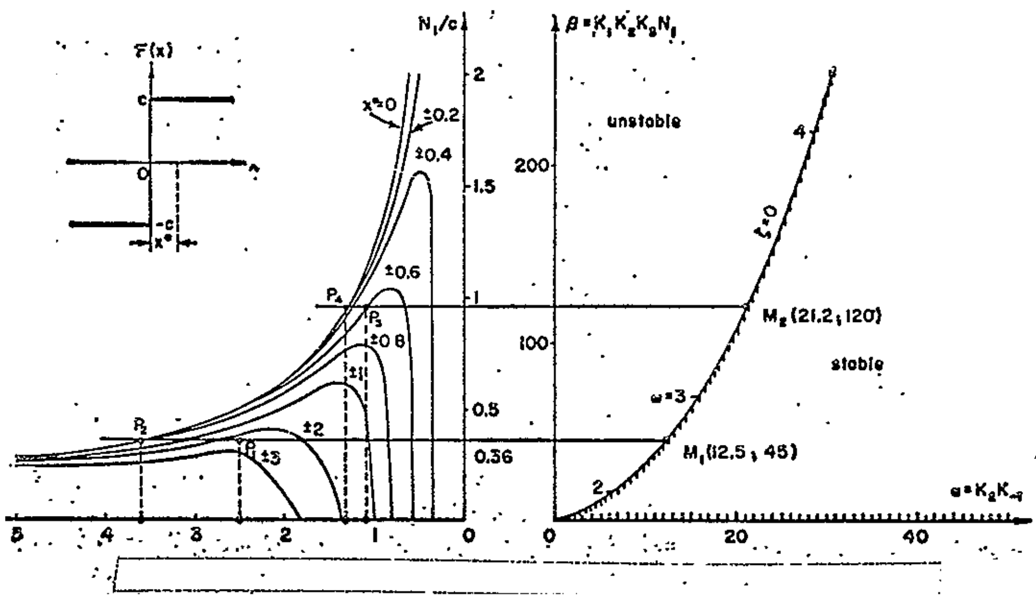


Fig. 3-14 - Parameter plane diagram

such a linearization is possible and then to construct the linearized characteristic of the nonlinearity. This linearization has several practical aspects discussed in Section 3.1, which are based upon a general property of the linearized system that, for a limited magnitude of the reference signal, behaves like a linear system. Therefore, results of the nonlinearities, such as dead-zone, hysteresis, backlash, etc., are eliminated. The procedure to achieve this will be best illustrated in the following examples.

Consider the system on Fig. 3.13 with the transfer functions

$$G_1(s) = K,\ G_2(s) = \frac{K_2}{s^2+0.8s+1},\ G_3(s) = \frac{K_3}{s(s+1)},\ G_{-1}(s) = K_{-1} \tag{3.38}$$

and the nonlinearity n as shown in Fig. 3.14. The input to the system is a slowly-varying reference signal $r = r(t)$.

The equation which describes the system is

$$[s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1)]x + K_1K_2K_3F(x) = K_1K_2s(s+1)r \tag{3.39}$$

where the signal $x = x(t)$ is the input to the nonlinearity. Equation 3.39 can be rewritten in terms of equations 3.9 as

$$[s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1)]x^o+K_1K_2K_3F^o = K_1K_2s(s+1)r$$

$$[s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1)]+K_1K_2K_3N_1\ x^* = 0 \tag{3.40}$$

The characteristic equation of the second equation 3.40 is

$$s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1) + K_1K_2K_3N_1 = 0 \tag{3.41}$$

Substituting $K_2K_{-1} = \alpha$, $K_1K_2K_3N_1 = \beta$, and $s = j\Omega$ into equation 3.41, one obtains the parameter plane equations of the $\zeta = 0$ curve

110

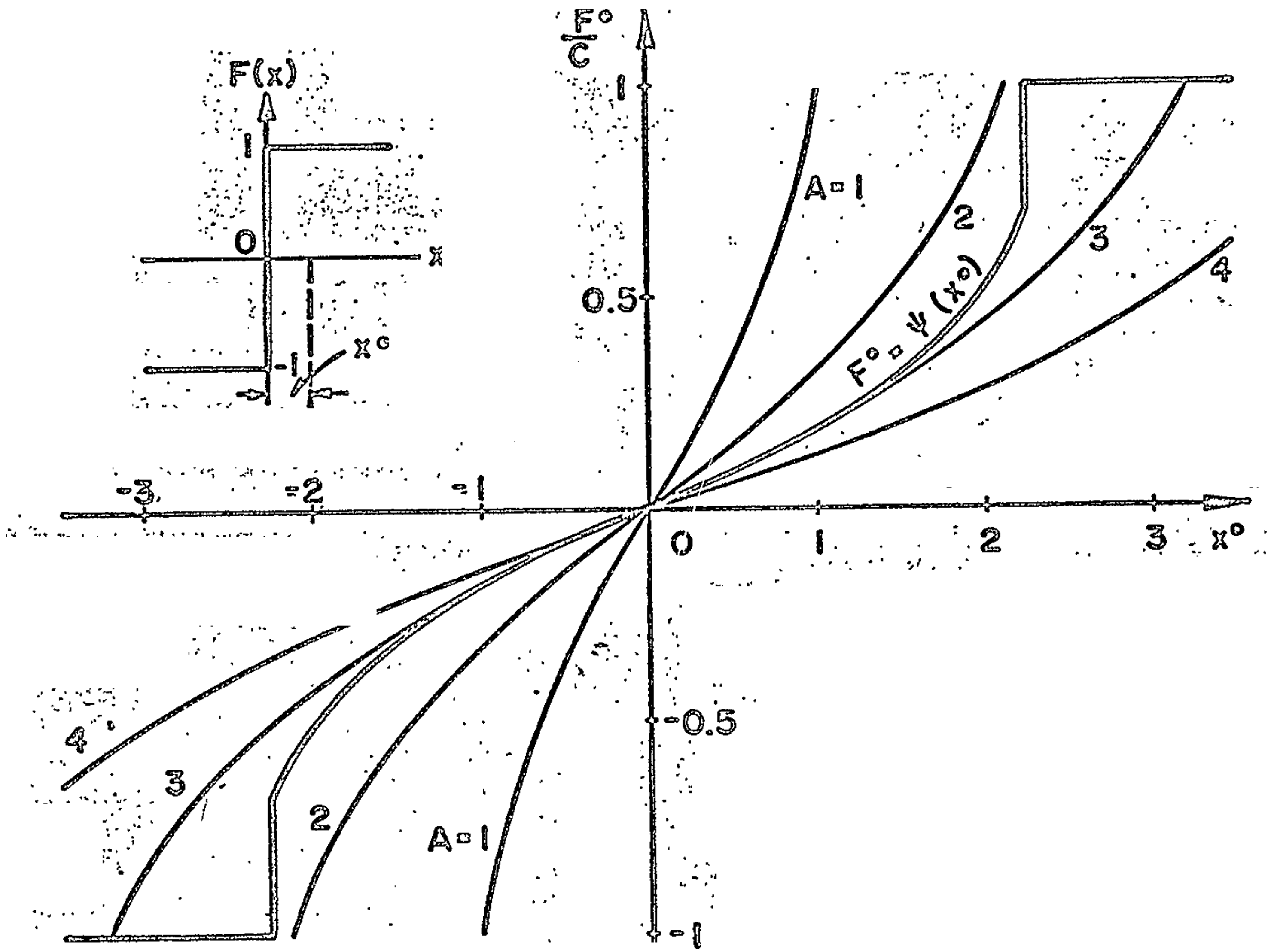


Fig. 3.13 – Function ψ(x)

as

$$\alpha = 1.8\ \Omega^2 - 1$$
$$\beta = 0.8\ \Omega(\Omega + 1). \qquad (3.42)$$

The curve $\zeta = 0$ is plotted in Fig. 3.14. The variations of the M point are plotted also in Fig. 3.14 according to

$$N_1 = \frac{4c}{\pi A}\sqrt{1 - (\frac{x}{A})^2}, \quad A \geq |x^o| \qquad (3.43)$$

The system parameters $K_1 = 1$, $K_2 = 12.5$, $K_3 = 10$, $K_{-1} = 1$ result in the point $M_1(12.5; 45)$. If $c = 1$, this point M_1 gives $N_1 = \beta/K_1K_2K_3 = 0.36$, and the straight line P_1P_2 is plotted on the diagram of function $N_1 = N_1(x^o, A)$. After the diagram $F^o = F^o(x^o, A)$ is plotted in Fig. 3.15 using

$$F^o = \frac{2c}{\pi} \text{arc sin} \frac{x^o}{A}, \quad A \geq |x^o| \qquad (3.44)$$

the replotting of the straight line P_1P_2 on the diagram $F^o(x^o, A)$ yields the function (x^o) of Fig. 3.15. The replotting procedure is the same as that used in the previous section; i.e., for each pair of values (x^o, A) read on the straight line P_1P_2, the corresponding pair exists in the diagram $F^o(x^o, A)$, which determines one point on the curve $\psi(x^o)$.

Function $\psi(x^o)$ of Fig. 3.15 is smooth and represents the nonlinearity for the slowly-varying signal x^o. The shape of $\psi(x^o)$ explains the smoothing effect of the high frequency limit cycle which has a slowly-varying amplitude, the value of which is located between the points Q_1 and Q_2 on the A axis of Fig. 3.14. The frequency Ω is approximately constant and has the value $\Omega \simeq 2.7$ rad/sec. According to $\psi(x^o)$, the smoothing effect of the

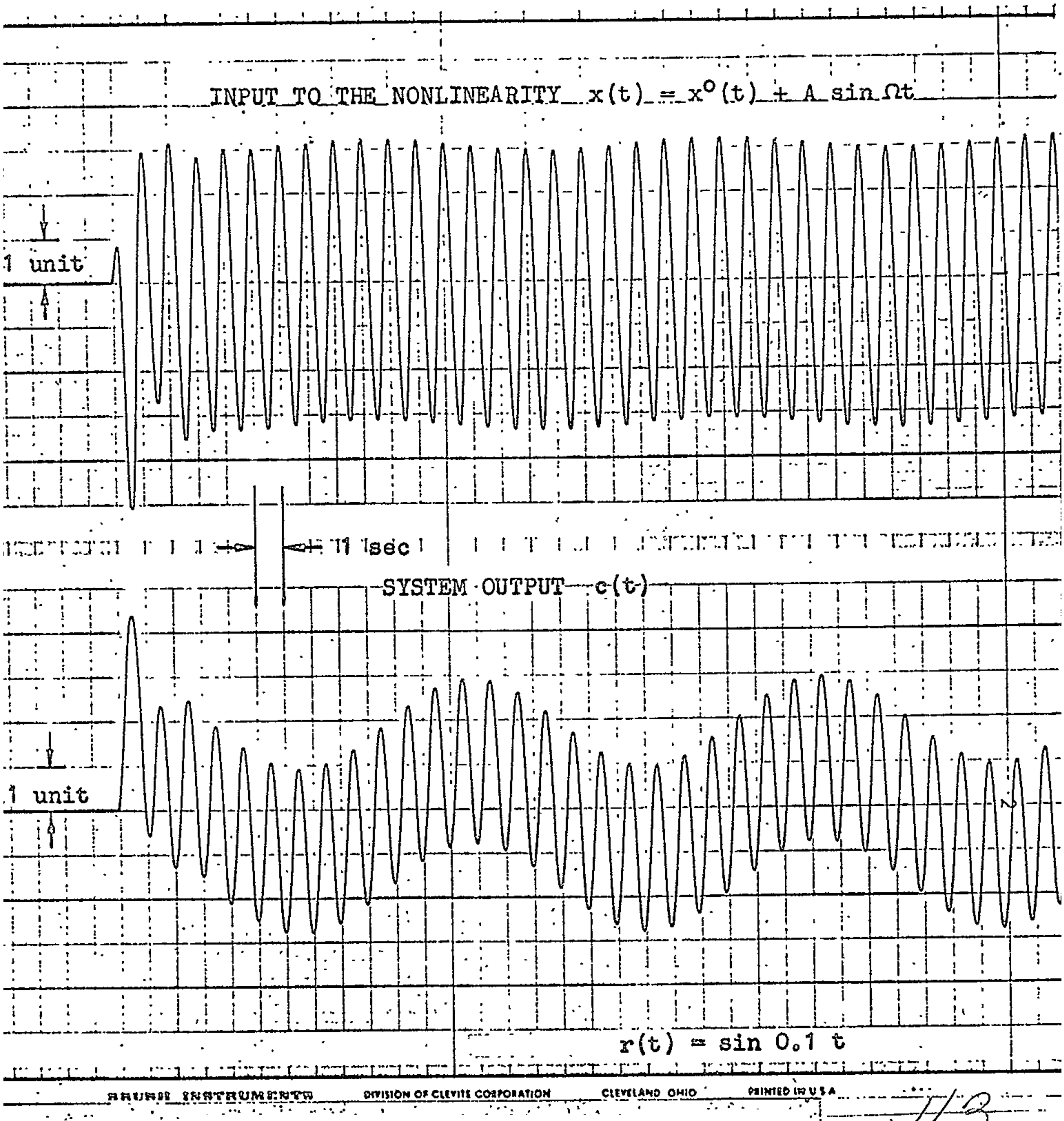


Fig. 3.16 - Computer solution

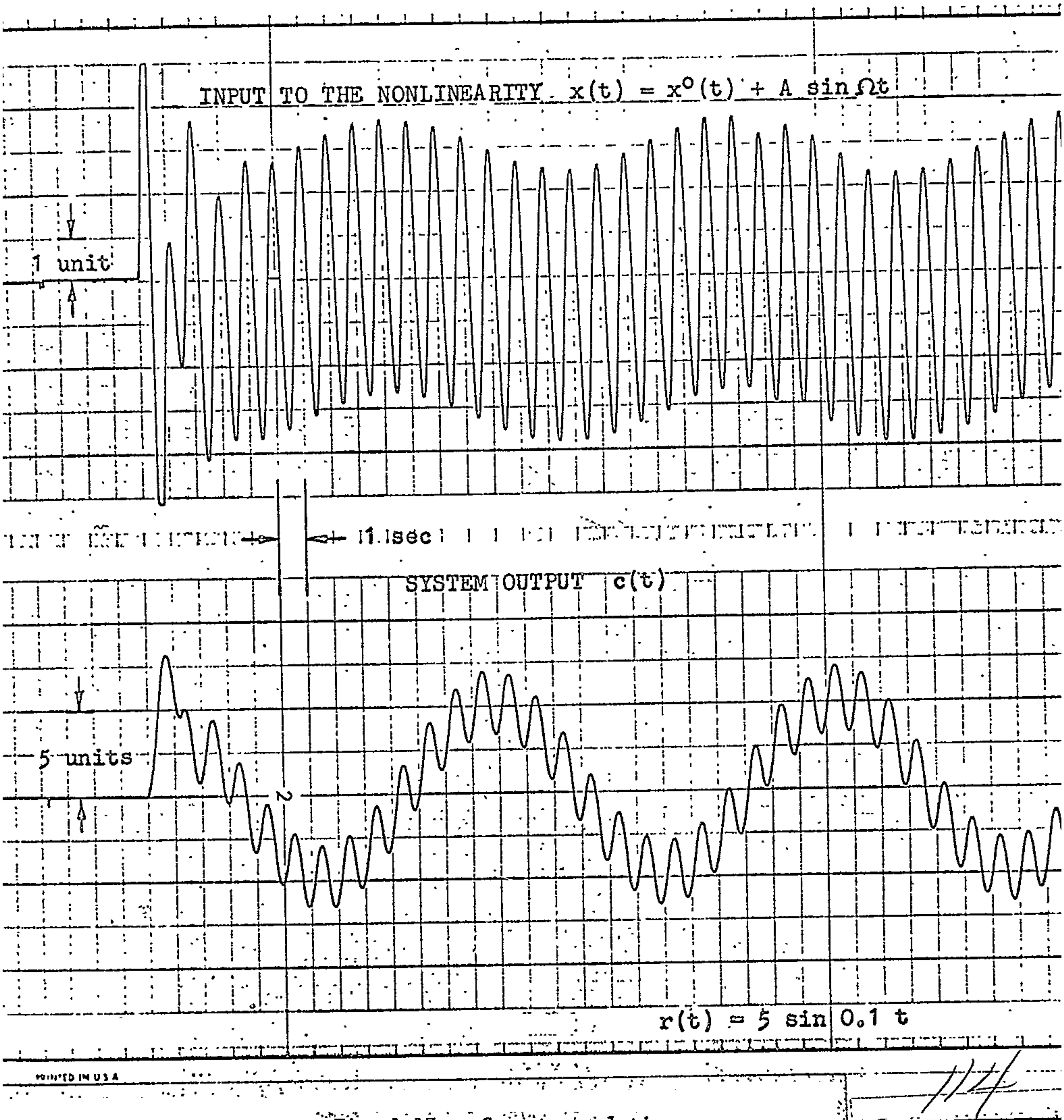


Fig. 1-17 - Computer solution

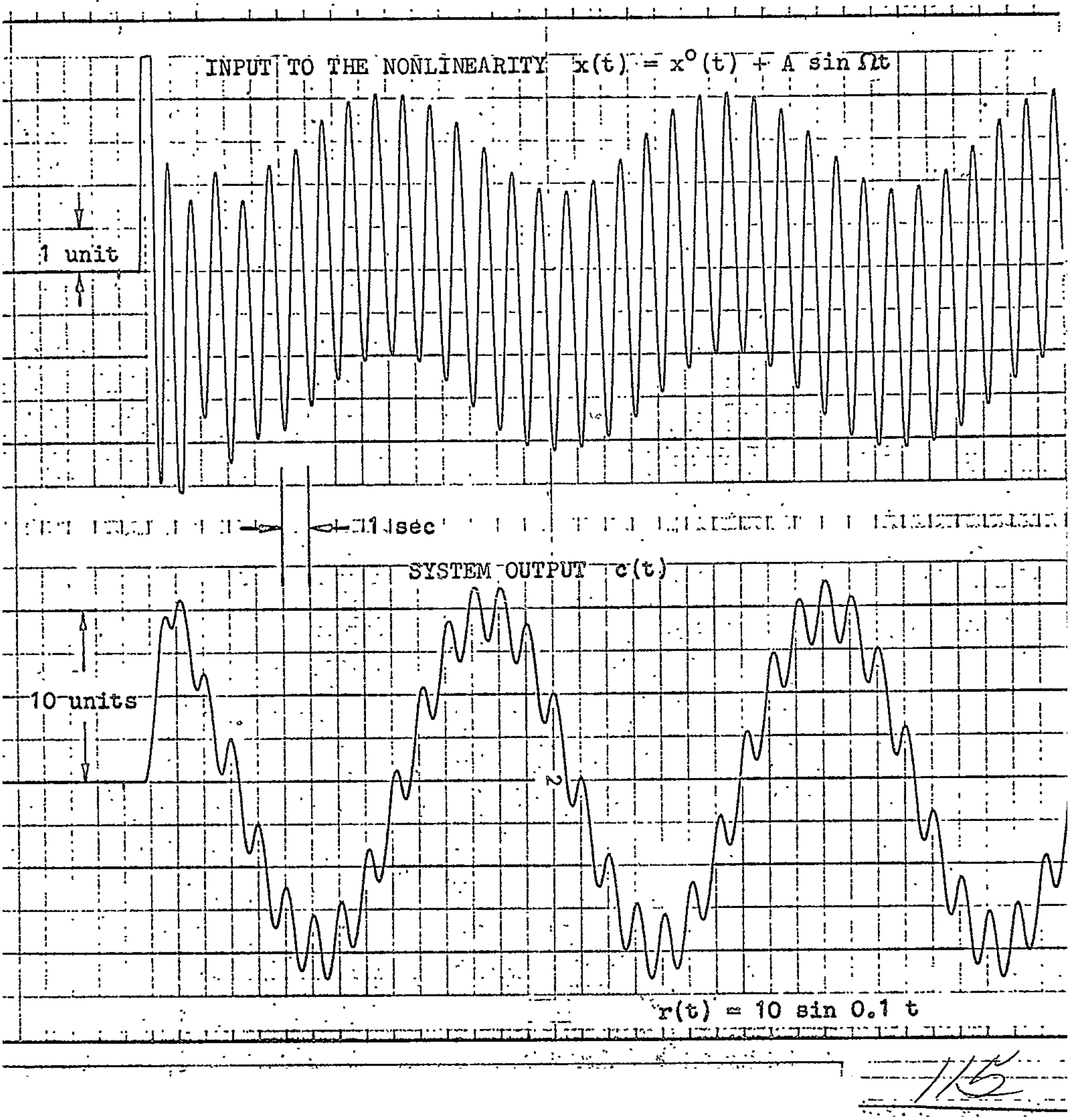


Fig. 3.18 - Computer solution

limit cycle is present under the condition that $|x^o| \le 2.25$. For small values of x^o, it is possible to consider $\psi(x^o) = Kx^o$ where K = const. Then the stability of the system with respect to slowly-varying signals may be investigated by well-known linear methods outlined in Chapter II. In the specific example, the equation of interest is

$$s(s+1)(s^2+0.8s+1) + K_2K_{-1}s(s+1) + K' K_1 K_2 K_3 = 0 \qquad (3.45)$$

Finally, it is to be noted that for the smoothing effect to take place, the amplitude A should be $A \ge |x^o|$, as stated in equations 3.43 and 3.44.

The results of the above analysis are checked by simulating the system on an analog computer. Three cases are considered. In Fig. 3.16, the input to the nonlinearity $x = x^o + A \sin \Omega t$ and the system output $x = x(t)$ are shown when the input signal is $r = \sin 0.1t$. The obtained computer solution agrees with the prediction. The output c(t) exhibits a smaller amplitude limit cycle with the same frequency. When the input amplitude is increased five times, the diagram of Fig. 3.17 is obtained. This change increased x^o, but the amplitude A remained almost the same. The frequency Ω did not change. Similar results occurred when the input amplitude increased ten times except that the amplitude A became slightly smaller, which agrees with the diagram of Fig. 3.14. The third case is given in Fig. 3.18. It should be noted from these computer solutions that the output signal c(t) represents the input signal r(t) except for the superimposed limit cycle. It can be eliminated by introducing

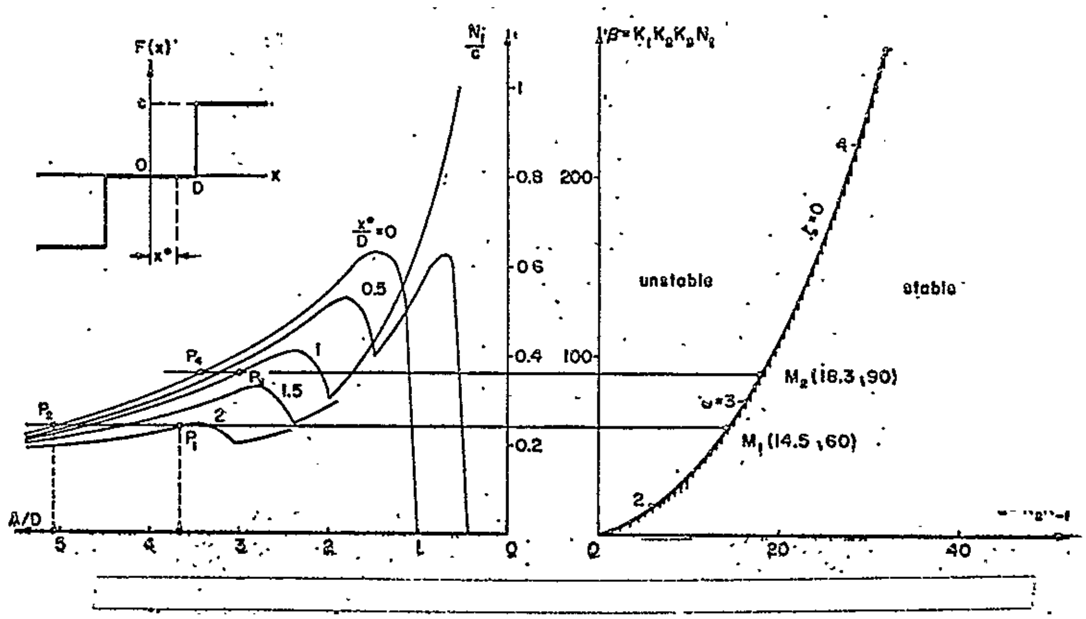


Fig. 3-19 - Parameter plane diagram.

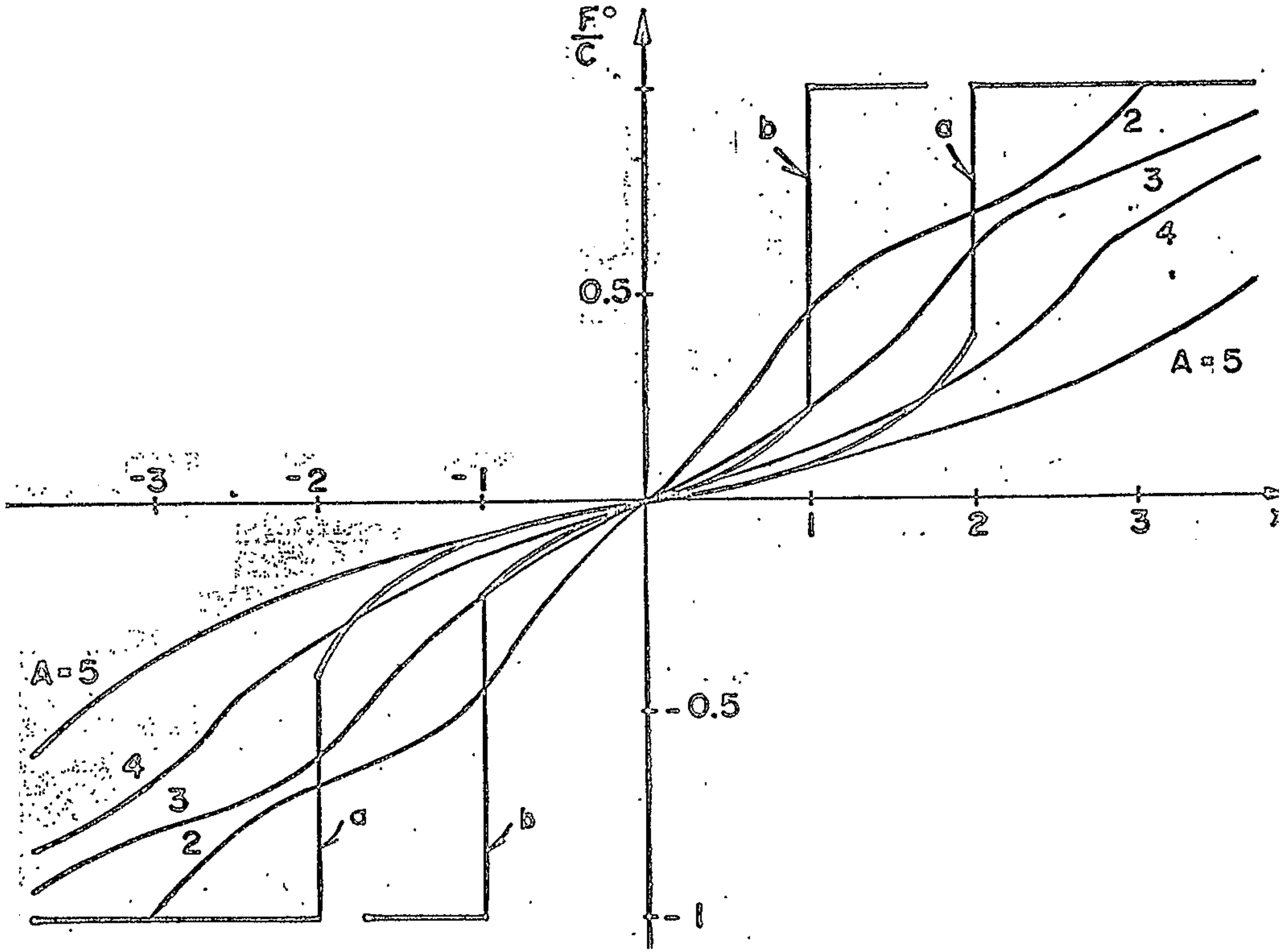


Fig. 3.20 - Linearized characteristic

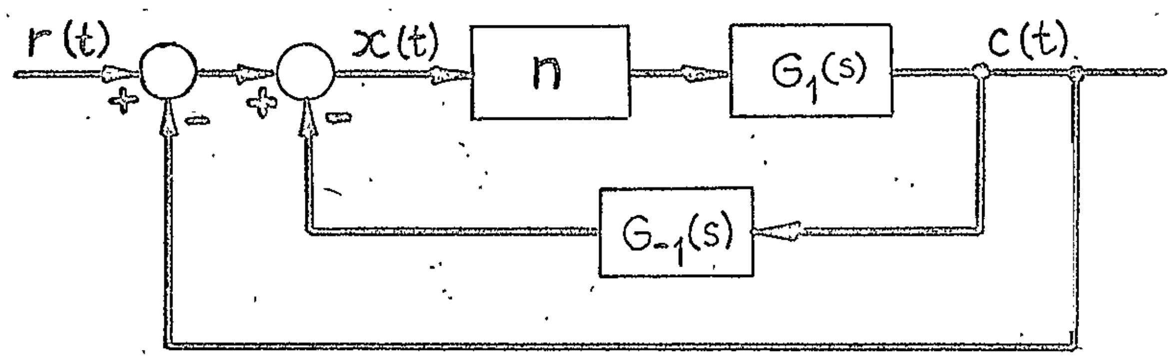


Fig. 3.21 - System block diagram

sufficient filtering in the block $G_3(s)$ of the system of Fig. 3.13, or by readjusting the system parameters to obtain a higher frequency limit cycle.

If the values of the system parameters are chosen so that the operating point is $M_2(21.2; 120)$ of Fig. 3.14, the frequency of the limit cycle becomes higher. However, the corresponding range of variations of x^o is decreased to $|x^o| < 0.7$, together with the range of the amplitude A which is between Q_3 and Q_4. This indicates that the presented procedure is convenient to apply when the system parameters and operating conditions are changed.

If the nonlinearity h is changed in the system of Fig. 3.13 by introducing a considerable dead zone, D, a diagram of Fig. 3.19 is obtained. The variation of the M point is calculated by using equation 3.6b for the given nonlinearity of Fig. 3.19. Two cases should be considered separately; i.e.,

$$N_1 = \frac{2c}{\pi A}\left[\sqrt{1-\left(\frac{x^o+D}{A}\right)^2} + \sqrt{1-\left(\frac{x^o-D}{A}\right)^2}\right], \quad A \geq |x^o|+D \qquad (3.46a)$$

$$N_1 = \frac{2c}{\pi A}\sqrt{1-\left(\frac{x^o-D}{A}\right)^2}, \quad |x^o|-D \leq A \leq |x^o|+D \qquad (3.46b)$$

and the diagram $N_1(x^o, A)$ is shown in Fig. 3.19. By using equation 3.6a, the corresponding diagram $F(x^o, A)$ of Fig. 3.20 is plotted according to

$$F^o = \frac{c}{\pi}\left(\arcsin\frac{x^o+D}{A} + \arcsin\frac{x^o-D}{A}\right), \quad A \geq |x^o| + D \qquad (3.47a)$$

$$F^o = \frac{c}{\pi}\left(\frac{\pi}{2} + \arcsin\frac{|x^o|-D}{A}\right)\sin x^o, \quad |x^o|-D \leq A \leq |x^o|+D \qquad (3.47b)$$

120

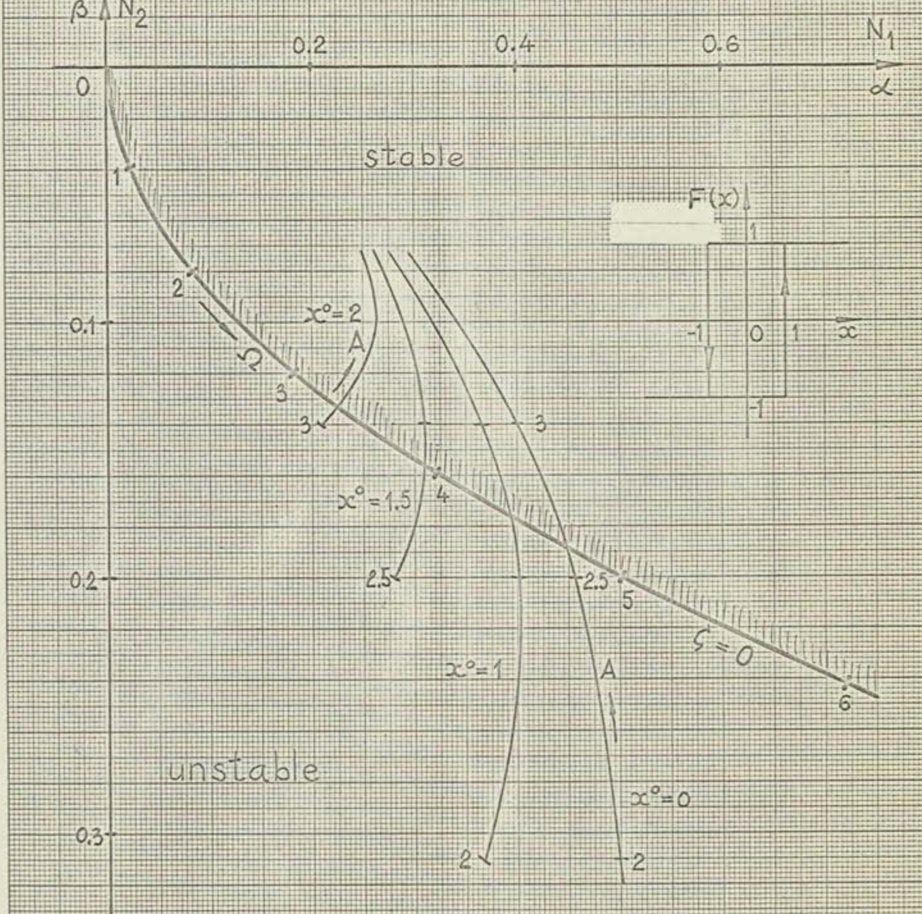


Fig. 3.22 - Parameter plane diagram

121

If the points M_1 and M_2 are chosen in Fig. 3.19 as operating points, the replotting of the straight lines P_1P_2 and P_3P_4 results in the two linearized characteristics a and b of Fig. 3.20, respectively. They are constructed for the values of nonlinear parameters $c = D = 1$. As can be seen from Fig. 3.20 the dead zone is eliminated as far as the slowly-varying signals are concerned. For this to take place, it is necessary to choose operating conditions such that equation 3.47a is valid. This means that the amplitude A of the limit cycle must be greater than $|x^o|+D$. Otherwise the linearized characteristic $\psi(x^o)$ does not go to zero when $x^o = 0$ since F^o does not go to zero for $x^o = 0$. This is indicated in Fig. 3.20 whereby $F^o = 0$ for $x^o = 0$ and the dead zone is eliminated.

By the outlined technique, it is possible to eliminate the hysteresis and backlash in systems with multi-valued nonlinearities. The linearization yields a single-valued function $\psi(x^o)$ which is linear in a certain limited range of values of the variable x^o about the origin. To illustrate this, consider a nonlinear system with the block diagram of Fig. 3.21 and the transfer functions

$$G_1(s) = \frac{K_1}{s(s+1)(s+2)}, \quad G_{-1}(s) = K_{-1}s \tag{3.48}$$

The nonlinear function F(x) of the nonlinearity n is given in Fig. 3.22.

The equation describing the system is

$$s(s+1)(s+2)s + (K_{-1}s_{-1})K_1F(x) = 0 \tag{3.49}$$

After harmonic linearization of 3.49, the corresponding characteristic equation is

$$s(s+1)(s+2) + K_1(K_{-1}s+1)(N_1 + \frac{N_2}{\Omega}s) = 0 \qquad (3.50)$$

If $K_1 = 50$, $K_{-1} = 1$

$$\alpha = N_1$$
$$\beta = N_2 \qquad (3.51)$$

and $s = j\Omega$, one obtains the $\zeta = 0$ curve as

$$\alpha = \frac{1}{50}\Omega^2$$
$$\beta = \frac{1}{25}\Omega. \qquad (3.52)$$

The curve is plotted in Fig. 3.22. On the same plot, the variation of the point $M(N_1; N_2)$ is constructed according to

$$N_1 = \frac{2c}{\pi A}\left(\sqrt{1-(\frac{D-x^o}{A})^2} + \sqrt{1-(\frac{D+x^o}{A})^2}\right) \quad A \geq D + |x^o|$$

$$N_2 = -\frac{4cD}{\pi A^2} \qquad (3.53)$$

and the nonlinearity $F(x)$ of Fig. 3.22 for which $c = D = 1$. From the intersections of the $\zeta = 0$ curve and the variation of the M point, one can determine the amplitude A and the frequency Ω as function of x^o; i.e.,

$$A = A(x^o)$$
$$\Omega = (x^o) \qquad (3.54)$$

Then, by using the expression

$$F^o = \frac{c}{\pi}(\text{arc sin}\,\frac{D+x^o}{A} - \text{arc sin}\,\frac{D-x^o}{A}), \quad A \geq D + |x^o| \qquad (3.55)$$

for $c = D = 1$, a family of curves with constant amplitude A is plotted on Fig. 3.23. If the first equation 3.54 is mapped onto

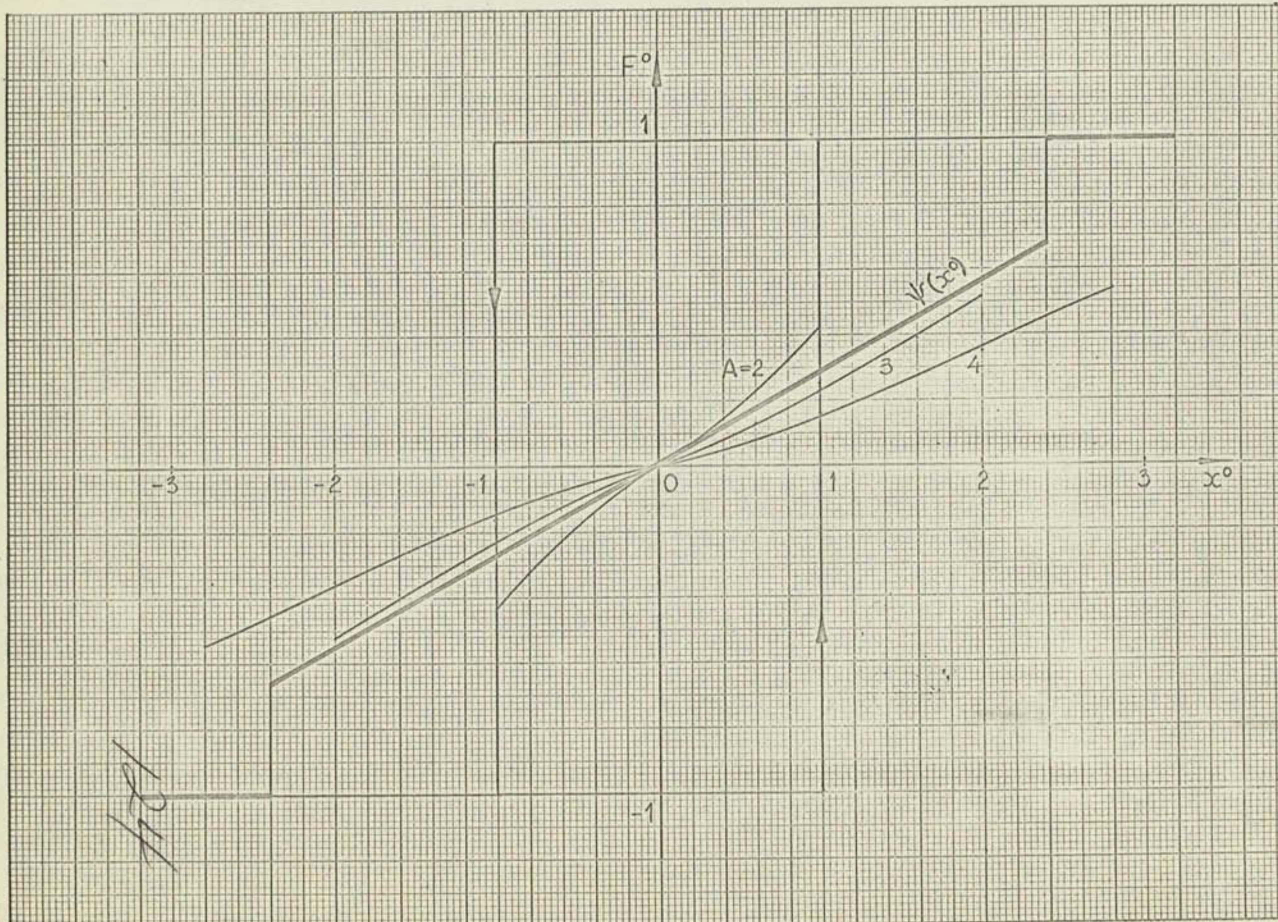
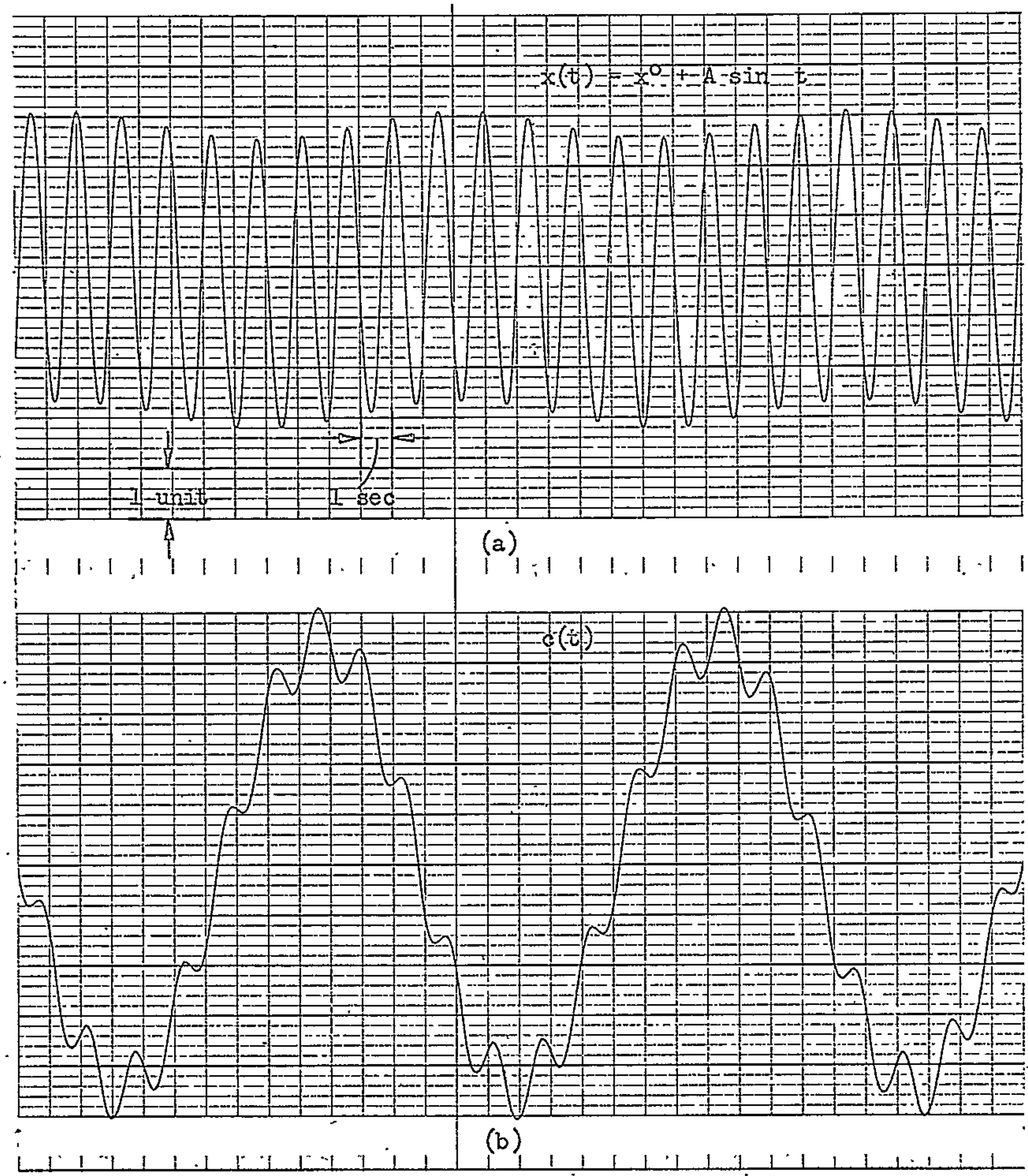


Fig. 3.23 - Function $\psi(x^{0})$

3-117



125

Fig. 3.24 - Computer solution

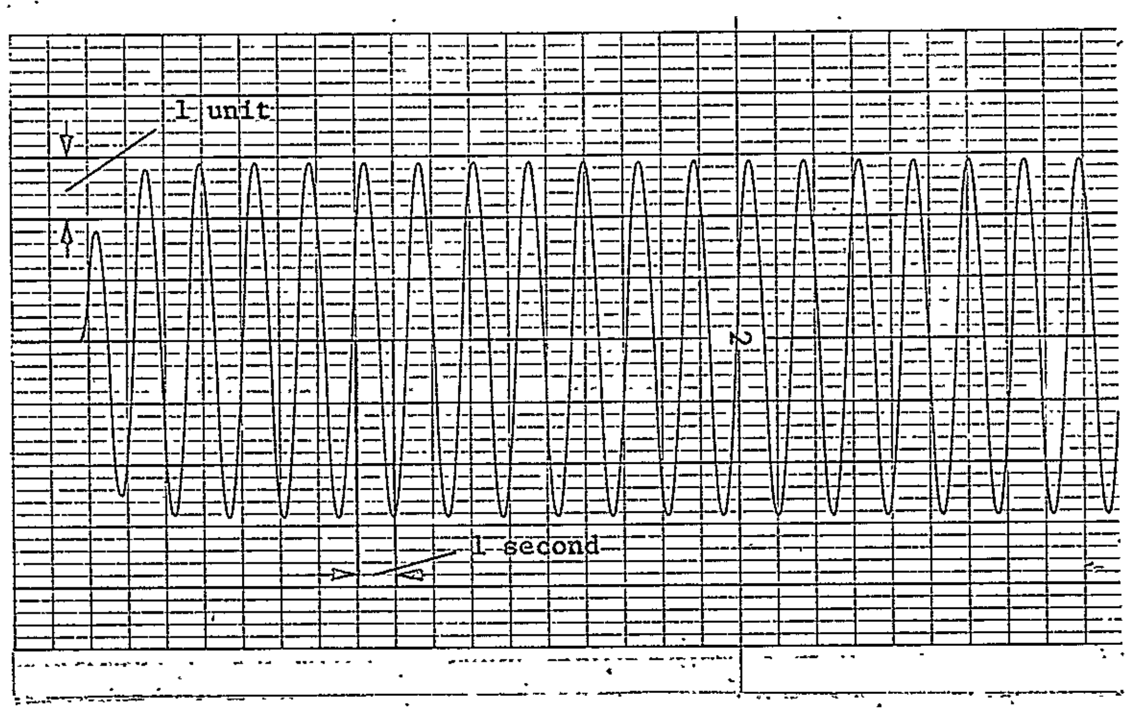


Fig. 3-25 - Computer solution for $x = 2.6 \sin 4.8t$

the family of constant amplitude A, the function $\psi(x^o)$ is obtained as shown in Fig. 3.23. The function $\psi(x^o)$ as a single-valued function of x^o, which is linear in the range $0 \le |x^o| \le 2.4$.

For an input $r(t) = 5 \sin 0.5t$, the computer solution is shown in Fig. 3.24. The amplitude A and the frequency Ω of the limit cycle are slowly-varying quantities according to equations 3.54 and the slowly-varying variable x^o. Their average values, however, are close to that which can be predicted from the parameter plane diagram of Fig. 3.22; i.e., $A = 2.8$ and $\Omega \simeq 4.5$ rad/sec. This can be concluded from the diagram (a) of Fig. 3.24. On the diagram (b), the output signal $c(t)$ is shown whereby the limit cycle is largely attenuated by the block $G_1(s)$ of Fig. 3.21. The low-frequency component in the signal $c(t)$ represents the input $r(t) = 5 \sin 0.5t$ at the output of the system.

Of course, if the input $r(t)$ is not present, the system will exhibit a limit cycle which can be determined from the intersection of the M locus $x^o = 0$ and the $\zeta = 0$ curve on Fig. 3.22 as $x = A \sin t$, $A = 2.6$, $\Omega = 4.8$. This is checked by the analog computer simulation and the obtained solution is shown on Fig. 3.25.

3. 6 Conclusion

The parameter plane method has been used to indicate existence of asymmetrical oscillations in nonlinear control systems. A procedure has been developed to determine the oscillations for different values of system parameters and input signals. It has been shown how a limit cycle can modify the nonlinear characteristic for slowly-varying signals. This modification may be of importance when a high-accuracy control system has to be designed in the presence of nonlinearities with excessive dead zone, hysteresis, backlash, etc. The design technique can be directly applied to a large class of plant-adaptive control systems where a sinusoidal signal is used as an identification signal.

In a future study, the technique may be extended to the investigation of transient asymmetrical oscillations. Thus, to study how these oscillations are established after certain amplitude perturbation, this study should be largely based upon the material presented in the following chapter.

It may also be shown [16, 17] that the presented analysis can be extended to the case when the signal superimposed on a sinusoid is not only a constant or slowly-varying sinusoid, but also when the additional signal is described as a Gaussian process, provided that the amplitude or standard deviation of the additional signal is of no consequence in the analysis. This further generates the idea of applying the dual-input describing function [15,17] along with the parameter plane method, and investigates the case when the input to a nonlinearity of the system is a combination of two similar sinusoidal signals.

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SOME APPLICATIONS OF ALGEBRAIC METHODS

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CONTENTS

INTRODUCTION

Classical techniques for analysis and design of dynamic systems are largely restricted to cases in which only one parameter of the system is adjustable. As a consequence complex systems cannot be treated adequately with classical techniques. Algebraic methods, as developed in NASA CR-617*, are capable of treating systems in which two parameters are adjustable, and thus permit analysis and synthesis of systems which are too complex for treatment with classical methods.

The treatment of algebraic methods presented in CR-617 develops the fundamental theoretical basis for the coefficient plane and parameter plane methods. It also applies these methods to basic problems such as stability analysis, cascade compensation of systems, and related topics. The applications indicated in CR-617 are rather elementary, i.e., the problems considered illustrated the procedures to be used but were not very complex problems. This report is based on the findings of CR-617, and extends the applications of the algebraic methods to problems of a more complex nature.

When cascade compensation is used in a feedback control system, more than one filter section may be required to achieve desired performance. Frequency response methods involving trial and error are often used, but parameter plane methods permit analysis and design without trial and error if it is permassible

*Algebraic Methods for Dynamic Systems by G. J. Thaler, D. D. Siljak and R. C. Dorf, Nasa Contractor Report NASA CR-617, Nov., 1966.

to use two identical filter sections. This problem is treated in Chapter I of this report. The applicable parameter plane equations are derived and a digital computer program based on these equations is presented. The program is used to study the effects of compensation on several systems.

Chapters II and III are concerned with nonlinear systems. Conventional methods such as frequency domain analysis of systems with the Describing function have proven useful when the system contains only one nonlinearity (or several nonlinearities conveniently located so that they can be incorporated in one describing function). These techniques can define stability and estimate relative stability for fairly complex systems as long as the conditions of nonlinearity are not too complex. Such cases are easily treated using algebraic methods, the effect of the nonlinearity being represented as a movement of the operating point on the parameter plane, which in turn represents a variation of the characteristic roots as a function of signal amplitude. The algebraic methods are capable of extending such analysis to systems containing two distinct nonlinear components, and can be used to predict the transient response of the system rather accurately. Techniques for such problems are developed in Chapter II.

Chapter III is concerned with a much more difficult nonlinear problem, that of asymmetrical nonlinear oscillations. These are oscillations consisting of a limit cycle superimposed on another signal. The problems studied on the parameter plane involve steady-state operating conditions (rather than transient conditions), and permit analysis of the existence of oscillations as well as their dependence on parameter values and input signal values. Extension to linearization with either signals is included, as well as some design considerations.

It is felt that the results presented here indicate the capabilities of the algebraic methods in dealing with complex linear and nonlinear problems. It is also felt that the results presented here will be directly applicable to a number of practical problems, and will point out avenues of approach to still additional problems.

I

SOLUTION OF EQUATIONS WITH COEFFICIENTS THAT ARE QUADRATIC IN α and β

1.1 INTRODUCTION

It has been shown that the characteristic equation can be solved for $\alpha = \alpha(\zeta, \omega_n)$ and $\beta = \beta(\zeta, \omega_n)$ when the coefficients of the characteristic equation are of the forms:

$$\begin{aligned}
&a)\quad a_k = b_k\alpha + c_k\beta + d_k \\
&b)\quad a_k = b_k\alpha + c_k\beta + h_k\alpha\beta + d_k \qquad (1)\\
&c)\quad a_k = b_{k2}\alpha^2 + b_{k1}\alpha + h_k\alpha\beta + c_{k1}\beta + c_{k2}\beta^2 + d_k \\
&d)\quad a_k = b_{kn}\alpha^n + b_{k(n-1)}\alpha^{n-1} + \ldots h_{k(n-1)}\alpha^{n-1}\beta + \ldots \\
&\qquad\quad c_{k(n-1)}\beta^{n-1} + c_{kn}\beta^n + d_k
\end{aligned}$$

In addition practical solutions have been obtained for the first two of these coefficient forms, i.e., computer programs have been written for them and successfully applied. The development to be presented here is a particular solution for case 1-1c, particularly in the sense that a computer program has been obtained which solves the equations of a third order system for which the coefficients are quadratic in α and β, but which do not contain all of the α and β combinations indicated. At the same time the solution is a general solution in the sense that the program can be modified to solve the equations of an n^{th} order system, and can also be modified to accept all of the α and β forms indicated in

$$a_k = b_{k2}\alpha^2 + b_{k1}\alpha + h_k\alpha\beta + c_{k1}\beta + c_{k2}\beta^2 + d_k$$

The modifications to be made in the program are discussed, but the necessary programming has not been done.

1.2 THE PROBLEM: Cascade Compensation with two identical filter sections.

In the design of feedback control systems it is common to use compensators which are filters placed in cascade with the main transmission path. Frequently two sections of filter are needed, and if identical sections are used with an isolation amplifier so that their transfer functions can be multiplied, then manipulation of the transfer function equation provides a characteristic equation in which the coefficients are quadratic in z and p, the zero and pole of the compensators. For example let:

$$G = \frac{K}{s^3+Xs^2+Ys} \qquad (1\text{-}2)$$

$$G_c = \left(\frac{s+z}{s+p}\right)^2 = \frac{s^2+2zs+z^2}{s^2+2ps+p^2} \qquad (1\text{-}3)$$

$$1+G_cG = 0 = 1 + \frac{K(s^2+2zs+z^2)}{(s^3+Xs^2+Ys)(s^2+2ps+p^2)} \qquad (1\text{-}4)$$

from which the characteristic equation is

$$s^5 + (X+2p)s^4 + (p^2+2Xp+Y)s^3 + (Xp^2+2Yp + K)s^2 + + (Yp^2 + 2Kz)s + Kz^2 = 0 \qquad (1\text{-}5)$$

Letting $p \triangleq \alpha$ and $z \triangleq \beta$ it is noted that all of the forms specified in the quadratic case definition of a_k do appear in at least some of the coefficients except that there is no $\alpha\beta$ product term.

The formulation just given does not conform to normal control system practice, however, in that an important restriction on the design of the compensator is the usual requirement that steady state accuracy must be maintained by keeping the error coefficient *unchanged. To do this* the physical adjustment is to alter the gain of the amplifier, but in the mathematical analysis it is more convenient to include this restriction in the transfer function of the compensator by defining (for this case)

$$G_c = \left(\frac{p}{z}\right)^2 \left(\frac{s+z}{s+p}\right)^2 \qquad (1\text{-}6)$$

This alters the algebraic form of the characteristic equation which becomes:

$$0 = 1 + \frac{K\left(\frac{p}{z}\right)^2 (s^2+2zs+z^2)}{(s^3+Xs^2+Ys)(s^2+2ps+p^2)}$$

$$= (s^3+Xs^2+Ys)(s^2+2ps+p^2) + K\frac{p^2}{z^2}(s^2+2zs+z^2)$$

$$= s^5+(X+2p)s^4 + (p^2+2Xp+Y)s^3 + \left[Xp^2+2Yp+K\left(\frac{p}{z}\right)^2\right]s^2$$

$$+\left[Yp^2+2Kp\left(\frac{p}{z}\right)\right]s + Kp^2 \qquad (1\text{-}7)$$

Choosing $p \triangleq \beta$ and $\frac{p}{z} \triangleq \alpha$ this becomes

$$0 = s^5 + (X+2\beta)s^4 + (\beta^2+2X\beta+Y)s^3 + (X\beta^2+2Y\beta+K\alpha^2)s^2$$

$$+ (Y\beta^2+2K\beta\alpha)s + K\beta^2 \qquad (1\text{-}8)$$

In equation 1-8 the coefficients are quadratic in α and β, but there is no term of the form $b_{k1}\alpha$, and the program as written does not make provision for such a term, though modification of the program to include it is not difficult. The problem to be studied, then is that *of a third order system compensated* with two cascaded identical sections of filter, and with the added requirement that the error coefficient be maintained constant at a predetermined value.

1.3 DERIVATION OF THE GENERAL THIRD ORDER SYSTEM RELATIONSHIPS

The general third order system is defined by the transfer function

$$G(s) = \frac{K}{(s+A)(s+B)(s+C)} \qquad (1\text{-}9)$$

which is a Type Zero system, but which can be changed to Type 1, 2, or 3 by setting one or more of the poles to zero. The compensator transfer function, *including the gain* multiplier which maintains the error coefficient is

$$G_c = \left(\frac{p}{z}\right)^2 \left(\frac{s+z}{s+p}\right)^2 = \frac{p^2(s^2+2zs+z^2)}{z^2(s^2+2ps+p^2)} \qquad (1\text{-}10)$$

From 1-9 and 1-10 the characteristic equation is

$$\left[s^3 + (A+B+C)s^2 + (AB+BC+AC)s + ABC\right](s^2+2ps+p^2) +$$

$$+ K\frac{p^2}{z^2}(s^2+2zs+z^2) = 0 \qquad (1\text{-}11)$$

This expands to

$$s^5 + (A+B+C+2p)s^4 + \left[AB+BC+AC+2p(A+B+C)+p^2\right]s^3$$

$$+ \left[ABC+2p(AB+BC+CA) + p^2(A+B+C) + K\frac{p^2}{z^2}\right]s^2 +$$

$$+ \left[2pABC+p^2(AB+BC+AC) + 2Kp\left(\frac{p}{z}\right)\right]s+p^2(ABC+2K) = 0$$

Let $p \triangleq \beta \qquad \frac{p}{z} \triangleq \alpha$

$A+B+C \triangleq \sum r_i$ = sum of roots (poles)

$AB+BC+AC \triangleq \sum \prod_2 r_i$ = sum of root products taken 2 at a time

$\sum \prod_n r_i \triangleq$ sum of root products taken n at a time

$ABC\ldots \triangleq \prod r_i \triangleq$ products of the roots

Then equation 1-12 becomes:

$$s^5 + (\sum r_i + 2\beta)s^4 + (\sum \prod_2 r_i + 2\sum r_i\beta + \beta^2)s^3 +$$

$$(\prod r_i + 2\sum \prod_2 r_i\beta + \sum r_i\beta^2 + K\alpha^2)s^2 + \quad (1\text{-}13)$$

$$(2\prod r_i\beta + \sum \prod_2 r_i\beta^2 + 2K\alpha\beta)s + (\prod r_i + 2K)\beta^2 = 0$$

Collecting like terms in α and β:

$$\alpha^2(Ks^2) + \alpha\beta(2Ks) + \beta^2(s^3 + \sum r_i s^2 + \sum \prod_2 r_i s + \prod r_i + K) +$$

$$+ \beta(2s^4 + 2\sum r_i s^3 + 2\sum \prod_2 r_i s^2 + 2\prod r_i s)$$

$$+ (s^5 + \sum r_i s^4 + \sum \prod_2 r_i s^3 + \prod r_i s^2) = 0 \quad (1\text{-}14)$$

Using the basic parameter plane relationships:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k \overline{U}_{k-1}(\zeta) = 0 \quad (1\text{-}15)$$

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k \overline{U}_k(\zeta) = 0 \quad (1\text{-}16)$$

and defining:

$$B_{21} = K\omega^2 U_1(\zeta) \quad (1\text{-}17)$$

$$B_{22} = K\omega^2 U_2(\zeta) \quad (1\text{-}18)$$

$$D_1 = -2K\omega U_o(\zeta) \quad (1\text{-}19)$$

$$D_2 = -2K\omega U_1(\zeta) \quad (1\text{-}20)$$

$$E_{11} = 2\omega^4 U_3(\zeta) - 2\sum r_i\omega^3 U_2(\zeta) + 2\sum \prod_2 r_i\omega^2 U_1(\zeta) -$$

$$- 2\prod r_i\omega U_o(\zeta) \quad (1\text{-}21)$$

$$E_{12} = 2\omega^4 U_4(\zeta) - 2\sum r_i\omega^3 U_3(\zeta) + 2\sum \prod_2 r_i\omega^2 U_2(\zeta)$$

$$-2\prod r_i\omega U_1(\zeta) \quad (1\text{-}22)$$

$$F_{21} = -\omega^3 U_2(\zeta) + \sum r_i\omega^2 U_1(\zeta) - \sum \prod_2 r_i\omega U_o(\zeta)$$

$$+(\prod r_i + K)U_{-1}(\zeta) \quad (1\text{-}23)$$

$$F_{22} = -\omega^3 U_3(\zeta) + \sum r_i\omega^2 U_2(\zeta) - \sum \prod_2 r_i\omega U_1(\zeta)$$

$$+ (\prod r_i + K)U_o(\zeta) \quad (1\text{-}24)$$

$$G_1 = -\omega^5 U_4(\zeta) + \sum r_i\omega^4 U_3(\zeta) - \sum \prod_2 r_i\omega^3 U_2(\zeta)$$

$$+ \prod r_i\omega^2 U_1(\zeta) \quad (1\text{-}25)$$

$$G_2 = -\omega^5 U_5(\zeta) + \sum r_i\omega^4 U_4(\zeta) - \sum \prod_2 r_i\omega^3 U_3(\zeta)$$

$$+\prod r_i\omega^2 U_2(\zeta) \quad (1\text{-}26)$$

$$P_1 = \beta D_1 \quad (1\text{-}27)$$

$$P_2 = \beta D_2 \quad (1\text{-}28)$$

$$Q_1 = \beta E_{11} + \beta^2 F_{21} + G_1 \quad (1\text{-}29)$$

$$Q_2 = \beta E_{12} + \beta^2 F_{22} + G_2 \quad (1\text{-}30)$$

This results in

$$\alpha^2 B_{21} + \alpha P_1 + Q_1 = 0 \quad (1\text{-}31)$$

$$\alpha^2 B_{22} + \alpha P_2 + Q_2 = 0 \quad (1\text{-}32)$$

which are two non-linear algebraic equations completely generalized in terms of the uncompensated system poles and root locus gain, ζ, ω and the first kind of Chebyshev Functions. These must be solved simultaneously for the correct values of α and β. To do this, the method with the best chance of success appears to be Sylvester's Method in which we form a set of four equations by taking the original Equations (1-31) and (1-32) and forming two more by a multiplication with α giving:

$$\alpha^2 B_{21} + \alpha P_1 + Q_1 = 0 \quad (1\text{-}33)$$

$$\gamma^2 B_{22} + \alpha P_2 + Q_2 = 0 \quad (1\text{-}34)$$

$$\alpha^3 B_{21} + \alpha^2 P_1 + \alpha Q_1 = 0 \tag{1-35}$$

$$\alpha^3 B_{22} + \alpha^2 P_2 + \alpha Q_2 = 0 \tag{1-36}$$

Now placing these equations in matrix form:

$$\begin{bmatrix} 0 & B_{21} & P_1 & Q_1 \\ 0 & B_{22} & P_2 & Q_2 \\ B_{21} & P_1 & Q_1 & 0 \\ B_{22} & P_2 & Q_2 & 0 \end{bmatrix} \begin{bmatrix} \alpha^3 \\ \alpha^2 \\ \alpha \\ 1 \end{bmatrix} = 0 \tag{1-37}$$

If the α's are not zero then:

$$\begin{vmatrix} 0 & B_{21} & P_1 & Q_1 \\ 0 & B_{22} & P_2 & Q_2 \\ B_{21} & P_1 & Q_1 & 0 \\ B_{22} & P_2 & Q_2 & 0 \end{vmatrix} = 0 \tag{1-38}$$

Expanding this determinant

$$-B_{21}^2 Q_2^2 + B_{21}B_{22}Q_1Q_2 + P_1P_2B_{21}Q_2 - P_1^2Q_2B_{22} + Q_1Q_2B_{21}B_{22}$$
$$-Q_1^2B_{22}^2 - Q_1P_2^2B_{21} + Q_1P_1P_2B_{22} = 0 \tag{1-39}$$

Substituting equations (1-27) through (1-30) in equation (1-39) provides a fourth order equation in β:

$$\beta^4(-F_{22}^2B_{21}^2 + 2F_{21}F_{22}B_{21}B_{22} + D_1D_2F_{22}B_{21} - F_{22}D_1^2B_{22} -$$
$$F_{21}^2B_{22}^2 - F_{21}D_2^2B_{21} + D_1D_2F_{21}B_{22}) +$$
$$\beta^3(-2E_{12}F_{22}B_{21}^2 + 2E_{11}F_{22}B_{21}B_{22} + 2F_{21}E_{12}B_{21}B_{22} +$$
$$D_1D_2E_{12}B_{21} - D_1^2E_{12}B_{22} - 2E_{11}F_{21}B_{22}^2 -$$
$$D_2^2E_{11}B_{21} + D_1D_2E_{11}B_{22}) +$$
$$\beta^2(-E_{12}^2B_{21}^2 - 2F_{22}G_2B_{21}^2 + 2E_{11}E_{12}B_{21}B_{22} + 2F_{21}G_2B_{21}B_{22}$$

$$2G_1F_{22}B_{21}B_{22} + D_1D_2G_2B_{21} - D_1^2G_2B_{22} - E_{11}^2B_{22}^2 -$$
$$2F_{21}G_1B_{22}^2 - D_2^2G_1B_{21} + D_1D_2G_1B_{22}) +$$
$$\beta(-2E_{12}G_2B_{21}^2 + 2E_{11}G_2B_{21}B_{22} + 2G_1E_{12}B_{21}B_{22} - 2E_{11}G_1B_{22}^2) +$$
$$(-G_2^2B_{21}^2 + 2G_1G_2B_{21}B_{22} - G_1^2B_{22}^2) = 0 \tag{1-40}$$

from which the coefficients may be determined by a substitution of (1-17) through (1-26) and the values of the first kind of *Chebyshev functions in terms of* ζ and ω. Since the solution of a fourth order equation is at best difficult, it is at this point a digital computer becomes a necessity.

The major problem is not the actual solution of the quartic itself, but rather the proper choice of one of the four solutions. There are two marked characteristics, however, which help in the selection. These are:

a) Complex answers to the quartic have no physical significance and may therefore be discarded as erroneous.

b) The definition of α requires that α and β be of the same sign so that p and z will be of identical sign.

Using this information and that available from the Ross-Warren[2] method as to compensator pole and zero location, it is found that the solution to the β quartic is the largest, positive, real value.

Now entering equation (1-27) with this value, and evaluating the other coefficients

$$\alpha = [-Q_1/B_{21}]^{1/2} \tag{1-41}$$

for in the third order case P_1 is always identically zero.

Thus, with the programming of the appropriate equations, the digital computer could give all of the values and plot the constant zeta and constant omega loci on the Parameter Plane for any desired values.

1.4 SOME APPLICATIONS OF THE PROGRAM

Several third order systems were investigated by the application of the generalized equations and the Parameter Plane curves, Figures 1-1 through 1-8 were plotted. Of these, the K/s^3 family appears the most interesting. Further investigation of three of the curves in this family, Figures 1-1, 1-2 and 1-3 shows that there is a relationship between K, the root locus gain, α and β.

These relations are:

a) Choose a point on the $1/s^3$ α-β plane.

b) Zeta reads directly.

c) Determine the actual omega at that point by multiplying the value read by the cube root of the uncompensated system gain.

d) Read the value of α directly from the point chosen.

e) Read the value of β from the point chosen.

f) Obtain the true value of β by multiplying this value by the cube root of the uncompensated system gain.

By this method, the values of α and β may be determined for all $\frac{K}{s^3}$ systems from one universal curve.

1.5 BANDWIDTH CURVES ON THE α-β Plane

In many instances, there is also a bandwidth criterion

imposed on the engineer as well as an optimal operating point for the plant under consideration. With this in mind, equations for the plotting of constant bandwidth curves on the α-β plane are developed. For the purpose of this development a constant bandwidth curve will be defined as:

> A constant bandwidth curve for $G(j\omega_b) = M$ is a curve drawn upon the parameter plane which specifies the relation between the parameters necessary if the transfer function $G(s)$, which is a function of the parameters, is to have magnitude M at the real frequency ω_b.[5]

Once these curves are obtained they may be superimposed on the parameter plane thus indicating what values of the parameters are necessary in order to meet the specifications.

Taking the rational transfer function and defining it:

$$G(s) = \frac{P(s)}{Q(s)} = \frac{p_m s^m + p_{m-1}s^{m-1} + \dots p_1 s + p_o}{q_n s^n + q_{n-1}s^{n-1} + \dots q_1 s + q_o} \qquad (1\text{-}42)$$

where the p_m's and q_n's are of the form:

$$p_u = g_u\alpha^2 + h_u\alpha + i_u\alpha\beta + j_u\beta + k_u\beta^2 + l_u$$

$$u = 0,1,2,\dots,m \qquad (1\text{-}43)$$

$$q_v = a_v\alpha^2 + b_v\alpha + c_v\alpha\beta + d_v\beta + e_v\beta^2 + f_v$$

$$v = 0,1,2,\dots,n \qquad (1\text{-}44)$$

Therefore

$$G(s) = \frac{\sum_{u=0}^{m} p_u s^u}{\sum_{v=0}^{n} q_v s^v} \qquad (1\text{-}45)$$

Employing Equation 1-45 in the parameterized form the generalized compensated third order transfer function is:

$$G(s) = \frac{P(s)}{Q(s)} \qquad (1\text{-}46)$$

where:

$$P(s) = \alpha^2Ks^2 + 2\alpha\beta Ks + \beta^2K \qquad (1\text{-}47)$$

and:

$$\begin{aligned} Q(s) = {} & \beta^2\Big[s^3 + \sum\prod_1 r_i s^2 + \sum\prod_2 r_i s + \sum\prod_3 r_i\Big] + \\ & \beta\Big[2s^4 + 2\sum\prod_1 r_i s^3 + 2\sum\prod_2 r_i s^2 + 2\sum\prod_3 r_i s\Big] + \\ & \Big[s^5 + \sum\prod_1 r_i s^4 + \sum\prod_2 r_i s^3 + \sum\prod_3 r_i s^2\Big] \end{aligned} \qquad (1\text{-}48)$$

Making the definitions:

$$A_r = \sum_{\substack{v=0\\ \text{even}}}^{n} (-1)^{\frac{1}{2}v}\omega_b^v a_v; \text{ etc. for } B_r, C_r, D_r, E_r, F_r \qquad (1\text{-}49)$$

$$A_i = \sum_{\substack{v=0\\ \text{odd}}}^{n} (-1)^{\frac{1}{2}(v-1)}\omega_b^v a_v; \text{ etc. for } B_i, C_i, D_i, E_i, F_i \qquad (1\text{-}50)$$

$$G_r = \sum_{\substack{u=0\\ \text{even}}}^{m} (-1)^{\frac{1}{2}u}\omega_b^u g_u; \text{ etc. for } H_r, I_r, J_r, K_r, L_r \qquad (1\text{-}51)$$

$$G_i = \sum_{\substack{u=0\\ \text{odd}}}^{m} (-1)^{\frac{1}{2}(u-1)}\omega_b^u g_u; \text{ etc. for } H_i, I_i, J_i, K_i, L_i \qquad (1\text{-}52)$$

and substituting in Equation 1-46,

$$G(j\omega_b) = \frac{(\alpha^2G_r + K_r) + j(\alpha\beta I_i)}{(\beta^2D_r + \beta E_r + F_r) + j(\beta^2D_i + \beta E_i + F_i)} \qquad (1\text{-}53)$$

Setting the magnitude of $G(j\omega_b) = M$:

$$M^2 = |G(j\omega_b)^2| = \frac{(\alpha^2G_r + K_r)^2 + (\alpha\beta I_i)^2}{(\beta^2D_r + \beta E_r + F_r)^2 + (\beta^2D_i + \beta E_i + F_i)^2} \qquad (1\text{-}54)$$

Manipulating Equation (1-54) algebraically

$$\phi(\alpha,\beta) - M^2\theta(\alpha,\beta) = 0 \qquad (1\text{-}55)$$

where

$$\phi(\alpha,\beta) = \alpha^4G_r^2 + 2\alpha^2K_rG_r + K_r^2 + \alpha^2\beta^2I_i^2 \qquad (1\text{-}56)$$

$$\begin{aligned} \theta(\alpha,\beta) = {} & \beta^4D_r^2 + 2\beta^3D_rE_r + 2\beta^2D_rF_r + \beta^2E_r^2 + \\ & 2\beta E_rF_r + F_r^2 + \beta^4D_i^2 + \beta^2E_i^2 + F_i^2 + \\ & 2\beta^3E_iD_i + 2\beta^2D_iF_i + B^2E_i^2 + F_i^2 + \\ & 2\beta^3E_iD_i + 2\beta^2D_iF_i + 2\beta E_iF_i \end{aligned} \qquad (1\text{-}57)$$

Substituting Equations (1-56) and (1-57) in Equation (1-55) and defining:

$$P_1 = D_r^2 + D_i^2 \qquad (1\text{-}58)$$

$$Q_1 = 2D_rE_r + 2E_iD_i \qquad (1\text{-}59)$$

$$R_1 = 2D_rF_r + E_r^2 + E_i^2 + 2D_iF_i \qquad (1\text{-}60)$$

$$R_2 = \alpha^2I_i^2 \qquad (1\text{-}61)$$

$$V_1 = 2E_rF_r + 2E_iF_i \qquad (1\text{-}62)$$

$$W_1 = F_r^2 + F_i^2 \qquad (1\text{-}63)$$

$$W_2 = \alpha^4G_r^2 + 2\alpha^2K_rG_r + K_r^2 \qquad (1\text{-}64)$$

It follows that

$$M^2P_1\beta^4 + M^2Q_1\beta^3 + (M^2R_1 - R_2)\beta^2 + M^2V_1\beta +$$

$$(M^2W_1 - W_2) = 0 \qquad (1\text{-}65)$$

Since the Parameter Plane for compensation purposes has already been determined it is now a matter of taking the computed α values and substituting them along with a constant value of omega and M into Equation (1-65) and then solving the β quartic. This has as its solution the largest, real and positive value of the four roots as before.

1.5 EXTENSIONS TO HIGHER ORDER SYSTEMS

Although the work presented to this point has been limited to third order systems and the program written for this specific case, investigation shows that generalized equations may be written which will allow the extension of the program to higher ordered systems. It can be shown for a given n^{th} order system with no zeros to be compensated with two identical sections of cascade compensation, that the characteristic equation of the system may be generally written as:

$$s^{n+2} + 2ps^{n+1} + p^2s^n + (z^2s^2 + 2pzs + p^2)K +$$

$$2p \sum_{\substack{k=n\\ j=1}}^{\substack{j=n\\ k=1}} \left(\sum_j \Pi\, r_i\right)s^k + p^2 \sum_{\substack{k=n-1\\ j=1}}^{\substack{j=n\\ k=0}} \left(\sum_j \Pi\, r_i\right)s^k +$$

$$\sum_{\substack{k=n+1\\ j=1}}^{\substack{k=2\\ k=n}} \left(\sum_j \Pi\, r_i\right)s^k = 0 \qquad (1\text{-}66)$$

where for n=4 the equation would be written:

$$s^6 + 2ps^5 + p^2s^4 + (z^2s^2 + 2pzs + p^2)K +$$

$$2p\left(\sum_1 \Pi\, r_is^4 + \sum_2 \Pi\, r_is^3 + \sum_3 \Pi\, r_is^2 + \sum_4 \Pi\, r_is\right) +$$

$$p^2\left(\sum_1 \Pi\, r_is^3 + \sum_2 \Pi\, r_is^2 + \sum_3 \Pi\, r_is + \sum_4 \Pi\, r_i\right) +$$

$$\left(\sum_1 \Pi\, r_is^5 + \sum_2 \Pi\, r_is^4 + \sum_3 \Pi\, r_is^3 + \sum_4 \Pi\, r_is^2\right) = 0$$

(1-67)

It may be further shown that the parameters defined by Equations (1-17) through (1-26) may be written:

$$B_{21} = K\omega^2U_1(\xi) \qquad (1\text{-}68)$$

$$B_{22} = K\omega^2U_2(\xi) \qquad (1\text{-}69)$$

$$D_1 = -2K\omega U_o(\xi) \qquad (1\text{-}70)$$

$$D_2 = -2K\omega U_1(\xi) \qquad (1\text{-}71)$$

$$E_{11} = 2(-1)^{n+1}\omega^{n+1}U_n(\xi) + 2\sum_{\substack{k=n\\ j=1}}^{\substack{j=n\\ k=1}} \left[(-1)^k\omega^kU_{k-1}(\xi)\left(\sum_j \Pi\, r_i\right)\right] \qquad (1\text{-}72)$$

$$E_{12} = 2(-1)^{n+1}\omega^{n+1}U_{n+1}(\xi) + 2\sum_{\substack{k=n\\ j=1}}^{\substack{j=n\\ k=1}} \left[(-1)^k\omega^kU_k(\xi)\left(\sum_j \Pi\, r_i\right)\right] \qquad (1\text{-}73)$$

$$F_{21} = (-1)^n\omega^nU_{n-1}(\xi) + \sum_{\substack{k=n-1\\ j=1}}^{\substack{j=n\\ k=0}} \left[(-1)^k\omega^kU_{k-1}(\xi)\left(\sum_j \Pi\, r_i\right)\right] + KU_{-1}(\xi)$$

(1-74)

$$F_{22} = (-1)^n \omega^n U_n(\xi) + \sum_{\substack{k=n-1 \\ j=1}}^{\substack{j=n \\ k=0}} \left[(-1)^k \omega^k U_k(\xi) \left(\sum \prod_j x_i\right) \right] + K U_o(\xi) \qquad (1\text{-}75)$$

$$G_1 = (-1)^{n+2} \omega^{n+2} U_{n+1}(\xi) + \sum_{\substack{k=n+1 \\ j=1}}^{\substack{j=n \\ k=2}} \left[(-1)^k \omega^k U_{k-1}(\xi) \left(\sum \prod_j x_i\right) \right] \qquad (1\text{-}76)$$

$$G_2 = (-1)^{n+2} \omega^{n+2} U_{n+2}(\xi) + \sum_{\substack{k=n+1 \\ j=1}}^{\substack{j=n \\ k=2}} \left[(-1)^k \omega^k U_k(\xi) \left(\sum \prod_j x_i\right) \right] \qquad (1\text{-}77)$$

These then are the recursive equations required for the complete generalization to a n^{th} order system. By employing the above equations and replacing in PROGRAM PROJECT cards 100 through 150 and 300 through 540 with the appropriate programming, the program may be used for any given n^{th} order system.

In like manner by generally defining:

$$P(s) = \alpha^2 K s^2 + \alpha\beta K s + \beta^2 K \qquad (1\text{-}78)$$

and:

$$Q(s) = s^{n+2} + 2\beta s^{n+1} + \beta^2 s^n + 2 \sum_{\substack{k=n \\ j=1}}^{\substack{j=n \\ k=1}} \left(\sum \prod_j x_i\right) s^k +$$

$$\beta^2 \sum_{\substack{k=n-1 \\ j=1}}^{\substack{j=n \\ k=0}} \left(\sum \prod_j x_i\right) s^k + \sum_{\substack{k=n+1 \\ j=1}}^{\substack{j=n \\ k=2}} \left(\sum \prod_j x_i\right) s^k \qquad (1\text{-}79)$$

and using Equations (1-49) through (1-52) we may replace in the program cards 2860 and 2880 through 2920, thus adapting this part of the program to a general n^{th} order application.

1.6 COMMENTS

Throughout this development of the Parameter Plane quadratic extension, the c_k's in the generalized coefficient form:

$$\sum_{k=0}^{n} (b_k\alpha^2 + c_k\alpha + d_k\alpha\beta + e_k\beta + f_k\beta^2 + g_k) = 0 \qquad (1\text{-}80)$$

have been identically zero. This at first appearance might seem to detract from the generalization. The inclusion of this parameter does not however introduce any great difficulty in the solution. The change in the development would be to the value of P_1 and P_2 which would become:

$$P_1 = C_1 + \beta D_1 \qquad (1\text{-}81)$$

$$P_2 = C_2 + \beta D_2 \qquad (1\text{-}82)$$

and the final solution for α which would change to:

$$\alpha = -\frac{P_1}{2B_{21}} \pm \left[\frac{P_1^2 - 2B_{21}Q_1}{4B^2{}_{21}}\right]^{\frac{1}{2}} \qquad (1\text{-}83)$$

For this case, new selection rules for acceptable values of α would be used, and would be much like those presented for β.

Though the extension of the Parameter Plane to include the $\alpha - \beta$ quadratic case makes this tool even more useful, further work is still to be done in this field. Not only must the equations for the solutions of the Parameter Plane curves for such cases as:

$$a_k = b_k\alpha^2\beta^2 + c_k\alpha^2\beta + d_k\alpha\beta^2 + e_k\alpha^2 + f_k\beta^2 +$$

$$g_k\alpha\beta + h_k\alpha + q_k\beta + t_k \qquad [5]\ (1\text{-}84)$$

and higher ordered combinations of the parameters be developed, but more efficient programming techniques must be developed. In the use of PROGRAM PROJECT, for instance, as the location of the system poles on the σ axis of the S-plane move to the left, the computational time becomes excessive due to present programming technique and computer speed.

Another major problem in further extensions of these techniques, and indeed even other applications of the curves from the proceeding development, will be interpretation. In this case, the initial substitution of variables immediately allowed interpretation of the curves sight unseen. Here then, will be most likely the one single drawback to further extension, for as the parameters α and β are used as representations of other variables in control systems, each application will have its own unique interpretation.

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APPENDIX I

PROGRAM PROJECT is designed to solve the α quadratic and β quartic. The program is divided into two main sections, the first for the computation of the α-β points and the second for the bandwidth points.

The first section computes an 80 by 80 matrix of the α and β points corresponding to set values of ζ and ω. The computational part is followed by two distinct graphing sections, one for lag and the other for lead compensation.

The lag graphing section is set up so that during the plotting of the curves each value of α is tested to determine if its value is $10^{-7} \le \alpha \le 1.0001$. If no points are found within this range then a print out is made:

NO LAG COMPENSATION POSSIBLE

For the lead section graphs, α is again tested by the criterion $1.0001 \le \alpha \le$ (X-graph scale)(X graph width). Again if there are no values of α within this region the statement:

NO LEAD COMPENSATION POSSIBLE

is printed. In this case however a study of the printed values of α must be made to insure that the points are indeed non-existant or rather just lie outside the range of the graph.

The second main section of the program computes the value of β for a given value of α is determined by the X graph scale. Here the plotting routine is set up so as to not plot zero points and to stop the curve when either the α or β value exceeds the range of the graph.


```
      PROGRAM PROJECT                                                   000000
C   THIS PROGRAM COMPUTES THE VALUES OF BETA(POLE LOCATION) AND ALFA(POLE
C      -ZERO RATIO) BASED ON PARAMETER PLANE TECHNIQUES.  THE COMPUTED  .
C       ALFA AND BETA VALUES ARE THOSE REQUIRED TO PLACE THE ROOTS OF ANY
C       THIRD ORDER SYSTEM, TYPE 0,1,2 OR 3, AT A DESIRED ZETA AND OMEGA
C      LOCATION WHILE MAINTAINING A CONSTANT VELOCITY COEFFICIENT.  AFTER
C       COMPUTING THE VALUES IT WILL PLOT THE PARAMETER PLANE CONSTANT ZETA
C      CURVES FROM 0.0 TO 0.9 AND THE CONSTANT OMEGA CURVES FOR EVEY ONE
C       TENTH OF THE VALUE OF THE MAXIMUM VALUE OF OMEGA USED.  A 9 BY 15
C       INCH GRAPH IS OUTPUT BY THE ROUTINE.  THIS IS DONE
C       ON TWO SEPARATE GRAPHS, ONE FOR POSSIBLE LAG COMPENSATION AND ONE
C      FOR LEAD COMPENSATION.  THE PROGRAM THEN HAS THE ADDITIONAL FEATURE
C       OF COMPUTING AND PLOTTING THE CONSTANT BANDWIDTH CURVES.
C   THE FOLLOWING FEATURES ARE AVAILABLE WITH THE PROPER USE OF THE DATA
C       CARDS.
C      1. THE ALFA-BETA COMPUTATIONS MAY OR MAY NOT BE DONE.
C      2. LAG COMPENSATION MAY OR MAY NOT BE PLOTTED.
C      3. LEAD COMPENSATION MAY OR MAY NOT BE PLOTTED.
C      4. BANDWIDTH COMPUTATIONS MAY OR MAY NOT BE COMPLETED.
C      5. BANDWIDTH CURVES MAY OR MAY NOT BE PLOTTED. (AVAILABLE ONLY IF
C          THE BANDWIDTH COMPUTATIONS HAVE BEEN MADE)
C   THE FOLLOWING DATA CARDS ARE REQUIRED.
C***CARD ONE - A,B,C,G - TEN COLUMNS PER NUMBER IN FLOATING POINT.
C         THESE ARE THE LOCATIONS OF THE UNCOMPENSATED POLES AND THE
C      UNCOMPENSATED ROOT LOCUS GAIN.
C***CARD TWO - WFIN - TEN COLUMNS IN FLOATING POINT.
C      THIS IS THE MAXIMUM VALUE OF OMEGA TO BE USED.
C***CARD THREE - IABCMP - COLUMN ONE IN FIXED POINT
C      0 - THE ALFA-BETA COMPUTATIONS WILL BE DONE
C      1 - THE ALFA-BETA COMPUTATIONS WILL NOT BE DONE
C                             *****
C    IF IABCMP=1 CARDS FOUR THROUGH THIRTEEN ARE OMITTED
C                             *****
C***CARD FOUR - ILGPLT - COLUMN ONE IN FIXED POINT
C      0 - THE LAG ZONE CURVES WILL BE PLOTTED
```

```
C      1 - THE LAG ZONE CURVES WILL NOT BE PLOTTED
C                   ******
C    IF ILGPLT=1 THE NEXT FOUR CARDS ARE OMITTED
C                   ******
C***CARD FIVE -IT(1)-IT(6) - COLUMNS 1-48 IN ALFANUMERIC CHARACTERS
C     THIS IS THE FIRST LINE OF THE LAG GRAPH TITLE
C***CARD SIX - IT(7)-IT(12) - COLUMNS 1-48 IN ALFANUMERIC CHARACTERS
C     THIS IS THE SECOND LINE OF THE LAG GRAPH TITLE.
C***CARD SEVEN - LBL(11)-LBL(20) - FOUR COLUMNS PER LABEL (TEN LABELS IN
C      CONSECUTIVE COLUMNS) IN ALFANUMERIC CHARACTERS.
C     THESE ARE THE LABELS TO BE PUT ON THE CONSTANT OMEGA CURVES.   TO
C      DETERMINE WHICH VALUES WILL BE PLOTTED, DIVIDE WFIN BY 10 .  THIS
C      VALUE AND INTEGER MULTIPLES OF IT TO 10 WILL BE PLOTTED.
C***CARD EIGHT - XLGZ,YLGZ - TEN COLUMNS PER NUMBER IN EXPONENTIAL OR
C      FLOATING POINT.
C     THESE ARE THE X AND Y SCALES FOR THE LAG GRAPH.  ONLY ONE SIGNI-
C     FICANT NUMBER IS TO BE USED.
C***CARD NINE - ILDPLT - COLUMN ONE IN FIXED POINT
C      0 - THE LEAD CURVES WILL BE PLOTTED
C      1 - THE LEAD CURVES WILL NOT BE PLOTTED
C                   ******
C    IF ILDPLT=1 THE NEXT FOUR CARDS ARE OMITTED
C                   ******
C***CARD TEN - THE SAME AS CARD FIVE EXCEPT FOR THE LEAD GRAPH
C***CARD ELEVEN - THE SAME AS CARD SIX EXCEPT FOR THE LEAD GRAPH
C***CARD TWELVE - THE SAME AS CARD EIGHT EXCEPT FOR THE LEAD GRAPH
C***CARD THIRTEEN - A***DUPLICATE*** OF CARD SEVEN
C***CARD FOURTEEN - IBWCMP - COLUMN ONE FIXED POINT
C      0 - BANDWIDTH COMPUTATIONS AND GRAPHING WILL NOT BE DONE.
C      1 - BANDWIDTH COMPUTATIONS WILL BE DONE
C                   ******
C    IF IBWCMP=0 THE REMAINING CARDS ARE OMITTED
C                   ******
C***CARD FIFTEEN - BWX,BWY - THE SAME AS CARD EIGHT EXCEPT FOR THE
C     BANDWIDTH CURVES.
C      BWY IS ALSO USED TO DETERMINE WHICH VALUES OF ALFA WILL BE USED IN
C      THE BANDWIDTH COMPUTATIONS.
C***CARD SIXTEEN - WEND - TEN COLUMNS IN FLOATING POINT
C      THIS IS THE MAXIMUM VALUE OF OMEGA FOR WHICH THE BANDWIDTH
C      COMPUTATIONS WILL BE DONE
C***CARD SEVENTEEN - IBWPLT - COLUMN ONE IN FIXED POINT
C      0 - THE BANDWIDTH CURVES WILL BE PLOTTED
C      1 - THE BANDWIDTH CURVES WILL NOT BE PLOTTED
C                   ******
C    IF IBWPLT=1 THE REMAINING CARDS ARE OMITTED
C                   ******
C***CARD EIGHTEEN - THE SAME AS CARD FIVE EXCEPT FOR THE BANDWIDTH CURVES
C***CARD NINETEEN - THE SAME AS CARD SIX EXCEPT FOR THE BANDWIDTH CURVES
C***CARD TWENTY - BANDWIDTH CURVE LABELS
C      TO DETERMINE WHICH CURVES WILL BE PLOTTED, DIVIDE  WEND  BY 10  .
C      THE PROGRAM PLOTS THIS CURVE AND INTEGER MULTIPLES OF IT UP TO 10.
C
C
C    IT IS RECOMMENDED THAT FOR THE INITIAL RUN THE FOLLOWING DATA CARDS
C      BE USED.
C         CARDS 1,2,3(IABCMP=0),4(ILGPLT=1),9(ILDPLT=1),14(IBWCMP=0)
C
C    THESE DATA CARDS WILL ALLOW ONLY THE ALFA-BETA COMPUTATIONS TO BE
C      COMPLETED.  A PRINT OUT OF THE VALUES WILL BE OUTPUT WHICH WILL ALLOW
C      YOU TO CHOOSE THE PROPER CURVES AND SCALES.  CAREFUL SELECTION
C      OF CURVE SCALES IS IMPORTANT, FOR THE PROGRAM WILL NOT ALLOW POINTS
C      OUTSIDE THE AXIX LIMITS TO BE PLOTTED.
C
      DIMENSION AFIN(80,80),BFIN(80,80),XAZ(80),YBZ(80),XAW(80),        000010
     1 YBW(80),IT(12),LBL(20),BCOFI(5),ROOTR(4),ROOTI(4),ACOFI(3),      000020
     2 U(10),AROOTI(4),ACOFR(3),BCOFR(5),WLAB(80),ZLAB(80),AROOTR(4)    000030
      COMMON BCOFR,BCOFI,ROOTR,ROOTI,BFINAL,IFLAG,AFIN,BFIN             000040
 9999 PRINT 140                                                         000050
  140 FORMAT (1H1)                                                      000060
      DO 60 JK=1,6400                                                   000070
      AFIN(JK) =0.0                                                     000080
   60 BFIN(JK) = 0.0                                                    000090
```

```
      READ 1,A,B,C,G
    1 FORMAT(4F10.0)
      PROD = A*B*C
      SUM = A+B+C
      SMPRD = A*B + A*C + B*C
      PRDGN = PROD + G
      ZETA = 0.0
      READ 2,WFIN
    2 FORMAT (F10.0)
      READ 9, IABCMP
    9 FORMAT (I1)
      IF(IABCMP-1)23,24,24
   23 STP = WFIN/80..
      DO 12 L = 1,80
      LJ = 80*(L-1)
      W = STP
   30 U(1)=-1.
      U(2)=0.
      U(3)=1.
      DO 10 N=2,6
   10 U(N+2)=2.*ZETA*U(N+1)-U(N)
      DO 11 J=1,80
      LJ = LJ + 1
      W2=W*W
      W3=W2*W
      W4=W2*W2
      W5=W2*W3
      CONN = G*W2
      CON = -2.*G*W
      CON1 = 2.*W4
      CON2 = -2.*SUM*W3
      CON3 = 2.*SMPRD*W2
      CON4 = -2.*PROD*W
      CON5 = SUM*W2
      CON6 = -SMPRD*W
      CON7 = SUM*W4
```

```
      CON8 = -SMPRD*W3
      CON9 = PROD*W2
      B21 = CONN*U(3)
      B22 = CONN*U(4)
      D1 = CON*U(2)
      D2 = CON*U(3)
      E11 = CON1*U(5) + CON2*U(4) + CON3*U(3) + CON4*U(2)
      E12 = CON1*U(6) + CON2*U(5) + CON3*U(4) + CON4*U(3)
      F21 = -W3*U(4) + CON5*U(3) + CON6*U(2) + PRDGN*U(1)
      F22 = -W3*U(5) + CON5*U(4) + CON6*U(3) + PRDGN*U(2)
      G1 = -W5*U(6) + CON7*U(5) + CON8*U(4) + CON9*U(3)
      G2 = -W5*U(7) + CON7*U(6) + CON8*U(5) + CON9*U(4)
      COF1 = B21*F22*(2.*F21*B22-F22*B21)-F21*(F21*B22*B22+D2*D2*B21)
      COF2 = E11*(2.*B22*(F22*B21-F21*B22)-D2*D2*B21)+2.*E12*B21*(F21*B2
     12-F22*B21)
      COF3 = B21*(-B21*(E12*E12+2.*F22*G2)-D2*D2*G1+2.*B22*(E11*E12+F21*
     1 G2+G1*F22))-B22*B22*(E11*E11+2.*F21*G1)
      COF4=2.*G2*B21*(E11*B22-E12*B21)+2.*B22*G1*(E12*B21-E11*B22)
      COF5= -(G2*B21-G1*B22)*(G2*B21-G1*B22)
      DO 50 I =1,5
   50 BCOFI(I) = 0.0
      BCOFR(1) = 1.0
      BCOFR(2) = COF2/COF1
      BCOFR(3) = COF3/COF1
      BCOFR(4) = COF4/COF1
      BCOFR(5) = COF5/COF1
      CALL ABETART
      IFLAG = 0
      CALL SORT
      IF (IFLAG-1)300,11,11
  300 BFIN(LJ) = BFINAL
      Q1 = BFIN(LJ)*(E11+BFIN(LJ)*F21)+G1
      ACOFR(1)=1.0
      ACOFR(2)=0.0
      ACOFR(3) = Q1/B21
      ALFASQ = ABSF(ACOFR(3))
```

```
      AFIN(LJ) = SQRTF(ALFASQ)                                          000790
   11 W = W+STP                                                         000800
   12 ZETA = ZETA + .0125                                               000810
      LBL(1) = 4HZ=.0                                                   000820
      LBL(2) = 4HZ=.1                                                   000830
      LBL(3) = 4HZ=.2                                                   000840
      LBL(4) = 4HZ=.3                                                   000850
      LBL(5) = 4HZ=.4                                                   000860
      LBL(6) = 4HZ=.5                                                   000870
      LBL(7)   4HZ=.6                                                   000880
      LBL(8) = 4HZ=.7                                                   000890
      LBL(9) = 4HZ=.8                                                   000900
      LBL(10) = 4HZ=.9                                                  000910
      READ 7, ILGPLT                                                    000920
    7 FORMAT (I1)                                                       000930
      IF(ILGPLT-1)8,67,67                                               000940
    8 READ 3, (IT(I),I=1,12)                                            000950
    3 FORMAT (6A8)                                                      000960
      READ 6, (LBL(I),I=11,20)                                          000970
    6 FORMAT (10A4)                                                     000980
      READ 4, XLGZ,YLGZ                                                 000990
    4 FORMAT (2E10.0)                                                   001000
      XLGLM = 9.*XLGZ                                                   001010
      YLGLM = 15.*YLGZ                                                  001020
      MODE = 1                                                          001030
      IL = 0                                                            001040
      DO 62 K=1,80,8                                                    001050
      LL = 1                                                            001060
      KJ = (K-1)*80                                                     001070
      DO 61 J=1,80                                                      001080
      KJ = KJ+1                                                         001090
      IF(AFIN(KJ)-.0000001)61,6110,6110                                 001095
 6110 IF(AFIN(KJ)-1.0001)6113,61,61                                     001100
 6113 IF(AFIN(KJ) - XLGLM)6114,61,61                                    001110
C         CARDS 1120 - 1130 ARE MISSING
 6114 XAZ(LL) = AFIN(KJ)                                                001140

      IF(BFIN(KJ) - YLGLM)6112,61,61                                    001150
C         CARDS 1160 - 11.0 ARE MISSING
 6112 YBZ(LL) = BFIN(KJ)                                                001180
      LL = LL + 1                                                       001190
   61 CONTINUE                                                          001200
      JJ = LL - 1                                                       001210
      IL = IL + 1                                                       001220
      IF(JJ-1)62,62,6116                                                001230
 6116 LAL = LBL(IL)                                                     001240
      CALL DRAW(JJ,XAZ,YBZ,MODE,0,LAL,IT,XLGZ,YLGZ,0,0,0,0,9,15,0,LAST) 001250
 6111 MODE = 2                                                          001260
   62 CONTINUE                                                          001270
      IF(MODE-1)65,65,6120                                              001280
 6120 DO 66 K=8,80,8                                                    001290
      LL = 1                                                            001300
      DO 63 J=1,80                                                      001310
      JK = (J-1)*80 + K                                                 001320
      IF(AFIN(JK)-.0000001)63,6127,6127                                 001325
 6127 IF(AFIN(JK)-1.0001)6123,63,63                                     001330
 6123 IF(AFIN(JK) - XLGLM)6124,63,63                                    001340
C         CARDS 1350 - 1360 ARE MISSING
 6124 XAW(LL) = AFIN(JK)                                                001370
      IF(BFIN(JK) - YLGLM)6122,63,63                                    001380
C         CARDS 1390 - 1400 ARE MISSING
 6122 YBW(LL) = BFIN(JK)                                                001410
      LL = LL + 1                                                       001420
   63 CONTINUE                                                          001430
      JJ = LL - 1                                                       001440
      IL = IL + 1                                                       001450
      IF(JJ-1)6121,6121,6126                                            001460
 6121 IF(K-80)66,6125,6125                                              001470
 6125 MODE = 3                                                          001480
      LAL = 4H                                                          001490
      JJ = 2                                                            001500
      XAW(1) = XLGLM                                                    001510
      XAW(2) = XLGLM                                                    001520
```

```
      YBW(1) = 0.0                                                         001530
      YBW(2) = YLGZ                                                        001540
      GO TO 2000                                                           001550
 6126 LAL = LBL(IL)                                                        001560
 2000 CALL DRAW(JJ,XAW,YBW,MODE,0,LAL,IT,XLGZ,YLGZ,0,0,0,0,9,15,0,LAST)    001570
      MODE = 2                                                             001580
      IF(K-7 )66,64,64                                                     001590
   64 MODE = [illegible]                                                   001600
   66 CONTINUE                                                             001610
      GO TO 67                                                             001620
   65 PRINT 130                                                            001630
  130 FORMAT (1X,33H NO LAG COMPENSATION IS POSSIBLE  ,///)                001640
   67 READ 20, ILDPLT                                                      001650
   20 FORMAT (I1)                                                          001660
      IF(ILDPLT-1)68,1000,1000                                             001670
   68 READ 5, (IT(I),I=1,12)                                               001680
    5 FORMAT (6A8)                                                         001690
      READ 21, XLDZ,YLDZ                                                   001700
   21 FORMAT (2E10.0)                                                      001710
      READ 22, (LBL(I),I=11,20)                                            001713
   22 FORMAT (10A4)                                                        001716
      XLDLM = 9.*XLDZ                                                      001720
      YLDLM = 15.*YLDZ                                                     001730
      IL = 0                                                               001740
      MODE = 1                                                             001750
      DO 72 K = 1,80,8                                                     001760
      KJ = (K-1)*80                                                        001770
      KK = 1                                                               001780
      DO 71 J = 1,80                                                       001790
      KJ = KJ + 1                                                          001800
      IF(AFIN(KJ)-1.0001)71,7111,7111                                      001810
 7111 IF(AFIN(KJ) - XLDLM)7117,71,71                                       001820
C       CARDS 1830 - 1840 ARE MISSING
 7117 XAZ(KK) = AFIN(KJ)                                                   001850
      IF(BFIN(KJ) - YLDLM)7118,71,71                                       001860
C       CARDS 1870 - 1880 ARE MISSING


 7118 YBZ(KK) = BFIN(KJ)                                                   001890
      KK = KK + 1                                                          001900
   71 CONTINUE                                                             001910
      MM = KK-1                                                            001920
      IL = IL + 1                                                          001930
      IF(MM-1)72,72,7119                                                   001940
 7119 LAL = LBL(IL)                                                        001950
      CALL DRAW(MM,XAZ,YBZ,MODE,0,LAL,IT,XLDZ,YLDZ,0,0,0,0,9,15,0,LAST)    001960
 7110 MODE = 2                                                             001970
   72 CONTINUE                                                             001980
      IF(MODE-1)70,70,78                                                   001990
   78 DO 76 K=8,80,8                                                       002000
      KK = 1                                                               002010
      DO 73 J = 1,80                                                       002020
      JK = (J-1)*80 + K                                                    002030
      IF(AFIN(JK)-1.0001)73,7121,7121                                      002040
 7121 IF(AFIN(JK) - XLDLM)7127,73,73                                       002050
C       CARDS 2060 - 2070 ARE MISSING
 7127 XAW(KK) = AFIN(JK)                                                   002080
      IF(BFIN(JK) - YLDLM)7128,73,73                                       002090
C       CARDS 2100 - 2110 ARE MISSING
 7128 YBW(KK) = BFIN(JK)                                                   002120
      KK = KK + 1                                                          002130
   73 CONTINUE                                                             002140
      MM = KK-1                                                            002150
      IL = IL + 1                                                          002160
      IF(MM-1)7120,7120,7129                                               002170
 7120 IF(K-80)76,7122,7122                                                 002180
 7122 MODE = 3                                                             002190
      LAL = 4H                                                             002200
      MM = 2                                                               002210
      XAW(1) = XLDLM                                                       002220
      XAW(2) = XLDLM                                                       002230
      YBW(1) = 0.0                                                         002240
      YBW(2) = YLDZ                                                        002250
      GO TO 2001                                                           002260
```

```
7179 LAL = LBL(II)
2001 CALL DRAW(I,XAW,YBW,MODE,0,LAL,IT,XLD7,YLD7,0,0,0,0,9,15,0,LAST)
     MODE = 2
     IF(K-72)76,75,75
  75 MODE = 3
  76 CONTINUE
     GO TO 1000
  70 PRINT 131
 131 FORMAT (1X, 4H NO LEAD COMPENSATION IS POSSIBLE  ,///)
1000 CONTINUE
     ZLAB(1) = 0.0
     DO 81 I=2,10
  81 ZLAB(I) = ZLAB(I-1) + .1
     WLAB(8) = 8.*STP
     DO 82 N=16,80,8
  82 WLAB(N) = WLAB(N-8) + WLAB(8)
     PRINT 100
 100 FORMAT (1H1)
     PRINT 101
 101 FORMAT(2X,20H THE ALFA VALUES ARE  ,//)
     PRINT 102, (ZLAB(I) I=1,10)
 102 FORMAT (1X,6H ZETA    10F11.6)
     PRINT 103, (WLAB(J),( IN(J,I),I=1,73,8)J=8,80,8)
 103 FORMAT (/,1X,F6.2,10E11.5)
     PRINT 111
 111 FORMAT (/////2X,20H THE BETA VALUES ARE      ,//)
     PRINT 112, (ZLAB(I) I=1,10)
 112 FORMAT (1X,6H ZETA    ,10F11.6)
     PRINT 113, (WLAB(J),(BFIN(J,I),I=1,73,8)J=8,80,8)
 113 FORMAT (/,1X,F6.2,10E11.5)
  74 PRINT 114
 114 FORMAT (1H1)
     READ 218, IBWCMP
 218 FORMAT (I1)
     IF(IBWCMP-1)1002,219,219
 219 READ 217, BWX,BWY
 217 FORMAT (2E10.0)
     READ 223, WEND
 223 FORMAT (F10.0)
     YBWLM = 15.*BWY
     XLM = 9.*BWX
     AM2=.5
     STEP = WEND/20.
     W = STEP
     ALFASP = XLM/20.
     XAW(1) = ALFASP
     DO 200 K=2,20
 200 XAW(K) = XAW(K-1) + XAW(1)
     DO 203 N=1,20
     ALFA = ALFASP
     DO 210 M=1,20
     XAZ(M) = 0.0
 210 YBZ(M) = 0.0
     DO 207 I=1,20
     YBW(N) = W
     W2=W*W
     W3=W*W2
     W4=W2*W2
     W5=W2*W3
     ACR = -W2*G
     AKR = G
     ADR=2.*W4-2.*W2*SMPRD
     AII = 2.*G*W
     AER=-W2*SUM + PROD
     AFR=W4*SUM - W2*PROD
     ADI=-2.*W3*SUM + 2.*W*PROD
     AEI = -W3 + W*SMPRD
     AFI = W5 - W3*SMPRD
     P1 = ADR*ADR+ADI*ADI
     Q1 = 2.*ADR*AER + 2.*AEI*ADI
     R1 = 2.*ADR*AFR + AER*AER + AEI*AEI + 2.*ADI*AFI
     V1 = 2.*AER*AFR + 2.*AEI*AFI
```

```
      W1 = AFR*AFR + AFI*AFI                                           002970
      A2 = ALFA*ALFA                                                   002980
      A4 = A2*A2                                                       002990
      R2 = A2*AII*AII                                                  003000
      WW2 = A4*AGR*AGR + 2.*A2*AKR*AGR + AKR*AKR                       003010
      DO 51 N = 1,5                                                    003020
   51 BCOFI(N) = 0.0                                                   003030
      BCOFR(1) = 1.0                                                   003040
      BCOFR(2) = Q1/P1                                                 003050
      BCOFR(3) = (AM2*R1-R2)/(AM2*P1)                                  003060
      BCOFR(4) = V1/P1                                                 003070
      BCOFR(5) = (AM2*W1-WW2)/(AM2*P1)                                 003080
      CALL ABETART                                                     003090
      IFLAG = 0                                                        003100
      CALL SORT                                                        003110
      IF(IFLAG-1)900,206,206                                           003120
  206 BFIN(N,I) = 0.0                                                  003130
      GO TO 207                                                        003140
  900 BFIN(N,I) = BFINAL                                               003150
  207 ALFA = ALFA + ALFASP                                             003160
  203 W = W + STEP                                                     003170
      READ 216, IBWPLT                                                 003180
  216 FORMAT (I1)                                                      003190
      IF(IBWPLT-1)214,1001,10(                                         003200
  214 MODE = 1                                                         003210
      READ 202,(IT(K),K=1,12)                                          003220
  202 FORMAT (6A8)                                                     003230
      READ 201, (LBL(N),N=2,20,2)                                      003240
  201 FORMAT (10A4)                                                    003250
      DO 211 N=2,20,2                                                  003260
      KK = 1                                                           003270
      IF(N-20)204,205,20!                                              003280
  205 MODE = 3                                                         00 .290
  204 CONTINUE                                                         003300
      DO 212 I=1,20                                                    003310
      IF(BFIN(N,I) - .000001)212,212,209                               003320



  209 IF(BFIN(N,I) - YBWLM)905,906,906                                 003330
  905 YBZ(KK) = BFIN(N,I)                                              003340
      XAZ(KK) = XAW(I)                                                 003350
       CARD 3360 IS MISSING
      KK = KK + 1                                                      003370
  212 CONTINUE                                                         003380
  906 JJ = KK - 1                                                      003390
      IF(JJ-1)221,221,220                                              003400
  221 IF(N-20)211,225,225                                              003410
  225 IF(MODE-1)224,224,215                                            003415
  215 MODE = 3                                                         003420
      LAL = 4H                                                         003430
      XAZ(1) = 0.0                                                     003440
      XAZ(2) = 0.0                                                     003450
      YBZ(1) = 0.0                                                     003460
      YBZ(2) = BWY                                                     003470
      JJ = 2                                                           003480
      GO TO 2002                                                       003490
  220 LAL = LBL(N)                                                     003500
 2002 CALL DRAW (JJ,XAZ,YBZ,MODE,0,LAL,IT,BWX,BWY,0,0,0,0,9,15,0,LAST) 003510
  222 IF(N-20)208,211,211                                              003520
  208 MODE = 2                                                         003530
  211 CONTINUE                                                         003540
      GO TO 1001                                                       003541
  224 PRINT 226                                                        003542
  226 FORMAT (1H1,1X,76HTHERE ARE NO POSSIBLE BANDWIDTH CURVES FOR THE F 003543
     1INAL VALUE OF OMEGA STATED                                       003544
      GO TO 1002                                                       003545
 1001 PRINT 120                                                        003550
  120 FORMAT (1H1,20X,38H VALUES OF BETA FOR CONSTANT BANDWIDTH        003560
      PRINT 122, (YBW(K),K=2,20,2)                                     003570
  122 FORMAT (/,9X,10F11.6)                                            003580
      PRINT 121, (XAW(K),(BF W(J,K),J=2,20,2),K=1,20)                  003590
  121 FORMAT (/,1. 6,2,2X,10E11.6)                                     003600
 1002 CONTINUE                                                         003610
      GO TO 9999                                                       003615
```

```
      END

      SUBROUTINE ARETART
      DIMENSION A(5),YIMAG(5),U(4),V(4),H(50),B(50),C(50),D(50),E(50)
     1 ,CONV(50)
      DIMENSION AFIN(80,80),BFIN(80,80)
      COMMON A,YIMAG,U,V,DUMMY1,DUMMY2,AFIN,BFIN
      N = 4
      F=10.0
      L=25
      IER=0
      IF(N) 54,54,52
   54 IER=1
   52 NP3=N+3
  100 B(2)=0.0
      B(1)=0.0
      C(2)=0.0
      C(1)=0.0
      D(2)=0.0
      E(2)=0.0
      H(2)=0.0
      DO 101 J=3,NP3
  101 H(J)=A(J-2)
      T=1.0
      SK=10.0**F
  150 IF(H(NP3)) 200,151,200
  151 U(NP3)=0.0
      V(NP3)=0.0
      CONV(NP3)=SK
      NP3=NP3-1
      IF(NP3)152,152,150
  152 IER=1
  200 IF(NP3-3)205,51,201
  205 IER=1
  201 PS=0.0
      QS=0.0
      PT=0.0
      QT=0.0
      S=0.0
      REV=1.0
      SK=10.0**F
      IF(NP3-4)206,202,203
  206 IER=1
  202 R=-H(4)/H(3)
      GO TO 500
  203 DO 207 J=3,NP3
      IF(H(J))204,207,204
  204 S=S+LOGF(ABSF(H(J)))
  207 CONTINUE
      FPN1=N+1
      S=EXPF(S/FPN1)
      DO 208 J=3,NP3
  208 H(J)=H(J)/S
  210 IF(ABSF(H(4)/H(3))-ABSF(H(NP3-1)/H(NP3)))250,252,252
  250 T=-T
      M=(NP3-4)/2 + 3
      DO 251 J=3,M
      S=H(J)
      JJ=NP3-J+3
      H(J)=H(JJ)
  251 H(JJ)=S
  252 IF(QS) 253,254,253
  253 P=PS
      Q=QS
      GO TO 300
  254 HH2=H(NP3-2)
      IF(HH2) 256,255,256
  255 Q=1.0
      P=-2.0
      GO TO 257
  256 Q=H(NP3)/HH2
```

```
      P=(H(NP3-1)-Q*H(NP3-3))/HH2
  257 IF(NP3-5)258,550,258
  258 R=0.0
  300 DO 490 I=1,L
  350 DO 351 J=3,NP3
      B(J)=H(J)-P*B(J-1)-Q*B(J-2)
  351 C(J)=B(J)-P*C(J-1)-Q*C(J-2)
      IF(H(NP3-1))352,400,352
  352 IF(B(NP3-1))353,400,353
  353 AVHB1=ABSF(H(NP3-1)/B(NP3-1))
  356 IF(AVHB1-SK)450,354,354
  354 B(NP3)=H(NP3)-Q*B(NP3-2)
  400 IF(B(NP3))401,550,401
  401 AVHB2=ABSF(H(NP3)/B(NP3))
  403 IF(SK-AVHB2)550,450,450
  450 DO 451 J=3,NP3
      D(J)=H(J)+R*D(J-1)
  451 E(J)=D(J)+R*E(J-1)
      IF(D(NP3))452,500,452
  452 AVHD3=ABSF(H(NP3)/D(NP3))
  430 IF(SK-AVHD3)500,453,453
  453 CC2=C(NP3-2)
      CC3=C(NP3-3)
      C(NP3-1)=-P*CC2-Q*CC
      CC1=C(NP3-1)
      S=CC2*CC2-CC1*CC3
      IF(S)455,454,455
  454 P=P-2.0
      Q=Q*(Q+1.0)
      GO TO 456
  455 P=P+(B(NP3-1)*CC2-B(NP3)*CC3)/S
      Q=Q+(-B(NP3-1)*CC1+B(NP3)*CC2)/S
  456 IF(E(NP3-1))458,457,458
  457 R=R-1.0
      GO TO 490
  458 R=R-D(NP3)/E(NP3-1)
```

```
  490 CONTINUE
      PS=PT
      QS=QT
      PT=P
      QT=Q
      IF(REV)491,492,492
  491 SK=SK/10.
  492 REV=-REV
      GO TO 250
  500 IF(T)501,502,502
  501 R=1.0/R
  502 NP=NP3-3
      U(NP)=R
      V(NP)=0.0
      CONV(NP)=SK
      NP3=NP3-1
      DO 503 J=3,NP3
  503 H(J)=D(J)
      IF(NP3-3)300,51,300
  550 IF(T)551,552,552
  551 P=P/Q
      Q=1.0/Q
  552 PP2=P/2.0
      QMPSQ=Q-PP2*PP2
  560 IF(QMPSQ)554,554,553
  553 NP=NP3-3
      U(NP)=-PP2
      U(NP-1)=-PP2
      S=SQRTF(QMPSQ)
      V(NP)=S
      V(NP-1)=-S
      GO TO 561
  554 S=SQRTF(-QMPSQ)
      NP=NP3-3
      IF(P)555,556,556
  555 U(NP)=-PP2+S
```

```
      GO TO 557
      U(NP)=-PP?-S
      U(NP-1) OAU(NP)
      V(NP)=0.0
      V(NP-1) 0.0
      CONV(NP)=SK
      CONV(NP-1)=SK
      NP3=NP3-2
      DO 558 J=3,NP3
    3 H(J)-B(J)
      GO TO 200
   51 RETURN
      END

      SUBROUTINE SORT
      DIMENSION RR(4),RI(4),REAL(5),YIMAG(5)
      DI ENSION AFIN(80,80),BFIN(80,80)
      COMMON REAL,YIMAG,RR,RI,BF,IFLAG,AFIN,BFIN
      B = 0.0
      DO 800 I=1,4
      IF(ABSF(RI(I))-1.E-7)801,800,800
  801 B = MAX1F(B,RR(I))
  800 CONTINUE
      IF(B) 802,802,803
  802 PRINT 804
  804 FORMAT (34H THERE ARE NO POSITIVE REAL ROOTS
      IFLAG = 1
       RETURN
  803 BF = B
       RETURN
      END
      END
```

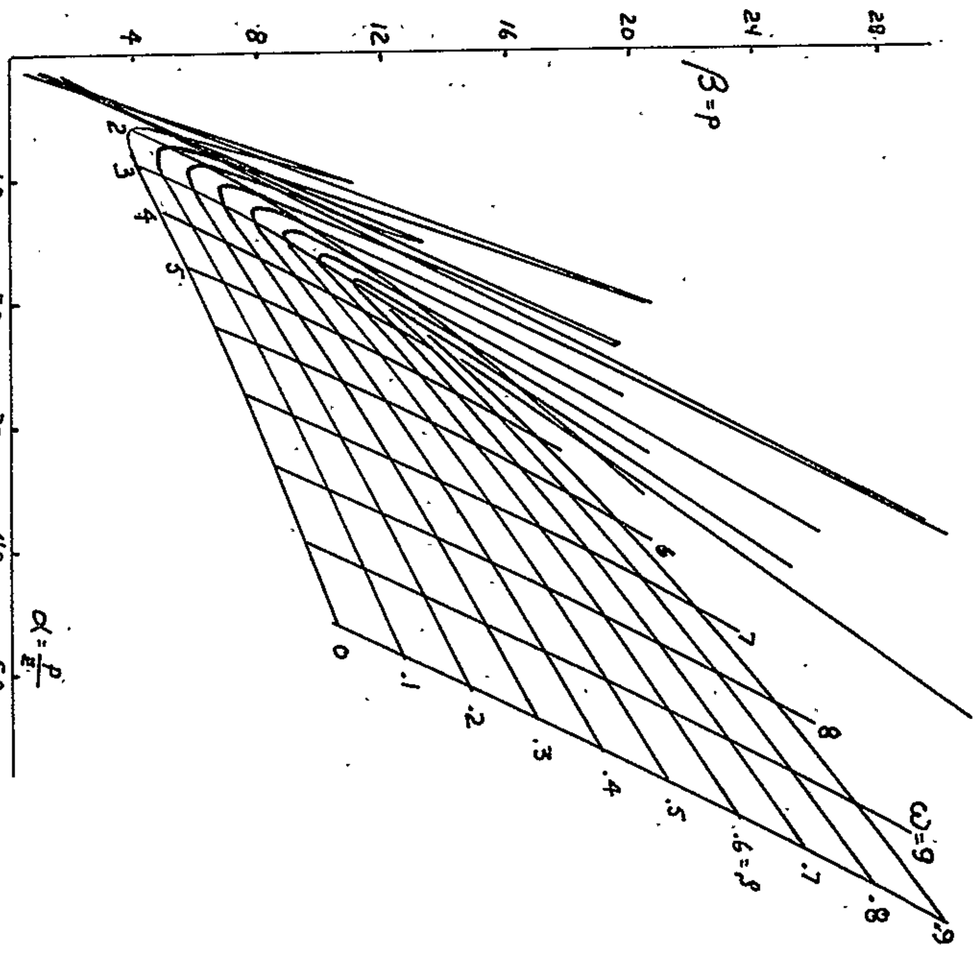

Fig. 1-1. Double Lead Compensation of Plant with $G(s) = \frac{1}{s^3}$

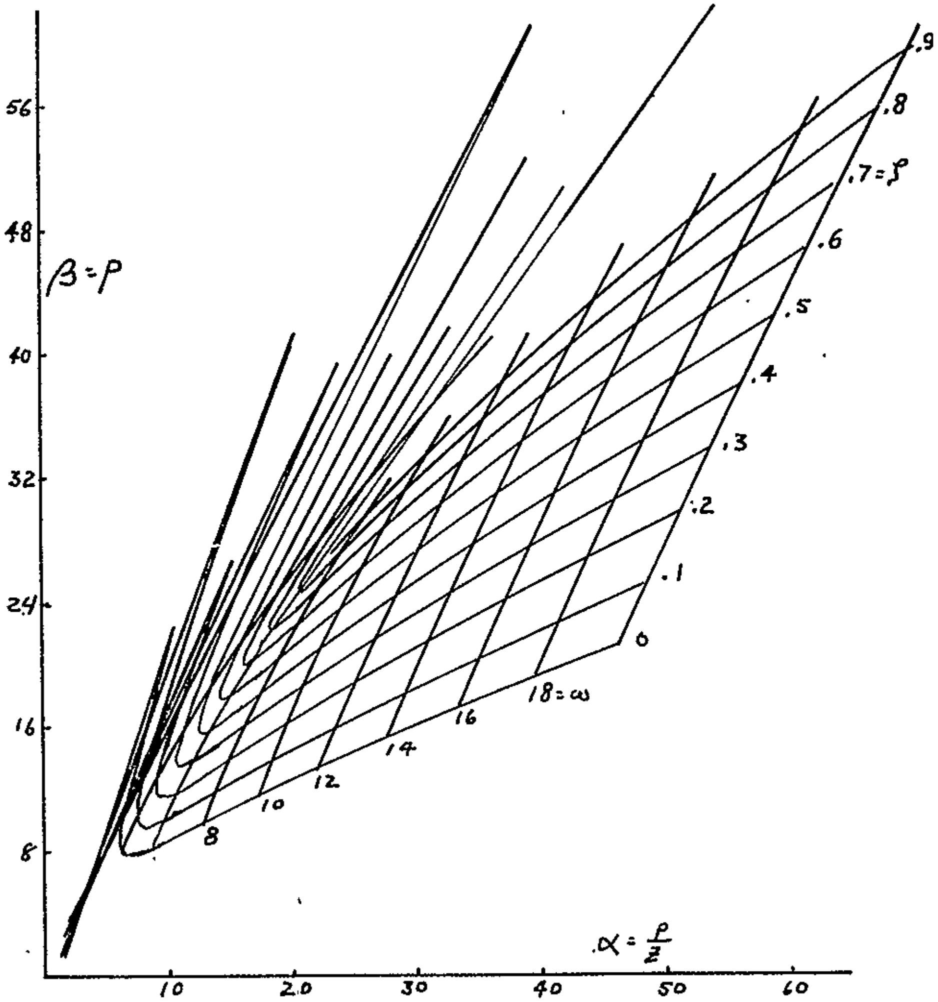

Fig. 1-2. Double Lead Compensation of Plant with $G(s) = \frac{8}{s^3}$

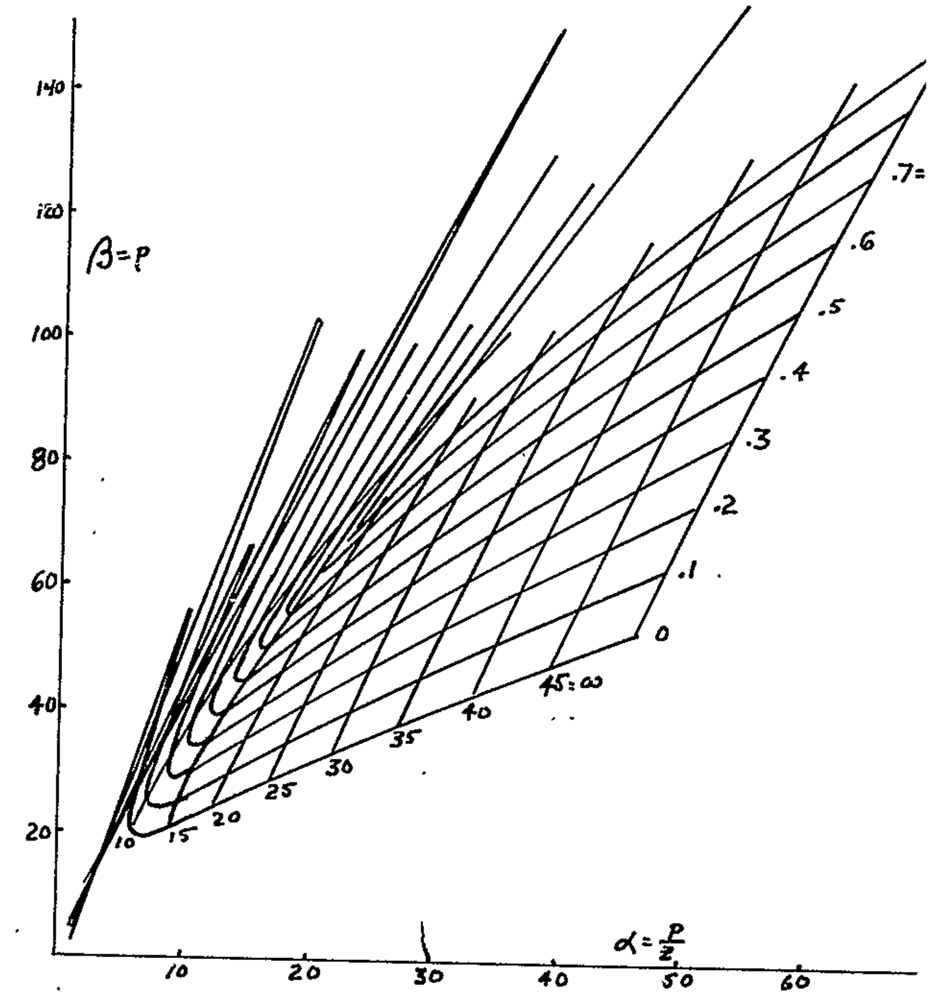

Fig. 1-3. Double Lead Compensation of Plant with $G(s) = \frac{125}{s^3}$

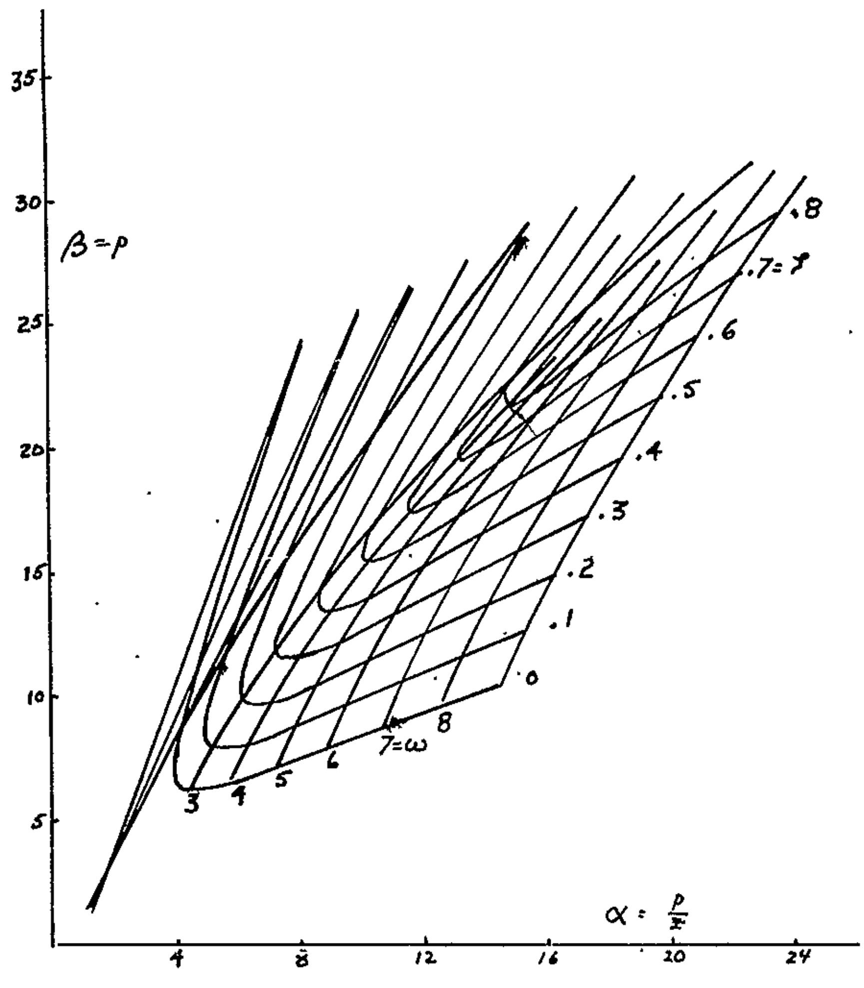

Fig. 1-4. Double Lead Compensation of Plant with $G(s) = \dfrac{10}{s^2(s+1)}$

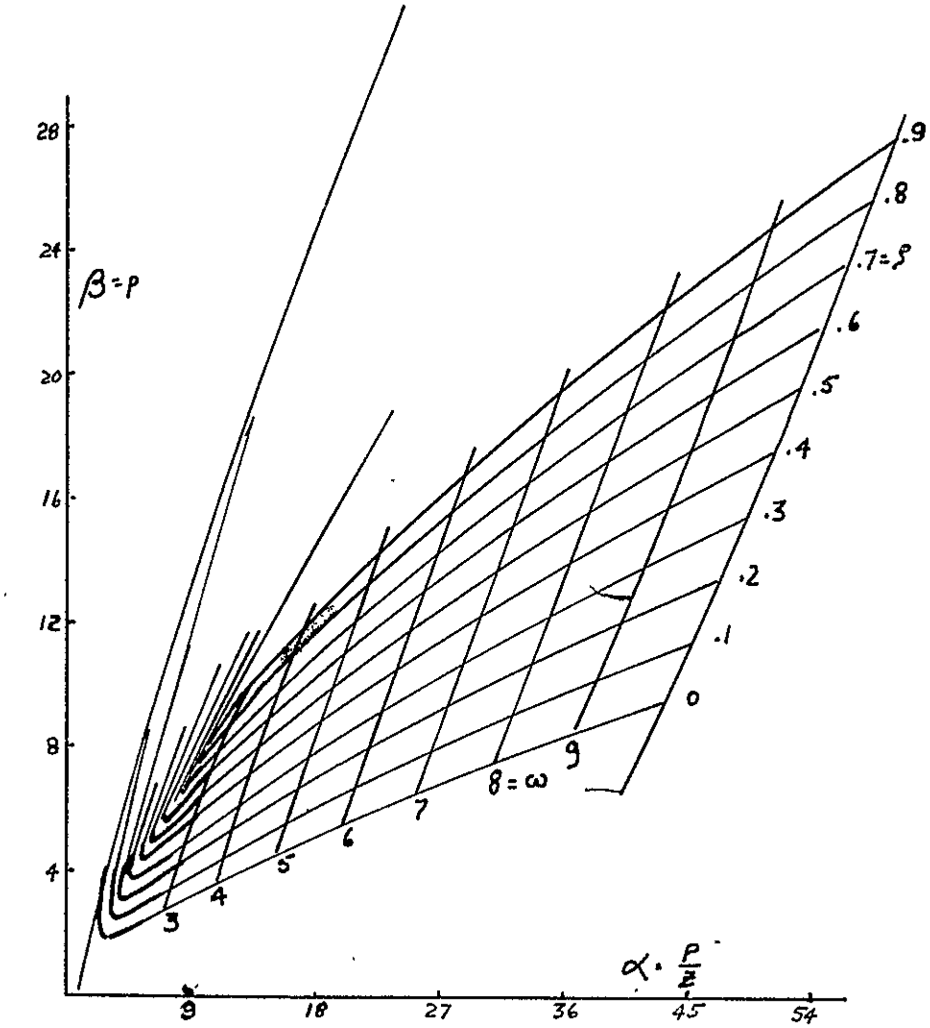

Fig. 1-5. Double Lead Compensation of Plant with $G(s) = \dfrac{1}{s^2(s+1)}$

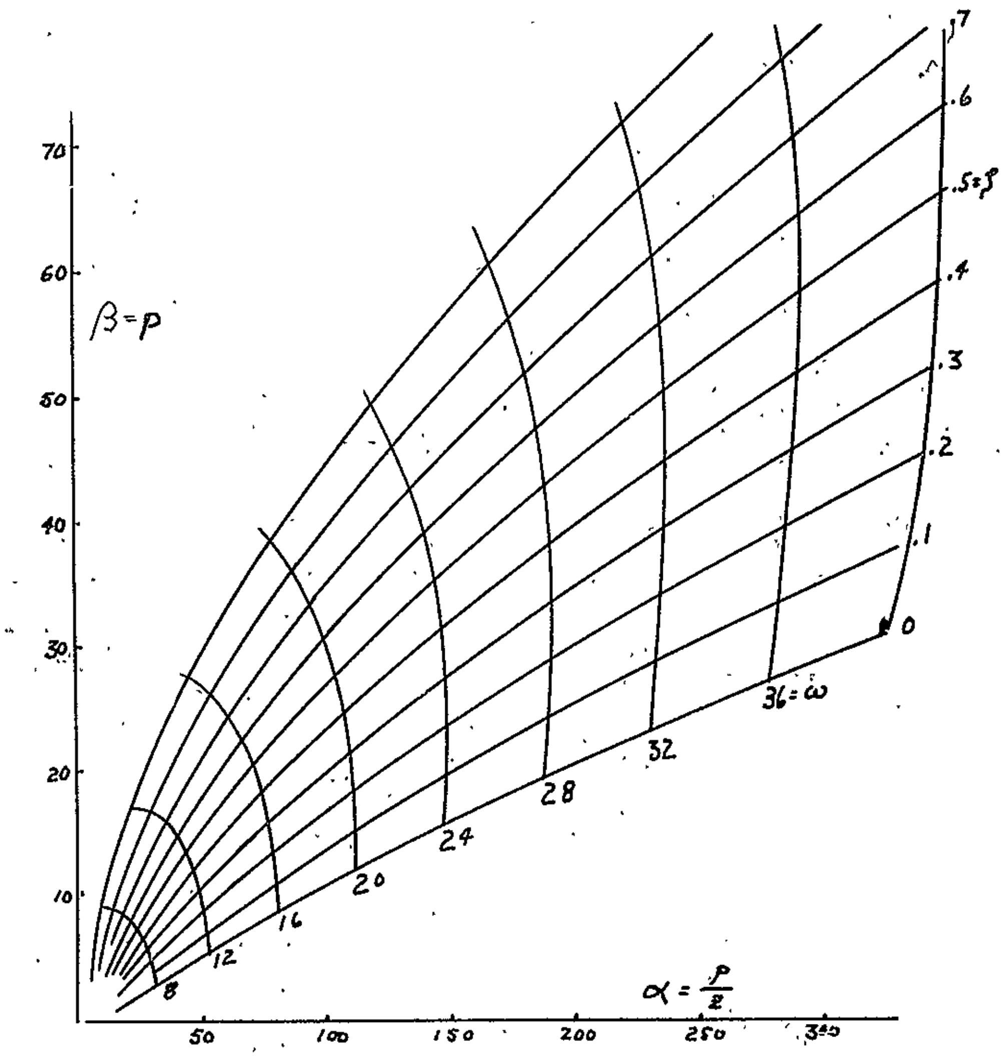

Fig. 1-6. Double Lead Compensation of Plant with $G(s) = \frac{1}{s^2(s+10)}$

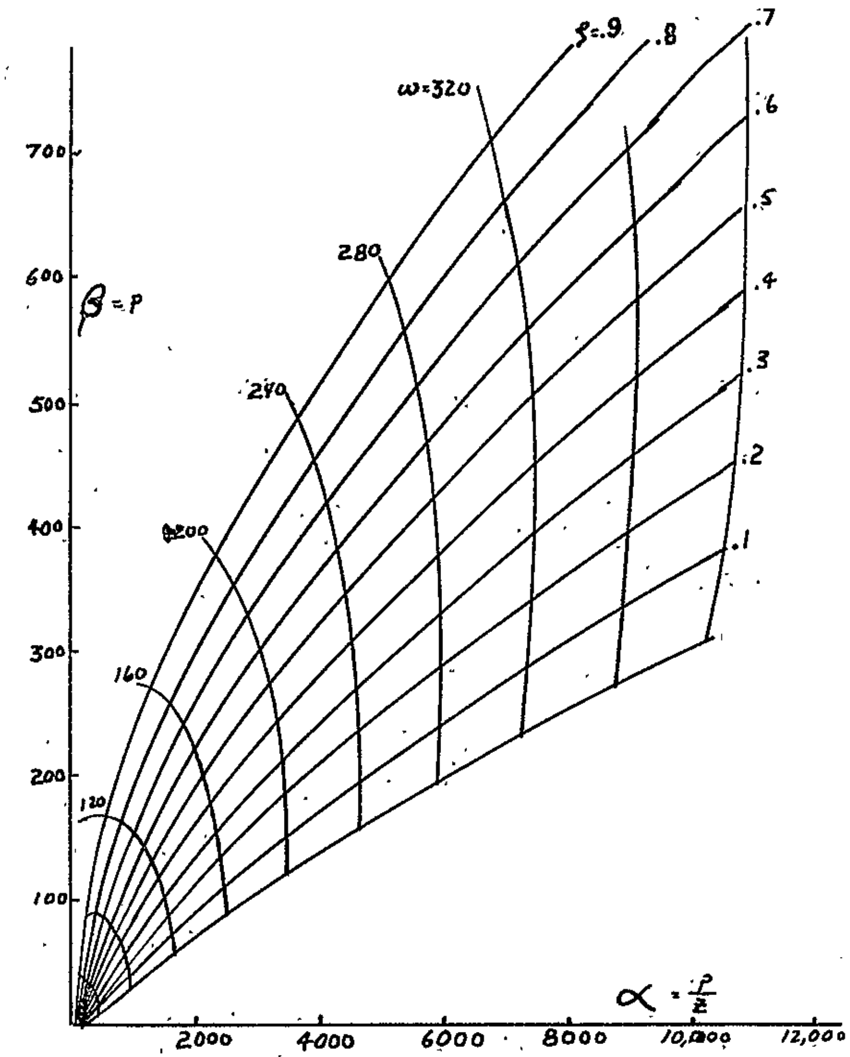

Fig. 1-7. Double Lead Compensation of Plant with $G(s) = \frac{1}{s^2(s+100)}$

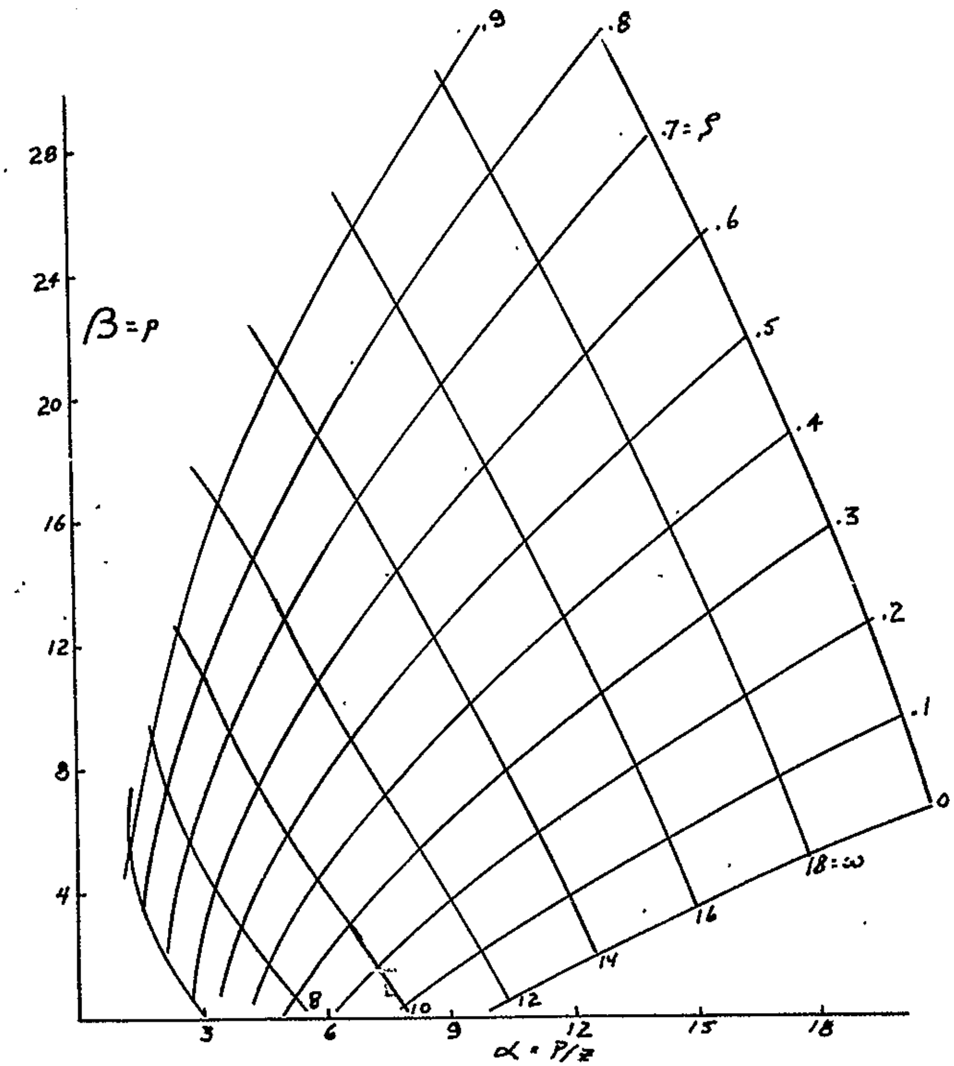

Fig. 1-8. Double Lead Compensation of Plant with with $G(s) = \dfrac{25}{(s+5)^2(s+10)}$

TRANSIENT RESPONSE OF NONLINEAR SYSTEMS

2.1 INTRODUCTION

When a system has one nonlinear element that is single valued and non-frequency dependent, analysis of the system is conveniently accomplished using the parameter plane methods. The nonlinear element is represented by a describing function, which is a function of signal amplitude only. The describing function is designated as one of the parameters, α or β. This designation removes the nonlinear parameter from the functions that determine the parameter plane curves* so that these may be plotted on the α-β plane. The M-point is located on the α-β plane in the usual way, but for the case of one nonlinear element one coordinate of the M-point is the numerical value of the describing function of the nonlinear parameter. For linear systems the M-point is stationary on the α-β plane, but for a nonlinear system the M-point moves because the numerical value of the describing function is a function of signal amplitude. For a system with one single valued nonlinearity, N, where N is designated as β, the locus followed by the M-point is a straight line parallel to the β-axis. This locus of M-point motion can be said to start at the value of β corresponding to very small (zero) signal amplitude into the nonlinear element. The displacement of the M-point along this locus is determined by the way in which β varies as a function of signal amplitude, and this is determined by using the

*Constant $-\zeta$ and constant $-\omega_n$ curves, or constant $-\sigma$ and constant $-\omega$ curves.

the describing function of the nonlinear element.

Previous work has shown how to predict limit cycles using M-point locus on the parameter plane. If this locus crosses the stability boundary (ζ=0 curve or σ=0 curve) the intersection of these curves defines the frequency of the limit cycle. If an amplitude scale can be determined for the location of the M-point on the M-locus, then this scale is used to define the amplitude of the limit cycle.

The concept of a moving M-point on the parameter plane can be used to calculate the transient response of nonlinear systems. As the M-point moves along the M-locus, each point defines both signal amplitude and all roots of the characteristic equation. This information can be used to determine the amplitude vs time relationship which is the transient response. Computations are based on Siljak's extension of some basic work by Krylov and Bogoliubov, and details are given in the following paragraphs.

Assume* that the system is second order, and that the nonlinear element is represented by its describing function. Then for an initial signal amplitude A_o, the transient response is defined by

$$X(t) = A_o \epsilon^{\sigma t} \cos(\omega t + \phi) \qquad (2\text{-}1)$$

where σ and ω are both functions of the signal amplitude.

*These assumptions restrict use of this method to systems in which a pair of complex roots dominate the transient response, and these systems must have low pass filter characteristics to justify use of a describing function.

$$\sigma \triangleq \sigma(A)$$
$$\omega \triangleq \omega(A) \qquad (2\text{-}2)$$

The parameter plane curves are prepared, the M-locus is superimposed on them, and the describing function is used to associate an amplitude scale with the M-locus. Then the values of $\sigma(A)$ and $\omega(A)$ may be read from the parameter plane for any X.

The transient response of the system from any initial displacement, A_o, is determined in two steps, the first of which is to calculate the envelope of the transient. Assuming that ϕ (in eqn. 2-1) is zero, the envelope is defined by

$$X(t) = A_o \epsilon^{\sigma(A)t} \qquad (2\text{-}3)$$

which may be approximated over a short time interval by a straight line tangent to the exponential curve. Thus at t = 0, $X=A_o$ and from the parameter plane $\sigma(A_o)=\sigma_o$ is evaluated. Then $X(t)=A_o\epsilon^{\sigma_o t}$ is approximated by a short straight line segment on the X vs t plane. This straight line is terminated at $t = t_1$ and at t_1 a new amplitude A_1 is read from the curve. Entering the M-locus on the parameter plane with A_1 values are obtained for σ_1 and ω_1. The envelope of the transient is extended from t_1 to t_2 with another straight line segment defined by $X = A_1\epsilon^{\sigma_1 t}$. This procedure is repeated until the envelope is defined over an acceptable time interval.

As a by-product of this procedure, ω has been determined quantitatively as a function of amplitude and also as a function of time. Using the definition

$$\Phi = \int_0^t \omega(A)\ dt \qquad (2\text{-}4)$$

the phase can be determined at any t by graphical integration (i.e., evaluation of the area under the $\omega(A)$ vs t curve). If ϕ in eqn. 2-1 is zero, then $X(t) = 0$ for $\Phi = (2n-1)(\pi/2)$. Values of t corresponding to $\Phi = 90^\circ$, 270°, 450°, etc., are determined by graphical integration, are marked on the axis of the X vs t plane, and the transient response is drawn tangent to the envelope and intersecting the $X = 0$ axis at the indicated values of t.

The above procedures are readily applied to systems with one nonlinearity, and correlation with simulation results is excellent. Since such applications are elementary no illustrations are given here, and the study is extended to systems with two single valued nonlinear elements. In general no other methods exist for predicting the transient response of systems with two nonlinear elements, so the results obtained here represent a significant advance in the state of the art.

2.2 CLASSIFICATION OF SYSTEMS WITH TWO NONLINEARITIES

When a system contains two nonlinear elements, N_1 and N_2, that are single valued and are not frequency dependent, parameter plane representation may be used but both α and β become functions of N_1 and N_2. Computation of the parameter plane curves presents no difficulty, but determination of the M-locus may be difficult. As a result it is convenient to classify nonlinear systems according to the structural conditions which complicate the evaluation of the M-locus. The following classes are proposed:

CLASS 1. Identical signal excitation to both nonlinear elements.

In Fig. 2-1a, the signal X is the input to both nonlinear elements N_1 and N_2. For every value of X corresponding values of N_1 and N_2 are uniquely defined and are independent of frequency so evaluation of the M-locus is easy.

CLASS 2. The input signals to the two nonlinear elements are related by a linear differential equation.

In Fig. 2-1b the signal X is the input to N_2, but the input to N_1 is $X\ G_{-1}(s)$. Thus the input to N_2 is a function of amplitude only, but the input to N_1 is a function of both amplitude and frequency. For a given amplitude of the signal X, the describing function for N_2 provides one unique value, but for each amplitude of X the describing function for N_1 has an infinite number of possible values, one for each possible value of frequency. As a result the evaluation of the M-locus is considerably more difficult than for Class 1.

CLASS 3. The input signals to the two nonlinear elements are related by a nonlinear differential equation.

Fig. 2-1c illustrates this class of nonlinear systems. The signal X is the input to N_1, but the input to N_2 is $X\{N_1\}\{G_2(s)\}$ where the brackets are intended to represent some functional relationship rather than a multiplication. Evaluation of the M-locus can be very difficult for such systems.

2.3 EVALUATION OF THE M-LOCUS. THE DYNAMIC DESCRIBING FUNCTION.

When a system with one single valued nonlinear element is represented on the parameter plane the M-locus is clearly a straight line parallel to one of the coordinate axes. Thus the M-locus itself is readily found but the amplitude scale associated with this locus must be evaluated. For systems with two nonlinearities (especially Class 2 or 3) the path of the M-point on the parameter plane cannot be predicted by inspection. It can be calculated, however, using the ordinary describing function to define the amplitude relationships.

To justify the choice of the describing function as a tool, consider the fact that parameter plane predictions of limit cycles are defined on the basis of a single point where the M-locus interests the stability boundary. This single point defines both the fundamental frequency of the oscillation and also the amplitude of this fundamental component. It is clear that the location of the M-point represents some sort of average value of amplitude, since the instantaneous value of amplitude varies cyclically during a limit cycle. The describing function of a nonlinear element effectively averages the response of the element to a sinusoidal input over one cycle of operation. Thus its use is clearly justified when system operation is periodic and lightly damped. While not so clearly justified for other operating conditions it has given surprisingly accurate results and therefore will be used until a better technique becomes available.

Using the describing functions of the two nonlinearities in a system, a family of decribing function curves are computed and plotted on the α-β parameter plane. When these curves are superimposed on the regular parameter plane curves, the M-locus can

be determined. The M-locus represents the curve along which the M-point moves when the system is in dynamic operation, and it consists of the locus of all points at which the describing function curves and the parameter plane curves have common frequency intersections. We choose to call this curve the "Dynamic Describing Function Locus". The procedure and also a justification is as follows:

a) Assume a constant amplitude, constant ω signal at X, the input to one nonlinear element. Using the describing function compute the equivalent gain of that element; also compute the signal amplitude at the input to the second nonlinear element, and the equivalent gain of this second element.

b) The two equivalent gains evaluated in (a) determine one point on a describing function curve on the α-β plane. Repetition using the same value of ω but different amplitudes at X determines a describing function curve for a constant ω signal.

c) Repetition of a) and b) for other values of ω provides a family of describing function curves, each curve being for a designated value of ω.

d) These curves are then superimposed on the usual* parameter plane curves. The constant $-\omega$ describing function

*Curves for constant $-\sigma$ and constant ω are most convenient, but constant $-\zeta$ and constant ω_n curves can be used if it is noted that $\omega = \omega_n\sqrt{1-\zeta^2}$.

curves will intersect the constant -ω parameter plane curves, and those intersections for which the ω is the same. Define the Dynamic Describing Function locus.

The nonlinear system is described by one nonlinear differential equation. The procedures used here effectively partition this equation into two parts, a linear part represented by the parameter plane curves, and a nonlinear part represented by the describing function curves. Then parts are "coupled" by the parameters α and β which are the coordinates of both plots. If the system is in steady state periodic motion at a given frequency the nonlinear differential equation of the system must be satisfied, so the linear and nonlinear partitions must be satisfied at that frequency. This condition can exist only at the intersection of the common frequency curves. The points thus defined on the "Dynamic Describing Function Locus" are determined on the basis of steady state sinusoidal operation (unforced). Under transient conditions the M-point moves along some locus on the parameter plane, and we assume that the points on The Dynamic Describing Function locus apply to transient operation although they are determined by means of steady state sinusoidal concepts. Experimental results indicate that this is a good assumption.

2.4 CALCULATED AND EXPERIMENTAL RESULTS

In order to verify the correctness and the applicability of the dynamic describing function and the graphical transient response calculations, specific examples of each of the three general cases of Fig. 2-1 were investigated. The details of some of

these examples, and the corresponding calculated results are presented here. Simulation of the systems provided experimental results which are also presented to permit comparison between theory and experiment.

System 1. Two nonlinear elements with identical excitation:

The block diagram is given in Fig. 2-2. The characteristic equation is

$$s^3 + 10s^2 + (10N_1 + 10N_2)s + 100N_1 = 0 \qquad (2\text{-}5)$$

and it is convenient to let $N_1 = \alpha$, $N_2 = \beta$. Fig. 2-3 gives the parameter plane plot (in σ- and ω- curves). Since the two nonlinear elements have identical excitation a single dynamic describing function curve is obtained which is independent of frequency. However, the dynamic describing function is dependent on the specific numerical characteristics of the nonlinearities, and Fig. 2-3 contains three dynamic describing function curves (dotted) for three different sets of characteristics in N_1 and N_2. These three curves were chosen to illustrate different root variations. For curve 1 a real root becomes dominant early in the transient, for curves 2 and 3 complex roots are dominant, the system being moderately damped for curve 2 but going to a stable limit cycle for curve 3.

Calculated and analog computer results are given on Figs. 2-4, 5,6. It is seen from Fig. 2-4 that the dominant real root condition cannot be handled accurately with the graphical computations. It is not known whether the discrepancy lies solely in the graphical method which is based on complex roots, or whether the dynamic describing function also contributes to the errors. Research on this point is continuing. For the cases of Fig. 2-5 and 2-6 the calculated results compare well with the computer results.

<u>System 2</u>. Two nonlinear elements related to a common signal by a linear differential equation.

The block diagram is given in Fig. 2-7, and the parameter plane curves with dynamic describing function curve shown dotted are given on Fig. 2-8. Fig. 2-9 gives the describing function grid needed to obtain the dynamic describing function curve. To obtain the grid of Fig. 2-9 the point A_o on Fig. 2-7 was chosen as a reference point, and at each value of ω the amplitude of the (assumed) sinusoidal signal at A_o was varied to obtain the N_1 vs N_2 values for a constant ω curve on Fig. 2-9. The dynamic describing function curve on Fig. 2-8 is obtained by superimposing the parameter plane curves of Fig. 2-8 on the describing function net of Fig. 2-9 and locating intersections of constant ω curves of the same ω value.

Limit cycle predictions of the dynamic describing function curve on the parameter plane agree with analog computer simulation results. In addition Figs. 2-10, 11,12 compare predicted transient response with simulation results.

Additional checks were run using different values for the deadzone and saturation limits in the two nonlinearities, but the detailed data is not given here. In general the predicted and simulated results were in good agreement except when a <u>real</u>

root became dominant during the transient response, in which case the frequency of the oscillatory component was usually predicted with reasonable accuracy, but amplitudes were not, nor was the total response time due to the influence of this real root.

The calculations and simulations were also repeated with the nonlinearities interchanged (i.e., in Fig. 2-7, N_1 becomes a saturated element and N_2 a dead zone element). Using the same techniques the results obtained were always in agreement with about the same degree of accuracy and with the shortcomings as previously noted.

<u>System 3</u>. Two nonlinear elements related to a common signal by nonlinear differential equation.

The classification described as System 3 can contain a wide variety of combinations of linear and nonlinear elements, of which the parameter plane method may be applicable to only a small subset. A specific system which belongs in this class is shown in Fig. 2-13. The characteristic equation of this system is

$$s^3 + 3s^7 + 2s + 40KN_1(N_a + jN_b) \tag{2-6}$$

where $N_2 \triangleq N_a + jN_b$ for the hysteretic nonlinearity, and we define $\alpha = N_1N_a$; $\beta = N_1N_b$. The parameter plane equations are still applicable and the parameter plane curves can be computed. For the purposes of this study only $\zeta = 0$ curve was calculated, and only the limit cycle predictions were checked. The describing function net is required, and in this case relates the N_1N_a and N_1N_b pairs to the common signal at A on Fig. 2-13. The results of these computations are given on Fig. 2-14, which shows the $\zeta = 0$ curve from the parameter plane equations and the describing function net for the case where K = 0.15. Only one point is defined on the dynamic describing function curve, and this is marked on the $\zeta = 0$ curve at the point where the ω value on the $\zeta = 0$ curve is the same as the value of the constant ω describing function curve passing through that point. This defines the frequency and amplitude of the limit cycle, and the results agree with simulation results.

Note that a change in the value of K changes the differential equation of the system, thus requiring a new set of curves. Results were obtained with other values of K and again the predictions agreed with simulation results.

2.5 COMMENTS

The results obtained thus far indicate that the parameter plane is a useful tool in predicting the stability and response of nonlinear systems. The accuracy available is only fair, but is more than adequate for many engineering applications. The transient response predictions - in particular for systems containing two nonlinearities, - are better than are available with any other method.

The graphical presentation of the dynamic describing function curve on the parameter plane is potentially a valuable <u>design</u> tool. It indicates at a glance the range of variations of the roots, and thus permits prediction of a <u>desired</u> location of the describing function curve, which in turn implicitly defines the required characteristics of the nonlinear element. Further research is required in this area.

The technique becomes inaccurate when the transient response is influenced by more than two complex roots. Again more research is required to evaluate this situation.

It is too early to assess the true value of studying nonlinear systems on the parameter plane. Without question it does make possible many types of analyses that are not readily available otherwise. However, the limitations of the technique are not clearly defined, and it obviously is important to know under what conditions the methods are not applicable, or should be applied with care.

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5. Wilson, P.L., "Use of Mitrovic's Method in Predicting Limit Cycles in Feedback Control Systems with One and Two Gain Variable Nonlinearities." M.S. Thesis, U.S. Naval Postgraduate School, 1964

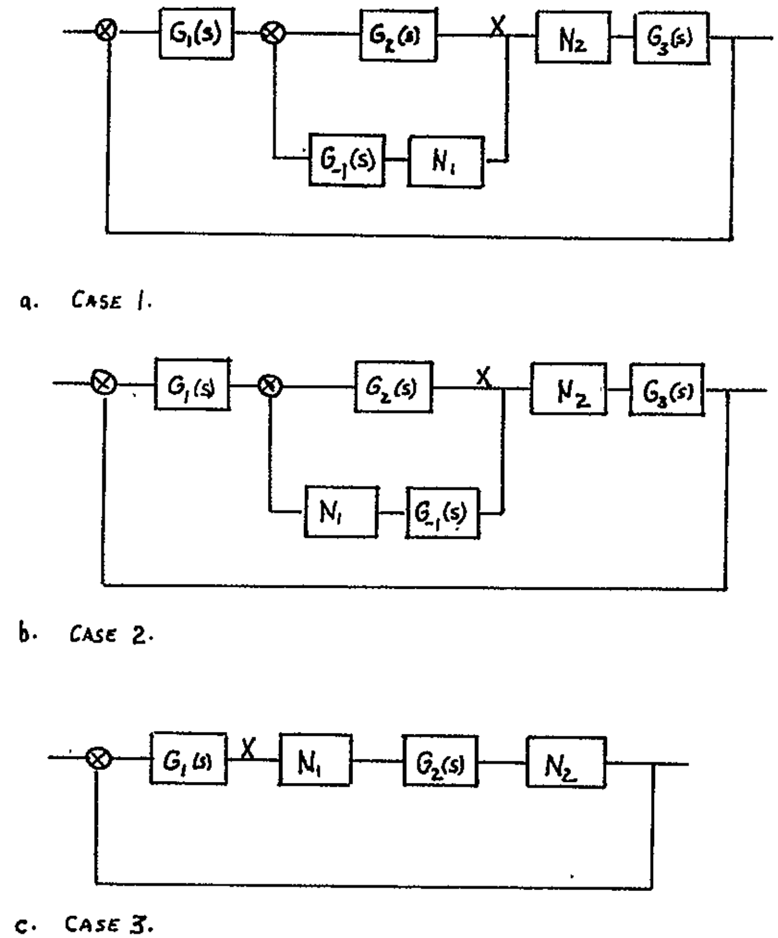


Fig. 2-1. General Classification of Control Systems with Two Nonlinear Elements.

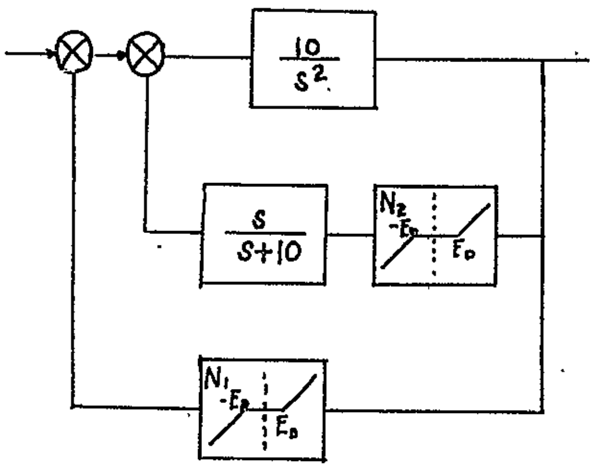


Figure 2-2. Block Diagram of Third Order System with Two Nonlinear Elements.

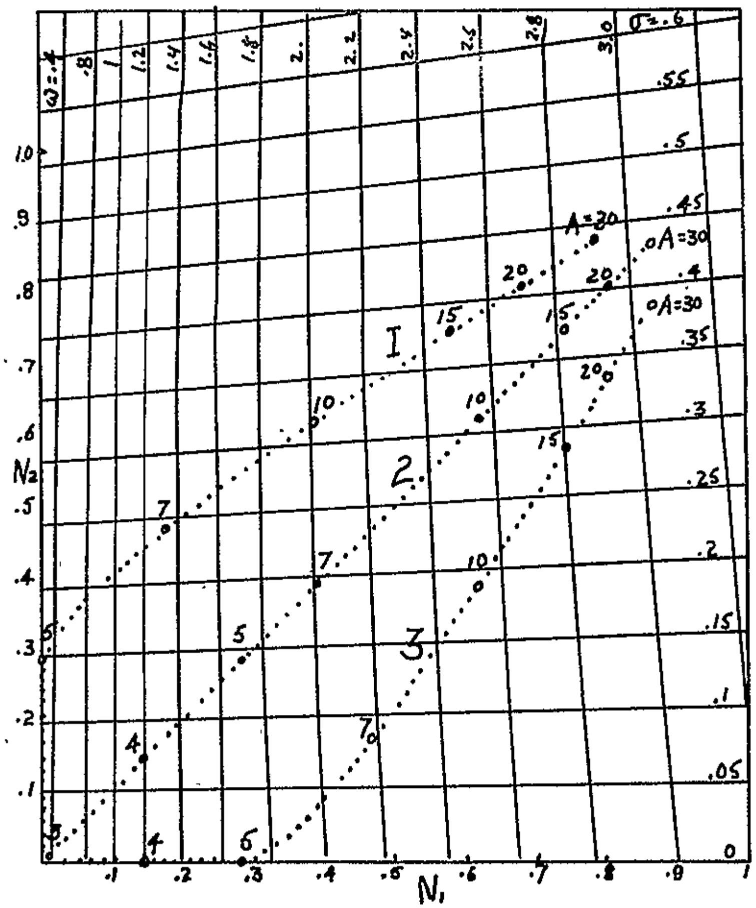


Fig. 2-3. Dynamic Describing Function Curve on Sigma-Omega Curves.

	N_1 - Dead Zone	N_2 - Dead Zone
1	$E_d = \pm 5$ Volts	$E_d = \pm 3$ Volts
2	$E_d = \pm 3$ Volts	$E_d = \pm 3$ Volts
3	$E_d = \pm 3$ Volts	$E_d = \pm 5$ Volts

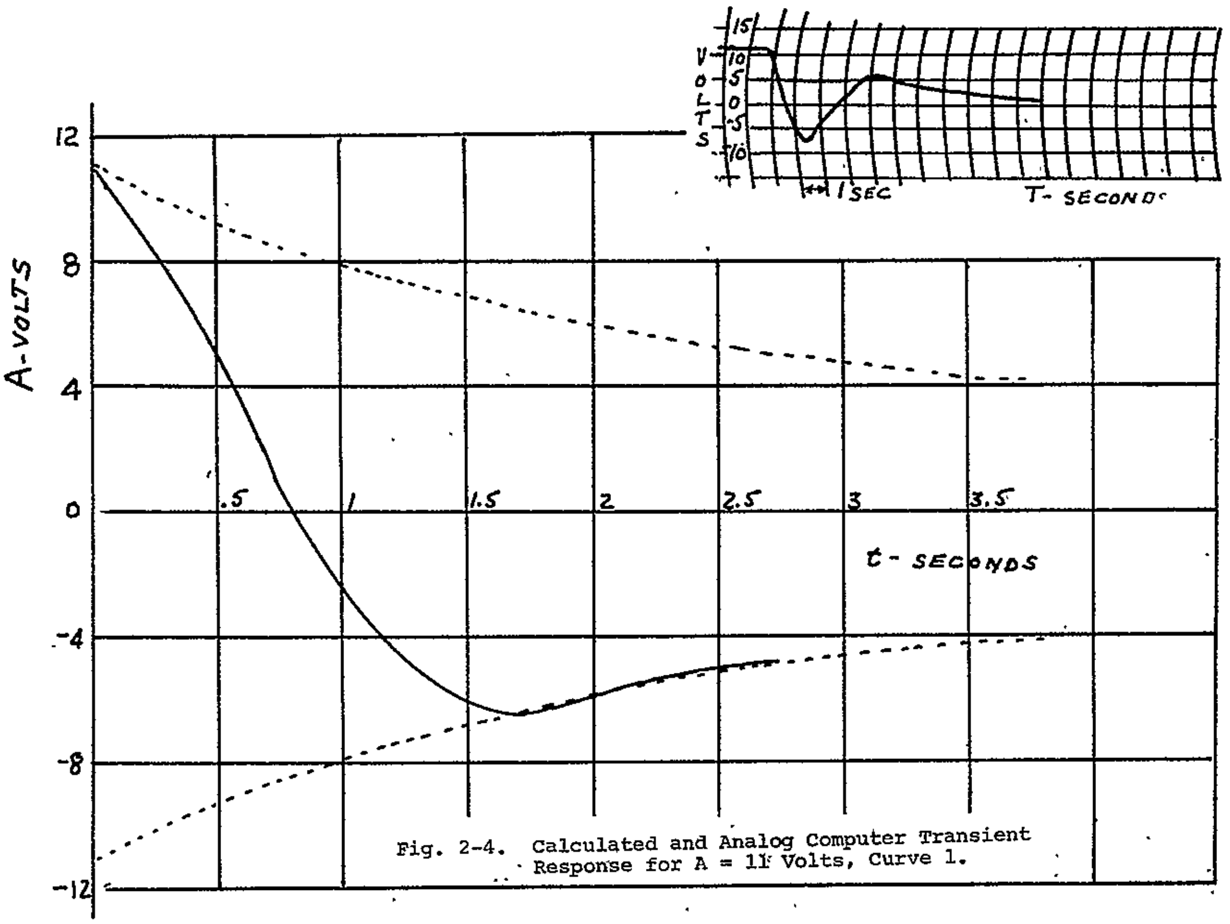


Fig. 2-4. Calculated and Analog Computer Transient Response for A = 11 Volts, Curve 1.

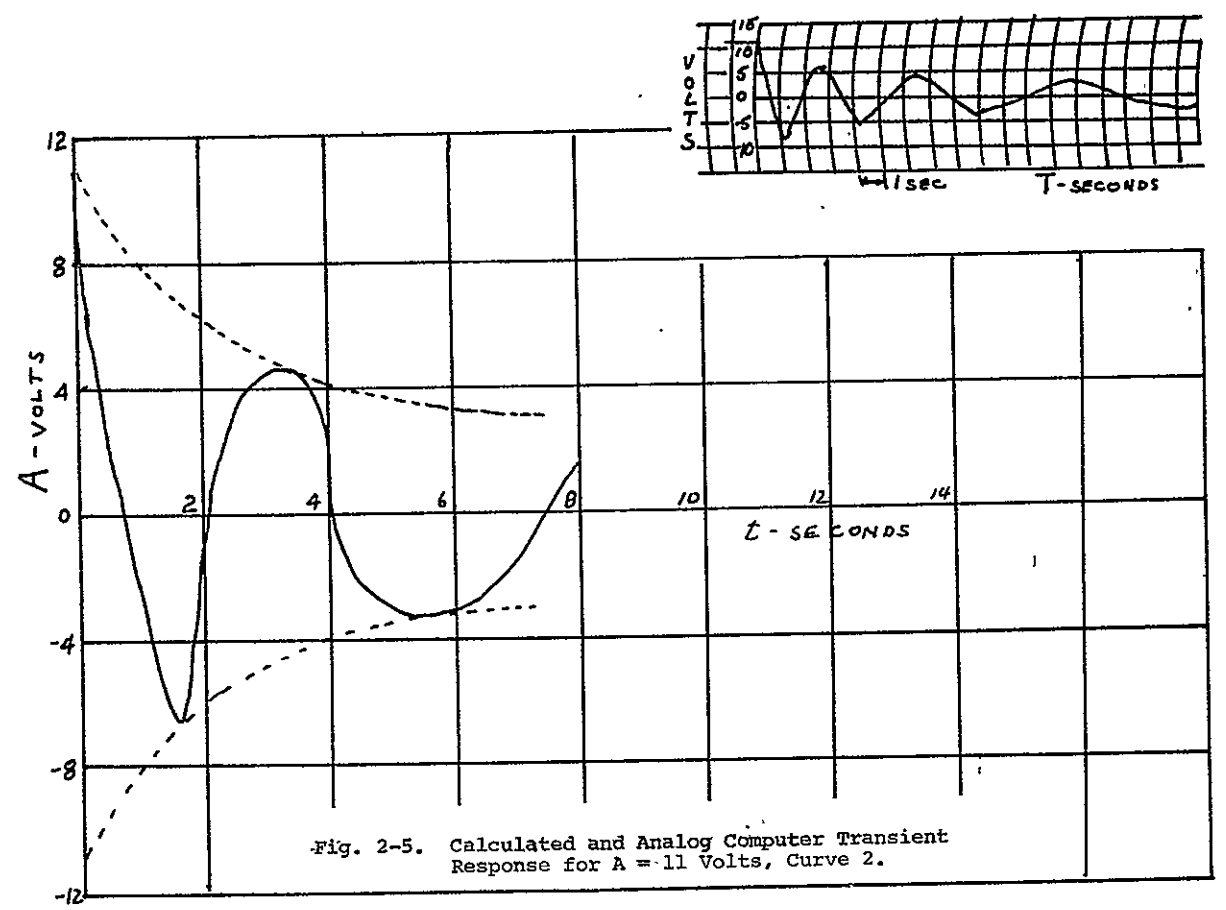


Fig. 2-5. Calculated and Analog Computer Transient Response for A = 11 Volts, Curve 2.

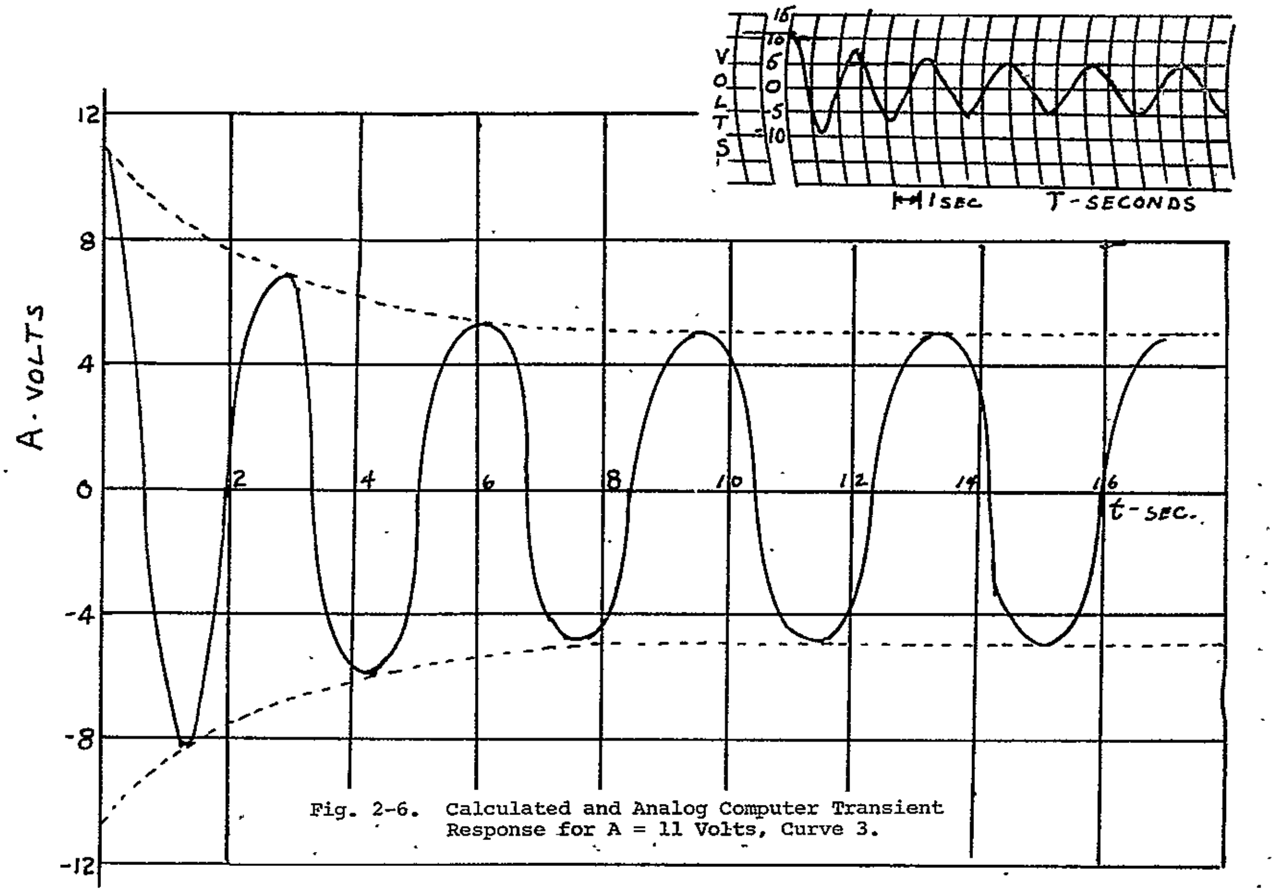


Fig. 2-6. Calculated and Analog Computer Transient Response for A = 11 Volts, Curve 3.

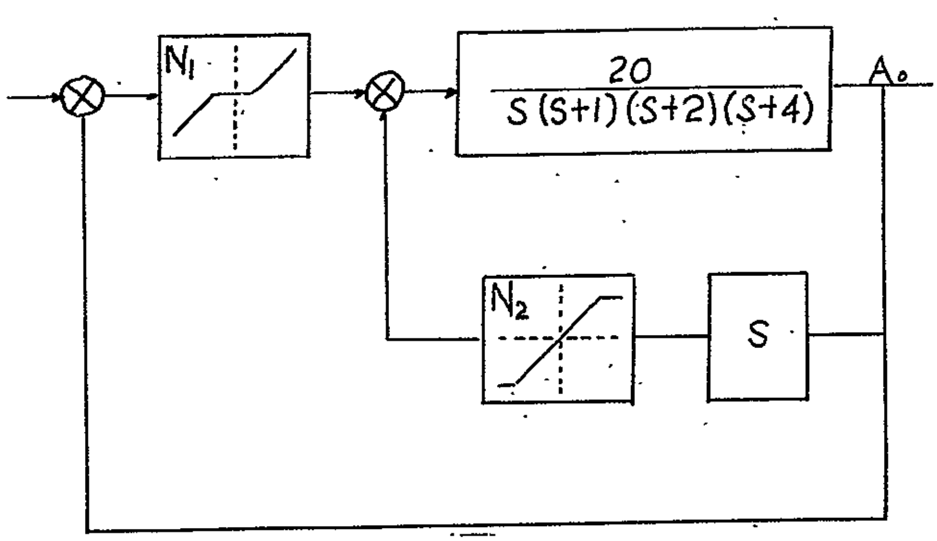


Fig. 2-7. Block Diagram of Fourth Order System with Two Nonlinear Elements.

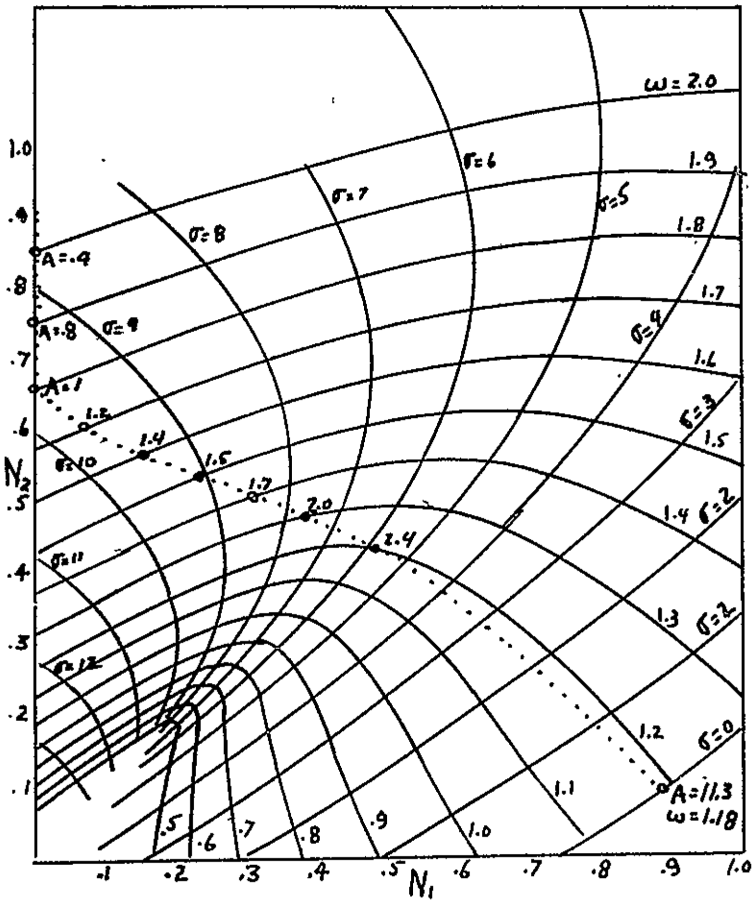


Fig. 2-8. Dynamic Describing Function Curve on Sigma-Omega Curves.
N_1 - Dead Zone $E_d = \pm 1$ Volt
N_2 - Saturation $E_{sat} = \pm 1$ Volt

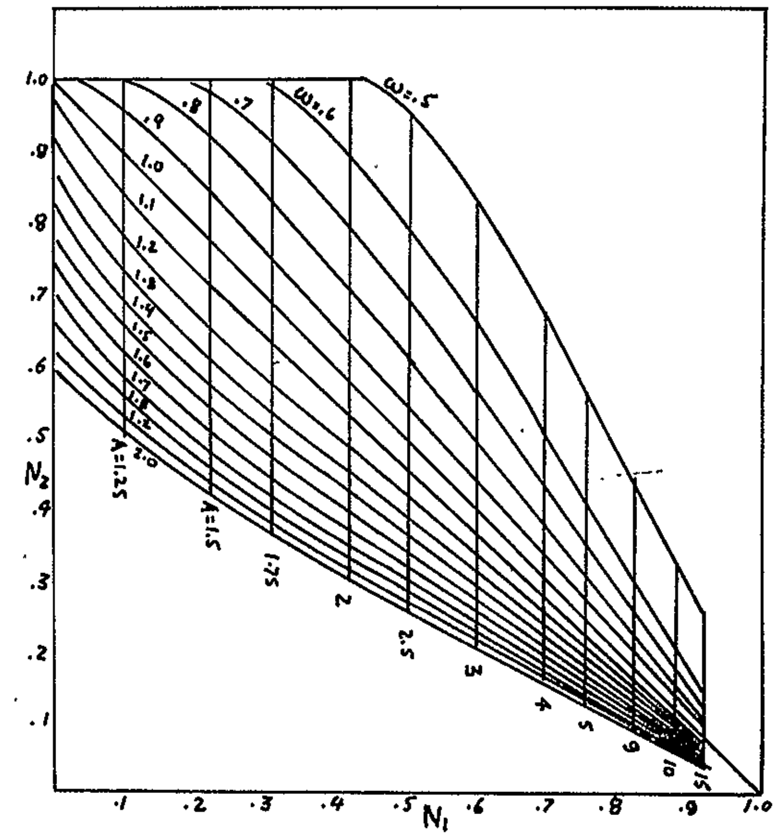


Fig. 2-9. ω-A Grid for Figures 4-2 and 4-3.
N_1 - Dead Zone $E_d = \pm 1$ Volt
N_2 - Saturation $E_{sat} = \pm 1$ Volt

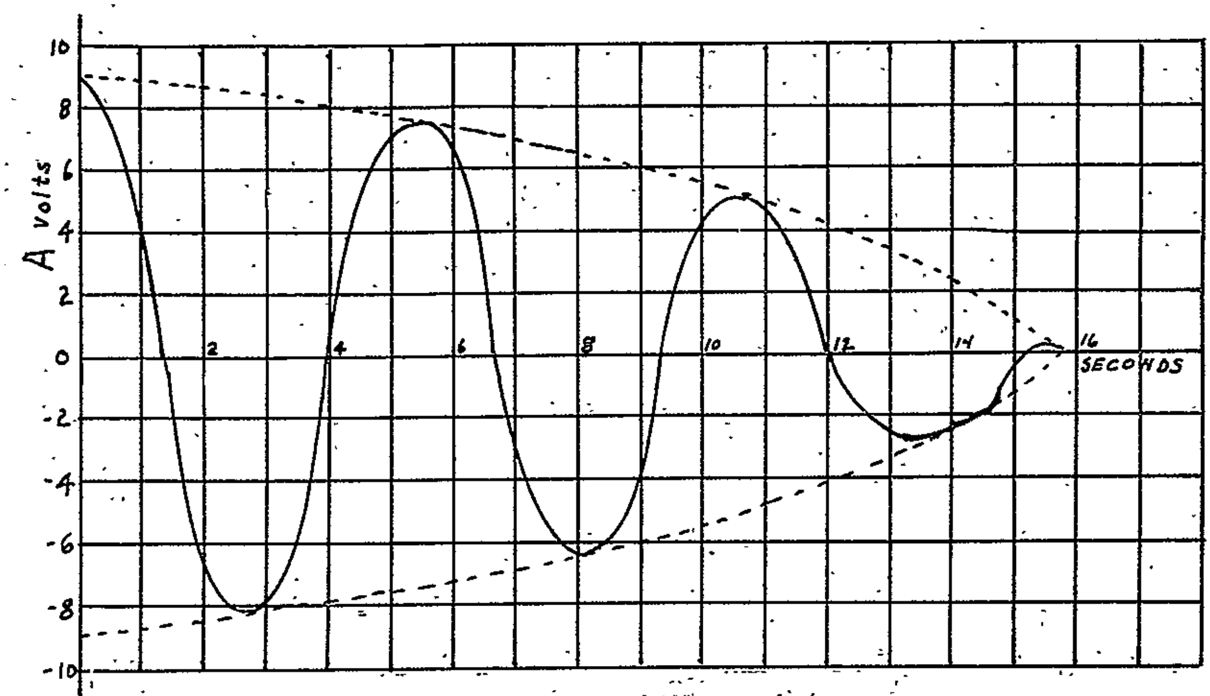


Figure 2-10a. Calculated Transient Response for
A = 9 Volts.

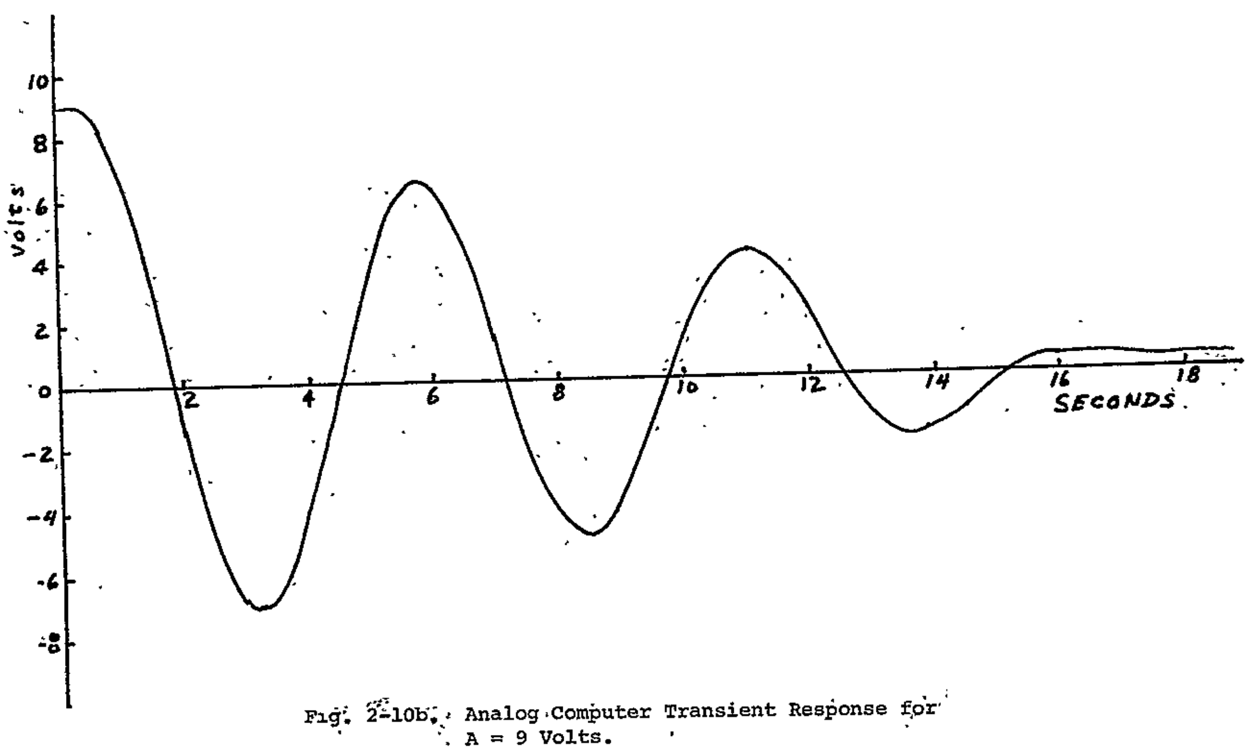


Fig. 2-10b. Analog Computer Transient Response for
A = 9 Volts.

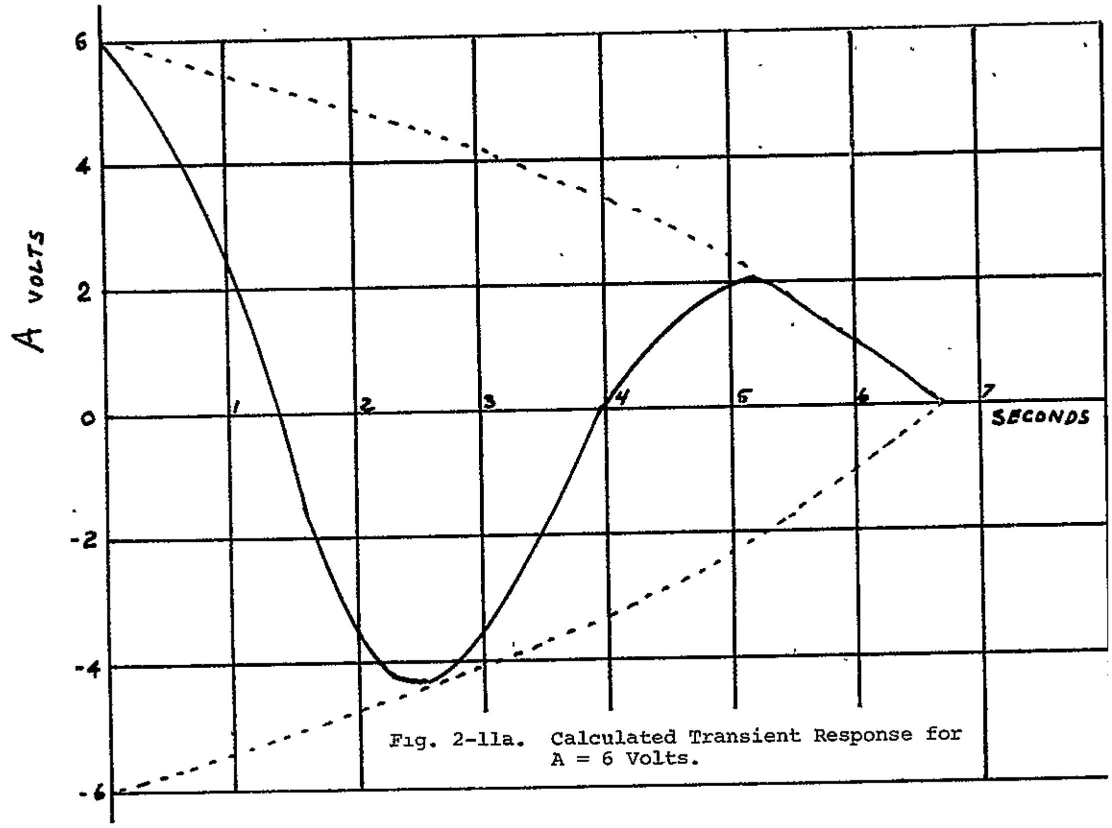


Fig. 2-11a. Calculated Transient Response for A = 6 Volts.

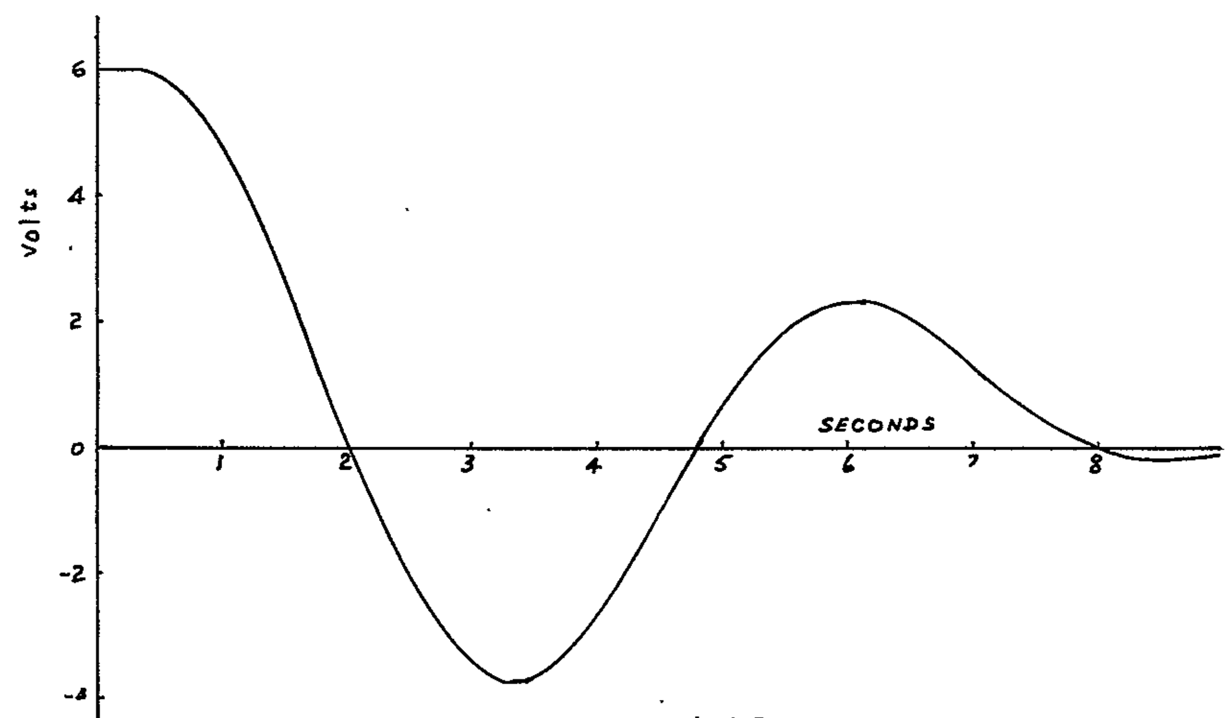


Fig. 2-11b. Analog Computer Transient Response for A = 6 Volts.

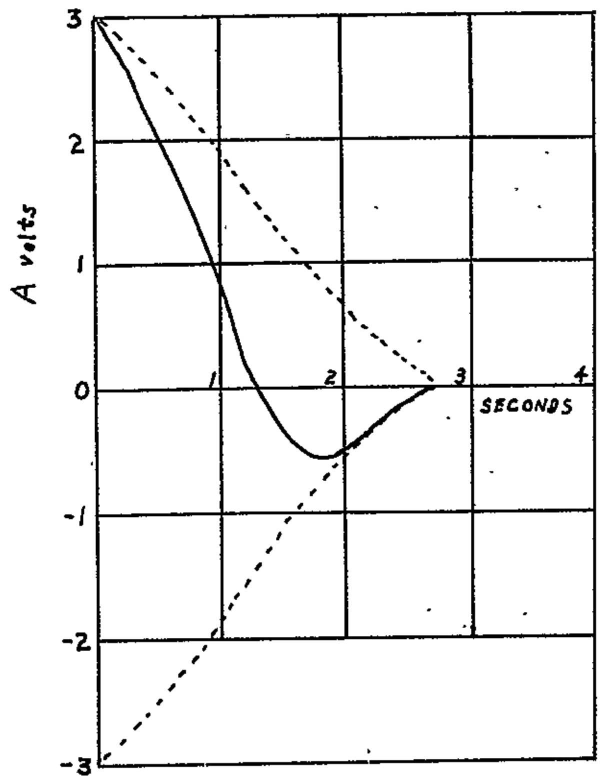


Fig. 2-12a. Calculated Transient Response for A = 3 volts.

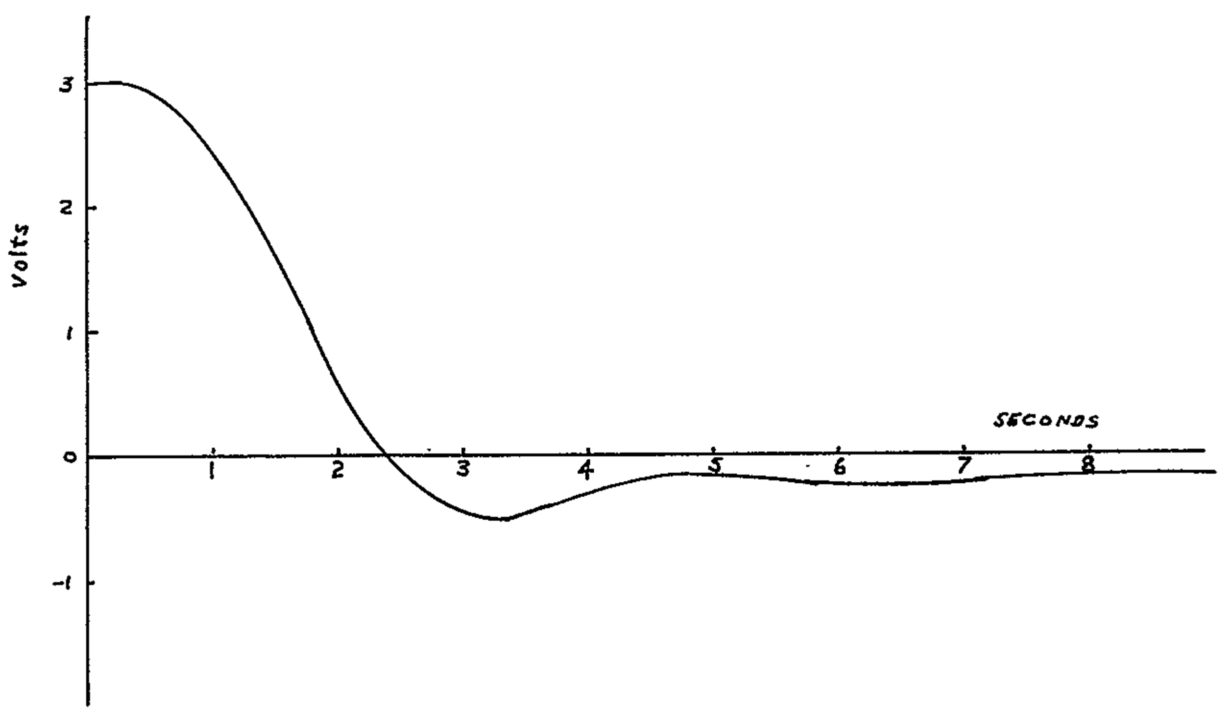


Fig. 2-12b. Analog Computer Transient Response for A = 3 Volts.

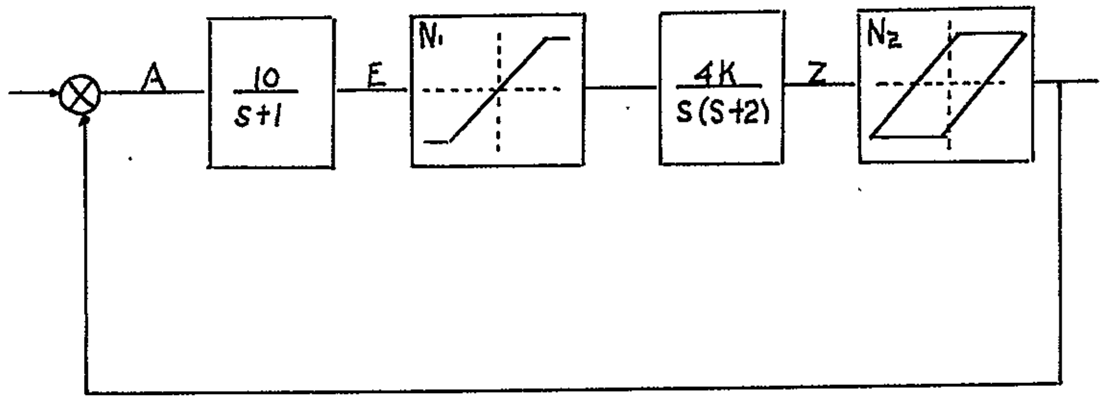


Fig. 2-13. Block Diagram of Test Example.

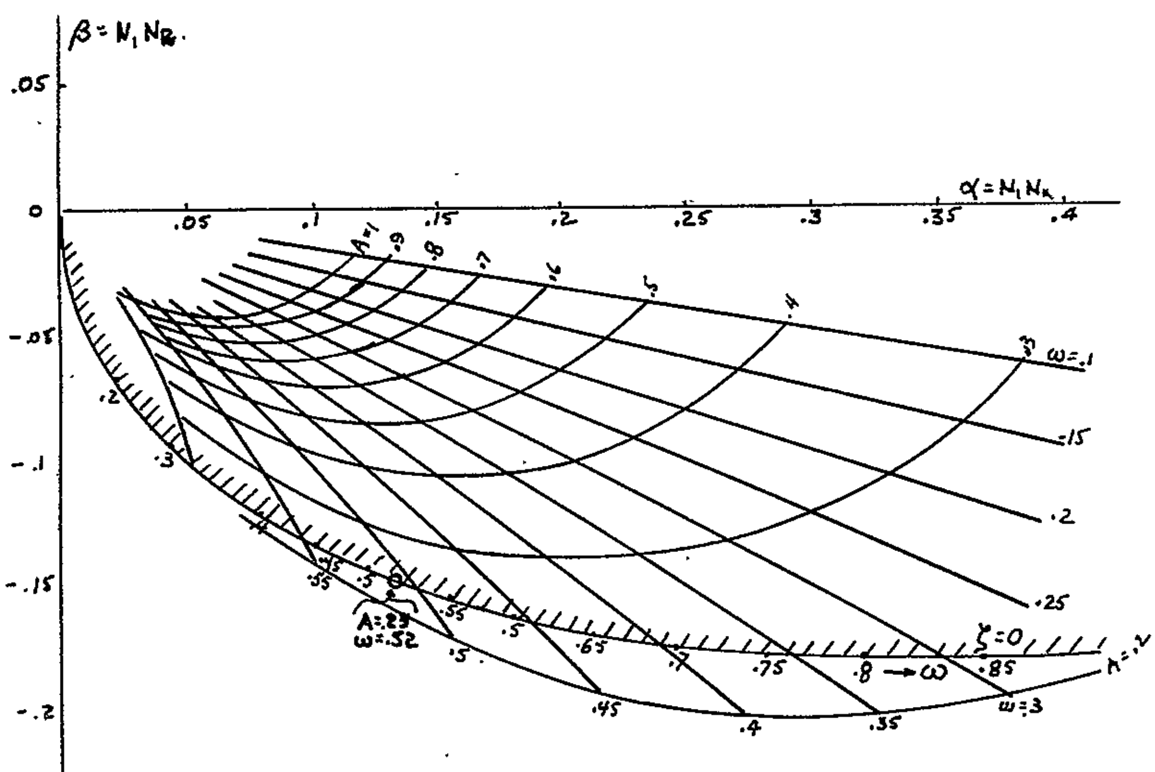


Fig. 2-14. Parameter Plane Diagram for Figure 2-13, K = .15.

CHAPTER III

ASYMMETRICAL NONLINEAR OSCILLATIONS

3.1 Introduction.

In certain classes of nonlinear systems, oscillations may consist of a limit cycle superimposed on a constant or slow-varying signal. These oscillations are referred to as asymmetrical oscillations since the center of the limit cycle is shifted according to the corresponding value of the constant or slow-varying signal. In general, asymmetrical oscillations may occur when the input-output characteristic of the nonlinearity in the system is not symmetrical about the origin, or when the system is subject to forcing signals. When the nonlinear characteristic is asymmetric, the output of the nonlinearity may contain a constant term even though the corresponding input is a single sinusoidal wave. If the nonlinear characteristic is symmetric, asymmetrical oscillations can arise whenever the system is subject to forcing input signals. Evidently these oscillations may take place at certain points of the system if both conditions are present. Before the analysis of asymmetrical oscillations in the parameter plane is presented, the previous work and results in considering these oscillations and related problems are reviewed.

It has been shown first by MacColl [3.1] that the introduction of an external sinusoidal signal at the input to an on-off servomechanism yields a system that behaves like a linear one for small inputs superimposed on the sinusoidal signal. This phenomena has been later investigated under various names, such as "dither effect", "signal stabilization", etc. Asymmetrical nonlinear oscillations has been found by a majority of authors as the most appropriate term for the mentioned phenomena.

In analyzing a carrier-controlled relay servo, Lozier [3.2] has introduced an idea to accomplish the linearization of the relay by a limit cycle existing in the system and without an external signal. This idea has been further developed by several authors [3.3-3.9] and a detailed treatment of the problem has been given by Popov and Palitov [3.8]. On the other hand, the external signal application has been developed by Loeb [3.9] and Oldenburger with his associates [3.10-3.12]. The latter introduced the name "signal stabilization" to indicate that the nonlinear system is stabilized in the state of sustained oscillations with sufficiently high frequency. The stabilization is actually a consequence of the linearizing effect discovered by MacColl. The concept of signal stabilization has been extended by Sridhar [3.13-3.14] to the case of a nonlinear system which has one single-valued nonlinearity in the loop, and the stabilizing signal is a stationary random process with a Gaussian distribution and obeys the ergodic hypothesis.

The above defined problem can be treated by dual-input describing functions as proposed by West [3.15]. This approach has been significantly simplified by Boyer [3.16] as outlined by Gibson [3.17]. A similar approach is used by Gelb and Van der Velde [3.18], and significant results have been obtained by Atherton and others [3.19-3.20] who made a comparison of the utilized concept with the Tsypkin method [3.21].

The study of asymmetrical nonlinear oscillations has been extensively performed in the analysis and design of a large class of plant adaptive control systems. This class of system is

sometimes called the limit cycling adaptive systems because of the fact that the existing limit cycle is used as an identification signal. Some of the references on this subject are listed here [3.22-3.26]. A majority of the authors proposed an external sinusoidal signal for identification. More recently, Gelb and Van der Velde [3.18] have examined to a limited extent and in a quantitative manner the properties of self-oscillating adaptive systems which have several advantages over the external adaptation, such as simplicity, cost, reliability, etc. The following analysis of asymmetrical nonlinear oscillations in the parameter plane can be applied directly to self-oscillating adaptive systems.

In the following developments, the asymmetrical nonlinear oscillations are analyzed in the parameter plane [3.27]. The control systems with asymmetrical nonlinear characteristics are considered to determine stability and sustained oscillations. The same type of oscillations is investigated in nonlinear control systems subject to constant reference and perturbing input signals. The procedure is further extended to the analysis of systems with slow-varying input signals. In this case, it is shown how a nonlinear characteristic can be modified for the slow-varying signals. The presented analysis is performed with respect to both input signals and the values of adjustable system parameters. The analysis procedure is illustrated by examples in which multiloop feedback structures with several adjustable parameters are considered. In addition, various nonlinear characteristics are used in either the forward or the feedback path. The obtained results are checked by computer simulations which indicate a sufficient accuracy of the presented procedure.

3.2 Basic Developments

Consider a nonlinear system described by the nonlinear differential equation

$$B(s)x + C(s)\ F(x,sx) = H(s)f, \quad s \equiv \frac{d}{dt} \tag{3.1}$$

where B(s), C(s), and H(s) are polynomials in s and the degree of the polynomial B(s) is greater than the degree of the polynomials C(s) and H(s). The function F(x,sx) describes the nonlinearity. Function $f = f(t)$ is a forcing signal, which may be either a reference input or a perturbing signal, and it is assumed to be a constant or a slowly-varying function of time.

As a first approximation, the steady-state solution $x = x(t)$ of equation 3.1 which represents the input to the nonlinearity, is assumed to be

$$x = x^{o} + x^{*} \tag{3.2}$$

where $x^{o} = x^{o}(t)$ is either a slowly-varying function of time or is constant, and x^{*}, which is

$$x^{*} = A \sin \phi, \qquad \phi = \Omega t + \theta, \tag{3.3}$$

represents the periodic component of the solution x(t). Since θ in (3.3) merely corresponds to a shift in t, one can put $\theta = 0$ and use $x^{*} = A \sin \Omega t$.

The forcing function f(t) is considered as a slowly-varying function of time if it can be assumed approximately as constant over any cycle of the periodic component x^{*}; i.e.,

$$|f(t+T) - f(t)| << |f(t)| \tag{3.4}$$

where the period $T = 2\pi/\Omega$. In the frequency domain, equation 3.4 means that the frequency Ω of the periodic component x^{*} is much greater (practically ten times or more) than the highest frequency of the slowly-varying component x^{o}. In this case, no harmonic relation between the components x^{o} and x^{*} nonlinear system subject to forcing signals, such as jump-resonance, generation of subharmonics, etc., cannot take place. The forced nonlinear oscillations for which the condition (3.4) is not satisfied necessarily, are considered in other works.

Under the condition (3.4), the values of x^{o}, A, and Ω, which appear in the solution $x = x^{o} + A \sin \Omega t$, are slowly-varying quantities in time. This enables the extension of the conventional harmonic linearization in which the describing function is defined for the signal $x = x^{o} + x^{*}$ as an input to the nonlinear element. Thus, the nonlinear function F(x,sx) is approximately expressed by the first terms of the Fourier series as

$$F(x,sx) = F^{o} + N_1 x^{*} + \frac{N_2}{\Omega} s x^{*} \tag{3.5}$$

where

$$F^{o} = \frac{1}{2\pi}\int_{0}^{2\pi} F(x^{o} + A \sin \phi,\ A\Omega \cos \phi)\, d\phi \tag{3.6a}$$

$$N_1 = \frac{1}{\pi A}\int_{0}^{2\pi} F(x^{o} + A \sin \phi,\ A\Omega \cos \phi) \sin \phi\, d\phi \tag{3.6b}$$

$$N_2 = \frac{1}{\pi A}\int_{0}^{2\pi} F(x^{o} + A \sin \phi,\ A\Omega \cos \phi) \cos \phi\, d\phi \tag{3.6c}$$

and $\phi = \Omega t$.

As can be seen from equations 3.5 and 3.6a, the component F^{o} of the output of the nonlinearity F(x,sx) is not considered zero as was the case in the analysis of symmetrical nonlinear oscillations presented in the previous chapter. This results from the fact that either the nonlinear function $F(x,sx)$ is not symmetric or the system is subject to an external input signal, or that both facts are present in the system.

According to equations 3.6, all coefficients F^o, N_1, and N_2 are generally functions of x^o, A, and Ω, i.e.,

$$F^o+F^o(x^o,A,\Omega),\quad N_1 = N_1(x^o,A,\Omega),\quad N_2 = N_2(x^o,A,\Omega) \quad (3.7)$$

For a majority of the nonlinear functions $F(x,sx)$ encountered in practical applications, the above functions (3.7) are obtained once and for all.

By applying the linearization of the function $F(x,sx)$ given in equation 3.5, the solution $x = x^o + x^*$ of (3.1) can be obtained by considering the following linearized differential equation

$$B(s)(x^o+x^*) + C(s)(F^o+N_1x^* + \frac{N_2}{\Omega} sx^*) = H(s)f \qquad (3.8)$$

instead of equation 3.1. If x^o, A, and Ω are slowly-varying functions of time as a consequence of the same property associated with the forcing function f, equation 3.8 can be rewritten as two simultaneous equations corresponding to the slowly-varying signal x^o and the periodic signal x^* as follows:

$$B(s)x^o + C(s)F^o = H(s)f \qquad (3.9a)$$

$$B(s)x^* + C(s)(N_1x^* + \frac{N_2}{\Omega} sx^*) = 0 \qquad (3.9b)$$

Equations 3.9, however, cannot be solved independently since they are related to each other by the nonlinear equations 3.7. This

fact indicates that the applied linearization preserves the essential feature of nonlinear systems and that the superposition principle from linear analysis is not valid.

An analytical solution of equations 3.9 is difficult to obtain since F^o in(3.9a) is usually a trancendental function with respect to x^o. A graphical procedure is presented for solving equations 3.9 in the parameter plane. A necessary condition for equation 3.1 to have a solution $x(t)$ close to 3.2 is that the characteristic equation

$$B(s) + C(s)(N_1 + \frac{N_2}{\Omega}s) = 0. \qquad (3.10)$$

corresponding to the linearized differential equation 3.9b, have a pure imaginary root $s = j\Omega$.

By using the parameter plane approach, equation 3.10 can be solved for α and β as

$$\alpha = \alpha(\Omega) \qquad (3.11)$$
$$\beta = \beta(\Omega)$$

where α and β are N_1 and N_2 or some other system adjustable parameter. Equations 3.11 represent the $\Sigma = 0$ (or $\zeta = 0$) curve for which $s = j\Omega$. The $\Sigma = 0$ curve determines the stable region in the $\alpha\beta$ plane in the usual manner. After the stable region is found, the loci of points $M(\alpha,\beta)$ are plotted according to the variations of α and/or β representing N_1 and/or N_2. The M loci incorporates the additional variable x^o, and a family of the loci should be constructed for different values of x^o. Then the stability of the nonlinear system is determined by the relative location of the Σ curve and the M loci and the limit cycles are found at their intersections. The stability of the limit cycles is determined in the usual manner. This part of the solution process will be best described by the examples that follow.

The presence of a limit cycle in the system can modify the nonlinear characteristic for the slowly-varying input signal. In order to determine the modified characteristic, the intersections of the $\Sigma = 0$ curve and the M loci are considered to evaluate the amplitude A and the frequency Ω of the limit cycle as functions of the slowly-varying component x^o; i.e.,

$$A = A(x^o), \quad \Omega = \Omega(x^o) \tag{3.12}$$

These functions, when substituted into the function $F^o(x^o, A, \Omega)$ yield the modified nonlinear characteristic for the slowly-varying signal

$$F^o = \psi(x^o) \tag{3.13}$$

The function $\psi(x^o)$ is continuous in a limited range of x^o, which indicates the smoothing effect due to the presence of the limit cycle.

Substitution of equation 3.13 into equation 3.9a gives

$$B(s)x^o + C(s)\ \psi(x^o) = H(s)f \tag{3.14}$$

Equation 3.14 is a nonlinear differential equation in x^o, which can be solved graphically for x^o after the function $\psi(x^o)$ is obtained. This, in turn, yields the related values of the functions $A(x^o)$ and $\Omega(x^o)$ of equations 3.12, and the solution $x = x^o + A \sin \Omega t$ is thereby determined.

The function $\psi(x^o)$ is a continuous function of x^o and it can

be assumed approximately linear for small variations of x^o. Then the stability problem related to equation 3.14 can be solved by known linear methods. If it is regarded as a nonlinear function of x^o, it can be linearized by harmonic linearization and the results of the previous chapter can be applied.

It should be noted here that the same parameter plane procedure can be used when the right side of equation 3.1 has more than one forcing function; i.e., the right-hand side is expressed by $\sum_{i=1}^{r} H_i(s) f_i$. The solution x, however, must be found by considering all existing inputs simultaneously since the superposition principle of linear analysis is not valid. Furthermore, if the polynomial H(s) of equation 3-1 can be factored in the form $sH_1(s)$, the procedure applied to the case in which the rate sf of the function f is considered as a slowly-varying signal; i.e., $|sf(t+T) - sf(t)|$.

The presented graphical procedure can be extended to nonlinear control systems with two nonlinear functions $F_1(s)$ and $F_2(x)$, whereby the following nonlinear differential equation is investigated:

$$B(s)x + C(s)\ F_1(x) + D(s)\ F_2(x) = H(s)f. \tag{3.15}$$

In this case, a procedure similar to that given in Section can be extended to determine the solution $x = x^o + x^*$.

The general procedure outlined in this section is modified depending on the actual problem involved. These problems may be divided into three major groups: asymmetrical nonlinearities;

constant forcing signals; and slow-varying signals. In the following, each group is considered separately.

3.3 Asymmetrical Nonlinearities.

In an autonomous nonlinear system, which is described by the differential equation 3.1 and where $f \equiv 0$, the asymmetrical oscillations may occur whenever the function $F(x, sx)$ is not symmetrical to the origin. Then, under the conditions discussed in the previous section, the system may be described by equations 3.9 which has the form

$$B(o)\ x^{o} + C(o) F^{o} = 0 \qquad (3.16a)$$

$$[B(s) + C(s)(N_1 + \frac{N_2}{\Omega}s)]x^{*} = 0 \qquad (3.16b)$$

In equation 3.16a, which corresponds to equation 3.9a, there is no forcing slowly-varying function ($f \equiv 0$), and in the steady-state solution $x = x^{o} + x^{*}$, the x^{o} is constant and hence s is replaced by zero in $B(s)$ and $C(s)$.

In practical situations, $B(o)$ or $C(o)$ can be zero. Also, the nonlinearity in the system is often described by a single-valued function $F(x)$ and $N_2=0$. Thus, an adjustable parameter appearing in $B(s)$ or $C(s)$ can be chosen as one of the axes in the parameter $\alpha\beta$ plane, while the other axes is related to the describing function coefficient N_1. Some of these situations are discussed in the following examples.

Consider a feedback control system with the block diagram of Fig. 3.1 in which the transfer functions are

$$G_1(s)=K_1,\ G_2(s)=\frac{K_2}{s(s+1)},\quad G_3=\frac{K_3}{s+2},\ G_{-1}(s)=K_{-1}s. \qquad (3.17)$$

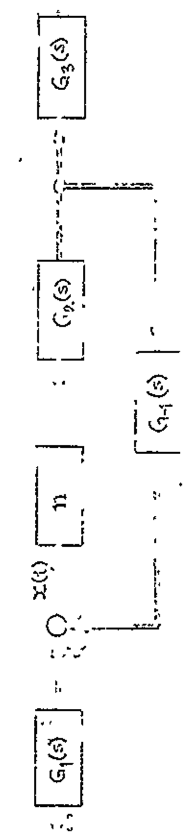


Fig. 3.1. System block diagram

The nonlinearity n has the form shown in the upper left corner of Fig. 3.2.

Equations 3.16, for the system under investigation, have the form

$$F^{o} = 0 \tag{3.18a}$$

$$\{s(s+1)(s+2) + [K_2K_{-1}s(s+2) + K_1K_2K_3]N_1\}x^* = 0 \tag{3.18b}$$

where, according to the function F(x) of Fig. 3.2 and equations 3.6, one has

$$F^{o} = \frac{(1-m)c}{2} + \frac{(1+m)c}{\pi} \text{ arc sin } \frac{x^{o}}{A} \tag{3.19a}$$

$$N_1 = \frac{2(1-m)c}{A}\sqrt{1 - (\frac{x^{o}}{A})^2} \tag{3.19b}$$

$$N_2 = 0 \tag{3.19c}$$

and $x = x(t)$ is the input signal to the nonlinearity n as indicated in Fig. 3.1.

The characteristic equation of equation 3.18b is

$$s(s+1)(s+2) + [K_2K_{-1}s(s+2) + K_1K_2K_3]N_1 = 0 \tag{3.20}$$

By denoting $K_2K_{-1}N_1 = \alpha$ and $K_1K_2K_3N_1 = \beta$, the $\zeta = 0$ curve is obtained as

$$\alpha = \frac{1}{2}(\Omega^2 - 2) \tag{3.21}$$

$$\beta = \frac{1}{2}\Omega^2(\Omega^2 + 4)$$

and the stable region is determined in the $\alpha\beta$ plane in the usual fashion as shown in Fig. 3.2.

From equations 3.18a and 3.19a, one obtains

$$x^{o} = A \cos \frac{\pi}{1+m} \tag{3.22}$$

and N_1 of equation 6.19b becomes

$$N_1 = \frac{2(1+m)c}{A} \sin \frac{\pi}{1+m} \tag{3.23}$$

By using equation 3.23 and the expressions $\alpha = K_2K_{-1}N_1$, $\beta = K_1K_2K_3N_1$, three M loci (a), (b), and (c), are drawn in Fig. 3.2. They correspond to the parameter values $m = 0.5$, $c = 1$, $K_2 = 1$ and (a) K_1K_3, $K_{-1} = 0.125$; (b) $K_1K_3 = 8.39$, $K_{-1} = 0.28$; (c) $K_1K_3 = 26$, $K_{-1} = 1.75$. The stable asymmetrical oscillations are found at the point M_1 and M_2 where the M loci (a) and (b) intersect the $\zeta = 0$ curve. The amplitudes of the oscillations are approximately $A_1 = 0.85$ and $A_2 = 0.8$, which is read from the M loci (a) and (b) at the intersections M_1 and M_2. The corresponding frequencies $\Omega_1 = 1.5$ and $\Omega_2 = 1.6$ are indicated on the $\gamma = 0$ curve. The related values of x^{o} in the solution $x = x^{o} + \alpha \sin \Omega t$ is calculated for each point M_1 and M_2 using equation 3.22, namely, $x_1^{o} = -0.42$ and $x_2^{o} = -0.39$.

In Fig. 3.3, the solution $x_1 = 0.42 + 0.85 \sin 1.5t$ for the case (a) is shown as obtained by a digital computer simulation. The calculated results are sufficiently close to that obtained by the simulation. From Fig. 3.3, it can be seen that an initial condition $x_1(o) = 4.25$ is used and the variable $x_1(t)$ approached a stable limit cycle. That the limit cycle is stable and will be reached by $x_1(t)$ starting from $x_1(o) = 4.25$ can be concluded from the relative location of the $\zeta = 0$ curve and the M locus (a), as explained in the preceding chapter on the symmetrical oscillations. The additional component x^{o} of the solution $x(t)$ does not alter the stability analysis of the oscillations.

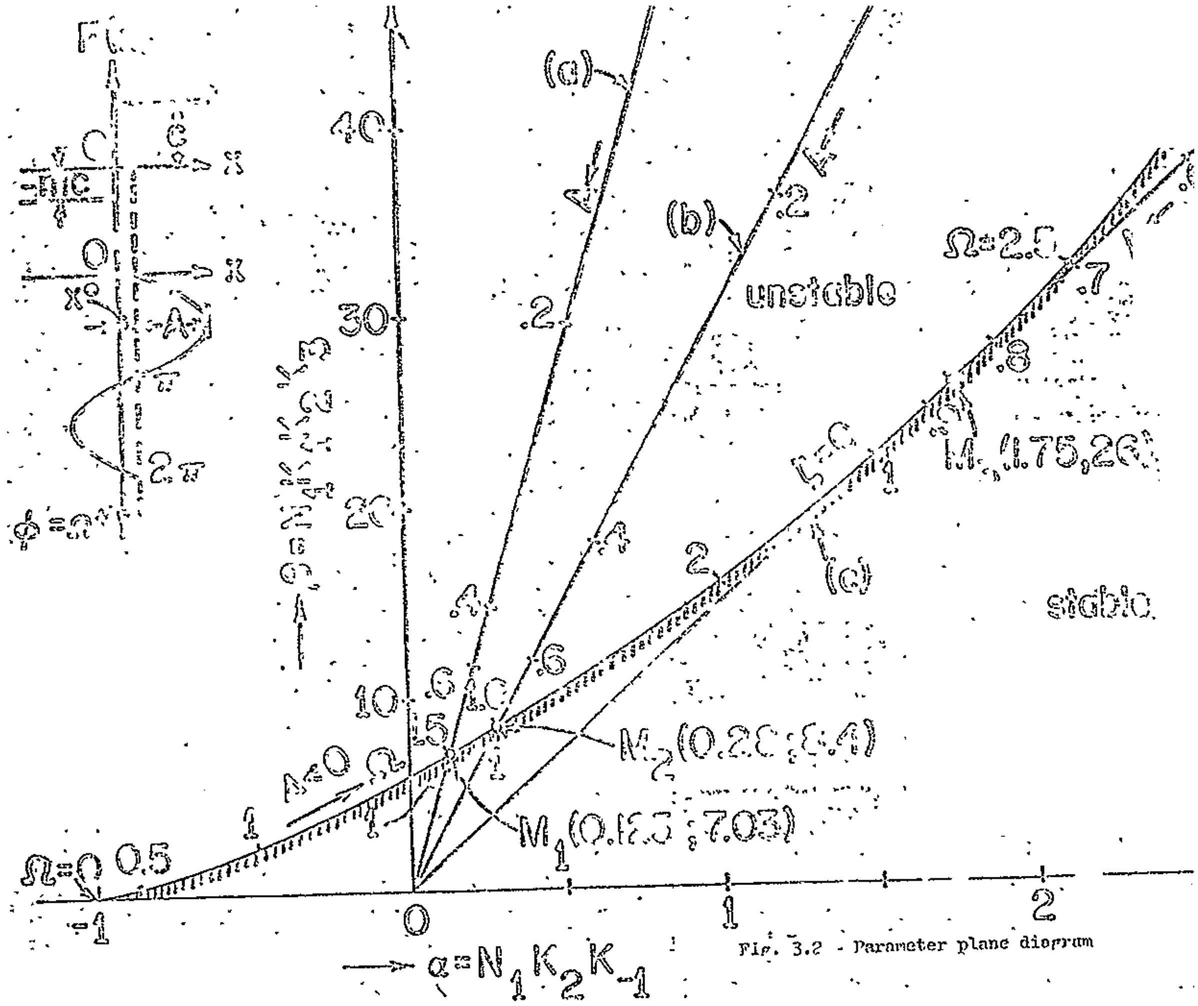


Fig. 3.2 - Parameter plane diagram

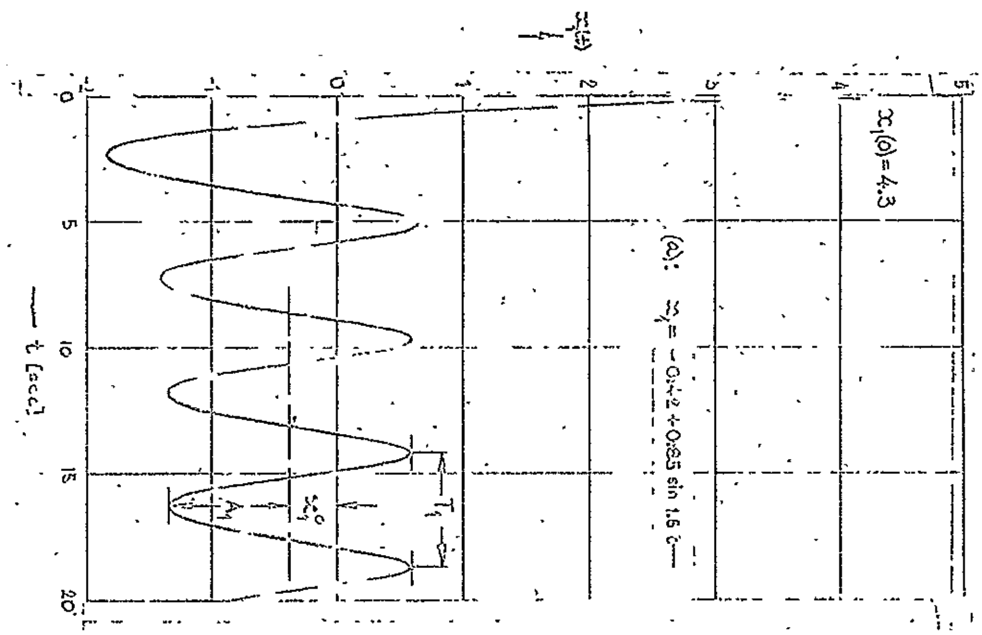
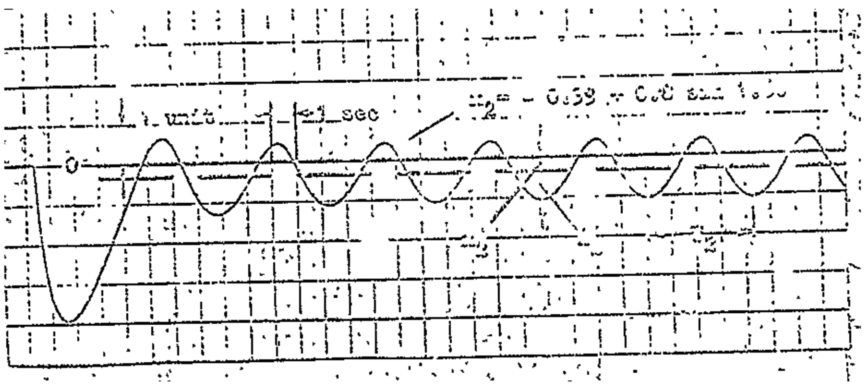


Fig. 3.3 - Digital computer solution in case (a)



Analog computer solution in case (c)

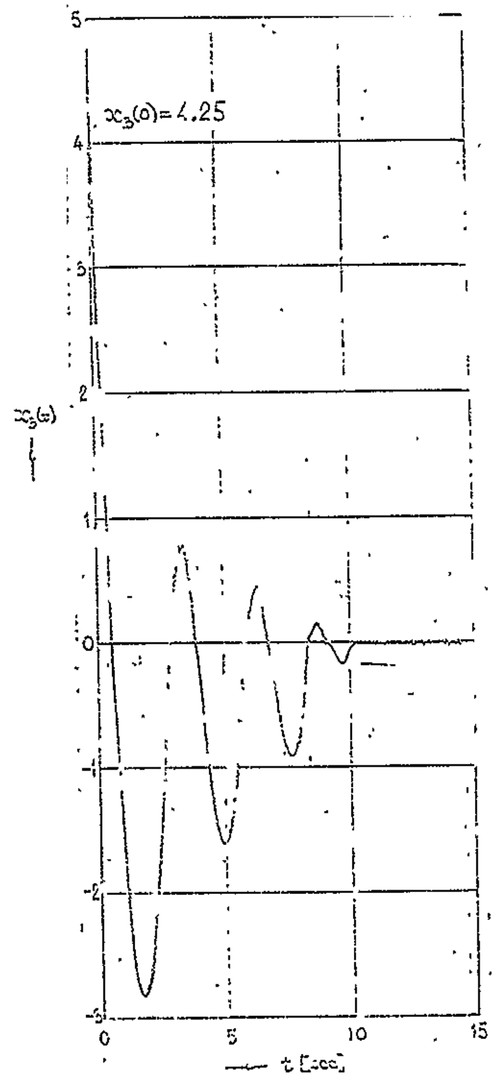


Fig. 3.5 – Digital computer solution in case (c)

An analog computer simulation of the case (b) gives the solution $x_2 = -0.39 + 0.8 \sin 1.6t$ as shown in Fig. 3.4. A sufficient accuracy is indicated. The initial condition $x_2(o) = 0$ and $x_2(t)$ reached a limit cycle. This could be concluded from Fig. 3.2 as previously noted.

It is of particular interest to consider the case (c) of Fig. 3.2. The M locus (c) is tangent to the = 0 curve and corresponds to the ratio $\alpha/\beta = K_1K_3/K_{-1} = 14.8$. If this ratio is higher than 14.8, then there is a limit cycle as shown by cases (a) and (b). On the other hand, if this ratio is less than 14.8, the entire M locus is situated in the stable region and the corresponding system is always stable. The tangent case (c): $m = 0.5$, $c = 1$, $K_2 = 1$, $K_1K_3 = 26$, $K_{-1} = 1.75$, is simulated on a digital computer and the obtained solution $x_3(t)$ is shown in Fig. 3.5, which indicates that the system is stable.

3.4 Constant Forcing Signals

When the forcing signal at certain points of a nonlinear system is constant, the solution $x = x^o + A \sin \Omega t$ (if it exists) will have x^o, A, and Ω as constant values. To determine these values, note that the equations to solve in the presence of a constant forcing signal f^o have the form

$$B(o)x^o + C(o)F^o = H(o)f^o \tag{3.24a}$$

$$[B(s) + C(s)N_1 + \frac{N_2}{\Omega} s]x^* = 0 \tag{3.24b}$$

In general B(o), C(o), and H(o) are constants different from zero, and the solution procedure is somewhat more complicated to perform than in the previous section where the rig

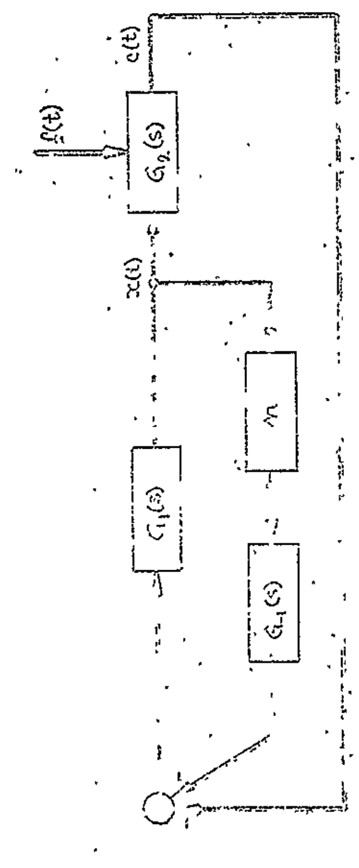


Fig. 3.6 - Syst. a block diagram

side of equation 3.24a was zero.

To illustrate the solution procedure, consider a nonlinear feedback system with the block diagram of Fig. 3.6 and the transfer functions

$$G_1(s) = \frac{2}{0.2s^2+0.8s+1}, \quad G_2(s) = \frac{0.5(s+1)}{0.2s+1}, \quad G_{-1} = \frac{K_{-1}}{T_{-1}s+1} \tag{3.25}$$

The nonlinearity n is given in Fig. 3.7a. The input to the system is a perturbation signal $f = f(t)$ which is related to the signal $x = x(t)$ and $c = c(t)$ of Fig. 3.6 as

$$(0.2s+1)c = 0.5(s+1)x-f \tag{3.26}$$

If the perturbation signal is $f(t) = f^o = \text{const.}$, equations 3.24 have the form

$$x^o + K_{-1}F^o = f^o \tag{3.27a}$$

$$(0.04s^4 + 0.36s^3 + 2s^2 + 2s)T_{-1}+(0.4s+2)K_{-1}N_1 + \\ + 0.04s^3 + 0.36s^2 + 2s + 2 = 0 \tag{3.27b}$$

where equation 3.27b represents the characteristic equation of the linearized equation 3.24b. By substituting $T_{-1} = \alpha$ and $K_{-1}N_1 = \beta$, the parameter plane diagram is plotted in Fig. 3-7b according to the parameter plane equations

$$\alpha = \frac{0.64\Omega^2 + 3.2}{0.016\Omega^4 - 0.08\Omega^2 - 4}$$

$$\beta = \frac{0.016\Omega^6 - 0.03\Omega^4 + 2.56\Omega^2 + 4}{0.016\Omega^4 - 0.08\Omega^2 - 4} \tag{3.28}$$

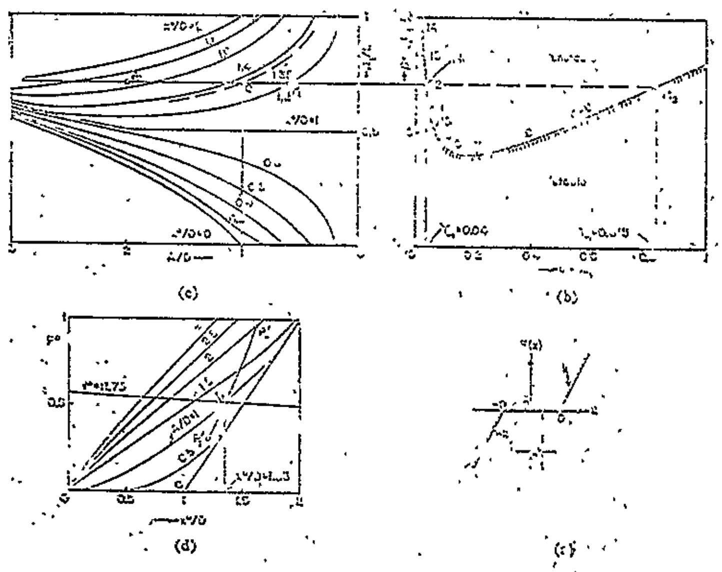


Fig. 3.7 - Parameter plane diagram

The variation of the M point due to the function $N_1 = N_1(x^o, A)$ given as

$$N_1 = k - \frac{k}{\pi}(\arc\sin\frac{D-x^o}{A} + \arc\sin\frac{D+x^o}{A} + \frac{D-x^o}{A}\sqrt{1-(\frac{D-x^o}{A})^2} + \frac{D+x^o}{A}\sqrt{1-(\frac{D+x^o}{A})^2}, \quad A \geq D + |x^o| \tag{3.29}$$

is plotted in Fig. 3.7c. (The expression (3.29), corresponds to the nonlinearity of Fig. 3.7a). In order to find a solution $x = x^o + x^*$ of equations 3.27, the parameter k is assumed equal to one, and the function $F^o(x^o, A)$ is plotted in Fig. 3.7d by using

$$F^o = \frac{kA}{\pi}\sqrt{1-(\frac{D-x^o}{A})^2} - \sqrt{1-(\frac{D+x^o}{A})^2} + kx^o + \frac{k}{\pi}[D(\arc\sin\frac{D-x^o}{A} - \arc\sin\frac{D+x^o}{A}) - x^o(\arc\sin\frac{D-x^o}{A} + \arc\sin\frac{D+x^o}{A})], \quad A \geq D + |x^o| \tag{3.30}$$

For $T_{-1} = 0.04$, the point $M_1(0.04; 14.3)$ corresponds to a solution $x = x^o + x^*$ which will have $\Omega = 12$ rad/sec as indicated on the curve $\zeta = 0$. If $K_{-1} = 20$, from M_1 it follows that $N_1 = \beta/K_{-1} = 0.715$. This value of N_1 determines the relationship between the values of x^o and A for a possible solution x. This relationship, expressed as a function $A = A(x^o)$, can be graphically obtained from the diagram $N_1 = N_1(x^o, A)$ by plotting the straight line P_1P_2 corresponding to the value $N_1 = 0.715$.

The function $A = A(x^o)$ represents the solution of equation 3.27b only. The pair of values (x^o, A) which enter into the actual solution of equation 3.27, is replotted on the diagram

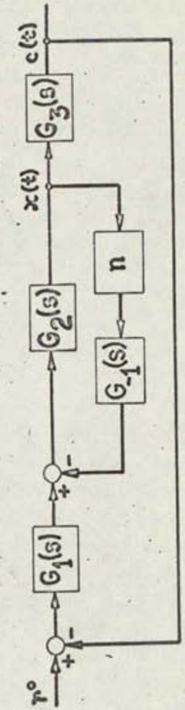


Fig. 3.8 - System block diagram

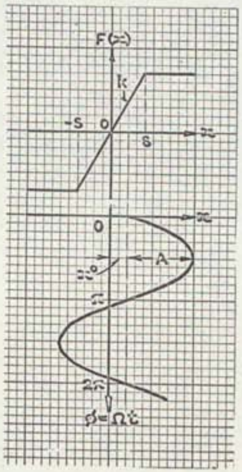


Fig. 3.9 - Nonlinear characteristic

$F^o = F^o(x^o, A)$ of Fig. 3.7d into the curve $P_1'P_2'$. Suppose that the constant perturbing signal has a value of $f^o = 11.75$; then equation 3.27a determines the straight line $f^o = 11.75$ plotted in the diagram $F^o = F^o(x^o, A)$ of Fig. 3.7d. The intersection R of that straight line and the curve $P_1'P_2'$ gives the pair (x^o, A) of the solution x(t) which satisfies equation 3.27 simultaneously. At this point R, the values are $x^o/D = 1.35$ and $A/D = 1$. The same values are obtained at the point Q on the diagram $N_1 = N_1(x^o, A)$ and the solution $x = x^o + A \sin \Omega t$ of equations 3.27 is found. If $D = 1$, it is $x = 1.35 + \sin 12t$. Note that the same solution is obtained if the point M_2 of Fig. 3.7b is considered save that the frequency Ω is lower (approximately $\Omega = 5.5$ rad/sec).

Simpler situations may occur if one of the values B(o) or C(o) is zero. To illustrate, consider the nonlinear system of Fig. 3.8. The transfer functions are

$$G_1(s) = K_1,\ G_2(s) = \frac{K_2}{s(s+1)},\ G_3(s) = \frac{K_3}{s+2},\ G_{-1}(s) = K_{-1}s \tag{3.31}$$

and the nonlinearity n in the system is given by the function F(x) of Fig. 3.9. The input to the system is the reference constant input signal $r(t) = r^o$.

The nonlinear differential equation describing the above system is

$$[s(s+1)(s+2)+K_1K_2K_3]x+K_2K_{-1}s(s+2)F(x) = K_1K_2(s+2)r^o \tag{3.32}$$

which may be rewritten according to equations 3.24 as

$$K_1K_2K_3x^o = 2r^o \tag{3.33a}$$

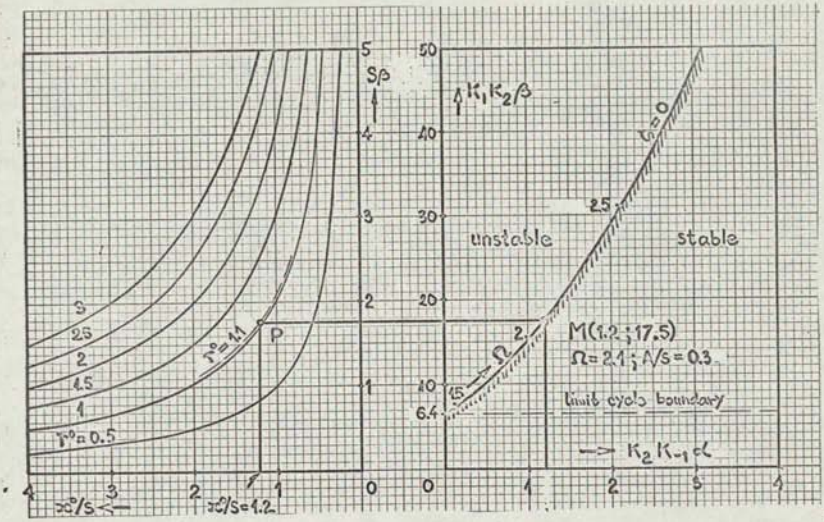


Fig. 3-10 - Parameter plane diagram

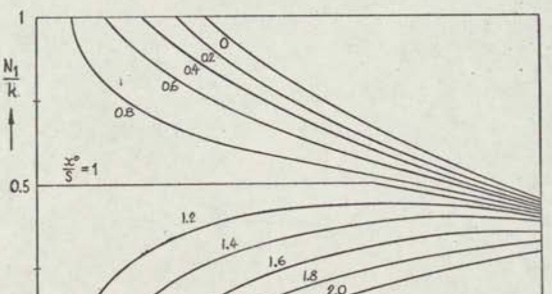


Fig. 3.11 - Function $N_1(A, x^o)$

$$[s(s+1)(s+2) + K_1K_2K_3 + K_2K_{-1}s(s+2)N_1]x^* = 0 \qquad (3.33b)$$

The characteristic equation of the equation 3.33b is evidently

$$s(s+1)(s+2) + K_1K_2K_3 + K_2K_{-1}s(s+2)N_1 = 0. \qquad (3.34)$$

By denoting

$$\alpha = N_1 \qquad \beta = K_3 \qquad (3.35)$$

the parameter plane diagram is plotted in Fig. 3.10 in the usual fashion. The function $N_1 = N_1(A, x^o)$, which appears as a variation of α in the point $M(a;\ \beta)$ is plotted in Fig. 3.11 by using general formula 3.6b.

From equation 3.33a, one can derive the following relationship between the input r^o, the constant term x^o, and the parameter $\beta = k_3$,

$$S\beta = \frac{2r^o}{x^o/S} \qquad (3.36)$$

where S is the parameter of the nonlinearity F(x) of Fig. 3.9. The function $S\beta$ given in (3.36) is plotted in Fig. 3.10.

Now, by using Fig. 3.10 and 3.11, it is possible to determine the sustained oscillations and their stability for various values of system parameters K_1, K_2, K_3, K_{-1}, S, k, and the input r^o. For example, if $K_1 = 1$, $K_2 = 10$, $K_3 = 1.75$, $K_{-1} = 1$, $S = 1$, $k = 1$, and $r^o = 1.1$, then the solution of equation 3.33 is determined by the values $x^o = 1.2$, $A = 0.3$, and $\Omega = 2.1$ rad/sec to be approximately

$$x = 1.2 + 0.3 \sin 2.1t \qquad (3.37)$$

For a given value of $\beta = K_3 = 1.75$, $r^o = 1.1$, and $S = 1$, the value of $x^o = 1.2$ is read from the left part of Fig. 3.10. Then the

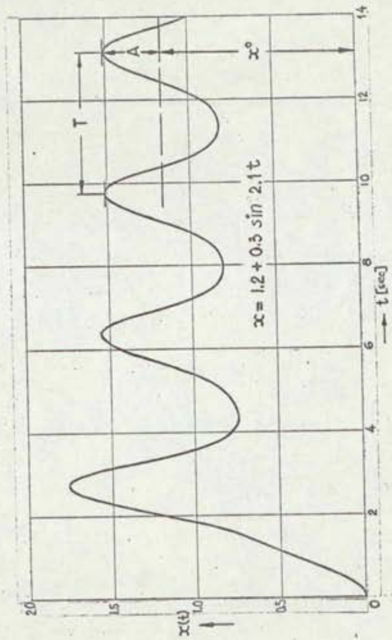


Fig. 3.12 - Computer solution

value of $K_1K_2\beta = 17.5$ determines the point M(1.2; 17.5) on the $\zeta = 0$ curve where $\Omega = 2.1$ rad/sec. At this point, $K_2K_{-1}\alpha = 1.2$ which gives $N_1 = \alpha = 0.12$. Fig. 3.11 is used to evaluate the amplitude A = 0.3 from the curve $x^o/S = 1.2$. The value A = 0.3 is read directly from the diagram $N_1(A, x^o)$ of Fig. 3.11, since K = S = 1 are the parameters of the given nonlinearity in Fig. 3.9.

The solution (3.37) is stable since an increase in the amplitude A causes the point M to move into the stable region; while a decrease in the amplitude A places the point M inside the unstable region of the parameter plane (Fig. 3.10). It is of interest to note that if the produce $K_1K_2\beta$ where $\beta = K_3$ is such that it is less than 6.4, the system is always stable since there is no intersections of the variation of the M point and the $\zeta = 0$ curve.

The above solution (3.37) is checked by computer simulation to obtain the curve on Fig. 3.12. The accuracy of the calculated solution is sufficiently high and, for calculated values of x^o, A, and Ω, is approximately 10%. On the other hand, the computer solution indicates a distortion of the assumed solution $x = x^o + A \sin \Omega t$ which is due to the higher harmonics present in the actual solution.

3.5 Slowly-varying Signals

In this section, the problem of linearizing a nonlinear system by a high-frequency limit cycle is considered in more detail. The objective is to determine the conditions under which

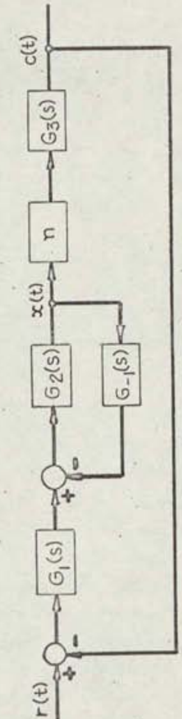


Fig. 3.13 - System block diagram

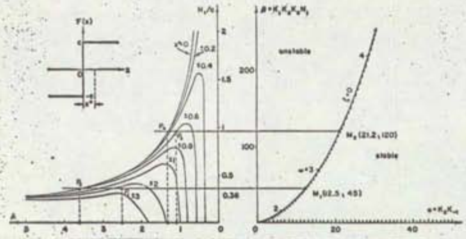


Fig. 3-14 - Parameter plane diagram

such a linearization is possible and then to construct the linearized characteristic of the nonlinearity. This linearization has several practical aspects discussed in Section 3.1, which are based upon a general property of the linearized system that, for a limited magnitude of the reference signal, behaves like a linear system. Therefore, results of the nonlinearities, such as dead-zone, hysteresis, backlash, etc., are eliminated. The procedure to achieve this will be best illustrated in the following examples.

Consider the system on Fig. 3.13 with the transfer functions

$$G_1(s) = K, \quad G_2(s) = \frac{K_2}{s^2+0.8s+1}, \quad G_3(s) = \frac{K_3}{s(s+1)}, \quad G_{-1}(s) = K_{-1} \tag{3.38}$$

and the nonlinearity n as shown in Fig. 3.14. The input to the system is a slowly-varying reference signal $r = r(t)$.

The equation which describes the system is

$$[s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1)]x + K_1K_2K_3F(x) = K_1K_2s(s+1)r \tag{3.39}$$

where the signal $x = x(t)$ is the input to the nonlinearity. Equation 3.39 can be rewritten in terms of equations 3.9 as

$$[s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1)]x^o+K_1K_2K_3F^o = K_1K_2s(s+1)r$$

$$[s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1)]+K_1K_2K_3N_1\ x^* = 0 \tag{3.40}$$

The characteristic equation of the second equation 3.40 is

$$s(s+1)(s^2+0.8s+1)+K_2K_{-1}s(s+1) + K_1K_2K_3N_1 = 0 \tag{3.41}$$

Substituting $K_2K_{-1} = \alpha$, $K_1K_2K_3N_1 = \beta$, and $s = j\Omega$ into equation 3.41, one obtains the parameter plane equations of the $\zeta = 0$ curve

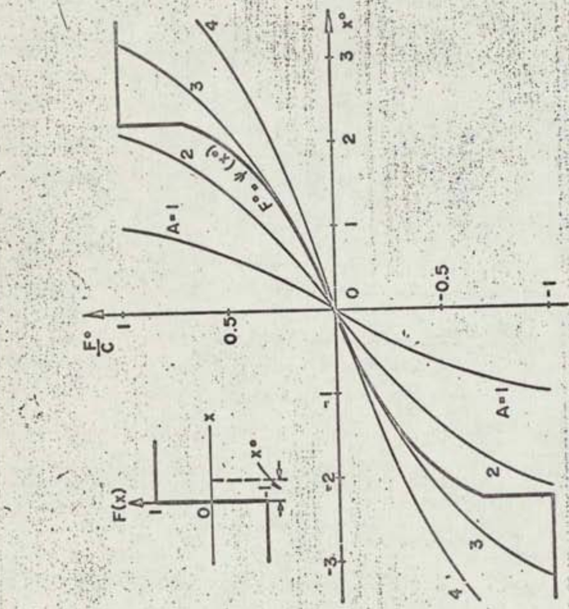


Fig. 3.13 - Function ψ(x°)

as

$$\alpha = 1.8\ \Omega^2 - 1 \qquad (3.42)$$
$$\beta = 0.8\ \Omega(\Omega + 1).$$

The curve $\zeta = 0$ is plotted in Fig. 3.14. The variations of the M point are plotted also in Fig. 3.14 according to

$$N_1 = \frac{4c}{\pi A}\sqrt{1 - (\frac{x}{A})^2}, \quad A \geq |x^o| \qquad (3.43)$$

The system parameters $K_1 = 1$, $K_2 = 12.5$, $K_3 = 10$, $K_{-1} = 1$ result in the point $M_1(12.5;\ 45)$. If $c = 1$, this point M_1 gives $N_1 = \beta/K_1K_2K_3 = 0.36$, and the straight line P_1P_2 is plotted on the diagram of function $N_1 = N_1(x^o, A)$. After the diagram $F^o = F^o(x^o, A)$ is plotted in Fig. 3.15 using

$$F^o = \frac{2c}{\pi} \text{ arc sin } \frac{x^o}{A}, \ A \geq |x^o| \qquad (3.44)$$

the replotting of the straight line P_1P_2 on the diagram $F^o(x^o, A)$ yields the function (x^o) of Fig. 3.15. The replotting procedure is the same as that used in the previous section; i.e., for each pair of values (x^o, A) read on the straight line P_1P_2, the corresponding pair exists in the diagram $F^o(x^o, A)$, which determines one point on the curve $\phi(x^o)$.

Function $\phi(x^o)$ of Fig. 3.15 is smooth and represents the nonlinearity for the slowly-varying signal x^o. The shape of $\phi(x^o)$ explains the smoothing effect of the high frequency limit cycle which has a slowly-varying amplitude, the value of which is located between the points Q_1 and Q_2 on the A axis of Fig. 3.14. The frequency Ω is approximately constant and has the value $\Omega \simeq 2.7$ rad/sec. According to $\phi(x^o)$, the smoothing effect of the

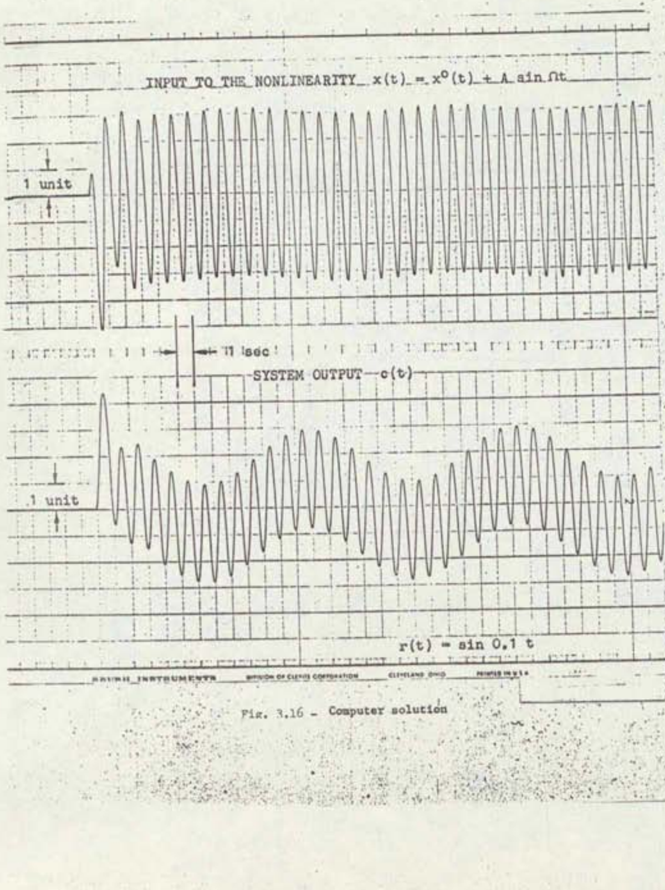


Fig. 3.16 - Computer solution

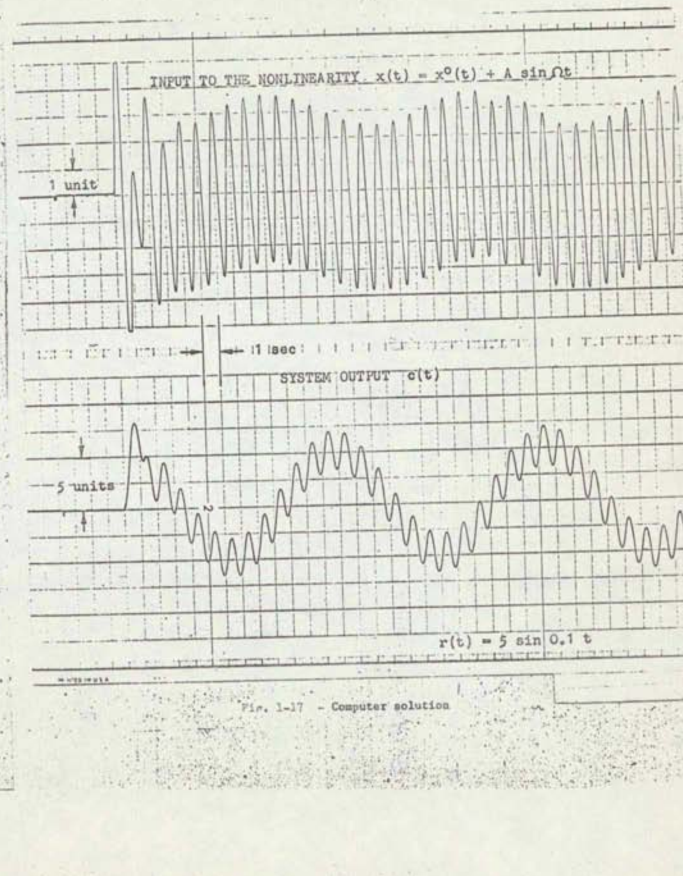


Fig. 1-17 - Computer solution

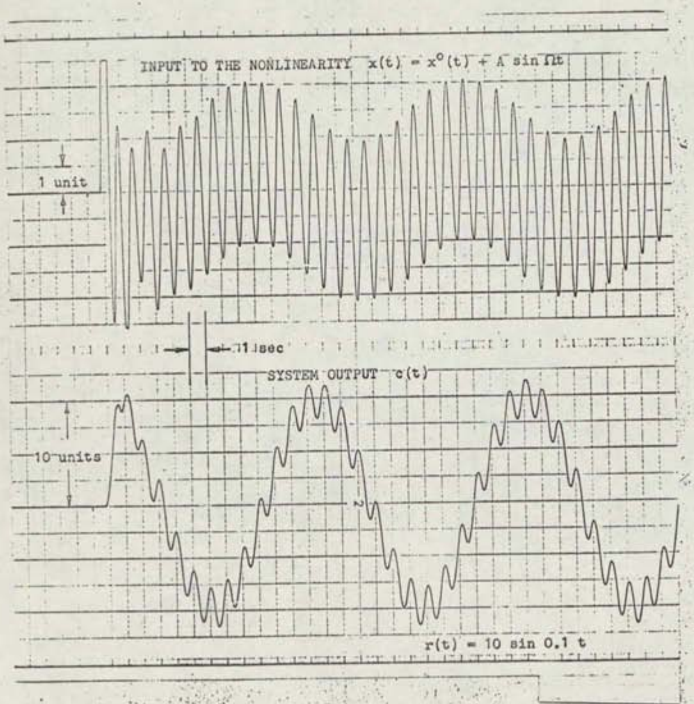


Fig. 3.18 - Computer solution

limit cycle is present under the condition that $|x^o| \leq 2.25$. For small values of x^o, it is possible to consider $\phi(x^o) = Kx^o$ where K = const. Then the stability of the system with respect to slowly-varying signals may be investigated by well-known linear methods outlined in Chapter II. In the specific example, the equation of interest is

$$s(s+1)(s^2+0.8s+1) + K_2K_{-1}s(s+1) + K\,K_1K_2K_3 = 0 \qquad (3.45)$$

Finally, it is to be noted that for the smoothing effect to take place, the amplitude A should be $A \geq |x^o|$, as stated in equations 3.43 and 3.44.

The results of the above analysis are checked by simulating the system on an analog computer. Three cases are considered. In Fig. 3.16, the input to the nonlinearity $x = x^o + A \sin \Omega t$ and the system output $x = x(t)$ are shown when the input signal is $r = \sin 0.1t$. The obtained computer solution agrees with the prediction. The output c(t) exhibits a smaller amplitude limit cycle with the same frequency. When the input amplitude is increased five times, the diagram of Fig. 3.17 is obtained. This change increased x^o, but the amplitude A remained almost the same. The frequency Ω did not change. Similar results occurred when the input amplitude increased ten times except that the amplitude A became slightly smaller, which agrees with the diagram of Fig. 3.14. The third case is given in Fig. 3.18. It should be noted from these computer solutions that the output signal c(t) represents the input signal r(t) except for the superimposed limit cycle. It can be eliminated by introducing

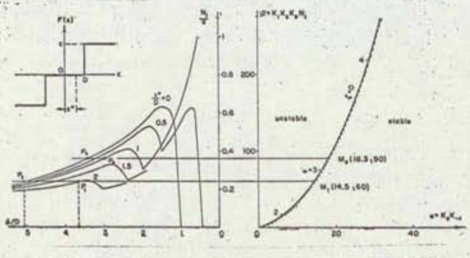


Fig. 3-19 - Parameter plane diagram

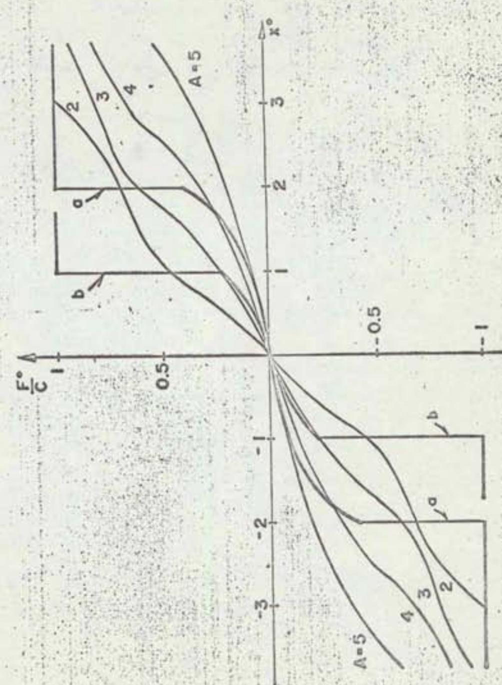


Fig. 3.20 - Linearized characteristic

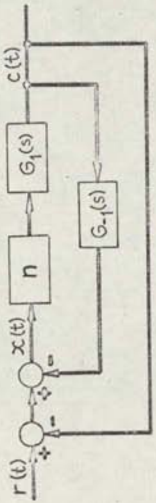


Fig. 3.21 - System block diagram

sufficient filtering in the block $G_3(s)$ of the system of Fig. 3.13, or by readjusting the system parameters to obtain a higher frequency limit cycle.

If the values of the system parameters are chosen so that the operating point is $M_2(21.2;\ 120)$ of Fig. 3.14, the frequency of the limit cycle becomes higher. However, the corresponding range of variations of x^o is decreased to $|x^o| < 0.7$, together with the range of the amplitude A which is between Q_3 and Q_4. This indicates that the presented procedure is convenient to apply when the system parameters and operating conditions are changed.

If the nonlinearity n is changed in the system of Fig. 3.13 by introducing a considerable dead zone D, a diagram of Fig. 3.19 is obtained. The variation of the M point is calculated by using equation 3.6b for the given nonlinearity of Fig. 3.19. Two cases should be considered separately; i.e.,

$$N_1 = \frac{2c}{\pi A}\left[\sqrt{1-\left(\frac{x^o+D}{A}\right)^2} + \sqrt{1-\left(\frac{x^o-D}{A}\right)^2}\right],\quad A \geq |x^o|+D \qquad (3.46a)$$

$$N_1 = \frac{2c}{\pi A}\sqrt{1-\left(\frac{x^o-D}{A}\right)^2},\quad |x^o|-D \leq A \leq |x^o|+D \qquad (3.46b)$$

and the diagram $N_1(x^o,\ A)$ is shown in Fig. 3.19. By using equation 3.6a, the corresponding diagram $F(x^o,\ A)$ of Fig. 3.20 is plotted according to

$$F^o = \frac{c}{\pi}\left(\arcsin\frac{x^o+D}{A} + \arcsin\frac{x^o-D}{A}\right),\quad A \geq |x^o| + D \qquad (3.47a)$$

$$F^o = \frac{c}{\pi}\left(\frac{\pi}{2} + \arcsin\frac{|x^o|-D}{A}\right)\sin x^o,\quad |x^o|-D \leq A \leq |x^o|+D \qquad (3.47b)$$

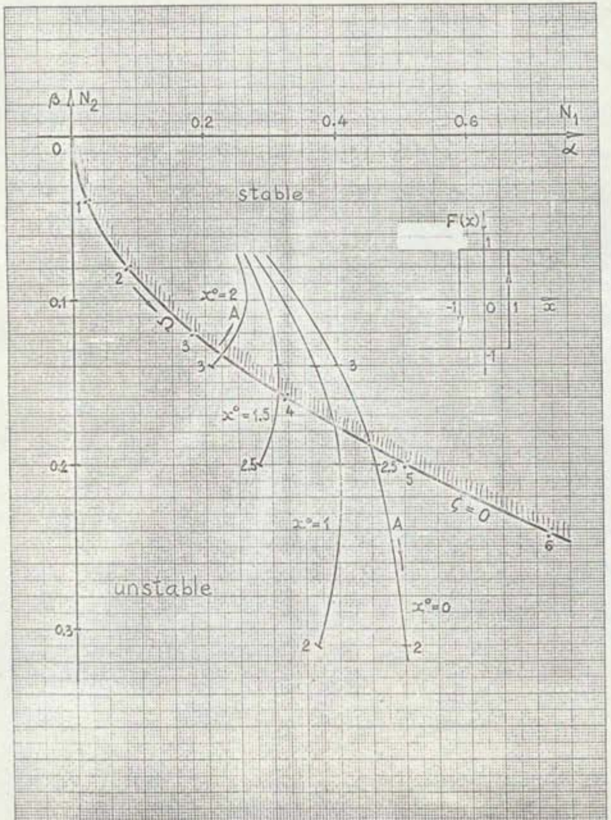


Fig. 3.22 - **Parameter plane diagram**

If the points M_1 and M_2 are chosen in Fig. 3.19 as operating points, the replotting of the straight lines P_1P_2 and P_3P_4 results in the two linearized characteristics a and b of Fig. 3.20, respectively. They are constructed for the values of nonlinear parameters c = D = 1. As can be seen from Fig. 3.20 the dead zone is eliminated as far as the slowly-varying signals are concerned. For this to take place, it is necessary to choose operating conditions such that equation 3.47a is valid. This means that the amplitude A of the limit cycle must be greater than $|x^o|+D$. Otherwise the linearized characteristic $\phi(x^o)$ does not go to zero when $x^o = 0$ since F^o does not go to zero for $x^o = 0$. This is indicated in Fig. 3.20 whereby $F^o = 0$ for $x^o = 0$ and the dead zone is eliminated.

By the outlined technique, it is possible to eliminate the hysteresis and backlash in systems with multi-valued nonlinearities. The linearization yields a single-valued function $\phi(x^o)$ which is linear in a certain limited range of values of the variable x^o about the origin. To illustrate this, consider a nonlinear system with the block diagram of Fig. 3.21 and the transfer functions

$$G_1(s) = \frac{K_1}{s(s+1)(s+2)}, \quad G_{-1}(s) = K_{-1}s \qquad (3.48)$$

The nonlinear function F(x) of the nonlinearity n is given in Fig. 3.22.

The equation describing the system is

$$s(s+1)(s+2)s + (K_{-1}s_{-1})K_1F(x) = 0 \qquad (3.49)$$

After harmonic linearization of 3.49, the corresponding characteristic equation is

$$s(s+1)(s+2) + K_1(K_{-1}s+1)(N_1 + \frac{N_2}{\Omega}s) = 0 \qquad (3.50)$$

If $K_1 = 50$, $K_{-1} = 1$

$$\alpha = N_1$$
$$\beta = N_2 \qquad (3.51)$$

and $s = j\Omega$, one obtains the $\zeta = 0$ curve as

$$\alpha = \frac{1}{50}\Omega^2$$
$$\beta = \frac{1}{25}\Omega. \qquad (3.52)$$

The curve is plotted in Fig. 3.22. On the same plot, the variation of the point $M(N_1;\ N_2)$ is constructed according to

$$N_1 = \frac{2c}{\pi A}\left(\sqrt{1-\left(\frac{D-x^o}{A}\right)^2} + \sqrt{1-\left(\frac{D+x^o}{A}\right)^2}\right) \quad A \geq D + |x^o|$$

$$N_2 = -\frac{4cD}{\pi A^2} \qquad (3.53)$$

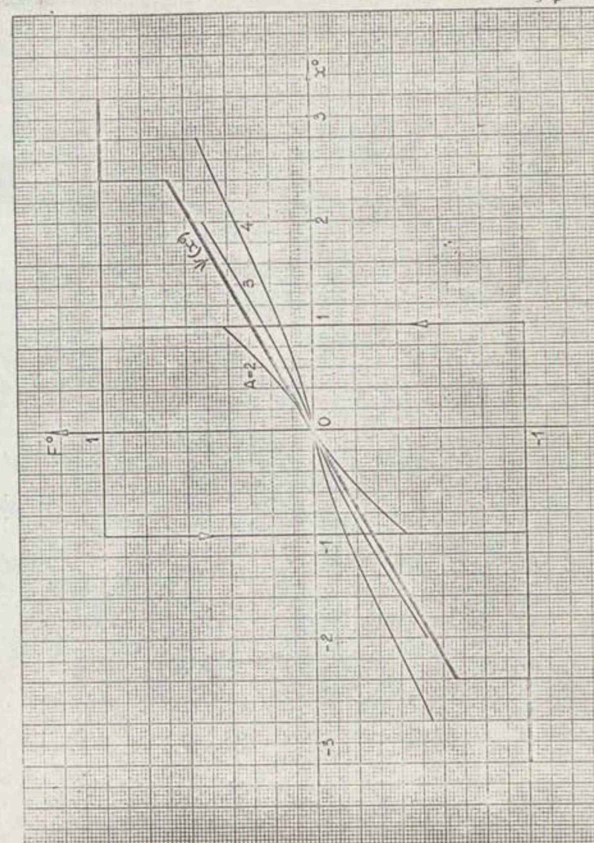
and the nonlinearity $F(x)$ of Fig. 3.22 for which $c = D = 1$. From the intersections of the $\zeta = 0$ curve and the variation of the M point, one can determine the amplitude A and the frequency Ω as function of x^o; i.e.,

$$A = A(x^o)$$
$$\Omega = (x^o) \qquad (3.54)$$

Then, by using the expression

$$F^o = \frac{c}{\pi}\left(\arcsin\frac{D+x^o}{A} - \arcsin\frac{D-x^o}{A}\right),\ A \geq D + |x^o| \qquad (3.55)$$

for $c = D = 1$, a family of curves with constant amplitude A is plotted on Fig. 3.23. If the first equation 3.54 is mapped onto



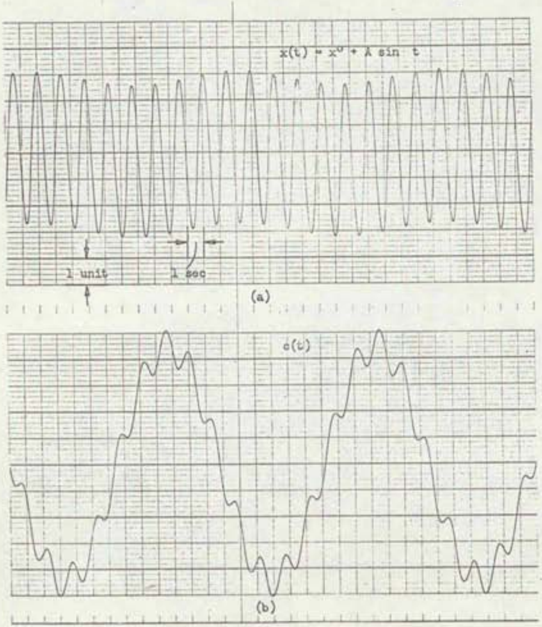


Fig. 3.24 - Computer solution

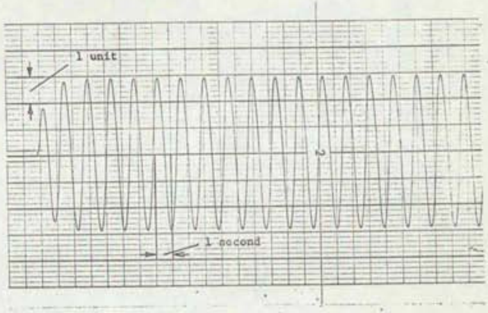


Fig. 3-25 - Computer solution for $x = 2.6 \sin 4.8t$

the family of constant amplitude A, the function $\psi(x^o)$ is obtained as shown in Fig. 3.23. The function $\psi(x^o)$ as a single-valued function of x^o, which is linear in the range $0 \le |x^o| \le 2.4$.

For an input $r(t) = 5 \sin 0.5t$, the computer solution is shown in Fig. 3.24. The amplitude A and the frequency Ω of the limit cycle are slowly-varying quantities according to equations 3.54 and the slowly-varying variable x^o. Their average values, however, are close to that which can be predicted from the parameter plane diagram of Fig. 3.22; i.e., $A = 2.8$ and $\Omega \simeq 4.5$ rad/sec. This can be concluded from the diagram (a) of Fig. 3.24. On the diagram (b), the output signal $c(t)$ is shown whereby the limit cycle is largely attenuated by the block $G_1(s)$ of Fig. 3.21. The low-frequency component in the signal $c(t)$ represents the input $r(t) = 5 \sin 0.5t$ at the output of the system.

Of course, if the input $r(t)$ is not present, the system will exhibit a limit cycle which can be determined from the intersection of the M locus $x^o = 0$ and the $\zeta = 0$ curve on Fig. 3.22 as $x = A \sin t$, $A = 2.6$, $\Omega \doteq 4.8$. This is checked by the analog computer simulation and the obtained solution is shown on Fig. 3.25.

3. 6 Conclusion

The parameter plane method has been used to indicate existence of asymmetrical oscillations in nonlinear control systems. A procedure has been developed to determine the oscillations for different values of system parameters and input signals. It has been shown how a limit cycle can modify the nonlinear characteristic for slowly-varying signals. This modification may be of

importance when a high-accuracy control system has to be designed in the presence of nonlinearities with excessive dead zone, hysteresis, backlash, etc. The design technique can be directly applied to a large class of plant-adaptive control systems where a sinusoidal signal is used as an identification signal.

In a future study, the technique may be extended to the investigation of transient asymmetrical oscillations. Thus, to study how these oscillations are established after certain amplitude perturbation, this study should be largely based upon the material presented in the following chapter.

It may also be shown [16, 17] that the presented analysis can be extended to the case when the signal superimposed on a sinusoid is not only a constant or slowly-varying sinusoid, but also when the additional signal is described as a Gaussian process, provided that the amplitude or standard deviation of the additional signal is of no consequence in the analysis. This further generates the idea of applying the dual-input describing function [15,17] along with the parameter plane method, and investigates the case when the input to a nonlinearity of the system is a combination of two similar sinusoidal signals.

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