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RECIPROCITY IN NONLINEAR NETWORKS

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1. Introduction.

There are two contrasting attitudes which are rather prevalent among engineers when faced with nonlinear network problems: the naive belief that nonlinear problems can be handled by extensions of linear techniques, and the pessimistic view that no known systematic approach will work, and therefore it is best to avoid these problems. However, naiveté will often lead to trouble, and pessimism will not solve today's problems. While there can be no simple and general quantitative approaches to nonlinear analysis (quantitative questions are, in any case, best left to the computer), the purpose of this paper is to show that there are important qualitative questions which can be answered in a systematic fashion. Moreover, these questions should be resolved before any intelligent quantitative analysis can proceed.

As an example of the type of qualitative problem which arises consider the equilibrium equations for a network of resistors and sources. In the linear case existence and uniqueness of the solutions of these equations are usually taken for granted. In the nonlinear case, however, these issues cannot be ignored, and they are not always easy to resolve. Furthermore, once it is known that one or more solutions exist, the problem of computation remains.

As another example, consider an RLC network, possibly containing constant sources but no other excitation. In the linear case we expect that whatever initial conditions we apply to the reactive elements, the network variables will tend toward an equilibrium as $t \to \infty$, or else "blow up." (The borderline case of oscillatory solutions is hardly worth considering in a realistic situation.) There are rather simple means of determining which of these two conditions holds. In the nonlinear case, network variables may settle down to any one of several equilibria, they may tend asymptotically toward certain oscillatory conditions (fundamentally different from linear oscillations), or they may grow without bound. If the first possibility occurs, let us say that the network has a "constant limiting regime". Unfortunately, this important attribute of nonlinear networks is not easily ascertained in general.

Finally, consider an RLC network driven by time-varying sources. In the linear case one generally expects that, after transients have died away, each excitation waveform will lead to a unique response. We shall call this a "unique limiting regime". In the nonlinear case this is more the exception than the rule. In fact, one of the most useful features of nonlinear networks is that they can be designed so that transients do <u>not</u> die out. It is therefore pertinent to ask: what sorts of networks do or do not possess unique limiting regimes? This too, is far from a trivial question.

The qualitative questions raised above and many more like them are fundamental ones to the nonlinear network analyst. Yet, for very general types of networks and systems they are largely unanswerable. There are, however, certain special characteristics of <u>reciprocal</u> networks which greatly simplify the problem of determining qualitative network behavior. In the following sections we shall show by means of simple examples that reciprocity as well as certain other properties allow us to obtain rather simple and direct answers to the questions posed above. Furthermore, these answers will be obtained on the basis of little or no specific information regarding the network.

Since this is basically a tutorial exposition, it will draw on the results of several different authors. Also, certain mathematical assumptions will sometimes be omitted in the interest of clarity and brevity, and proofs will merely be outlined.

2. One-element-kind Networks.

In this section certain properties of one-element-kind networks will be discussed, using the all-resistive network as an example. (Most of the results on the resistive case can be carried over to the capacitive case by replacing currents by charges, and to the inductive case by replacing voltages by flux-linkages.) By resistive networks we mean collections of branches defined by terminal relations of the form,[1]

$$f(e_r, i_r) = 0 \tag{1}$$

where the vectors[2] e_r and i_r represent terminal voltages and currents respectively, and the vector f represents a set of implicit functions relating these variables (one relation for each branch). There may or may not be coupling among the relations. A collection of branches of this type may be thought of as a single multi-port resistive element or, if certain sets of relations are "decoupled" from other sets, they can be considered as a collection of several elements. In all practical

[1] Everywhere except in section 4, all elements will be time-invariant.

[2] The same notation will be used throughout for vectors and scalars. The context will make the meaning clear. Similarly, in diagrams, a single port labelled with a vector variable will represent a set of ports.

cases, it will be possible to write the terminal relations in one or more of the explicit forms,

$$e_r = E_r(i_r) \tag{2a}$$

$$i_r = I_r(e_r) \tag{2b}$$

$$\begin{cases} i_{r_1} = I_{r_1}(e_{r_1}, i_{r_2}) \\ e_{r_2} = E_{r_2}(e_{r_1}, i_{r_2}) \end{cases} \tag{2c}$$

depending upon which variables in (1) can be expressed in terms of which others. The linear resistor, independent voltage or current source and low frequency model of a diode are all examples of one-port (i.e., non-coupled) resistive elements. Examples of multi-port resistive elements would be the gyrator, the ideal transformer and the low frequency model of the transistor.

The most important attributes of network elements are those which remain invariant under interconnection. Two properties which exhibit this invariance and which are of fundamental importance in nonlinear network analysis are reciprocity and quasilinearity.

Reciprocity.

Reciprocity is defined in terms of the symmetry of certain incremental parameter matrices. In particular, if a resistive element is characterized by terminal relations (1a), (1b) or (1c) respectively, an incremental resistance matrix[1]

$$R(i_r) = \partial E_r / \partial i_r \tag{3a}$$

an incremental conductance matrix

$$G(e_r) = \partial I_r / \partial e_r \tag{3b}$$

or a hybrid incremental resistance matrix

$$H(e_{r_1}, i_{r_2}) = \partial (I_{r_1}, -E_{r_2}) / \partial (e_{r_1}, i_{r_2}) \tag{3c}$$

[1]When x and y are vectors, $\partial y/\partial x$ denotes the matrix $[a_{ij}] = [\partial y_i / \partial x_j]$

can be defined. An element whose incremental parameter matrices are symmetric (for all values of their arguments) is said to be reciprocal. (The negative sign in (3c) is necessary to assure the required symmetry.) Obviously one-port elements are reciprocal by definition.

An extremely useful property of reciprocal elements is the fact that they can be characterized in terms of (scalar) state functions. In the resistive case these are called "dissipation functions" and they are completely analogous to the energy functions associated with reactive elements. Thus, for a reciprocal element defined by (2a) the dissipation function[1]

$$\varphi(i_r) = \int^{i_r} E_r^t(u)\,du \qquad (4a)$$

can be defined. If the terminal relations are given in the form (2b) one can define

$$\varphi'(e_r) = \int^{e_r} I_r^t(w)\,dw \qquad (4b)$$

and if they are given in the form (2c) the "hybrid" function

$$\varphi''(e_{r_1}, i_{r_2}) = \int^{e_{r_1}, i_{r_2}} I_{r_1}^t(w,u)\,dw - E_{r_2}^t(w,u)\,du \qquad (4c)$$

can be defined. Each of these is a line integral and is independent of path of integration because of the reciprocity condition. The lower limit on the integral has been omitted since it merely introduces an unimportant arbitrary constant. The usefulness of the dissipation function is that it completely defines the terminal characteristics of the element in a very compact fashion. Note that equations (2) can be obtained from (4) by differentiation.

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[1] In all that follows, vectors are treated as columns and $(\cdot)^t$ denotes "transpose".

Quasilinearity.

Among the reciprocal elements there is a special class which shares certain qualitative features with the common linear elements, and is therefore called quasilinear. (See Duffin [1]). A one-port resistive element is called quasilinear (QL) if its incremental resistance and conductance are bounded below by positive constants. A multi-port element is QL if it is reciprocal and if the least eigenvalues of its incremental resistance and conductance matrices are bounded below by positive constants. Thus these elements are always "incrementally" passive, although they need not be passive in the usual sense of the term. The non-hybrid functions ϕ and ϕ' associated with a QL element have special properties:

1) They tend radially to $+\infty$.

2) They are strictly concave.

As an example, a linear multi-port with symmetric positive definite resistance matrix R is QL. Its dissipation function, $\phi(i_r) = \frac{1}{2} i_r^t R i_r$ clearly has the above properties.

Interconnection.

Now, consider a network containing only reciprocal resistive elements, and suppose soldering-iron or pliers entries are made into the network at arbitrary points, but with the following restrictions: (1) The voltages and currents at all entries are topologically independent, and (2) If e_s represents the voltages at the soldering-iron entries and i_p the currents at the pliers entries, the values of all other voltages and currents in the network can be expressed uniquely in terms of (e_s, i_p). If these restrictions are fulfilled we shall call the set (e_s, i_p) and the associated set of entries "complete". In designating a particular complete set of entries of a network we are in effect viewing it as a single multi-port element which can be described by terminal relations of the form

$$i_s = I_s(e_s) \quad \text{or} \quad e_p = E_p(i_p) \quad \text{or} \quad \begin{cases} i_s = I_s(e_s, i_p) \\ e_p = E_p(e_s, i_p) \end{cases} \tag{5}$$

depending upon which types (soldering iron or pliers) of entries are selected. (See Fig. 1). Now it can be shown that nay resistive network containing only reciprocal (or QL) elements will be reciprocal (or QL) when viewed from any complete set of entries. It is in this sense that reciprocity and quasilinearity are invariant under interconnection.

Equilibrium Equations.

Most systematic procedures for formulating equilibrium equations for resistive networks reduce to selecting a complete set of entries (and variables) for the network, constructing the functions appearing in (5) and setting these functions equal to zero (that is, reducing the external excitation to zero). Thus, in these systematic formulations, the reciprocal and QL properties are preserved.

As an example, consider the node equations for a connected network whose elements are defined by the terminal relations,

$$i_r = I_r(e_r) \tag{6}$$

Letting e_s be the node-to-datum voltages, and A be the incidence matrix for the network (reduced by the deletion of the datum node), we have

$$Ai_r = 0 \quad \text{(KCL)} \quad \text{and} \quad e_r = A^t e_s \quad \text{(KVL)}$$

which, when combined with (6), give the node equations

$$i_s = I_s(e_s) = A I_r(A^t e_s) = 0 \tag{7}$$

If independent sources are added to some or all of the branches in the manner of Fig. 2, eq.(7) becomes

$$i_s' = I_s(e_s) = A\left[I_r(A^t e_s - V) + J\right] \tag{8}$$

where the vector V represents th voltage sources and J the current sources. It can be seen by inspection of (7) and (8) that the incremental conductance matrix $G(e_r)$ for the original elements transforms to the matrix AGA^t for the network equations, and hence it follows that both reciprocity and quasilinearity are preserved.

Let us now turn to questions of existence, uniqueness and computation of solutions of the equilibrium equations of resistive networks, with particular reference to reciprocal and QL networks. In general, equations of the form (8) can have one, many, or no solution. (Of course, the network itself will always have at least one equilibrium, but the models chosen to represent it, being only approximate, may not display this property.) Therefore, one must first determine conditions under which a solution of the equilibrium equations exists.

EXAMPLE 1: Let us determine under what conditions the node equations of a reciprocal network will have at least one solution. Because of reciprocity, Eq.(7) or (8) can be expressed in the form

$$i_s' = \left(\frac{\partial \varphi'}{\partial e_s}\right)^t \tag{9}$$

where

$$\varphi'(e_s) = \int^{e_s} I_s^t(u)\,du \tag{10}$$

Equation (9) indicates that <u>the equilibria of a reciprocal network correspond to the stationary points of its dissipation function</u>.

Now, for ϕ' to have at least one stationary point, all that is required is that it tend radially to ∞. This will be the case, for example, if the network is composed of one-port elements each of whose e-i characteristics tend toward $+(-)\ \infty$ as their arguments tend toward $+(-)\ \infty$.

Having established existence it is equally important to determine if the equilibrium equations have more than one solution. In general a network of reciprocal elements may possess many equilibria. (For example, tunnel diodes are often used to obtain several equilibria.) If, however, the network contains only QL elements, we can establish both existence and uniqueness.

EXAMPLE 2: Assume now that the network elements of Ex. 1 are all QL. Condition (1) on page 6 implies that a solution of the node equations exists, and condition (2) implies that ϕ' will possess one and only one stationary point (a minimum). Hence the network will have a unique equilibrium.

The above discussion with respect to the node equations generalizes to other formulations of the equilibrium equations and shows that <u>any network of QL elements has exactly one equilibrium.</u>

Even when it is known that a unique solution of the equilibrium equations exists, it generally cannot be computed exactly. Successive approximation is usually the only reasonable computational procedure. However, it is important to be able to predict when such a procedure will converge. Here again reciprocity and quasilinearity are helpful. Since the equilibria of a reciprocal network can be identified with the stationary points of a dissipation function, it is natural to try some sort of "descent" method which, in successive iterations, will seek out the stationary points of the dissipation function. Although it was not described as such by its originators, the "relaxation" method of Birkhoff and Diaz [2] is an interesting example of such a procedure.

EXAMPLE 3: Consider a connected network with node-to-datum voltages $e_1, \ldots, e_n$. One way of finding a set of equilibrium values of these voltages is to assume arbitrary values of all but one of them, and then solve the KCL equation about the remaining node for the remaining voltage. This is referred to as "relaxing" that node and is a well-known procedure in <u>linear</u> equations. By relaxing each node in succession and repeating the procedure long enough, we might hope to converge on a set of equilibrium values. It is known that this procedure does <u>not</u> converge in general, but it <u>always</u> does in QL networks. To see why, let us return to the node equations (8) and the associated dissipation function ϕ'. Relaxing node j corresponds to moving along the surface ϕ' in a direction parallel to the e_j-axis until a stationary point <u>with respect to e_j</u> is found. (This will be a minimum in the QL case.) Thus, relaxation is equivalent to descent on the surface ϕ' moving parallel to one coordinate axis at a time, and continuing this process until the minimum is reached. (See Fig. 3). The form of the function ϕ' for QL networks assures the convergence of this procedure. (Refer to conditions (1) and (2) on page 6.)

3. Autonomous RLC Networks.

In this section we shall examine the question of constant limiting regimes in RLC networks which are autonomous; i.e., containing only constant excitation and time-invariant elements. To simplify the development it is convenient to think of an RLC network as having been constructed by starting with a general resistive network, making an arbitrary number of entries into this network, and inserting a capacitive branch into each soldering-iron entry and an inductive branch into each pliers entry. The network can then be viewed as three multi-port subnetworks connected as shown in Fig. 4. Assuming that the set of entries is complete, the dynamic equations for the system will take the form,

$$\begin{aligned} C(e_s)\,\dot{e}_s &= -I_s(e_s, i_p) \\ L(i_p)\,\dot{i}_p &= -E_p(e_s, i_p) \end{aligned} \qquad (11)$$

where $C(e_s)$ is the incremental capacitance matrix $\partial q/\partial e_s$ of the elements facing the soldering-iron entries (q = capacitor charge) and $L(i_p)$ is the incremental inductance matrix $\partial\lambda/\partial i_p$ of the elements facing the pliers entries (λ = flux-linkages). The state variables (e_s, i_p) comprise the voltages on each capacitor and the currents through each inductor.[1] Now, assuming that all elements are reciprocal, Eq.(11) can be represented in the more compact form,

$$\left[\; J \;\right]\begin{bmatrix} \dot{e}_s \\ \dot{i}_p \end{bmatrix} = -\left[\frac{\partial \varphi''}{\partial(e_s, i_p)}\right]^t \qquad (12)$$

where

$$J = \begin{bmatrix} C(e_s) & 0 \\ 0 & -L(i_p) \end{bmatrix} \text{ (symmetric)}$$

and ϕ'' is a hybrid dissipation function derived from the right hand side of (11):

$$\varphi''(e_s, i_p) = \int^{e_s, i_p} I_s^t(w,u)\,dw - E_p^t(w,u)\,du$$

[1] Eq.(11) is valid for much less restrictive classes of networks than we have implied. The restrictive assumptions were imposed to simplify the exposition.

Equation (12) reduces to

$$C(e_s)\,\dot{e}_s = -\left(\frac{\partial \varphi'}{\partial e_s}\right)^t \quad \text{where} \quad \varphi'(e_s) = \int^{e_s} I_s^t(w)\,dw \qquad (13)$$

in the RC case and to the dual of (13) in the RL case.

This formulation, first used by Brayton and Moser [3], offers much insight into the question of a constant limiting regime. For example, consider the relatively simple case of a reciprocal RC network in which all capacitors are QL. We note first that (13) implies that all solutions of the network equations flow "downhill" on the dissipation function ϕ'. This is deduced by evaluating $d\phi'/dt$ along solutions of (13), giving

$$\frac{d\varphi'}{dt} = \left(\frac{\partial \varphi'}{\partial e_s}\right)\dot{e}_s = -\dot{e}_s^t\, C(e_s)\,\dot{e}_s \qquad (14)$$

Since the capacitors are QL, $C(e_s)$ is positive definite for all e_s and thus the rate of change of ϕ' is always negative except at points where $\dot{e}_s = 0$, that is, the equilibria of the network (singular points of (13)). Now, if ϕ' tends radially to $+\infty$ (which it normally does in real networks), then the "downhill" condition (14) will assure that no solution can grow without bound, and that all initial conditions lead to solutions which tend toward singular points of (13) as $t \to \infty$. Those singular points corresponding to minima of the function ϕ' are stable equilibria of the network equations. Other stationary points represent unstable equilibria. Thus, under these conditions, a constant limiting regime exists.

It is interesting to note that these conclusions were reached without the slightest bit of <u>quantitative</u> information on the network. The essential assumptions required were merely that all resistors were reciprocal and all capacitors were QL.

EXAMPLE 4: To illustrate the usefulness of the above result, consider the following problem: A resistive network is composed of QL elements plus an arbitrary number of (one-port) voltage-controlled negative resistance (VCNR) elements. (The VCNR's are assumed to have the property: $i \rightarrow +(-)\ \infty$ as $e \rightarrow +(-)\ \infty$.) It is well-known that such a network may possess several equilibria, and that some will be stable and others unstable. Assuming that stability is governed primarily by the stray capacitances in the network, how can we determine which equilibria will be stable, and whether the network has a constant limiting regime? To investigate this problem, let us augment the resistive network by adding linear positive capacitances of arbitrary values to the network at arbitrary soldering-iron entries. The only restriction which need be placed on the augmentation is that each VCNR is shunted by some capacitance. Under this very reasonable restriction, we find that the network equations reduce to the form (13), and that all the requirements for a constant limiting regime are satisfied. Any equilibrium corresponding to a minimum of ϕ' is stable, where ϕ' is the dissipation function for the network as viewed from the capacitor terminals. Since this result is based on the resistive element characteristics alone, <u>it is completely independent of the location and values of the augmenting capacitances</u>.

To obtain similar results for networks containing both L's and C's is considerably more difficult, the reason being that the solutions of (12) will not run downhill on the function ϕ'' because the matrix J is not positive definite. However, it has been shown that the pair J, ϕ''

in (12) can be replaced by various equivalent pairs, $\hat{J}$, $\hat{\phi}'$ without changing the solutions of the equations. Thus, to demonstrate the existence of a constant limiting regime in a reciprocal RLC network we must seek an equivalent representation of (12) in which solutions run downhill on the dissipation function. Without going into the details of this procedure we shall illustrate its application in a particular case.

EXAMPLE 5: Let us consider the effects of stray inductances as well as stray capacitances in the network of example 4. We shall augment the resistive network by adding linear positive capacitors at arbitrary soldering-iron entries and linear positive inductors at arbitrary pliers entries, subject only to the same restriction as before: that each VCNR be shunted by some capacitance. Such a network can be described by (12) where C and L will be constant symmetric and positive definite. Furthermore, it turns out that an equivalent pair $\hat{J}$, $\hat{\phi}''$ having the desired "downhill" properties can be found (see [3]) provided that

$$m\left(L^{-\frac{1}{2}} R L^{-\frac{1}{2}}\right) + m\left(C^{-\frac{1}{2}} G\, C^{-\frac{1}{2}}\right) > 0 \tag{15}$$

where

$m(A)$ = glb of the eigenvalues of A.

and

$$G = \frac{\partial}{\partial e_s}\left(\frac{\partial \varphi''}{\partial e_s}\right)^t, \quad R = -\frac{\partial}{\partial i_p}\left(\frac{\partial \varphi''}{\partial i_p}\right)^t$$

To interpret condition (15) physically, note that the matrix G represents the incremental conductance matrix at the capacitive ports when the inductor currents are held constant (Refer to Fig. 4), and thus the first term in (15) represents the least RC "incremental" natural frequency. Because of the presence of the VCNR's this may be negative. Similarly, R represents the incremental resistance matrix at the inductive terminals when the capacitor voltages are held constant. Because of the way in which the augmentation was performed, the resistive network will be QL when viewed from the inductor ports, and hence R will always be positive definite. The second term in (15) represents the smallest RL incremental natural frequency. Because of the QL condition just mentioned, this will be positive. For arbitrary values of capacitors and inductors it is clear that condition (15) may not be fulfilled and thus the network may not have a constant limiting regime. However, condition (15) can always be fulfilled by making the inductances sufficiently small and/or the capacitances sufficiently large. The conclusion therefore , is that in a network of QL resistances and VCNR's, increasing the stray capacitances and decreasing the stray inductances tends to "stabilize" the network. (Under constant limiting regime conditions the equilibria which were stable in the RC case will also be stable in the RLC case.)

4. RLC Networks with Excitation.

In order to explore the behavior of forced RLC networks, let us associate an arbitrary set of independent, possibly time-varying sources with the resistive elements in the manner of Fig. 2. Insertion of the sources modifies (11) to the form

$$\begin{aligned} C\dot{e}_s &= -I_s(e_s, i_p, u) \\ L\dot{i}_p &= -E_p(e_s, i_p, u) \end{aligned} \qquad (16)$$

where u (a vector time function, in general) represents the effects of sources. There are many common examples of networks of this type which do not exhibit a unique limiting regime. For example, many simple circuits containing nonlinear reactive elements will, when driven by a sinusoidal excitation, produce either a harmonic or a subharmonic response depending upon initial conditions. Such circuits clearly do not exhibit a unique limiting regime. Neither reciprocity nor quasilinearity is enough to insure a unique limiting regime, but the following example shows that a few additional restrictions will assure this condition.

EXAMPLE 6: Consider a network describable by (16) where all capacitances and inductances are linear and positive, each capacitance is shunted by a resistive branch, and each inductance is in series with a resistive branch. These assumptions imply that C and L are constant symmetric and positive definite, and that the network is QL when viewed from the entries associated with (e_s, i_p), for any fixed value of u. From these conditions it follows easily that a bounded excitation u(t) produces a bounded response. What we wish to show is that for a given input u(t), solutions of (16) starting from any initial state will all tend asymptotically to a unique response. Consider any two solutions (e_s, i_p) and (e'_s, i'_p) corresponding to the same excitation. Using

$$\delta = (e_s - e'_s)^t C (e_s - e'_s) + (i_p - i'_p)^t L (i_p - i'_p)$$

as a measure of the difference between these solutions, we find from (16) that

$$\dot{\delta} = -(e_s - e'_s)^t [\mathcal{I}_s(e_s, i_p) - \mathcal{I}_s(e'_s, i'_p)] - (i_p - i'_p)^t [\mathcal{E}_p(e_s, i_p) - \mathcal{E}_p(e'_s, i'_p)] \qquad (17)$$

But it can be shown [4] that the QL condition implies that the right hand side of (17) is $\leq -k\delta$ where k is a positive constant. This is enough to insure that any two solutions approach each other as $t \to \infty$; that is, a unique limiting regime exists.

We have given several illustrations of the way in which reciprocity and quasilinearity can be used to predict the qualitative behavior of networks. Of course, many other illustrations exist, and in many cases the same results could have been proved with slightly less restrictive assumptions. In any case, the status of "qualitative" network theory today suggests that much remains to be learned, and that this is a very fertile area for current research.

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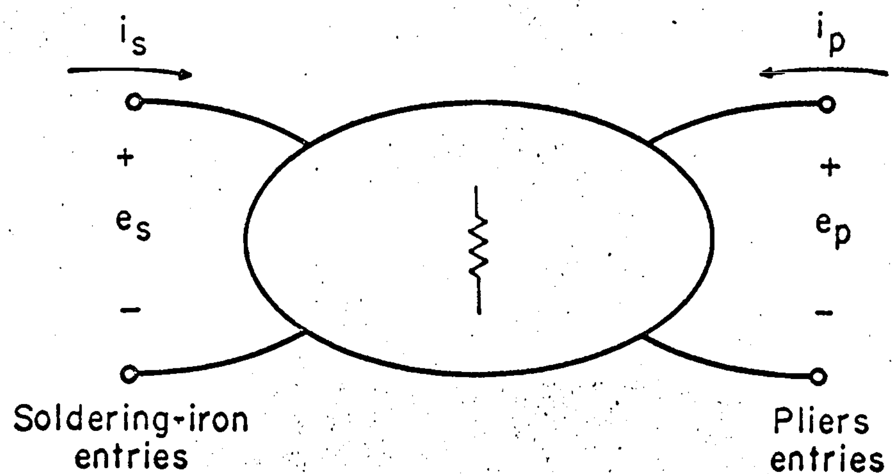

Fig. 1 Entries to a resistive network.

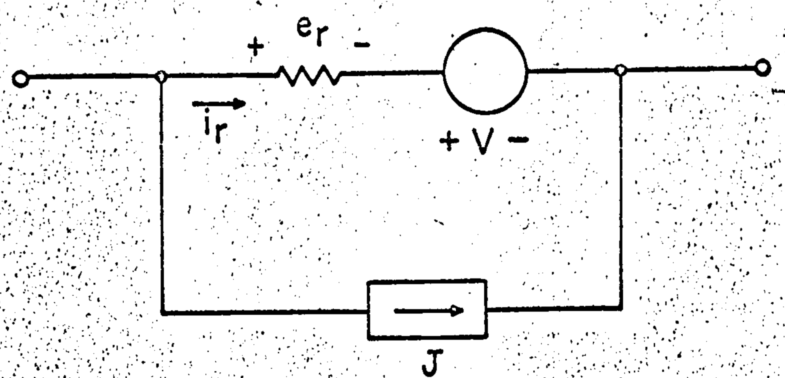

Fig. 2 Sources

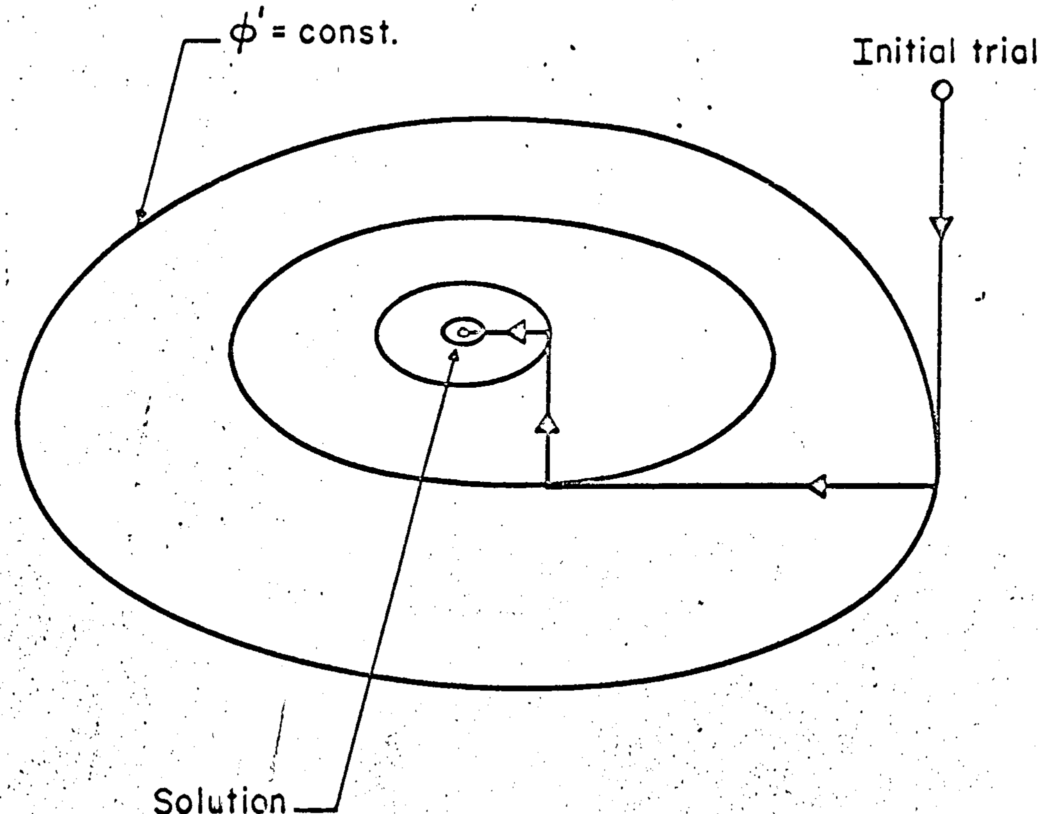

Fig. 3 Relaxation

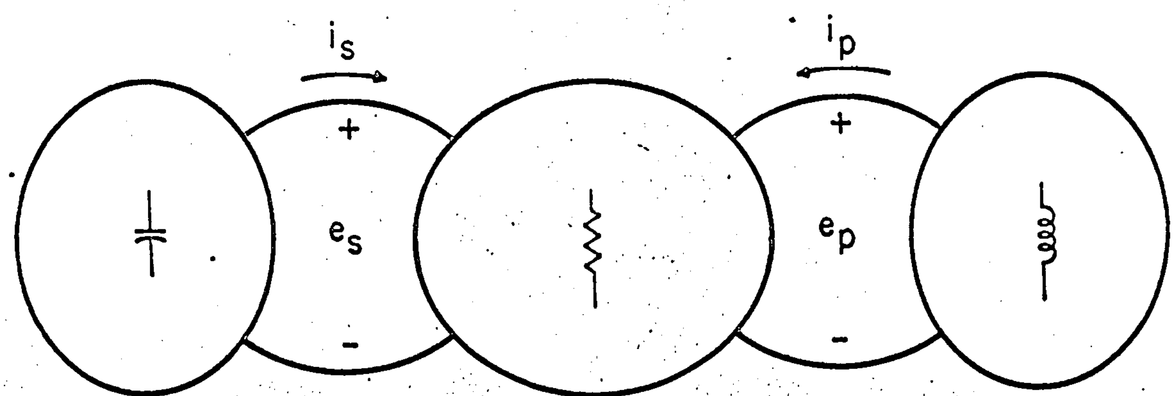

Fig. 4 RLC Network