THE INFLUENCE OF THE IONIZED MEDIUM ON SYNCHROTRON EMISSION OF INTERMEDIATE ENERGY ELECTRONS

REUVEN RAMATY*

NASA-GODDARD SPACE FLIGHT CENTER, GREENBELT, MD.

*National Academy of Science - National Research Council
Resident Research Associate

ABSTRACT

The suppression of synchrotron emission at low frequencies due to the influence of the ionized medium is investigated. Explicit expressions and detailed numerical values of the emission spectra are presented for various electron energies and plasma and cyclotron frequencies. Unlike previous studies of this suppression effect, which were applicable only to ultrarelativistic electrons, the present treatment is valid for electrons of arbitrary energies and in particular for intermediate energy electrons such as those presumably accelerated in solar flares.

INTRODUCTION

The suppression of the low frequencies of synchrotron emission due to the influence of the ambient ionized medium was investigated by Tsytovich (1951), Ginzburg (1953) and Razin (1957, 1960). This suppression effect is essentially the result of the phase velocity of light becoming greater than c at frequencies above the plasma frequency but sufficiently near to it, where the index of refraction of the ambient medium is less than unity. Additional studies of this effect and its application to cosmic radio emission were given by Ginzburg and Syrovatskii (1964, 1965), Scheuer (1965) and McCray (1966, 1967).

The influence of the ambient coronal plasma on the spectra of solar Type IV radio bursts, which are generally believed to be synchrotron emission of energetic electrons accelerated in solar flares, was investigated by Ramaty and Lingenfelter (1967). They showed that the low frequency cutoffs observed for these bursts (Takakura and Kai, 1961) could result from the suppression of the synchrotron emissivity at low frequencies rather than from absorption during propagation through the solar corona, and that from the study of the observed cutoff it is possible to deduce the magnetic field or the ambient density of the emitting region. This possibility was recently substantiated by Boischot and Clavelier (1967) who pointed out that the very sharp low frequency cutoff of a Type IV solar burst, which they observed to originate at a large distance from the solar surface, could indeed be a suppression effect rather than absorption, since at the frequency of the observed cutoff, the ambient coronal plasma could hardly affect the propagation of radio waves. From these considerations, they then provided an estimate of the coronal magnetic field at the site of the emission.

The straight-forward application of this suppression effect to solar radio emission, however, must be treated with some caution because all the available treatments of the influence of the ionized medium on synchrotron emission are valid only for ultrarelativistic electrons whereas the bulk of the Type IV bursts could originate from electrons of only mildly relativistic energies (Takakura, 1960; Holt and Cline, 1968).

The radiation in vacuum of electrons of arbitrary energies spiraling about the lines of force of a constant magnetic field was first derived by Schott (1912) and discussions of his treatment were given by Schwinger (1949) and Landau and Lifshitz (1962). Takakura (1960) used these results to calculate the synchrotron emission from intermediate energy solar electrons. His formalism was applied to various Type IV bursts (Takakura and Kai, 1961; Takakura, 1967; Holt and Cline, 1968), but, as was pointed out recently (Takakura, 1967; Holt and Cline, 1968) this formalism suffers from the neglect of the effects of the ionized medium. Therefore, in order to improve our understanding of the nature of these solar bursts it would be of considerable importance to relax the ultrarelativistic approximation used in previous treatments and to evaluate synchrotron emission spectra from electrons of arbitrary energies in the presence of an ionized medium.

The general problem of the radiation from charged particles moving in a magnetoplasma was treated by Eidman (1958), Liemohn (1965) and Mansfield (1967). These treatments provide explicit expressions for the spectral and angular distribution of the power radiated into both ordinary and extraordinary modes by a single electron moving in a helical orbit in a cold, collisionless plasma permeated by a static uniform magnetic field. The resulting spectra were numerically analyzed by Liemohn (1965) at frequencies in the vicinity of the plasma and gyro-frequencies of the ambient electrons for which the indices of refraction of either the ordinary or extraordinary modes are real and greater than unity. Although part of the solar radio emission, especially in the microwave band, may be produced at these frequencies, in the present paper we shall only investigate the radiation at frequencies above the plasma frequency where the indices of refraction of both ordinary and extraordinary modes are real and less than unity. For such frequencies the phase velocity of light is greater than c, and, as pointed out above, this results in the suppression of the low frequencies of synchrotron emission. In order to assess the importance of this effect for intermediate energy electrons, an expression is derived for the frequency spectrum, integrated over all angles, of the emission from an electron of arbitrary energy moving in a circular orbit in a homogeneous and isotropic electron plasma. This derivation is based on the more general treatments mentioned above, as well as on an independent derivation appropriate for circular motion and frequencies for which both indices of refraction are isotropic, real, and less than unity. The resultant emission spectra are evaluated numerically for various electron energies and plasma parameters and it is shown that for large plasma frequencies there is a significant suppression of emission for electrons of all energies. Finally, by comparing these spectra with those obtained from the ultrarelativistic approximation mentioned above, the validity of the high-energy formulas used in the previous treatments of the effects of the medium is investigated at various electron energies which could be of importance for solar radio emission.

Radiation From Arbitrary Energy Electrons

The angular and frequency distribution of the electromagnetic emission from an electron moving with an arbitrary velocity v in a circular orbit in a homogeneous ionized medium permeated by a static uniform magnetic field $\vec{B}$ is given by (Liemohn, 1965):

$$\frac{dI_\pm(\nu,\theta)}{d\Omega} = \sum_{s=1}^{\infty} \frac{s^2}{2\pi} \frac{e^4 B^2}{m^2 c^3} \frac{1-\beta^2}{n_\pm(1+\alpha_{\theta\pm}^2)} \cdot$$

$$\cdot\left[(\alpha_{\theta\pm} \cot\theta + \alpha_{K\pm}) J_s(s\beta n_\pm \sin\theta) - n_\pm \beta J_s'(s\beta n_\pm \sin\theta)\right]^2 \cdot$$

$$\cdot\left[1 + \frac{\partial \ln n_\pm}{\partial \ln \nu}\right] \delta\left(\nu - \frac{s\nu_B}{\gamma}\right) \quad \frac{\text{erg}}{\text{ster sec c/s}} \qquad (1)$$

The subscripts (+) or (-) indicate emission into the ordinary or extraordinary modes respectively; β is the ratio of v to c; γ is the electron Lorentz factor; θ is the angle between $\vec{B}$ and the radius vector from the electron's guiding center to the point of observation; ν_B is the cyclotron frequency of the ambient electrons; $n_\pm(\nu,\theta)$ is the index of refraction of the ionized medium and is in general frequency dependent and anisotropic with respect to the direction of the magnetic field; and J_s is a Bessel function of order s. The polarization coefficients $\alpha_{\theta\pm}(\nu,\theta)$ and $\alpha_{k\pm}(\nu,\theta)$ (Liemohn, 1965) are defined in terms of the components of the electric vector of the radiation field

$$i\alpha_\theta = \frac{E_\theta}{E_x} \quad ; \quad i\alpha_K = \frac{E_K}{E_x}$$

where E_x and E_θ are the transverse components of $\vec{E}$ and E_k is the component of $\vec{E}$ along the θ direction defined above.

The index of refraction and the polarization coefficients can be determined from the properties of the ambient ionized medium. For a cold collisionless electron plasma these quantities are (Ginzburg, 1961; Liemohn, 1965).

$$n_\pm^2 = 1 + \frac{2P^2(P^2-F^2)}{\pm\left[F^4\sin^4\theta + 4F^2(P^2-F^2)^2\cos^2\theta\right]^{1/2} - 2F^2(P^2-F^2) - F^2\sin^2\theta} \tag{2}$$

$$\alpha_{\theta\pm} = -\frac{2F(P^2-F^2)\cos\theta}{-F^2\sin^2\theta \pm\left[F^4\sin^4\theta + 4F^2(P^2-F^2)^2\cos^2\theta\right]^{1/2}} \tag{3}$$

$$\alpha_{K\pm} = -\frac{P^2F\sin\theta - \alpha_{\theta\pm}P^2\cos\theta\sin\theta}{P^2(\cos^2\theta - F^2) - F^2(1-F^2)} \tag{4}$$

The dimensionless quantities F and P are defined in terms of the cyclotron and plasma frequencies

$$F = \nu/\nu_B \qquad ; \qquad P = \nu_p/\nu_B$$

where ν_B and ν_p are given in terms of the magnetic field B and the electron density n_e by

$$\nu_B = 1/2\pi\ eB/mc \qquad ; \qquad \nu_p = \left(e/\sqrt{\pi m}\right)\sqrt{n_e}$$

The indices of refraction, $n+$ and $n-$, have several cutoffs (n=o) and resonances $(n \to \infty)$. For $F > P$, or $\nu > \nu_p$, however, $n+$ is real and less than 1, and for $F > (P^2 + \frac{1}{4})^{1/2} + \frac{1}{2}$, or $\nu > (\nu_p^2 + \nu_B^2/4)^{1/2} + \nu_B/2$, n_-

is also real and less than 1. Assuming that $\nu_p \gg \nu_B$ and limiting our discussion to frequencies greater than $(\nu_p^2 + \nu_B^2/4)^{\frac{1}{2}} + \nu_B/2 \approx \nu_p$ we see that for $\nu > \nu_p$ both indices of refraction are real and therefore the total emission is the sum of the emissions in the ordinary and extraordinary modes given by equation (1). Since $\nu > \nu_p \gg \nu_B$, we can neglect second order terms in 1/F and 1/P, and therefore the index of refraction and the polarization coefficients given by equations (2), (3) and (4) reduce to

$$n^2 = 1 - \nu_p^2/\nu^2 \tag{5}$$

$$\alpha_{\theta\pm} = \pm 1 \tag{6}$$

$$\alpha_{K\pm} = 0 \tag{7}$$

Using equations (5), (6) and (7), the total emission is obtained by adding the contributions from the two modes in equation (1). This results in

$$\frac{dI(\nu,\theta)}{d\Omega} = \sum_s \frac{s^2}{2\pi}\frac{e^4 B^2}{m^2 c^3}\frac{1-\beta^2}{n} \cdot \left[\cot^2\theta\, J_s^2(s\beta n\sin\theta) + \beta^2 n^2 J_s'^2(s\beta n\sin\theta)\right]\delta\left(\nu - \frac{s\nu_B}{\gamma}\right) \tag{8}$$

In order to substantiate the validity of the limiting process used to derive equation (8), we provide an independent derivation of this equation, appropriate for the physical conditions mentioned above. This is done in Appendix I, where we show that equation

(8) can indeed be derived directly from Maxwell's equations for an electron of an arbitrary velocity moving in a circular orbit in a static uniform magnetic field immersed in an isotropic electron plasma with index of refraction given by equation (5).

Since the radiation has azimuthal symmetry with respect to the direction of $\vec{B}$, the total radiation, $I(\nu)$, integrated over all angles is given by

$$I(\nu) = 2\pi \int_0^{\pi} \frac{dI(\nu,\theta)}{d\Omega} \sin\theta \, d\theta \tag{9}$$

This integral can be evaluated by the same methods as those used by Schott (1912) for the evaluation of a similar integral for radiation in vacuum. The presence of the ionized medium, however, introduces some complications and therefore in Appendix II we provide an outline of the integration process suitable for the present problem. The resulting integrated emission is given by

$$I(\nu) = \sum_s \frac{2e^4B^2}{m^2c^3} \frac{1-\beta^2}{\beta} \frac{1}{n^2} \Big[s\beta^2n^2 J'_{2s}(2s\beta n) - s^2(1-\beta^2n^2) \int_0^{\beta n} J_{2s}(2sx)\,dx \Big] \delta\left(\nu - \frac{s\nu_B}{\gamma}\right) \ \frac{\text{erg}}{\text{sec c/s}} \tag{10}$$

For n=1 equation (10) directly reduces to the emission spectrum in vacuum given by Schott (1912) and Landau and Lifschitz (1962). In order to compare equation (10) with previous treatments of the effects of the medium on synchrotron emission we introduce the emissivity function, $F(\nu/\nu_c)$, given by

$$F(\nu/\nu_c) = \frac{4\pi}{\sqrt{3}} \frac{1}{\gamma\beta n^2} \Big[s\beta^2n^2 J'_{2s}(2s\beta n) - s^2(1-\beta^2n^2) \int_0^{\beta n} J_{2s}(2sx)\,dx \Big] \tag{11}$$

where

$$\frac{\nu}{\nu_c} = \frac{2\nu}{3\nu_B\gamma^2} = \frac{2s}{3\gamma^3} \quad , \tag{12}$$

in terms of which equation (10) can be written as

$$I(\nu) = \frac{\sqrt{3}}{2\pi}\ \frac{e^4 B^2}{m^2 c^3}\frac{1}{\gamma}\sum_s F\left(\frac{\nu}{\nu_c}\right)\delta\left(\nu - \frac{s\nu_B}{\gamma}\right) \tag{13}$$

For certain limiting conditions equation (11) can be considerably simplified by using asymptotic forms of the Bessel functions. In Appendix III we show that:

a) If $s/\gamma_1^3 \gg 1$ equation (11) reduces to

$$F\left(\frac{\nu}{\nu_c}\right) = \sqrt{\frac{\pi}{3}}\ \frac{\sqrt{s}}{n}\ \frac{1}{\gamma\gamma_1^{1/2}}\left[\frac{\gamma_1 - 1}{\gamma_1 + 1}\ e^{2/\gamma_1}\right]^s \tag{14}$$

b) If $s \gg 1$ and $\gamma_1 \gg 1$ (but s/γ_1^3 not necessarily larger than 1) equation (11) also reduces to

$$F\left(\frac{\nu}{\nu_c}\right) = \frac{\gamma_1}{\gamma}\left(\frac{\nu}{\nu_c'}\right)\int_{\nu/\nu_c'}^{\infty} K_{5/3}(y)\,dy \tag{15}$$

where

$$\frac{\nu}{\nu_c'} = \frac{\nu}{\nu_c}\left(\frac{\gamma}{\gamma_1}\right)^3 = \frac{2s}{3\gamma_1^3} \tag{16}$$

and

$$\gamma_1 = \frac{1}{\sqrt{1-\beta^2 n^2}} = \frac{\gamma}{\sqrt{1 + \frac{\gamma^2(\gamma^2-1)}{s^2}\frac{\nu_p^2}{\nu_B^2}}} \tag{17}$$

From equation (17) we see that, depending on the value of ν_p/ν_B, condition (a) can be satisfied for small as well as for large values of s and γ. For example, for $\nu_p/\nu_B = 6$, $s/\gamma_1^3 \geq 3$ for all values of s and γ of interest. On the other hand, condition (b) will be satisfied in general only by ultrarelativistic electrons. Indeed, equation (15) is identical to the emissivity function used in the previous treatments of the influence of the ionized medium (Ginzburg and Syrovatskii, 1964, 1965; Ramaty and Lingenfelter, 1967.) The validity of this approximation, however, can be best assessed by the numerical evaluations which are discussed in the next section.

Numerical Results

The emissivity function $F(\nu/\nu_c)$ can be evaluated by using equation (11), or if either conditions (a) or (b) are satisfied, $F(\nu/\nu_c)$ can be directly obtained from equations (15) or (16) respectively. The use of these asymptotic equations is necessary in certain cases owing to the difficulties involved in the numerical evaluation of Bessel functions of large orders and arguments. By simultaneously evaluating equations (11) and (15) we find that the values of $F(\nu/\nu_c)$ given by these equations do not differ by more than 20% if $S/\gamma_1^3 \geq 3$. Similarly, we also find that equation (15) is accurate to within 20% if $\gamma_1 \geq 2.5$. Using these transitional conditions we have evaluated $F(\nu/\nu_c)$ as a function of ν/ν_c for various values of γ and ν_p/ν_B .

In figure 1 $F(\nu/\nu_c)$ is plotted for ν_p=o (emission in vacuum.) The circles correspond to the first 10 harmonics (s=1,...,10) and it can be seen that as the electron energy increases the harmonics cluster together and the emission spectrum can be approximated by a smooth distribution. It can also be seen that the emissivity at high frequencies increases with increasing energy and that for Lorentz factors greater than about 3, $F(\nu/\nu_c)$ can be well approximated by the ultrarelativistic spectrum $(\gamma \to \infty)$ calculated from equation (15) with n=1.

In figure (2) $F(\nu/\nu_c)$ is plotted as a function of ν/ν_c for $\nu_p/\nu_B = 6$. (The assumption that $\nu_p \gg \nu_B$ made in the derivation of equation (8) is quite well satisfied in this case.) As in the previous ultrarelativistic treatments of the effect of the medium (e.g., Ramaty and Lingenfelter, 1967, Figure 1), there is a significant suppression of the emission at low frequencies. Moreover, for low values of γ, the total radiated power is also strongly suppressed. This total suppression is the combined result of the low frequency cutoff due to the influence of the ionized medium and the small emissivity of the mildly relativistic electrons at high frequencies.

In order to assess the deviations of the ultrarelativistic approximation mentioned above from the present results we have plotted in figure (3) the ratio, F_{UR}/F, of the spectra obtained from equation (15) to those shown in figure (2). We can see that the ultrarelativistic approximation is quite adequate at large values of γ since for all values of ν/ν_c where there is significant emission, F_{UR}/F does not deviate significantly from unity. At lower values of γ, however, the ultrarelativistic approximation appreciably overestimates the intensity of the emission at all values of ν/ν_c. Because of this, for sufficiently large plasma frequencies, the cutoffs in the emission spectra of intermediate energy electrons due to the influence of the medium will be even sharper than the low frequency suppressions obtained in the previous ultrarelativistic treatments of this effect. As pointed out in the introduction, this may have interesting implications on Type IV solar radio bursts.

In figures (4) and (5) we plotted $F(\nu/\nu_c)$ for ν_p/ν_B of 3 and 1.5 respectively. It can be seen that the influence of the medium diminishes with decreasing plasma frequency. The approximations based on the assumption that $\nu_p \gg \nu_B$, mentioned above, probably break down for the spectra shown in figure 5. However, as can be seen by comparing figures 1 and 5, for ν_p/ν_B =1.5 the influence of the medium becomes quite small and at frequencies where both modes propagate freely the emission is much like that in vacuum.

Appendix I

We consider the radiation produced by an electron moving in a circular orbit in a static and uniform magnetic field $\vec{B}$ immersed in a homogeneous and isotropic medium characterized by a dielectric constant ε and a magnetic permeability equal to that of free space. The electric and magnetic vectors of the radiation field satisfy Maxwell's equations:

$$\nabla\cdot\vec{E} = 4\pi\rho/\varepsilon \qquad \text{(I1)}$$

$$\nabla\cdot\vec{H} = 0 \qquad \text{(I2)}$$

$$\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t} \qquad \text{(I3)}$$

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{H}}{\partial t} = 0 \qquad \text{(I4)}$$

where ρ and $\vec{J}$ are the charge and current densities associated with the electron's motion. A wave equation for the magnetic vector $\vec{H}$ can be obtained by taking the curl of equation (I3) and by using equations (I2) and (I4):

$$\nabla^2\vec{H} - \frac{\varepsilon}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = -\frac{4\pi}{c}\nabla\times\vec{J} \qquad \text{(I5)}$$

Since the motion of the electron is periodic, with gyroperiod T, both $\vec{H}$ and $\vec{J}$ can be decomposed into Fourier series.

$$\vec{H}(\vec{r},t) = \sum_s \vec{H}_s(\vec{r})\, e^{-\frac{2\pi i s t}{T}} \qquad \text{(I6)}$$

$$\vec{J}(\vec{r},t) = \sum_s \vec{J}_s(\vec{r})\, e^{-\frac{2\pi i s t}{T}} \qquad \text{(I7)}$$

the coefficients of which satisfy Poisson's equation:

$$\nabla^2 \vec{H}_s + k_s^2 \vec{H}_s = -\frac{4\pi}{c} \nabla \times \vec{J}_s \qquad (I8)$$

where

$$k_s = \frac{2\pi s\sqrt{\varepsilon}}{cT} \qquad (I9)$$

By using the inverse transform of (I7), the solution of (I8) can be written as

$$\vec{H}_s(\vec{r}) = \frac{2}{T}\int_0^T dt\, e^{\frac{2\pi i s t}{T}} \int d^3r' \, \nabla \times \vec{J}(\vec{r}\,') \frac{e^{ik_s|\vec{r}-\vec{r}\,'|}}{c|\vec{r}-\vec{r}\,'|} \qquad (I10)$$

Since the current density can be written as

$$\vec{J}(\vec{r},t) = e\vec{v}(t)\delta(\vec{r}-\vec{r}_e) \qquad (I11)$$

where $\vec{r}_e(t)$ and $\vec{v}(t)$ are the electron's instantaneous position and velocity, equations (I10) reduces to

$$\vec{H}_s(\vec{r}) = \frac{2e}{T}\int_0^T dt\, e^{\frac{2\pi i s t}{T}} \nabla\left[\frac{e^{ik_s|\vec{r}-\vec{r}_e|}}{c|\vec{r}-\vec{r}_e|}\right] \times \vec{v}(t) \qquad (I12)$$

where the gradient is taken with respect to $\vec{r}_e$.

We now define a coordinate system such that its origin is at the electron's guiding center and that the z axis is parallel to the direction of the static field $\vec{B}$. Since the radiation is symmetric with respect to $\vec{B}$, we can define the azimuthal orientation of the system so that the yz plane contains the point of observation .

Let R_0 be the distance from the origin to the point of observation and $\hat{k}$ a unit vector along this direction. If R_0 is much larger than both the wavelength of the radiation and the gyroradius r_e, $|\vec{r}-\vec{r}_e|$ can be approximated by

$$|\vec{r}-\vec{r}_e| \approx R_0 - \vec{r}_e \cdot \hat{k}$$

Using this relation and neglecting $\vec{r}_e \cdot \hat{k}$ in the denominator of the term in square brackets, equation (I12) can be written as

$$\vec{H}_s(\vec{r}) = i\vec{A}_s(\vec{r}) \times \vec{k}_s \qquad \text{(I 13)}$$

where $\vec{k}_s = \hat{k}\, k_s$

and where $\vec{A}_s(\vec{r})$ is given by

$$\vec{A}_s(\vec{r}) = \frac{2e}{T}\frac{e^{ik_sR_0}}{cR_0}\int_0^T dt\, \vec{v}(t)\, e^{i\left[\frac{2\pi st}{T} - \vec{k}_s\cdot\vec{r}_e\right]} \qquad \text{(I 14)}$$

In the coordinate system defined above, the z component of $\vec{A}_s$ vanishes and its x and y components are given by

$$A_{sx} = -\frac{ev}{\pi c R_0}\, e^{ik_sR_0}\int_0^{2\pi} d\varphi \sin\varphi\, e^{i[s\varphi - \sqrt{\varepsilon}\,\beta \sin\theta \sin\varphi]} \qquad \text{(I15)}$$

$$A_{sy} = \frac{ev}{\pi c R_o} e^{ik_s R_o} \int_0^{2\pi} d\varphi \cos\varphi \, e^{i[s\varphi - \sqrt{\varepsilon}\beta \sin\theta \sin\varphi]} \tag{I16}$$

where v is the magnitude of $\vec{v}$, β is the ratio of v to c and θ is the angle between $\hat{k}$ and $\vec{B}$. Using some well known integrals from the theory of Bessel functions, (I15) and (I16) reduce to

$$A_{sx} = \frac{2eiv}{cR_o} e^{ik_s R_o} J_s'(s\beta\sqrt{\varepsilon}\sin\theta) \tag{I17}$$

$$A_{sy} = \frac{2e}{\sqrt{\varepsilon} R_o \sin\theta} e^{ik_s R_o} J_s(s\beta\sqrt{\varepsilon}\sin\theta) \tag{I18}$$

In an isotropic dielectric medium the electric and magnetic vectors of plane electromagnetic waves and the propagation vector, $\hat{k}$, are mutually orthogonal and related by

$$\vec{E} = \frac{1}{\sqrt{\varepsilon}} \vec{H} \times \hat{k} \tag{I19}$$

As discussed in the main text, however, this relation is valid only for sufficiently high frequencies where the anisotropy introduced by the static field becomes negligible. Using (I19), the pointing vector $\vec{S}$ can be written as

$$\vec{S} = \frac{c|\vec{H}|^2}{4\pi\sqrt{\varepsilon}} \hat{k} \tag{I20}$$

and therefore, the instananeous power radiated into the solid angle element $d\Omega$ is given by

$$dI = |\vec{S}| R_o^2 d\Omega = \frac{c|\vec{H}|^2}{4\pi\sqrt{\varepsilon}} R_o^2 d\Omega \tag{I21}$$

The average power radiated into the s harmonic is obtained by using (I6) and by averaging (I21) over one gyroperiod T. This results in

$$dI_s = \frac{c|\vec{H}_s|^2}{8\pi\sqrt{\varepsilon}} R_o^2 \, d\Omega \qquad \text{(I22)}$$

Making use of equations (I13), (I17) and (I18), equation (I22) can finally be written as

$$\frac{dI_s}{d\Omega} = \frac{2\pi e^2 s^2}{c T^2} \frac{1}{\sqrt{\varepsilon}} \left[\cot^2\theta \, J_s^2(s\beta\sqrt{\varepsilon}\sin\theta) + \beta^2 \varepsilon \, J_s'^{\,2}(s\beta\sqrt{\varepsilon}\sin\theta) \right] \qquad \text{(I23)}$$

Since $\sqrt{\varepsilon}$ is equal to the index of refraction n, and the gyroperiod T is given by

$$T = \frac{2\pi m c}{e B} \sqrt{1-\beta^2}$$

equation (I23) is equivalent to equation (8) given in the main text.

Appendix II

We now provide an outline of the evaluation of the integral in equation (9). Using the identities (Schott, 1912; Watson, 1966)

$$J_s^2(x) = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\varphi)\cos 2s\varphi\, d\varphi \qquad (\text{II}1)$$

$$J_s'^2(x) = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\varphi)\left(\cos 2\varphi - \frac{s^2}{x^2}\right)\cos 2s\varphi\, d\varphi \qquad (\text{II}2)$$

the term in square brackets in equation (8) reduces to

$$\frac{1}{\pi}\int_0^\pi J_0(2s\beta n\sin\theta\sin\varphi)\cos 2s\varphi\left[\beta^2 n^2 - 1 - 2\beta^2 n^2\sin^2\varphi\right]d\varphi\ .$$

Using this expression and the integral (Gradshteyn and Ryzhik, 1965)

$$\int_0^1 \frac{x\, J_0(y x)\, dx}{\sqrt{1-x^2}} = \frac{\sin y}{y}$$

equation (9), with $dI/d\Omega$ given by equation (8), reduces to

$$I(\nu) = -\frac{s e^4 B^2}{m^2 c^3}\,\frac{1-\beta^2}{\pi\beta n^2}\cdot$$

$$\cdot\left\{2\beta^2 n^2\int_0^\pi \sin(2s\beta n\sin\varphi)\cos 2s\varphi\sin\varphi\, d\varphi\ +\right.$$

$$\left.(1-\beta^2 n^2)\int_0^\pi \frac{\sin(2s\beta n\sin\varphi)}{\sin\varphi}\cos 2s\varphi\, d\varphi\right\}\delta\left(\nu - \frac{s\nu_B}{\gamma}\right) \qquad (\text{II}3)$$

Making use of the identities (Schott, 1912; Schwinger, 1949)

$$J_{2s}'(z) = -\frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\cos 2s\varphi \sin\varphi \, d\varphi$$

$$\int_0^z J_{2s}(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(z\sin\varphi)\cos 2s\varphi}{\sin\varphi}\, d\varphi$$

equation (III3) becomes equivalent to equation (10).

Appendix III

We shall first derive equation (14) by using an asymptotic formula from the theory of Bessel functions (Watson, 1966), namely that for $s(1-\beta^2 n^2)^{3/2} \gg 1$

$$J_{2s}(2s\beta n) \approx \frac{1}{\sqrt{4\pi s}} \frac{1}{(1-\beta^2 n^2)^{1/4}} \left[\frac{\beta n}{1+\sqrt{1-\beta^2 n^2}}\, e^{\sqrt{1-\beta^2 n^2}} \right]^{2s}$$

By differentiating and integrating this expression and by using the identity

$$\frac{d}{d(\beta n)} \left[\frac{\beta n}{1+\sqrt{1-\beta^2 n^2}}\, e^{\sqrt{1-\beta^2 n^2}} \right]^{2s} = 2s \frac{\sqrt{1-\beta^2 n^2}}{\beta n} \left[\frac{\beta n}{1+\sqrt{1-\beta^2 n^2}}\, e^{\sqrt{1-\beta^2 n^2}} \right]^{2s}$$

,which can be derived by direct differentiation, and the condition $s(1-\beta^2 n^2)^{3/2} \gg 1$, mentioned above, we find that

$$J'_{2s}(2s\beta n) \approx \frac{1}{\sqrt{4\pi s}} \frac{(1-\beta^2 n^2)^{1/4}}{\beta n} \left[\frac{\beta n}{1+\sqrt{1-\beta^2 n^2}}\, e^{\sqrt{1-\beta^2 n^2}} \right]^{2s} \qquad (\mathrm{III}\,1)$$

and

$$\int_0^{\beta n} J_{2s}(2sx) \approx \frac{1}{2s\sqrt{4\pi s}} \frac{\beta n}{(1-\beta^2 n^2)^{3/4}} \left[\frac{\beta n}{1+\sqrt{1-\beta^2 n^2}}\, e^{\sqrt{1-\beta^2 n^2}} \right]^{2s} \qquad (\mathrm{III}\,2)$$

Using equations (III1) and (III2), equation (11) reduces to

$$F\left(\frac{\nu}{\nu_c}\right) \approx \sqrt{\frac{\pi}{3}}\, \frac{\sqrt{s}}{n} (1-\beta^2)(1-\beta^2 n^2)^{1/4} \left[\frac{\beta n}{1+\sqrt{1-\beta^2 n^2}}\, e^{\sqrt{1-\beta^2 n^2}} \right]^{2s} \qquad (\mathrm{III}\,3)$$

Finally, making use of equation (17), we obtain that

$$F\left(\frac{\nu}{\nu_c}\right) \approx \sqrt{\frac{\pi}{3}}\ \frac{\sqrt{s}}{n}\ \frac{1}{\gamma\gamma_1^{1/2}}\left[\frac{\gamma_1-1}{\gamma_1+1}\ e^{2/\gamma_1}\right]^{s} ;\ s/\gamma_1^3 \gg 1 \qquad (\text{III}\,4)$$

As can be seen, this expression is identical to equation (14) in the main text.

We now consider the derivation of equation (15). Making use of the asymptotic formula (Watson, 1966)

$$J_s(x) \approx \frac{1}{\pi}\sqrt{\frac{2(s-x)}{3x}}\ K_{1/3}\left[\frac{2^{3/2}(s-x)^{3/2}}{3x^{1/2}}\right] \qquad (\text{III}\,5)$$

which is valid for large s and x, and $x \lesssim s$, we find that for $1-\beta^2n^2 \ll 1$

$$J_{2s}(2s\beta n) \approx \frac{1}{\pi}\sqrt{\frac{1}{3}(1-\beta^2n^2)}\ K_{1/3}\left[\frac{2s}{3}(1-\beta^2n^2)^{3/2}\right] \qquad (\text{III}\,6)$$

By integrating and differentiating equation (III6) we directly obtain

$$\int_0^{\beta n} J_{2s}(2sx)\,dx \approx \frac{1}{2\sqrt{3}\,\pi s}\int_{\nu/\nu_c^1}^{\infty} K_{1/3}(y)\,dy \qquad (\text{III}\,7)$$

and

$$J_{2s}'(2s\beta n) \approx \frac{\sqrt{3}}{2\pi s}\ \frac{1}{\sqrt{1-\beta^2n^2}}\left(\frac{\nu}{\nu_c^1}\right)K_{2/3}\left(\frac{\nu}{\nu_c^1}\right) \qquad (\text{III}\,8)$$

where ν/ν_c^1 is given by equation (16).

In the process of differentiating equation (III6), we made use of the formula (Schwinger, 1949)

$$3x\,K'_{1/3}(x) + K_{1/3}(x) = -3x\,K_{2/3}(x)$$

Substituting (III7) and (III8) into equation (11) and again making use of the condition $1-\beta^2 n^2 \ll 1$, we obtain

$$F\left(\frac{\nu}{\nu_c}\right) \approx \frac{2s(1-\beta_1^2)}{3\gamma}\left[2K_{2/3}\left(\frac{\nu}{\nu_c^1}\right) - \int_{\nu/\nu_c^1}^{\infty} K_{1/3}(y)\,dy\right] \qquad (\text{III}\,9)$$

Using the identity (Schwinger, 1949)

$$2K'_{2/3}(x) + K_{1/3}(x) = -K_{5/3}(x)$$

equation (III9) can be reduced to

$$F\left(\frac{\nu}{\nu_c}\right) \approx \frac{\gamma_1}{\gamma}\left(\frac{\nu}{\nu_c^1}\right)\int_{\nu/\nu_c^1}^{\infty} K_{5/3}(y)\,dy \quad ; \quad s \gg 1,\ \gamma_1 \gg 1 \qquad (\text{III}\,10)$$

which is identical to equation (15) given in the main text. As pointed out above, equation (III4) is valid for all values of s and γ_1 as long as $s/\gamma_1^3 \gg 1$ is satisfied. If, however, γ_1 also is much larger than unity, equation (III4) should reduce to an appropriate asympotic form of equation (III10). Indeed, if $\gamma_1 \gg 1$, the expression in square brackets in equation (III4) can be expanded in powers of $1/\gamma_1$:

$$\frac{\gamma_1-1}{\gamma_1+1}\,e^{2/\gamma_1} \approx 1 - \frac{2}{3\gamma_1^3} \approx e^{-\frac{2}{3\gamma_1^3}}$$

Using this expression, and assuming that $n \approx 1$, equation (III4) reduces to

$$F\left(\frac{\nu}{\nu_c}\right) \approx \frac{\gamma_1}{\gamma}\sqrt{\frac{\pi}{2}}\left(\frac{\nu}{\nu_c'}\right)^{1/2} e^{-\nu/\nu_c'} \qquad (\mathrm{III}\,11)$$

By making use of an asymptotic expression given by <u>Ginzburg</u> and <u>Syrovatskii</u> (1964), namely that for $x \gg 1$

$$x\int_x^{\infty} K_{5/3}(y)\,dy \approx \sqrt{\frac{\pi}{2}}\, x^{1/2} e^{-x}$$

we see that equations (III4) and (III10) have indeed the same asymptotic forms for $\gamma_1 \gg 1$ and $S > \gamma_1$.

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FIGURE CAPTIONS

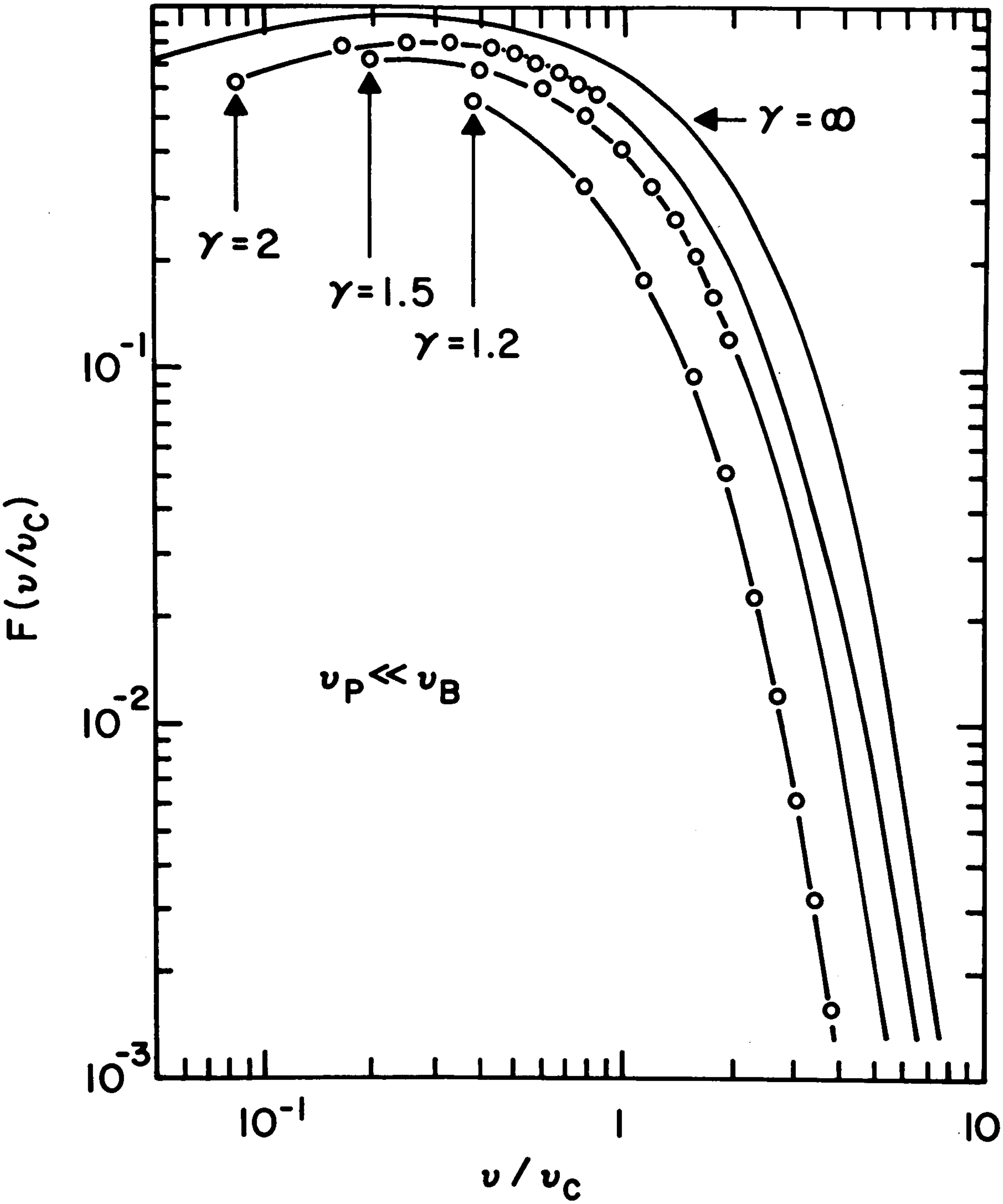
Figure 1: The emissivity function $F(\nu/\nu_c)$ in vacuum for electrons of various Lorentz factors. The circles denote the frequencies corresponding to the first 10 harmonics. The actual emission consists of discrete spikes at the individual harmonics and can be obtained from $F(\nu/\nu_c)$ by using equation (13).

Figure 2: The emissivity function $F(\nu/\nu_c)$ in an ionized medium for $\nu_p/\nu_B = 6$ and for electrons of various Lorentz factors. The actual emission consists of discrete spikes at the individual harmonics and can be obtained from $F(\nu/\nu_c)$ by using equation (13).

Figure 3: The ratio between the values of $F(\nu/\nu_c)$ calculated by using the ultrarelativistic approximation to those shown in figure 2. F_{UR} corresponds to the values of $F(\nu/\nu_c)$ obtained from equation (15) for $\nu_p/\nu_B = 6$.

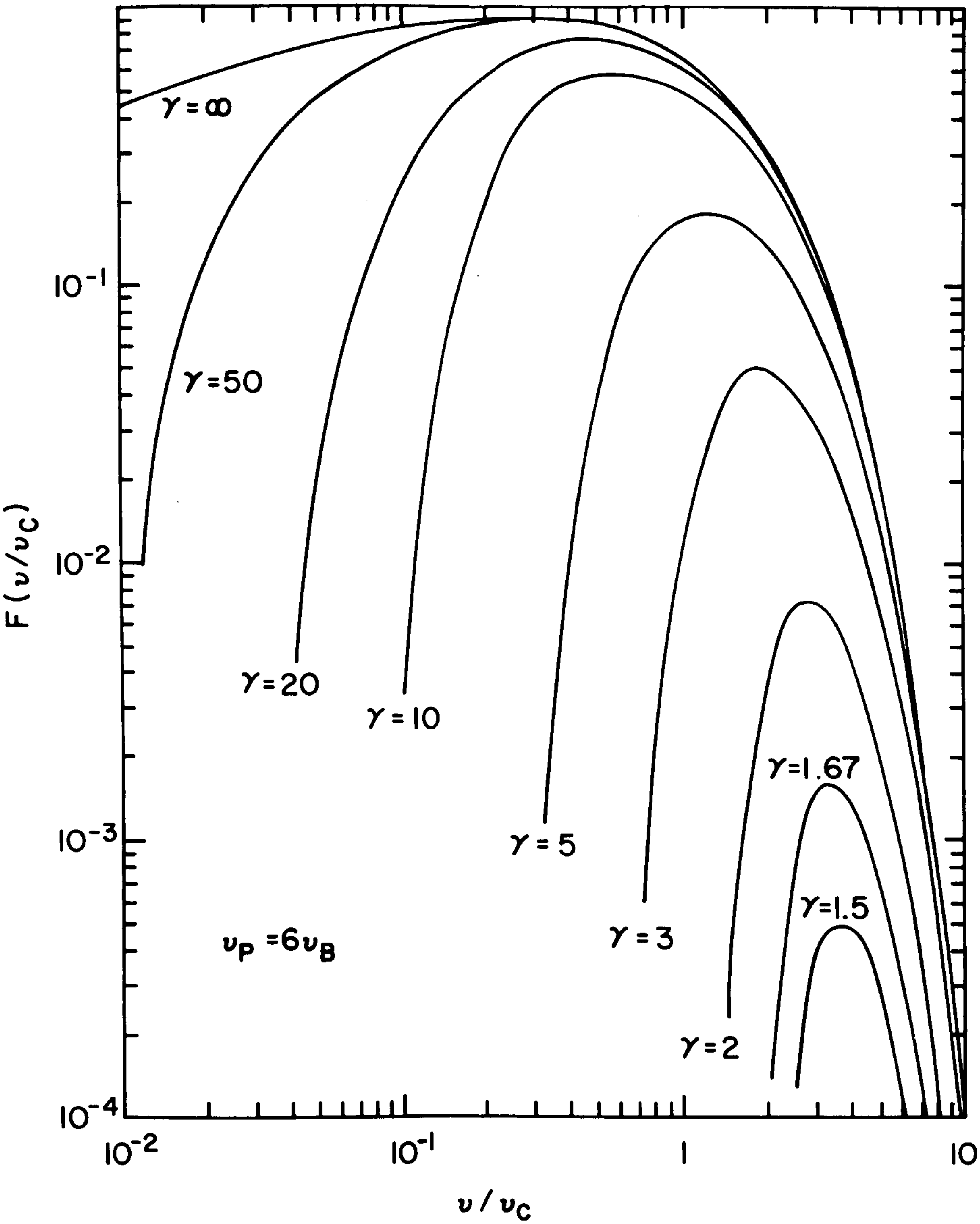
Figure 4: The emissivity function $F(\nu/\nu_c)$ in an ionized medium for $\nu_p/\nu_B = 3$ and for electrons of various Lorentz factors. The actual emission consists of discrete spikes at the individual harmonics and can be obtained from $F(\nu/\nu_c)$ by using equation (13).

Figure 5: The emissivity function $F(\nu/\nu_c)$ in an ionized medium for $\nu_p/\nu_B = 1.5$ and for electrons of various Lorentz factors. The actual emission consists of discrete spikes at the individual harmonics and can be obtained from $F(\nu/\nu_c)$ by using equation (13).



$\gamma = \infty$
$\gamma = 2$
$\gamma = 1.5$
$\gamma = 1.2$
10^{-1}
10^{-2}
10^{-3}
$F(\nu/\nu_C)$
$\nu_P \ll \nu_B$
10^{-1}
1
10
ν / ν_C

Figure 1



F (ν/ν_C)
ν/ν_C
10^-1
10^-2
10^-3
10^-4
10^-2
10^-1
1
10
γ=∞
γ=50
γ=20
γ=10
γ=5
γ=3
γ=2
γ=1.67
γ=1.5
ν_P =6ν_B

Figure 2

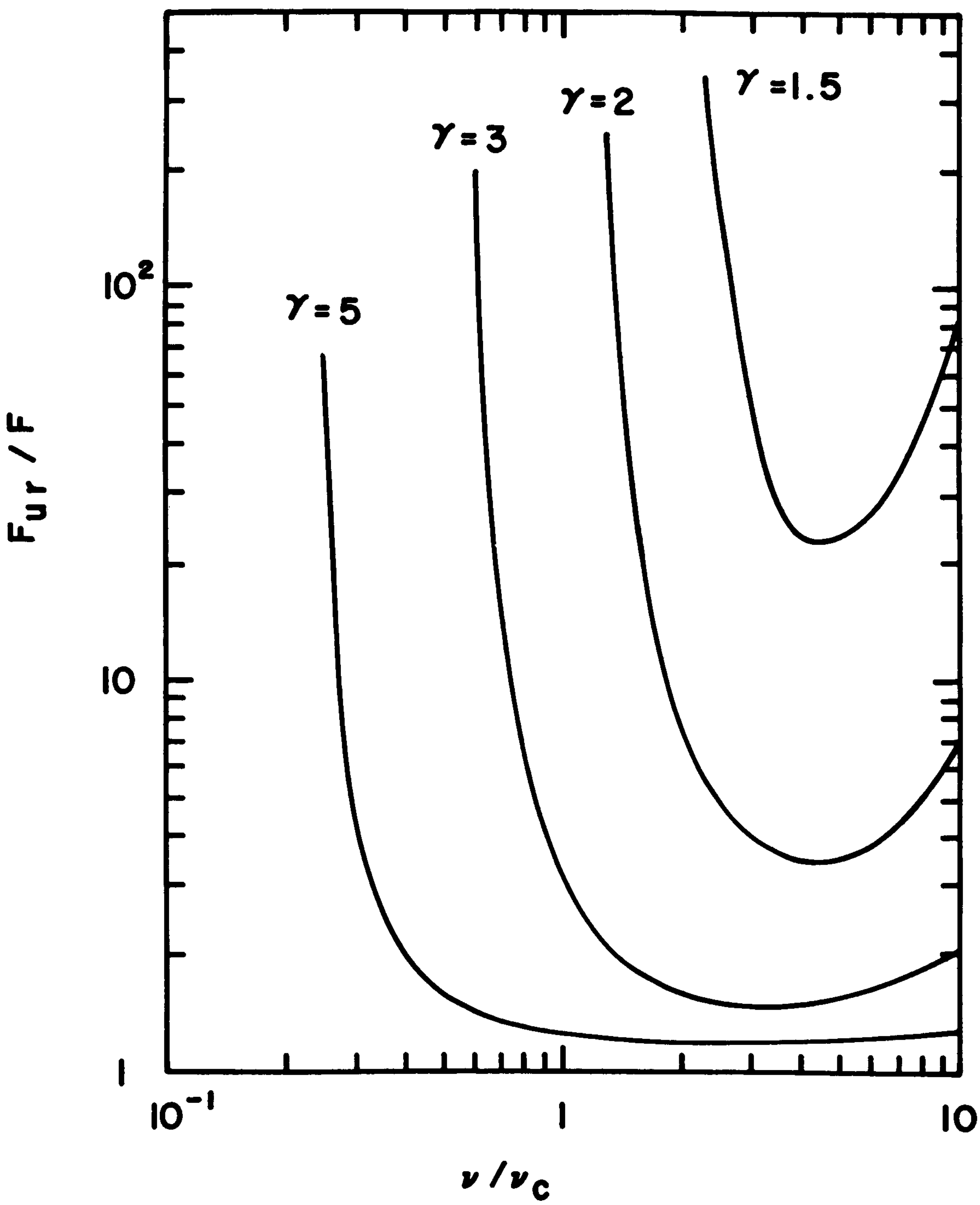


Figure 3

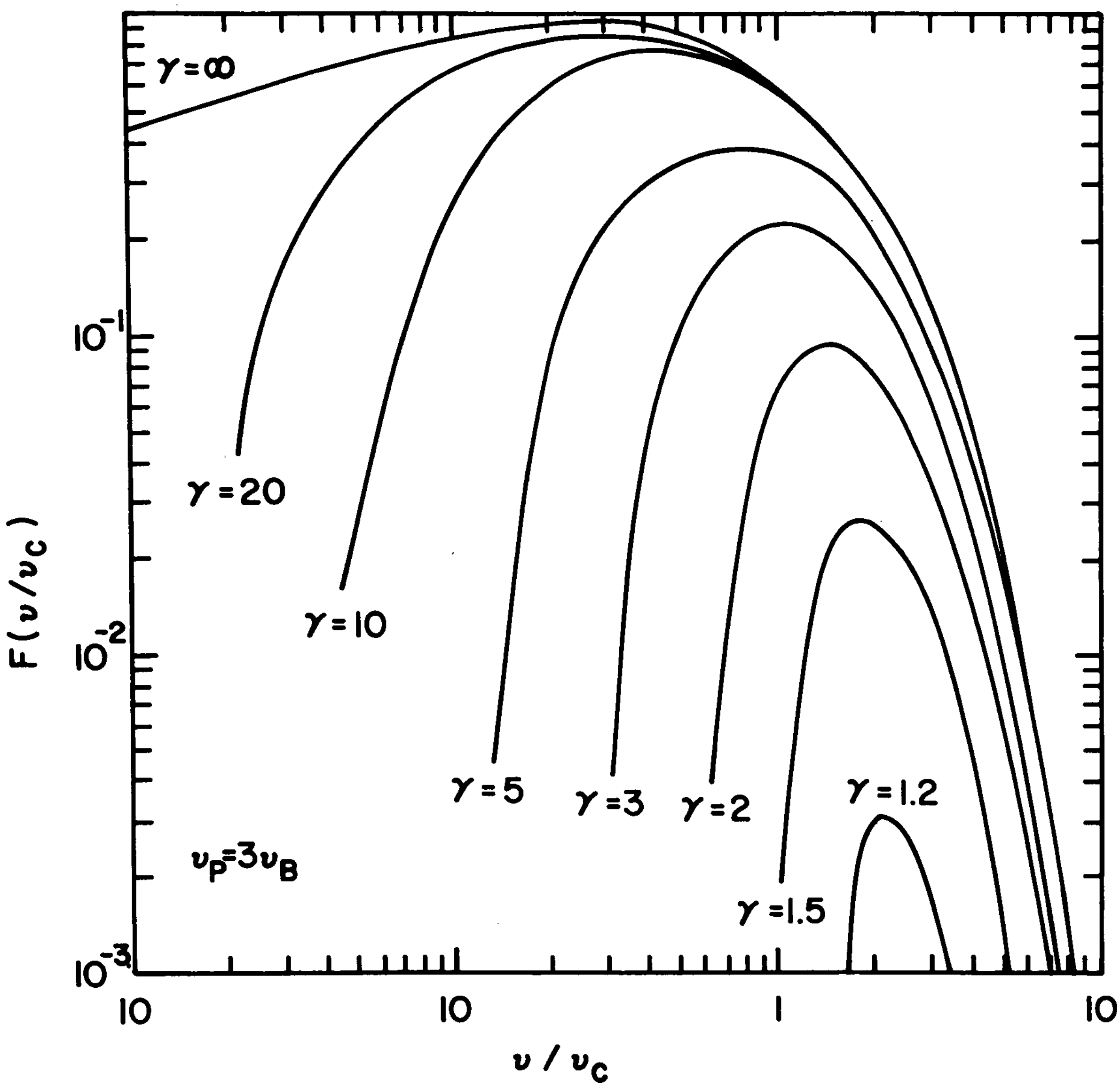
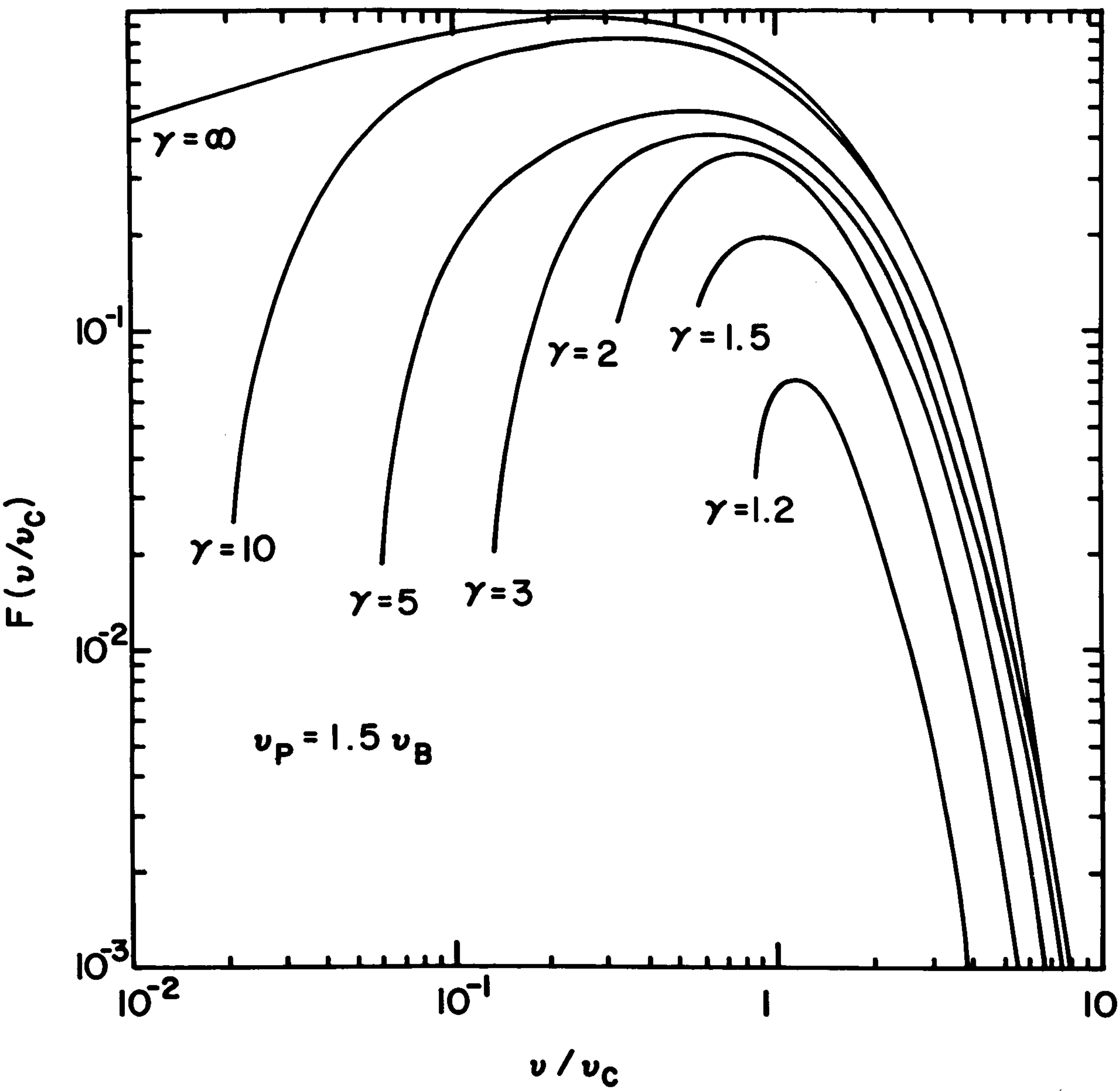

γ = ∞
γ = 20
γ = 10
γ = 5
γ = 3
γ = 2
γ = 1.5
γ = 1.2
$\nu_P = 3\nu_B$
$F(\nu/\nu_C)$
10^{-1}
10^{-2}
10^{-3}
10
10
1
10
ν / ν_C

Figure 4



$F(\nu/\nu_C)$
ν/ν_C
$\gamma = \infty$
$\gamma = 10$
$\gamma = 5$
$\gamma = 3$
$\gamma = 2$
$\gamma = 1.5$
$\gamma = 1.2$
$\nu_P = 1.5\ \nu_B$
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
1
10

Figure 5