On the Range of Unbounded Vector Valued Measure

by

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Introduction

Consider a vector valued measure (S,Σ,μ), that is a space S, a σ-field Σ of subsets of S and a countably additive function μ defined on Σ and taking values from R^n if finite or infinity denoted by ∞. We assume that $\infty+a = \infty$, $\sum_{i=1}^{\infty} a_i = \infty$ if $\|\sum_{i=1}^{n} a_i\| \to \infty$ as $n \to \infty$.

The purpose of this note is to describe the range of such measure. In the case $\mu(S)$ is finite and the measure μ is non-atomic, then a result due to A.A. Liapunov [4] says that the range of μ is compact and convex. In the case we consider, the range remains convex as can be easily seen from the Liapunov theorem but need not be closed. However, we have the following:

Theorem 1. Consider a non-atomic n-vector valued measure (S,Σ,μ). Then the range $P = \mu(\Sigma)$ of the measure μ has the following properties: (i) P is convex, (ii) the closure $\overline{P}$ of P does not contain a line, (iii) each compact extreme face of $\overline{P}$ is contained in P.

Let us recall that a subset A of a convex set B is an extreme face of B if for any three points $p_1,p_2,p_3 \in B$ such that $p_1 = \lambda p_2+(1-\lambda)p_3$, where $0 < \lambda < 1$ we have the implication: if $p_1 \in A$ then $p_2,p_3 \in A$. Since B itself is an extreme face of B, thus in the case $\overline{P}$ is compact, the part (iii) of Theorem 1 implies that $P = \overline{P}$, thus compactness of the range. Hence Theorem 1 contains as a special case Liapunov's theorem.

A proof of Theorem 1 is given in section 2 and is preceded by a Lemma given in section 1. Let us point out here that a convex closed subset of R^n, which does not contain a line has not empty profile, that is, the set of extreme points. In particular from (ii) it follows that there exist compact extreme faces of $\overline{P}$. In section 3 we discuss in more details some geometrical properties of P implied by (ii) and (iii).

1. Denote by $|\mu|$ the total variations of μ. (If $E \in \Sigma$ then $|\mu|(E) = \sup \sum \|\mu(E_i)\|$, where supremum is taken over all decomposition of E into disjoint subsets $\{E_i\} \subset \Sigma$.) The total variation $|\mu|(E)$ of E is finite if and only if $\mu(E)$ is finite. Denote by Σ_o the subset of Σ on which μ is finite. For any two $E,F \subset \Sigma_o$ denote by $\rho(E,F) = |\mu|(E \triangle F)$, where $E \triangle F$ is the symmetric difference. The function ρ is a metric function on Σ_o, provided the equality $E = F$ is meant modulo $|\mu|$, that is, $E = F$ if and only if $|\mu|(E \triangle F) = 0$. The metric space (Σ_o, ρ) is complete (cf. P. Halmos [2], p. 169) and the map $\mu: \Sigma_o \to R^n$ is continuous. Consider the inverse map $\mu^{-1}: P \subset R^n \to 2^{\Sigma_o}$. In [6], the author noticed that (in the case P compact) $\mu^{-1}(e)$ is a singelton {E} if and only if e is an extreme point of P. The lemma which follows generalizes this by showing that if e is close to an extreme point then the diameter of $\mu^{-1}(e)$ is small.

Lemma. Let (S,Σ,μ) be as in Theorem 1. Let e be an extreme point of $\overline{P} = \overline{\mu(\Sigma)}$. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\|\mu(E_i)-e\| < \delta$, $i = 1,2$, then $\rho(E_1,E_2) < \varepsilon$.

Proof: There is a basis in R^n such that e is the lexicographical maximum of $\overline{P}$ with respect to this basis (cf. [5]). Without any loss of generality we may assume this basis to be the natural basis of R^n. Thus if $(e_1,\ldots,e_n)$ are coordinates of e then we have

$$(1.1) \qquad e_1 = \max\{x_1 | (x_1,\ldots,x_n) \in \overline{P}\}$$

$$(1.2) \qquad e_i = \max\{x_i | e_1,\ldots,e_{i-1},x_i,x_{i+1},\ldots,x_n) \in \overline{P}\}$$

Since $\overline{P}$ is convex and closed the maximum in (1.2) is continuous function of $e_1,\ldots,e_{i-1}$, therefore for each $k \leqq n$ and any $\gamma > 0$ there are $\gamma_{1k},\ldots,\gamma_{k-1,k}$ positive such that the following implication holds:

(1.3k) if $(x_1,\ldots,x_n) \in \overline{P}$ and $|x_i - e_i| < \gamma_{ik}$ $i = 1,\ldots,k-1$ then $x_k < e_k + \gamma$.

Suppose $\mu = (\mu_1,\ldots,\mu_n)$, μ_i are real valued countably additive functions defined on Σ_o. For each $i = 1,2,\ldots,n$ $|\mu_i|(E)$ is the total variation of $\mu_i(E)$ and is defined in the same way as

$|\mu|$. It is easy to check the inequality $|\mu|(E) \leqq \sum_i |\mu_i|(E)$ for each $E \in \Sigma_o$. Thus we will prove the lemma if we show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any two $E_1, E_2 \in \Sigma_o$ the inequalities

$$(1.4) \qquad |\mu_i(E_j) - e_i| < \delta \quad i = 1,2,\ldots,n,\ j = 1,2$$

imply

$$(1.5) \qquad |\mu_i|(E_1 \triangle E_2) < \varepsilon \quad i = 1,2,\ldots,n.$$

Suppose that the above implication does not hold for some i and let $i=k$ be the smallest one. Hence, for each $\varepsilon > 0$ there is $\delta_o > 0$ such that for each $\delta \leqq \delta_o$ the inequalities (1.4) implies (1.5) if $i = 1,\ldots,k-1$ while for $i=k$ there is $\varepsilon_o > 0$ such that for each $\delta \leqq \delta_o$ there exists $E_j(\delta) \in \Sigma_o$ such that (1.4) holds but

$$(1.6) \qquad |\mu_k|(E_1(\delta) \triangle E_2(\delta)) \geqq \varepsilon_o$$

Set γ, in (1.3k) to be smaller than $\varepsilon_o/4$, choose $\varepsilon <$ $(1/2)\min\{\gamma_{ik}, \varepsilon_o/4\}$ and $\delta < \varepsilon$.

From (1.6) we conclude that there is $G \in \Sigma_o$ such that either $G \subset E_1 \setminus E_2$ or $G \subset E_2 \setminus E_1$ and $|\mu_k(G)| \geqq \varepsilon_o/2$. To fix the idea suppose $G \subset E_1 \setminus E_2$ and $\mu_k(G) \geqq \varepsilon_o/2$. Then $E_2 \cap G = \emptyset$, $\mu_k(E_2 \cup G) = \mu_k(E_2)+\mu_k(G) \geqq e_k-\delta + \varepsilon_o/2 > e_k+ \varepsilon_o/4 > e_k+\delta$. While by (1.4) and (1.5) $|\mu_i(E_2 \cup G)-e_i| \leqq |\mu_i(E_2)-e_i| + |\mu_i(G)| \leqq \delta + |\mu_i|(E_2 \triangle E_1) < \delta+\varepsilon \leqq 2\varepsilon \leqq \gamma_{ik}$. Thus we have a contradiction with (1.3k) applied to $x = \mu(E_2 \cup G) \in \overline{P}$. Hence (1.4) implies (1.5) if δ is small enough and the proof of the Lemma is completed.

Corollary. If $e \in \overline{P}$ is an extreme point of $\overline{P}$ then there is $E \in \Sigma_o$ such that $\mu(E) = e$. Thus $e \in P$.

Proof: Since $e \in \overline{P}$ there is a sequence $\{E_i\} \subset \Sigma_o$ such that $\mu(E_i) \to e$. By the Lemma it follows that this sequence $\{E_i\}$ is a Cauchy sequence in the metric space (Σ_o, ρ) and since the latter is complete $\{E_i\}$ has the limit $E \in \Sigma_o$. By continuity of μ it follows that $\mu(E) = e$, what was to be proved.

2. Proof of Theorem 1. If $E \in \Sigma_o$ then $\Sigma_E = \{F \cap E | F \in \Sigma\}$ is a σ-field of subsets of E and μ restricted to Σ_E is totally finite and of course non-atomic. Thus Liapunov's theorem can be applied and $\mu(\Sigma_E)$ is concluded to be convex and of course contained in P. Let now $E_1, E_2 \in \Sigma_o$. Then $E = E_1 \cup E_2 \in \Sigma_o$ also and $\mu(E_i) \in \mu(\Sigma_E) \subset P$. But $\mu(\Sigma_E)$ is convex and therefore $\lambda\mu(E_1) + (1-\lambda)\mu(E_2) \in P$ for each $0 < \lambda < 1$, which proves convexity of P.

To prove part (ii) let us suppose the contrary; that is, suppose there is $a \in R^n$, $\|a\| = 1$ such that

$$\lambda a \in \overline{P} \quad \text{for each } \lambda \text{ real.} \tag{2.1}$$

(Note that if $\overline{P}$ contains a line then the parallel line through any point of $\overline{P}$ is also contained in $\overline{P}$. Since $0 \in \overline{P}$ thus the contradiction of (ii) implies (2.1)).

In this case we can choose a sequence $\{E_i\}$ $i = \pm 1, \pm 2, \ldots$ of disjoint sets from Σ such that

$$\mu(E_i) = (\operatorname{sgn} i)a + \varepsilon_i \tag{2.2}$$

where

$$\|\varepsilon_i\| \leq 2^{-|i|} \tag{2.3}$$

This clearly will contradict the additivity of the measure μ, since the measure of $\bigcup_i E_i$ could not be uniquely determined.

By (2.1) it is clear that E_1 can be chosen. To use induction argument assume that $E_1, E_{-1}, \ldots, E_n, E_{-n}$ are chosen and (2.2) and (2.3) hold for $i = \pm 1, \ldots, \pm n$. Put $E = \bigcup_{|i|=1}^{n} E_i$ manifestly $E \in \Sigma_o$, thus $\mu(\Sigma_E)$ is compact. It is also easy to see that $P = \mu(\Sigma_E) + \mu(\Sigma_{S\setminus E})$ and therefore

$$\overline{P} = \mu(\Sigma_E) + \overline{\mu(\Sigma_{S\setminus E})}. \tag{2.4}$$

From (2.1) and (2.4) it follows that $\lambda a \in \overline{\mu(\Sigma_{S\setminus E})}$ for each real λ. Indeed, let us fix λ. For each integer k there is $f_k \in \mu(\Sigma_E)$ and $g_k \in \overline{\mu(\Sigma_{S\setminus E})}$ such that $f_k + g_k = k\lambda a$. But

$\mu(\Sigma_E)$ is compact therefore $f_k/k \to 0$ as $k \to \infty$. Hence $g_k/k \to \lambda a$ as $k \to \infty$. Since $\mu(\Sigma_{S\setminus E})$ is convex and contains 0 thus $g_k/k \in \mu(\Sigma_{S\setminus E})$ if g_k does and the closedness of the latter set gives us the desired conclusion. Therefore we may choose $E_{n+1} \subset S\setminus E$ such that (2.2) and (2.3) are satisfied also for $i = n+1$. This completes the proof of (ii).

To prove (iii) suppose B is a compact extreme face of $\overline{P}$. In particular B is compact convex subset of R^n and each extreme point of B is an extreme point of $\overline{P}$. Thus the set $\ddot{B}$ of all extreme points of B, by the Corollary, is contained in P. Since P is convex, therefore the convex hull of $\ddot{B}$ is also contained in P. But the convex hull of $\ddot{B}$ is B. Hence $B \subset P$ and the proof of (iii) is completed.

Remark. In proving (ii) we made use of Liapunov theorem for finite measure but only of convexity part. Part (iii) of Theorem 1 or rather the Corollary of Section 1 in the bounded case were obtained by Blackwell (cf. [ ], Theorem 4)

3. Denote by C the asymptotic cone of $\overline{P}$; that is

$$C = \{c \in R^n \mid p+\lambda c \in \overline{P} \text{ for each } p \in \overline{P} \text{ and } \lambda \geqq 0\} \tag{3.1}$$

Since $\overline{P}$ does not contain a line, is closed and convex therefore C is a proper closed convex cone. Consider the polar C^o of C; that is

$$(3.2) \qquad C^{o} = \{d \in R^{n} \mid \langle d,c \rangle \leqq 0 \text{ for each } c \in C\}$$

Suppose $d \in R^{n}$ is such that $\sup_{p \in \overline{P}} < d,p > < +\infty$ then $d \in C^{o}$. Indeed if d does not belong to C^{o} then there is $c \in C$ such that $< d,c > > 0$ and this together with (3.1) implies that $\sup_{p \in \overline{P}} < d,p > = +\infty$. It is easy to see that if C is proper, closed convex cone then $\operatorname{int} C^{o}$ is not empty, for each $d \in \operatorname{int} C^{o}$ there exists $\max_{p \in \overline{P}} < d,p >$ and the set $B(d) = \{p \in \overline{P} \mid < d,p > = \max_{p \in \overline{P}} < d,p >$ is compact (cf. for example [3]). On the other hand if the $\max_{p \in \overline{P}} < d,p >$ exists for a d from the boundary of C^{o} then the corresponding set $B(d)$ is unbounded. In fact, one can show that the asymptotic cone $C(d)$ of $B(d)$ is given by $\{c \in C \mid < d,c > = 0\}$. Manifestly $B(d)$ is an extreme face of $\overline{P}$ for each $d \in C^{o}$.

In particular, it follows from the above discussion that (ii) implies the existence of a compact extreme face of $\overline{P}$.

With each convex cone C in R^{n} we can associate an order in R^{n} by defining: $x \leqq y$ iff $y-x \in C$. Let $A \subset R^{n}$, a point $a \in A$ is called a minimal point of A if for each $b \in A$ the inequality $b \leqq a$ implies the equality $a = b$.

We can prove now a theorem which describes the range of a vector-valued measure from some other point of view.

<u>Theorem 2</u>. Let (S,Σ,μ) be like in Theorem 1. Then the range $P = \mu(\Sigma)$ is convex, the asymptotic cone C of $\overline{P}$ is proper, convex and closed and for each $p \in \overline{P}$ there is $p_{*} \in P$, $p_{*} \leqq p$, that is

$p-p_* \in C$. In particular, each minimal point of $\overline{P}$ belongs to P.

Proof: Consider the set
$Q = \overline{P} \cap (\{p\}-C)$. We claim it is nonempty (since $p \in \overline{P}$), convex and compact. It clearly is closed and convex. If it were unbounded there would exist $a \neq 0$ such that $\{p+\lambda a | \lambda \geqq 0\} \subset Q$. That would mean that $c \in C$ (cf. (3.1)) as well as $c \in -C$, which is impossible since $C \cap (-C) = \{0\}$. Take now any $d \in \text{int } C^{o}$ and define

$$Q_1 = \{q \in Q | < q,d > = \max_{q \in Q} < q,d > \}.$$

For each $p_* \in Q_1$ we have the inequality $p_* \leqq p$.

On the other hand let B be the smallest closed extreme face of $\overline{P}$ containing p_*. For each $b \in B$ there is $\varepsilon > 0$ such that $p = p_* + \lambda(b-p_*) \in B$ for $\lambda \in [-\varepsilon,+\varepsilon]$. In fact, if $I = \{p_*\lambda(b-p_*) | -\varepsilon \leqq \lambda < 0\}$ were disjoint with B then there would exist a hyperplane π separating I and B. But in that case $\pi \cap B$ would be an extreme face of $\overline{P}$ containing p_* but not containing b, hence smaller than B. Suppose now B is unbounded, then there exist $c \in C$ such that $p_*+\lambda c \in B$ for $\lambda \geqq 0$, thus there exist $\varepsilon > 0$ such $p_*-\varepsilon c = p_1 \in B$. Since $d \in \text{int } C^{o}$, therefore $< d,c > < 0$. Hence $< d,p_1 > = < d,p_* > - \varepsilon < d,c > > < d,p_* >$. The latter is impossible because $p_1 \leqq p_* < p$ and thus $p_1 \in Q$. So B is compact extreme face of $\overline{P}$ and as such by (iii) is contained in P. Hence $p_* \in P$ what was to be proved. If p is a minimal point of $\overline{P}$ then

$p_* = p$, therefore $p \in P$.

We will finish with a few examples.

Example 1. Let $S = [0,1]$, Σ the Lebesgue measurable subsets of $[0,1]$ and $\mu = (\mu_1,\mu_2)$ defined by

$$\mu(E) = (\int_E (1-2\tau)d\tau/\tau(1-\tau),\ \int_E d\tau/\tau(1-\tau))$$

The range P in this case is $\{(x_1,x_2) \mid |x_1| < x_2$ if $x_2 > 0$, $x_1 = 0$ if $x_2 = 0\}$. Therefore only $(0,0)$ belongs to P from the boundary of P and P is not closed. The cone $C = \overline{P}$ in this case and $(0,0)$ is the unique minimal point of P.

Example 2. Let S,Σ be as above.

$$\mu(E) = (\int_E \text{sign}(1-2\pi)d\tau/\tau(1-\tau),\ \int_E d\tau/\tau(1-\tau))$$

In this case P is closed and equal $\{(x_1,x_2) \mid |x_1| \leqq x_2\}$

Example 3. Again S,Σ are as in Example 1

$$\mu(E) = (\int_E (1-2\tau)d\tau/\tau^{1/2}(1-\tau)^{1/2},\ \int_E d\tau/\tau(1-\tau))$$

Now P is contained in $\{(x_1,x_2) \mid x_2 \geqq 0,\ |x_1| \leqq \alpha\}$ and is unbounded, therefore $C = \{(x_1,x_2) \mid x_1 = 0,\ x_2 \geqq 0\}$ and $C^o =$

$\{(x_1,x_2) | x_2 \leqq 0\}$. On the boundary of C^o we have two different directions $d_1 = (-1,0)$ and $d_2 = (1,0)$

$$\sup_{p \in P} < d_2, p > = \sup_{E \subset [0,1]} \int_E (1-2\tau) d\tau / \tau^{1/2}(1-\tau)^{1/2} =$$

$$= \int_0^{1/2} (1-2\tau) d\tau / \tau^{1/2}(1-\tau)^{1/2} < +\infty.$$

Since $\int_0^{1/2} d\tau/\tau(1-\tau) = \infty$, thus $B(d_2)$ is empty and so is $B(d_1)$ for the same reason.

Therefore we can conclude in this case that P is closed, since each point of the boundary of P belongs to a $B(d)$ for $d \in \text{int } C^o$, thus belongs to a compact extreme face of $\overline{P}$. Hence is in $\overline{P}$ and $P = \overline{P}$.

In general we have the following

Theorem 3. Let (S, Σ, μ) and P be like in Theorem 1. If the set $D = \{d | d \neq 0$ and the $\max_{p \in \overline{P}} < d,p >$ exists$\}$ is open, then P is closed.

Proof: It follows from the discussion of this section that $\text{int } C^o \subset D \subset C^o$. Thus if D is open then $D = \text{int } C^o$. On the other hand, like in example 3, if $p_o \in \partial\overline{P}$ then there is $d \in D$ such that $< p_o, d > = \max_{p \in \overline{P}} < p,d >$. Hence $p_o \in B(d)$. But $d \in \text{int } C^o$ thus $B(d)$ is compact and by (iii) of Theorem 1 $B(d) \subset P$ and consequently $p_o \in P$. Thus $\overline{P} = P$ what was to be proved.

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