X-643-66-463

NASA TM X-63084

THE USE OF RESONANT SATELLITE ORBITS TO DETERMINE LONGITUDE VARIATIONS IN THE EARTH'S GRAVITY FIELD

BY

C. A. WAGNER

SEPTEMBER 1966

NASA

GODDARD SPACE FLIGHT CENTER

GREENBELT, MARYLAND

THE USE OF RESONANT SATELLITE ORBITS TO DETERMINE LONGITUDE VARIATIONS IN THE EARTH'S GRAVITY FIELD

C. A. Wagner
Special Projects Branch
Goddard Space Flight Center
Greenbelt, Maryland

ABSTRACT

A satellite whose orbital rate is commensurate (or rational) with the earth's rotation rate will describe a constant ground track over a whole number of sidereal days. On such a satellite, the small orbit averaged along track force from certain longitude harmonics of gravity can build up over many commensurate (synodic) periods to produce appreciable changes in the semimajor axis of the orbit and longitude placement of the ground track. The libratory nature of these changes and the specific harmonics which can cause them for any resonance case are discussed with reference to a circular orbit. General formulas are presented for these librations, with typical errors determined by numerical integration. The specific theory of the 24 hour, nearly circular orbit, satellite is presented in detail with actual orbit data from three such satellites used to determine the resonant earth gravity sectorial harmonics through third order. These are found to be: $10^6\overline{C}_{22} = 2.42 \pm 0.03$, $10^6\overline{S}_{22} = -1.44 \pm 0.03$, $10^6\overline{C}_{33} = 0.32 \pm 0.11$, $10^6\overline{S}_{33} = 1.16 \pm 0.11$.

It is shown that a number of subsynchronous circular resonant orbits offer similar promise in discriminating easily and uniquely other low and high order longitude harmonics; among them H_{32}, H_{43}, H_{54}, H_{55}, H_{76} and H_{98}.

TABLE OF CONTENTS

INTRODUCTION

It has been known for some time that satellites whose periods are commensurate with the earth's rotation period can suffer long term perturbations in their elements due to longitude variations in the earth's gravity field [1]. The reason this is so is that viewed in a system of coordinates rotating with the earth, the satellite with commensurate period maintains a fixed relationship to the underlying longitude gravity field (Figure 1). Assume there is net change in an element over the period of commensurability, due to a gravity perturbation. Then, no matter how small the periodic perturbation is, over many commensurate or synodic periods a large change in that element can occur as long as the geographic configuration of the orbit is relatively unchanged. The large, or secular, change is merely the sum of the many small net periodic changes. In this paper I will be concerned only with developing the secular effects of longitude gravity harmonics on the semimajor axis of circular orbits with constant or nearly constant ground track. The orbit period of such a satellite is said to be commensurate or resonant with the earth's rotation period. In the circular orbit case, without inclination restriction, it is possible to derive the secular change of the semimajor axis from a very primitive viewpoint by a straightforward orbit averaging of the perturbation forces. A simple investigation of the symmetries of the harmonics with respect to the constant ground track will be shown to be almost sufficient to discriminate those earth perturbations which can cause secular changes of the semimajor axis and those which cannot as long as the orbit remains circular. As might be inferred, the long term change of the semimajor axis with respect to the longitude harmonics of gravity is strongly dependent on the longitude orientation of the ground track. This may be specified, for example, by one of the equator crossings of the satellite. Since a change in semi-major axis is accompanied by a change in orbit period, the longitude orientation of the ground track will also change in response to the long term perturbation. The residual perturbations themselves are strongly dependent on the orientation of the ground track with respect to each longitude harmonic. Thus, we may expect the evolution of the orbit under resonant earth gravity to be essentially described by a system of two coupled equations in the longitude orientation and semi-major axis of the satellite's orbit. We will easily derive close approximations to these equations for any nearly resonant orbit. One of the critical questions in the use of these equations is how many relevant earth harmonics must be included in any application to get a good representation of the long term evolution. Where no knowledge of the longitude harmonics exists, an infinite number of them must be carried. But in the case of the earth, it is shown that for the high altitude 24 hour satellite (with period equal to the earth's rotation period) it is highly probable that only resonant harmonics of second and third order need be retained to maintain acceptable accuracy in the equations over long periods of time.

As a reverse application of the resonant orbit evolution equations, we solve for the underlying resonant longitude gravity field of the earth from actual position data on three synchronous satellites over a three year period. Further investigation of these equations using an earth model based on recent data reveals that a number of other resonant orbits of less than 24 hours may show sufficiently strong long term effects to be useful in discriminating precisely other longitude gravity harmonics.

THE GROUND TRACK OF RESONANT ORBITS

Figure 1 shows the ground track for a general resonant orbit of moderate inclination.

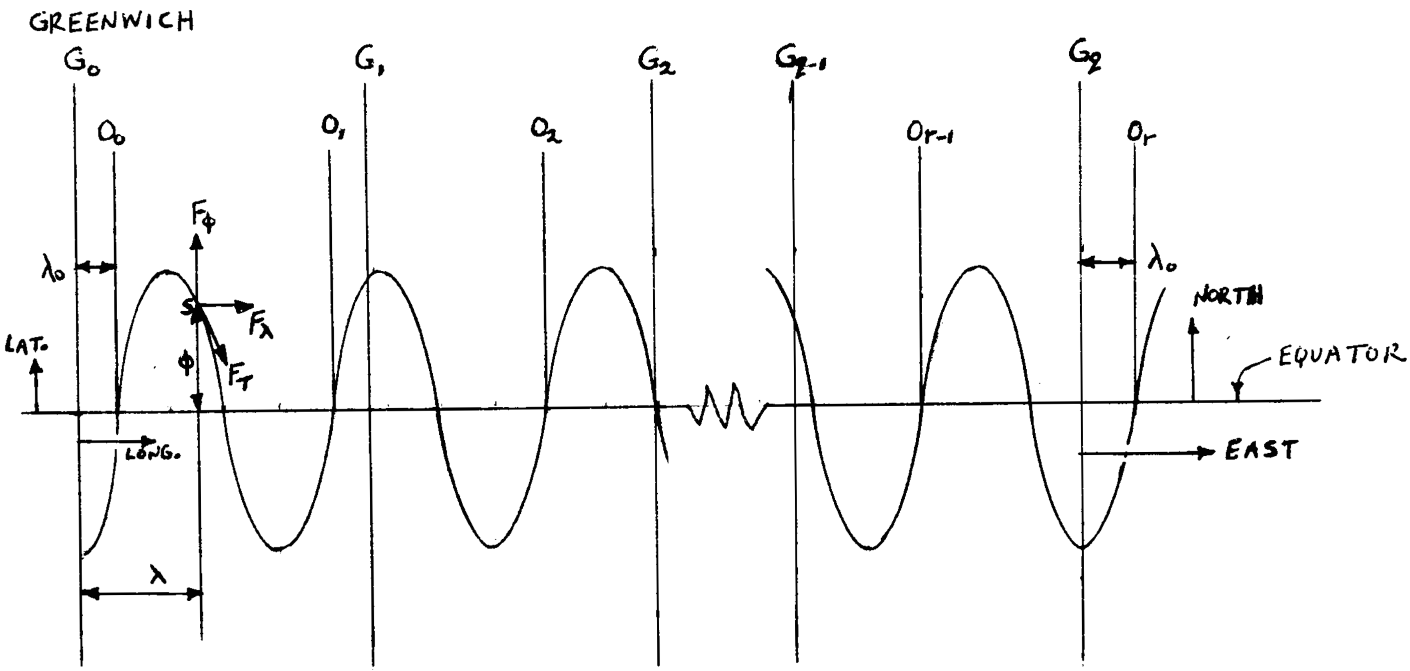

Figure 1. Ground track of a Resonant Circular Orbit

The resonant satellite makes r orbits (O_r) over $2\pi q$ radians of geographic longitude (with r and q whole numbers) before the ground track begins to repeat at, say, an ascending equator crossing of from Greenwich. Now consider the geometry of the orbit in inertial space (Figure 2). At time t, the satellite is at S , θ_s from its ascending node. The inertial longitude from its node, ΔL , is given by

$$\Delta L = \tan^{-1}(\cos i \tan \theta_s)$$

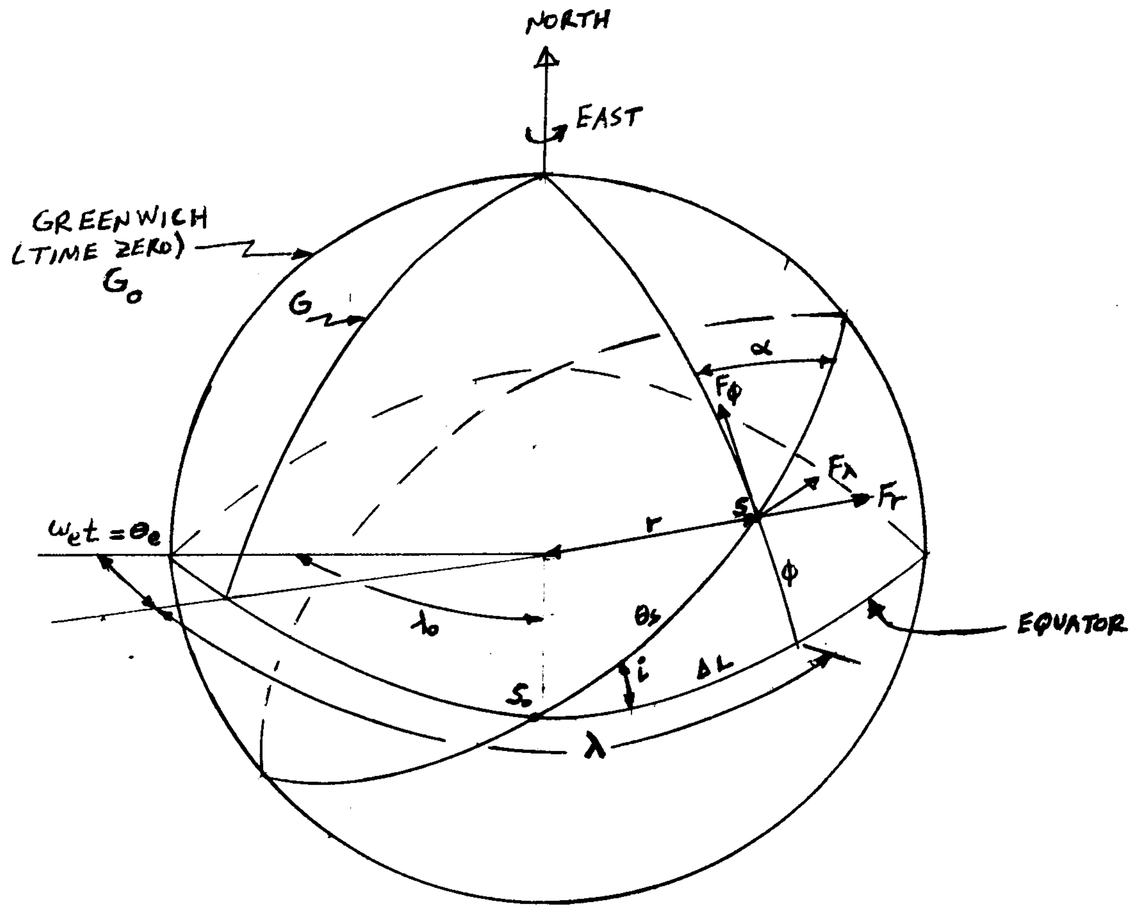

Figure 2. Orbital Geometry of Resonant Satellite in Inertial Space

Let us assume a prograde orbit, with ω_e the earth's orbital rate. Thus, the geographic longitude at s is:

$$\lambda = \lambda_o + \Delta L - \omega_e t = \lambda_o + \tan^{-1}(\cos i \tan\theta_s) - \omega_e t \tag{1}$$

In the resonant or constant ground track orbit; when $\lambda - \lambda_o = \Delta\lambda = 2\pi q$, $\theta_s = 2\pi r$. Let ω_s be the satellite's orbital rate, and $q_1\omega_s = q_2\omega_e$, where q_1 and q_2 are arbitrary real numbers at this point in the analysis. Also, let $\omega_e t = \theta_e$, the longitude turned by the earth since time zero. Between ground track repetions, $\theta_e = 2\pi n'$ with n' in days. Then, since:

$$\theta_s = \omega_s t = \frac{q_2\omega_e t}{q_1} = \frac{q_2\theta_e}{q_1}, \tag{1a}$$

(1) becomes:

$$\Delta\lambda = \left[\tan^{-1}\left(\cos i \tan\frac{\theta_e q_2}{q_1}\right) - \frac{q_2\theta_e}{q_1}\right] + \theta_e\left(\frac{q_2}{q_1} - 1\right). \tag{2}$$

$[\delta\lambda]$ in (2) is zero and periodic every $\frac{q_2\theta_e}{q_1} = 2\pi r_i$, $r_i = 1, 2, 3, \dots r$.

Therefore, the ground track repetion condition:

$\Delta\lambda = 2\pi q$, when $\theta_s = \frac{q_2\theta_e}{q_1} = 2\pi r$, implies

From (2):

$\Delta\lambda$ (at repetition of ground track)

$$= 2\pi q = 2\pi n'\left(\frac{q_2}{q_1} - 1\right) \quad \text{, or}$$

$$q = n'\left(\frac{q_2}{q_1} - 1\right) . \tag{3}$$

Also since $\frac{q_2(2\pi n')}{q_1} = 2\pi r$ at ground track repetition,

$$\frac{r}{n'} = \frac{q_2}{q_1} = \frac{\omega_s}{\omega_e} , \tag{4}$$

which means that q_1 and q_2 must be whole numbers for resonant orbits. With a given q_2/q_1 ratio, n' is the resonant, repetition, or synodic period in sidereal days and r is the number of orbits in the resonant period. Equation (4) in (3) gives

$$q = r - n' \tag{5}$$

Thus, from (5) and (4), a specification of q_2 and q_1 giving the commensurate orbital frequency, determines r from n' as a least common denominator of (4), and then q , the global circuits in the resonant period. If $r > n'$, $q_2 > q_1$ and the satellite period is less then 24 hours. If $r < n'$, $q_2 < q_1$, and the satellite period is greater than 24 hours.

RESONANT ORBIT EVOLUTION

Consider the perturbation forces on the resonant circular orbit. The only component of these forces which can effect a change in the semimajor axis of this orbit is the along track component F_T (Figure 3). This is so because the instantaneous semimajor axis of an orbit is only a function of its total energy, kinetic and two-body potential. The only force which does work on and thus effects the total energy of the circular orbit satellite, is the along track force.

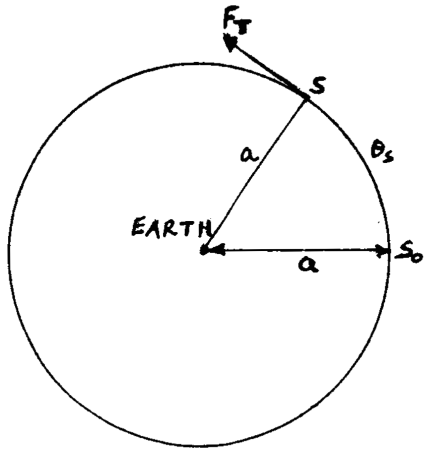

Figure 3. Along track perturbation force on a circular resonant orbit

In an elliptic orbit of the earth the total two body energy of a satellite is:

$$E = -\frac{\mu_e}{2a} \tag{6}$$

where a is the semimajor axis. To first order, then, the change in the semimajor axis due to a small change of energy ΔE is, From (6):

$$\Delta a = \frac{2a^2 \Delta E}{\mu_e} . \qquad (7)$$

But for a circular orbit satellite, the work done by a small, orbit varying, along track force F_T adds energy ΔE to the satellite each r orbits, amounting to:

$$\Delta E \text{ (per r orbits)} = 2\pi a r \bar{F}_T , \qquad (8)$$

where $\bar{F}_T$ is the orbit averaged force defined by:

$$\bar{F}_T = \frac{1}{2\pi r} \int_0^{2\pi r} F_T(\theta_s) d\theta_s \qquad (9)$$

Combining Equations (8) and (7), the evolution of the semimajor axis of a nearly circular orbit under a small perturbing force, must be governed by the difference equation:

$$\Delta a \text{ (per r orbits)} = \frac{4\pi a^3 r}{\mu_e} \bar{F}_T$$

In the resonant case, r orbits are made in n' sidereal days so that in time units ΔT of n' sidereal days (the synodic period):

$$\Delta a = \frac{4\pi a^3 r}{\mu_e} \bar{F}_T \Delta T . \qquad (10)$$

Considering Δa and $\Delta\tau$ small in terms of the long term orbit evolution, (10) can be written as a differential equation with respect to time Δt in sidereal days ($\Delta\tau = \Delta t/n'$) as:

$$\dot{a} = \frac{4\pi a^3 r \bar{F}_T}{\mu_e n'} . \qquad \frac{\text{length units}}{\text{sidereal days}} \tag{11}$$

It is to be understood, of course, that equation (11) is an r orbit averaged differential equation. To derive the orbit averaged longitude motion of the initially constant ground track under a sustaining orbit expanding force $\bar{F}_T$, we resort to Kepler's period law:

$$T = \frac{2\pi a^{3/2}}{\mu_e^{1/2}} . \tag{11a}$$

Thus, to first order, if the semimajor axis changes by Δa, the period changes by

$$\Delta T = \frac{3\pi a^{1/2} \Delta a}{\mu_e^{1/2}} . \tag{12}$$

But (10) in (12) gives the period change due to $\bar{F}_T$ after one synodic day as:

$$\Delta T \text{ (in one synodic day)} = \frac{12\pi^2 a^{7/2} r \bar{F}_T}{\mu_e^{3/2}} . \tag{13}$$

If, in r orbits, the satellite's period changes by $+\Delta T_d$, where is in units of sidereal days, then the satellite returns to its ascending equator crossing in $r(T_d + \Delta T_d)$ days. The earth meanwhile has turned 2π (rad/day) $\left[r(T_d + \Delta T_d\right]$ (days) $= 2\pi r T_d + 2\pi r \Delta T_d$ radians. But since $r T_d = n'$, a whole number, the equator crossing is $2\pi r \Delta T_d$ west of where it was at the beginning of the resonant period. Thus the drift rate of a resonant orbit after 1 synodic period is

$$\frac{\Delta\lambda}{\Delta\tau} = -2\pi r \Delta T_d .$$

But;

$$\Delta T_d = \frac{\Delta T}{T} \text{ (any units) } \times T_d = \frac{\Delta T}{T}\frac{n'}{r} .$$

Thus, $$\frac{\Delta\lambda}{\Delta\tau} \text{ (at one synodic day from resonance) } = -2\pi n' \frac{\Delta T}{T} . \tag{14}$$

Substituting (11a) and (13) in (14) gives the drift rate with respect to the synodic period at the first synodic day as:

$$\frac{\Delta\lambda}{\Delta\tau} = -\frac{12\pi^2 r n' \overline{F}_T}{(\mu_e/a^2)} \quad \text{rad/synodic day} \tag{15}$$

In the second and succeeding synodic days, if the drift rate and perturbations are small, the continually acting residual force will build up the drift rate linearly with respect to the synodic period, according to (15) whose elements may be treated as constants. Thus (15) also represents, approximately, the resonant, long term acceleration of the geographic longitude of the initially constant or nearly constant ground track of the satellite. Or, treated as a differential equation with respect to the elements at successive synodic periods:

$$\ddot{\lambda} = \frac{-12\pi^2 r n' \overline{F_T}}{(\mu_e/a^2)} \qquad \frac{\text{radians}}{\text{synodic day}^2} \qquad (16)$$

But n' sidereal days = 1 synodic day.

Thus (16) can be rewritten:

$$\ddot{\lambda} = \frac{-12\pi^2 (r/n') \overline{F_T}}{(\mu_e/a^2)} \qquad \frac{\text{radians}}{\text{sidereal day}^2} \qquad (17)$$

The relevant (longitude and latitude) earth perturbation forces (per unit mass) can be conventionally written as $F = \frac{\mu_e}{a^2} F'$, where a is the initial or instantaneous radius of the circular resonant orbit. Writing $\dot{a}_n = \dot{a}/a$ we can simplify (11) and (17) to:

$$\dot{a}_n = 4\pi (r/n') \overline{F'_T} \qquad \text{1/sid day} \qquad (18)$$

$$\ddot{\lambda} = -12\pi^2 (r/n') \overline{F'_T} \qquad \text{rad./sid.day}^2 \qquad (19)$$

EVALUATION OF THE RESONANT PERTURBATION FORCE IN TERMS OF GROUND TRACK COORDINATES

We can now proceed to evaluate the orbit averaged perturbation forces. The simplest way of doing this appears to be in terms of the ground track coordinates latitude and longitude which are the coordinates of the conventional spherical harmonic expansion of gravity given below.

We take the gravity potential on the circular orbit as the conventional associated legendre expansion:

$$V_e = \frac{\mu_e}{a}\left\{1 - \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(R_o/a\right)^n P_n^m(\sin\phi)\, J_{nm}\cos m(\lambda - \lambda_{nm})\right\}, \quad (20)$$

where R_O is the mean equatorial radius of the earth, ϕ is the latitude of the satellite and λ is the geographic longitude. By convention: [2]

$J_{nm} \leq 0,\ m \neq 0$. The associated legendre polynomials are defined as follows [2]:

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}, \quad \text{where} \qquad (20a)$$

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left\{x^n - \frac{n(n-1)x^{n-2}}{(2^1)(1!)(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{(2^2)(2!)(2n-1)(2n-3)} - \cdots\right\} \quad (20b)$$

A. Zonal Gravity Forces

The zonal gravity potential ($m=0$) is not dependent on longitude. The latitude dependence of this part of the potential can be expressed as:

$$V_n(\sin\phi) = a_1(\sin\phi)^n + a_2(\sin\phi)^{n-2} + a_3(\sin\phi)^{n-4} + \cdots , \tag{21}$$

With the series terminating at the term in $\sin\phi$ if n is odd, and in a constant term if n is even. Consider a typical single orbit ground trace of the resonance case in Figure 1.

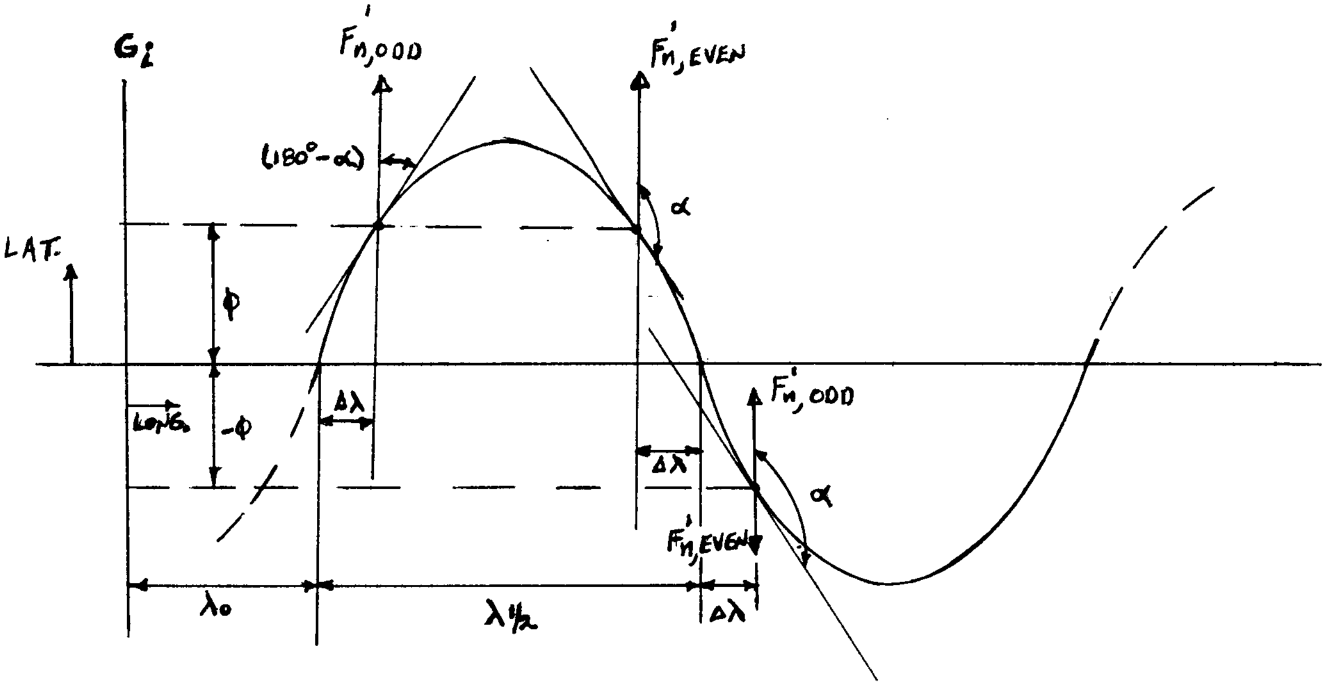

Figure 4. Orbital Ground Track with Zonal Perturbation Forces

The latitude force arising from (21) is $\frac{\hat{\phi}}{a}\frac{\partial V_n}{\partial\phi}$. Its latitude dependence is:

$$F'_{n,ODD} = \cos\phi\left\{a'_1(\sin\phi)^{n-1} + a'_2(\sin\phi)^{n-3} + \cdots + a'_i\right\}, \tag{22a}$$

$$F'_{n,EVEN} = \cos\phi\left\{a'_1(\sin\phi)^{n-1} + a'_2(\sin\phi)^{n-3} + \cdots a'_j(\sin\phi)\right\}. \tag{22b}$$

Thus, $F'_{n,EVEN}$ is an odd function and $F'_{n,ODD}$ is an even function with respect to the equator. With respect to the longitude of the descending equator crossing of the track, the azimuth α is an even function, but $F'_{n,EVEN}$ is "odd" (See fig. 4). Thus the incremental work done by $F'_{n,EVEN}$ on the satellite ($\Delta W = a \Delta \theta_s F'_T = a F'_{n,EVEN} \cos\alpha \Delta\theta_s$) is an odd function with respect to the longitude of the descending equator crossing. The integral of ΔW , or the total work done by $F'_{n,EVEN}$ over an orbit must therefore be zero, as it must be over r orbits, or the "resonance" or synodic period . As long as the orbit remains circular the even zonals can have no long term influence on the longitude acceleration of the constant or nearly constant ground track. On the other hand, for the odd zonals, we can use the property of the ground track that cos(azimuth) is antiperiodic over $\lambda_{1/2}$, or half the orbital track. Since $F'_{n,ODD}$ is periodic over $\lambda_{1/2}$; $F'_{T,n\,ODD} = F'_{n,ODD}$ cos(azimuth), is antiperiodic over $\lambda_{1/2}$. Thus the incremental work due to the odd zonals is also antiperiodic over $\lambda_{1/2}$, or the total work due to them is zero over any number of single orbits.

Concluding; as long as the orbit remains circular, the perturbing zonal harmonics of gravity can have no long term influence on the longitude acceleration of the resonant orbit.

B. Longitude Gravity Forces

With respect to the periodic longitude gravity forces, since $\cos\alpha$ is antiperiodic and $\sin\alpha$ is periodic over a half orbital track, we would like to find those longitude harmonics H_{nm} whose latitude force components are antiperiodic and whose longitude force components are periodic over $\lambda_{1/2}$. With such forces, the long track component is period every half orbit, orbit, and so forth to r orbits or the synodic period (See Figure 5). There does not seem to be a possibility of the along track force having periodicities or antiperiodicities over spans less than $\lambda_{1/2}$ throughout the resonant ground track (but see Appendix A). Therefore, in the general case for an arbitrarily inclined orbit and longitude placement it appears that the mean value of these $\lambda_{1/2}$ periodic fluctuations (when found) will not be zero. In fact every detailed calculation of the general case of a specific longitude harmonic for a resonant orbit has shown so far that only when there is periodicity in $F'_{T,nm}$ over $\lambda_{1/2}$, can the orbit average of $F'_{T,nm}$ be different from zero. An example of this calculation, for the orbit averaged effect of the H_{32} harmonic on the 12 hour resonant orbit ($r=2, n'=1$), is found in Appendix C. These detailed calculations suggest that $F'_{T,nm}$ can always be decomposed into the product of orthogonal functions in the intervals $\lambda_{1/2}$ or $2\lambda_{1/2}$ (a full orbit). Such a decomposition would allow us to make firm, instead of merely suggestive conclusions as to the relevancy of any particular harmonic on a particular resonant orbit. However, in those cases where we can show <u>anti</u>periodicity of $F'_{T,nm}$ in $\lambda_{1/2}$ or $2\lambda_{1/2}$, it is evident that those particular harmonics are <u>not</u> relevant to the orbit evolution we are interested in.

Figure 5 illustrates the method we will use to show periodicity or antiperiodicity of $F'_{T,nm}$ over $\lambda_{1/2}$. In achieving periodicity of $F'_{T,nm}$ over $\lambda_{1/2}$ (from ① to ③ in Figure 5), we can follow the changes of the longitude harmonic forces first from ① to ②, a constant latitude path, then from ② to ③, a constant longitude path. This scheme is suggested by the form of the longitude and latitude force components of the longitude harmonic $H_{nm}(m \neq 0)$. It may be readily shown (following the development in Ref. [2]) that the longitude and latitude dependencies of these force components (derived from the gradient of Eq. (20)) are:

$$F'_{\phi,nm}(\phi,\lambda) = \Big[\cos^{m+1}\phi \left\{ b_1 \sin^{n-m-1}\phi + b_2 \sin^{n-m-3}\phi + \cdots + b_i \sin\phi \right\} + \cos^{m-1}\phi \sin\phi \left\{ b'_1 \sin^{n-m}\phi + b'_2 \sin^{n-m-2}\phi + \cdots + b'_i \right\}\Big] \cos m(\lambda-\lambda_{nm}), \tag{23a}$$

and

$$F'_{\lambda,nm}(\phi,\lambda) = \Big[\cos^{m}\phi \left\{ c_1 \sin^{n-m}\phi + c_2 \sin^{n-m-2}\phi + \cdots + c_i \right\}\Big] \sin m(\lambda-\lambda_{nm}), \tag{23b}$$

FOR n-m EVEN, and:

$$F'_{\phi,nm}(\phi,\lambda) = \Big[\cos^{m+1}\phi \left\{ b_1 \sin^{n-m-1}\phi + b_2 \sin^{n-m-3}\phi + \cdots + b_i \right\} + \cos^{m-1}\phi \sin\phi \left\{ b'_1 \sin^{n-m}\phi + b'_2 \sin^{n-m-2}\phi + \cdots + b'_i \sin\phi \right\}\Big] \cos m(\lambda-\lambda_{nm}), \tag{24a}$$

AND

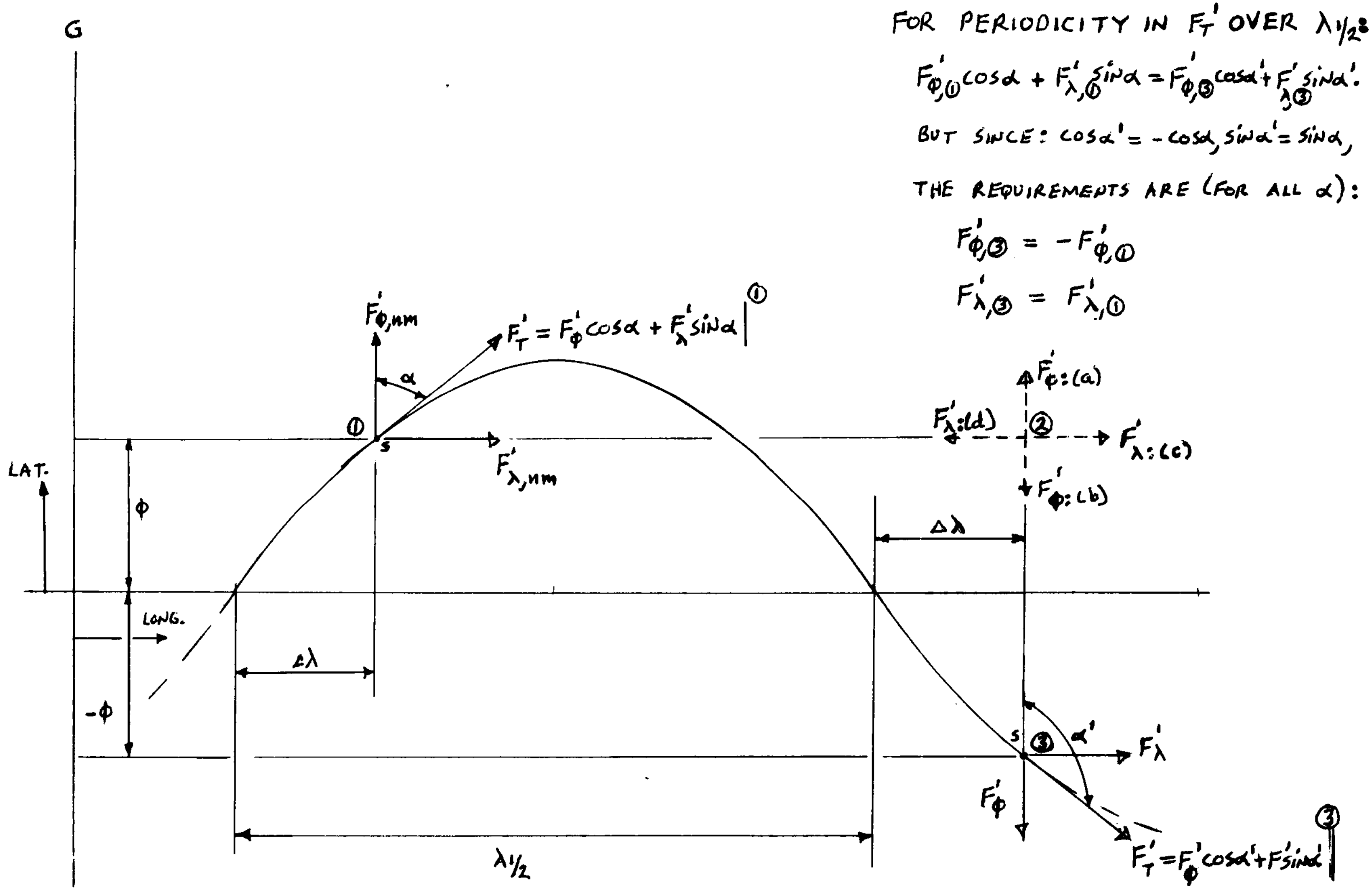

Figure 5. Half Orbital Ground Track With Longitude Harmonic Perturbation Forces

$$F'_{\lambda,nm}(\phi,\lambda) = \cos^{m-1}\phi\left[c_1 \sin^{n-m}\phi + c_2 \sin^{n-m-2}\phi + \cdots + c_j \sin\phi\right]\sin m(\lambda-\lambda_{nm}),$$

(24b)

For n-m odd.

Now, $\cos m(\lambda-\lambda_{nm})$ and $\sin m(\lambda-\lambda_{nm})$ are periodic over $\lambda_{1/2}$ [i.e., $\cos m(\lambda+\lambda_{1/2}-\lambda_{nm}) = \cos m(\lambda-\lambda_{nm})$] only if $\cos m\lambda_{1/2} = 1$ and $\sin m\lambda_{1/2} = 0$. These conditions (for $m \neq 0$) are satisfied only for $\lambda_{1/2} = 0$ (all positive m), or $\lambda_{1/2}$ arbitrary and:

$$m = \frac{\pi}{\lambda_{1/2}}\left[\pm 2, \pm 4, \pm 6, \cdots\right]. \tag{25a}$$

Similarly, $\cos m(\lambda-\lambda_{nm})$ and $\sin m(\lambda-\lambda_{nm})$ are antiperiodic [i.e., $\cos m(\lambda+\lambda_{1/2}-\lambda_{nm}) = -\cos m(\lambda-\lambda_{nm})$] only if $\cos m\lambda_{1/2} = -1$ and $\sin m\lambda_{1/2} = 0$. These conditions are satisfied for $\lambda_{1/2}$ arbitrary, only if:

$$m = \frac{\pi}{\lambda_{1/2}}\left[\pm 1, \pm 3, \pm 5, \cdots\right]. \tag{25b}$$

Further, examination of (23) shows that if n-m is even:

$$F'_{\phi,nm}(\phi) = -F'_{\phi,nm}(-\phi), \tag{26a}$$

and

$$F'_{\lambda,nm}(\phi) = F'_{\lambda,nm}(-\phi). \tag{26b}$$

Similarly, (24) shows that if n-m is odd:

$$F'_{\phi,nm}(\phi) = F'_{\phi,nm}(-\phi), \tag{27a}$$

and (27b)

$$F'_{\lambda,nm}(\phi) = -F'_{\lambda,nm}(-\phi).$$

Thus, if n-m is even, and we want periodicity of $F'_{T,nm}$ over $\lambda_{1/2}$, (26) shows we can only achieve state 3 from 2 by way of the a) and c) forces. But since the a) and c) forces can only be achieved from state 1 if (25a) holds, we have the following general result: periodicity of $F'_{T,nm}$ over a half orbit is achieved for n-m even, if and only if $\lambda_{1/2} = 0$ or $m = \frac{\pi}{\lambda_{1/2}}[\pm 2, \pm 4, \dots]$. Similarly, if n-m is odd, (27) shows we can only achieve the periodic state 3 from 2 by way of the b) and d) forces, which can be achieved from 1 only if (25b) holds. Thus we have the alternate result that if n-m is odd, periodicity of $F'_{T,nm}$ over $\lambda_{1/2}$ is assured if and only if $m = \frac{\pi}{\lambda_{1/2}}[\pm 1, \pm 3, \pm 5, \dots]$. But from (3) and (4), the half orbit excursion is:

$$\lambda_{1/2} = \frac{2\pi n'}{r}\left(\frac{q_2}{q_1} - 1\right) = \pi(1 - n'/r). \qquad (28)$$

Thus the strong presumption is that for a satellite which repeats its ground track every r circular orbits over n' sidereal days, the gravity harmonics H_{nm} which are capable of producing a long term change in the period of the satellite are given by:

n-m even, for $n' = r$

$$m = \frac{(\pm 2, \pm 4, \pm 6, \dots)}{(1 - n'/r)}, \text{ for n-m even and all integral positive m } (n' \neq r) \qquad (29a)$$

(29b)

and $$m = \frac{(\pm 1, \pm 3, \pm 5, \dots)}{(1 - n'/r)}, \text{ for n-m odd and all integral positive m } (n' \neq r). \qquad (29c)$$

Condition (29a) is just that for the 24 hour satellite (see equation (4)). In fact, for this satellite with a simple reversal of the periodicity arguments above, we can show rigorously that all the remaining longitude harmonics, H_{nm}, $n-m$ odd , can have no long term effects on the period of this orbit. Similarly, to guarantee antiperiodicity of $F'_{T,nm}$ over $\lambda_{1/2}$ for any resonant circular orbit, (enabling us to ignore the influence of H_{nm} for these purposes) the necessary and sufficient conditions are that:

$$n-m \text{ be odd, for } n'=r, \tag{30a}$$

$$m = \frac{(\pm 2, \pm 4, \pm 6, \ldots)}{(1-n'/r)}, \text{ for } n-m \text{ odd and all integral positive } m, (n' \neq r), \tag{30b}$$

and
$$m = \frac{(\pm 1, \pm 3, \pm 5, \ldots)}{(1-n'/r)}, \text{ for } n-m \text{ even and all integral positive } m, (n' \neq r). \tag{30c}$$

We can go at least one step further in eliminating additional harmonics from consideration by utilizing the periodicities of both $\cos\alpha$ and $\sin\alpha$ over a full orbit $2\lambda_{1/2}$. Since a full orbit returns the ground track to the same latitude, we only need examine the antiperiodicities of the longitude dependent factors of the harmonic forces, $\cos m(\lambda-\lambda_{nm})$ and $\sin m(\lambda-\lambda_{nm})$. Rewriting (25b), we see such antiperiodicities are possible for $2\lambda_{1/2}$ arbitrary, only if:

$$m = \frac{\pi}{2\lambda_{1/2}}[\pm 1, \pm 3, \pm 5, \ldots]$$

. But since $2\lambda_{1/2} = 2\pi(1-n'/r)$ from (28), we have the general result that additional "non resonant" harmonics on a constant ground track orbit (in the sense considered here) are those H_{nm} for which:

$$m = \frac{[\pm 1, \pm 3, \pm 5, \ldots]}{2(1-n'/r)} \quad , \quad n' \neq r. \tag{31}$$

In passing we can note that (31) should also be a valid discriminant of non resonant harmonics on a constant ground track elliptical orbit as well. This is because (assuming negligible change of all elements in a single orbit) the flight path angle is still periodic over $2\lambda_{1/2}$. The tangential component of the radial gravity perturbation forces [all with longitude dependence cos $m(\lambda - \lambda_{nm})$] are thus antiperiodic in $2\lambda_{1/2}$ for all m given by (31).

The criteria (30) and (31) enable us to ignore a large class of harmonics on the resonant orbit (specified by n'/r with the lowest common denominator). But detailed calculation shows they appear to <u>exhaust</u> the ignorable harmonics for only the 12 and 24 and 36 hour orbits. It can be verified that a single simple formula equivalent to the presumably resonant harmonic criteria (29a, b, c) , at least for $r > n'$ (synchronous and subsynchronous orbits) is:

$$m = \frac{r}{n'}(n - 2p) : \quad p = 0, 1, 2, \ldots \tag{32}$$

where m and n are positive numbers and p is such that $n \geq m$ (see appendix A for a discussion of the supersynchronous , $r < n'$, resonance selector). But (32) is just the resonance condition given by Allan [3] for subsynchronous satellites of daily ground track repetition $(n' = 1)$. Allan's theory (in terms of Lagrange's planetary equations, orbit averaged over a resonant period of a day) thus appears to be readily extendable for a resonant orbit of any commensurate period by way of (32). (see Section 5). Previous to Allan's development the author had calculated $\overline{F'_{T,nm}}$ through $n=m=4$ for the synchronous satellite directly from the spherical harmonic potential (20), λ [4]. The results were (for the n-m even, relevant harmonics):

$$\overline{F'_{T,22}} = \frac{6J_{22}}{a_*^2}\left[(1+\cos i)/2\right]^2 \sin 2(\lambda-\lambda_{22})$$

$$\overline{F'_{T,31}} = \frac{-3J_{31}}{2a_*^3}\left\{(1+\cos i)/2 - \frac{5\sin^2 i}{8}(1+3\cos i)\right\}\sin(\lambda-\lambda_{31})$$

$$\overline{F'_{T,33}} = \frac{45J_{33}}{a_*^3}\left\{(1+\cos i)/2\right\}^3 \sin 3(\lambda-\lambda_{33})$$

$$\overline{F'_{T,42}} = -\frac{15J_{42}}{a_*^4}\left\{\frac{(1+\cos i)^2}{4} - \frac{7\sin^2 i}{4}(1+\cos i)\right\}\sin 2(\lambda-\lambda_{42})$$

$$\overline{F'_{T,44}} = \frac{420J_{44}}{a_*^4}\left[(1+\cos i)/2\right]^4 \sin 4(\lambda-\lambda_{44}) \qquad (33)$$

(where a_* = semimajor axis in earth radii).

Utilizing the same technique of direct trigonometric reduction for $\overline{F'_{T,nm}}$ on the 12 hour satellite $(r=2,\ n'=1)$, the author has shown (see Appendix C) that through 4th order (n=4), the only relevant harmonics are H_{32} and H_{44} with:

$$\overline{F'_{T,32}}(12\ \text{HOUR}) = -\frac{15}{8}\frac{J_{32}\sin i(1-2\cos i-3\cos^2 i)}{a_*^3}\cos 2(\lambda-\lambda_{32}), \qquad (34a)$$

$$\overline{F'_{T,44}}(12 \text{ HOUR}) = \frac{105 J_{44}}{2 a_*^4} \left[\sin i (1+\cos i) \right]^2 \sin 4(\lambda - \lambda_{44}) \tag{34b}$$

Similarly, for the 8 hour satellite through 3rd order (n=3), the only relevant harmonic was found to be H_{33} with:

$$\overline{F'_{T,33}}(8 \text{ HOUR}) = \frac{45 J_{33}}{8 a_*^3} \sin^2 i \,(1+\cos i) \sin 3(\lambda - \lambda_{33}) . \tag{35}$$

EVALUATION OF THE GENERAL RESONANT, ORBIT AVERAGED, ALONG TRACK FORCE IN TERMS OF KEPLERIAN COORDINATES

In the previous section, though we developed a number of criteria by which we could reject a class of nonresonant harmonics, we could not accomplish the two necessary and sufficient tasks for a truly successful resonance theory:

1. To find the single necessary and sufficient criteria which determines H_{nm} to be either resonant or nonresonant on a given commensurate orbit,
2. To determine efficiently by a simply evaluated formula, the inclination function or integrals representing the orbit averaged behavior of the resonant satellite.

We will see in this section that through the expression of the perturbation force in terms of Keplerian elements, instead of ground track elements, these major tasks can be efficiently accomplished.

We have seen from the previous development (Equation 18) that the nondimensional semimajor axis (a_n) of the resonant orbit evolves at a rate, in units of 1/sidereal days, given by:

$$\dot{a}_n = 4\pi (r/n') \overline{F_T'} ,$$

where $\overline{F_T'}$ is the r orbit averaged along track force due to a disturbing harmonic as a fraction of the principal, radial, gravitational attraction. Over one synodic period (n' days), the evolution of the semimajor axis is thus given by:

$$\dot{a} = 4\pi a r \overline{F_T'} . \qquad \frac{\text{semimajor axis units}}{\text{synodic days}} \qquad (36)$$

But the instantaneous evolution of the semimajor axis is given also from Lagrange's planetary equations [3],[5] as:

$$\frac{da}{dt} = (2/\omega_s a)\frac{\partial U}{\partial X^*} . \tag{37}$$

In Equation (37), is any disturbing <u>potential</u> function, whose gradient, by definition, is the disturbing force. X^* is the mean anomaly at t=0 (epoch) (or the modified mean anomaly), defined from the mean anomaly M by:

$$M = \int_0^t \omega_s \, dt + X^* \tag{38}$$

Since the orbit we are considering is circular, we can arbitrarily put w (the argument of perigee) equal to zero. Then M is also the true anomaly counted from the ascending node, or θ (see Figure 6).

Thus,

$$\theta \text{ (circular resonant orbit)} = \int_0^t \omega_s \, dt + X^*$$

We note in passing that from its definition above, for the resonant circular orbit

$$\frac{\partial}{\partial X^*} = \frac{\partial}{\partial \theta} , \tag{39}$$

At any time in the dynamics, all other elements being held fixed. Separating (37) and integrating over r orbits in n' days, under the assumption that $(2/\omega_s a)$ changes negligibly in this time, we have

$$\Delta a \text{ (in } n' \text{ days)} = (2/\omega_s a) n' \overline{\frac{\partial U}{\partial X^*}} , \tag{40}$$

where $\overline{\frac{\partial U}{\partial X^*}} = \frac{1}{n'}\int_0^{n'} \frac{\partial U}{\partial X^*}\,dt$, is the r orbit averaged derivative of the disturbing potential.

But from Kepler's period law:

$$\omega_s = \left(\mu/a^3\right)^{1/2}, \text{ and} \tag{41}$$

$$n' = r\left(\frac{2\pi a^{3/2}}{\mu^{1/2}}\right), \tag{42}$$

with the time units of μ in sidereal days. Equations (41) and (42) in (40) gives $\dot{a}$ over a resonant time period as:

$$\dot{a} = (4\pi a r)\frac{a}{\mu}\overline{\frac{\partial U}{\partial X^*}} \qquad \frac{\text{semimajor axis units}}{\text{synodic days}} \tag{43}$$

Equating (43) with (36), the nondimensional r orbit averaged along track force is given from the averaged potential derivative, by;

$$\overline{F'_T} = \frac{a}{\mu}\overline{\frac{\partial U}{\partial X^*}}. \tag{44}$$

Now, in order to find $\overline{\frac{\partial U}{\partial X^*}}$, we have to express the disturbing potential, in our case, that arising from a longitude gravity harmonic H_{nm}, in terms of a set of mutually exclusive orbitial elements including X^*. Allan [3] has summarized previous developments [6], [7] of the longitude harmonics into series in terms of the ordinary Keplerian elements a, e, i, Ω, w and M or f (the true anomaly). Here we only state the result for zero eccentricity. Each longitude spherical harmonic H_{nm} $(m \neq 0)$ in the expansion (20), gives rise to a potential function, in terms of the elements of a circular orbit (see Fig. 6).

$$U_{nm} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \sum_{p=0}^{n} R \left\{ F_{nmp}(i) \exp j\left[(n-2p)\theta + m(\Omega - \omega_e t - \lambda_{nm})\right] \right\} \tag{45}$$

where $j = \sqrt{-1}$, $R\{\ \}$ signifies the real part of $\{\ \}$ and the dynamics (epoch) begins at zero sidereal time, or when the Greenwich Meridian passes through the Vernal Equinox.

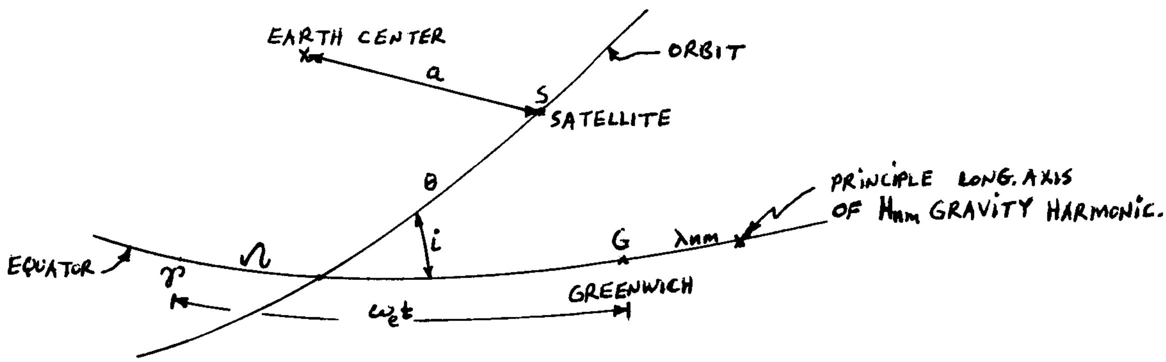

Figure 6. Circular Orbital Elements at time t

The inclination functions $F_{nmp}(i)$ are given as:

$$F_{nmp}(i) = \frac{j^{n-m}(n+m)!}{2^n p!(n-p)!} \sum_{k=0}^{k_{MAX}} (-1)^k \binom{2n-2p}{k} \binom{2p}{n-m-k} \cos^{3n-m-2p-2k}(i/2) \cdot \sin^{m-n+2p+2k}(i/2) \tag{46}$$

where $k_{MAX} = n - m$.

Now, since $\frac{\partial U}{\partial X^*} = \frac{\partial U}{\partial \theta}$ for the circular orbit, From (45):

$$\frac{\partial U_{nm}}{\partial X^*} = -\frac{\mu}{a} J_{nm} (R_o/a)^n \sum_{p=0}^{n} R\left\{ F_{nmp}\ j(n-2p) \exp j\left[(n-2p)\theta + m(\Omega - \omega_e t - \lambda_{nm})\right]\right\}. \tag{47}$$

We can include the imaginary part of $j F_{nmp}$ in the exp. term of (47) by defining a phase angle δ_{nm} such that:

$$j \cdot j^{n-m} = j^{n-m+1} = e^{j\delta_{nm}} = \cos\delta_{nm} + j \sin\delta_{nm}. \tag{48a}$$

We see from its definition in (48a) that:

$$\delta_{nm} = \frac{\pi}{2}(n-m+1). \tag{48b}$$

With the result of (48), the real part of (47) is readily evaluated and the potential derivative becomes

$$\frac{\partial U_{nm}}{\partial X^*} = -\frac{\mu}{a} J_{nm}\left(\frac{R_o}{a}\right)^n \sum_{p=0}^{n} F'_{nmp}(i)(n-2p) \cos\left[(n-2p)\theta + m(\Omega - \omega_e t - \lambda_{nm}) + \delta_{nm}\right], \tag{49}$$

where $$F'_{nmp}(i) = F_{nmp}/j^{n-m} \tag{49a}$$

Taking the average of $\frac{\partial U_{nm}}{\partial X^*}$ over r orbits in n' days:

$$\overline{\frac{\partial U_{nm}}{\partial X^*}} = \frac{1}{n'}\int_0^{n'} \left[\frac{\partial U}{\partial \theta}\right] dt = -\frac{\mu}{a} J_{nm} (R_o/a)^n \left(\frac{1}{n'}\right)\int_0^{n'} \left[\sum_{p=0}^{n} (F'_{nmp})(n-2p)\cos\left\{(n-2p)\theta + m(\Omega - \omega_e t - \lambda_{nm}) + \delta_{nm}\right\}\right] dt, \tag{50}$$

assuming constant a over the resonant period. Let θ_o be the nodal argument at t=0. Then $\theta = \theta_o + \omega_s t$, assuming a constant orbital rate over the resonant period. In terms of θ_o , the n+1 integrals of (50) are:

$$\frac{1}{n'}\int_0^{n'}[\ \ ]dt = \frac{1}{n'}\int_0^{n'}\Big[\Big[\sum_{p=0}^{n} F'_{nmp}(n-2p)\cos\{(n-2p)\theta_o + m(\Omega-\lambda_{nm}) + \delta_{nm} + t[\omega_s(n-2p) - m\omega_e]\}\Big]\Big]dt\ . \tag{51}$$

Let $$\delta'_{1p} = (n-2p)\theta_o + m(\Omega - \lambda_{nm}) + \delta_{nm}, \text{ and} \tag{52a}$$

$$\delta'_{2p} = \omega_s(n-2p) - m\omega_e\ . \tag{53b}$$

Under the basic assumption of constant orbital elements over the relatively short resonant period n', δ'_{1p} and δ'_{2p} will be constants and the integrals of (51) can be evaluated as:

$$\frac{1}{n'}\int_0^{n'}[\ \ ]dt = +\sum_{p=0}^{n}\frac{1}{n'\delta'_{2p}}\Big| F'_{nmp}(n-2p)\sin(\delta'_{1p} + \delta'_{2p}t)\Big|_{t=0}^{t=n'}$$

$$= \sum_{p=0}^{n}\frac{1}{n'\delta'_{2p}} F'_{nmp}(n-2p)\left[\sin(\delta'_{1p}+n'\delta'_{2p}) - \sin\delta'_{1p}\right] \tag{53}$$

But ω_s and ω_e are simply related by the constant ground track condition, Equation (4):

$$\omega_s/\omega_e = r/n' \tag{54a}$$

$$n' = \frac{2\pi r}{\omega_s} \text{ days} \tag{54b}$$

with ω_s in units of radians/day.

From (54a), (54b), and (52b),

$$\begin{aligned} n'\delta'_{2p} &= 2\pi\left[r(n-2p) - mn'\right] \\ &= 2\pi K', \end{aligned} \tag{55}$$

where

$$K' = r(n-2p) - mn' = 0, \pm 1, \pm 2, \ldots \tag{56}$$

depending on the integral values of n, m, p, r and n' for the orbit and harmonic in question.

If $K' \neq 0$, $n'\delta'_{2p} = 2\pi(\pm 1, \pm 2, \ldots)$, and $\sin(\delta'_{1p} + n'\delta'_{2p}) = \sin\delta'_{1p}$.

In this case then <u>all</u> the n+1 integrals of (50) are <u>zero</u>. However, when $K' = 0$, $n'\delta'_{2p} = 0$ and each integral solution from (53) is an indeterminate form. To resolve this case we seek formally:

$$\begin{aligned} &\lim_{n'\delta'_{2p}\to 0} \frac{1}{n'\delta'_{2p}}\left[\sin(\delta'_{1p} + n'\delta'_{2p}) - \sin\delta'_{1p}\right] \\ &= \lim_{n'\delta'_{2p}\to 0} \frac{1}{(n'\delta'_{2p})}\left\{\sin\delta'_{1p}\left[\cos(n'\delta'_{2p}) - 1\right] + \cos\delta'_{1p}\sin(n'\delta'_{2p})\right\}. \end{aligned}$$

By L'Hopital's rule, this last limit is the limit of the ratio of the derivative of the numerator to the derivative of the denominator with respect to $(n'\delta'_{2p})$ as $n'\delta'_{2p} \to 0$ or:

$$\lim_{(n'\delta'_{2p})\to 0} \left[-\sin(n'\delta'_{2p})\sin\delta'_{1p} + \cos\delta'_{1p}\cos(n'\delta'_{2p})\right] = \cos\delta'_{1p} \quad (57)$$

From (53) we note that the average force integrals can also be zero if $n-2p=0$, but then from (56), $k'\neq 0$, which case has already been covered.

We thus have the general result that for a resonant or constant ground track circular orbit, only those harmonics which satisfy the condition:

$$k' = 0 = r(n-2p) - mn' \quad \text{or}$$

$$m = (r/n')(n-2p) , \quad (58)$$

for $p = 0, 1, 2, \ldots$ can produce residual or long term orbit averaged along track forces.

But this is precisely the resonant harmonic condition presumed in (32) and here shown rigorously. We have thus shown that Allan's one day resonance theory [3] , $n'=1$, for circular orbits is readily extendable to any number of synodic days. (The extension is made by substituting r/n' for β in Allan's results).

To complete the calculation of the averaged along track force for any constant ground track orbit (r, n') , H_{nm} is resonant on it if a single positive (or zero) p exists which satisfies (58). Given n, m, r and n', of course, there can be only one p which does satisfy (58). Thus only one of the n+1 integrals of (50) can be non-zero and with this single resonant n, m, p combination [satisfying (58)] in mind, the r orbit averaged disturbing function due to H_{nm} (resonant) is [with the results of (57), and (58) in the form $n-2p = mn'/r$, in (53) and then (50)]:

$$\frac{\partial \overline{U_{nm}}}{\partial \chi^*} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \frac{mn'}{r} F'_{nmp} \cos\left\{\frac{mn'}{r}\theta_0 + m(\Omega - \lambda_{nm}) + \delta_{nm}\right\}. \tag{59}$$

Finally we would like to express this disturbing function in terms of the longitude of the constant equator crossings λ_ℓ in the resonant ground track. (see Fig. 7).

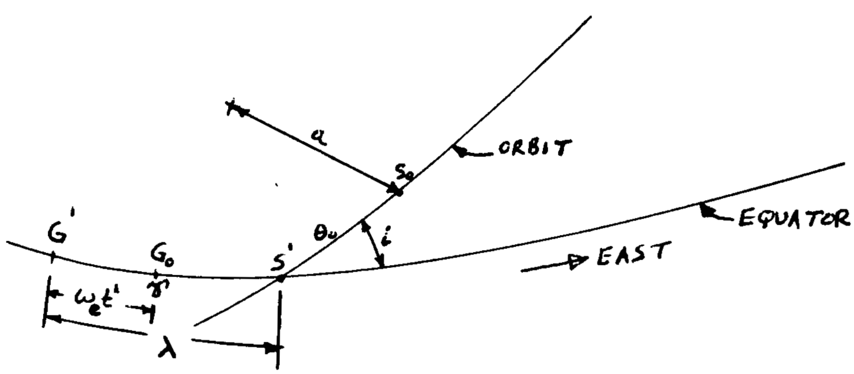

Figure 7. Orbital Parameters at t=0.

The satellite was at its ascending equator crossing just previous to the epoch at S' when the Greenwich Meridian was at G', $\omega_e t'$ west of Υ. We assume, as before, that the orbital parameters (with the modified mean anomaly) suffer negligible change over the resonance period. Then since $\theta_o = \omega_s t'$ and $\omega_s/\omega_e = r/n'$,

$$t' = \theta_o/\omega_s\,, \quad \omega_e t' = \theta_o \omega_e/\omega_s \text{ and } \omega_e t' = \theta_o (n'/r)$$

Thus λ, the longitude of the ascending equator crossing just previous to the epoch is given by:

$$\lambda = \omega_e t' + \Omega = (n'/r)\theta_o + \Omega \,. \tag{60}$$

Equation (60) in (59) gives the averaged disturbing function as:

$$\overline{\frac{\partial U_{nm}}{\partial X^*}} = -\frac{\mu}{a} J_{nm} \left(\frac{R_o}{a}\right)^n m(n'/r) F'_{nmp} \cos\left\{m(\lambda - \lambda_{nm}) + \delta_{nm}\right\} \,. \tag{61}$$

Finally, it makes no physical difference in our arguments whether we reference the orbit to any of the r distict, constant, ascending or descending, equator crossings ($2r$ longitudes in total). The r orbit averaged disturbing force on the constant ground track is obviously independent of the specific longitude or orbit time reference.

We can find the representation of the force with respect to the other $2r-1$ crossings by finding the relationship of these crossings to λ, the ascending equator crossing just prior to the epoch.

From (3) and (4), the constant ground track extends $2\pi n'[(r/n')-1]$ $= 2\pi(r-n')$ radians in geographic longitude. Thus, since there are $2r$ equally spaced equator crossings in this span, they will be separated by $\frac{\pi}{r}(r-n')$ radians. Thus the successive equator crossings, from the original one calculated in (60), are given by

$$\lambda_\ell = \lambda + \pi\ell(1-n'/r)\,, \tag{62}$$

Where $\ell = 1,2,3,\ldots 2r-1$, (odd for descending and even for ascending crossings). Solving (62) for λ and substituting this in the cos. part of (61), this part becomes:

$$\begin{aligned}\cos\{\ \} &= \cos\{m[\lambda_\ell - \pi\ell(1-n'/r) - \lambda_{nm}] + \delta_{nm}\}\\ &= \cos[m(\lambda_\ell - \lambda_{nm}) + \delta_{nm}]\cos m\pi\ell(1-n'/r)\\ &\quad - \sin[m(\lambda_\ell - \lambda_{nm}) + \delta_{nm}]\sin m\pi\ell(1-n'/r)\,.\end{aligned}$$

But from the resonance condition (58), $mn'/r = n-2p$, which must be a positive integer. Thus $m\pi\ell(1-n'/r) = \pi\ell(m-n+2p)$, which must, for the resonant orbit, be some integral (negative or positive) multiple of π.

We see from this that for ℓ even, or the ascending equator crossings, $\cos m\pi\ell(1-n'/r)=1$, $\sin m\pi\ell(1-\frac{n'}{r})=0$ and the force is expressed exactly as in (61). Furthermore, if $|m-n+2p|$ is <u>even</u> (or zero) the force is also given by (61) with respect to all the descending equator crossings. Only when $|m-n'+2p|$ is <u>odd</u> will the form for the force with respect to the descending equator crossings be different than (61). Since $2p$ is always even, and $n \geq m$, this latter condition is fulfilled if n-m is odd. We can cover all cases by the form (61) if we write a new phase angle $\delta'(n,m,\ell)$ which varies not only with (n,m) but with whether the descending or ascending node λ is being considered.

Let

$$\begin{aligned} \delta'(n,m,\ell) &= \delta_{nm} + \pi \ell (n-m) \\ &= \frac{\pi}{2}\left\{(n-m)(2\ell+1)+1\right\} . \end{aligned} \tag{63}$$

Then we can write (61) in the general form:

$$\overline{\frac{\partial U_{nm}}{\partial X^*}} = -\frac{\mu}{a} J_{nm}\left(\frac{R_o}{a}\right)^n m(n'/r)\, F'_{nmp} \cos\left\{m(\lambda_\ell - \lambda_{nm}) + \delta'(n,m,\ell)\right\}, \tag{64}$$

where $\ell = 1$ refers to the geographic longitude of any descending equator crossing, and $\ell = 2$ refers to any ascending equator crossing of the resonant orbit. Summarizing these results, from (64) and (44), the nondimensional r orbit averaged along track force on the resonant orbit (n',r) is given as the sume of all relevant n, m terms discriminated by (58):

$$\overline{F'_T} = -\sum_{n,m \text{ relevant}} J_{nm}\left(\frac{R_o}{a}\right)^n m(n'/r)\, F'_{nmp} \cos\left\{m(\lambda_\ell - \lambda_{nm}) + \delta'(n,m,\ell)\right\}, \tag{65}$$

Where:

$$F'_{nmp}(i) = \frac{(n+m)!}{2^n p!(n-p)!} \sum_{k=0}^{k_{MAX}} (-1)^k \binom{2n-2p}{k} \binom{2p}{n-m-k} \cos^{3n-m-2p-2k}(i/2) \sin^{m-n+2k+2p}(i/2) . \tag{65a}$$

$$k_{MAX} = n-m , \tag{65b}$$

$$p = \frac{n(\text{relevant}) - m(\text{relevant})(n'/r)}{2} = 0, 1, 2, \ldots , \tag{65c}$$

there being only one integral or zero value of p [from (65c)] for each relevant H_{nm} on a resonant circular orbit, and:

$\ell = 1$ for descending equator crossings,

$\ell = 2$ for ascending equator crossings, and

$$\delta'(n, m, \ell) = \frac{\pi}{2}\left[(n-m)(2\ell+1)+1\right] . \tag{65d}$$

ORBIT EVOLUTION EQUATIONS FOR CIRCULAR EARTH RESONANT ORBITS

Using the results of the last section on the r orbit averaged forces due to the longitude harmonics, the long term semimajor axis change of the resonant orbit is given from (65) in (18) as:

$$\dot{a} = -4\pi a \sum_{n,m \text{ relevant}} J_{nm} (R_0/a)^n m F'_{nmp}(i) \cos\{m(\lambda_\ell - \lambda_{nm}) + S'(n,m,\ell)\},$$

$$(a \text{ units})/\text{sidereal day}. \tag{66}$$

Similarly, the r orbit averaged geographic longitude (λ_ℓ) drift resulting from the resonant forces is given from (65) in (19) as:

$$\ddot{\lambda}_\ell = 12\pi^2 \sum_{n,m \text{ relevant}} J_{nm} (R_0/a)^n m F'_{nmp}(i) \cos\{m(\lambda_\ell - \lambda_{nm}) + S'(n,m,\ell)\},$$

$$\text{rad./sid days}^2 \tag{67}$$

The parameters ℓ, p and (n,m) are defined in (65a-d), as is the inclination factor $F'_{nmp}(i)$ and $S'(n,m,\ell)$.
It should be noted that (66) and (67) apply strictly, only to an always circular orbit satellite. This condition is, of course, immediately violated as soon as the perturbations which drive these equations (as well as other perturbations) are introduced. In addition, the equations were derived with reference specifically to the perturbations from the condition of constant ground track. They may be thought of as giving the leading term in an expansion for the drift about the resonance condition. With respect to orbits whose equator crossings have

an initial drift rate, these equations, as stated, are in error even at time zero. However, numerical trajectories have confirmed their adequacy in predicting the orbit evolution of many near resonant satellites with appreciable drift rate and moderately small eccentricities [8]. (Allan's paper [3] gives more information on the magnitude of the orbit acceleration arising from nonzero eccentricty, nonzero drift rate, the oblateness of the earth, as well as sun and moon accelerations.) The most important feature of these equations is that, considered either as evolving from a constant orbit or near constant orbit state, only the variables time Δa (or $\dot{\lambda}$) and λ change appreciably from any initial values. This can only be appreciated by integrating (66) and (67) from initial conditions under this assumption and checking the change in the other variables (principally a) of the equations. This has been done in the case of the 24 hour satellite under H_{22} perturbations [9]. These tests show (67) to be essentially a nonlinear equation in λ with <u>constant</u> <u>coefficients</u> determinable from initial conditions. The coupled equation in the nondimensional semimajor axis change a_n ($a_n = \Delta a/a$) also may be thought of as having essentially constant coefficients. With these remarks, (67) can be reduced to a simple pendulum equation [for $\lambda(t)$] for each relevant harmonic H_{nm}. Depending on the initial conditions of the near constant ground track, with respect to each relevant harmonic, the possibility of either circulatory or libratory drift of λ exists, coupled with long period oscillations of the semimajor axis described through (66). The characteristics of this "long term resonance" or libratory behavior is discussed in Section 9.

It has also been discussed in the literature previously in references [10], [11] and [12] for synchronous satellites and Ref. [13] for other resonant orbits. Here, we would just like to estimate the maximum long term effects the longitude harmonics may have on near circular subsynchronous resonant orbits which may have future geodetic applications (see section 8). Precise measurement of the long term longitude accelerations on three operating synchronous satellites has already given an excellent discrimination of H_{22} and H_{33} [8], as will be described shortly. The implementation of these effects to determine longitude variations in the earth's gravity field is extremely simple. Aside from a rough determination of the initial inclination of the orbit, all that is necessary is to obtain good data on the geographic longitudes of the equator crossings of the near resonant satellite. In general, the additional disturbances of the sun, moon and earth zonal gravity cannot be ignored where the longitude gravity effects (which are strongly dependent on inclination) are likely to be very weak.

DETERMINATION OF H_{22} AND H_{33} FROM 24 HOUR SATELLITE DATA

In the case of the 24 hour satellites, numerical integration has shown that additional gravity perturbations to equation (67) are within the standard deviations of the nodal acceleration observations presented in Table 1[8]. This data has been derived from almost three years of tracking record on the free gravity drift of syncom 2, 3 and the early bird satellite, all with nearly circular orbits. The gravity harmonics C_{nm}, S_{nm} in Table 1 are defined from the J_{nm}, λ_{nm} of (20) as:

$$C_{nm} = -J_{nm} \cos m\lambda_{nm}$$
$$S_{nm} = -J_{nm} \sin m\lambda_{nm},$$

to conform to the common way of expressing the series of longitude harmonics. The normalized longitude coefficients $\overline{C_{nm}}$, $\overline{S_{nm}}$ are defined from C_{nm}, S_{nm} by:

$$\overline{C_{nm}}, \overline{S_{nm}} = C_{nm}, S_{nm} \left[\frac{(n-m)!\,(2n+1)2}{(n+m)!}\right]^{-1/2},$$

from which it can be shown that if all $\overline{C_{nm}}$, $\overline{S_{nm}}$ are 1, the integral of the square of all the normalized harmonic functions over the unit sphere is 4π. The harmonics H_{22} and H_{33} reported in Table 1 were derived by a weighted least squares solution of (67) [more specifically, using the orbit averaged coefficients of (33) in (19) with $r = n' = 1$] according to the data in the table. The data was also tested for H_{31}, H_{42} and H_{44}. These tests did not produce a significant change in the results for H_{22} and H_{33} or an improvement in their standard errors. It is concluded that the 24 hour data is not yet accurate enough or sufficiently widespread in longitude to allow a good determination of these other resonant harmonics through 4th order.

TABLE 1

LONG TERM ACCELERATION DATA AND LONGITUDE HARMONICS FROM 24 HOUR SATELLITES

Satellite	Arc	a_* (E.R.)	i (Deg's)	λ (Deg's)	$\ddot{\lambda}$ (10^{-5} Rad./Day^2)	$\sigma(\ddot{\lambda})$ (10^{-5} Rad./Day^2)	C_{22} (10^{-6})	S_{22} (10^{-6})	C_{33} (10^{-6})	S_{33} (10^{-6})	S 10^{-7} Rad./Day^2
Syncom 2	1	6.611	33.02	-55.22	-2.253	0.0500					
Syncom 2	2	6.612	32.83	-60.94	-2.291	0.0572	1.56 ± 0.02	-0.93 ± 0.02	0.044 ± 0.015	0.165 ± 0.015	4.54
Syncom 2	4	6.620	32.58	-140.00	2.138	0.0842					
Syncom 2	5A	6.617	32.40	161.00	-0.199	0.0661	$\bar{C}_{22}$	$\overline{S_{22}}$	$\bar{C}_{33}$	S_{33}	
Syncom 2	5-18	6.618	32.15	104.5	-2.278	0.0656	(10^{-6})	(10^{-6})	(10^{-6})	(10^{-6})	
Syncom 3	6	6.611	0.11	178.71	1.707	0.0591	2.42 ± 0.03	-1.44 ± 0.03	0.32 ± 0.11	1.18 ± 0.11	
Syncom 3	7	6.612	0.27	176.80	1.550	0.175	(Recent data from lower altitude satellites)				
Syncom 2	8	6.611	31.87	66.12	0.950	0.0616	2.49	-1.40	0.41	1.36 ①	
Early Bird	9	6.611	0.20	-28.70	-1.441	0.0500	2.45	-1.52	0.58	1.62 ②	
Syncom 3	10	6.611	0.00	172.75	1.072	0.0934					
Syncom 2	11-1	6.61	31.4	69.0	0.676	0.0500	① From camera data-Private communication				
Syncom 2	11-2	6.61	31.4	77.6	-0.152	0.0500	(Due to W. M. Kaula) From W. M. Kaula (1966)				
Early Bird	12-1	6.611	0.43	28.61	-1.430	0.0500	② From Doppler Data-Private communication				
Early Bird	12-2	6.612	0.55	31.24	-1.597	0.0531	(Due to R. J. Anderle) From W. M. Kaula (1966)				
Early Bird	12-3	6.61	0.6	35.91	-1.901	0.0500					
Early Bird	12-4	6.61	0.74	36.20	-1.961	0.0500					
Syncom 3	13-1	6.61	0.53	169.1	0.831	0.0500	S = Standard error of least squares fit				
Syncom 3	13-2	6.61	0.53	168.29	0.699	0.0500					

SENSITIVITY AND DISCRIMINATION OF LONG TERM HARMONIC EFFECTS ON NON-SYNCHRONOUS RESONANT ORBITS

The fact is that only a limited number of harmonics are in long term resonance on a given commensurate orbit [i.e., Eq. (65c)]. The possibility thus arises of using the long term evolution of these orbits to sense and discriminate these particular harmonics only.

On a general orbit, and considering the periodic perturbations of all the satellite's elements, an infinity of longitude harmonics are active. In any gravitational analysis of limited satellite tracking data, a major problem is the discrimination of one harmonic effect from another. The use of resonant orbits in satellite geodesy helps to solve this problem by reducing the set of gravity harmonics which are theoretically active. In addition, resonance phenomena, characterized by accentuated and persistant effects, act as natural amplifiers of inherently very weak harmonic forces. Resonance increases many fold the sensitivity of these orbits to the weak perturbations, and thus enables the analyst to discriminate readily these weak forces from a background of much stronger (but not resonant) perturbation forces and observation noise. To judge the required sensitivity of the long term effects, we can use the accurate tracking period of the equator crossings of "early bird". This has shown [8] that we may expect to achieve resolution of the geographic acceleration of a near constant ground track to the order of 0.01×10^{-5} rad./sidereal day^2 in about 30 synodic days of nodal tracking on the lower-altitude sub-synchronous resonant satellites. With complete reduction of the relatively strong sun-moon effects of bi-weekly, monthly and longer periods, we should be able to reduce this resolution further. Table 2 gives a list of commensurate orbits of less than 24 hours and up to three sidereal days resonance period to illustrate the sensitivity and harmonic discrimination problem with respect to the likely long term effects. It is noted that for each resonant orbit except the 12 hour, the second relevant harmonic [from (65c)] is two orders of n higher than the first. Since the strength of each potential term falls off as a_*^{-n}, we can expect in general that the leading (lowest n) harmonic will also be dominant for that resonant orbit. However, the long term effects are strongly dependent on inclination, with a different dependence for each harmonic. Thus it is necessary to consider a range of inclinations to determine both the sensitivity and discriminating powers of the resonant orbit with respect to the relevant harmonics. The evaluation in Table 2 is intended to be indicative of the likely order of magnitude of the resonant accelerations. It is based on the curve in Figure 8 which, in turn, is based on recent satellite-gravity results. The form of this "planning curve" was suggested by W.M. Kaula [14], and appears to be a fair fit for the relevant resonant harmonics H_{nm} $(n=m, m+1)$ through $H_{15,15}$ as far as is known to date. The normalized harmonics $\overline{J}_{nm}$ are defined from the non-normalized J_{nm} by:

$$\overline{J}_{nm} = +J_{nm}\left[\frac{2(2n+1)(n-m)!}{(n+m)!}\right]^{-1/2}$$

TABLE 2

LONG TERM GEOGRAPHIC ACCELERATIONS ON LONGITUDE GRAVITY RESONANT ORBITS(1)

Synodic Period: n' (Sid. Days)	Orbit(2) Period: 24 n'/r (Sid. Hours)	Dominant Resonant Harmonic (D, H_{nm})	Subdominant Resonant Harmonic (SD, H_{nm})	Other Resonant Harmonics (H_{nm})	$\ddot{\lambda}_{max}$ (D, H_{nm}) at i = 30° (10^{-5} Rad./Sid. Day^2)	$\ddot{\lambda}_{max}$ (SD, H_{nm}) at i = 30° (10^{-5} Rad./S.D.^{2})	$\ddot{\lambda}_M$ (D) at i = 60° (10^{-5} R./S.D.^{2})	$\ddot{\lambda}_M$ (SD) at i = 60° (10^{-5} R./S.D.^{2})	$\ddot{\lambda}_M$ (D) at i = 90° (10^{-5} R./S.D.^{2})	$\ddot{\lambda}_M$ (SD) at i = 90° (10^{-5} R./S.D.^{2})	Orbits Per Synodic Period: r	Approximate(3) Semimajor Axis a_* (E.R.)
1	12.00	H_{32}	H_{44}	$H_{52}, H_{64}\cdots$	0.3481	0.0476 ✓	0.1516 ✓	0.0922 ✓	-0.2335 ✓	0.0546 ✓	2	4.164
1	8.00	H_{33}	H_{53}	$H_{66}, H_{73}\cdots$	0.4502	0.0336 ✓	1.086	-0.0040	0.9650	-0.0143 ✓	3	3.178
1	6.00	H_{54}	H_{74}	$H_{88}, H_{94}\cdots$	0.0545 ✓	0.0075 ✓	0.1025	-0.0046	-0.0702	0.0049	4	2.623
1	4.80	H_{55}	H_{75}	$H_{95}, H_{10,10}\cdots$	0.0340 ✓	0.0137 ✓	0.2462	0.0115 ✓	0.2918	-0.0144 ✓	5	2.261
1	4.00	H_{76}	H_{96}	$H_{11,6}, H_{12,12}\cdots$	0.0122 ✓	0.0056 ✓	0.0753	-0.0017	-0.0412 ✓	0.0062 ✓	6	2.002
1	3.43	H_{77}	H_{97}	$H_{11,7}, H_{13,7}\cdots$	0.0054 ✓	0.0060 ✓	0.1170 ✓	0.0238 ✓	0.1848	-0.0185 ✓	7	1.806
1	3.00	H_{98}	$H_{11,8}$	$H_{13,8}, H_{15,8}\cdots$	0.0039	0.0040	0.0752	0.0048	-0.0392 ✓	0.0100 ✓	8	1.653
1	2.67	H_{99}	$H_{11,9}$	$H_{13,9}, H_{15,9}\cdots$	0.0014	0.0032	0.0900 ✓	0.0453 ✓	0.1897 ✓	-0.0309 ✓	9	1.528
1	2.40	$H_{11,10}$	$H_{13,10}$	$H_{15,10}, H_{17,10}\cdots$	0.0017	0.0032	0.0992 ✓	0.0232 ✓	-0.0536 ✓	0.0202 ✓	10	1.424
1	2.18	$H_{11,11}$	$H_{13,11}$	$H_{15,11}, H_{17,11}\cdots$	0.0005	0.0021	0.0990 ✓	0.0962 ✓	0.2782 ✓	-0.0657 ✓	11	1.336
1	2.00	$H_{13,12}$	$H_{15,12}$	$H_{17,12}, H_{19,12}\cdots$	0.0009	0.0028	0.1661 ✓	0.0833 ✓	-0.0980 ✓	0.0503 ✓	12	1.261
1	1.846	$H_{13,13}$	$H_{15,13}$	$H_{17,13}\cdots$	0.0002	0.0016	0.1445 ✓	0.2358 ✓	0.5414 ✓	-0.1720 ✓	13	1.196
1	1.714	$H_{15,14}$	$H_{17,14}$	$H_{19,14}\cdots$	0.0006	0.0029	0.3422 ✓	0.3009 ✓	-0.2277 ✓	0.1509 ✓	14	1.138
1	1.600	$H_{15,15}$	$H_{17,15}$	$H_{19,15}\cdots$	0.0002	0.0014	0.2667 ✓	0.6662 ✓	1.332 ✓	-0.5410 ✓	15	1.087
1	1.500	$H_{17,16}$	$H_{19,16}$	$H_{21,16}\cdots$	0.0005	0.0004	0.8447 ✓	0.1493 ✓	-0.6495 ✓	0.0686 ✓	16	1.041
2	16.00	H_{43}	H_{63}	$H_{66}, H_{83}\cdots$	0.0343	0.0003	0.0000	-0.0002	-0.0269	0.0003	3	5.044
2	9.60	H_{65}	H_{85}	$H_{10,5}, H_{10,10}\cdots$	0.0035	0.0002	0.0037	-0.0002	-0.0050	0.0002	5	3.588
2	6.86	H_{87}	$H_{10,7}$	$H_{12,7}, H_{14,7}\cdots$	0.0004	0.0001	0.0018	-0.0001	-0.0016	0.0001	7	2.867
2	5.33	$H_{10,9}$	$H_{12,9}$	$H_{14,9}, H_{16,9}\cdots$	0.0001	0.0000	0.0009	-0.0000	-0.0008	0.0001	9	2.425
3	18.00	H_{54}	H_{74}	$H_{88}, H_{94}\cdots$	0.0039	0.0000	-0.0013	-0.0000	-0.0027	0.0000	4	5.456
3	14.40	H_{55}	H_{75}	$H_{95}, H_{10,10}\cdots$	0.0061	0.0002	0.0095	-0.0001	0.0037	0.0001	5	4.702
3	10.29	H_{77}	H_{97}	$H_{11,7}\cdots$	0.0003	0.0000	0.0012	-0.0000	0.0007	0.0000	7	

NOTES:

(1) Based on maximum effects for a given inclination, from Eq. (67) and the $\overline{J_{nm}}$ curve of Figure 8.

(2) Nodal period.

(3) Assumes the nodal period is equal to the Keplerian period.

✓: Checked effects are reasonably strong but use of such single orbits probably will not permit unambiguous discrimination of these harmonics.

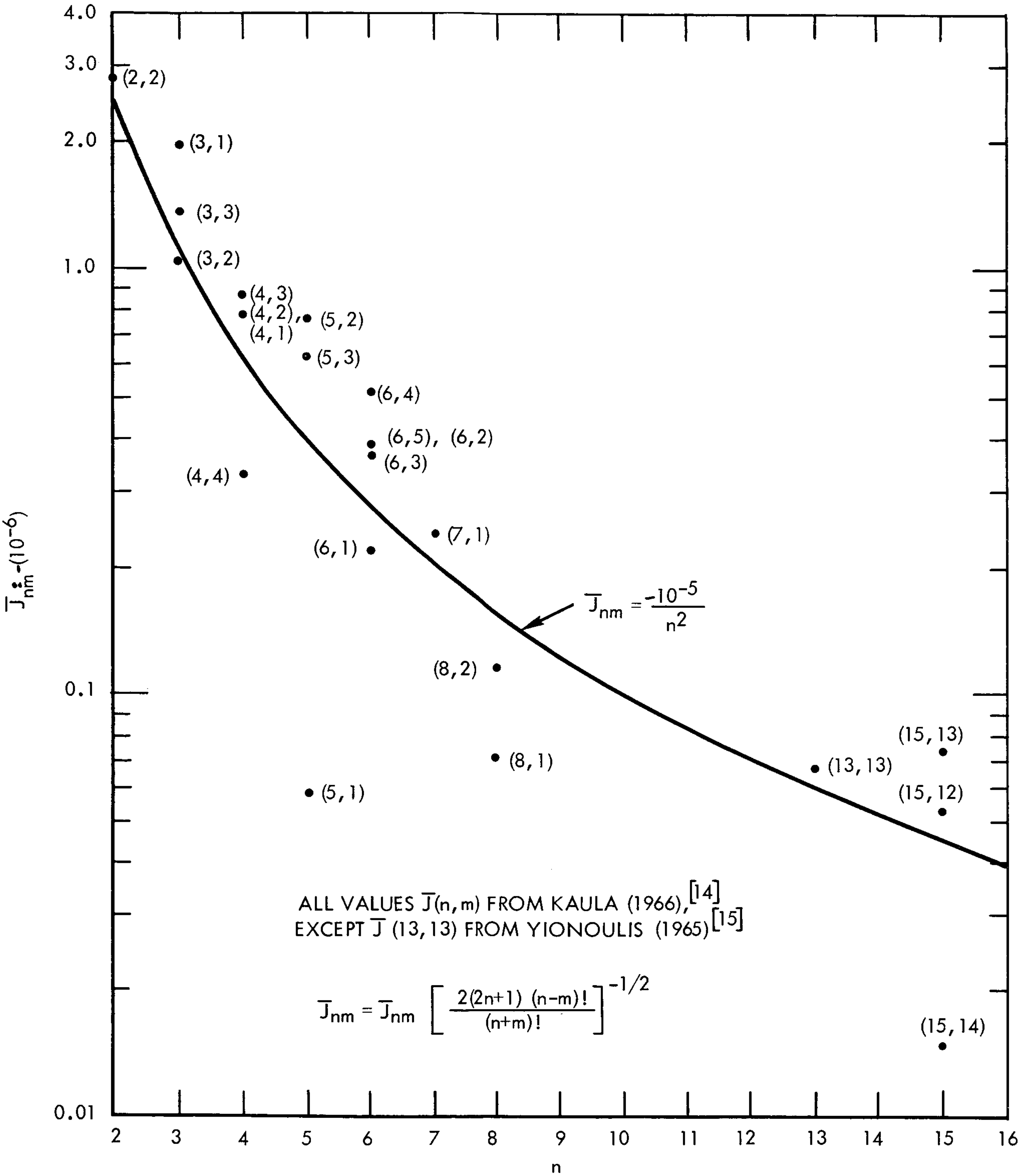

Figure 8. $\bar{J}_{nm}$ as a Function of n From Recent Satellite Results.

As stated previously it can be shown [16] that the normalized harmonics all have a mean square amplitude of unity over the surface of the geoid (actually a sphere of unit radius) if $\overline{J_{NM}} \equiv 1$. The underlined values of $\ddot{\lambda}_{MAX}$ in Table 2 indicate those resonant orbits where both sensitivity (to the 0.01×10^{-5} rad./sid.day^2 level in 30 synodic days) and discrimination of a single harmonic effect appear to be good. Discrimination is considered good if the dominant (underlined) effect is likely to be an order of magnitude greater than the subdominant. It can be seen from Table 2 that only for the 8, 12 and 16 hour satellites, will 30° inclined (or maximum payload) orbits be sufficiently sensitive and discriminating to harmonic effects in the above sense. (the 14.4 hour orbit may be marginal). Admittedly, these two criteria are somewhat arbitrary. One would like to use the already strongly discriminating properties of the resonant orbits to the maximum extent possible to be able to define unambiguously as many relevant harmonics as possible, with a minimum number of observations. It is noted that for all the orbits in Table 2 except the 12 hour one an infinite series of subdominant harmonics have the same m or longitude frequency as the dominant harmonic. Thus, investigating only a single resonant orbit (of constant inclination and major axis) will never allow an unambiguous discrimination of these harmonics no matter how wide a longitude survey of this orbit is made (by natural libration or onboard propulsion longitude-shift manuevers). The technical reason is that there is no longitude discrimination between such m constant harmonics in equation (67) which governs the evolution of such orbits.

It has been proposed [17] that the gravitationally sensitive and discriminating properties of resonant orbits discussed here, be used to define specific longitude variations in the earth's gravity field. If only one satellite of medium (30°) inclination is available for such purposes, Table 2 shows that H_{33}, H_{32} and H_{43} may be detectable unambiguously, in a reasonably short time, by simple single nodal longitude observations on 8, 12 and 16 hour satellites respectively. It is a property of the latitude dependence of the potential function [in particular, $F^{1}_{nmp}(i)$ in (65a)] that for the $H_{nm=n}$ (dominant) resonances, the leading subdominant effect ($H_{n+2,m}$) can always be "tuned out" (made zero) at one or more orbit inclinations above 30° [3], [4]. Such higher inclination resonant orbits may be particularly useful, geodetically. In addition, the inclination functions $F^{1}_{nmp}(i)$ for $H_{n,m\neq n}$ increase in sensitivity to i as n increases. The effect is that "tuning out" of the subdominant to the dominant tesseral effect ($H_{n,m\neq n}$) is <u>also</u> possible at specific inclinations closer to 30° than the first "tuning out" inclination for the dominant tesseral. Thus, for example, a single geodetic 60° inclined satellite of 3, 4, 4.8, 6 and 8 hour periods probably could unambiguously define H_{98}, H_{76}, H_{55}, H_{54} and H_{33} respectively in a reasonable length of time. It appears from Table 2 [and a study of (65a)] that the strong resonances of higher order n (on orbits between 1.5 and 3 hours) demand carefully tuned inclinations of greater than 60° to allow for sufficient one-orbit discrimination of effects. Of course, if other data on the harmonics can be used reliably as supplementary information, non (or weekly) discriminating single orbit information may still be valuable. Thus it appears that all the one day resonances with periods greater than 1.6 hours (reasonably drag free), may be geodetically useful if $i > 45^\circ$. We conclude with the observation from Table 2 that no resonant orbit of 2 or more day synodic period seeds to have geodetically useful long term effects except the 16 and 14.40 hour satellites. These resonances are generally weaker than the one day ones, since they generally involve longer period, higher altitude orbits of high order H_{nm} effects.

LIBRATORY CHARACTERISTICS OF NEAR RESONANT ORBIT SATELLITES

It should be clear from (67) that if the inclination and semi-major axis undergo only small changes, the long term longitude drift of a nearly constant ground track orbit due to any relevant harmonic H_{nm} will be analogous to that of a circular pendulum. In fact many integrations of the complete equations of motion have confirmed this assumption for a wide variety of resonant and near resonant orbits [9], [11], [12]. As such, the drift as a function of time (more precisely the time as a function of drfit) can only be given exactly (and only for a single harmonic effect) in terms of an elliptic integral [9], [12]. However, we can easily find the drift rate regime and also determine the critical or equilibrium points of the pendulum drift directly from (67). Additional information about the libratory drift, easily found, will be the minimum oscillation period (about the stable equilibrium points) and the maximum drift rate possible for "capture" in the libratory regime due to any relevant harmonic.

We will take our standard pendulum equation for the drift due to H_{nm} in the form:

$$\ddot{\lambda} + K_{nm}^2 \sin m(\lambda - \lambda_e) = 0, \tag{68}$$

where, for convenience, λ refers to the ascending equator crossing longitude. λ_e is seen to be a position of stable equilibrium $(\lambda_{e,s})$ if $K_{nm}^2 > 0$, and unstable equilibrium $(\lambda_{e,u})$ if $K_{nm}^2 < 0$.

From (68) these positions are separated by: $\lambda_{e,s} - \lambda_{e,u} = \pm\pi/m$. (69)

Equation (67) can be put in the form of (68) by rewriting (67) as

$$\ddot{\lambda} = K_{nm}^2 \cos\left\{m(\lambda-\lambda_{nm}) + m(\lambda_{nm}-\lambda_e) + \pi/2\right\}, \tag{70}$$

where

$$K_{nm}^2 = 12\pi^2 J_{nm}\left(\frac{R_o}{a}\right)^n m\, F'_{nmp}(i)\,, \quad \text{Rad./sid. day}^2. \tag{71}$$

Now, since $J_{nm} \leq 0$, the positions of <u>unstable</u> equilibrium are given from (70) and (67) by writing:

$$m(\lambda_{nm} - \lambda_{e,u}) + \pi/2 = \delta'(n,m,2), \quad \text{for } F'_{nmp} > 0,$$

or, equivalently [using (63) with $\ell = 2$ and the fact that n-m is an integer or zero]:

$$\lambda_{e,u} = \lambda_{nm} - \frac{\pi}{2m}(n-m) \tag{72}$$

Similarly, those of <u>stable</u> equilibrium for $F'_{nmp} > 0$ are given from (72) and (69) by

$$\lambda_{e,s} = \frac{\pi}{2m}\left[2-(n-m)\right] + \lambda_{nm}. \tag{73}$$

It may be noted that for low inclination satellites ($i \lessdot 30°$), F'_{nmp} is always positive [3], and for these orbits (72) and (73) will be the equilibrium criteria. For example, for the 24 hour satellite the relevant H_{nm} are all those for which n-m is even [see (65)] or $n-m = 2K;\ K = 0,1,2,\cdots$. Thus, for the low inclination 24 hour satellite, the unstable equilibrium ascending equator crossing longitudes are at:

$$\lambda_{e,u} = \lambda_{nm} - \frac{\pi K}{m} , \tag{74}$$

and the stable equilibrium longitudes for low inclination 24 hour satellites are at:

$$\lambda_{e,s} = \lambda_{nm} + \left(\frac{\pi}{m}\right)(1-K) , \tag{75}$$

where : $$K = \frac{n-m}{2} = 0, 1, 2, \ldots \tag{75a}$$

The situation is just the reverse for inclinations (at medium and high inclinations for example) where $F'_{nmp} < 0$. In these cases the stable equilibrium longitudes are given by (72) and the unstable by (73). Thus:

$$\lambda_{e,s} = \lambda_{nm} - \frac{\pi}{2m}(n-m) , \tag{76}$$

$$\lambda_{e,u} = \lambda_{nm} + \frac{\pi}{2m}\left[2 - (n-m)\right] , \tag{77}$$

for resonant orbits where $F'_{nmp} < 0$.

Of course, since the unstable and stable positions are equally spaced around the globe and separated by $\frac{\pi}{m}$ radians, there will be a total of $2m$ of these around the equator. There will thus be m unstable longitudes and m stable longitudes in all for each relevant H_{nm} . The maximum libratory excursion of the ground track will thus be $360^\circ/m$.

Proceeding with the integration of the pendulum equation (68), the variables $(\dot{\lambda})^2$ and λ can be separated by means of the differential reduction:

$$\ddot{\lambda} = \frac{d(\dot{\lambda})^2}{2\dot{\lambda}\,dt} = \frac{d(\dot{\lambda})^2}{2\,d\lambda} . \tag{78}$$

First we rewrite (68) to apply to $\lambda_e = \lambda_{e,s}$:

$$\ddot{\lambda} + |K^2_{nm}| \sin m(\lambda - \lambda_{e,s}) = 0 . \tag{79}$$

(78) in (79) permits us to separate variables:

$$d(\dot{\lambda})^2 = -2|K^2_{nm}| \sin m(\lambda - \lambda_{e,s})\, d\lambda . \tag{80}$$

The integral of (80) is the energy, or first, integral of the pendulum equation:

$$(\dot{\lambda})^2 = C_0 + \frac{2|K^2_{nm}|}{m} \cos m(\lambda - \lambda_{e,s}) . \tag{81}$$

If the initial condition $\dot{\lambda} = \dot{\lambda}_0$, at $\lambda = \lambda_0$ is given, then (81) may be written as:

$$(\dot{\lambda})^2 = (\dot{\lambda}_0)^2 + \frac{2|K^2_{nm}|}{m} \left\{ \cos m(\lambda - \lambda_{e,s}) - \cos m(\lambda_0 - \lambda_{e,s}) \right\} . \tag{82}$$

From (82) we can easily find the maximum drift rate change possible in a resonant orbit due to each relevant harmonic. This will occur with nearly exact resonance $(\dot{\lambda}_0 \doteq 0)$ momentarily over a position of unstable equilibrium. Then the drift rate passing the stable point (π/m radians away from $\lambda_{e,u}$) will be the maximum possible libratory drift.

For $\dot{\lambda}_0 = 0, \lambda_0 = \lambda_{e,u} = \lambda_{e,s} - \pi/m$, (82) reduces to:

$$(\dot{\lambda})^2 = \frac{2|K_{nm}^2|}{m}\left\{\cos m(\lambda - \lambda_{e,s}) + 1\right\}.$$

The maximum libratory drift rate (at $\lambda = \lambda_{e,s}$) will then be:

$$(\dot{\lambda})_{MAX,nm} = 2\sqrt{\frac{|K_{nm}^2|}{m}} . \tag{83}$$

Since the maximum acceleration in the H_{nm} libratory regime, from (79) is $|\ddot{\lambda}_{MAX,nm}| = |K_{nm}^2|$; we can write (83) as:

$$(\dot{\lambda})_{MAX,nm} = 2\sqrt{\frac{|\ddot{\lambda}_{MAX,nm}|}{m}} . \tag{84}$$

Another parameter of interest with respect to the regime of the resonant satellite is the minimum libration period. This is the period of small "pendulum" oscillations of the ground track about the stable equilibrium points defined in (73) and (76). If $\lambda - \lambda_{e,s}$ is always sufficiently small (as in a simple pendulum oscillation), $\sin m(\lambda - \lambda_{e,s})$ can be replaced by $m(\lambda - \lambda_{e,s})$ and (79) becomes a simple harmonic equation

$$\ddot{\lambda} + |K_{nm}^2|\, m(\lambda - \lambda_{e,s}) = 0. \tag{85}$$

From (85) we can easily find the small oscillations or minimum period libration about the stable equilibrium point. This minimum period is:

$$T_{MIN, nm} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{m|K^2_{nm}|}} = \frac{2\pi}{\sqrt{m|\ddot{\lambda}_{nm, max.}|}} . \tag{86}$$

Table 3 below gives these 3 libratory characteristics $\ddot{\lambda}_{MAX}$, $(\dot{\lambda})_{MAX}$ and $T_{MIN.}$ for the most geodetically promising (underlined) orbits in Table 2. The $\overline{J}_{nm}$ coefficients in this evaluation are taken from the "average" curve of Figure 8 and not from any specific set of coefficients. A comparative case, from actual data on 24 hour satellites, is provided at the bottom of Table 3.

As mensioned previously, the complete analytic treatment of the drift regime due to a single resonant harmonic (for a circular orbit) can be carried through by solving (82) in terms of elliptic integrals. This will not be done here. The form of the general solution has been shown previously, [9], as has the specific solution for the long period of libration of one day resonant orbits [3], [18]. As in the circular pendulum, the actual libration period is strongly dependent on the librational amplitude. In addition, Gideon et al. (1966) [18] has also obtained complete analytic solutions for the one day librational resonances of an eccentric orbit due to single harmonics.

We note only that since the basic averaged equation of librational motion, (67), is nonlinear, the complete solution of the quasi-libration due to the total, infinite set of resonant harmonics, cannot be a simple superposition of the individual harmonic solutions.

TABLE 3

CHARACTERISTICS OF GEODETICALLY PROMISING RESONANT CIRCULAR ORBITS

Orbit Period (SID. Hours)	Approximate Semimajor Axis (Earth Radii)	Clearly Dominant Resonant Harmonic (H_{nm})	Inclination (Degrees)	$\ddot{\lambda}_{max}$ (10^{-5} Rad./Day^2)	$\dot{\lambda}_{max}$ (Deg./Day)	T_{min} (Years)	$T_{min/4}$ (Days)
3.00	1.653	H_{98}	60	0.0752	0.035	7.01	640
4.00	2.002	H_{76}	60	0.0753	0.041	3.11	740
4.80	2.261	H_{55}	60	0.2462	0.080	4.90	447
6.00	2.623	H_{54}	60	0.1025	0.058	8.43	775
8.00	3.178	H_{33}	60	1.086	0.22	3.02	275
8.00	3.178	H_{33}	30	0.4502	0.14	4.69	428
12.00	4.164	H_{32}	30	0.3481	0.15	6.52	595
14.40	4.702	H_{55}	30	0.0060	0.0125	31.4	2860
14.40	4.702	H_{55}	60	0.0096	0.0158	24.8	2270
16.00	5.044	H_{43}	30	0.0342	0.038	17.0	1550
24.00*	6.61	H_{22}	0	2.92	0.438	2.25	205

*From actual data[8].

In fact there is no second integral to (67) which is known in a simple closed form. However the first, or energy integral to (67) can be simply written down analogously to (82). In terms of $\ell = 2$ or the longitudes of the ascending equator crossing, (67) may be rewritten with the differential reduction (78), as:

$$d(\dot{\lambda})^2 = 2\,d\lambda \sum_{n,m\ \text{relevant}} K_{nm}^2 \cos\left[m(\lambda-\lambda_{nm}) + \delta'(n,m,2)\right]. \tag{87}$$

The integral of (87) in terms of the initial conditions

$\dot{\lambda} = \dot{\lambda}_o$ at $\lambda = \lambda_o\ (t=0)$ is:

$$(\dot{\lambda})^2 = (\dot{\lambda}_o)^2 + 2\sum_{n,m\ \text{rel.}} \frac{K_{nm}^2}{2}\Big\{\sin\left[m(\lambda-\lambda_{nm}) + \delta'(n,m,2)\right] - \sin\left[m(\lambda_o-\lambda_{nm}) + \delta'(n,m,2)\right]\Big\}. \tag{88}$$

Then, formally, we can separate variables in (88) and find the time as a function of drift from:

$$t = \int_{\lambda_o}^{\lambda} d\lambda \Big\{(\dot{\lambda}_o)^2 + 2\sum_{n,m\ \text{rel.}} \frac{K_{nm}^2}{m}\Big[\sin\left(m[\lambda-\lambda_{nm}] + \delta'(n,m,2)\right) - \sin\left(m[\lambda_o-\lambda_{nm}] + \delta'(n,m,2)\right)\Big]\Big\}^{-1/2}. \tag{89}$$

Of course, in all of this simple derivation, we have assumed the elements of the orbit to be essentially constant (in particular K_{nm}) not only over the resonance period n' but over the long librational time t. While this will undoubtedly be a very good assumption for the semimajor axis (which in most cases will suffer small relative changes over long periods of time), it may not be so with regards to the inclination. Due to the gravitational action of the sun and moon combined with earth zonal gravity, the orbit plane of a satellite (otherwise unperturbed) will precess about a plane between the ecliptic and the equator causing a long period sinusoidal like change in the satellites inclination [19]. The period of this inclination change (with an amplitude of the order of about 10^{o}) can be as low as 10 years for close earth satellites. However, for the close earth satellites ($a_* \lesssim 4$, for example) zonal gravity equatorial precession predominates so that the total inclination change is small. Thus, it will probably be found that only the relatively weak far earth resonances can be seriously disturbed by these plane precession effects.

DISCUSSION

First, the use of the term "resonance" and "resonant orbit" in this report should be made clear. In mechanical systems resonant phenomena occur when the periodicity of applied loads are close to the natural (unforced) periodicities of the system. Resonance in this sense is characterized by greatly amplified vibrations of the system. Viewed internally, the applied loads are always in step, or in the same direction as the internal parts upon which they act and so continually increase the vibrational energy of those parts. But viewed externally, the loads are purely sinusoidal and so the mean translational energy of the center of mass of the system is unchanged. In the case of the satellite whose orbit yields a constant ground track, we have seen that, with respect to certain harmonics, the applied loads are not purely sinusoidal but contain a bias component which drives the long period libration. In the case of the constant ground track orbit, what we have called resonance implies a change in the translational or orbit energy of the satellite. The vibrational, or internal energy of the orbit can perhaps best be characterized by the eccentricity. In an eccentric orbit the satellite can be imagined as vibrating continually about its mean anomaly and semimajor axis, or mean radius. In this report we have chosen to ignore this vibrational aspect of the orbit by orbit averaging the disturbing force from the outset. We do this because we are here only interested in long term orbit energy changes. In terms of the mechanical resonance analogy, the full effect would be equivalent to a mass-spring assembly on wheels vibrating and translating under the action of a biased, sinusoidally varying external force. We have only concerned ourselves with the steady, translational, part of the complete motion. In celestial mechanics these changes are called secular, or effects which increase in time without apparent limit. In the case of commensurate orbits, the bias of the disturbing force will <u>also</u> change sinusoidally over a long period of time. This is a phenomenon which we call libration, having as its proper analogy the circular pendulum.

Resonant orbits as they are used here are synonomous with commensurate or constant ground track orbits; those whose rates are rational with respect to the earth's rotation rate. In addition, of course, we have specified the orbits to be near circular. The "resonance" phenomenon that we have, by our orbit averaging method, limited to ourselves to, might better be called secular or librational-circular pendulum effects. (In particular we have only really dealt with the secular changes in the semimajor axis). But the "in step action" of the longitude harmonics on these commensurate orbits give rise also to true vibrational resonances (in radius and anomaly) following the mechanical analogy [13], [20], [21]. Therefore, though the effect we analyse for should not commonly be thought of as a resonance effect, it does arise most strongly for those orbits which do show true "resonance" phenomenon and thus these orbits may be justifiably called resonant orbits. But when we speak of a "resonant gravity harmonic" in this paper, we use the word resonant only in the sense of being capable of producing the <u>amplified</u> changes of <u>libration</u> which are also properties of the resonant orbits.

It may be pointed out that the true vibrational resonances in a commensurate orbit imply a buildup, or amplification, of the orbit eccentricity. This amplification would be without bound if true commensurability could be maintained. Of course it cannot, since the resonant orbit generally librates. In fact, as might be inferred from its dependence on the direction of the disturbing force, the eccentricity itself in a resonant orbit appears to go through the same long period libration that the semimajor axis does [18]. However, these true "resonances of eccentricity" may not be as useful geodetically since all the longitude harmonics H_{nm} appear to contribute to them.

Finally, we would point out that the pendulum like changes of the semimajor axis [describable from a seaprated solution of the coupled equations (66) and (67), see Ref. [9] for example] have a periodicity of the order of years only for near commensurate conditions. These are the librating orbits first discussed in Ref. 10 for the 24 hour case. In cases where the ground track moves at a considerable rate, the basic circular pendulum equations [(66) and (67)] with small modification (along the lines outlined in Refs. 3 and 18) may still serve as the model of the evolution. In these cases of fast global circulation of the ground track, evaluation of (66) and (67) shows that the crossing rate and semimajor axis oscilate with a more rapid frequency but much diminished amplitude as compared to a libration case. For example, in a gast circulation case the period is typically of the order of days and the amplitude may be of the order of hundreds of meters (for the semimajor axis change). In full libration cases well away from the equilibrium positions, the amplitudes of the semimajor axis change may be of the order of tens of kilometers [9]. The fast world circulation regime (far from resonance) has been called dynamic resonance by Blitzer [21]. But by the discussion here it is actually a circulating pendulum and not a true resonance phenomenon. The reduction in amplitude of the pendulum effect in a fast circulation regime is due to the rapidity with which the orbit averaged bias force averages out over a global circulation.

But in spite of the small amplitudes, "circulating pendulum resonance" has been used to good effect in discriminating a number of very high order coefficients in the earth's field from a dense global tracking of the Navy's Doppler-Transity Satellites [15].

The limitations of the model used here to derive the key orbit evolution equations (66) and (67) should be re-emphasized. Though strictly speaking these equations apply only to circular orbits of exact resonant period, they have been shown, by numerous examples [8] and in a rigorous way [3], [18], to be applicable with only slight correction or modification to a much wider class of satellites. This class includes those with drift rates considerably exceeding the maximum permissible for libration as well as for noncircularorbits with eccentricities no higher than about 0.01. The equations, furthermore, give only the gross orbit averaged effects due to the earth's longitude gravity field. But over a period of the order of month's, it may be presumed that, well away from any strong influence of the earth's atmosphere or the moon's gravity, these equations will describe the dominating long term effect on the geographic configuration of all near resonant near circular orbits with the exception of inclination change due to the sun and moon.

The extension of these regime equations to the class of eccentric resonant orbits is indicated by Gedeon, et. al. [18].

SUMMARY AND CONCLUSIONS

The long term geographic evolution of circular orbit satellite ground tracks which are originally near stationary, has been found.

The equation governing the motion of the near stationary ground track is essentially that of a circular pendulum for each relevant earth longitude harmonic. The relevant harmonics for circular resonant orbits form a tenuous but infinite subset of the infinite set of longitude harmonics

For orbit periods very close to resonant, or rational with respect to the earth's rotation rate, the sinusoidal ground track librates with a maximum excursion of $360^{o}/m$ due to an individual relevant (resonant) earth longitude harmonic. The complete period of libration is the order of 2-10 years for the strongest resonances and depends strongly on the libration amplitude and inclination.

These libratory-like evolution equations can be extended to far-from-stationary conditions to cover the smaller, but detectable, drift oscillations of higher frequency, called "dynamic resonance" by Blitzer [21].

Circular resonant orbits of periods 3, 4, 4.8, 6, 8, 12, 14.4, and 16 hours appear to be particularly suited to discriminating unambiguously the longitude harmonics H_{98}, H_{76}, H_{55}, H_{54}, H_{33}, H_{32}, H_{55} and H_{43} respectively, in a reasonably short period of time. The 24 hour satellites have already provided the best discrimination of the two leading sectorial harmonics H_{22} and H_{33} as of 1966. It appears that a single controllable test satellite, "hopping" from low to high circular resonant orbits and "sliding" a number of times within each orbit could provide unique discrimination of most of the above harmonics in a year or two. The orbit determination for the resonant satellite used as a gravity probe would not have to be elaborate. It would have to insure that a reasonably circular orbit has been achieved. The inclination must also be determined to about 0.1^{o} accuracy. During free drift periods the longitudes of all the equator crossings is all the data that is necessary for a rapid harmonic determination.

REFERENCES

[1] Cook, A. H., "Resonant Orbits of Artificial Satellites and Longitude Terms in the Earth's External Gravitational Potential", Geophys. J., 4, 53-72, 1961.

[2] Wagner, C. A., "The Gravity Potential and Force Field of the Earth Through Fourth Order," NASA-TN-D3317, 1966.

[3] Allan, R. R., "Resonance Effects for Satellites with Nominally Constant Ground Track," R. A. E. Internal Document (Private Communication), Royal Aircraft Establishment; Farnborough, England, August 1965.

[4] Wagner, C. A., "The Drift of an Inclined Orbit 24 Hour Satellite in an Earth Gravity Field Through Fourth Order", NASA-TN-D3316, 1966.

[5] Brouwer, D., and Clemence, G M., Methods of Celestial Mechanics, Chapter 11; Academic Press, New York (1961).

[6] Izsak, I. G., "Tesseral Harmonics of the Geopotential and Corrections to Station Coordinates," J. Geophy. Res. 69, 2621-2630 (1964).

[7] Kaula, W. M., "Analysis of Gravitational and Geometric Aspects of Geodetic Utilization of Satellites," Geophys. Jour. R. A. S., 5, 104-133, (1961).

[8] Wagner, C. A., "The Earth's Longitude Gravity Field as Sensed by the Drift of Three Synchronous Satellites," NASA-TN-D3318, 1966.

[9] Wagner, C. A., "Determination of the Ellipticity of the Earth's Equator from Observations on the Drift of the Syncom 2 Satellite", NASA-TN-D2759, 1965.

[10] Blitzer, L., "Effect of Ellipticity of the Equator on 24 Hour Nearly Circular Satellite Orbits," Jour. of Geophys. Research, Vol. 67, No. 1, 329-335, 1962.

[11] Frick, R. H. and Garber, T. B. "Perturbations of a Synchronous Satellite Due to the Triaxiality of the Earth", Rand Corp. Memo. RM-2996-NASA, Jan. 1962, Santa Monica, California.

[12] Allan, R. R. "Perturbations of a Geostationary Satellite by the Longitude-Dependent Terms in the Earth's Gravitational Field", Planet. Space Sci. Vol. II, 1325-1334, 1963.

[13] Blitzer, L., "Resonances and Librations Associated with Tesseral Harmonics of the Geopotential," TRW Systems Memorandum No. 9883.10-5, 10 Sept. 1965; Redondo Beach, California.

[14] Kaula, W. M. "Variations of the Earth's Gravitational Field From Camera Tracking of Satellites", Publication No. 487, Inst. of Geophys. and Pl. Phys., U. C. L. A., Los Angeles, California, 1966.

[15] Yionoulis, S. M., "A Study of the Resonance Effects due to the Earth's Potential Function," J. Geophys. Res. 70, 5991-5996, 1965.

[16] Kaula, W. M. "Statistical and Harmonic Analysis of Gravity", J. Geophys. Res. 64 (12): 2401-2421, 1959.

[17] Greene, R. H. "Unsolicited Proposal (to NASA) For The Study of a Maneuverable Geodetic Satellite in a Variety of Resonant Orbits" (Private Communication, 1966).

[18] Gedeon, G. S., et. al., "Orbital Resonance Caused by Tesseral Harmonics", TRW Systems Internal Document (Private Communication); Redondo Beach, California, 1966.

[19] Allan, R. R., and Cook, G. E., "The Long Period Motion of the Plane of a Distant Circular Orbit," Proceedings of the Royal Society: 280; 97-109, 1964.

[20] Frick, R. H., "Perturbations of Equatorial Satellites due to Equatorial Ellipticity", Rand Corp. Memo. RM-3861-NASA; Santa Monica, California, 1963.

[21] Blitzer, L. "Synchronous and Resonant Satellite Orbits Associated with Equatorial Ellipticity", ARS Journal 1016-1021, July 1962.

APPENDIX A - THE RESONANCE SELECTOR FOR SUPERSYNCHRONOUS ORBITS

Introduction

It may be verified that, for the supersynchronous 48 hour satellite [since $r T_d = n'; n'/r = 2$, and $n' = 2, r = 1$ for the 48 hour satellite] the half orbit periodicity conditions (29a, b, c) suggest that the resonant harmonics are,

For n-m even:

$$m = 2, 4, 6, \ldots$$

For n-m odd:

$$m = 1, 3, 5, \ldots, \text{ or}$$

$H_{22}, H_{42}, H_{62} \ldots, H_{44}, H_{64}, H_{84} \ldots, H_{41}, H_{61}, H_{81} \ldots,$

$H_{43}, H_{63}, H_{83}, \ldots$

However, criteria (32), extended from Allen's work [3] and apparently equivalent to (29a, b, c) for $r \geq n'$, eliminates some of the leading harmonics in the above series. Thus, (32) for the 48 hour satellite is:

$$2m(\text{relevant}) = n(\text{relevant}) - 2p; \quad p = 0, 1, 2, \ldots,$$

which produces the following table of relevant harmonics:

m	n
1	4, 6, 8, ...
2	4, 6, 8, ...
3	6, 8, 10, ...
4	8, 10, 12, ...

For example, criteria (32) says that H_{22} is not resonant on the 48 hour satellite white criteria (29a, b, c) says that it is.

We will now attempt to verify by direct calculation, which of these two resonance criteria is correct for the 48 hour satellite and presumably for all supersynchronous orbits. Equation (B8) in (B6) gives the along track force on the 48 hour satellite $(q_2 = 1,\ q_1 = 2)$ due to H_{22} as:

$$
\begin{aligned}
\frac{F_{T,22}^{(\theta_s)}}{K'_{22}} = {} & \sin 2(\lambda_0 - \lambda_{22}) \Bigg\{ \cos i \left[1 - \frac{\sin^2 2\theta_s (1-\cos i)^2}{2\{1 - \sin^2\theta_s \sin^2 i\}} \right] \cos\theta_s \\
& - \frac{\sin^2 i}{2} \left[\sin^2 2\theta_s (\cos i - 1) \right] \left[\frac{1 - \sin^2\theta_s (1-\cos i)}{1 - \sin^2\theta_s \sin^2 i} \right] \cos\theta_s \\
& + \cos i \left[1 - \sin^2\theta_s (1 - \cos i) \right] \left[\frac{\sin 2\theta_s (\cos i - 1)}{1 - \sin^2\theta_s \sin^2 i} \right] \sin\theta_s \\
& + \frac{\sin^2 i \sin 2\theta_s}{2} \left[1 - \frac{\sin^2 2\theta_s (1-\cos i)^2}{2\{1 - \sin^2\theta_s \sin^2 i\}} \right] \sin\theta_s \Bigg\} \\
& + \cos 2(\lambda_0 - \lambda_{22}) \Bigg\{ \cos i \left[1 - \sin^2\theta_s (1-\cos i) \right] \left[\frac{\sin 2\theta_s (\cos i - 1)}{1 - \sin^2\theta_s \sin^2 i} \right] \cos\theta_s + \frac{\sin^2 i \sin 2\theta_s}{2} \left[1 - \frac{\sin^2 2\theta_s (1-\cos i)^2}{2\{1 - \sin^2\theta_s \sin^2 i\}} \right] \cos\theta_s \\
& - \cos i \left[1 - \frac{\sin^2 2\theta_s (1-\cos i)^2}{2\{1 - \sin^2\theta_s \sin^2 i\}} \right] \sin\theta_s + \frac{\sin^2 i}{2} \left[\sin^2 2\theta_s (\cos i - 1) \right] \left[\frac{1 - \sin^2\theta_s (1-\cos i)}{1 - \sin^2\theta_s \sin^2 i} \right] \sin\theta_s \Bigg\} . \qquad \text{(A1)}
\end{aligned}
$$

Note that $F_{T,22}$ in (A1) has been written in terms of $\theta_s = \frac{\theta_e}{2}$ for the 48 hour satellite. The orbit average of (A1) is to be taken over $0 \le \theta_s \le 2\pi$ since, for the 48 hour satellite, r = 1.

It can be seen from (A1) that the terms in $\sin 2(\lambda_0 - \lambda_{22})$ are all odd functions with respect to $\theta_s = \pi/2$ and $3\pi/2$. Similarly, the terms in $\cos 2(\lambda_0 - \lambda_{22})$ are all odd functions with respect to $\theta_s = \pi$. Thus the orbit average of $F_{T,22}^{(\theta_s)}$ for a 48 hour satellite is zero and H_{22} is not resonant on this orbit. Thus, even though $F_{T,22}$(48 hour) is periodic over a half orbit [from (29b)] it orbit averages to zero. This can only be the case if it also half-orbit averages to zero. It is interesting, but so far unexplained, that no half orbit periodicity in F_T has been found to orbit average to zero for subsynchronous resonant orbits. For resonant orbits higher than synchronous, however, we presume from the inference of the preceeding calculation that many such periodicities will half-orbit average to zero (in the manner of products of periodic orthogonal functions). In particular, the inference is strong that (32) is still the correct resonant harmonic selector for supersynchronous orbits.

APPENDIX B - ALONG TRACK FORCES FROM LONGITUDE GRAVITY HARMONICS THROUGH 4TH ORDER, FOR RESONANT ORBITS

It will be useful, in discussing the orbit averaged behavior of a resonant orbit under the action of a longitude gravity harmonic, to develope the along track force from this harmonic as a function of θ_e, the earth angle turned since the satellite was at its ascending node. Figure 2 illustrates the geometry applicable to the consideration here of a circular orbit satellite. From this figure we note the following spherical trigonometric relationships:

$$\begin{aligned} \sin\alpha &= \cos i/\cos\phi \\ \cos\alpha &= \tan\phi/\tan\theta_s \\ \sin\phi &= \sin i \sin\theta_s \\ \Delta L &= \tan^{-1}(\cos i \tan\theta_s) \end{aligned} \tag{B1}$$

The geographic longitude satellite at any time t after its nodal passage is:

$$\lambda = \lambda_0 + \Delta L - \theta_e \,. \tag{B2}$$

We note from (2) that for the resonant orbit specified by whole positive numers q_1 and q_2, (B2) becomes:

$$\lambda = \lambda_0 + \left[\tan^{-1}\left(\cos i \tan \frac{q_2\theta_e}{q_1}\right) - \frac{q_2\theta_e}{q_1}\right] + \theta_e\left(\frac{q_2}{q_1} - 1\right). \tag{B3}$$

We also note that for the resonant orbit θ_s can be replaced in (B1) by $q_2\theta_e/q_1$. The along track force on the circular orbit satellite, due to H_{nm} is (See Figure 5):

$$F_{T,nm} = F_{\lambda,nm}\sin\alpha + F_{\phi,nm}\cos\alpha . \tag{B4}$$

The longitude and latitude components of the earth gravity perturbation forces through 4th order are given in Section VIII of Ref. [B1]. It can be seen from (B1),(B4) and Ref. [B1] that we can write

$$F_{T,nm} = K'_{nm}\left\{\sin m(\lambda-\lambda_{nm})\,f_{1,nm}\left(i,\frac{q_2\theta_e}{q_1}\right) + \cos m(\lambda-\lambda_{nm})\,f_{2,nm}\left(i,\frac{q_2\theta_e}{q_1}\right)\right\}, \tag{B5}$$

where K'_{nm} is a constant over the circular orbit. Substituting (B3) into (B5), with $\delta\lambda\left(i,\frac{q_2\theta_e}{q_1}\right) = \tan^{-1}\left(\cos i \tan\frac{q_2\theta_e}{q_1}\right) - \frac{q_2\theta_e}{q_1}$, gives:

$$\begin{aligned} F_{T,nm}(\theta_e) = K'_{nm}\Bigg\{ & \sin m(\lambda_0-\lambda_{nm})\left[(g_1-g_3)\cos\left(\frac{q_2}{q_1}-1\right)\theta_e - (g_2+g_4)\sin\left(\frac{q_2}{q_1}-1\right)\theta_e\right] \\ & + \cos m(\lambda_0-\lambda_{nm})\left[(g_2+g_4)\cos\left(\frac{q_2}{q_1}-1\right)\theta_e + (g_1-g_3)\sin\left(\frac{q_2}{q_1}-1\right)\theta_e\right]\Bigg\}. \end{aligned} \tag{B6}$$

The g's are all functions of i, n, m, and $\frac{q_2\theta_e}{q_1}$ only:

$$\begin{aligned} g_{1,nm} &= f_{1,nm}\cos m\delta\lambda \\ g_{2,nm} &= f_{1,nm}\sin m\delta\lambda \\ g_{3,nm} &= f_{2,nm}\sin m\delta\lambda \\ g_{4,nm} &= f_{2,nm}\cos m\delta\lambda \end{aligned} \tag{B7}$$

For the 24 hour satellite $q_2 = q_1 = 1$ and the g's can be evaluated through 4th order (for $m \neq 0$) from equations (31), (36), (42), (45), (51), (54), (60) and (63) in Reference [B2]. On the right sides of these equations we replace θ by $\frac{q_2 \theta_e}{q_1}$ and find, (for H_{22}):

$$K'_{22} = 6 J_{22} \frac{\mu_e}{a^2} \left(R_o/a \right)^2$$

$$g_{1,22} = \cos i \left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}} \right]$$

$$g_{3,22} = \frac{\sin^2 i}{2}\left[\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]\left[\frac{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)}{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i}\right].$$

$$g_{2,22} = \cos i\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\frac{\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)}{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i}\right]$$

$$g_{4,22} = \frac{\sin^2 i \, \sin 2\left(\frac{q_2\theta_e}{q_1}\right)}{2}\left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}}\right]. \tag{B8}$$

For H_{31}, we find;

$$g_{1,31} - g_{3,31} = \frac{\cos i \left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\left[1-\cos i\right]\right\}\left\{5\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i - 1\right\}}{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i}$$

$$- \frac{\sin^2 i \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\left[15\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i - 11\right]}{4\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]}$$

$$g_{2,31} + g_{4,31} = \frac{\cos i\left[5\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i - 1\right]\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right.}$$

$$+ \frac{\sin^2 i \sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)\left[15\sin^2 i \sin^2\left(\frac{q_2\theta_e}{q_1}\right) - 11\right]\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \quad \text{(B9)}$$

For H_{32}, we find;

$$K'_{32} = 15 J_{32} \frac{\mu_e}{a^2}\left(R_0/a\right)^3$$

$$g_{1,32} - g_{3,32} = \sin 2i \sin\left(\frac{q_2\theta_e}{q_1}\right)\left\{1 - \frac{\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]}\right\}$$

$$- \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)\left[3\sin^2 i \sin^2\left(\frac{q_2\theta_e}{q_1}\right) - 1\right]\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]}{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i}$$

$$g_{2,32} + g_{4,32} = \frac{\sin 2i \sin\left(\frac{q_2\theta_e}{q_1}\right)\left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right\}\left\{\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right\}}{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i}$$

$$+ \cos\left(\frac{q_2\theta_e}{q_1}\right)\sin i\left[3\sin^2 i \sin^2\left(\frac{q_2\theta_e}{q_1}\right) - 1\right]\left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}}\right] \quad \text{(B10)}$$

For H_{33} we find;

$$K'_{33} = 45\, J_{33} \frac{\mu_e}{a^2} \left(R_0/a\right)^3$$

$$\begin{aligned}
g_{1,33} - g_{3,33} = {} & \cos i \left\{ 1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \right\} \left\{ 1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i) \right\} \\
& - \cos i \left\{ 1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i) \right\} \left\{ \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)^2}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \right\} \\
& - \frac{\sin^2 i \sin 2}{2}\left(\frac{q_2\theta_e}{q_1}\right) \left\{ \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]}{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i} \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right] \right\} \\
& - \sin^2 i \sin 2\left(\frac{q_2\theta_e}{q_1}\right) \left\{ 1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \right\} \left\{ \frac{\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)}{2} \right\} .
\end{aligned}$$

$$\begin{aligned}
g_{2,33} + g_{4,33} = {} & \frac{\cos i \left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]}{1 - \sin^2\left(q_2\theta_e/q_1\right)\sin^2 i} \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right] \\
& + \cos i \left\{ 1 - \frac{\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \right\} \left\{ \frac{\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)}{2} \right\} \\
& + \frac{\sin^2 i \sin 2}{2}\left(\frac{q_2\theta_e}{q_1}\right) \left\{ 1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2\left(q_2\theta_e/q_1\right)\sin^2 i\right]} \right\} \left\{ 1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i) \right\} \\
& - \frac{\sin^2 i \sin 2\left(\frac{q_2\theta_e}{q_1}\right)\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]}{4\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right] . \qquad \text{(B11)}
\end{aligned}$$

For H_{41}, we find;

$$K'_{41} = \frac{5}{2} J_{41} \frac{\mu_e}{a^2} \left(R_0/a\right)^4$$

$$\begin{aligned}
g_{1,41} - g_{3,41} = {} & \frac{\cos i \left[7\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i - 3\right]\left[\sin\left(\frac{q_2\theta_e}{q_1}\right)\sin i\right]}{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i} \left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right] \\
& - \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)\left[28\sin^4\left(\frac{q_2\theta_e}{q_1}\right)\sin^4 i - 27\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i + 3\right]}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right] .
\end{aligned}$$

$$\begin{aligned}
g_{2,41} + g_{4,41} = {} & \frac{\cos i \left[7\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i - 3\right]}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \left[\sin\left(\frac{q_2\theta_e}{q_1}\right)\sin i \sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right] \\
& + \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)\left[28\sin^4\left(\frac{q_2\theta_e}{q_1}\right)\sin^4 i - 27\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i + 3\right]}{1 - \sin^2\left(q_2\theta_e/q_1\right)\sin^2 i} \left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right] . \qquad \text{(B12)}
\end{aligned}$$

For H_{42} we find;

$$K'_{42} = 15\, J_{42} \frac{\mu_e}{a^2} \left(R_o/a \right)^4$$

$$g_{1,42} - g_{3,42} = \cos i \left[7 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 1 \right] \left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}} \right]$$

$$- \sin^2 i \sin 2\left(\frac{q_2\theta_e}{q_1}\right) \left[7 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 4 \right] \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]}{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i} .$$

$$g_{2,42} + g_{4,42} = \cos i \left[7 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 1 \right] \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]}{1 - \sin^2(q_2\theta_e/q_1)\sin^2 i}$$

$$+ \sin^2 i \sin 2\left(\frac{q_2\theta_e}{q_1}\right) \left[7 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 4 \right] \left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}} \right] . \tag{B13}$$

For H_{43}, we find;

$$K'_{43} = 315\, J_{43} \frac{\mu_e}{a^2} \left(R_o/a \right)^4$$

$$g_{1,43} - g_{3,43} = \frac{\sin 2i}{2} \sin\left(\frac{q_2\theta_e}{q_1}\right) \left\{ 1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]} \right\} \left\{ 1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i) \right\}$$

$$- \frac{\sin 2i \sin\left(\frac{q_2\theta_e}{q_1}\right)}{2} \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)^2\right]}{1 - \sin^2(q_2\theta_e/q_1)\sin^2 i}$$

$$- \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)}{3} \left[4 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 1 \right] \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]^2 \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]}{1 - \sin^2(q_2\theta_e/q_1)\sin^2 i}$$

$$- \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)}{6} \left[4 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 1 \right] \left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}} \right] \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right] .$$

$$g_{2,43} + g_{4,43} = \frac{\sin 2i \sin\left(\frac{q_2\theta_e}{q_1}\right)}{2} \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]^2 \left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)\right]}{1 - \sin^2(q_2\theta_e/q_1)\sin^2 i}$$

$$+ \frac{\sin 2i \sin\left(\frac{q_2\theta_e}{q_1}\right)}{4} \left\{ 1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left[1 - \sin^2(q_2\theta_e/q_1)\sin^2 i\right]} \right\} \left\{ \sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1) \right\} + \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)}{3} \Big[$$

$$4 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 1 \Big] \left[1 - \frac{\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1 - \sin^2(q_2\theta_e/q_1)\sin^2 i\right\}} \right] \left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i) \right] - \frac{\sin i \cos\left(\frac{q_2\theta_e}{q_1}\right)}{6} \Big[\tag{B14}$$

$$4 \sin^2\left(\frac{q_2\theta_e}{q_1}\right) \sin^2 i - 1 \Big] \frac{\left[1 - \sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i - 1)^2\right]}{1 - \sin^2(q_2\theta_e/q_1)\sin^2 i} .$$

For H_{44}, we find;

$$K'_{44} = 420 J_{44}\left(\frac{\mu_e}{a^2}\right)\left(\frac{R_0}{a}\right)^4 .$$

$$g_{1,44} - g_{3,44} = \cos i\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]\left\{1 - \frac{2\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]^2\left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i-1)\right]^2}{\left[1-\sin^2(q_2\theta_e/q_1)\sin^2 i\right]^2}\right\}$$

$$-\sin^2 i\,\sin 2\left(\frac{q_2\theta_e}{q_1}\right)\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\frac{\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i-1)}{1-\sin^2(q_2\theta_e/q_1)\sin^2 i}\right]\left[\frac{1-\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}}\right].$$

$$g_{2,44} + g_{4,44} = \cos i\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]2\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]\left[\frac{\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i-1)}{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i}\right]\left[\frac{1-\sin^2 2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)^2}{2\left\{1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right\}}\right]$$

$$+\frac{\sin^2 i\,\sin 2\left(\frac{q_2\theta_e}{q_1}\right)}{2}\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]\left\{1 - \frac{2\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)(1-\cos i)\right]^2}{\left[1-\sin^2\left(\frac{q_2\theta_e}{q_1}\right)\sin^2 i\right]^2}\left[\sin 2\left(\frac{q_2\theta_e}{q_1}\right)(\cos i-1)\right]^2\right\}.$$

(B15)

REFERENCES

[B1] Wagner, C. A. "The Gravity Potential and Force Field of the Earth through Fourth Order", NASA-TN-D-3317, 1966.

[B2] Wagner, C. A. "The Drift of an Inclined Orbit 24 Hour Satellite in an Earth Gravity Field Through Fourth Order", NASA-TN-D-3316, 1966.

APPENDIX C - THE THEORY OF THE 12 HOUR SATELLITE

We can derive the theory of the 12 hour (circular orbit) resonant satellite in two ways. Essentially they are both methods of evaluating the orbit averaged disturbing force. In the first we compute directly the average along track disturbing force from the associated Legendre harmonics as they appear in spherical coordinates (Appendix B). In the second, we compute the potential as a function of the Keplerian elements (see Section 5). The derivative of this with respect to the orbit argument also represents the along track force whose orbit average may be taken. Once the orbit averaged force is obtained, Equations (18) and (19) give the orbit evolution as long as conditions are reasonably close to resonance. The first method was used in Reference [C1] to derive the theory for the 24 hour satellite through 4th order in earth gravity. The results of the orbit-averaged computation appear in (34). These may be compared with the results of the second method of computation which is summarized in equations (66) and (67) for any resonant orbit. The two methods give identical results (as they should) for the 24 hour satellite.

It is noted that the straightforward method, while cumbersome to carry out, is capable of generating the orbit averaged force for any harmonic-resonance condition through evaluation of the $\mathcal{G}$ equations in Appendix B to any order. However, extensive evaluation of these equations for synchronous and nonsynchronous resonant orbits convinces us that calculation of the orbit averaged force through (65) will yield correct results in all resonance cases. In particular, where the resonant harmonic selector, Equation (32), has been violated for many harmonic-resonance conditions at, below, and above synchronous altitudes, the straightforward orbit averaged force, as given through 4th order in Appendix B, has always been zero.

We show here, for example, the correctness of (65) for evaluating the H_{32} (12 hour) resonant effect. In Appendix D we show by numerical integration of a partical trajectory that (67) is correct for a specific configuration of a 16 hour orbit $(r=3, n'=2)$. This latter case is important because it represents a check of the extension here of the formulae [similar to (66) and (67)] given by Allan [3] for the $n'=1$ day resonances.

Before beginning the evaluation of the leading resonant harmonic effect, we may note that in the case of the 12 hour satellite, <u>all</u> the nonresonant harmonics appear to be selected by the "antiperiodic" criteria (30) and (31). For example, the strictly nonresonant harmonics for the 12 hour orbit $(r=2, n'=1)$ are from (30) and (31): H_{nm}, where

$$m = 4, 8, 12, \ldots \text{ for } n\text{-}m \text{ odd,}$$
$$m = 2, 6, 10, \ldots \text{ for } n\text{-}m \text{ even}$$
and
$$m = 1, 3, 5, 7$$

It is seen, then, that from these strict negative criteria, the harmonics which <u>may</u> have long term resonance effects on the 12 hour satellite are those H_{nm} where

$$m = 4, 8, 12, \ldots \text{ for } n\text{-}m \text{ even,}$$
and
$$m = 2, 6, 10, \ldots \text{ for } n\text{-}m \text{ odd.}$$

Indeed, such harmonics satisfy the periodicity selector (29) which, we have seen, is strictly equivalent to the more general selector (32) or (65c), for $r>n'$ (subsynchronous resonant orbits).

Thus, the lowest order (n) resonant harmonic on the 12 hour orbit is H_{32}. It is presumed to be dominant both because of the a_*^{-n} potential decline with distance [See Equation (67)] and the apparent $\overline{J}_{nm}$ decline with increasing n. The normalized coefficients themselves give a correct interpretation of the mean realtive strengths of the harmonics at the surface of the earth. It should be noted, though, (see Table 2, for example) that the dominance of harmonic effects is strongly dependent on inclination as well. This dependence reflects the integrated effect of the satellite's motion in latitude.

The spherical harmonics themselves, of course, vary strongly in latitude. When we speak of general dominance here, we refer to the mean effect over all inclinations.

Proceeding with the evaluation of the long term effect due to H_{32} (12 hour), from Appendix B, Equations (B6) and (B10), we have (with $q_2 = 2, q_1 = 1$ from Eq. (4) for the 12 hour orbit):

$$
\begin{aligned}
F_{T,32}(12\ \text{HOUR}) = {} & K'_{32} \sin 2(\lambda_c - \lambda_{32}) \Bigg\{ \cos 2\theta_e \Bigg[\sin 2i \sin 2\theta_e \left\{ 1 - \frac{\sin^2 4\theta_e (1-\cos i)^2}{2(1-\sin^2 2\theta_e \sin^2 i)} \right\} \\
& - \sin i \cos 2\theta_e \frac{[3 \sin^2 i \sin^2 2\theta_e - 1]}{(1-\sin^2 2\theta_e \sin^2 i)} \left[1 - \sin^2 2\theta_e (1-\cos i) \right] \left\{ \sin 4\theta_e (\cos i - 1) \right\} \Bigg] \\
& - \sin 2\theta_e \Bigg[\sin 2i \sin 2\theta_e \frac{\{1-\sin^2 2\theta_e (1-\cos i)\}}{1-\sin^2 2\theta_e \sin^2 i} \{ \sin 4\theta_e (\cos i - 1) \} + \cos 2\theta_e \sin i (\\
& 3 \sin^2 i \sin^2 2\theta_e - 1) \left\{ 1 - \frac{\sin^2 4\theta_e (1-\cos i)^2}{2(1-\sin^2 2\theta_e \sin^2 i)} \right\} \Bigg] \Bigg\} + K'_{32} \cos 2(\lambda_c - \lambda_{32}) \Bigg\{ \sin 2\theta_e \Bigg[\\
& \sin 2i \sin 2\theta_e \left\{ 1 - \frac{\sin^2 4\theta_e (1-\cos i)^2}{2(1-\sin^2 2\theta_e \sin^2 i)} \right\} - \sin i \cos 2\theta_e \frac{(3 \sin^2 i \sin^2 2\theta_e - 1)}{1-\sin^2 2\theta_e \sin^2 i} \{ \\
& 1 - \sin^2 2\theta_e (1-\cos i) \} \{ \sin 4\theta_e (\cos i - 1) \} \Bigg] + \cos 2\theta_e \Bigg[\sin 2i \sin 2\theta_e \frac{\{1-\sin^2 2\theta_e (1-\cos i)\}}{1-\sin^2 2\theta_e \sin^2 i} \cdot \\
& \cdot \{ \sin 4\theta_e (\cos i - 1) \} + \cos 2\theta_e \sin i (3 \sin^2 i \sin^2 2\theta_e - 1) \{ \\
& 1 - \frac{\sin^2 4\theta_e (1-\cos i)^2}{2(1-\sin^2 2\theta_e \sin^2 i)} \} \Bigg] \Bigg\} . \qquad \text{(C1)}
\end{aligned}
$$

It can be seen that the factor of $\sin 2(\lambda_0-\lambda_{32})$ above is an "odd" function with respect to $\theta_e = \pi$. Thus, since the double-orbit average of (C1) [See Eq. (9)] is achieved in the interval $0 \le \theta_e \le 2\pi$ ($n' = 1$ sidereal day), the $\sin 2(\lambda_0-\lambda_{32})$ term will double orbit average to zero. Retaining only the relevant $\cos 2(\lambda_0-\lambda_{32})$ term of (C1):

$$F_{T,32}(12\text{ HOUR}) = K'_{32}\cos 2(\lambda_0-\lambda_{32})\Big\{\sin 2i\,\sin^2 2\theta_e\left[1-\frac{\sin^2 4\theta_e(1-\cos i)^2}{2(1-\sin^2 2\theta_e\sin^2 i)}\right]$$

$$-\frac{\sin^2 4\theta_e\sin i(\cos i-1)\left[3\sin^2 i\sin^2 2\theta_e-1\right]\left[1-\sin^2 2\theta_e(1-\cos i)\right]}{2(1-\sin^2 2\theta_e\sin^2 i)}$$

$$+\cos^2 2\theta_e\sin i(3\sin^2 i\sin^2 2\theta_e-1)\left[1-\frac{\sin^2 4\theta_e(1-\cos i)^2}{2(1-\sin^2 2\theta_e\sin^2 i)}\right]\Big\}. \qquad \text{(C2)}$$

Let:

$$A = \cos^2 2\theta_e\sin i(3\sin^2 i\sin^2 2\theta_e-1)+\sin 2i\sin^2 2\theta_e .$$

Then (C2) becomes:

$$F_{T,32}(12\text{ HOUR}) = K'_{32}\cos 2(\lambda_0-\lambda_{32})\Big\{A+\left[\frac{1}{2(1-\sin^2 2\theta_e\sin^2 i)}\right]\Big[-\sin^2 4\theta_e\sin^2 2\theta_e$$

$$\cdot\sin 2i(1-\cos i)^2-\sin^2 4\theta_e\sin i(\cos i-1)(3\sin^2 i\sin^2 2\theta_e-1)\{1-\sin^2 2\theta_e(1-\cos i)\}$$

$$+\sin^2 4\theta_e\sin 2i(\cos i-1)\{1-\sin^2 2\theta_e(1-\cos i)\}-\cos^2 2\theta_e\sin i(3\sin^2 i\sin^2 2\theta_e-1)\{$$

$$\sin^2 4\theta_e(1-\cos i)^2\}\Big]\Big\}$$

$$= K'_{32}\cos 2(\lambda_0-\lambda_{32})\Big\{A+\frac{(1-\cos i)\sin i\sin^2 4\theta_e}{2(1-\sin^2 2\theta_e\sin^2 i)}\Big[-2\sin^2 2\theta_e\cos i($$

$$(1-\cos i)+(3\sin^2 i\sin^2 2\theta_e-1)\{1-\sin^2 2\theta_e(1-\cos i)\}-2\cos i\{1-\sin^2 2\theta_e(1-\cos i)\}$$

$$-\cos^2 2\theta_e(3\sin^2 2\theta_e\sin^2 i-1)(1-\cos i)\Big]\Big\}. \qquad \text{(C3)}$$

Let B be the term in [ ] immediately above.

Expanding B, we have:

$$B = -2\sin^2 2\theta_e \cos i\,(1-\cos i) + 3\sin^2 i \sin^2 2\theta_e - 3\sin^2 i \sin^2 2\theta_e (1-\cos i)$$
$$-1 + \sin^2 2\theta_e (1-\cos i) - 2\cos i + 2\cos i \sin^2 2\theta_e (1-\cos i)$$
$$-3\cos^2 2\theta_e \sin^2 i \sin^2 2\theta_e (1-\cos i) + \cos^2 2\theta_e (1-\cos i).$$

Reducing B above:

$$B = 3\sin^2 i \sin^2 2\theta_e \Big\{1 - [\sin^2 2\theta_e (1-\cos i) + \cos^2 2\theta_e (1-\cos i)]\Big]$$
$$+ (1-\cos i)(\cos^2 2\theta_e + \sin^2 2\theta_e) - 1 - 2\cos i\Big\}.$$

Further reducing B:

$$B = 3\sin^2 i \sin^2 2\theta_e \cos i + (1-\cos i) - 1 - 2\cos i$$
$$= -3\cos i\,(1 - \sin^2 2\theta_e \sin^2 i).$$

Combining B (above) and A, (C3) becomes:

$$F_{T,32}(12\ \text{HOUR}) = K'_{32}\cos 2(\lambda_0 - \lambda_{32})\Big\{\cos^2 2\theta_e \sin i\,(3\sin^2 i \sin^2 2\theta_e - 1)$$
$$+ \sin 2i \sin^2 2\theta_e - \frac{3\cos i}{2}(1-\cos i)(\sin i \sin^2 4\theta_e)\Big\}. \tag{C4}$$

In the 12 hour orbit, $\theta_s = 2\theta_e$ [see Equation (1a)]. Thus,

$d\theta_s = 2d\theta_e$ and Equation (9) becomes:

(for the 12 hour resonant orbit, double orbit averaged):

$$\overline{F_T}(12\ \text{HOUR}) = \frac{1}{2\pi}\int_0^{2\pi} F_T(\theta_e)\,d\theta_e. \tag{C5}$$

We use the following integrals:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 2\theta_e \sin^2 2\theta_e\,d\theta_e = \frac{1}{8\pi}\int_0^{2\pi}\sin^2 4\theta_e\,d\theta_e = \frac{1}{16\pi}\Big|\theta_e - \frac{1}{4}\sin 4\theta_e \cos 4\theta_e\Big|_0^{2\pi} = \frac{1}{16\pi}(2\pi) = \frac{1}{8}.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 2\theta_e\,d\theta_e = \frac{1}{4\pi}\left|\theta_e + \frac{\sin 2\theta_e \cos 2\theta_e}{2}\right|_0^{2\pi} = \frac{1}{2}\,.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2 2\theta_e\,d\theta_e = \frac{1}{4\pi}\left|\theta_e - \frac{\sin 2\theta_e \cos 2\theta_e}{2}\right|_0^{2\pi} = \frac{1}{2}\,.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2 4\theta_e\,d\theta_e = \frac{1}{4\pi}\left|\theta_e - \frac{1}{4}\sin 4\theta_e \cos 4\theta_e\right|_0^{2\pi} = \frac{1}{2}\,.$$

With these integrals, (C4) in (C5) gives:

$$\begin{aligned}
F_{T,32}(12\text{ HOUR}) &= K'_{32}\cos 2(\lambda_0-\lambda_{32})\left[\frac{3\sin^3 i}{8} - \frac{\sin i}{2} + \sin i \cos i - \frac{3}{4}\cos i(1-\cos i)\sin i\right] \\
&= K'_{32}\cos 2(\lambda_0-\lambda_{32})\left[\frac{\sin i}{8}\left\{3\sin^2 i - 4 + 8\cos i - 6\cos i(1-\cos i)\right\}\right] \\
&= K'_{32}\cos 2(\lambda_0-\lambda_{32})\frac{\sin i}{8}\left[1 - 2\cos i - 3\cos^2 i\right].
\end{aligned}$$

(C6)

Equation (C6) in (19) gives the long term acceleration of the ascending equator crossings of the 12 hour satellite due to H_{32}. We note that $F = F'\frac{\mu}{a^2}$ and $\lambda = \lambda_0$ the unrestricted position of the crossing over many orbits, treated, effectively, as a continuous variable. Thus, this formulation gives:

$$\ddot{\lambda}_{12\text{ HOUR}}(\text{DUE TO } H_{32}\text{ ONLY}) = 24\pi^2\left[15\left(\frac{R_0}{a}\right)^3 J_{32}\left\{\frac{\sin i}{8}(1-2\cos i - 3\cos^2 i)\right\}\right]\cos 2(\lambda-\lambda_{32}),$$
$$\text{Rad./sid. day}^2.$$

(C7)

We now evaluate Equation (65) for the second formulation of this acceleration. Evaluating (65c) for n(relevant) = 3, m(relevant) = 2, $\frac{n'}{r}$ (12 hour) = 1/2:

$$\rho\,(H_{32},\ 12\ \text{hour resonance}) = \frac{3-1}{2} = 1.$$

Thus, (65a) gives the resonant inclination function for H_{32}(12 hour) as:

$$F'_{nmp}(i) = F'_{321}(i) = \frac{(5)!}{2^3 \cdot 1!\,(2)!} \sum_{k=0}^{1} (-1)^k \binom{4}{k}\binom{2}{1-k} \cos^{5-2k}(i/2)\,\sin^{1+2k}(i/2)$$

$$= \left(\frac{15}{2}\right)\left[2\cos^5(i/2)\sin(i/2) - 4\cos^3(i/2)\sin^3(i/2)\right].$$

But;

$$\sin(i/2) = \sqrt{1/2\,(1-\cos i)}, \quad \cos(i/2) = \sqrt{1/2\,(1+\cos i)}.$$

Thus:

$$F'_{321}(i) = \left(\frac{15}{2}\right)\left[\frac{2}{4}(1+\cos i)^2 \cdot \frac{1}{2}\sin i - \frac{4}{2}(1+\cos i)\cdot\frac{1}{2}(1-\cos i)\cdot\frac{1}{2}\sin i\right]$$

$$= \frac{15}{8}\sin i\left[(1+\cos i)\left\{1+\cos i - 2(1-\cos i)\right\}\right]$$

$$= \frac{-15\sin i}{8}\left[1 - 2\cos i - 3\cos^2 i\right]. \tag{C8}$$

From Equation (C8) in (67), the acceleration of the crossing longitude due to H_{32} in the 12 hour orbit is:

$$\ddot{\lambda} = 24\pi^2\left\{\frac{1}{a_*^3}\cdot J_{32}\cdot\frac{15}{8}\sin i\,(1-2\cos i-3\cos^2 i)\cos 2(\lambda-\lambda_{32})\right\},$$

rad./sid.day^2, (C9)

which is identical to the direct orbit averaging formulation, Equation (C7).

Similarly, from either the direct orbit averagining formulation, or from Equation (67), we have found that the effect due to H_{44} on the 12 hour satellite is:

$$\ddot{\lambda} \text{ (due to } H_{44} \text{ only)} = -24\pi^2 \left\{ \frac{1}{a_*^4} \cdot J_{44} \cdot \frac{105}{2} \left[\sin i (1 + \cos i) \right]^2 \sin 4(\lambda - \lambda_{44}) \right\},$$

$$\frac{\text{rad.}}{\text{sid.day}^2} \quad \text{(C10)}$$

Tables D2 and D5 give a comparison of the theoretical motion due to H_{32} and H_{44} from equations (C9) and (C10), with numerically integrated partical trajectories.

REFERENCES

[C1] Wagner, C. A. "The Drift of an Inclined-Orbit 24 Hour Satellite in an Earth Gravity Field Through Fourth Order", NASA-TN-D-3316, 1966.

APPENDIX D - ONE AND TWO DAY RESONANCE TESTS: 8, 12 AND 16 HOUR ORBITS

We can check the validity of the formulation of the orbit averaged librational equations of motion (67) by comparing them with a complete numerical solution of the equations of motion for a trajectory in the presence of the critical disturbing forces. Tables D1-D6, show the results of these numerical studies on near circular resonant orbits of 8, 12 and 16 hours.

The trajectories run about 16 sidereal days from near stationary ground track conditions. We hope to determine that even in this short time span, it will be possible to discriminate the relevant harmonic accelerations to reasonably high accuracy according to the simple theory presented here.

8 HOUR ORBIT

For the 8 hour (one day) resonant orbit the leading relevant harmonic is H_{33} (having the least n satisfying (32) for $r=3, n'=1$). The orbit averaged equation for the acceleration of the longitude of the ascending equator crossing due to H_{33} on an 8 hour orib [(67), with $\ell=2, p=1$] gives:

$$\ddot{\lambda} = -\frac{278.68}{a_*^3}\,\overline{J}_{33}\,\sin^2 i\,(1+\cos i)\,\sin 3(\lambda-\lambda_{33}),$$

rad./sid.day^2. (D1)

The normalized $\overline{J}_{nm}$ are given from the non-normalized [in (67)] by:

$$\overline{J}_{nm} = J_{nm}\left[\frac{2(2n+1)(n-m)!}{(n+m)!}\right]^{-1/2}. \qquad \text{(D2)}$$

Evaluating (D1) for: $\overline{J}_{33} = -1.1474 \times 10^{-6}$, a_*(8 hour orbit) $= 3.1781$ E.R., $i = 30^\circ$, $\lambda_{33} = 24.0^\circ$, $\lambda = 54.0^\circ$, the theoretical longitudinal acceleration in the neighborhood of $\lambda = 0^\circ$ should be:

$$\ddot{\lambda}_{TH.} = 0.4647 \times 10^{-5} \text{ rad./sid. day}^2. \tag{D3}$$

(see Table D1).

The following data is from a partical trajectory [generated by a modified Encke method [D1]] computed numerically, in the presence of perturbations due to H_{33} [with constants given above] H_{22} and H_{31} only. The specifications for H_{22} in the trajectory generator are: $\overline{J}_{22} = -1.9 \times 10^{-6}$, $\lambda_{22} = -21^\circ$. The specifications for H_{31} are: $J_{31} = -1.5 \times 10^{-6}$, $\lambda_{31} = 0^\circ$. From the theory, H_{31} and H_{22} should have no long term effect on the 8 hour satellite. By long term here, we mean over periods which are multiples of one sidereal day.

j	Time (Hrs.), T	Ascending Equator Crossing Longitude (Degrees East of Greenwich), λ	Semimajor Axis (Earth Radii)	Inclination (Degrees)	Eccentricity (10^{-6})
0	0.	54.0	3.1781	30.0	0.14
1	191.476	54.00290	3.1781	30.0	0.13
2	382.950	54.02279	3.1781	30.0	0.15

TABLE D1

Numerical Trajectory Data for an 8 Hour Earth Satellite Disturbed by H_{22}, H_{31} and H_{33} Only.

In the above trajectory, the longitude acceleration is given approximately as:

$$\ddot{\lambda}_{MEAS.} = \frac{1}{(\Delta T)^2} \left[\lambda_0 - 2\lambda_1 + \lambda_2 \right], \tag{D4}$$

where $\Delta T = T_{j+1} - T_j \doteq$ constant

Since $T_1 - T_0 \doteq T_2 - T_1 \doteq$ 8 sidereal days, the "measured" acceleration in the numerical trajectory is:

$$\ddot{\lambda}_{MEAS.} \doteq \frac{1}{64} \frac{\left[0.02279 - 0.00580\right]}{57.296} = 0.4633 \times 10^{-5} \qquad \frac{\text{rad}}{\text{sid.day.}^2} \tag{D5}$$

The discrepancy between the orbit averaged theory and the numerical trajectory data is only 0.0014 $\times 10^{-5}$ rad./sid.day^2 out of 0.4633$\times 10^{-5}$ rad/sid.day^2 or about 0.3%.

12 HOUR ORBIT

For the 12 hour (one day) resonant orbit, the leading relevant harmonic is H_{32} (having the least n satisfying (32) for $r = 2,\ n' = 1$). The orbit averaged equation for the acceleration of the longitude of the ascending equator crossing due to H_{32} on a 12 hour orbit [(67), with $\ell = 2,\ p = 1$] gives:

$$\ddot{\lambda} = \frac{151.702}{a_*^3} \overline{J}_{32} \sin i \left(1 - 2\cos i - 3\cos^2 i\right) \cos 2(\lambda - \lambda_{32}), \qquad \text{rad./sid.day}^2. \tag{D6}$$

We consider a partical trajectory with only H_{32} acting as a perturbation. The initial circular orbit has a period of 12 hours ($a_* = 4.16449$ E.R.). The inclination is 30° and the initial longitude of the ascending equator crossing is at 0°. The H_{32} constants are: $\overline{J}_{32} = -10^{-5}/9,\ \lambda_{32} = 0^\circ$.

From (D6) the long term acceleration should be:

$$\ddot{\lambda}_{TH.} = -0.3481 \times 10^{-5} \qquad \text{rad./sid.day}^2 \tag{D6}$$

(see Table D2)

The following ascending equator crossing data is from another numerically integrated particle trajectory, in the presence of perturbations due to H_{32} only (with constants given above).

Time (Hrs.)	Longitude (Degs.)	a_* (E.R.)	i (Deg's)	Eccentricity: e
0.	0.	4.1645	30.0	10^{-6}
191.476	-0.00660	4.1645	30.0	10^{-6}
382.953	-0.02598	4.1645	30.0	10^{-5}

TABLE D2

Numerical Trajectory Data for A 12 Hour Earth Satellite Disturbed By H_{32} Only.

In the above trajectory the longitude acceleration (over the 16 sidereal day period) is given approximately by (D4) or:

$$\ddot{\lambda}_{MEAS.} \doteq \frac{1}{64}\left[-0.02598 + 0.01320\right] \Big/ 57.296 = -0.3485 \times 10^{-5} \text{ rad./sid.day}^2 \tag{D7}$$

Once again the discrepancy between the orbit averaged theory and the numerical trajectory data is very small, amounting to only about 0.1%.

The simulated trajectory summarized in Table D3 has initial conditions precisely the same as in Table D2 except for the addition of earth zonal gravity and the attractions of the sun, moon and planets. The zonal gravity constants used are $J_{20} = 1.0823 \times 10^{-3}$, $J_{30} = -2.3 \times 10^{-6}$ and $J_{40} = -1.8 \times 10^{-6}$.

TIME (Hrs. from 1966.0)	LONG. (Degs.)	a_* (E.R.)	i (Degs.)	e
0.	0.	4.1645	30.00°	10^{-4}
191.434	0.14563	4.1645	30.02	10^{-4}
382.870	0.27110	4.1645	30.02	10^{-4}

TABLE D3 - Numerical Trajectory Data for A 12 Hour Earth Satellite Disturbed by H_{32}, Earth Zonal Gravity, the sun, moon and Planets

In the above trajectory, the longitude acceleration is given approximately as:

$$\ddot{\lambda}_{MEAS.} = \frac{1}{64}\left[0.27110 - 0.29126\right] / 57.296 = -0.549 \times 10^{-5},$$

$$\text{rad./sid.day}^2. \qquad \text{(D8)}$$

It is apparent from the 8 hour study that contributions to error in the simple theory from non resonant earth longitude gravity are negligible for these nearly stationary orbits. But from the previous two tables the error in the long term theory due to other gravitational effects is clearly not negligible over 16 sidereal days. In fact, other studies have shown ([D2], [D3]) that the sun and moon alone accounts for nearly all of this major discrepancy. But the presence of the strong nonresonant gravitational effects alone gives the particle trajectory a mean motion over only 16 days close to that of the fully perturbed motion. Therefore, it appears reasonable to try and evaluate the "sun and moon" effect independently of the full trajectory. Then, subtracting this from the full effect, we hope to get a reduced acceleration which can be assumed to be due to the relevant resonant gravity terms alone.

The simulated trajectories in Table D4 are a test of this error reduction idea. Initial conditions were the same as in Table D3 except no earth longitude gravity effects were included for the trajectory with bracketed data. The running time for both full and nonresonant trajectories was 14 days.

TIME (Hrs. from 1966.0)	LONGITUDE (Degrees)	a_* (E.R.)	i	e
0.	0.	4.1645	30.0^{o}	10^{-4}
(0.)	(0.)	(4.1645)	(30.0^{o})	(10^{-4})
167.505	0.12756	4.1645	30.0^{o}	10^{-4}
(167.505)	(0.13243)	(4.1645)	(30.0^{o})	(10^{-4})
335.011	0.24023	4.1645	30.0^{o}	10^{-4}
(335.010)	(0.25974)	(4.1645)	(30.0^{o})	(10^{-4})

TABLE D4: Numerical Trajectory Data in Two Sun and Moon Perturbed 12 Hour Orbits; Disturbed (unbracketed data) and Undisturbed (bracketed data) by H_{32}.

In the H_{32}-disturbed trajectory (unbracketed data), the measured acceleration is:

$$\ddot{\lambda}_{MEAS.}\text{(fully disturbed)} \doteq \frac{1}{49}\left[0.24023 - 0.25512\right]/57.296$$

$$= -0.5306 \times 10^{-5} \text{ rad./sid.day}^2 . \tag{D9}$$

In the H_{32}-undisturbed trajectory (bracketed data), the measured acceleration is:

$$\ddot{\lambda}_{MEAS.}\text{(partially disturbed)} \doteq \frac{1}{49}\left[0.25974 - 0.26486\right]/57.296$$

$$= -0.1830 \times 10^{-5} \text{ rad./sid.day}^2 . \tag{D10}$$

Equation (D10) subtracted from (D9) gives the reduced, or sun and moon corrected acceleration as:

$$\ddot{\lambda}_{MEAS.} \text{(corrected)} = -0.3476 \times 10^{-5} \text{ rad./sid.day}^2 . \qquad \text{(D11)}$$

But the theoretical acceleration due to H_{32} alone for this orbit is very close to (D6). Therefore we see that this simple method for evaluating the earth resonance acceleration is valid to within about 0.15% with this particular orbit. For relatively short trajectories, where the long period mean motion is still approximated very well by the nonresonant gravity effects, this simple correction technique should apply quite well in analyzing resonant orbit data. For longer trajectories, it may be necessary to alter the initial energy of the simulated trajectory without resonant gravity so it approximates the fully perturbed trajectory more closely. This may be done most simply by altering the initial semimajor axis, for example. At any rate, tests of the correction method, similar to that above, should always be performed to check the method. We may note that this is, if successful, a direct method for evaluating the long term contribution of nonresonant effects. It requires an additional trajectory simulation (one without the resonance terms) than the indirect method previously used by the author [D2] in analyzing 24 hour data. However, since it evaluates these effects directly, and without assuming a specific longitude gravity field, it should prove to be a valuable check on the previous method.

For the 12 hour (one day) resonant orbit, the next relevant harmonic H_{nm} (having the next to least n satisfying (32) for $r=2, n'=1$) is H_{44}. The orbit averaged equation for the acceleration of the longitude of the ascending equator crossing, due to H_{44} on a 12 hour orbit [(67), with $\ell = 2,\ p = 1$] gives:

$$\ddot{\lambda} = \frac{-75.073}{a_*^4} \bar{J}_{44} \left[\sin i(1+\cos i)\right]^2 \sin 4(\lambda - \lambda_{44}),$$

rad./sid.day^2 . (D12)

We consider a partical trajectory with only H_{44} acting as a perturbation. The initial conditions for the trajectory are otherwise the same as in the H_{32} studies. The H_{44} constants are: $\bar{J}_{44} = -10^{-3}/16,\ \lambda_{44} = 22.5^\circ$.

The $\bar{J}_{44}$ value is about 100 times a likely realistic value (see Figure 8) to provide an acceleration which will not be substantially "lost" in the roundoff error of the trajectory generator.

From (D12) the long term acceleration should be:

$$\ddot{\lambda}_{TH.} = -4.753 \times 10^{-5},$$

rad./sid.day^2 . (D13)

The following ascending equator crossing data is from a 16 day numerically integrated particle trajectory, in the presence of perturbations due to H_{44} only (whose constants are given above).

Time (Hrs.)	Longitude (Degs.)	a_* (E.R.)	i (Degs.)	e
0.	0.	4.1645	30.0	10^{-7}
191.482	-0.08736	4.1647	30.0	10^{-7}
382.975	-0.34907	4.1648	30.0	10^{-7}

TABLE D5 - Numerical Trajectory Data for a 12 Hour Earth Satellite Disturbed by H_{44} Only.

In the above trajectory, the longitude is given approximately as:

$$\ddot{\lambda}_{MEAS.} = \frac{1}{64}\left[-0.34907 + 0.17472\right]/57.296 = -4.755\times10^{-5},$$

rad./sid.day^2. (D14)

Comparison of (D13) with (D14) again shows a very small discrepancy between the theory and the simulated data; of the order of 0.05%.

16 HOUR ORBIT

The 16 hour resonant orbit will undoubtedly be the most promising one for geodetic purposes, of the two synodic day orbits (see TABLE 3). For this orbit, the leading relevant harmonic is H_{43} (having the least n satisfying (32) for $r = 3,\ n' = 2$). The orbit averaged equation for the ascending equator crossing due to H_{43} on a 16 hour orbit [(67), with $\ell = 2,\ p = 1$] gives:

$$\ddot{\lambda} = \frac{-2229.5}{a_*^4}\,\bar{J}_{43}\cos^5(i/2)\sin(i/2)\left\{\cos^2(i/2) - 3\sin^2(i/2)\right\}\cos 3(\lambda - \lambda_{43}),$$

rad/sid.day^2. (D15).

We consider a 16 hour orbit ($a_* = 5.0449$ E.R.) with only H_{43} acting as a perturbation. The initial circular orbit has an inclination of 30° and an initial ascending equator crossing at $\lambda = 0^\circ$. The H_{43} constants are: $\bar{J}_{43} = -2.2098\times10^{-5}$, $\lambda_{43} = 0^\circ$. Again, J_{43} is chosen about 100 times realistic in order to overcome roundoff error in the comparison with the numerically computed trajectory.

From (D15), the long term acceleration in the neighborhood of $\lambda = 0^\circ$ should be:

$$\ddot{\lambda}_{TH.} = 1.2118\times10^{-5}$$

rad./sid.day^2. (D16)

The following ascending equator crossing data is from a numerically integrated partical trajectory, in the presence of perturbations due to H_{43} only (with constants given above).

TIME (Hrs.)	LONGITUDE (Degs.)	a_* (E.R.)	i (Degs.)	e
0.	0.	5.0449	30.0	10^{-6}
191.474	0.02335	5.0449	30.0	10^{-6}
382.945	0.09107	5.0448	30.0	10^{-6}

TABLE D6 - Numerical Trajectory Data for a 16 Hour Earth Satellite Disturbed by H_{43} Only.

In the above trajectory the longitude acceleration (over the 16 day period) is given approximately by (D4), or:

$$\ddot{\lambda}_{MEAS.} = \frac{1}{64}\left[0.09107 - 0.04670\right] / 57.296 = 1.2100 \times 10^{-5} \text{ rad./sid.day}^2 \qquad \text{(D17)}$$

Again the discrepancy between measured (D17) and theoretical (D16) accelerations is about 0.15%.

We note that for the weak resonances ($\ddot{\lambda} \doteq 0.1 \times 10^{-5}$ rad./sid.day^2) the accuracy of the determination of the acceleration may be improved considerably by utilizing all 2r equator crossings. In doing so, of course, care must be taken in choosing the correct form of the condition equation (67); using $\ell = 1$ for descending and $\ell = 2$ for ascending equator crossings.

In summary, the results of this section combined with the reported accuracies of the tracking of "early bird", [D2], appear to make all the orbits in Table 3 amenable to simple geodetic analysis for resonant longitude gravity harmonics in the earth's field. (See conclusions.)

REFERENCES

[D1] Shaffer, F., Squires, R. K., Wolfe, H., "Interplanetary Trajectory Encke Method (ITEM) Program Manual", NASA-GSFC Document X-640-63-71, 1963.

[D2] Wagner, C. A. "The Earth's Longitude Gravity Field as Sensed by the Drift of Three Synchronous Satellites", NASA-TN-D-3318, 1966.

[D3] Frick, R. H. and Garber, T. B. "Perturbations of a Synchronous Satellite", Rand Corporation Report R-399-NASA, 1962.

APPENDIX E

LIST OF SYMBOLS

G	Location of the Greenwich Meridian
λ	Generally, geographic longitude of the ascending equator crossing of the satellite (also referred to as λ_o , with reference to the beginning or near the beginning of the dynamics). Can also refer to the geographic longitude of the descending equator crossing.
O	when subscripted, refers to orbit number
ϕ	Geographic latitude of the satellite.
S	Position of the satellite (S_o is its position at the start of the dynamics or at the reference ascending equator crossing).
r	Number of orbits in the synodic period of the resonant satellite (n' sidereal days); an integer.
q	An integer (positive, negative or zero) the number of global circuits of the resonant orbit's ground track, in its synodic period. When subscripted, also, integers of the rational fraction expressing the commensurability of the resonant orbit with the earth's rotation.
t	Time from the beginning of the dynamics or some arbitrary zero.
θ	Argument of the ascending node, central angle in the orbit from the ascending node to the satellite (also subscripted θ_o and θ_s).

ΔL	Inertial longitude excursion of the satellite from its ascending node
i	Inclination of the satellite's orbit (also used as a dummy integer-subscript).
ω_e	The earth's inertial rotation rate.
$\Delta\lambda$	Geographic excursion of the satellite from an initial position
ω_s	Orbital revolution rate of the satellite.
θ_e	The inertial longitude turned by the earth since the start of the dynamics
r	The distance from the center of the earth to the satellite.
F_r, F_λ, F_ϕ	Perturbation forces in the radial, longitudinal and latitudinal directions, acting on the satellite
α	The azimuth of the satellite's trajectory.
n'	The synodic period (in sidereal days) of the resonant orbit satellite, an integer. The number of integral days for the stationary ground track to begin to repeat itself.
a	The semimajor axis of the satellite's orbit, also subscripted a_s . (a_* is the semimajor axis in units of earth radii).
F_T	The tangential (along track) component of the perturbation force F.
F	A perturbation force (when superscripted, a non-dimensional force). When subscripted nmp, an inclination force function .
E	The total energy (kinetic and two-body gravitational potential) of a satellite.

Δ, δ	A small, but finite, change (δ not sub or superscripted)
μ_e, μ	The earth's Gaussian gravitational constant.
$\overline{(\ )}$	The r orbit average of the quantity ( ), except in reference to $\overline{C}_{nm}$, $\overline{S}_{nm}$, $\overline{J}_{nm}$ (see below).
$\Delta\tau$	An increment of time, in units of synodic days (not necessarily integral)
Δt	An increment of time, in units of sidereal days.
$\dot{(\ )}$	$\frac{d(\ )}{dt}$ with the time increment in units of sidereal days.
T	The period of the satellite's orbit (T_0 or T_d is the period in sidereal days).
$\ddot{(\ )}$	$\frac{d^2(\ )}{dt^2}$ with the time increment in units of sidereal days.
$\dot{a}_n$	A nondimensional semimajor axis change rate: $\dot{a}_n = \dot{a}/a$; the units of $\dot{a}_n$ are:(sidereal days)$^{-1}$.
V_e	The gravitational potential function of the earth.
H_{nm}	Signifying the gravitational harmonic term of order n and power m.
R_0	The mean equatorial radius of the earth.
$P_n^m(x)$	The associated Legendre polynomial of order n and degree m.
J_{nm}, λ_{nm}	The amplitude and phase of the non-normalized gravity harmonic H_{nm}.
$\overline{J}_{nm}, \overline{C}_{nm}, \overline{S}_{nm}$	The normalized amplitude, cosine coefficient and sine coefficient of the gravity harmonic H_{nm}.
$\lambda_{1/2}$	The ground track longitude span of half an orbit.

a_i', b_i'	When subscripted by integers and/or superscripted, a or b are constants
W or ΔW	Work done by perturbation forces.
(a), (b), (c)	Case numbers
p	A positive integer, or zero
x	A dummy variable
X^*	The mean anomaly of the satellite at the beginning of the dynamics.
e, w, M, f, Ω	Keplerian elements; eccentricity (not used as a subscript), argument of perigee, mean anomaly, true anomaly, and right ascension of the ascending node.
$\mathcal{R}\{\ \}$	The real part of $\{\ \}$
j	$\sqrt{-1}$, except where used as a dummy integer.
$F_{nmp}(i), F'_{nmp}$	An inclination force function (when superscripted, the real value of the force function).
U	A disturbing potential function
$\exp[\ ]$	$e^{[\ ]}$, e = 2.718...
Υ	The location of the Vernal Equinox.
K	A positive integer or zero.
$\delta_{nm}, \delta'(n,m,\ell)$	Phase angles of the disturbing function.
$\delta'_{1p}, \delta'_{2p}$	Phase and Frequency constants of the disturbing function.
K'	An integer or zero.
β	The number of orbits per day in Allan's Resonance Theory[3].
ℓ	As a subscript or in $\delta'(n,m,\ell)$; $\ell = 1$ or odd refers to a descending equator crossing, $\ell = 2$ or even refers to an ascending equator crossing.

C_{nm}, S_{nm}	Non-normalized cosine and sine gravity harmonic coefficients.
σ	Standard deviation (or estimate of it)
$(D), (SD)$	Dominant, subdominant.
K^2_{nm}	An inclination dependent harmonic forcing function usually assumed constant over a libration period.
$\lambda_e; \lambda_{e,s}; \lambda_{e,u}$	An equilibrium longitude; stable, and unstable.
$T_{MIN.}$	The minimum libration period of a resonant orbit.
ω	A frequency or angular rate.
K'_{nm}	An inclination independent harmonic forcing function.
$g_{i,nm}; f_{i,nm}$	Harmonic forcing functions.