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EFFECTS OF NON-GREY GAS BEHAVIOR AND NONEQUILIBRIUM IONIZATION IN RADIATING SHOCK LAYERS

by

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SUMMARY

An analysis of the effects of nonequilibrium ionization and non-grey radiation in hypersonic shock layers is formulated. Two essentially distinct models are used: (1) A Ferrari-Clarke Model for photoionization in an effectively "grey" gas and (2) A non-grey step function model for radiative nonequilibrium. Some difficulties associated with the numerical solution of the resulting two-point boundary-value problems are discussed.

I. INTRODUCTION

The problem of inviscid radiating flows over blunt bodies has been the subject of numerous studies in recent years.[1-7] Gross features of the phenomena are understood at present, although results obtained this far are limited by assumptions of local thermodynamic equilibrium (LTE), "grey" absorption coefficients, and radiatively inert gas in front of the shock. A more realistic approach for air, the gas of engineering interest, is quite cumbersome owing to numerous chemical species and reactions entering into the ionization kinetics and the fact that line radiation in air is not simply related to electron production rates. Therefore, the present approach treats a model (monatomic) gas where the relationship between chemical and radiative nonequilibrium is reasonably well understood.

The report essentially consists of two parts (Sections II and III), with emphasis on different aspects of the problem. Section II is concerned with the analysis of nonequilibrium radiative and collisional ionization; these effects have an important influence on the flow observables associated with vehicles in hypersonic flight at high altitudes. The problem is formulated for a monatomic gas obeying the radiative model of Clarke and Ferrari[8]; in this context a one-dimensional model of the shock layer around a blunt body is investigated. The analytic statement represents an extention of the work of Chien[9] for the case of a grey gas in local thermodynamic equilibrium.

Section III is concerned with the effects of actual (non-grey) radiative properties of high temperature gases on the heat transfer near the stagnation point of a blunt body. This class of effects is important for hypervelocity flight at altitudes lower than those considered in Section II, namely at altitudes where the absorption of radiation in the shock layer dominates the flow dynamics while the ionization has only moderate influence thereon. Inspection of absorption coefficient data for air in the temperature range between 10,000 and 15,000 K indicates that the actual coefficient may be approximated by step functions.[10] Accordingly, the differential (moment) approximation[11-13] of radiative transfer is extended to the case of non-grey gas with stepwise approximation in the frequency dependence. The approximate radiative transport equations are applied to the study of stagnation point flows, extending the treatment by Cheng and Vincenti[7] for a grey gas. It is emphasized that the motivation here differs from that of Section II in that the gas is considered thermally and calorically perfect and nonreacting; however, radiative transport and radiative heat flux are included in the formulation of the energy equation.

In view of the common nature of the two problems and of similar numerical difficulties associated therewith, a joint discussion and presentation is given in this report.

II. INVISCID RADIATING SHOCK LAYERS WITH NONEQUILIBIRUM RADIATIVE AND COLLISIONAL IONIZATION

It is well known that under conditions of low density and high temperature, prevailing in the flow field behind strong shocks generated by hypersonic vehicles at high altitude, the degree of the ionization lags considerably behind the equilibrium values. Although some recent papers[8,14] have considered nonequilibrium effects on shock wave structure, the related problem of radiating shock layers with nonequilibrium radiative and collisional ionization has not been treated analytically.

The governing equations for a gas with nonequilibrium radiative and collisional ionization are very complex.[8] In the following, we shall adopt the simplified model considered by Clarke and Ferrari.[8] Also, following Yoshikawa and Chapman[1], and Chien[9], we represent the shock layers by the inviscid flow through a normal shock wave into a cold, black, and porous planar wall. Such an analysis provides a qualitative picture of the flow field and of the nonequilibrium radiation field near the stagnation streamline of a blunt body.

In the absence of thermal conductivity and viscosity, the equations of continuity, momentum, energy, and state for the one-dimensional, steady flow of a monatomic gas are[8]

$$\rho u = \Gamma \tag{2.1}$$

$$p + \rho u^2 = \Gamma C_1 \tag{2.2}$$

$$\Gamma\left[\frac{5}{2}RT(1+\alpha) + \frac{u^2}{2}\right] + \Gamma\alpha RT_j + q_j = \Gamma W_1^2 \tag{2.3}$$

and

$$p = \rho RT(1+\alpha) \tag{2.4}$$

where ρ is the density, u the velocity, p the pressure, T the temperature, R the gas constant, q_j the radiative heat flux, and α the degree of ionization defined by $\alpha = \frac{n_1}{n_1+n_A}$, n_i and n_A the number densities of ions and atoms, respectively. Γ, c_1 and W_1 are the constants of integration with subscript 1 denoting condition at upstream infinity. T_j is the ionization temperature defined by the relation $E_j = kT_j$ where k is the Boltzmann constant, and E_j the energy required to remove an outmost electron from an atom is in the ground state.

If the wall is maintained at zero temperature and the stand-off distance is ℓ, the radiative heat flux is given by

$$q_j(\eta) = 2\pi \int_{-\infty}^{\eta_w} \mathrm{sgn}(\eta-\eta')\ S(\eta')\ E_2(|\eta-\eta'|)d\eta' \tag{2.5a}$$

where $E_2(|\eta-\eta'|) = \int_0^1 \exp\left[-\frac{(\eta-\eta')}{m}\right]dm$ is the integro-exponential function, η and η_w are the optical thickness defined by

$$\eta = \int_0^x \rho(1-\alpha)\chi dx \tag{2.5b}$$

$$\eta_w = \int_0^\ell \rho(1-\alpha)\chi dx \tag{2.5c}$$

χ is the absorption coefficient at the frequency ν_j per unit mass of atom[8] and S is the source function

$$S \equiv \left[\frac{\alpha^2}{1-\alpha}\right]\left[\frac{1-\alpha^*}{\alpha^{*2}}\right]\frac{2\nu_i^3 kT}{c^2}\exp\left(-\frac{T_i}{T}\right) \tag{2.5d}$$

In (2.5d) the asterisk denotes the equilibrium degree of ionization given by the Saha equation

$$\frac{\alpha^{*2}}{1-\alpha^*} = \frac{m_a}{\rho}\frac{2\Omega_i}{\Omega_o}\left(\frac{2\pi m_e kT}{h}\right)^{\frac{3}{2}}\exp\left(-\frac{T_i}{T}\right) \tag{2.5e}$$

The rate equation, as given by Clarke and Ferrari[8], is

$$\Gamma RT_j\frac{d\alpha}{d\eta} + \frac{dq_j}{d\eta} = \frac{RT_i\alpha}{\chi\tau_c}\left[\frac{\alpha^{*2}}{1-\alpha^*} - \frac{\alpha^2}{1-\alpha}\right] \tag{2.6}$$

where τ_c is the local characteristic time of the collisional ionization.

Eqs.(2.1-2.6) represent a determined set of integro-differential equation for six unknowns p, ρ, T, q_j and α. If the differential approximation[11-13] is employed, eq.(2.5d) can be replaced by

$$\frac{dq_j}{d\eta} = -(I_o - S) \tag{2.7}$$

and

$$\frac{dI_o}{d\eta} = -3q_j \tag{2.8}$$

where I_o is the average radiative intensity.

These approximate radiation-transport equations together with the remaining equations then constitute a determined set of purely differential equations.

For the purpose of solution it is convenient to recast the noted system of equations in terms of new depenedent and independent variables; in this way the problem can be reduced to the integration of two simultaneous equations of the first order. Thus, we first introduce a dimensionless velocity v such that

$$v = \frac{u}{c_1} \tag{2.9}$$

From Eqs. (2.1) and (2.2), it is immediately apparent that $c_1 = u_1\left[1 + \frac{1}{\gamma M_1^2}\right] \cong u_1$ for $M_1^2 \gg 1$. With the aid of Eqs. (2.1) and (2.2), the state equation (2.4) in terms of v becomes

$$T = \frac{c_1^2 v(1-v)}{R(1+\alpha)} \tag{2.10}$$

Substitution of Eqs. (2.9) and (2.10) into energy equation (2.3) leads to

$$v^2 - \frac{5}{4}v + \frac{W_1^2}{2c_1^2} = Z + \frac{q_i}{2c_1^2 p} \tag{2.11}$$

where $Z \equiv \frac{\alpha RT_i}{2c_1^2}$. Following Heaslet and Baldwin[15], we then introduce tne variable

$$\theta = \left(\frac{5}{8} - v\right)^2 \tag{2.12}$$

so that

$$v(\eta) = \frac{5}{8} - (\mathrm{sgn}\ \eta) \left[\theta(\eta)\right]^{\frac{1}{2}} \tag{2.13}$$

and the energy equation (2.11) takes the form

$$\theta - \theta_\infty = Z + q, \tag{2.14}$$

where

$$\theta_\infty \equiv \frac{25}{64} - \frac{W_1^2}{2c_1^2} = \frac{25}{64} - \frac{\gamma_1 M_1^2}{4} \frac{(5+\gamma M_1^2)}{(1+\gamma M_1^2)}$$

and q is the dimensionless radiative heat flux defined by $q \equiv \frac{q_1}{2c_1^2\Gamma}$.

It is apparent from Eq.(2.14) that θ_∞ is the value of θ in the limit condition $Z = q_j = 0$. When $M_1 \to \infty$, $\theta_\infty \cong \frac{9}{64} = 0.1406$; consequently $\theta_\infty \leq 0.1406$.

The process of recasting the governing equations is continued by introducing the dimensionless radiative quantities G and F

$$G \equiv \frac{I_o}{2c_1^2\Gamma}, \tag{2.15a}$$

$$F \equiv \frac{4\pi S}{2c_1^2 p} = \frac{K}{8}\left(\frac{T}{T_o}\right) \left[\frac{Z^2}{Z_j - Z}\right] \left[\frac{Z_j - Z^*}{Z^{*2}}\right] \exp\left[-T_j\left(\frac{1}{T} - \frac{1}{T_o}\right)\right] \tag{2.15b}$$

where $K \equiv \frac{4RT_j}{c_1^2 N_{B_o}}$ and

$$N_{B_o}^{-1} \equiv \frac{8\pi h v_j^4}{c^2 \rho_1 u_1 RT_1} \left(\frac{T_o}{T_j}\right) \exp\left(-\frac{T_j}{T_o}\right)$$

is the inverse Boltzmann number. The radiation transport equations (2.7) and (2.8) can then be written in the form

$$\frac{dq}{d\eta} = -(G - F), \tag{2.16}$$

$$\frac{dG}{d\eta} = - 3q \tag{2.17}$$

The rate equation (2.6) can be written as

$$\frac{dZ}{d\eta} + \frac{dq}{d\eta} = N_{D_a} g \frac{Z}{Z_j} \left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right) \right] \tag{2.18}$$

where

$$N_{D_a} \equiv \frac{2e^4}{\pi \chi m_a E_j^2} \sqrt{\frac{2\pi k T_o}{m_e}} \quad \rho_o \exp\left(- \frac{T_j}{T_o}\right)$$

$$g \equiv \frac{\rho}{\rho_o}\left(\frac{T}{T_o}\right)^{\frac{1}{2}} \exp\left[- T_j\left(\frac{1}{T} - \frac{1}{T_o}\right) \right]$$

ρ_o and T_o denote the density and temperature at some reference condition, and $Z_j \equiv \frac{RT_j}{2c_1^2}$ (consequently $Z = \alpha Z_j$ and $Z^* = \alpha^* Z_j$). The dimensionless quantity N_{D_a} is a Damkohler number corresponding to the ionization reaction, i.e., the ratio of the relaxation length to the radiation free path[14]. When $N_{D_a} \to \infty$, the radiation field is in local thermodynamic equilibrium.

Substitution of Eq. (2.16) into Eq.(2.18) leads to

$$\frac{dZ}{d\eta} = G - F + N_{D_a} g \frac{Z}{Z_j} \left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right) \right] \tag{2.19}$$

Elimination of q from Eqs.(2.14) and (2.17), yields

$$\frac{dG}{d\eta} = 3(\theta_\infty - \theta + Z) \tag{2.20}$$

Differentiation of Eq.(2.14) with respect to η with the aid of Eq.(2.16) leads to

$$\frac{d\theta}{d\eta} = \frac{dZ}{d\xi} + F - G \tag{2.21}$$

or, in view of (2.19),

$$\frac{d\theta}{d\eta} = N_{D_a} \, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z*}\right)^2 \left(\frac{Z_j - Z*}{Z_j - Z}\right)\right] \tag{2.22}$$

Eqs. (2.19), (2.20) and (2.22) provide three coupled ordinary differential equations for the three unknowns G, Z, and θ. The boundary conditions for the shock layers problem are

$$\eta \rightarrow \infty \ , \ G = F, \ Z = Z*, \ \theta = \theta_\infty + Z*, \tag{2.23}$$

and

$$\eta = \eta_w, \quad G = mq \tag{2.24}$$

Eq. (2.24) is the radiative boundary condition (differential approximation) when the wall is maintained at zero temperature. The values of m are $m = \sqrt{3}$ for the Marshak boundary condition and m = 2 for the Mark boundary condition[12,13]. With the aid of Eq.(2.14), the boundary condition (2.24) can be rewritten as

$$\eta = \eta_w \ , \qquad G = m(\theta_w - \theta_\infty - Z) \tag{2.25}$$

where θ_w, the value of θ at the wall, is a given parameter of the problem.

The structure of a normal shock was can also be studied on the basis of the noted equations together with the boundary condition (2.23) and

$$\eta \to \infty,\ G = F,\ Z = Z^*,\ \theta = \theta_\infty + Z^* \tag{2.26}$$

It is worth noting that Eqs.(2.19), (2.20) and (2.22) are a set of autonomous ordinary differential equations since the independent variable η does not appear explicitly on the right-hand side. We can, therefore, reduce the number of equations to two by using any of the variables G, Z and θ as the independent variable. For this purpose, we divide Eqs.(2.19) and (2.20) by Eq.(2.22) to obtain

$$\frac{dG}{d\theta} = \frac{3(\theta_\infty - \theta + Z)}{N_{D_a}\, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right)\right]} \tag{2.27}$$

$$\frac{dZ}{d\theta} = \frac{G - F + N_{D_a}\, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right)\right]}{N_{D_a}\, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right)\right]} \tag{2.28}$$

viz. two coupled differential equations, in the dependent variables G and Z, similar to the equations obtained by Emanuel[16] in the study of the structure of a normal shock wave with vibrational nonequilibrium. The boundary conditions associated with (2.27) and (2.28) are

$$G(\theta_1) = F(\theta_1) = \frac{K}{8}\left(\frac{T_1}{T_o}\right) \exp\left[-\,T_j\left(\frac{1}{T_1} - \frac{1}{T_o}\right)\right] , \tag{2.29}$$

$$Z(\theta_1) = \theta_1 - \theta_\infty = Z^*(\theta_1)$$

and

$$G(\theta_w) = m\left[\theta_w - \theta_\infty - Z(\theta_w)\right] \tag{2.30}$$

Thus the problem of radiating shock layers with nonequilibrium radiative and collisional ionization reduces to the solution two coupled first-order, non-linear, ordinary differential equations (2.27) and (2.28) with two-point boundary conditions (2.29) and (2.30).

A few comments are in order with regard to the numerical integration of equations (2.27) and (2.28). In the work reported here an iterative procedure has been investigated whereby the degree of ionization (i.e., Z) at the wall is chosen arbitrarily and the numerical integration of Eqs. (2.27) and (2.28) proceeds independently from the end points. It should be noted that the independent variable θ and the dependent variables G and Z are continuous functions. Thus, if the subscripts a and b to denote the two branches of G and Z starting respectively from upstream infinity (G_a and Z_a) and from the wall (G_b and Z_b), the unique solution is identified by the intersection of the two branches of G and Z at the same θs. The proper value of Z_b at the wall is the one that satisfies this condition. Once Z and G are obtained as a function of θ, the velocity, temperature, and radiative heat flux can be determined from Eqs. (2.10), (2.13) and (2.14), while the physical coordinate η can be found by the quadrature

$$\eta = \int \frac{d\theta}{N_{D_a}\, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right)\right]} + \text{Constant} \tag{2.31}$$

Eqs.(2.27) and (2.28) are singular at upstream infinity. To start the numerical integration from upstream, the usual method of linearization about the singular point must be employed. Accordingly, one set $G = G(\theta_1) + G^1$, $Z = Z^* + Z^1$, $\theta = \theta_1 + \theta^1$; substitution into Eqs.(2.19), (2.20) and (2.22) and linearization readily yield $G^1 = Z^1$, and $\theta^1 = 0$ near upstream infinity. Since upstream of the shock, the slopes $\frac{dG}{d\theta}$ and $\frac{dZ}{d\theta}$ are very large, Eqs. (2.27) and (2.28) are not suitable for numerical integration. In this region, it is convenient to use G as the independent variable. The equations to be integrated are then

$$\frac{dZ}{dG} = \frac{G - F\, N_{D_a}\, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right)\right]}{3(\theta_\infty - \theta + Z)} \tag{2.32}$$

and

$$\frac{d\theta}{dG} = \frac{N_{D_a}\, g\frac{Z}{Z_j}\left[1 - \left(\frac{Z}{Z^*}\right)^2 \left(\frac{Z_j - Z^*}{Z_j - Z}\right)\right]}{3(\theta_\infty - \theta + Z)} \tag{2.33}$$

On the downstream side of the shock, Eqs.(2.27) and (2.28) are to be integrated.

The parameters appearing in the governing equations and in the boundary conditions are N_{D_a}, N_{B_o}, θ_∞, and θ_w. Numerical integration has been attempted for the model blunt body problem assuming Helium gas ($R = 2.07 \times 10^7\, cm^2 sec^{-2}\, {}^\circ K$, $T_j = 285{,}000^\circ K$, $m_a = 6.692 \times 10^{-24}\, gm$)

$N_{D_a} = 0.0361$, $N_{B_o} = 36.38$, $\theta_\infty = 0.140137$, and $\theta_w = 0.24$; these conditions correspond to upstream temperature $T_1 = 300$ K, pressure $p = 10^{-2}$ atm, and Mach number $M_1 = 29.6$. Numerical integration of Eqs. (2.32) and (2.33) from upstream infinity and that of Eqs.(2.27) and (2.28) from the wall has been performed by means of an Adams-predictor-corrector routine. The integration from upstream infinity can be obtained once for all (for a particular set of parameters) without difficulty. Integration from the wall, subject to estimated values of Z at the wall, becomes unstable near the shock. Fig. 1 shows the unstable behavior of the numerical integration. For $Z_b(\theta_w) = 0.099115$, G_b continuously increases from the wall to the shock, whereas for $Z_b(\theta_w) = 0.99200$, G_b first increases and then decreases. In both cases, however, the two branches of G do not appear to intersect.

Considerable effort has been expended in determining the source of the difficulty. At first it was attributed to the "stiff" behavior of the equations. However, later considerations suggest that the difficulty is intrinsic to the proposed formulation. Specifically, in the present report both blunt body and normal shock flows are governed by essentially the same equations. Thus, the singular behavior of the two problems is essentially identical, i.e., the upstream "infinities" are singular points and numerical integration must pass through these points. But the body location in the blunt body problem does not correspond to the downstream singularity of the normal shock. Moreover the properties at the body surface, compatible with a solution of the governing equations, cannot be specified unless an integral curve passing through both infinities is known. Thus, by specifying body boundary conditions, the problem is ill posed and must be reformulated to take this effect into account.

III. RADIATING FLOW OF A NON-GREY GAS IN THE STAGNATION REGION OVER A BLUNT BODY

Although realistic optical properties of the gas (as opposed to a grey gas model) have been recognized and considered in recent analyses of radiating flows over blunt bodies (e.g. Ref.10), their influence on the structure of the shock layer has not been explored in detail. Intuitive physical considerations suggest an appreciable influence for the cases wherein the absorption coefficient of the gas varies importantly with frequency and the associated optical lengths vary (as a function of frequency) from much larger to much smaller than the shock layer thickness. Under these conditions one expects to observe regions of rapid temperature variation adjacent to the boundaries of the flow, with associated changes in the convective heat transfer as well as a strong frequency dependence of the radiative heat transfer. A marked frequency dependence of the absorption coefficient is quite common in practice; thus, the noted effects are of practical interest and significance. For illustrative purposes we will specifically consider in what follows a model gas with a step function dependence of the absorption coefficient upon frequency; such a model is representative, for example, of the optical properties of air in the temperature range 10,000 to 15,000°K.

The analysis is based on an extension of the moment equations (differential approximation) of radiative transfer to the case of non-grey gas with step-function approximation in the frequency dependence. We begin by dividing the frequency spectrum into a number of discrete frequency ranges. In any given range ($\nu_n < \nu < \nu_{n+1}$), the gas is assumed to behave according to certain average properties which give a good approximation to the actual behavior. The moment equations for each frequency group, $\nu_n < \nu < \nu_{n+1}$, are in the usual form[11-13]

$$\mathrm{div}\bar{q}_\nu = -\alpha_\nu(I_{o\nu} - 4\pi S_\nu) \tag{3.1}$$

and

$$\mathrm{grad}\ I_{o\nu} = -3\alpha_\nu \bar{q}_\nu \tag{3.2}$$

where q_ν is the radiative heat flux, $I_{o\nu}$ the average radiative intensity, and S_ν the source function. If local thermodynamic equilibrium is assumed, the source function is equal to the Planck function

$$S_\nu = B_\nu = \frac{2h_\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{KT}\right) - 1} \tag{3.3}$$

With the differential approximation, the radiative boundary condition on the wall of the blunt body is[12,13]

$$\frac{1}{2}\left[\frac{I_{o\nu}}{2} \pm \bar{n} \cdot \bar{q}_\nu\right] = q_\nu^\pm \tag{3.4}$$

where $\bar{n}$ is the outward normal of the wall and $q^\pm$ is the one-sided radiative heat flux from the wall*.

*The positive sign is taken when the outward normal of the wall is in the same direction of the positive coordinate whereas the negative sign is used if it is in the opposite direction of the positive coordinate.

For the considered model gas analysis is conveniently carried out in terms of frequency integrated quantities $\bar{q}_n$, J_n and B_n

$$\bar{q}_n = \int_{\nu_n}^{\nu_{n+1}} \bar{q}_\nu d\nu \tag{3.5}$$

$$J_n = \int_{\nu_n}^{\nu_{n+1}} I_{o\nu} d\nu$$

$$B_n = \int_{\nu}^{\nu_{n+1}} B_\nu d\nu = \frac{2h}{o^2} \int_{\nu_n}^{\nu_{n+1}} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{KT}\right) - 1}$$

Integrating the differential equations (3.1) and (3.2) and the boundary condition (3.4) over the frequency interval $\nu_n < \nu < \nu_{n+1}$(in each frequency range the absorption coefficient remains constant) we have

$$\text{div}\bar{q}_n = - \alpha_n (J_n - 4\pi B_n) \tag{3.6a}$$

and

$$\text{grad } J_n = - 3\alpha_n \bar{q}_n \quad , \tag{3.6b}$$

with the boundary condition

$$\frac{1}{4}\left[J_n \pm 2\bar{n} \cdot \bar{q}_n \right] = q_n^{\pm} \tag{3.7}$$

The overall frequency-integrated radiative quantities are then given by

$$\bar{q} = \int_0^\infty \bar{q}_\nu d\nu = \sum_{n=0}^{\infty} \bar{q}_n \tag{3.8}$$

$$I_o = \int_0^\infty I_{o\nu} d\nu = \sum_{n=0}^{\infty} J_n$$

$$B = \int_0^\infty B_\nu d\nu = \sum_{n=0}^{\infty} B_n = \frac{\sigma T^4}{\pi}$$

In neutron-transport theory, a set of equation similar to that of Eqs.(3.6) and (3.7) are called the multi-group diffusion equations[17].

The inverse problem of the axisymmetric flow field over the blunt body associated with a paraboloidal shock wave has been treated by Cheng and Vincenti[7] for a grey gas. We now extend Cheng and Vincenti's work taking into consideration the effects of non-grey gas behavior.

All variables in the following Eqs.(3.9)-(3.15) and in the boundary conditions [(3.16) - (3.17)] are dimensionless: the velocity components are referred to the free-stream velocity $\bar{U}_\infty$, the density to $\bar{\rho}_\infty$, the pressure to $\bar{\rho}_\infty \bar{U}_\infty^2$, the temperature to $\bar{T}_s$, and all radiative quantities to $\sigma \bar{T}_s^4$. (In the remainder of the report, dimensional quantities are denoted with bars and the corresponding dimensionless quantities without. Quantities pertaining to the point immediately behind the shock on the stagnation streamline, and to the free stream conditions are denoted respectively by subscripts s and ∞).

The equations of continuity, momentum, energy and state in the paraboloidal coordinates are

$$\left[\xi\eta\sqrt{\xi^2+\eta^2}\ \rho u\right]_\xi + \left[\xi\eta\sqrt{\xi^2+\eta^2}\ \rho v\right]_\eta = 0 \quad , \tag{3.9}$$

$$uu_\xi + v\left(u_n - \frac{\xi \nu - \eta u}{\xi^2 + \eta^2}\right) + \frac{p_\xi}{\rho v} = 0 \ , \tag{3.10}$$

$$u\left(\nu_\xi + \frac{\xi v - \eta u}{\xi^2 + \eta^2}\right) + \frac{p_n}{\rho} = 0 \quad , \tag{3.11}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\left(uT_\xi + vT_\eta\right)\frac{p}{T} - \left(up_\xi + vp_\eta\right) \tag{3.12}$$

$$+\sum_{n=1}^{\infty} \frac{\Gamma}{\xi\eta\left(\xi^2+\eta^2\right)^{\frac{1}{2}}} \left\{ \left[\xi\eta\left(\xi^2+\eta^2\right)^{\frac{1}{2}} q_n^3\right]\right.$$

$$\left. + \left[\xi\eta\left(\xi^2+\eta^2\right)^{\frac{1}{2}} q_n^\eta\right]_\eta \right\} = 0$$

$$p = \frac{1}{\gamma M_s^2}\ \rho T \tag{3.13}$$

where the parameter Γ is defined by $\Gamma \equiv \dfrac{\sigma T_s^4}{\rho_\infty U_\infty{}^3}$

The radiative transport eqs.(3.6) and (3.7) written in the paraboloidal coordinates are

$$\left[\xi\eta\sqrt{\xi^2+\eta^2}\; q_n^\xi\right]_\xi + \left[\xi\eta\sqrt{\xi^2+\eta^2}\; q_n^\eta\right]_\eta =$$

$$- \beta_n p^{a_n} T^{b_n} \xi\eta\left(\xi^2+\eta^2\right)\left(J_n - 4B_n\right) \tag{3.14}$$

$$J_{n\xi} = -3\beta_n p^{a_n} T^{b_n}\sqrt{\xi^2+\eta^2}\; q_n^\xi \tag{3.15}$$

$$J_{n\eta} = -3\beta_n p^{a_n} T^{b_n}\sqrt{\xi^2+\eta^2}\; q_n^\eta \tag{3.16}$$

where we have assumed that the absorption coefficient is of the form $\alpha_n = C_n(\rho_\infty U_\infty^2 p)^{a_n}(T_s T)^{b_n}$ and the parameter β_n is defined by $\beta_n = r_s C_n(\rho_\infty U_\infty^2)^{a_n} T_s^{b_n}$.

If we assume that the cold gas ahead of the shock is neither absorbing nor emitting, the boundary conditions immediately behind the shock ($\eta = 1$) are

$$u(\xi,1) = \xi/(1+\xi^2)^{\frac{1}{2}} \tag{3.17a}$$

$$v(\xi,1) = -(\gamma-1)/[(\gamma+1)(1+\xi^2)^{\frac{1}{2}}] \tag{3.17b}$$

$$p(\xi,1) = 2/[(\gamma + 1)\ (1 + \xi^2)] \quad , \tag{3.17c}$$

$$T(\xi,1) = 1/(1 + \xi^2) \tag{3.17d}$$

$$\rho(\xi,1) = (\gamma + 1)/(\gamma - 1) \tag{3.17e}$$

$$J_n(\xi,1) - 2q_n^{\eta}\ (\xi,1) = 0 \tag{3.17f}$$

where the strong shock approximation has been used in Eqs.(3.17a) - (3.17e). The boundary conditions at the wall are

$$v(\xi,\eta_w) - u(\xi,\eta_w)\ \left(\frac{d\eta_w}{d\xi}\right) = 0 \tag{3.18a}$$

and

$$\frac{1}{4}\left\{J_n(\xi,\eta_w) + 2\left[\ 1 + \left(\frac{d\eta_w}{d\xi}\right)^2\right]^{-\frac{1}{2}} \left[q_n^{\eta}(\xi,\eta_w) - q_n^{\xi}(\xi,\eta_w)\ \left(\frac{d\eta_w}{d\xi}\right)\ \right]\right\} = q_n^+ \tag{3.18b}$$

where η_w is the location of the wall to be determined. As in Ref. 7, we will now perform the following operations: (1) introduce the stream function Ψ, (2) interchange the roles of η and ψ by means of the Von Mises transformation, (3) introduce the normalized variables P, θ ,Q_n^{ξ}, Q_n^{η}, M_n such that

$$\omega \equiv 2\psi/\xi^2 \tag{3.19}$$

$$p \equiv 2(\gamma + 1)^{-1}\ (1 + \xi^2)^{-1}\ P(Z,\omega)$$

$$T \equiv (1 + \xi^2)^{-1} \quad \theta(Z,\omega)$$

$$q_n^{\xi} \equiv \frac{\xi}{(1 + \xi^2)} Q_n^{\xi} (Z,\omega)$$

$$q_n^{\eta} \quad \frac{1}{(1 + \xi^2)} Q_n^{\eta} (Z,\omega)$$

$$J_n = \frac{1}{(1 + \xi^2)} M_n(Z,\omega)$$

where $Z \equiv \xi^2/(1 + \xi^2)$ is a new independent variable in place of ξ. After all these operations, the governing equations in terms of the normalized variables are:

$$\begin{aligned}
&2Z(1 - Z)\eta_{Z\omega} - 2\omega\eta_{\omega\omega} - 2\eta_{\omega} + \frac{2\eta_{\omega}}{\eta}\Big[Z(1 - Z)\eta_Z - \omega\eta_{\omega}\Big] \\
&+ \frac{2\eta_{\omega}}{P}\Big[Z(1 - Z)\, P_Z - ZP - \omega P_{\omega}\Big] - \frac{2\eta_{\omega}}{\theta}\Big[Z(1 - Z)\theta_Z \\
&- Z\theta - \omega\theta_{\omega}\Big] + \eta_{\omega} + \frac{\eta_{\omega} Z}{[Z + \eta^2(1 - Z)]} \\
&+ \frac{4\eta_{\omega}(1 - Z)\Big[Z(1 - Z)\,\eta_Z - \omega\eta_{\omega}\Big]^2}{\Big[Z + \eta^2\,(1 - Z)} \\
&- 32\eta^2\Big[Z + \eta^2(1 - Z)\Big]\frac{\eta_{\omega}^2 P_s M_s^2(1 - Z)}{(\gamma + 1)^2\theta} \\
&\cdot\Big\{\Big[(1 - Z)P_Z - P\Big]\eta_{\omega} - (1 - Z)\eta_Z P_{\omega}\Big\} = 0
\end{aligned} \qquad (3.20)$$

$$2\eta_\omega(1 - Z)\left[2Z^2(1 - Z)\eta_{ZZ} + Z(1 - 4Z)\eta_Z\right] - 4Z(1 - Z)\left[\omega\eta_\omega + Z(1 - Z)\eta_Z\right]\eta_Z\omega \quad (3.21)$$

$$+ 4Z(1 - Z)\eta_Z\omega\eta_{\omega\omega} + 2\eta_\omega\left[\omega\eta_\omega + 2Z(1 - Z)\eta_Z\right] - \frac{4\eta_\omega}{\eta}\left[Z(1 - Z)\eta_Z - \omega\eta_\omega\right]^2$$

$$+ \frac{4\eta_\omega}{P}\left[\omega\eta_\omega - Z(1 - Z)\eta_Z\right]\left[Z(1 - Z)\ P_Z - ZP - \omega P_\omega\right]$$

$$- \frac{4\eta_\omega}{\theta}\left[\omega\eta_\omega - Z(1 - Z)\eta_Z\right]\left[Z(1 - Z)\theta_Z - Z\theta - \omega\theta_\omega\right] + 2\eta_\omega\left[\omega\eta_\omega - Z(1 - Z)\eta_\omega\right.$$

$$- \frac{\eta\eta_\omega Z}{\left[Z + \eta^2(1 - Z)\right]} - \frac{4\eta\eta_\omega(1 - Z)\left[\omega\eta_\omega - Z(1 - Z)\eta_Z\right]^2}{Z + \eta^2(1 - Z)}$$

$$+ \frac{16\eta^2\left[Z + \eta^2(1 - Z)\right]\eta_\omega^2 P_\omega P_s M_s^2}{(\gamma + 1)^2\theta} = 0$$

$$\gamma P\left[Z(1 - Z)\theta_Z - Z\theta - \omega\theta_\omega\right] - (\gamma - 1)\theta\left[Z(1 - Z)P_Z - ZP - \omega P_\omega\right] \quad (3.22)$$

$$- \sum_{n=1}^{\infty} \Gamma\beta_n\gamma M_s^2\left[\frac{2}{(\gamma + 1)}\right]^a (\gamma - 1)P^{a_n+1}\theta^{b_n}(1 - Z)^{a_n+b_n-1}$$

$$\cdot\left[Z + \eta^2(1 - Z)\right]\left[M_n - 4(1 - Z)^3 B_n\right]\eta\eta_\omega = 0 \quad ,$$

$$\eta\left[2Z(1-Z)Q^{\xi}_{nZ}\eta_\omega - (2Z-1)Q^{\xi}_{n}\eta_\omega - 2Z(1-Z)\eta_Z Q^{\xi}_{n\omega} + Q^{\eta}_{n\omega}\right] \tag{3.23}$$

$$+\left[\eta Q^{\xi}_n + Q^{\eta}_n\right]\eta_\omega + \frac{\eta\left[ZQ^{\xi}_n + \eta(1-Z)Q^{\eta}_n\right]\eta_\omega}{\left[Z+\eta^2(1-Z)\right]} + \beta_n\left(\frac{2P}{\gamma+1}\right)^{an}\theta^{bn}$$

$$\cdot\left[M_n - 4B_n\right]\left[\frac{Z+\eta^2(1-Z)}{1-Z}\right]^{\frac{1}{2}}\eta\eta_\omega = 0$$

$$\left[(1-Z)M_{nZ} - M_n\right]\eta_\omega - (1-Z)\eta_Z M_{n\omega} + \frac{3}{2}\beta_n\left(\frac{2P}{\gamma+1}\right)^{a_n}\theta^{b_n}(1-Z)^{a_n+b_n-1} \tag{3.24}$$

$$\cdot\left[\frac{Z+\eta^2(1-Z)}{1-Z}\right]^{\frac{1}{2}}\eta_\omega Q^{\eta}_n = 0 \qquad (n = 1,2,\ldots\infty)$$

$$M_{n\omega} + 3\beta_n\left(\frac{2P}{\gamma+1}\right)^{a_n}\theta^{b_n}(1-Z)^{a_n+b_n}\left[\frac{(Z+\eta^2(1-Z)}{1-Z}\right]^{\frac{1}{2}}Q^{\xi}_n\eta_\omega = 0 \tag{3.25}$$

$$(n = 1,2,\ldots\infty)$$

We now apply the method of series truncation. Each dependent variable is first expanded in a power series in Z of the form

$$F(Z,\omega) = F_1(\omega) + ZF_2(\omega) + \ldots \tag{3.26}$$

where subscripts 1 and 2 identify quantities associated with the first- and second-order problems, respectively. Substituting Eq.(3.26) into Eqs.(3.20) - (3.25), collecting the coefficient of Z, and setting to zero quantities with subscript 2, we obtain for the first-order problem(with the subscript 1 omitted)

$$\frac{d\eta}{d\omega} = \varkappa \tag{3.27}$$

$$\frac{dP}{d\omega} = \frac{\gamma P\theta\left(1 + \frac{2\varkappa}{\eta}\right) + \Gamma\gamma M_Y^2 \varkappa\left(\frac{2}{\gamma+1}\right)(\gamma-1)P^2\theta^{-1}\sum_{n=1}^{\infty}\beta_n(J_n - B_n)}{\left[\frac{4\gamma^2 M_Y^2 \eta^4 P^2}{(\gamma+1)^2} + \gamma\omega^2\right]} \tag{3.28}$$

$$\gamma\omega P\,\frac{d\theta}{d\omega} = (\gamma-1)\omega\,\frac{dP}{d\omega} - \Gamma\gamma M_Y^2\left(\frac{2}{\gamma+1}\right)(\gamma-1)P^2\theta^{-1}\eta^3\varkappa \cdot \left[\sum_{n=1}^{\infty}\beta_n(J_n - B_n)\right] = 0 \tag{3.29}$$

$$\frac{dM_n}{d\omega} = -\,3\beta_n\left(\frac{2P}{\gamma+1}\right)\theta^{-1}Q_n^{\eta}\,\eta\varkappa \qquad (n = 1,2,\ldots\infty) \tag{3.30}$$

and

$$\frac{dQ_n^{\eta}}{d\omega} = -\,\frac{1}{\eta}\left[2(\eta Q_n^{\xi} + Q_n^{\eta}) + \beta_n\left(\frac{2P}{\gamma+1}\right)\theta^{-1}(J_n - B_n)\eta^2\right]\varkappa \tag{3.31}$$

$$\frac{d\varkappa}{d\omega} = -\,\frac{1}{2\omega}\left[\varkappa - \frac{4\varkappa^3\omega^2}{\eta^2} + \frac{2\omega\varkappa^2}{\eta} - 32\eta^4\varkappa^2 P\,\frac{\gamma M_Y^2}{(\gamma+1)^2}\,\frac{P\varkappa}{\theta} + \frac{2\varkappa\omega}{P}\,\frac{dP}{d\omega} - \frac{2\varkappa\omega}{\theta}\,\frac{d\theta}{d\omega}\right] \tag{3.32}$$

The boundary conditions at the shock are

$$P(1) = \theta(1) = \eta(1) = 1 \tag{3.33}$$

$$\varkappa(1) = \frac{1}{2}\left(\frac{\gamma-1}{\gamma+1}\right) \tag{3.34}$$

$$M_n(1) = 2Q_n^{\eta}(1) \tag{3.35}$$

and on the wall

$$M_n(0) = -2Q_n^{\eta}(0) \tag{3.36}$$

Eqs. (3.27)-(3.32) are to be integrated numerically from the shock ($\omega = 1$) to the wall ($\omega = 0$). A sample calculation with $C_1 = 2.7 \times 10^{-7}$, $a_1 = 1$, $b_1 = -1$ for $\nu < 2.5 \times 10^{15}$ sec^{-1}, $C_2 = 0.108$, $a_2 = 1$, $b_2 = -1$ for $\nu > 2.5 \times 10^{15}$ sec^{-1} and with $\Gamma_s = 12{,}000°K$, $p_s = 1$ atm, and $\gamma = 5$ ft., corresponding to the parameters $\beta_1 = 5 \times 10^{-3}$, $\beta_2 = 1.8 \times 10^{3}$, and $\Gamma = 1.90$ was attempted.

Two methods were used to integrate the two-point boundary value problem numerically. The first method was the usual one: to convert it into an initial-value problem by prescribing two of the unknowns radiative quantities at the shock (for example $Q_1(1)$ corresponding to the small absorption coefficient α_1, and $Q_2(1)$ corresponding to the large absorption coefficient α_2); the initial-value problem is then solved numerically by means of an Adams predictor-corrector routine on an electronic computer. Although this procedure was used in Ref. 7 with no difficulty, the numerical integration becomes unstable in the present work. It is found that the solution is practically independent of the boundary value of $Q_1(1)$ (at the shock) but is extremely sensitive to the boundary value of $Q_2(1)$. Small changes in $Q_2(1)$ lead to drastic changes in the solution (see Fig. 2). The second approach to the two-point boundary value problem was to integrate numerically on the basis of quasilinearization[18]. It was found that the iteration did not converge and that small changes in initial distribution led to drastic changes in the solution.

Unstable behavior of a similar nature has been reported in the literature[19-21]. Specifically, Carrier and Averett[21] considered a non-grey radiative transport problem where the absorption coefficients had disparate frequency independent values on either side of some frequency bound separating these regimes. They found that the resulting transport equation was of the boundary layer type associated with singular perturbation problems. With respect to the radiative transport equations, the present problem is entirely analogous to that of Ref. 21. Thus, it may be concluded that the numerical instabilities could be removed, as in boundary layer theory, by proper matching of "inner" and "outer" solutions using techniques of asymptotic expansion theory.

IV. CONCLUDING REMARKS

Two facets of the problem of radiating flow over blunt bodies have been formulated: (1) the nonequilibrium ionization field and (2) non-grey effects. In both cases, a slightly modified differential approximation is employed to take proper account of the reabsorption of radiation, which is of crucial importance near the stagnation point.

The differential approximation leads to two-point boundary conditions. For numerical integration, the two-point boundary value problem is converted to an initial value problem. The difficulty arises as to how to make the successive iteration of the unknown boundary values converge. The integration is further complicated by the fact that the equations exhibit numerical instabilities. The difficulties associated with the two problems have been discussed in Sections II and III. It appears that in both cases, integration schemes appropriate to the mathematical nature of the respective governing equations would succeed in providing numerical solutions. Further work along these lines is indicated.

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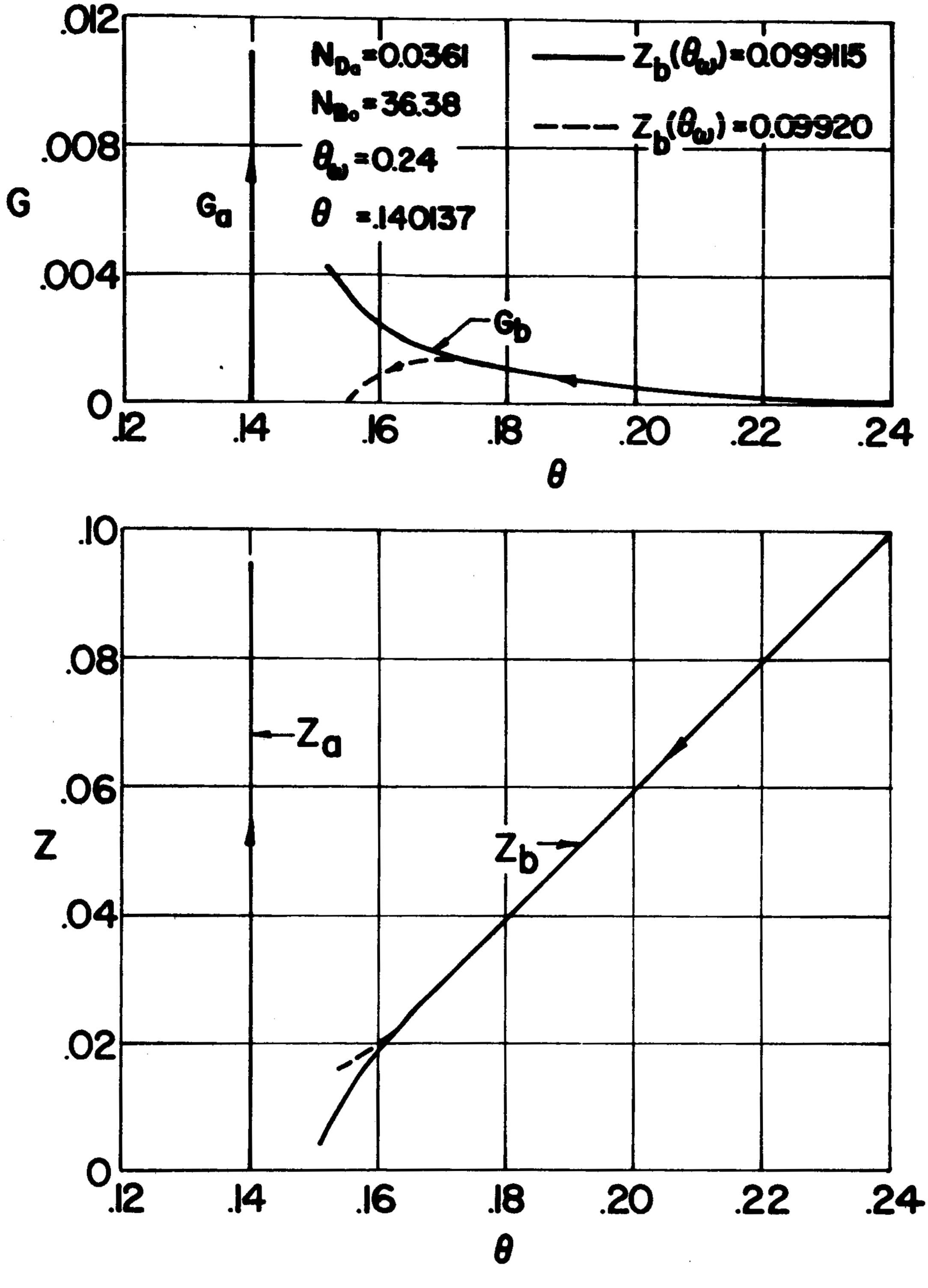

Figure 1 Standing normal shock in front of a wall with chemical and radiative nonequilibrium: Example of divergent iterative solutions of Eqs. (2.32) and (2.33) showing $G(\theta)$ and $Z(\theta)$ starting from wall initial conditions $Z_b(\theta_\omega) = 0.099115$ and $Z_b(\theta_\omega) = 0.09920$. G_a and Z_a are the asymptotic upstream values.

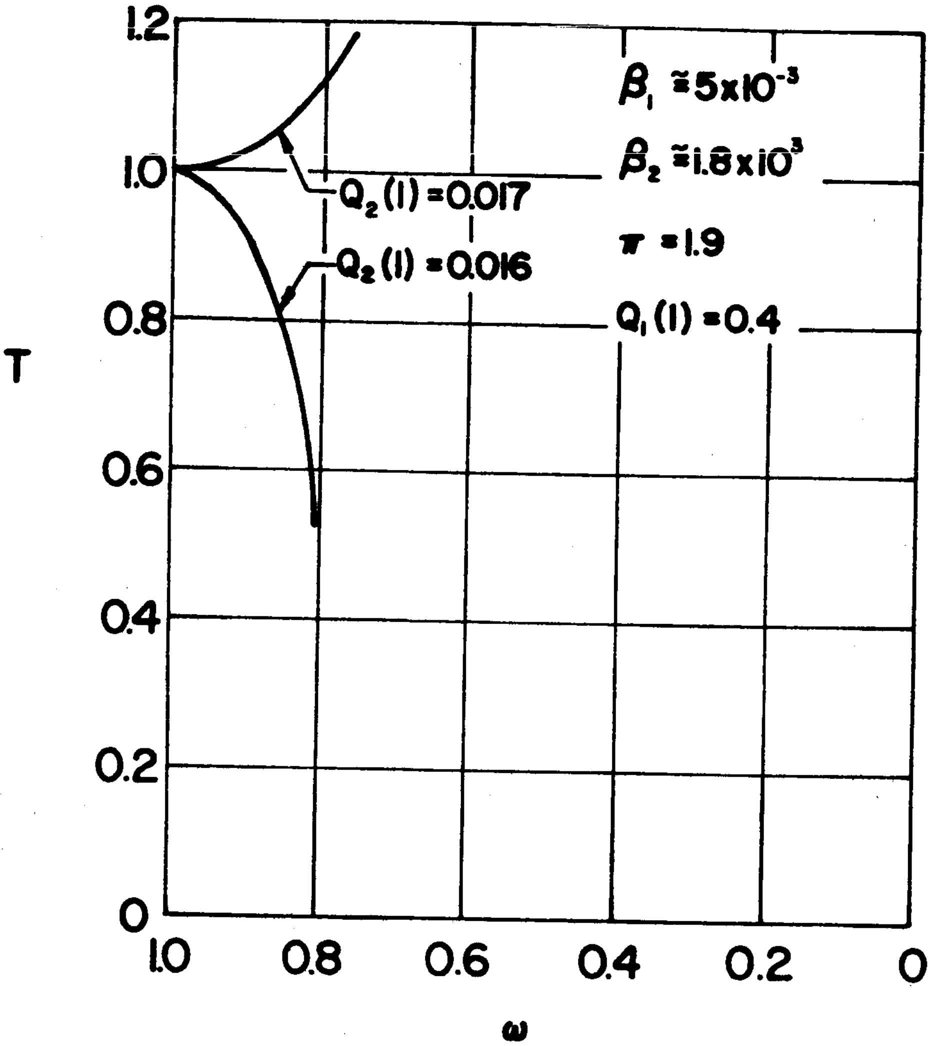

Figure 2 Blunt body stagnation line for a non-grey gas: Example of divergent iterative solutions of Eqs. (3.27) - (3.32) showing $T(\omega)$ for the shock ($\omega = 1$) initial conditions $Q(1) = 0.017$ and $Q(1) = 0.016$.