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DIGITAL ANALYSIS OF LIQUID PROPELLANT SLOSHING IN MOBILE TANKS WITH ROTATIONAL SYMMETRY

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GPO PRICE $ ____________

CFSTI PRICE(S) $ ____________

Hard copy (HC) 3.00

Microfiche (MF) .65

ff 653 July 65

FACILITY FORM 602

N68-16428	
(ACCESSION NUMBER)	(THRU)
40	1
(PAGES)	(CODE)
CR-61540	12
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

GENERAL DYNAMICS

Convair Division

[redacted]

A2136-1 (REV. 5-65)

CR-61540

DIGITAL ANALYSIS OF LIQUID PROPELLANT SLOSHING IN MOBILE TANKS WITH ROTATIONAL SYMMETRY

by

D. O. Lomen

TECHNICAL REPORT GD|A-DDE64-062

CONTRACT NAS8-11193

prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

30 November 1964

GENERAL DYNAMICS | ASTRONAUTICS

FOREWORD

This report explains the digital computer program used to determine the natural frequencies, mode shapes, forces, and moments for propellant sloshing in tanks of arbitrary shape. The description of this analysis has been prepared under the requirement of Part 1, Section A.1, of Contract NAS8-11193 and satisfies in part the provisions of this requirement. The remaining documentation requirements of the subject contract are fulfilled in the three companion reports GD|A-DDE64-059, GD|A-DDE64-060, and GD|A-DDE64-061.

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SECTION I

INTRODUCTION

The hydrodynamic forces and moments are derived (in Reference 1) for tanks possessing a longitudinal axis of symmetry. These forces and moments are given in terms of coefficients which depend only on the tank geometry.

This report explains the steps used to obtain these coefficients, given the tank geometry, and the procedures used in the program check-out. A description of the routines used in the program is included as well as instructions for use of the program. The output of the digital program gives the spring-mass parameters associated with the system.

The digital routines were developed and programed by Roger Barnes.

SECTION 2

SYNTHESIS OF EQUATIONS

2.1 HYDRODYNAMIC EQUATIONS. The equations which describe the sloshing of liquid propellant in a tank consisting of an arbitrary curve rotated about an axis of symmetry are derived in Reference 1 and are:

$$\eta = \sum_{n=1}^{\infty} \xi_n(t) \sin \theta \, \Psi_n(r,z) \tag{2.1}$$

$$F_3' = -M \alpha_3 \tag{2.2}$$

$$F_2' = -M \alpha_2 - M L_1 \ddot{\theta} - M \sum_{n=1}^{\infty} b_n \gamma_n \ddot{\xi}_n \tag{2.3}$$

$$T_1' = M L_1 \alpha_2 + I_{11}' \ddot{\theta} - M \sum_{n=1}^{\infty} \gamma_n \left\{ \alpha_3 b_n \xi_n + \left[L(b_n - h_n) - L_1 b_n \right] \ddot{\xi}_n \right\} \tag{2.4}$$

where

η is the surface wave height

F_3' and F_2' are forces in the x_3 and x_2 directions (see Figure 2-1)

T_1' is the moment about the x_1 axis

$\Psi_n(r,z)$ are eigenfunctions

and

$\xi_n(t)$ is given by the solution of (2.5).

$$\ddot{\xi}_n(t) + \alpha_3 \frac{K_n}{L} \xi_n(t) = -K_n b_n \alpha_2(t) - K_n \left[L_1 b_n - L(b_n - h_n) \right] \ddot{\theta}(t) \tag{2.5}$$

The $K_n = \dfrac{L \omega_n^2}{\alpha_3}$ are nondimensional frequencies. L is the distance from the center of gravity of the liquid to the undisturbed free surface, and L_1 is the distance from the center of gravity of the liquid to an arbitrary point along an extension of the line AC (see Figure 2-1).

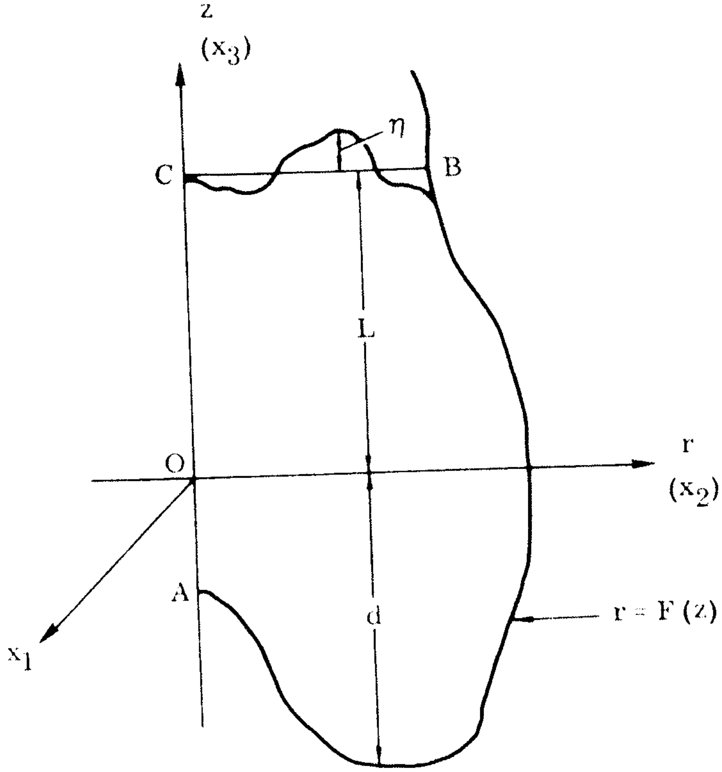


Figure 2-1. Tank of Arbitrary Shape

The component of acceleration along the x_3 (x_2) axis is denoted by α_3 (α_2). The constants in (2.3), (2.4), and (2.5) are given by

$$\gamma_n \quad \frac{\pi L}{V} \int_C^B r \left| \Phi_n (r, L) \right|^2 dr \tag{2.6}$$

$$b_n \quad \frac{\pi}{V \gamma_n} \int_C^B r^2 \Phi_n (r, L) \, dr \tag{2.7}$$

$$h_n \quad \frac{2\pi}{V \gamma_n K_n} \int_A^B z \, r \, \Phi_n (r, z) \, dz \tag{2.8}$$

and

$$I_{11}' = \rho \int_{UV} (x_2^2 + x_3^2)\, dV - 4\rho \int_{UV} x_3^2 dV + 2\rho L^2 \int_{US} x_3 \nu_2 \iota_1 dS$$

$$+ L_1^2 M \tag{2.9}$$

M is the total mass and V the total volume of the liquid.

$$\iota_1 = \sin\theta \left[A_o r + \sum_{n=1}^{\infty} A_n I_1 (\mu_n r) \cos \mu_n (z + d) \right], \quad \mu_n = \frac{n\pi}{h} \tag{2.10}$$

where A_n is to be determined from the equation

$$A_o + \sum_{n=1}^{\infty} A_n \mu_n \left\{ I_1' (\mu_n F(z)) \cos \mu_n (z + d) - \frac{dF}{dz} I_1 (\mu_n F(z)) \sin \mu_n (z + d) \right\}$$

$$= \frac{2z}{L^2}, \quad -d \le z \le L \tag{2.11}$$

To find an approximation for ι_1, truncate the series to include N + 1 terms. Evaluate (2.11) at N + 1 points (z_i) on the boundary to obtain the following set of linear equations for the determination of the A_n, i.e.

$$A_o + \sum_{n=1}^{N} A_n \mu_n \left\{ I_1' (\mu_n F(z_i)) \cos \mu_n (z_i + d) \right.$$

$$\left. - \frac{dF}{dz} (z_i) I_1 (\mu_n F(z_i)) \sin \mu_n (z_i + d) \right\} = \frac{2z_i}{L^2}, \quad i = 1, 2, \ldots, N + 1 \tag{2.12}$$

where r = F(z) is the equation of the boundary AB.

2.2 SYNTHESIS FOR DIGITAL SOLUTION. The boundary value problem for the determination of the eigenfunctions, Φ_n, and eigenvalues, K_n, is

$$\frac{\partial^2 \Phi_n}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_n}{\partial r} - \frac{\Phi_n}{r^2} + \frac{\partial^2 \Phi_n}{\partial z^2} = 0 \tag{2.13}$$

$$\frac{\partial \Phi_n}{\partial \nu} = 0 \qquad \text{(on AB)} \qquad (2.14)$$

$$\frac{\partial \Phi_n}{\partial z} = \frac{K_n}{L} \Phi_n \qquad \text{(on BC)} \qquad (2.15)$$

It has been shown (Reference 2) that the K_n/L may be found by minimizing the quotient

$$\frac{K}{L} = \frac{\int_{UA} \left[\left(\frac{\partial \Phi}{\partial r}\right)^2 + \left(\frac{\Phi}{r}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] r dr dz}{\int_{UFS} \left[\Phi (r, L)\right]^2 r dr} \qquad (2.16)$$

To reduce the magnitude of the variables in integration, make the problem described by (2.13) through (2.16) nondimensional by letting

$R = r/a$

$Z = z/a$

$\Phi (Ra, Za) = \phi (R, Z)$

and

$a =$ the distance from B to the z-axis.

If the free surface of the liquid does not intersect the center line of the tank, A and C coincide. In this situation, let the distance from the z-axis to point C be designated by R_1, and define $\epsilon = R_1/a$.

Thus

$$\frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} - \frac{\phi}{R^2} + \frac{\partial^2 \phi}{\partial Z^2} = 0 \qquad (2.17)$$

$$\frac{\partial \phi}{\partial \nu} = 0 \qquad \text{(on AB)} \qquad (2.18)$$

$$\frac{\partial \phi}{\partial Z} = \frac{Ka}{L} \phi \qquad \text{(on BC)} \qquad (2.19)$$

$$\frac{Ka}{L} = \frac{\int_{UA} \left[\left(\frac{\partial \varphi}{\partial R}\right)^2 + \left(\frac{\varphi}{R}\right)^2 + \left(\frac{\partial \varphi}{\partial Z}\right)^2 \right] RdRdZ}{\int_{C}^{B} \left[\varphi(R,\ L/a) \right]^2 RdR} \tag{2.20}$$

where the integrations are over nondimensional limits (see Figure 2-2).

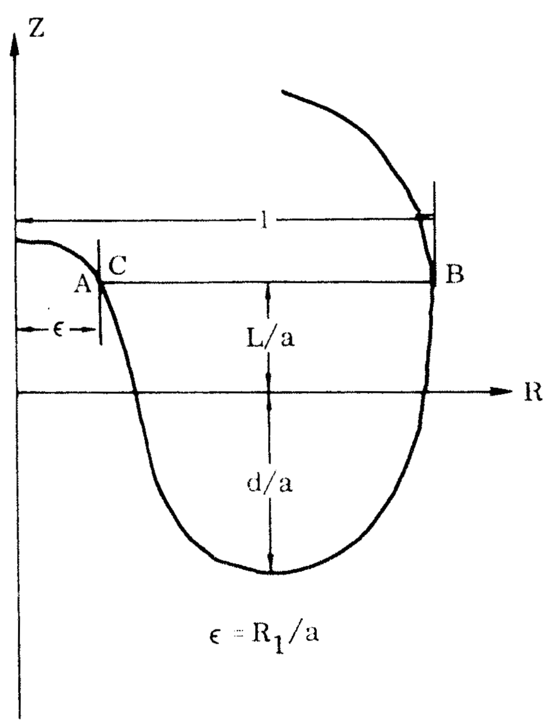


Figure 2-2. Nondimensional Geometry

Following the pattern of Reference 2, let φ be expressed as a linear combination of the eigenfunctions for a shallow tank and eigenfunctions for a deep cylindrical tank, and use a Rayleigh-Ritz technique. Let

$$\lambda = \frac{Ka}{L} \tag{2.21}$$

and substitute

$$\varphi = \sum_{n=1}^{N^*} c_n \varphi_n^* (R, Z) \tag{2.22}$$

into (2.20); differentiate with respect to c_m, and set the result equal to zero to obtain

$$\sum_{n=1}^{N^*} \left[a_{mn} - \lambda b_{mn}\right] c_{mn} = 0, \quad m = 1, 2, \ldots, N^* \tag{2.23}$$

where

$$a_{mn} = \int_{UA} \left[\frac{\partial \phi_m^*}{\partial R} \frac{\partial \phi_n^*}{\partial R} + \frac{\phi_m^* \phi_n^*}{R^2} + \frac{\partial \phi_m^*}{\partial Z} \frac{\partial \phi_n^*}{\partial Z} \right] RdRdZ \tag{2.24}$$

$$b_{mn} = \int_C^B \phi_m^*(R, L/a)\, \phi_n^*(R, L/a)\, RdR \tag{2.25}$$

The set of functions $\phi_n^*(R,Z)$ are chosen, for reasons mentioned previously, as

$$\phi_n^*(R, Z) = \begin{cases} R^{2n-1} & n = 1, 2, \ldots, M < N^* \\ J_1(j_n R)\, e^{j_n(Z-L/a)} & , n = M+1, \ldots, N^* \end{cases} \tag{2.26}$$

where

$$J_1'(j) = 0 \tag{2.27}$$

and the first root of (2.27) is called j_{M+1}.

The b_{mn} are calculated as

$$b_{mn} = \int_\epsilon^1 R^{2m-1} R^{2n-1} RdR = \frac{1 - \epsilon^{2n+2m}}{2n + 2m}, \quad m, n \le M \tag{2.28}$$

$$= \int_\epsilon^1 R\, J_1(j_n R)\, J_1(j_m R)\, dR, \quad m, n > M$$

$$b_{mn} = \frac{j_m \epsilon J_1(j_n \epsilon) J_1{}'(j_m \epsilon) - j_n \epsilon J_1(j_m \epsilon) J_1{}'(j_n \epsilon)}{j_m^2 - j_n^2}, \quad m, n > M, \quad m \neq n \tag{2.29}$$

$$b_{nn} = \frac{(j_n^2 - 1) J_1^2(x) - \left[j_n \epsilon J_1{}'(j_n \epsilon)\right]^2 - (j_n^2 \epsilon^2 - 1) J_1^2(j_n \epsilon)}{2 j_n^2}, \quad n > M \tag{2.30}$$

$$b_{mn} = \int_\epsilon^1 R^{2m} J_1(j_n R)\, dR, \quad m \leq M < n$$

$$= \frac{1}{j_n^2} \Big[(2m - 1) J_1(j_n) - \epsilon^{2m-1} \left\{ (2m - 1) J_1(j_n \epsilon) - j_n \epsilon J_1{}'(j_n \epsilon) \right\}$$

$$- 4m(m - 1)\, b_{(m-1)n} \Big] \tag{2.31}$$

where

$$b_{1n} = \frac{1}{j_n^3} \left[j_n J_1(j_n) - j_n \epsilon J_1(j_n \epsilon) + j_n^2 \epsilon^2 J_1{}'(j_n \epsilon) \right] \tag{2.32}$$

Substitute the value of ϕ_m^* from (2.26) into (2.24) to obtain the following expressions for a_{mn}. Thus, by Green's theorem

$$a_{mn} = \int_{UA} \left[\left\{ (2m - 1)(2n - 1) + 1 \right\} R^{2n+2m-4} \right] RdRdZ, \quad n, m \leq M$$

$$= - \left[4mn - 2(n + m - 1) \right] \oint_C R^{2n+2m-3}\, ZdR \tag{2.33}$$

(see Reference 3, p. 239)

In a similar manner

$$a_{mn} = \int_{UA} \left[(2m-1)\, R^{2m-2} \frac{d}{dR} J_1(j_n R)\, e^{j_n (Z-L/a)} + R^{2m-3} J_1(j_n R)\, e^{j_n (Z-L/a)} \right] RdRdZ, \quad m \le M < n$$

$$= -\frac{1}{j_n} \oint_C \left[(2m-1)\, R^{2m-1} j_n J_1'(j_n R) + R^{2m-2} J_1(j_n R) \right] e^{j_n (Z-L/a)}\, dR \qquad (2.34)$$

and

$$a_{mn} = \int_{UA} \left[R \frac{d}{dR} J_1(j_n R) \frac{d}{dR} J_1(j_m R) + R^{-1} J_1(j_n R)\, J_1(j_m R) + R\, j_n j_m J_1(j_n R)\, J_1(j_m R) \right] e^{j_n (Z-L/a)}\, e^{j_n (Z-L/a)}\, dRdZ, \quad n, m > M$$

$$= -\frac{1}{2} \oint_C \Big\{ R\, j_n j_m J_1'(j_n R)\, J_1'(j_m R) + R^{-1} J_1(j_n R)\, J_1(j_m R) + R\, j_n j_m J_1(j_n R)\, J_1(j_m R) \Big\} \frac{e^{(j_n + j_m)(Z-L/a)}}{j_n + j_m}\, dR \quad n, m > M \qquad (2.35)$$

Once the b_{mn} and a_{mn} are determined, the eigenvalue problem can be solved yielding λ_k and the corresponding c_n^k.

The k^{th} mode is given by

$$\phi^k = \sum_{n=1}^{N^*} c_n^k\, \phi_n^* (R, Z) \qquad (2.36)$$

Its dimensional form is

$$\Phi_k (r, z) = \sum_{n=1}^{N^*} c_n^k \phi_n^* (r/a, z/a). \tag{2.37}$$

It is this value which must be used in the evaluation of (2.6), (2.7) and (2.8). The K_n are given by

$$K_n = \frac{L}{a} \lambda_n \tag{2.38}$$

Thus

$$b_n = \frac{\pi}{V\gamma_n} \int_C^B r^2 \Phi_n (r, L) dr$$

$$= \frac{\pi}{V\gamma_n} \sum_{k=1}^{N^*} c_k^n \int_C^B r^2 \phi_k^* (r/a, L/a) dr$$

$$= \frac{\pi a^3}{V\gamma_n} \sum_{k=1}^{N^*} c_k^n \int_\epsilon^1 R^2 \phi_n^* (R, L/a) dR = \frac{\pi a^3}{V\gamma_n} \sum_{k=1}^{N^*} c_k^n b_{1k} \tag{2.39}$$

Similarly

$$\gamma_n = \frac{\pi L a^2}{V} \sum_{k=1}^{N^*} \sum_{i=1}^{N^*} c_k^n c_i^n b_{ki} \tag{2.40}$$

$$h_n = \frac{2\pi a^3}{V\gamma_n K_n} \sum_{k=1}^{N^*} c_k^n h_k^* \tag{2.41}$$

Where

$$h_n^* = \int_A^B ZR^{2n} dZ \quad n \leq M \tag{2.42}$$

$$h_n^* = \int_A^B ZR J_1 (j_n R) e^{j_n (Z-L/a)} dZ \quad M < n \leq N^* \tag{2.43}$$

Finally consider the evaluation of I_{11} .

$$\rho \int_{UV} (x_2^2 + x_3^2)\, dV = \rho\pi \int_{UA} (r^2 + 2z^2)\, r dr dz$$

$$= - \rho\pi a^5 \oint_C \left(R^3 Z + \frac{2RZ^3}{3} \right) dR \qquad (2.44)$$

$$- 4\rho \int_{UV} x_3^2 dV = \frac{\gamma\rho\pi a^5}{3} \oint_C RZ^3 dR \qquad (2.45)$$

$$2\rho L^2 \int_{US} x_3 \nu_2 \zeta_1 dS = 2\rho\pi L^2 \int_A^B zr \left[A_o r + \sum_{n=1}^{\infty} A_n I_1(\mu_n r) \cos\mu_n (z + d) \right] dz \qquad (2.46)$$

Once the preceding coefficients have been calculated, the parameters required for a mechanical model simulation may be obtained. For a spring-mass system the desired quantities are listed below (see Reference 1, page 3-15).

$$\ell_n = \left[L_1 b_n - L(b_n - h_n) \right] / b_n \qquad (2.47)$$

$$m_n = M \gamma_n b_n^2 K_n \qquad (2.48)$$

$$K_n = \alpha_3 M \gamma_n b_n^2 K_n^2 / L \qquad (2.49)$$

$$m_o = M \left(1 - \sum_{n=1}^{N} \gamma_n b_n^2 K_n \right) \qquad (2.50)$$

$$\ell_o = \frac{\left[L_1 - \sum_{n=1}^{N} \gamma_n b_n K_n \left[L_1 b_n - L (d_n + h_n) \right] \right]}{\left[1 - \sum_{n=1}^{N} \gamma_n b_n^2 K_n \right]} \qquad (2.51)$$

$$I_o = I'_{11} - m_o \ell_o^2 - \sum_{n=1}^{N} m_n \ell_n^2 \qquad (2.52)$$

SECTION 3

ROUTINES OF THE COMPUTER PROGRAM

The routines of the computer program are coded in the FORTRAN IV language, with the exception of parts of a SHARE simultaneous equation routine (SOLVE) and a SHARE eigenvalue routine (RWEG2F).

3.1 DRIVER ROUTINES. The boundary value problem requires solution of the eigenvalues of (2.23). Elements of the a_{mn} and b_{mn} matrices are computed for ten eigenfunctions, of which the first five are polynomials. The integration to evaluate the a_{mn} elements is described in Section 3.2. The Bessel functions needed for the elements of both matrices are computed by a routine described in Section 3.3. The resultant eigenvalues, and their eigenvectors, are found by the Jacobi method using the routine RWEG2F.

Routine RWEG2F solves an eigenvalue problem of the form:

$$\left[A\right]\left[X\right] = \lambda \left[B\right]\left[X\right] \tag{3.1}$$

where the B matrix must be positive definite. Now it was found that the b_{mn} matrix did not satisfy this requirement, though the a_{mn} matrix did. Therefore, letting $\left[A\right] = \left[b_{mn}\right]$ and $\left[B\right] = \left[a_{mn}\right]$, (3.1) is rewritten:

$$\left[b_{mn}\right]\left[X\right] = \lambda' \left[a_{mn}\right]\left[X\right] \tag{3.2}$$

where

$$\lambda' = \frac{1}{\lambda} \tag{3.3}$$

The λ' eigenvalues are found and then inverted to give the desired eigenvalues of (2.23). Each corresponding eigenvector is normalized by its largest element.

For each mode of oscillation entered as input data, the nondimensional frequency (K_n) is computed by (2.38). Force and moment coefficients (γ_n, b_n, and h_n) are computed using (2.39) through (2.43), where the $h_n{}^*$ factor of h_n employs a Bessel function, Section 3.3, and integration, Section 3.2. The above coefficients are then combined as in (2.47) through (2.52) to yield the spring-mass parameters (m_n, ℓ_n, $K_n{}^*/\alpha_3$, m_o, ℓ_o, and I_o). The center of rotation is assumed to be at the bottom of the tank. Therefore, $L_1 = d$ in the expressions for ℓ_n, ℓ_o, and I_o. The $I_{11}{}'$ term of I_o is found by (2.9). The first two terms of $I_{11}{}'$ are found by the integration described in Section 3.2.

Calculation of the third term of I_{11}' requires the solution of a set of simultaneous linear equations, yielding the coefficients, A_n. Equation (2.12) is evaluated at selected points on the rigid tank wall below the undisturbed free surface. The total number of points selected, n, must be such that:

$$(1) \quad \frac{n \pi R_{max}}{h} \leq 40.0$$

where R_{max} is the maximum radius of the rigid tank wall below the undisturbed free surface.

$$(2) \quad n \leq 50$$

Restriction (1) is necessary to prevent the modified Bessel function in the equation from exceeding about 10^{16}, making other terms of the equations completely insignificant. Restriction (2) is set by program core storage limitations. In general, points selected by the program will be end points of segments with equally spaced points between, point B, and point C (if $\epsilon \neq 0$) on Figure 2-2. The resultant set of simultaneous linear equations is solved by the routine SOLVE. Integration, again as in Section 3.2, is then done on (2.46) with the coefficients, A_n.

3.2 INTEGRATION. Integration is performed using Gauss's Quadrature Formula:

$$\int_a^b f(u)\,du = (b-a)\int_{-1/2}^{1/2} \varphi(t)\,dt = \left[S_1\,\varphi(t_1) + S_2\,\varphi(t_2) + \ldots + S_n\,\varphi(t_n)\right](b-a) \qquad (3.4)$$

where n is taken to be 16 and $u = (b-a)\,t + a + \frac{1}{2}(b-a)$.

The sum in (3.4) is taken by adding the smallest term to the next largest and so on, to obtain the most accuracy when f(u) is increasing from a to b.

Two integration routines were written:

1. INTG1 - - for line integrals of the form

$$\int_A^B f(z)\,dz$$

The line integral is taken in a counterclockwise direction along each segment of the rigid tank wall from A to B. (See Figure 2-1.)

$$\int_A^B f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz + \int_{z_2}^{z_3} f(z)\,dz + \ldots + \int_{z_{n-1}}^{z_n} f(z)\,dz \qquad (3.5)$$

where

$z_1, z_2, \ldots, z_{n-1}$ are the z-coordinate of end points of segments entered as input; and z_n is the z-coordinate at the outer radius of the undisturbed free surface.

2. INTG2 -- for line integrals of the form

$$\oint_C f(r)\,dr$$

The line integral is taken in a counterclockwise direction around the fluid of the cross section of the tank as in Figure 2-1.

$$\oint_C f(r)\,dr = \int_A^B f(r)\,dr + \int_B^C f(r)\,dr \tag{3.6}$$

With r-coordinate limits of integration for each segment of the rigid tank wall from A to B and the undisturbed free surface, (3.6) is rewritten

$$\oint_C f(r)\,dr = \left[\int_{r_1}^{r_2} f(r)\,dr + \int_{r_2}^{r_3} f(r)\,dr + \ldots + \int_{r_{n-1}}^{a} f(r)\,dr \right] + \int_a^{R_1} f(r)\,dr \tag{3.7}$$

where

r_1, r_2, r_3, . . . , r_{n-1} are the r-coordinate of end points of segments entered as input; a is the outer radius; and R_1 is the inner radius of the undisturbed free surface.

Equation (3.4) is then applied to each term of (3.5) and (3.7).

3.3 BESSEL FUNCTIONS. Two types of Bessel Functions are computed: the first kind of order one ($J_1(x)$) and its derivative ($J_1'(x)$), and the modified of order one ($I_1(x)$) and its derivative ($I_1'(x)$).

The BESSEL routine computes, in double precision, $J_1(x)$ and $J_1'(x)$ by the ascending series:

$$J_1(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+1}}{k! \quad (k+1)!} \tag{3.8}$$

$$J_1'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{(k!)^2} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{2\,k!\,(k+1)!} \tag{3.9}$$

Each term of the above series is found by multiplying the previous term by the appropriate factor:

$$\frac{(-1)\left(\frac{x}{2}\right)^2}{(k)\ (k+1)} \quad \text{for } J_1\ (x) \text{ and the second part of } J_1{}'\ (x) \tag{3.10}$$

$$\frac{(-1)\left(\frac{x}{2}\right)^2}{k^2} \quad \text{for the first part of } J_1{}'\ (x) \tag{3.11}$$

Since $0 \leq x \leq 14.863588$ in the program, the above technique is sufficiently accurate for the range of x.

Routine PMODBS computes, in double precision, the $I_1\ (x)$ and $I_1{}'\ (x)$ by polynomial approximations, with $t = x/3.75$.

For $-3.75 \leq x \leq 3.75$:

$$\begin{aligned} I_1\ (x) = x\ [\ & 1/2 + 0.87890594\ t^2 \\ & + 0.51498869\ t^4 + 0.15084934\ t^6 \\ & + 0.02658733\ t^8 + 0.00301532\ t^{10} \\ & + 0.00032411\ t^{12}\ ] \end{aligned} \tag{3.12}$$

$$I_1'\ (x) = I_o\ (x) - \frac{I_1(x)}{x} \tag{3.13}$$

where

$$\begin{aligned} I_o\ (x) = 1 & + 3.5156229\ t^2 \\ & + 3.0899424\ t^4 + 1.2067492\ t^6 \\ & + 0.2659732\ t^8 + 0.0360768\ t^{10} \\ & + 0.0045813\ t^{12} \end{aligned} \tag{3.14}$$

For $3.75 \leq x < \infty$:

$$I_1(x) = x^{-1/2} e^{x} \Big[0.39894228 - 0.03988024\, t^{-1}$$

$$- 0.00362018\, t^{-2} + 0.00163801\, t^{-3}$$

$$- 0.01031555\, t^{-4} + 0.02282967\, t^{-5}$$

$$- 0.02895312\, t^{-6} + 0.01787654\, t^{-7}$$

$$- 0.00420059\, t^{-8} \qquad (3.15)$$

$$I_1'(x) = I_0(x) - \frac{I_1'(x)}{x} \qquad (3.16)$$

where

$$I_0(x) = x^{-1/2} e^{x} \Big[0.39894228 + 0.01328592\, t^{-1}$$

$$+ 0.00225319\, t^{-2} - 0.00157565\, t^{-3} + 0.00916281\, t^{-4}$$

$$- 0.02057706\, t^{-5} + 0.02635537\, t^{-6} - 0.01647633\, t^{-7}$$

$$+ 0.00392377\, t^{-8} \Big] \qquad (3.17)$$

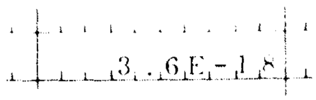
SECTION 4

UTILIZATION OF COMPUTER PROGRAM

4.1 <u>DESCRIPTION OF PROGRAM INPUT</u>. Three sets of numerical data are required as input for a computer run. Sections 4.1.1 through 4.1.3 describe this data, which is subject to limitations that are outlined in Section 4.1.4. All numbers are entered on 80-column input forms. Each digit of a number or a decimal point (if needed) occupies one column. Unless otherwise specified in the following sections, a number must be entered with a decimal point. The decimal point may be anywhere within the columns allocated for that number on the input form.

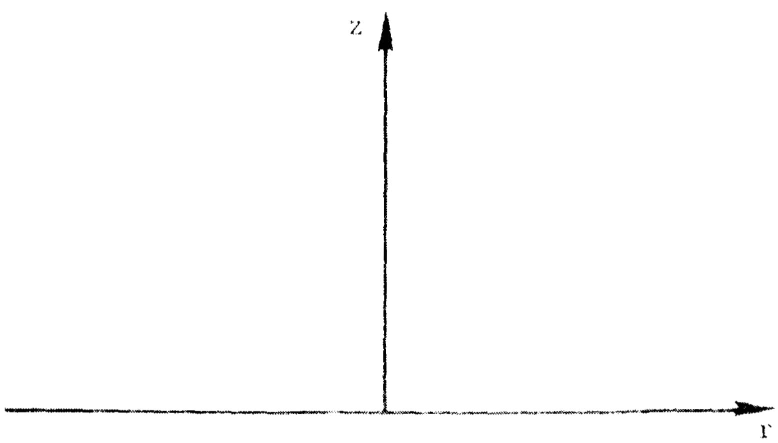
A number that must not have a decimal point must be right-adjusted. That is, all digits must occupy the right-most columns of those allocated for the number: there may be no blank columns between the number and the last column, inclusive.

Extremely large or small numbers may be entered as a coefficient times ten to a power:



Here a decimal point must be in the coefficient, and the last digit of the exponent must occupy the last column of those allocated for the number.

All coordinates required as input must be from a system in which the r-coordinate of the centerline is zero, and the z-coordinate increases upward.



4.1.1 Problem Input. The first line of the input form contains that set of data required to set up the problem for the computer.

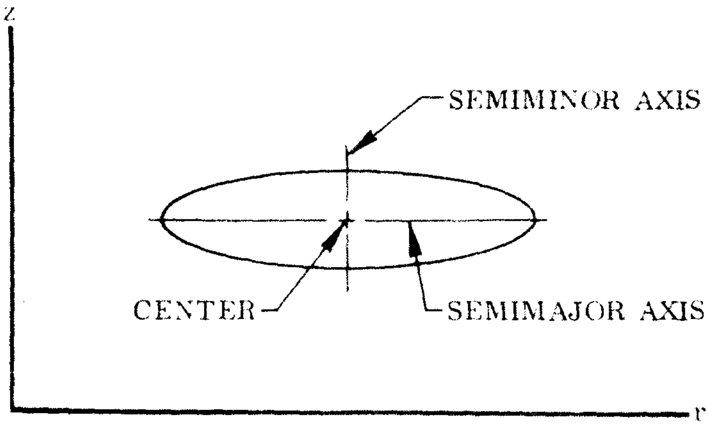
Columns	Data
1 through 10	Enter the number of segments that describe the rigid tank wall. The number must be right-adjusted, without a decimal point.
11 through 20	Enter the number of modes of oscillation desired. The number must be right-adjusted, without a decimal point.
21 through 30	Enter the liquid density (in pounds per cubic foot).
31 through 40	Enter the r-coordinate (in inches) of the beginning of the segments that describe the tank.
41 through 50	Enter the z-coordinate (in inches) of the beginning of the segments that describe the tank.

4.1.2 Tank Geometry Input. Beginning with the second line of the input form, a line of the data below is required for each segment that describes the rigid tank wall. Segments must be ordered in a continuous, counterclockwise path around the tank. The number of these lines of input must equal the number entered in columns 1 through 10 of the first line (i.e. number of segments).

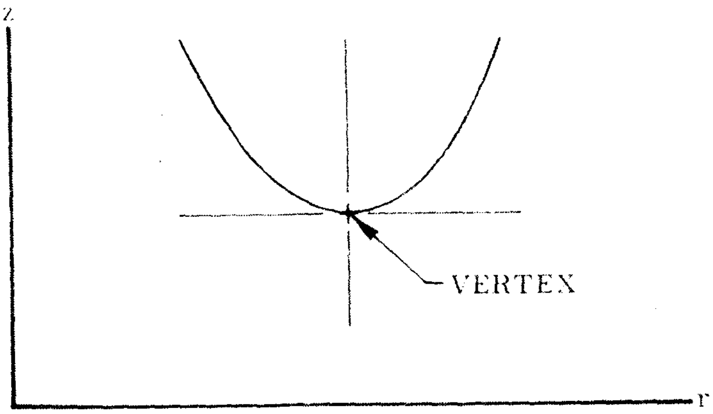
Columns	Data
1 through 10	Enter the r-coordinate (in inches) of the end of the particular segment, having proceeded along it in a counterclockwise direction around the tank.
11 through 20	Enter the z-coordinate (in inches) of the end of the particular segment, having proceeded along it in a counterclockwise direction around the tank.
21 through 30	Leave blank for a straight line segment; otherwise, enter as follows, depending on the type of segment: Elliptical. Enter the r-coordinate (in inches) of the center of the ellipse. Circular. Enter the r-coordinate (in inches) of the center of the circle.

Columns	Data
21 through 30 (Contd)	Parabolic. Enter the r-coordinate (in inches) of the vertex of the parabola.
31 through 40	Leave blank for a straight line segment; otherwise, enter as follows, depending on the type of segment: Elliptical. Enter the z-coordinate (in inches) of the center of the ellipse. Circular. Enter the z-coordinate (in inches) of the center of the circle. Parabolic. Enter the z-coordinate (in inches) of the vertex of the parabola.
41 through 50	Leave blank for a straight line segment; otherwise enter, depending on the type of the segment, as follows: Elliptical. Enter the semimajor axis (in inches) of the ellipse. Circular. Enter the radius (in inches) of the circle. Parabolic. Enter the directrix (in inches) of the parabola.
51 through 60	Leave blank for a parabolic or straight line segment; otherwise enter, depending on the type of the segment, as follows: Elliptical. Enter the semiminor axis (in inches) of the ellipse. Circular. Enter the radius (in inches) of the circle.
61 through 70	Leave blank for a circular or straight line segment; enter the amount of counterclockwise rotation from the norm (see below) of the ellipse if the segment is elliptical, or of the parabola if the segment is parabolic. It may be left blank for an angle of zero degrees.

Normal Position of an Ellipse



Normal Position of a Parabola



4.1.3 Case Input. Each case to be run requires a line of input as listed below. Any number of cases may be run.

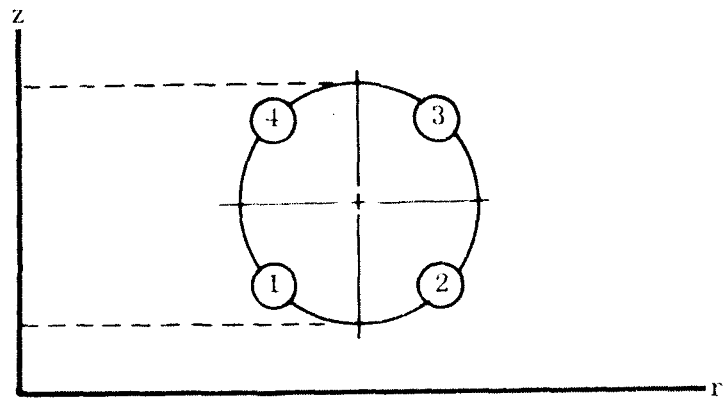
Columns	Data
1 through 10	Enter the z-coordinate (in inches) of the liquid level in the tank.

4.1.4 Limitations on Input. The following restrictions must be applied to the input data.

1. There can be no more than 50 segments.

2. Each segment must be such that for a given radius, there is only a single value of the height.

i.e. A toroid must be described by four segments of the same circle:



3. The liquid level must not indicate a completely full tank.

4. There can be no more than ten modes of oscillation.

5. Do not enter the centerline segment unless it is, in reality, a part of the rigid tank wall.

4.2 DESCRIPTION OF PROGRAM OUTPUT. Printed output on a computer run consists of the following:

1. Tank Geometry. A complete definition of all segments entered as input.

And for each liquid level case:

2. Liquid Level. The z-coordinate (in inches) of the undisturbed free surface in the tank.

3. Mass of Liquid. The mass (in pounds) of the liquid contained in the tank.

4. Center of Gravity. The z-coordinate (in inches) of the center of gravity of the liquid.

5. I_{11}' and its four terms (in lb-in.^{2}).

6. Eigenvalue Statistics. These, for each mode requested, are the eigenvalue and the eigenvector and its normalizing factor.

7. For each mode requested, the coefficients (K_n, γ_n, h_n, and h_n) of the force and moment equations.

8. For each mode requested, the following parameters for the spring-mass analogy:

   m_n (in pounds)

   ℓ_n (in inches)

   K_n^*/α_3 (in lb/in.)

9. Also, the following spring-mass parameters:

   m_o (in pounds)

   ℓ_o (in inches)

   I_o (in lb-in.^{2})

   These parameters are a summation for all modes of oscillation entered as input data, and printed out in item 8 above. Therefore, if a one-mode analysis is to be used, only one mode should be entered as input data.

4.3 SETUP FOR A COMPUTER RUN. A run on the computer requires a card deck containing system control cards, the program binary deck, and input data cards. The input data is punched on cards from the 80-column coding form described in Section 4.1. Arrangement of the deck will be as shown in Figure 4-1.

The program is designed to be run under the IBSYS monitor system. For the IBM 7094 computer, execution time estimates should be based on about 10 millihours per segment per liquid level case.

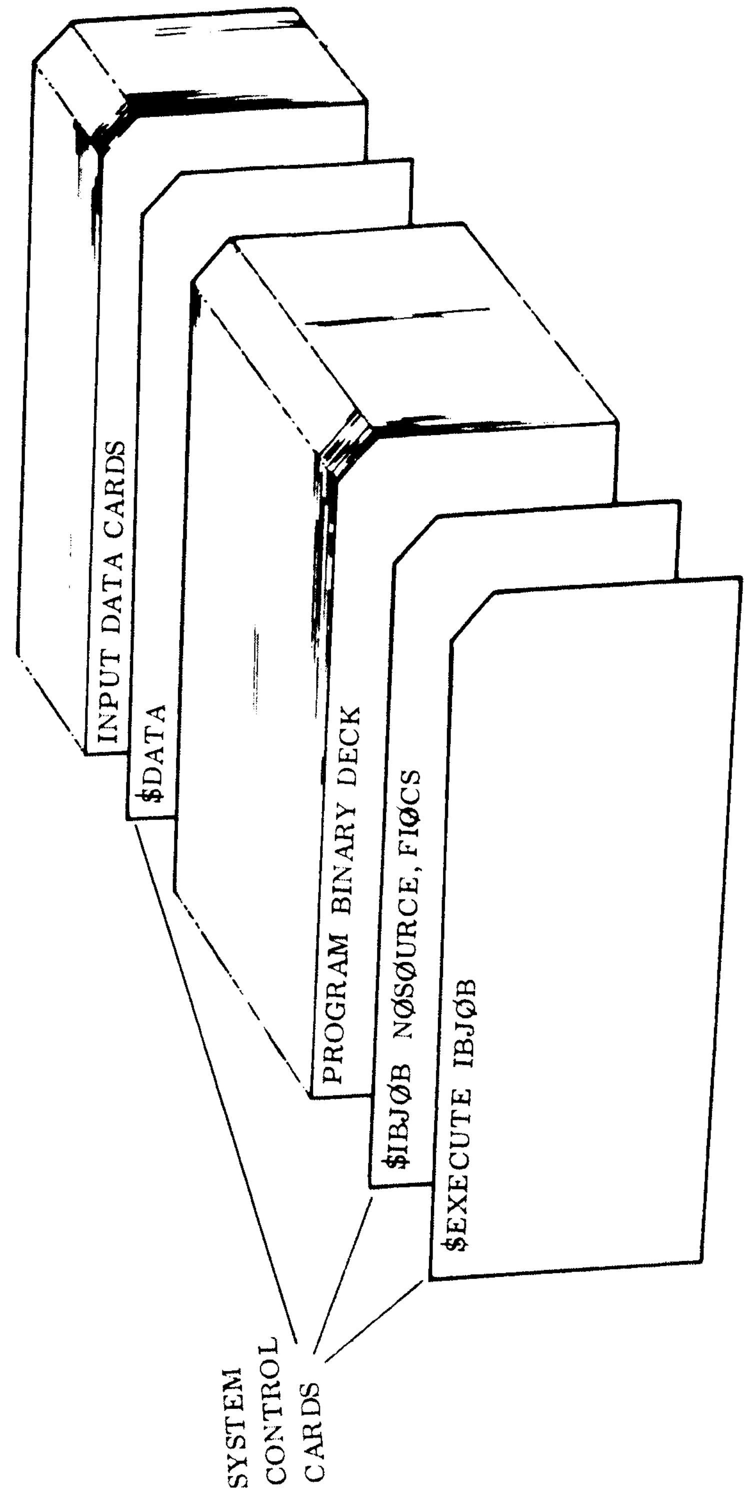


Figure 4-1. Card Deck Arrangement

SECTION 5

CHECK-OUT OF THE DIGITAL PROGRAM

The output of the computer program was checked against the experimental and (if possible) theoretical data available for tanks comprised of spheres, toroids, oblate spheroids, right circular cylinders, and concentric right circular cylinders.

The volume and center of gravity of the fluid as well as the first two integrals in (2.9) may be easily checked out by hand-calculation for any tank configuration. The nondimensional frequencies, K_n, may be checked for any tank configuration that has analytical or experimental results available. The operations involved in finding these five quantities constitute the major portion of the program. The rest of the program simply manipulates these terms. Thus if these five quantities are calculated properly and the multiplications and additions required for the evaluation of the rest of the terms are checked out for a particular case, one is assured that the entire program is working properly. The only term which must be checked out in a different manner is the third term of (2.9). The boundary value problem for ψ_1 (see Reference 1, Equations 3.10 and 3.11) may be solved explicitly for specialized cases only. The shapes checked out here were for a right circular cylinder at various liquid levels. For a cylinder of radius 60 inches and liquid level 50 inches the calculated value was 1.36×10^8, the computer obtained 1.3779×10^8. For a liquid level of 240 inches, note the comparison of I_0 on page 5-6. Since ψ_1 is approximated using Bessel functions, the routine is not very accurate for elliptical or spherical tanks. However if the tank is "close to cylindrical" in shape, the output is good. That is, for the two Atlas tanks the routine calculates I_0 properly. A rough check on this term for any tank shape is to compare the four terms comprising I_{11}'. If the third term is of different magnitude than the other three, it is not usable. The approximation then used is that I_{11}' equals the sum of the first and fourth terms.

The routines for a parabolic section were checked by using a cylindrical tank of radius 60 inches with a parabolic bottom. The parabola had a directrix of 5000 and thus approximated a straight line. The difference between the output for this shape and the output for a cylinder with a flat bottom were different only in the third significant digit.

Figure 5-1 shows the comparison of the lowest three frequencies of a spherical tank (radius of 60 inches) with a graph from Reference 3. In this figure, where R is the radius of the sphere and h the liquid height, note that the lowest frequencies fall on the dark line while the next two sets of frequencies appear at times to be closer to the experimental rather than the theoretical results.

The check for the lowest frequencies for an oblate spheroidal tank is given in Figure 5-2. The tank used in this program had a semimajor axis of 26.26 and a semiminor axis of 15.13 and was compared with the results in Reference 4.

The second linear mode shown in Figure 5-2 is the third mode in Reference 4 since the second mode in Reference 4 is due to nonlinear effects.

The comparison of frequencies for a toroidal tank, with results from Reference 6, is shown in Figure 5-3. Note the good agreement for the two lowest modes.

The frequencies for a ring tank, inner radius 10 inches and outer radius 60 inches, were checked against the analytical solution for a liquid height of 50 inches. (Use must be made of the tables in Reference 7.) The results appear in the table below.

	ANALYTICAL	COMPUTER
K_1	0.6618	0.6646
K_2	2.090	2.091
K_3	3.481	3.496
K_4	4.956	4.933

The program's entire output may also be checked out against the analytical solutions (Reference 1, p. 5-8) for a right circular cylindrical tank. Because of a difference in normalization in the computer program and the analytical work, the parameters γ_n, b_n, and h_n may not be directly compared. However the combinations $\gamma_n b_n^2$, $b_n h_n$, and $\gamma_n b_n h_n$ which appear in the spring-mass parameters are independent of the normalization. Thus, if the spring-mass parameters check out (using results of Reference 1, p. 5-8) the γ_n, b_n, h_n are being calculated properly. If desired, however, these combinations could be checked out by themselves. The comparison of parameters is given in the following table for a tank of radius 60 inches and height of 500 inches, filled to a depth of 240 inches with a liquid having a density of 1.

	ANALYTICAL RESULTS	COMPUTER OUTPUT
c.g.	120.	120.
K_1	3.682	3.6823
K_2	10.663	10.663
K_3	17.073	17.073
γ_1	3.729×10^4	0.5969
b_1	9.0965×10^{-4}	0.71901
h_1	13.259×10^{-4}	1.0486
ℓ_1	174.91	175.0
m_1	0.30837×10^6	0.30841×10^6
K_1^*/α_3	9.4628×10^3	9.4639×10^3
m_o	2.4059×10^6	2.4059×10^6
ℓ_o	112.96	112.95
M	2.7143×10^6	2.7143×10^6
I_o	0.14973×10^{11}	0.15081×10^{11}

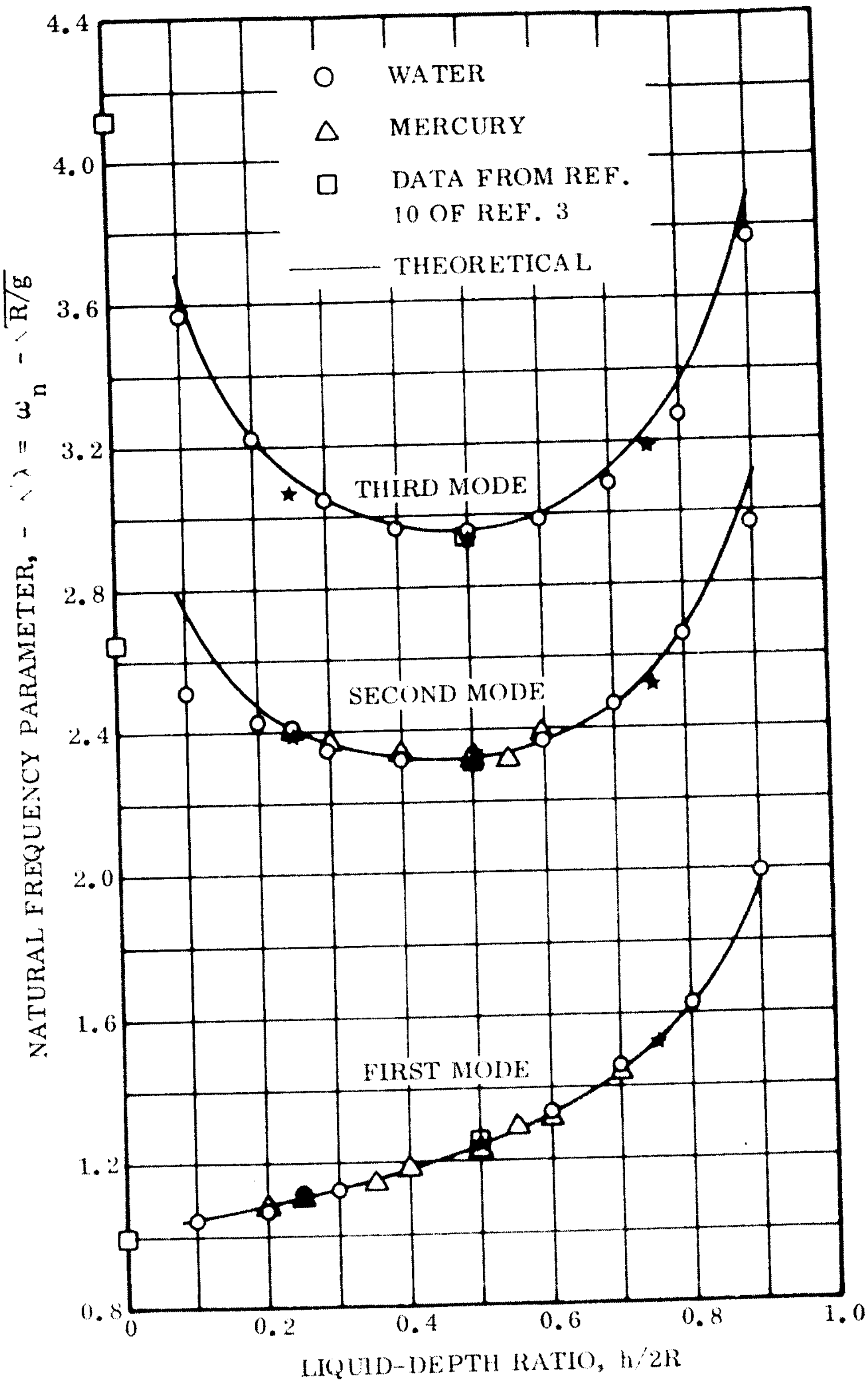


Figure 5-1. Theoretical and Experimental Values of Natural Frequency Parameters for the First Three Modes

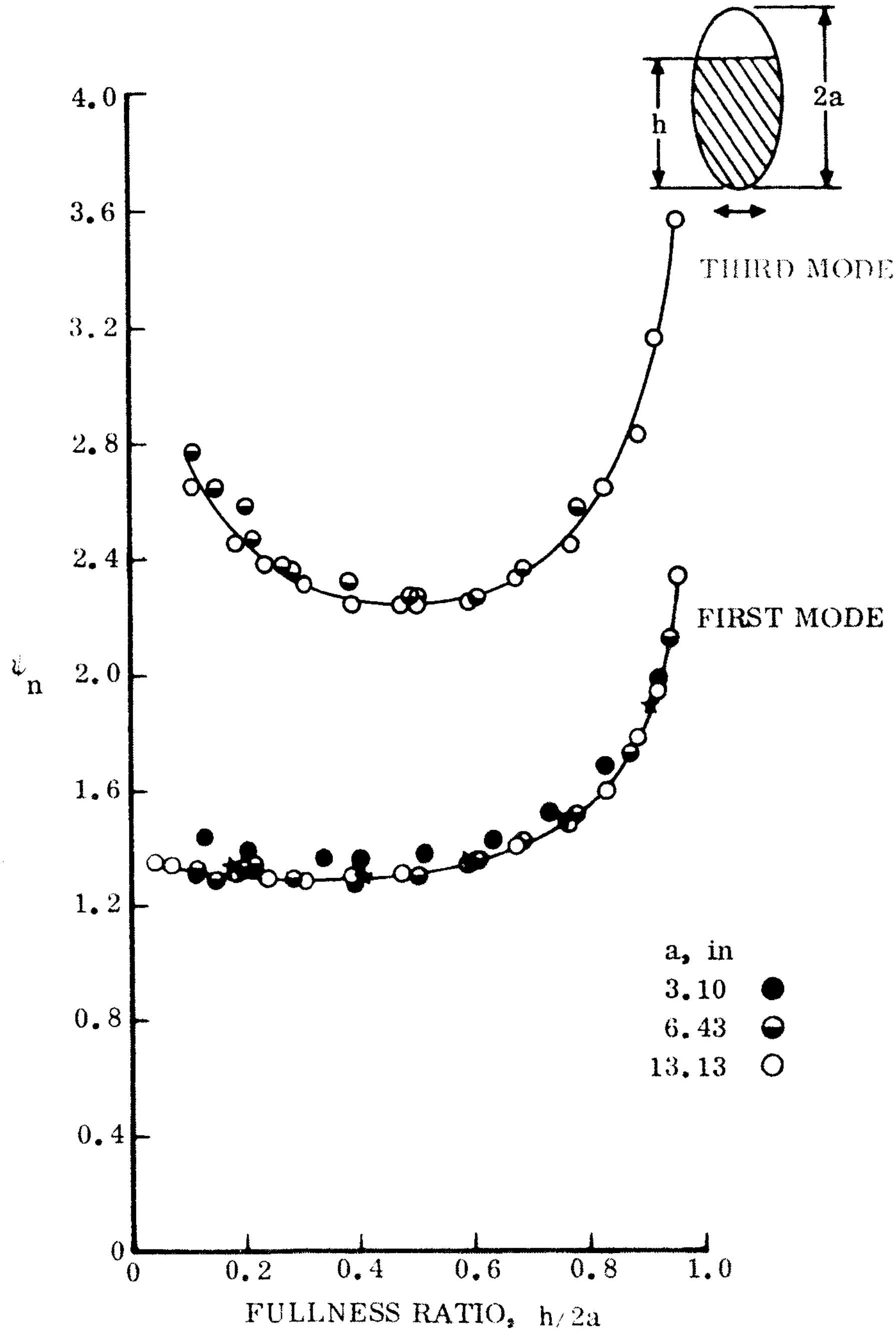


Figure 5-2. Variation of Frequency Parameter $\left(\psi_n = \omega_n \sqrt{\frac{b}{g}}\right)$ with Fullness Ratio for Liquid in Spheroids; Transverse Orientation; b/a = 0.50

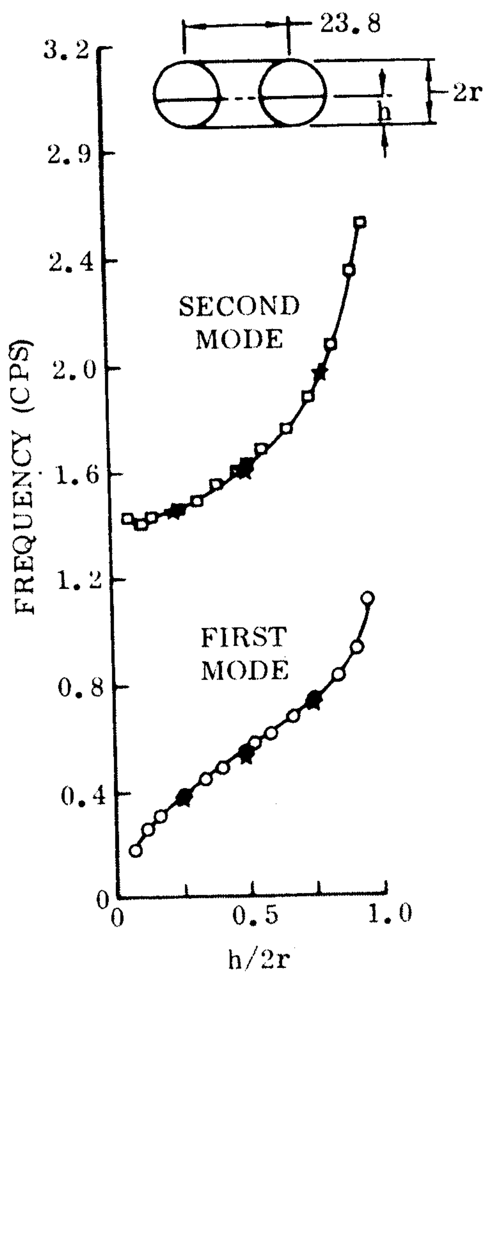


Figure 5-3. Variation of Liquid Natural Frequency with Fullness Ratio for a Horizontal Toroidal Tank; $r = 5.9$

SECTION 6

REFERENCES

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2. H. R. Lawrence, C. J. Wang, and R. B. Reddy, "Variational Solution of Fuel Sloshing Modes," Jet Propulsion Vol. 28 No. 11, November 1958, pp. 729-736.

3. A. J. Stofan and A. A. Armstead, Analytical and Experimental Investigations of Forces and Frequencies Resulting from Liquid Sloshing in a Spherical Tank, NASA TN D-1281, Cleveland, 1962.

4. H. W. Leonard and W. C. Walton, Jr., An Investigation of the Natural Frequencies and Mode Shapes of Liquids in Oblate Spheroidal Tanks, NASA TN D-904, Langley Field, 1961.

5. Rayleigh, Theory of Sound.

6. J. L. McCarty, H. W. Leonard, and W. C. Walton, Jr., Experimental Investigation of the Natural Frequencies of Liquids in Toroidal Tanks, NASA TN D-531, Langley Field, 1960.

7. H. F. Bauer, Theory of the Fluid Oscillations in a Circular Cylindrical Ring Tank Partially Filled with Liquid, NASA TN D-557, Huntsville, 1960.

APPENDIX

SAMPLE CASE

FOR

ATLAS TANK

650 PUNCH FORM — GENERAL PURPOSE

Atlas Lox Tank with Ring Baffles

3712

ENGINEER D. O. Lomen

ANALYST R. E. Barnes

PAGE 1 OF 1

DATE 14 Jan 1965

W-1	W-2	W-3	W-4	W-5	W-6	W-7	W-8
7.	3.	70.0	0.0	43.5			
60.0	0.0	0.0	0.0	60.0	43.5		
60.0	46.0						
56.0	46.0						
56.0	47.0						
60.0	47.0						
60.0	75.0						
0.0	75.0						
55.0							
47.0							

TANK GEOMETRY

SEGMENT	TYPE	FROM	TO	CHARACTERISTICS
1	ELLIPTICAL	R = 0. INCHES Z = 43.50 INCHES	R = 60.00 INCHES Z = 0. INCHES	SEMI-MAJOR AXIS = 60.00 INCHES ROTATED BY SEMI-MINOR AXIS = 43.50 INCHES -0. DEGREES CENTER AT R = 0. INCHES AND Z = 0. INCHES
2	STRAIGHT LINE	R = 60.00 INCHES Z = 0. INCHES	R = 60.00 INCHES Z = 46.00 INCHES	
3	STRAIGHT LINE	R = 60.00 INCHES Z = 46.00 INCHES	R = 56.00 INCHES Z = 46.00 INCHES	
4	STRAIGHT LINE	R = 56.00 INCHES Z = 46.00 INCHES	R = 56.00 INCHES Z = 47.00 INCHES	
5	STRAIGHT LINE	R = 56.00 INCHES Z = 47.00 INCHES	R = 60.00 INCHES Z = 47.00 INCHES	
6	STRAIGHT LINE	R = 60.00 INCHES Z = 47.00 INCHES	R = 60.00 INCHES Z = 75.00 INCHES	
7	STRAIGHT LINE	R = 60.00 INCHES Z = 75.00 INCHES	R = 0. INCHES Z = 75.00 INCHES	

FLUID LEVEL = 55.00 INCHES

MASS OF FLUID = 0.11853E 05 POUNDS

CENTER OF GRAVITY AT 33.95 INCHES

IIIP = 0.38326E 08 WHERE THE TERMS ARE 0.14347E 05 -0.13479E 07 0.10413E 08 0.18913E 08

EIGENVALUE = 0.10399E 01 EIGENVECTOR NORMALIZING FACTOR = -0.40130E 01

EIGENVECTOR
0.58747E 00
-0.89036E 00
0.10000E 01
-0.64695E 00
0.17990E-00
0.27145E-00
-0.27494E-01
0.17998E-02
-0.52338E-03
0.58549E-03

EIGENVALUE = 0.49273E 01 EIGENVECTOR NORMALIZING FACTOR = -0.12465E 02

EIGENVECTOR
0.56983E-01
-0.52531E 00
0.10000E 01
-0.83842E 00
0.26189E-00
-0.29865E-02
0.12824E-00
-0.30491E-02
0.19135E-02
0.33291E-03

EIGENVALUE = 0.83918E 01 EIGENVECTOR NORMALIZING FACTOR = -0.74354E 01

EIGENVECTOR
0.56994E-01
-0.42354E-00
0.10000E 01
-0.97686E 00
0.34110E-00
0.29292E-02
-0.14389E-01
0.23203E-00
0.20308E-02
0.25985E-02

COEFFICIENTS	MODE	KN	GAMMAN	BN	FN
	1	0.26087E-00	0.34716E-01	0.80704E 01	0.31913E 01
	2	0.12363E 01	0.76501E-03	-0.76267E 01	-0.39165E 01
	3	0.21055E 01	0.12542E-02	0.17448E 01	0.17941E 01

SPRING-MASS PARAMETERS	MODE	MN	LN	KNSTAR/ALPHA(3)
	1	0.69918E 04	0.30845E 02	0.12117E 03
	2	0.64778E 03	0.32623E 02	0.33197E 02
	3	0.95285E 02	0.40371E 02	0.13327E 02