A RE-EVALUATION OF THE THEORY FOR THE HYDROSTATIC FIGURE OF THE EARTH[1]

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ABSTRACT

The equations of the classical theory for the hydrostatic figure of the earth are modified in order to make their development independent of the external potential theory. These modifications are essential in order to study any discrepancy which may exist between the actual flattening f and the hydrostatic flattening f_h. An expression is obtained giving the hydrostatic flattening explicitly in terms of other parameters, which seems to have the advantage of easy numerical manipulation. The results obtained from the modified equations disagree with previous findings, giving a hydrostatic flattening of 1/296.70 ± 0.05 in contrast to the value of 1/299.86 ± 0.05 as found from the classical hydrostatic equations of *de Sitter* or the recently proposed method of *Jeffreys*. These results, if true, lead to the conclusion that the hydrostatic flattening of the earth is greater than its actual flattening, i.e., $f_h^{-1} - f^{-1} < 0$, in contradiction to the currently prevalent opinion that $f_h^{-1} - f^{-1} > 0$. Some difficulty is likely to arise with certain of the geophysical hypotheses that are constructed on the basis of the currently accepted belief that the actual flattening of the earth is greater than the corresponding hydrostatic flattening for it.

INTRODUCTION

The hydrostatic theory of the earth, in its present form, was first developed by *Clairaut* (1743). *Radau* (1885) simplified Clairaut's differential equation by making an important substitution, and the theory has since been used primarily in estimating the flattening of the earth spheroid, on the assumption that the value of flattening obtained from it would give the best available approximation to the real flattening of the earth spheroid. Doubt was cast on this assumption, however, when *Tisserand* (1891) and *Poincaré* (1910), on the basis of the hydrostatic theory, derived $f_h^{-1} = 297.3$, in contrast to the then-accepted value of $f^{-1} = 293.5$ which A. R. Clarke had obtained from arc measurements in 1880 (*Jeffreys*, 1962, p. 152). Later, *Hayford* (1909, 1910), using more extensive data, obtained $f^{-1} = 297.0$, which was quite consistent with the value obtained from hydrostatic theory within the limits of accuracy. This reaffirmed the then-prevalent opinion that the hydrostatic theory method for computing the flattening of the earth was more accurate than any other available at that time (*de Sitter*, 1924, 1938; *Jeffreys*, 1948, 1962; *Bullard*, 1948; *Jones*, 1954) and was an interesting finding--in view of the apparent departure of the shape of the physical surface of the earth from that which hydrostatic conditions would require (*Jeffreys*, 1952, p. 145). However, it was based on the fact that since hydrostatic equilibrium probably existed throughout the earth's interior except for the upper crust (which forms only a negligible part of the whole earth), the assumption of hydrostatic equilibrium for the whole earth was not unjustified. This belief persisted until artificial satellites made possible the direct determination of the geopotential coefficient J, from which the flattening f of the best-fitting spheroid could be accurately computed using the external potential theory, without recourse either to geodetic or gravity data or to the hydrostatic equilibrium theory. The currently accepted value of f computed in this way is 1/298.25.

At about the same time *Henriksen* (1960), using *de Sitter's* (1924) equations, found $f_h^{-1} = 300.0$ corresponding to[2] $J = 1622.4 \times 10^{-6}$. Later, *Jeffreys* (1964), using $J = (1624.17 \pm 0.05)10^{-6}$, as obtained by *King-Hele, Cook,* and *Rees* (1963), found $f_h^{-1} = 299.67 \pm 0.05$, confirming the discrepancy between f and f_h. In a previous paper (*Khan,* 1967a) I used the improved values of the constant of precession (*Rabe,* 1950) and of moon-earth mass ratio (1/81.303) and obtained $f_h^{-1} = 299.86 \pm 0.05$ by employing *Jeffreys'* (1964) ~~and~~ approach, which has both the advantages of speed and simplicity. Although $f_h^{-1} - f^{-1}$ is greater than the uncertainty in the

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[2] Actually it is erroneous to use the satellite-determined J in the hydrostatic equations and the problem of hydrostatic equilibrium of the earth can be and should be solved with any knowledge of J. However, this possibility does not seem to have been explored and the trend of using a pre-determined J (either from satellite motion as in post-satellite times or from internal density distribution as in pre-satellite times) in the solution of hydrostatic equations persists up till now. The results reported in this paper are obtained accepting this trend as correct (which it is not). The flaws of this method will be discussed in a subsequent paper (*Khan,* 1967b) in which a correct solution of the problem will be given.

determination of either f or f_h, yet in magnitude it is so small that it is worthwhile to ensure that the discrepancy is genuine and that it does not arise merely from an error in some of the parameters involved in the calculations or even from some undetected flaw in the hydrostatic theory itself, before attempting to construct the possible geophysical hypotheses to explain it. For this reason I have re-evaluated the hydrostatic theory and give first a brief summary of de Sitter's method followed by a very brief account of *Henriksen's* (1960) post-satellite approach. When *de Sitter* (1924) worked out his hydrostatic equations, the geopotential coefficient J was not known accurately and he eliminated it from the right hand side of his equations by means of a well-known relation derived from external potential theory, hopefully presuming that there was no significant difference between the value of J for the real earth and the hydrostatic J. *Henriksen* (1960) seems to have used these equations without either detecting this fact or pointing it out. The satellite determination of J has proved the above assumption to be incorrect, and its elimination is no longer desirable (as explained later) because it tends to introduce an implicit $f_h = f$ (in case of the presently adopted approach to compute f_h). For this reason I have rewritten de Sitter's equation so that J appears explicitly in them, rather than being eliminated as was done earlier (*Darwin*, 1900; *de Sitter*, 1924). I call these equations 'the modified equations of hydrostatic theory' in this paper. As will be shown later, the use of these modified equations has led to a value of hydrostatic flattening which differs considerably from the previous results. These modified equations are solved simultaneously in order to obtain an explicit expression for f_h. It is important to note that usually the hydrostatic flattening is defined as the flattening which the earth, with its actual values of geopotential coefficient J and the dynamical flattening H, would have if it were in hydrostatic equilibrium. Since H and J for the real earth are not compatible on the basis of hydrostatic equilibrium, this situation is physically impossible and, consequently, hydrostatic flattening is a hypothetical number. It is in this sense that the term 'hydrostatic flattening' has been used throughout this paper. This is obviously not a realistic definition of the hydrostatic flattening in terms of the actual situation and the mathematical realities but this point will not be discussed any further in this paper lest it lead to confusion. I will discuss this aspect of the problem in detail in another paper (*Khan*, 1967b).

DYNAMICAL FLATTENING H

As in my previous paper (*Khan*, 1967a), I use *Rabe's* (1950) value for the constant of precession which he obtained from the 1950 opposition of Eros and the recently improved value of the moon-earth mass ratio obtained from Ranger-shot data. Taking

$$\text{moon-earth mass ratio} = 1/81.303$$

and

$$\text{constant of precession} = 5493.791'' \text{ (1900 epoch)},$$

the value of H can be calculated using well-known formulas (*de Sitter*, 1938). In Table 1 my value for H is compared with those of other investigators. It may be pointed out at this stage that the slight difference in my value of H and that of others, does not critically affect the value of the hydrostatic flattening and hence does not change in any way the nature of the discussion to follow.

CLASSICAL EQUATIONS OF HYDROSTATIC THEORY

Pre-satellite method. The classical hydrostatic theory is given in detail in a number of papers (*Darwin*, 1900; *de Sitter*, 1924, 1938; *Jeffreys*, 1953, 1962, 1964; *Bullard*, 1948; *Jones*, 1954). We, therefore, mention here only those points which are necessary for convenience of reference without giving a full account of the theory.

We can readily obtain the following relation from the external potential theory.

$$J = f - \frac{1}{2} m - \frac{1}{2} f^2 + \frac{1}{7} m f + \frac{4}{7} \kappa \tag{1}$$

where

$$m = \frac{\omega^2 r_m^3}{GM}$$

f = actual flattening of the best-fitting spheroid (Such a spheroid is sometimes called normal surface.)

κ = a constant indicating the departure of the spheroidal surface from the corresponding ellipsoidal surface

J = $3/2J_2$ where J_2 is the coefficient of the second harmonic term in the spherical harmonic expansion of the geopotential

ω = the angular velocity and

r_m = the mean radius of the earth.

Equation (1) gives f if J and κ are known. However, since direct determinations of the precise value of J were not available in pre-satellite times, J had to be found by some other means. The earth was assumed to be in hydrostatic equilibrium so that the resulting outer surface, often referred to as the "ideal surface," had the same flattening as the "normal surface," i.e., $f_h = f$. One could then use the well-known relation

$$J = q\ H \text{ where } q = \frac{3}{2}\ \frac{C}{Ma_e^2}\ . \tag{2}$$

In the above formula C denotes the polar moment of inertia, M the mass and a_e the equatorial radius of the earth.

H could be calculated from the constant of precession using formulas given by *de Sitter* (1938).

By considering the expansion of potential at a point inside the earth and then assuming that the surfaces of equal density coincide with the equipotential surfaces (which is the condition of hydrostatic equilibrium), the equation for q could be written as (*de Sitter*, 1924):

$$q = 3\left(1 - \frac{2}{3} f_h\right) \int_0^1 \delta\ \beta^4\ d\beta + \frac{2}{3} J \tag{3}$$

which, with the help of equation (1) could be rewritten[3] as

$$q = 1 - \frac{1}{3} m - 2\left(1 - \frac{2}{3} f_h\right) \int_0^1 D\ \beta^4\ d\beta \tag{4}$$

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[3] Note the introduction of an implicit $f_h = f$ if the observed value of J is used to compute f_h.

and finally transformed into

$$q = 1 - \frac{1}{3} m - 2 \left(1 - \frac{2}{3} f_h \right) \frac{\sqrt{1 + \eta_s}}{1 + \lambda_s} \tag{5}$$

where use was made of the relation

$$\int_0^1 D \, \beta^4 \, d\beta = \frac{1}{5} \, \frac{\sqrt{1 + \eta_s}}{1 + \lambda_s} \tag{6}$$

In the above expressions, δ is the density expressed in terms of the mean density as a unit, D the mean density within the surface β expressed in the same units, and η_s and λ_s are the surface values of the parameters η and λ which depend upon the internal density distribution of the earth.

Equation (5) will give q if η_s and λ_s are known. The considerations which led to equation (3), also gave a boundary condition for the determination of η_s. This can be written as

$$\eta_s \, f' \left(1 + \frac{4}{7} f_h - \frac{4}{21} m \right) = 3f' \left(1 + \frac{2}{7} f_h \right) - 5J \left(1 + \frac{2}{3} f_h \right) \tag{7}$$

which with the help of equation (1) can be transformed into[4]

$$\eta_s \, f' = \frac{5}{2} m - 2f' + \frac{10}{21} m^2 + \frac{4}{7} f_h^2 - \frac{6}{7} m \, f_h \tag{8}$$

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[4] Note again the introduction of an implicit $f_h = f$ if observed value of J is used to compute f_h.

where

$$f' = f_h - \frac{5}{42} f_h^2 + \frac{4}{7} \kappa$$

Note that the transformation of equation (3) into (5) and of (7) into (8) is only possible through the use of equation (1).

Equations (5) and (8) are the same as *de Sitter's* equations numbered (21) and (22) in his 1924 paper. In the process of deriving equation (8) de Sitter made an algebraic error (*Jeffreys*, 1953) but his final result[5] [stated in equation (8)] does not appear to have been affected by it.

In the pre-satellite procedure, equation (8) was used to find the best value of η_s by an iteration procedure (in conjunction with some other equations not given here, see *de Sitter*, 1924; *Bullard*, 1948; or *Jeffreys*, 1962, 1964 for a complete account). The corresponding value of λ_s was obtained from equation (6). These values were used to compute q from equation (5). J could then be computed from equation (2) and f_h from equation (1). Thus derived, f_h was then regarded as the best available approximation to the real flattening f. This method is discussed in further detail from a slightly different point of view in another paper (*Khan*, 1967b).

<u>Classical post-satellite method</u>. Using the satellite determination of J, *Henriksen* (1960) and *O'Keefe* (1960) were the first to compute hydrostatic

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[5] He has also omitted the term $\frac{4}{9} f^2$ in deriving equation (5) but this term ultimately becomes $O(f^3)$ by a subsequent multiplication and drops out anyway. However, it is desirable to include it to show where it really drops out. Note that none of the errors seems to be important to our discussion.

flattening. It appears they found q from equation (2), put $\lambda_s = 0$ in equation (5), substituted the value of η_s found therefrom in equation (8) and solved the resulting quadratic in f_h. *Jeffreys* (1964) also developed an excellent numerical procedure for computing f_h using a simplified density model. Details of these methods are available in the above-quoted references.

MODIFIED EQUATIONS OF THE HYDROSTATIC THEORY

Since *de Sitter* (1924) had derived equations (5) and (8) from equations (3) and (7) respectively [by eliminating J with the help of equation (1)], *Henriksen* (1960) appears to have used equation (1) indirectly in the computation of f_h. Note that there is nothing wrong with using equation (1) in hydrostatic theory if the true hydrostatic value of J (*Khan*, 1967b) is used in the computations, but since the observed value of J is usually used in equations (5) and (8) to find the corresponding f_h, and since the observed value of J in equation (1) relates the real flattening to the observed J (in case of an earth not in hydrostatic equilibrium), it is apparent that equations (5) and (8) contain an implicit $f_h = f$.

Jeffreys' (1964) numerical approach is valid if hydrostatic value of J is used but is not valid when non-hydrostatic J is used.

Since in our present method, in order to study any discrepancy between hydrostatic flattening f_h and actual flattening f, it is important to distinguish between f and f_h, we must correct for any intermixing of these two in *de Sitter's* equations (such as the one pointed out above) before using them to obtain numerical results. For this purpose I have modified *de Sitter's* equations, expressing them explicitly in terms of J, thus completely avoiding the use of any relation derived from the external potential theory.

To get the modified equations, we write equation (3), correct to the second order, in the form

$$q = 3\left(1 - \frac{2}{3} f_h + \frac{4}{9} f_h^2\right)\int_0^1 \delta\, \beta^4\, d\beta + \frac{2}{3} J \qquad (9)$$

As stated before, if D is the mean density within the surface β expressed in terms of the mean density as a unit,

$$D = \frac{3}{\beta^3}\int_0^\beta \delta\, \beta^2\, d\beta$$

which on differentiation with respect to β gives

$$\frac{-\beta}{D}\frac{dD}{d\beta} = 3\left(1 - \frac{\delta}{D}\right)$$

or

$$\delta = D\left(1 + \frac{1}{3}\frac{\beta}{D}\frac{dD}{d\beta}\right)$$

Substituting this value of δ in equation (9), we obtain

$$q = 3\left(1 - \frac{2}{3} f_h + \frac{4}{9} f_h^2\right)\int_0^1 D\left(1 + \frac{1}{3}\frac{\beta}{D}\frac{dD}{d\beta}\right)\beta^4\, d\beta + \frac{2}{3} J$$

which, with the help of equation (6), can be finally transformed into

$$q = 1 - \frac{2}{3} f_h + \frac{2}{3} J + \frac{4}{9} f_h^2 - \frac{2}{5} \left(1 - \frac{2}{3} f_h + \frac{4}{9} f_h^2 \right) \frac{\sqrt{1 + \eta_s}}{1 + \lambda_s} \qquad (10)$$

Equation (7) can be simplified to the form

$$\eta_s f' = 3f' - \frac{6}{7} f_h^2 + \frac{4}{7} m f_h - J \left(5 + \frac{10}{21} f_h + \frac{20}{21} m \right) \qquad (11)$$

λ is defined as the departure from unity of the average value of a certain function $F(\eta)$ over the range of integration. This function $F(\eta)$ which occurs in the classical hydrostatic theory is given by

$$F(\eta) = \frac{1 + \frac{1}{2} \eta - \frac{1}{10} \eta^2 + \frac{2}{105} \zeta \xi}{\sqrt{1 + \eta}} \qquad (12)$$

where

$$\zeta = 3 \left(1 - \frac{\delta}{D} \right)$$

and

$$\xi = 7(m/D)\,(1 + \eta) - 3f_h\,(1 + \eta)^2 - 4f_h$$

The function $F(\eta)$ has the remarkable property that its value always lies very near unity, the maximum deviation being of the order of 10^{-4}. λ_s, which is the value of λ for the outer surface, is given by *Bullard* (1948) as

$$\lambda_s = (1.6 \pm 1.8)10^{-4}$$

This estimate, however, is based on the density distribution suggested by *Bullen* (1940, 1942). *Jeffreys* (1964) using a simplified density model, finds $\lambda_s = 1.3 \times 10^{-4}$ and points out that if $\lambda_s = 0$ instead, the resulting f_h is greater by 6×10^{-7} only. It can be shown that $F(\eta) < 1$ for $\eta >$ 0.53. As will be shown later, my value of η_s (the value of η for the outer surface) is considerably greater than that of *Bullard* (1948) or *Jeffreys* (1964). Since $F(\eta) < 1$ for $\eta > 0.53$, it is logical to expect that the mean value of the function $F(\eta)$ for this new range of integration that is extending further into the domain $\eta - 0.53 > 0$, will be still closer to unity. This will revise the estimate of λ_s even further downwards and its effect on the value of f_h will be even less important.[6] It seems legitimate, therefore, to take $\lambda_s = 0$ for initial calculations. Note that *Henriksen* (1960) also took $\lambda_s = 0$.

From equation (10) one obtains

$$\eta_s = (1 + \lambda_s)^2 \left[\frac{\left(1 - q - \frac{2}{3} f_h + \frac{2}{3} J + \frac{4}{9} f_h^2\right)^2}{\frac{2}{5}\left(1 - \frac{2}{3} f_h + \frac{4}{9} f_h^2\right)} \right]^2 - 1 \qquad (13)$$

With $\lambda_s = 0$, one can substitute η_s from equation (13) into equation (11) and solve the resulting equation for f_h. This value of f_h can then be used to find η_s from equation (13) corresponding to $\lambda_s = 0$. Following this

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[6] Slight variations in the value of λ_s do not affect the value of f_h significantly. This is apparent from Table 2 and Figure 1 given later.

procedure, one can compute a set of values for f_h corresponding to arbitrary values of λ_s.

Although the procedure described above is adequate, it is instructive to derive an expression which gives f_h explicitly in terms of other parameters. This can be done by writing equation (13) as

$$n_s = \frac{25}{4} F^2 q'^2 \left(\frac{1 - \Delta_1}{1 - \Delta_2} \right)^2 -1 \tag{14}$$

where

$$\left. \begin{aligned} q' &= 1 - q \\ \Delta_1 &= \frac{\frac{2}{3}\left(f_h - J - \frac{2}{3} f_h^2 \right)}{q'} \\ \Delta_2 &= \frac{2}{3} \left(f_h - \frac{2}{3} f_h^2 \right) \\ F &= 1 + \lambda_s \end{aligned} \right\} \tag{15}$$

It is instructive to note that Δ_1 and Δ_2 are both of the order of flattening.

Simplifying equation (14), one obtains

$$\begin{aligned} n_s &= \frac{25}{4} F^2 q'^2 \left[1 + 2(\Delta_2 - \Delta_1) + \left(\Delta_1^2 + 3\Delta_2^2 - 4\Delta_1 \Delta_2 \right) \right] -1 \\ &= n_0 + n_1 + n_2 \end{aligned} \tag{16}$$

where

$$\eta_0 = \frac{25}{4} F^2 q'^2 - 1$$

$$\eta_1 = \frac{25}{2} F^2 q'^2 (\Delta_2 - \Delta_1) = \frac{25}{3} F^2 q' \left(J - q f_h + \frac{2}{3} q f_h^2 \right) \qquad (17)$$

and

$$\eta_2 = \frac{25}{4} F^2 q'^2 \left(\Delta_1^2 + 3\Delta_2^2 - 4\Delta_1 \Delta_2 \right)$$

Note that the quantity η_1 is of the order of f_h while η_2 is of the order of f_h^2.

Using this value of η_s, equation (11) can be written as

$$A f_h^2 + (\eta_0 - 3 + \delta_1) f_h + 5J + \delta_2 = 0 \qquad (18)$$

where we have put

$$\left.\begin{aligned} A &= \frac{17}{14} - \frac{5}{42} \eta_0 - \frac{25}{3} F^2 q q' \\ \delta_1 &= \frac{25}{3} F^2 q' J - \frac{4}{7} m + \frac{10}{21} J \\ &\text{and} \\ \delta_2 &= \frac{4}{7} \eta_0 \kappa - \frac{12}{7} \kappa + \frac{20}{21} m J \end{aligned}\right\} \qquad (19)$$

Note that δ_1 is approximately of the order of f_h while δ_2 is of the order of f_h^2.

Equation (18) gives the required expression for f_h which, correct to the second order of small quantities, is

$$f_h = \frac{1}{(n_0 - 3)} \left[- (5J + \delta_2) + \frac{(5J + \delta_2)\ \delta_1}{(n_0 - 3)} - \frac{25\ A\ J^2}{(n_0 - 3)^2} \right] \tag{20}$$

It is interesting to see that in this development, the expression for f_h corresponding to the first order theory is

$$f_h = \frac{5J}{3 - n_0} \tag{21}$$

Equation (20) is relatively easier to manipulate numerically. It also seems more straightforward to compute f_h corresponding to arbitrary values of λ_s than to compute it from the usual equations. This can be done by writing the equation (20) as

$$f_h = \frac{Q}{n_0 - 3}$$

where

$$Q = \left[- (5J + \delta_2) + \frac{(5J + \delta_2)\ \delta_1}{n_0 - 3} - \frac{25\ A\ J^2}{(n_0 - 3)^2} \right] .$$

Q appears to have the remarkable property that, with other parameters fixed, it remains paractically constant, (changing only in the ninth place after decimal) for a reasonable range of variation of λ_s. In fact, with λ_s ranging from $+8 \times 10^{-4}$ to -8×10^{-4}, the corresponding range of variation of Q (with all other parameters fixed) is only from -6×10^{-9} to $+6 \times 10^{-9}$. Thus f_h practically depends only on η_0 which can be easily computed for arbitrary values of λ_s. We have tabulated below f_h corresponding to a few values of λ_s.

The data of Table 2 are plotted in Figure 1. It can be seen that the value of f_h is not very sensitive to reasonable changes in the value of λ_s. Also note that for a reasonable range of variation of λ_s, f_h varies linearly with λ_s.

The value of κ has been obtained from the following equation:

$$K = \frac{24}{7} \kappa + 3f^2 - \frac{15}{7} m f \tag{22}$$

where K is satellite-determined. The value of K used in these calculations was obtained from *Kozai's* (1964) value of $J_4 = -1.649 \times 10^{-6}$. Consequently, κ becomes

$$\kappa = 78 \times 10^{-8}$$

This gives a maximum depression of 4.9 meters (at latitude 45°) of the earth spheroid from the corresponding ellipsoidal surface. This value of κ should be more reliable than that derived by *de Sitter* (1924) or *Bullard* (1948) from the solution of a second-order differential equation in κ (originally given by *Darwin*, 1899) which involved assumptions about the internal density distribution. However, any reasonable variations in κ do not appear to change f_h noticeably. Note that in equation (22), f is the actual flattening.

Table 3 summarizes some of the more important values of λ_s, κ, and n_s as obtained by different investigators.

In Table 4 I have compared different values of the hydrostatic flattening obtained both in the pre-satellite and the post-satellite times. The correct value of the hydrostatic flattening obtained from the modified hydrostatic equations is 1/296.70 corresponding to $J_2 = 1082.645 \times 10^{-6}$ (*Kozai*, 1964), as opposed to 1/299.86 obtained (with m and H fixed in both cases) from *de Sitter's* (1924) classical equations (used by *Henriksen*, 1960). The error in the previous value of hydrostatic flattening arose from the fact (pointed out earlier) that *de Sitter* (1924) had eliminated the geopotential coefficient J from the right-hand side of his hydrostatic equations because accurate determinations of the value of J were not available when he worked out his equations. This elimination, most probably, was not detected in the post-satellite investigations (*Henriksen*, 1960; *O'Keefe*, 1960; *Khan*, 1967a) in which *de Sitter's* hydrostatic equations appear to have been used as such without correcting for the error introduced by these eliminations. The modified equations of course, contain the geopotential coefficient J explicitly and hence are not susceptible to this source of error. It may be emphasized that the difference between my value of f_h and the previous ones cannot be attributed to the slightly different values of H and m used by me. To show this I have listed two values of f_h in Table 4 using two different sets of data which are given in the footnote of Table 4.

The change of sign in the value of $f_h^{-1} - f^{-1}$ will create some difficulty with some of the geophysical hypotheses (*Munk* and *Macdonald*, 1960; *Wang*, 1966; *Takeuchi*, 1963; *Takeuchi* and *Hasegawa*, 1964) constructed on the previous belief that the hydrostatic flattening is smaller than the real flattening. As is obvious from Table 4, the hydrostatic flattening in actuality turns out to be greater than the real flattening. I must emphasize, however, that this definition of the hydrostatic flattening (used throughout this paper and hitherto used by all the post-satellite investigators on hydrostatic theory) is not a realistic one. In the case of the earth, the true hydrostatic equilibrium will exist only when the hydrostatic and external potential solutions (independently obtained) converge to a single solution corresponding to a given rate of rotation of the earth, because equation (1) neither assumes nor discounts the existence of hydrostatic equilibrium in the earth's interior and hence should be valid for both hydrostatic or non-hydrostatic conditions. However, even with this definition the hydrostatic flattening turns out to be greater than the real flattening of the earth. This point along with some other related aspects of the hydrostatic equilibrium theory is discussed in detail in another paper (*Khan*, 1967b).

Table 5 gives the values of the geopotential coefficients J_2 and J_4, corresponding to my value of the hydrostatic flattening, the flattening of the international reference ellipsoid, and the currently accepted value of the hydrostatic flattening. Note that the satellite-determined value of J_2 is $J_2 = 1082.645 \times 10^{-6}$ (*Kozai*, 1964). It does not seem to be very realistic to use the J_2 (corresponding to the hydrostatic flattening) given in this table, to estimate the hydrostatic stresses existing in the earth. This point will be discussed in detail in another paper (*Khan*, 1967b).

SUMMARY AND CONCLUSIONS

The equations of hydrostatic theory are modified in order to make their development independent of the external potential theory. The value of hydrostatic flattening found from these modified equations is $f_h^{-1} = 296.70 \pm 0.05$ as opposed to the currently accepted value of $f_h^{-1} = 299.86 \pm 0.05$ found from *de Sitter's* equations in which the geopotential coefficient J was eliminated by *de Sitter* by making use of a relation derived from the external potential theory. The value of η_s is found to be $\eta_s = 0.5369$ from these modified equations and is greater (refer to Table 3) than that of *Bullard* (1948) or *Jeffreys* (1964). This, if correct, appears to revise even further downwards the value of the parameter λ_s which *Bullard* (1948) has given as $\lambda_s = (1.6 \pm 1.8)10^{-4}$ and *Jeffreys* (1964) as $\lambda_s = 1.3 \times 10^{-4}$. Small changes in the value of λ_s do not effect critically (refer to Table 2 and Fig. 1) the value of f_h.

In order to make the hydrostatic flattening coincide with the actual flattening by merely varying the value of λ_s, one has to assume very high negative values of λ_s which seem improbable. Hence these results appear to confirm that there does exist a discrepancy between the actual flattening of the earth and the flattening it would have if it were in hydrostatic equilibrium. However, contrary to the currently held belief, it appears that the hydrostatic flattening is in fact <u>greater than</u> the actual flattening. The error in the previous values of the hydrostatic flattening (given by *Henriksen*, 1960; *O'Keefe*, 1960; *Jeffreys*, 1964; *Khan*, 1967a) appears to have arisen from the elimination of J by means of a relation derived from the external potential theory. This can now be avoided by modifying the classical equations to make their development independent of the external potential theory.

These results would appear to create some difficulty with some of the geophysical hypotheses constructed on the previous belief, which attempted to explain the then-assumed excess of the actual flattening of the earth over the corresponding hydrostatic flattening as being due to a time-lag in the adjustment of the rocky earth to its decreased rate of rotation (*Munk* and *Macdonald*, 1960), or to the additional flattening of the earth because of the heavy load of polar ice caps (*Wang*, 1966), or to a rigid lower mantle and a viscous layer in the upper mantle, implying a long recovery-time because of the confinement of the flow to the upper mantle (*Takeuchi*, 1963; *Takeuchi* and *Hasegawa*, 1964). More work on some other aspects of the problem of the hydrostatic equilibrium of the earth is in progress and the results will be reported shortly in another paper (*Khan*, 1967b).

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TABLE 1. Dynamical Flattening, H*

Jeffreys	(1964)	0.0032730
Henriksen	(1960)	0.00327293
Jeffreys	(1950)	0.0032729
Bullard	(1948)	0.00327237
de Sitter	(1924)	0.00327942
Khan	(1967a)	0.00327364

*The uncertainty in these values is in the seventh place after the decimal.

TABLE 2. Hydrostatic Flattening Corresponding to Some Arbitrary Values of λ_s

λ_s	f_λ	f_λ^{-1}
$+8 \times 10^{-4}$	0.00337397	296.387
$+6 \times 10^{-4}$	0.00337308	296.465
$+4 \times 10^{-4}$	0.00337219	296.543
$+2 \times 10^{-4}$	0.00337131	296.621
0	0.00337042	296.699
-2×10^{-4}	0.00336953	296.777
-4×10^{-4}	0.00336865	296.855
-6×10^{-4}	0.00336776	296.933

TABLE 3. Comparison of Some Hydrostatic Theory Parameters

		λ_s	κ	η_s
de Sitter	(1924)	$(4.4 \pm 1.5)10^{-4}$	50×10^{-8}	0.5589
Bullard	(1948)	$(1.6 \pm 1.8)10^{-4}$	68×10^{-8}	0.565
Jeffreys	(1964)	1.3×10^{-4}	64×10^{-8}	0.5587
My values		---	78×10^{-8}	0.5869(3)*

*The use of modified equations does not affect the value of η_s to any appreciable degree. If we use *de Sitter's* classical equations as *Henriksen* (1960) did, $\eta_s = 0.5869(5)$. Thus the value of η_s does not seem to be the cause of the discrepancy in *Henriksen's* and my val[illegible] of E_n (mentioned later).

TABLE 4. Comparison of Hydrostatic Flattening Values

Results obtained prior to satellite-determination of J:

$$f = f_h$$

		f_h^{-1}
de Sitter	(1924)	296.92 ± 0.136
de Sitter	(1938)	296.753 ± 0.086
Bullard	(1948)	297.338 ± 0.050
Jeffreys	(1952)	297.299 ± 0.071
Jeffreys	(1964)*	296.75 ± 0.05

Results obtained after the satellite-determination of J:

$$f^{-1} = 298.25$$

		f_h^{-1}		$f_h^{-1} - f^{-1}$
Henriksen	(1960)	300.0		+1.75
O'Keefe	(1960)**	299.8		+1.55
Jeffreys	(1964)	299.67	± 0.05	+1.42
Khan	(1967)	299.86	± 0.05	+1.61
My present values		296.70	± 0.05†	-1.55
		297.04	± 0.05††	-1.21

*Using the pre-satellite approach.

***Henriksen's* calculations.

†Based on m = 0.00344980 (*Khan*, 1967a; *Jeffreys*, 1964), H = 0.00327364 (*Khan*, 1967a) and J_2 = .001082645 (*Kozai*, 1964).

††Based on m = 0.00344992 (*Henriksen*, 1960), H = 0.00327070 and J_2 = .00108270.

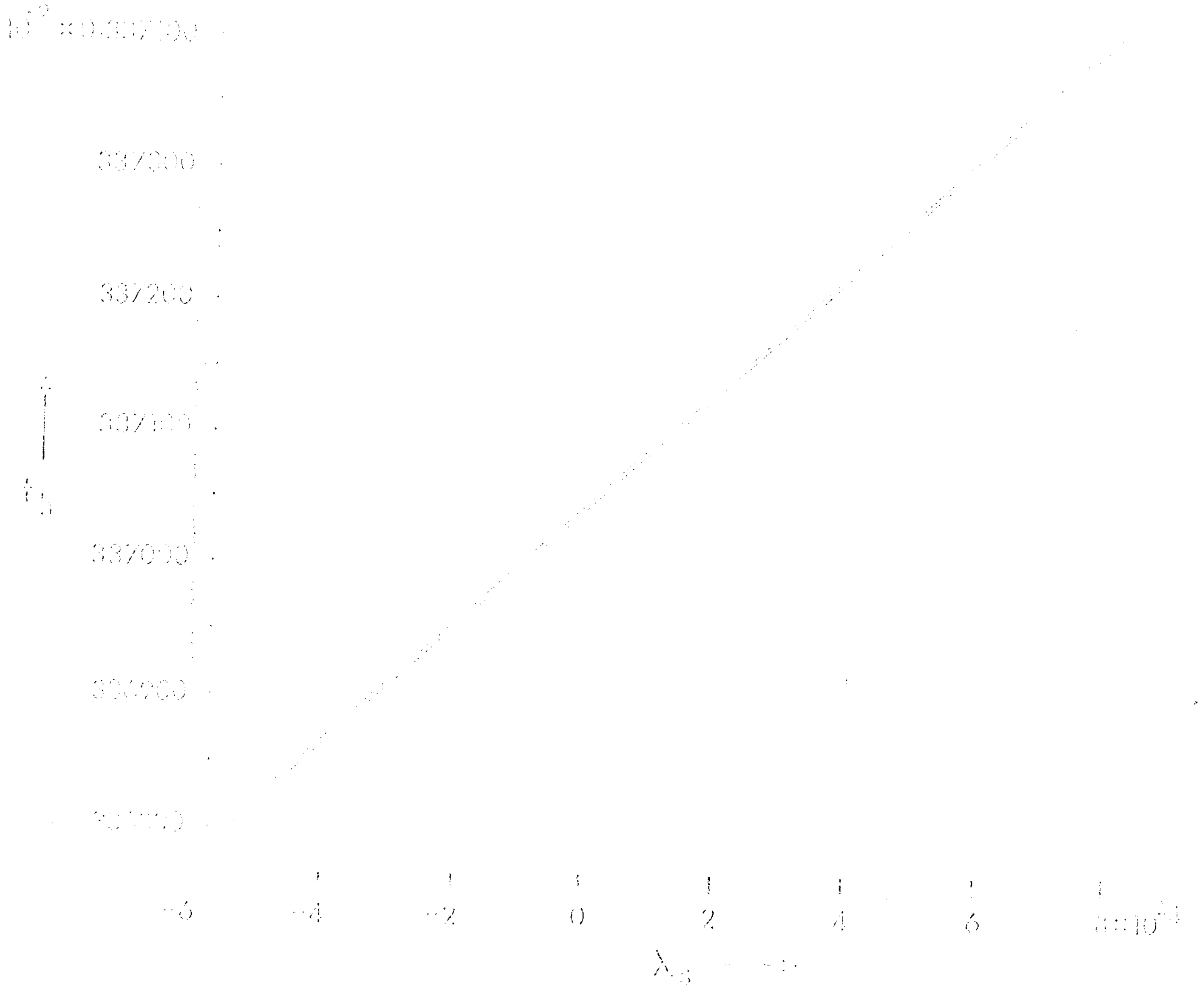
TABLE 5. Geopotential Coefficients Corresponding to Different Values of Flattening

Flattening^{-1}	Flattening	J_2	J_4	
296.70	0.00337040	1094.321×10^{-6}	-2.444×10^{-6}	(corresponding to my value of hydrostatic flattening)
297.0	0.00336700	1092.06×10^{-6}	-2.412×10^{-6}	(corresponding to flattening of the international reference ellipsoid)
299.80	0.00333556	1071.158×10^{-6}	-2.325×10^{-6}	(corresponding to the currently accepted value of the hydrostatic flattening)
298.25	0.00335299	1082.645×10^{-6}	-2.380×10^{-6}*	(corresponding to the actual flattening)

*Compare this value with observed $J_4 = -1.449 \times 10^{-6}$ (*Kozai*, 1964).

Legends for Figures

Fig. 1. Variation of Hydrostatic Stretching, S_H, with the parameter λ_2.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337000
-4
-2
0
2
4
6