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TRAPPED MODES IN PLASMA SLABS AND CYLINDERS

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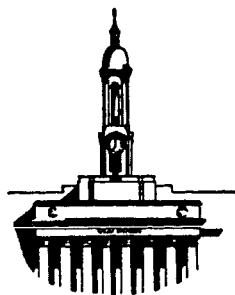
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Scientific Report

on

"Trapped Modes in Plasma Slabs and Cylinders"

by

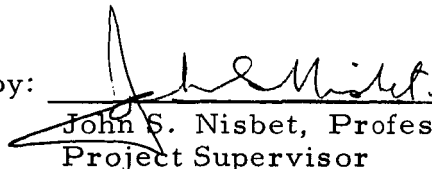
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February 1, 1968

Scientific Report No. 314

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ABSTRACT

An analysis of trapped modes is given for uniform, cold magnetoplasma slabs and rods. The full set of Maxwell's equations is used, and arbitrary magnetic field strength is allowed. Two dispersion relations are obtained for the slab, characterizing symmetric and antisymmetric field solutions. The general dispersion relation for the plasma rod geometry is obtained, simplified using an analogy with the slab case, and analyzed. For each dispersion relation, three kinds of interior waves are found: (1) waves which damp as you move toward the center; (2) oscillating or body waves; and (3) damped oscillating waves. The field characteristics of the waves are analyzed for the slab geometry and for the dipolar modes of the plasma rod. A new approximation, the "one wave" approximation is introduced and discussed. Dispersion curves and field solutions are given for $\Omega = .5\omega_p$ and $\Omega = 1.5\omega_p$, and for different thicknesses.

1 INTRODUCTION

1.1 STATEMENT OF THE PROBLEM; RELATED STUDIES

In the past two decades an extensive body of literature has developed on the subject of wave propagation in bounded plasmas. The origin of this interest comes from a number of different sources. For example, with the advent of manned spacecraft and the need for communication with it, the plasma sheath problem has become of vital importance (Owyang and Seshadri, 1966). In such a case, the energy of the propagating electromagnetic wave may be lost to classes of waves such as the "trapped modes." These are waves which are exponentially damped as a function of distance outside the plasma.

Such modes can also be found in ionospheric physics. Hodara and Cohn (1962) point out that trapped modes may propagate in the ion ducts formed by the lines of the earth's magnetic field. Also, plasma layers in the ionosphere may support these modes (Owyang and Seshadri, 1966).

A third application of such studies is the laboratory experiment, where characteristic properties of the trapped modes may be of importance in designing new methods for plasma diagnostics and millimeter wave generation (Trivelpiece and Gould, 1959; Formato and Gilardini, 1962; Akao and Ida, 1964).

Except for the laboratory devices, where plasma cylinders are used, the geometry of the structures supporting these trapped modes is usually complicated. Tamir and Oliner (1963) emphasize the fact that the plasma slab configuration in free space serves as an appropriate

simplified model for many situations. As a consequence, the plasma slab, half space and cylinder have received the most attention for bounded plasma studies.

The trapped modes existing in such plasmas can be classified as follows:

1.) Surface wave (S W)

Barlow (1962) defined the surface waves in the following terms: "A surface wave is one that propagates along an interface between two different media without radiation, such radiation being construed to mean energy converted from the surface wave field to some other form." We will use the term "surface wave" for a solution of Maxwell's equations propagating along an interface between two different media such that on both sides of the interface the fields decay exponentially with an increase of distance from the interface.

2.) Body wave (B W)

The body wave also propagates along the axes of the column. It has exponentially damped behavior outside the plasma, but it is essentially oscillatory in nature inside the plasma. Body waves of different order can be obtained, depending upon the number of oscillations inside the column. For the plasma rod the order of the body wave mode is given by half the number of nodes in the radial direction of the axial fields components for the symmetric case and half the number of nodes in the radial direction of the radial fields components for the antisymmetric case. In the plasma slab, the radial and axial fields components are

changed to the appropriate geometric counterpart. These waves have been first discussed by Trivelpiece and Gould (1959).

3.) Body wave-surface wave (B W S W)

In addition to the above two types of waves, there exist what we call "body wave-surface waves" which are a combination of body waves and surface waves. The propagation constant inside the plasma in the direction perpendicular to the direction of propagation is complex, implying damped oscillating waves. Outside the plasma these waves are damped. These waves have been discussed previously by Owyang and Seshadri (1966).

Previous analyses of these problems have been of varying degrees of complexity depending upon what approximations have been made in the calculations and whether magnetic fields are present or not. One of the most important papers related to this subject is the one by Trivelpiece and Gould (1959) who analyzed the trapped modes in a magnetized plasma cylinder using the quasistatic approximation, and then experimentally verified the existence of the symmetric surface wave. At about the same time, Fainberg and Gorbatenko (1959) obtained the dispersion relation for the symmetric solution for the magnetized plasma cylinder using the full set of Maxwell's equations. However, no dispersion curves were presented and the dispersion relation was kept in a very complicated form.

A more complete discussion of the modes in a plasma rod was given by Granatstein, Schlesinger and Vigant (1963) who also analyzed

the dipolar modes. These authors only considered the extremes of infinite and zero magnetic fields, however.

A semi quantitative analysis of the cutoff and resonance loci of all the possible trapped modes in magnetized plasma rods was given by Granatstein and Schlesinger (1964) with the help of a mapping in a 3 parameter space. However, they avoided the numerical solution of the exact dispersion relation. Numerical solutions of the exact symmetric dispersion relation have been given by Formato and Gilardini (1962), Camus (1965), and Yip and Seshadri (1967). Bevc (1962) solved the problem of the plasma-filled wave guide in a magnetic field. However, due to different boundary conditions, different results are obtained.

Numerous experiments have been performed on plasma rods without magnetic fields, in order to verify the existence and the properties of the surface waves. Akao and Ida (1963, 1964) demonstrated the backward character of the wave for the dipole mode. Breakdown of quasistatic theory and verification of the more exact theory was shown in the work of Granatstein and Schlesinger (1965), Leprince and Pommier (1966) and Granatstein, Korn, Ojo and Schlesinger (1967).

Trapped modes on plasma slabs without a magnetic field were originally investigated by Shuman (1948) and Oehrl (1957). However, their solutions were not complete since they considered only a symmetric distribution of the fields. Full details on the spectrum of electromagnetic waves guided by a plasma layer without a magnetic field was first presented by Tamir and Oliner (1963). Hodara and Cohn (1962) studied the propagation characteristics of electromagnetic waves in the

limit of zero and infinite magnetic field. Bertoni and Hessel (1966) also obtained the dispersion relation for an infinite magnetic field and studied power flow and group velocity. In these cases the dielectric tensor has only diagonal terms which greatly simplifies the analysis.

A complete discussion of the trapped modes in a magnetized plasma slab has not previously been given. Adiachi and Mushlake (1963) present a few brief remarks on this case, but no details are published. In addition, there has not been a critical or complete analysis of the field behavior of either the magnetized plasma rod or slab in any of the previous studies.

In the present analysis, the plasma slab and cylinder problems are studied assuming a finite D. C. magnetic field present along the axis of the column. It is further assumed that the distribution of electrons in the plasma is homogeneous. The full set of Maxwell's equations is used and the dielectric tensor of the cold plasma theory assuming no ion motion relates the electric displacement $\vec{D}$ to the electric field $\vec{E}$. The varying parameters of the problem are the intensity of the applied magnetic field and the slab thickness or the column diameter. The behavior of both the dispersion relation and the fields is analyzed.

In Chapter 2 the slab geometry is studied. In section 2.1 we obtain the complete dispersion relation in the presence of a finite magnetic field. Different limits are considered in section 2.2 where we verify that our expressions reduce to the result previously obtained

in simplified analyses. In section 2.3 we study the cutoffs and resonances of the trapped modes. The dispersion curves for different values of the parameters (intensity of the applied magnetic field and slab thickness) are shown in section 2.4 and the different types of wave solutions of the problem are analyzed. In section 2.5 the behavior of the fields is studied. A new approximation, called the "one wave" approximation, is developed and studied in detail in section 2.6.

Chapter 3 is devoted to the analysis of the plasma rod problem. In section 3.1 we obtain the general dispersion relation. Using an analogy between the slab problem and the rod problem, a simplification of the coefficients is indicated which leads to a relatively simple dispersion relation, especially in the case of the circularly symmetric mode and the dipolar mode. The dispersion relations are analyzed in section 3.2 for different values of magnetic field strength and rod thickness. Finally, in section 3.3 the behavior of the fields is studied, and the "one wave" approximation is discussed for both the circularly symmetric and dipolar modes.

1.2 DEFINITION OF SYMBOLS

For convenience we define here all the symbols used in the theory.

e absolute value of the charge of the electron

m mass of the electron

n_o number density of the electrons

c speed of light

$\omega_p = \frac{(4\pi n_o e^2)^{1/2}}{m}$ electron plasma frequency

B_o	applied magnetic induction
$\Omega = \frac{e B_o}{m}$	electron cyclotron frequency
$\alpha = \frac{\Omega}{\omega_p}$	normalized cyclotron frequency
ω	frequency of the wave
$Y = \frac{\omega}{\omega_p}$	normalized wave frequency
$T = \frac{Y^2 - 1}{Y^2}$	component of the dielectric tensor
$S = \frac{Y^2 - \alpha^2 - 1}{Y^2 - \alpha^2}$	component of the dielectric tensor
$D = \frac{-\alpha}{Y(Y^2 - \alpha^2)}$	component of the dielectric tensor
$\left[\begin{array}{l} E_{\zeta i} \\ H_{\zeta i} \\ E_{\xi i} \\ E_{\eta i} \\ H_{\xi i} \\ H_{\eta i} \end{array} \right]$	coefficients of the sinh or cosh functions for the fields inside the plasma
K_ζ	propagation constant in the ζ direction
K_ξ	propagation constant in the ξ direction
$X = \frac{c K_\xi}{\omega_p}$	normalized propagation constant in the ξ direction
$Z = \frac{c K_\zeta}{\omega_p}$	normalized propagation constant in the ζ direction
$K_{\xi o}$	propagation constant in the ξ direction outside the plasma

a half the slab thickness or radius of the plasma column

$2b = \frac{2a \omega_p}{c}$ normalized thickness or normalized diameter

$x = \frac{r \omega_p}{c}$ normalized distance from the center of the column

$$B = -\frac{1}{S} \left[Z^2 (S + T) - Y^2 (ST + S^2 - D^2) \right]$$

$$C = \frac{T}{S} \left[(Z^2 - SY^2)^2 - D^2 Y^4 \right]$$

$$\Delta = B^2 - 4C$$

$$X_{1,2}^2 = \frac{B \pm \sqrt{B^2 - 4C}}{2}$$

$$\beta_i = TZ^2 - STY^2 - SX_i^2 \quad i = 1, 2$$

2 PLASMA SLAB

2.1 DERIVATION OF THE DISPERSION RELATION

2.1.1 The Model

We introduce a rectangular coordinate system ξ, η, ζ . A plasma slab imbedded in a uniform magnetic field in the ζ direction is located between the two planes defined as $\xi = +a$ and $\xi = -a$. The slab is infinite in the η and ζ direction. A monochromatic wave is assumed to be propagating in the ζ direction. Because of the geometry of the problem, there is no field variation in the η direction. We can thus write the electric and the magnetic vectors in the following forms.

$$(1) \quad \vec{E}(\xi, \zeta, t) = \vec{E}(\xi) e^{-i\omega t + i K_{\zeta} \zeta}$$

$$(2) \quad \vec{H}(\xi, \zeta, t) = \vec{H}(\xi) e^{-i\omega t + i K_{\zeta} \zeta}$$

The dielectric properties of the plasma are represented by the cold plasma dielectric tensor:

$$(3) \quad \vec{\epsilon} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & T \end{bmatrix}$$

where

$$S = (Y^2 - \alpha^2 - 1) / (Y^2 - \alpha^2)$$

$$D = -\alpha / Y(Y^2 - \alpha^2)$$

$$T = (Y^2 - 1) / Y^2$$

and

$$Y = \frac{\omega}{\omega_p}, \quad \alpha = \frac{\Omega}{\omega_p}$$

Using the Equations 1, 2, and 3 we can write Maxwell's equations in the form:

$$(4) \quad -\frac{\partial E_{\eta}}{\partial \zeta} = \frac{i\omega}{c} H_{\xi}$$

$$(5) \quad \frac{\partial E_{\xi}}{\partial \zeta} - \frac{\partial E_{\zeta}}{\partial \xi} = \frac{i\omega}{c} H_{\eta}$$

$$(6) \quad \frac{\partial E_{\eta}}{\partial \xi} = \frac{i\omega}{c} H_{\zeta}$$

$$(7) \quad -\frac{\partial H_{\eta}}{\partial \zeta} = \frac{i\omega S}{c} E_{\xi} - \frac{\omega D}{c} E_{\eta}$$

$$(8) \quad \frac{\partial H_{\xi}}{\partial \zeta} - \frac{\partial H_{\zeta}}{\partial \xi} = \frac{\omega D}{c} E_{\xi} - \frac{i\omega S}{c} E_{\eta}$$

$$(9) \quad \frac{\partial H_{\eta}}{\partial \xi} = -\frac{i\omega}{c} T E_{\zeta}$$

The above set of equations yields two independent solutions. The first solution, referred to as the symmetric solution, is such that E_{ζ} and H_{ζ} are even functions of ξ and all the other field components are odd functions of ξ . For the other solution, E_{ζ} and H_{ζ} are odd functions of ξ and all the other components are even functions of ξ . This solution will be referred to as the antisymmetric solution.

The separation of the solution into symmetric and antisymmetric parts is possible because of the particular form of the dielectric tensor and because the functions are independent of η . This method can also be applied to the problem of the magnetized ferrite slab. It enables a simplification in the calculation of the dispersion relation. Furthermore, it emphasizes the differences between the two possible modes which have quite different properties as will be shown later.

2.1.2 The Dispersion Relation

2.1.2.1 Symmetric Dispersion Relation

To obtain the dispersion relation in the symmetric case we

substitute the assumed form for E_ζ and H_ζ into the Equations 4 to 9. Since they are symmetric with respect to ξ and since we are mainly interested in surface waves and using the fact that Maxwell's equations are linear with constant coefficients we can take the ξ dependence of E_ζ and H_ζ inside the plasma to be of the form:

$$(10) \quad \begin{aligned} E_\zeta &= E_{\zeta i} \cosh K_\xi \xi \\ H_\zeta &= H_{\zeta i} \cosh K_\xi \xi \end{aligned}$$

The other components of the fields are written:

$$(11) \quad \begin{aligned} E_\xi &= E_{\xi i} \sinh K_\xi \xi \\ E_\eta &= E_{\eta i} \sinh K_\xi \xi \\ H_\xi &= H_{\xi i} \sinh K_\xi \xi \\ H_\eta &= H_{\eta i} \sinh K_\xi \xi \end{aligned}$$

It is then found that for Maxwell's equations to yield a nontrivial solution it is necessary that

$$X^4 - B X^2 + C = 0$$

or, equivalently

$$(12) \quad X_{1,2}^2 = \frac{B \pm \sqrt{B^2 - 4C}}{2}$$

where

$$(13) \quad B = \frac{1}{S} \left[Z^2 (S + T) - Y^2 (ST + S^2 - D^2) \right]$$

$$(14) \quad C = \frac{T}{S} \left[(Z^2 - SY^2)^2 - D^2 Y^4 \right]$$

$$Z = \frac{c K_\zeta}{\omega p}, \quad X_i = \frac{c K_{\xi i}}{\omega p} \quad i = 1, 2$$

The quantities X and Z are the normalized propagation constants in the ξ and ζ direction while Y is the normalized frequency.

The general symmetric solution can now be written

$$(15) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} E_1 \cosh K_{\xi_1} \xi + E_2 \cosh K_{\xi_2} \xi \\ \frac{i E_1 \beta_1}{D Y Z} \cosh K_{\xi_1} \xi - \frac{i E_2 \beta_2}{D Y Z} \cosh K_{\xi_2} \xi \\ \frac{i E_1}{X_1 Z} (TY^2 + X_1^2) \sinh K_{\xi_1} \xi - \frac{i E_2}{X_2 Z} (TY^2 + X_2^2) \sinh K_{\xi_2} \xi \\ \frac{E_1 \beta_1}{D X_1 Z} \sinh K_{\xi_1} \xi + \frac{E_2 \beta_2}{D X_2 Z} \sinh K_{\xi_2} \xi \\ \frac{E_1 \beta_1}{D Y X_1} \sinh K_{\xi_1} \xi - \frac{E_2 \beta_2}{D Y X_2} \sinh K_{\xi_2} \xi \\ \frac{i E_1 T Y}{X_1} \sinh K_{\xi_1} \xi - \frac{i E_2 T Y}{X_2} \sinh K_{\xi_2} \xi \end{bmatrix} \times \exp(i K_{\zeta} \zeta - i \omega t)$$

$- a \leq \xi \leq a$

where E_1 and E_2 are two constants and

$$(16) \quad \beta_i = T Z^2 - S T Y^2 - S X_i^2 \quad i = 1, 2$$

We can calculate the components of the electric and magnetic vectors outside the plasma in the same way. The propagation constant in the ζ direction has to be the same inside and outside the plasma in order to match the boundary conditions. Denoting the propagation constant in the ξ direction as K_{ξ_0} , we obtain:

$$(17) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} E_0 \\ H_0 \\ E_{\xi_0} \operatorname{sgn}(\xi) \\ E_{\eta_0} \operatorname{sgn}(\xi) \\ H_{\xi_0} \operatorname{sgn}(\xi) \\ H_{\eta_0} \operatorname{sgn}(\xi) \end{bmatrix} \times \exp(-K_{\xi_0} |\xi| + i K_{\zeta} \zeta - i \omega t)$$

$- a \geq \xi \geq a$

where $\text{sgn}(\xi)$ is the signature of ξ and is equal to +1 if ξ is positive and to -1 if ξ is negative.

In order to have a nontrivial solution of Maxwell's equations,

$$X_o = \frac{c K_{\xi_o}}{\omega p} \text{ must satisfy}$$

$$(18) \quad X_o^2 = Z^2 - Y^2$$

$$\text{or} \quad X_o = \sqrt{Z^2 - Y^2} \quad Z \geq Y$$

$$X_o = -i \sqrt{Y^2 - Z^2} \quad Y \geq Z$$

The minus sign is chosen to satisfy the radiation condition. We are restricting our analysis to the case $Z > Y$.

In terms of the two constants E_o and H_o , the fields outside the plasma can be written

$$(19) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} E_o \\ H_o \\ \frac{i Z E_o \text{sgn}(\xi)}{X_o} \\ -\frac{i Y H_o \text{sgn}(\xi)}{X_o} \\ \frac{i Z H_o \text{sgn}(\xi)}{X_o} \\ \frac{i Y E_o \text{sgn}(\xi)}{X_o} \end{bmatrix} \times \exp(-K_{\xi_o} |\xi| + i K_{\zeta} \zeta - i \omega t)$$

The dispersion relation is now easily obtained by requiring that E_{ζ} , E_{η} , H_{ζ} , H_{η} and D_{ξ} be continuous across the boundaries. Since we already expressed the symmetry or the antisymmetry of the fields with respect to the plane $\xi = 0$, we need only write the boundary conditions for $\xi = +a$ and obtain:

$$(20) \quad \begin{bmatrix} \cosh X_1 b + \frac{T X_0}{X_1} \sinh X_1 b & \cosh X_2 b + \frac{T X_0}{X_2} \sinh X_2 b \\ \beta_1 (\cosh X_1 b + \frac{X_0}{X_1} \sinh X_1 b) & \beta_2 (\cosh X_2 b + \frac{X_0}{X_2} \sinh X_2 b) \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = 0$$

where $b = \frac{a \omega_p}{c}$ is the normalized thickness. The dispersion relation is the condition for nontrivial solution and is obtained by setting the determinant of the above square matrix equal to zero.

2.1.2.2 Antisymmetric Dispersion Relation

The ξ dependence of E_ζ and H_ζ is now taken to be of the form $\sinh K_\xi \xi$ inside the plasma and $\text{sgn}(\xi) \exp(-K_{\xi 0} |\xi|)$ outside the slab. From a similar analysis we obtain the fields inside and outside the slab to be

$$(21) \quad \begin{bmatrix} E_\zeta \\ H_\zeta \\ E_\xi \\ E_\eta \\ H_\xi \\ H_\eta \end{bmatrix} = \begin{bmatrix} E_1 \sinh K_{\xi 1} \xi + E_2 \sinh K_{\xi 2} \xi \\ -\frac{i E_1 \beta_1}{D Y Z} \sinh K_{\xi 1} \xi - \frac{i E_2 \beta_2}{D Y Z} \sinh K_{\xi 2} \xi \\ -\frac{i E_1}{X_1 Z} (TY^2 + X_1^2) \cosh K_{\xi 1} \xi - \frac{i E_2}{X_2 Z} (TY^2 + X_2^2) \cosh K_{\xi 2} \xi \\ \frac{E_1 \beta_1}{D X_2 Z} \cosh K_{\xi 1} \xi + \frac{E_2 \beta_2}{D X_2 Z} \cosh K_{\xi 2} \xi \\ -\frac{E_1 \beta_1}{D Y X_1} \cosh K_{\xi 1} \xi - \frac{E_2 \beta_2}{D Y X_2} \cosh K_{\xi 2} \xi \\ -\frac{i E_1 TY}{X_1} \cosh K_{\xi 1} \xi - \frac{i E_2 TY}{X_2} \cosh K_{\xi 2} \xi \end{bmatrix} \times \exp(i K_\zeta \zeta - i \omega t)$$

- $a \leq \xi \leq a$

$$(22) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} E_0 \operatorname{sgn}(\xi) \\ H_0 \operatorname{sgn}(\xi) \\ \frac{i Z}{X_0} E_0 \\ - \frac{i Y}{X_0} H_0 \\ \frac{i Z}{X_0} H_0 \\ \frac{i Y}{X_0} E_0 \end{bmatrix} \times \exp(-K_{\xi_0} |\xi| + i K_{\zeta} \zeta - i \omega t) \quad -a \geq \xi \geq a$$

The dispersion relation is obtained by setting the determinant of the following 2 x 2 matrix equal to zero:

$$(23) \quad \begin{vmatrix} \sinh X_1 b + \frac{T X_0}{X_1} \cosh X_1 b & \sinh X_2 b + \frac{T X_0}{X_2} \cosh X_2 b & E_1 \\ \beta_1 \left(\sinh X_1 b + \frac{X_0}{X_1} \cosh X_1 b \right) & \beta_1 \left(\sinh X_2 b + \frac{X_0}{X_2} \cosh X_2 b \right) & E_2 \end{vmatrix} = 0$$

The only difference between the symmetric and the antisymmetric case for the fields component inside the plasma and for the dispersion relation is that the sinh and cosh functions are interchanged.

2.2 APPROXIMATE LIMITS

2.2.1 Quasistatic Limit

In the quasistatic theory, the term $\frac{\partial \bar{B}}{\partial t}$ is neglected in comparison with the term $\nabla \times \bar{E}$. Bers (1963) points out that this approximation is valid only if

$$(24) \quad Z \gg Y$$

$$(25) \quad X \gg Y$$

If the limit $Z \gg Y$ is taken in the Equation 12, we obtain for the propagation constant in the ξ direction

$$(26) \quad X_1 \sim Z$$

$$(27) \quad X_2 \sim Z \sqrt{\frac{T}{S}} \quad Y \neq 1, Y \neq \alpha$$

Consequently, if Y is not equal to 1 or α , Equation 25 is also satisfied and we obtain the result of the quasistatic theory explicitly.

1.) Symmetric

$$(28) \quad \tanh b Z \sqrt{\frac{T}{S}} = - \frac{1}{T \sqrt{\frac{S}{T}}}$$

2.) Antisymmetric

$$(29) \quad \tanh b Z \sqrt{\frac{T}{S}} = - T \sqrt{\frac{S}{T}}$$

In the limit $Y \rightarrow 1$, it can be shown easily that the propagation constants X_1 and X_2 tend toward the same limits as Equations 26 and 27. Now, however, Equation 25 is not satisfied since as $Y \rightarrow 1$, $T \rightarrow 0$. The quantity X_2 then has the limit

$$X_2 = \pi/2 + k \pi/2 \quad k = 0, 1, 2 \dots$$

The dispersion relations obtained in this limit are the same as Equations 28 and 29, i.e. the quasistatic dispersion relation. The reason the quasistatic result is correct even though Equation 25 is not satisfied is that the magnetic vector goes to zero when $Z/Y \rightarrow \infty$ and Equations 28 and 29 become the exact dispersion relations.

In the limit $Y \rightarrow \alpha$, the propagation constants in the ξ direction have the following limits:

$$(30) \quad X_1^2 = Z^2$$

$$X_2^2 = \frac{Z^2 T}{S} \left(1 - \frac{2 SY^2}{Z^2} \right)$$

When Z goes to infinity, S goes to infinity in such a way that X_2 remains finite. This must be true in order to have a solution of the dispersion relations. The dispersion relations then take the form:

1.) Symmetric

$$(31) \quad \sqrt{\frac{T}{S}} \left[1 - \frac{2 SY^2}{Z^2} \right]^{\frac{1}{2}} \coth b Z \sqrt{\frac{T}{S}} \left[1 - \frac{2 SY^2}{Z^2} \right]^{\frac{1}{2}} + T = 0$$

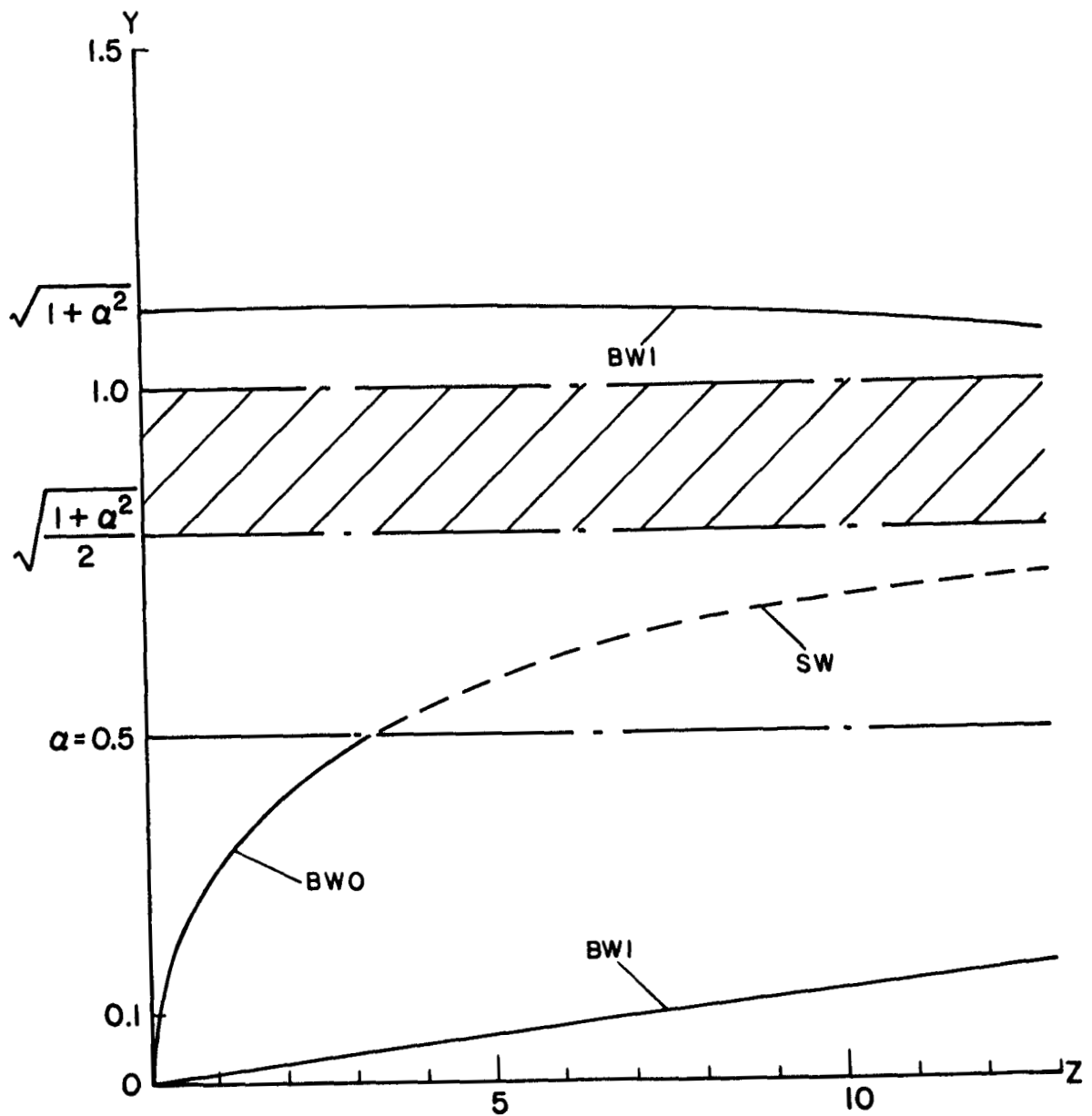
2.) Antisymmetric

$$(32) \quad \sqrt{\frac{T}{S}} \left[1 - \frac{2 SY^2}{Z^2} \right]^{\frac{1}{2}} \tanh b Z \sqrt{\frac{T}{S}} \left[1 - \frac{2 SY^2}{Z^2} \right]^{\frac{1}{2}} + T = 0$$

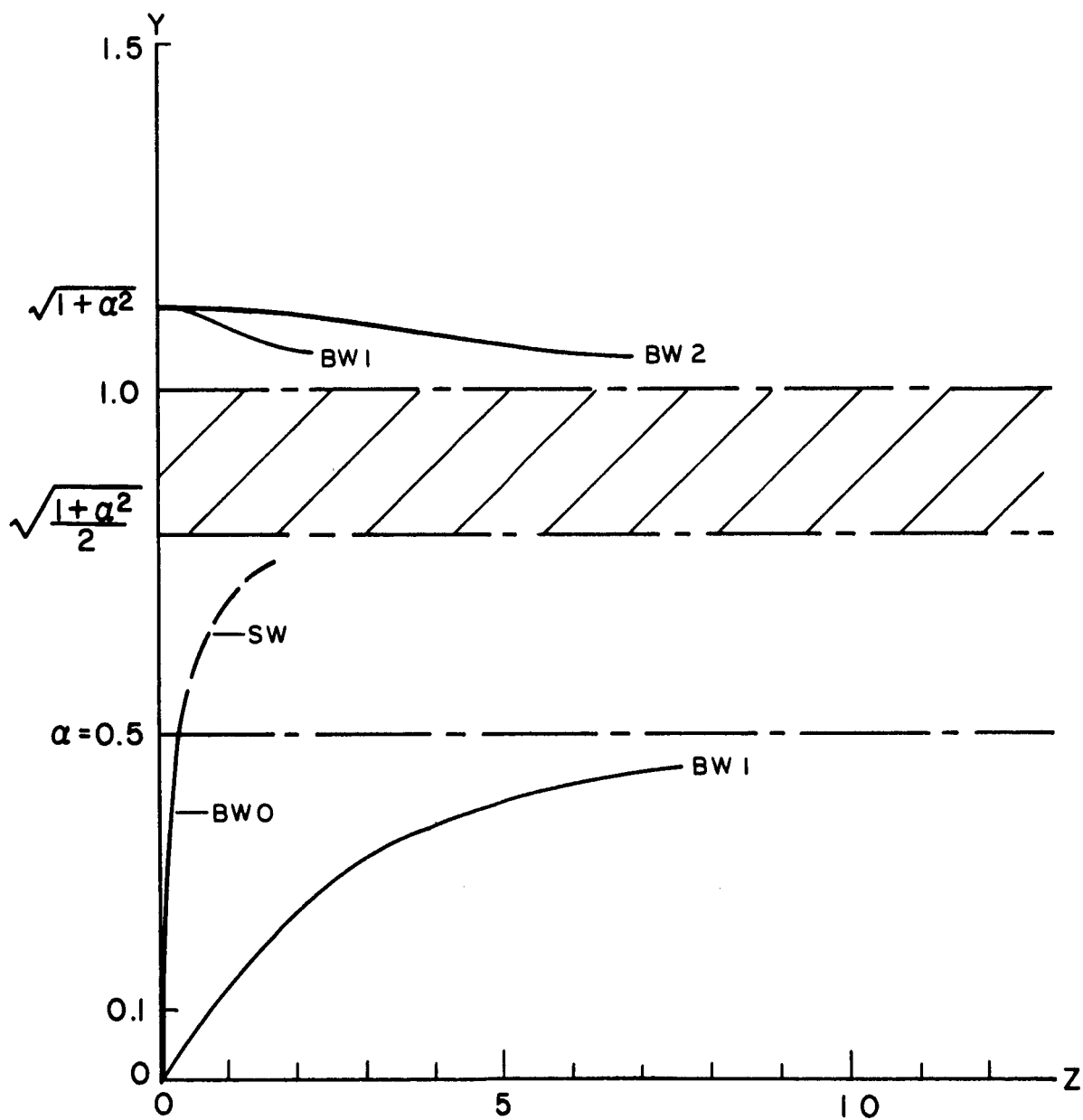
The same resonant frequency $Y = \alpha$ is obtained as in the quasistatic theory.

Except for the case $Y = \alpha$, the exact dispersion relation reduces to the quasistatic theory in the limit of large Z/Y . However, in the quasistatic theory, only one wave with propagation constant $X_2 = Z \sqrt{\frac{T}{S}}$ propagates. In the theory using the full set of Maxwell's equations, there are still two waves even when $Z \gg Y$. The explanation for this apparent discrepancy, as will be shown later, is that the wave with propagation constant X_1 has a much smaller amplitude than the wave with propagation constant X_2 .

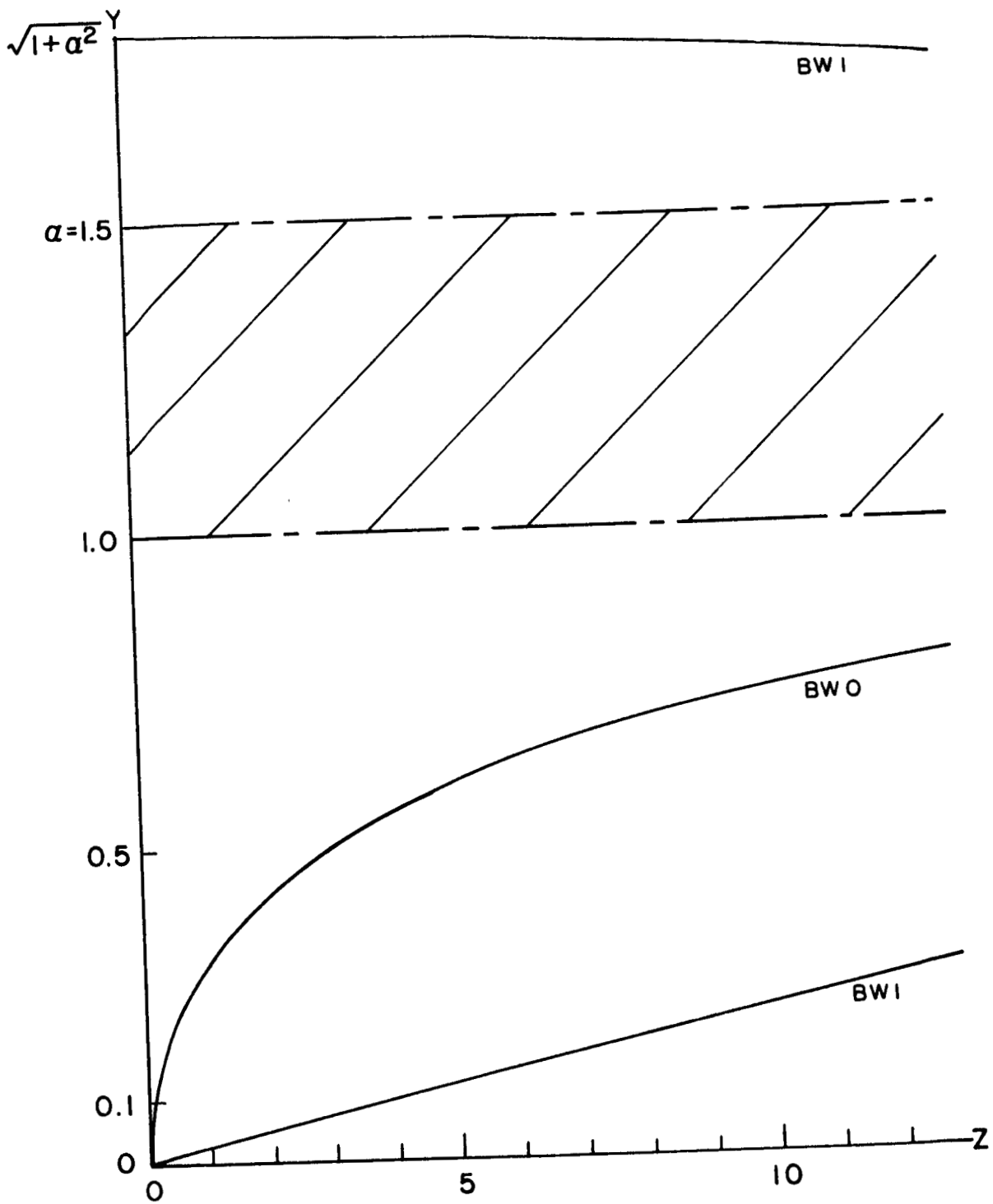
The quasistatic theory contains many of the features of the complete theory, but in a simpler form. Consequently, we will study briefly some of the results of the quasistatic theory.



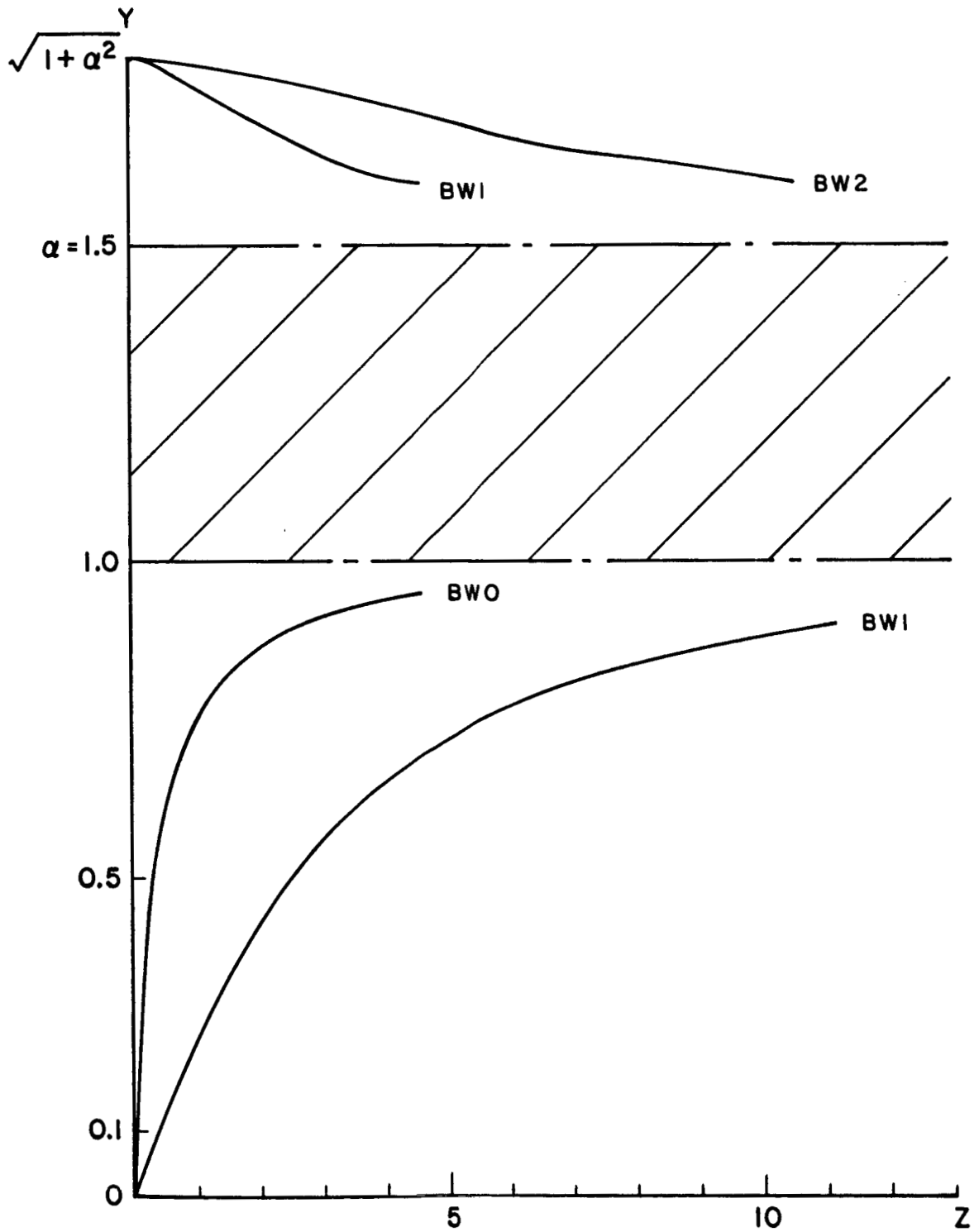
SOLUTIONS OF THE QUASISTATIC SYMMETRIC DISPERSION
RELATION FOR $\alpha=0.5$ AND $b=0.1$
FIGURE 1



SOLUTIONS OF THE QUASISTATIC SYMMETRIC DISPERSION
RELATION FOR $\alpha=0.5$ AND $b=1.0$
FIGURE 2



SOLUTIONS OF THE QUASISTATIC SYMMETRIC DISPERSION
RELATION FOR $\alpha=1.5$ AND $b=0.1$
FIGURE 3



SOLUTIONS OF THE QUASISTATIC SYMMETRIC DISPERSION
RELATION FOR $\alpha = 1.5$ AND $b = 1.0$
FIGURE 4

The solutions of the symmetric and antisymmetric dispersion relation are shown on Figures 1 to 8 for different values of α and b . These values are chosen so that we are able to pick up all the different kinds of waves contained in this theory. Also, these values represent typical order of magnitudes encountered experimentally. The waves for which the phase velocity and the group velocity have the same sign are referred to as forward waves, whereas the waves with phase and group velocity of opposite sign are referred to as backward waves.

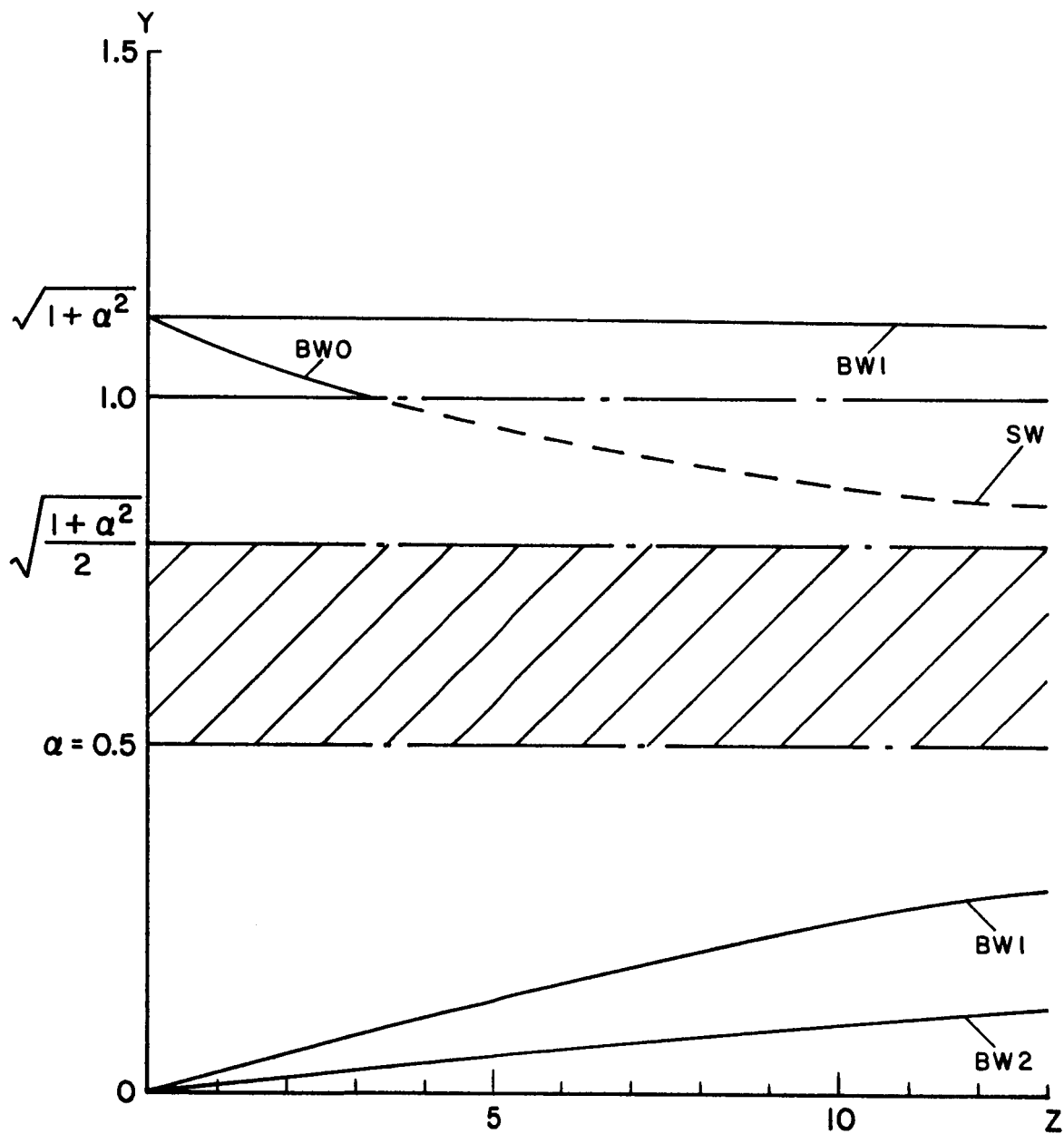
1.) Symmetric case

When α is smaller than one, the waves are always forward in the region of the $Y Z$ plane where Y is smaller than $\sqrt{\frac{1 + \alpha^2}{2}}$ and backward in the region of the plane where $1 < Y < \sqrt{1 + \alpha^2}$. As seen from Figures 1 and 2, a surface wave propagates for the frequencies between α and $\sqrt{\frac{1 + \alpha^2}{2}}$. All the other curves correspond to body waves of different order. No propagation is possible if $\sqrt{1 + \alpha^2} < Y < 1$. As b increases, the slope of the curves situated below the line $Y = 1$ increases. The asymptote $Y = 1$ is also reached more rapidly. (Compare the Figures 1 and 2.)

On Figures 3 and 4, dispersion curves for $\alpha > 1$ are presented. It is seen that only body waves propagate. There are no surface waves. The gap of frequency where no waves propagate is now between 1 and α . The effect of increasing b is again that of displacing the curves to the left.

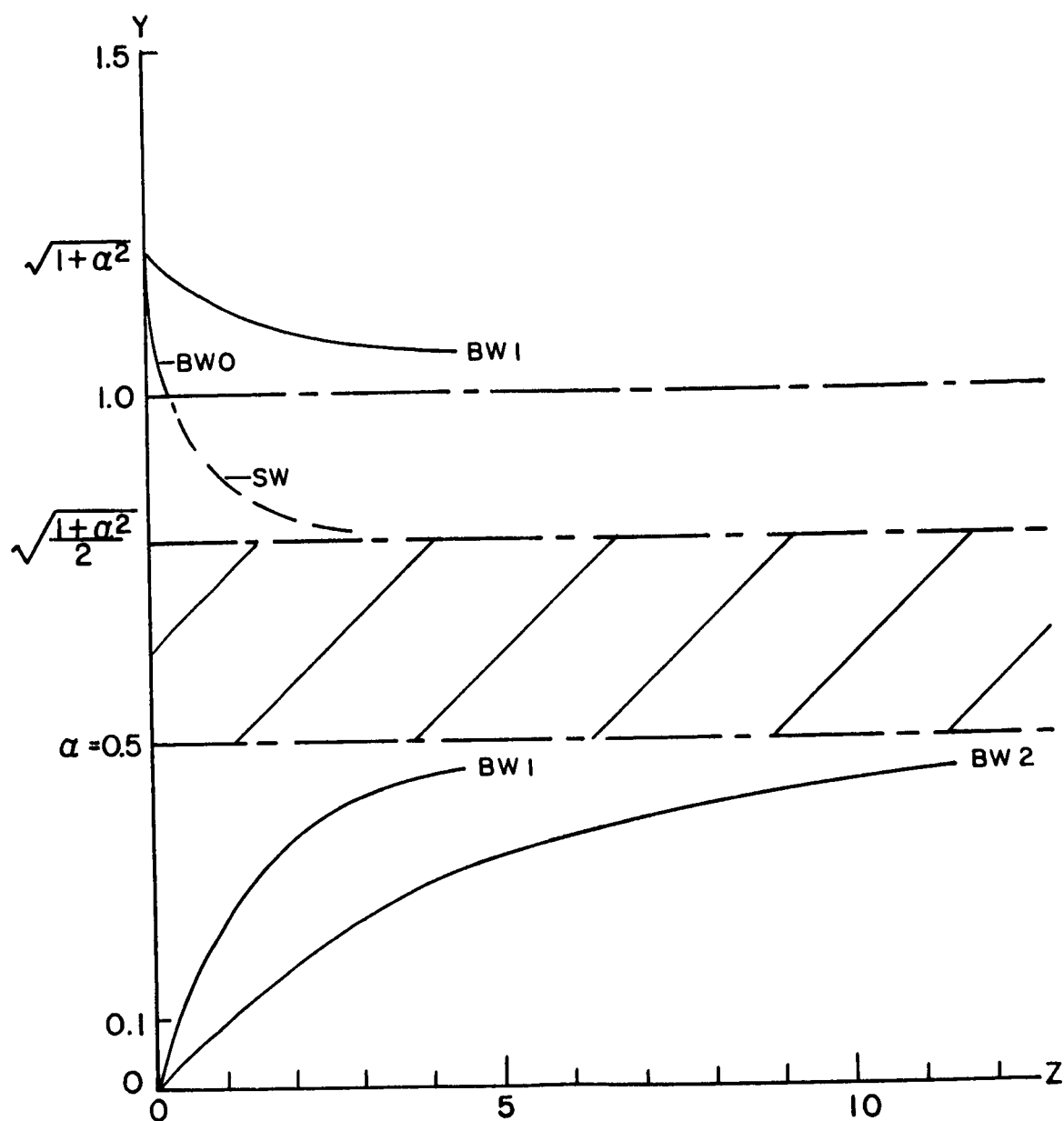
2.) Antisymmetric case

Figures 5 to 8 show the dispersion curves for $\alpha = 0.5$ and 1.5



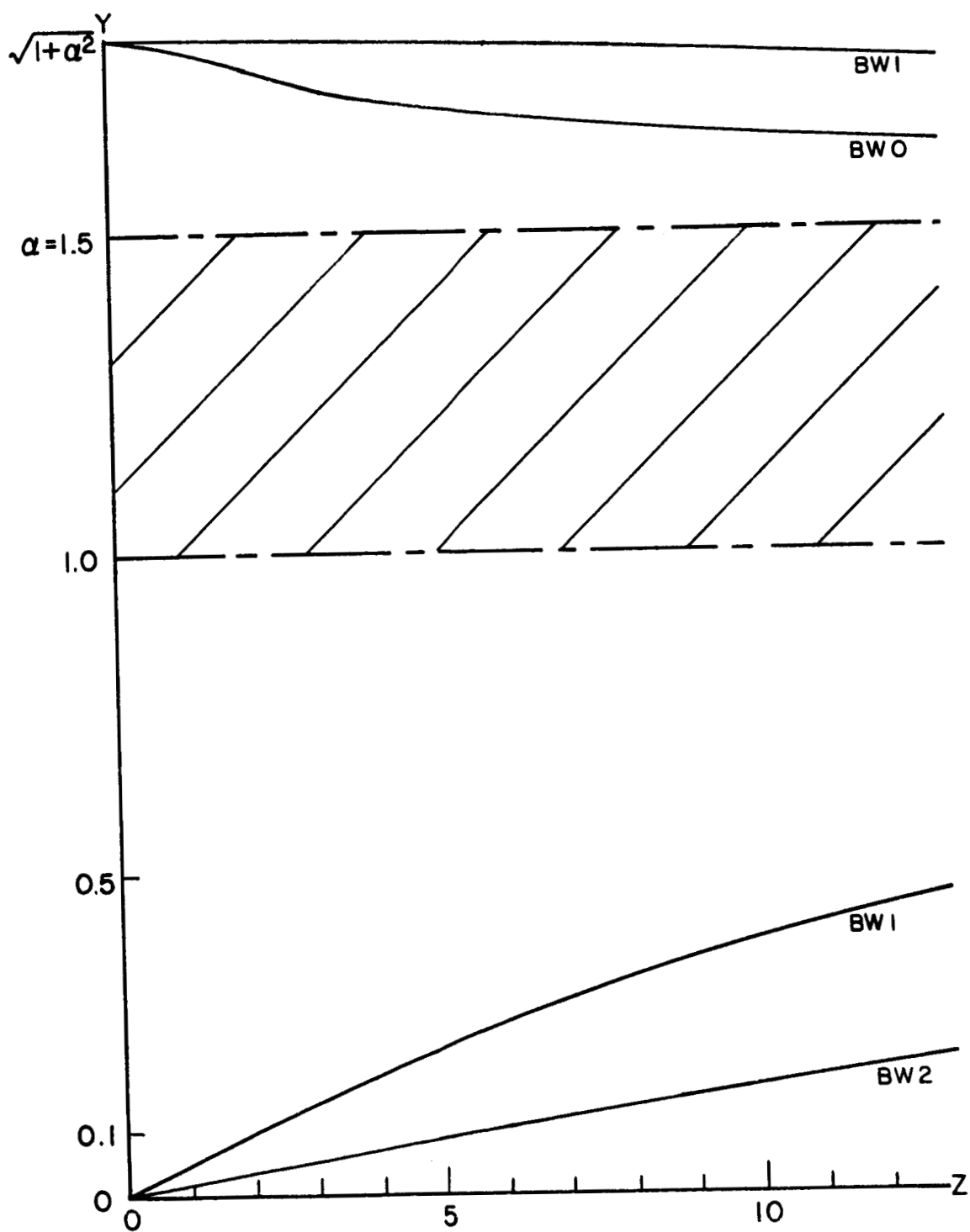
SOLUTIONS OF THE QUASISTATIC ANTISYMMETRIC
DISPERSION RELATION FOR $\alpha = 0.5$ AND $b = 0.1$

FIGURE 5



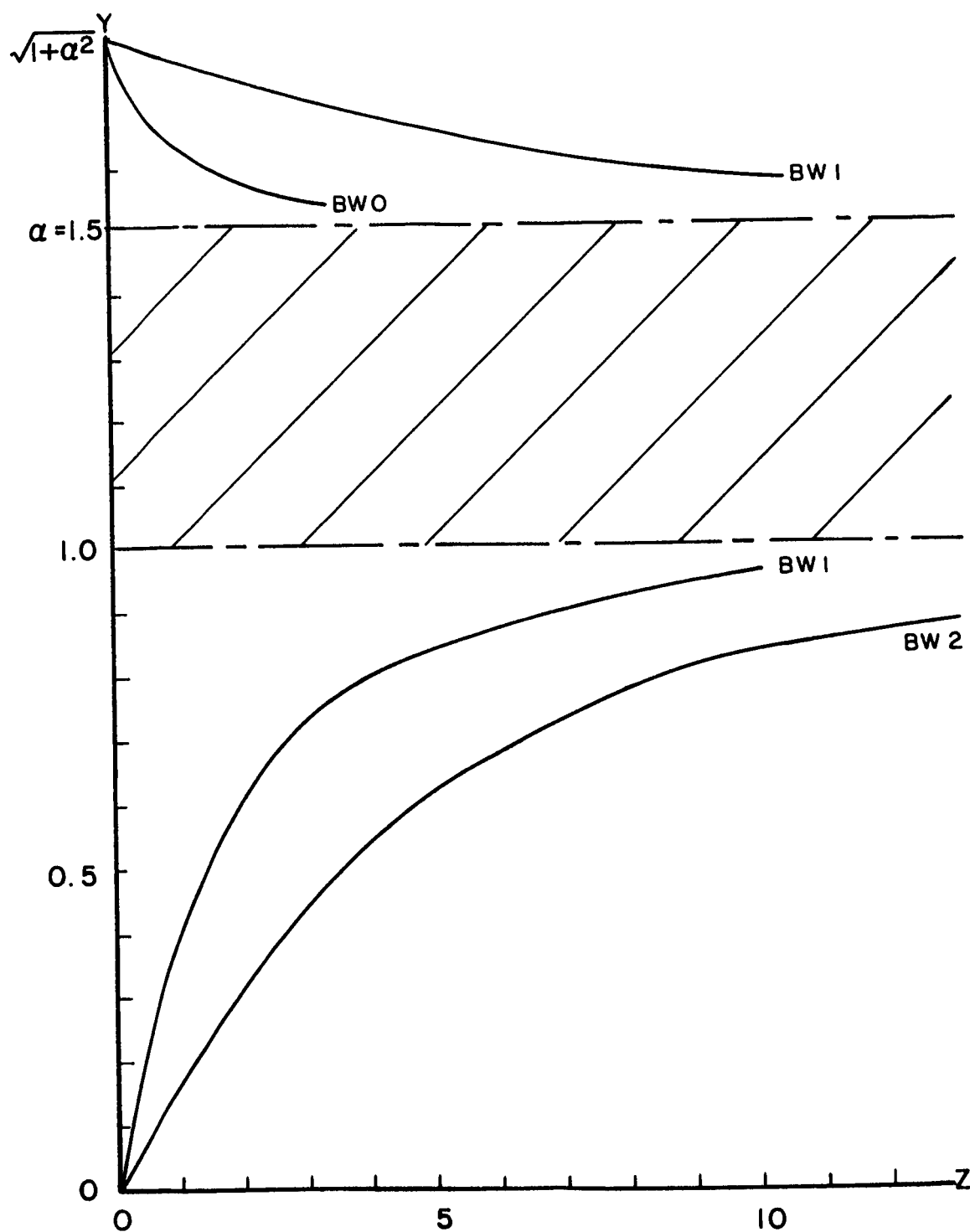
SOLUTIONS OF THE QUASISTATIC ANTISYMMETRIC
DISPERSION RELATION FOR $\alpha = 0.5$ AND $b = 1.0$

FIGURE 6



SOLUTIONS OF THE QUASISTATIC ANTISYMMETRIC
DISPERSION RELATION FOR $\alpha = 1.5$ AND $b = 0.1$

FIGURE 7



SOLUTIONS OF THE QUASISTATIC ANTISYMMETRIC
DISPERSION RELATION FOR $\alpha = 1.5$ AND $b = 1.0$

FIGURE 8

with $b = .1$ or 1 . When α is smaller than one, a backward surface wave exists for frequencies between 1 and $\sqrt{\frac{1+\alpha^2}{2}}$. When $Y < \alpha$ the body waves are forward, while for $Y > 1$, the body waves are backward. Note that BWO is also backward, which differs from the symmetric case (Figures 1 and 2). As b is increased, the curves are shifted to the left.

For $\alpha > 1$, only body waves propagate. These waves are forward for $Y < 1$, and backward for $Y > \alpha$. No waves propagate in the frequency gap $1 \leq Y \leq \alpha$. Again, increasing b corresponds to shifting the dispersion curve to the left. The type of wave is not changed and the characteristics are the same.

The quasistatic theory is not expected to be valid for $Z \sim Y$ or for $Y = \alpha$. In the region where quasistatic theory breaks down, we will find some important differences between these results just discussed and those obtained from the complete dispersion relation.

2.2.2 Infinite Magnetic Field Limit

In the limit of magnetic field strength going to infinity, the plasma dielectric tensor has only a uniaxial anisotropy. Equation 12 for the propagation constants inside the plasma reduces to

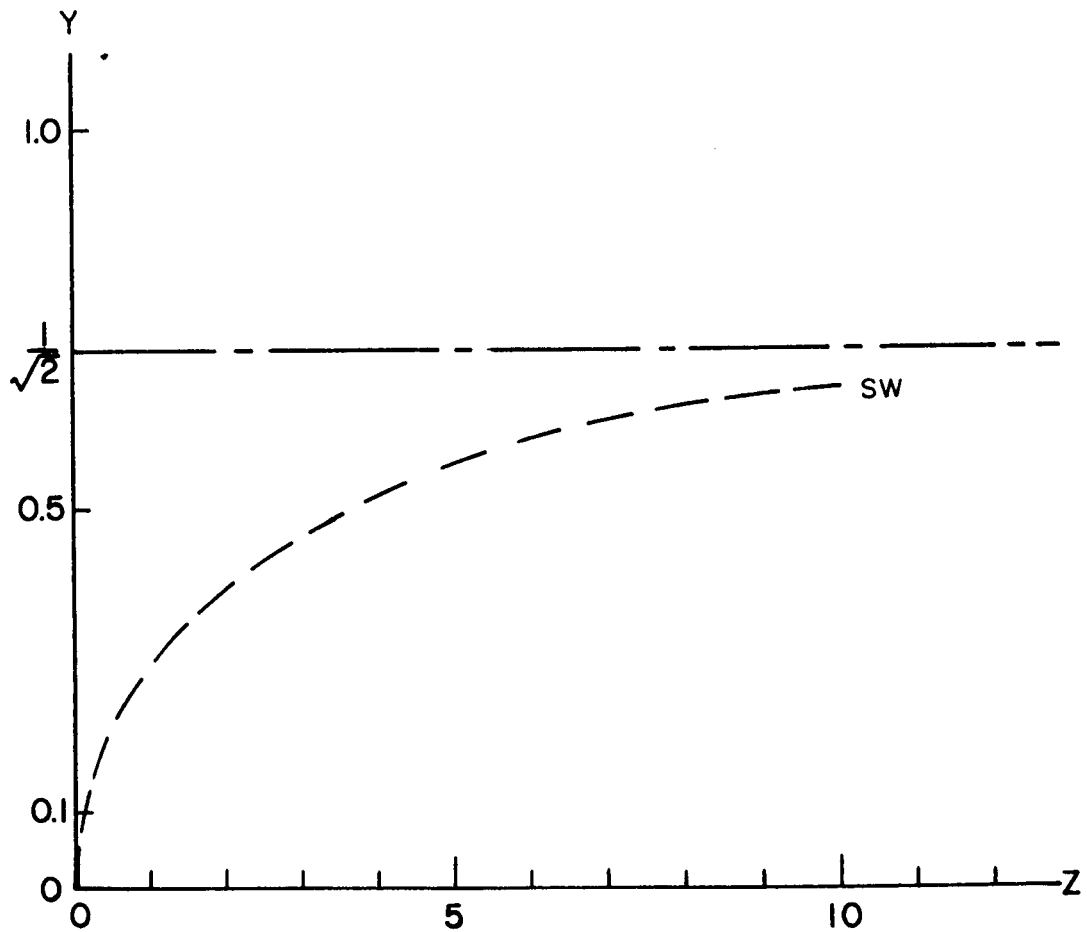
$$(33) \quad X_1^2 = Z^2 - Y^2$$

$$X_2^2 = (Z^2 - Y^2) T$$

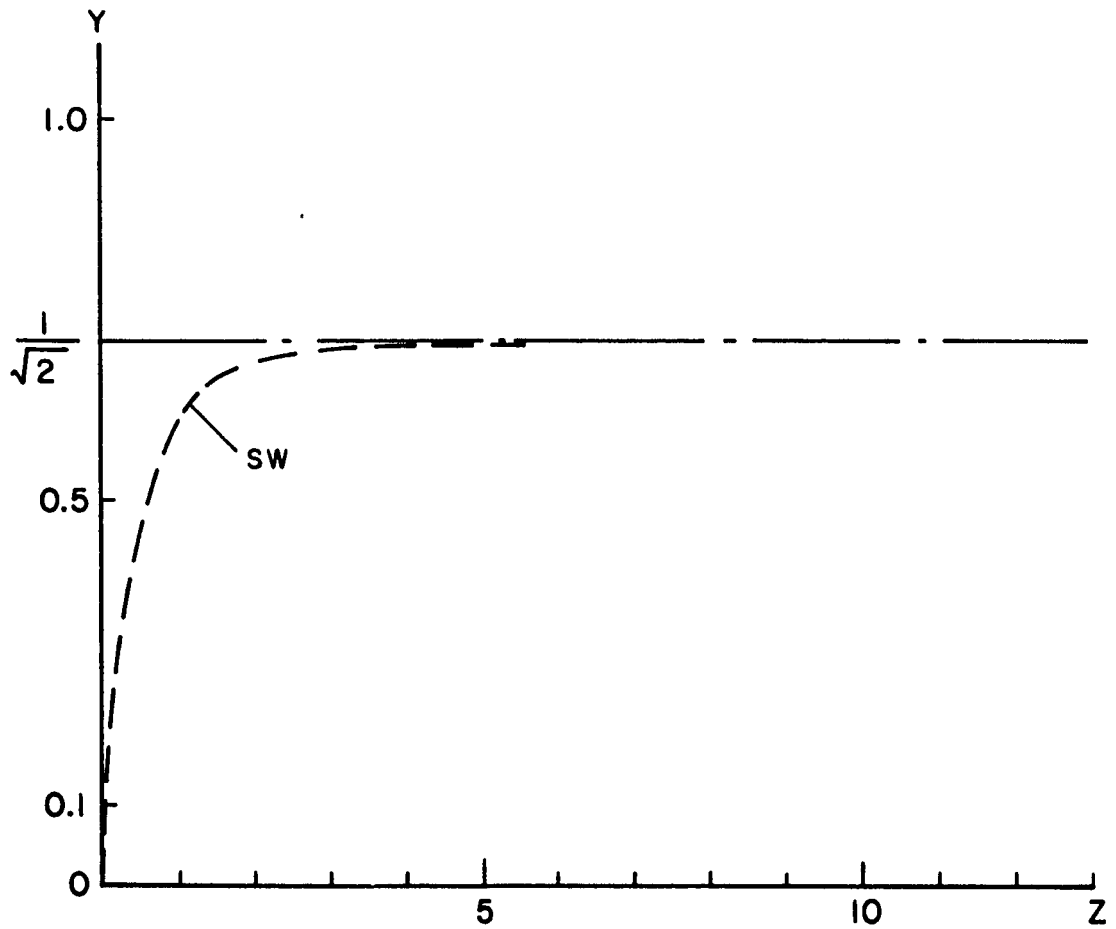
while the coefficients β_i become

$$(34) \quad \beta_1 = (T - 1) (Z^2 - Y^2)$$

$$\beta_2 = 0$$



SOLUTION OF THE SYMMETRIC DISPERSION RELATION
FOR $\alpha = 0$ AND $b = 0.1$
FIGURE 9



SOLUTION OF THE SYMMETRIC DISPERSION RELATION
FOR $\alpha = 0$ AND $b = 1.0$
FIGURE 10

Substituting these values in the dispersion relations (Equations 20 and 23) yields

1.) Symmetric

$$(35) \quad \tanh b \sqrt{T (Z^2 - Y^2)} = - \frac{1}{\sqrt{T}}$$

2.) Antisymmetric

$$(36) \quad \tanh b \sqrt{T (Z^2 - Y^2)} = - \sqrt{T}$$

These expressions were given previously by Hodara and Cohn (1962).

The value for E_1 is found in both cases to be zero, so that only the wave with propagation constant X_2 propagates.

2.2.3 Zero Magnetic Field Limit

In the limit of B going to zero, X_1 and X_2 tend to the same value, namely

$$(37) \quad X = \sqrt{Z^2 - SY^2}$$

Only one wave is needed to match the boundary conditions. The dispersion relation becomes

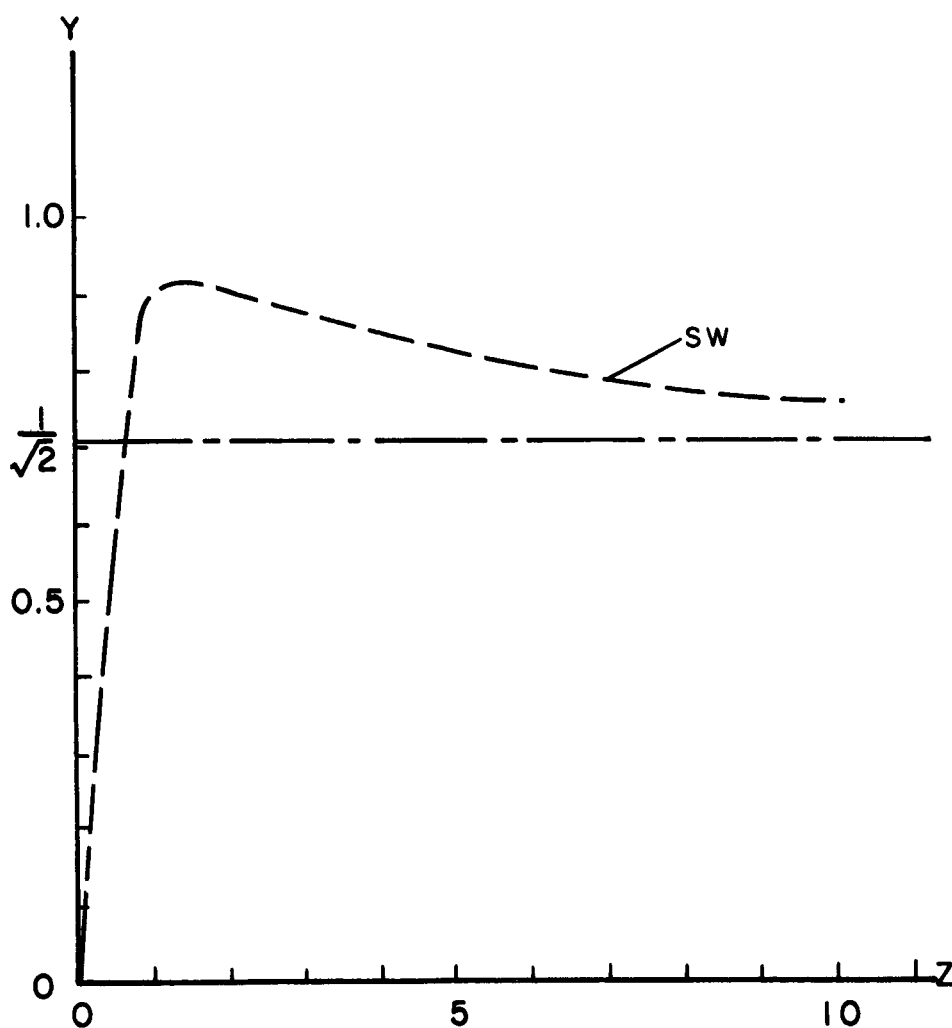
1.) Symmetric

$$(38) \quad \tanh b \sqrt{Z^2 - Y^2 + 1} = - \frac{\sqrt{Z^2 - Y^2 + 1}}{T \sqrt{Z^2 - Y^2}}$$

2.) Antisymmetric

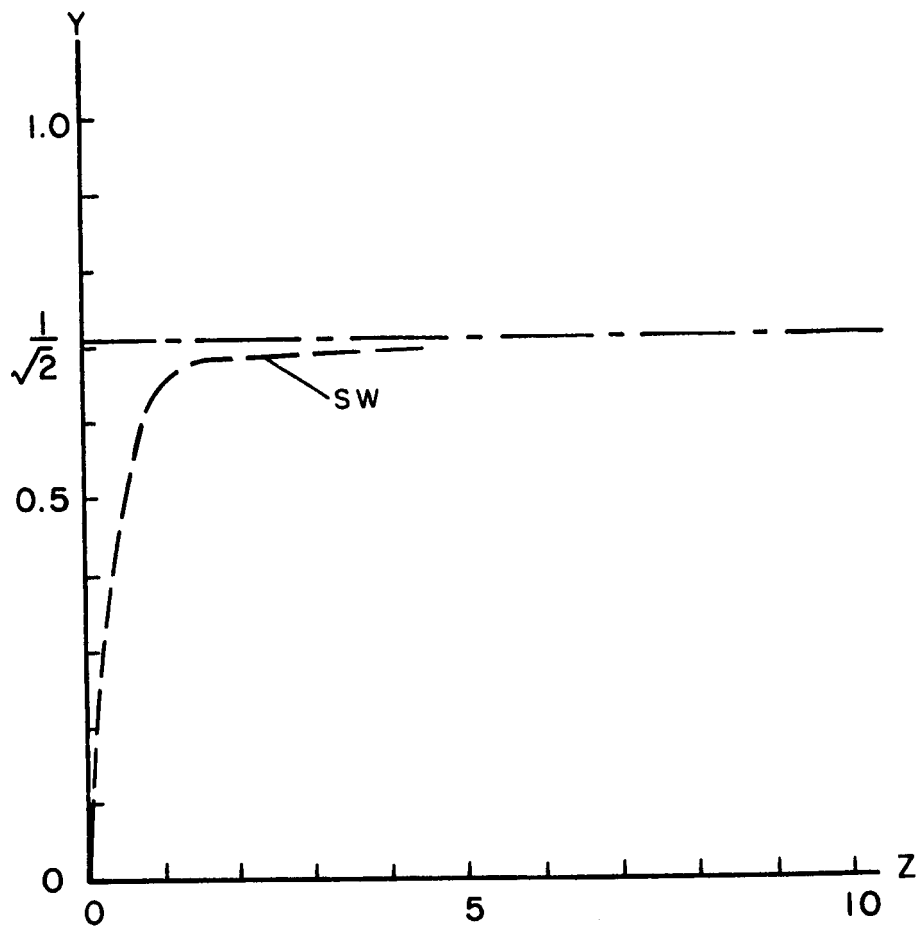
$$(39) \quad \tanh b \sqrt{Z^2 - Y^2 + 1} = - \frac{T \sqrt{Z^2 - Y^2}}{\sqrt{Z^2 - Y^2 + 1}}$$

These results are also in agreement with those obtained by Hodara and Cohn (1962). The dispersion curves for the symmetrical case always has a positive slope, as shown on Figures 9 and 10. Only a forward surface wave propagates. For the antisymmetrical case, the surface wave is



SOLUTION OF THE ANTISYMMETRIC
DISPERSION RELATION FOR $\alpha = 0$ AND $b = 0.1$

FIGURE 11



SOLUTION OF THE ANTISYMMETRIC
DISPERSION RELATION FOR $\alpha = 0$ AND $b = 1.0$
FIGURE 12

purely forward if b is greater than $b_{\min}$, and is forward and then backward if b is smaller than $b_{\min}$, where $b_{\min}$ is given by

$$\tanh \sqrt{U_0} \times b_{\min} = \sqrt{1 - \frac{1}{U_0}}$$

and U_0 is the solution of the equation

$$\tanh \sqrt{\frac{U_0}{U_0 - 1}} = \sqrt{\frac{U_0 - 1}{U_0}}$$

Figures 11 and 12 show the two different kinds of behavior that may be obtained. As Z increases, the absolute value of the group velocity decreases and becomes zero for $Y = \frac{1}{\sqrt{2}}$ in both cases.

2.2.4 Limit when $Y = 1$ ($\omega = \omega_p$)

The line $Y = 1$ plays a very important role in the analysis of the results. As will be shown in section 2.3.2, $Y = 1$ is a resonant frequency whether α is smaller or greater than one.

We take the limit of Y going to 1 in the dispersion relation and assume Z different from 1 but finite. ($Y = Z = 1$ cannot in general be a solution of the dispersion relation. We show in section 2.3 that $Y = Z = 1$ is a solution only when the thickness becomes infinite). We obtain:

1.) Symmetric

$$(40) \quad S(Z^2 - 1) + X_0 \sqrt{Z^2 - 1} \tanh b \sqrt{Z^2 - 1} = 0$$

This equation does not have a solution. Consequently, the dispersion curves never cross the line $Y = 1$ for finite values of Z .

2.) Antisymmetric

$$(41) \quad \tanh b \sqrt{Z^2 - 1} = \frac{S(Z^2 - 1)^2 + b\sqrt{Z^2 - 1}(Z^4 - 2SZ^2 + S)}{-b\sqrt{Z^2 - 1}(Z^4 - 2SZ^2 + S) - S(Z^2 - 1)^2 + Z^2(1 - S)}$$

Depending upon the values of the parameters α and b , this equation has zero, one or two solutions.

For $\alpha > 1$ there is always one solution. For $\alpha < 1$ there exists a value $b = b_m$ such that if $b < b_m$ Equation 41 has two solutions Z_1 and Z_2 ($Z_1 < Z_2$). This implies the existence of a forward wave for $0 < Z < Z_1$ and a backward wave for $Z > Z_2$. Figure 23 shows an example of such a situation. For $b > b_m$, Equation 41 does not have any solution.

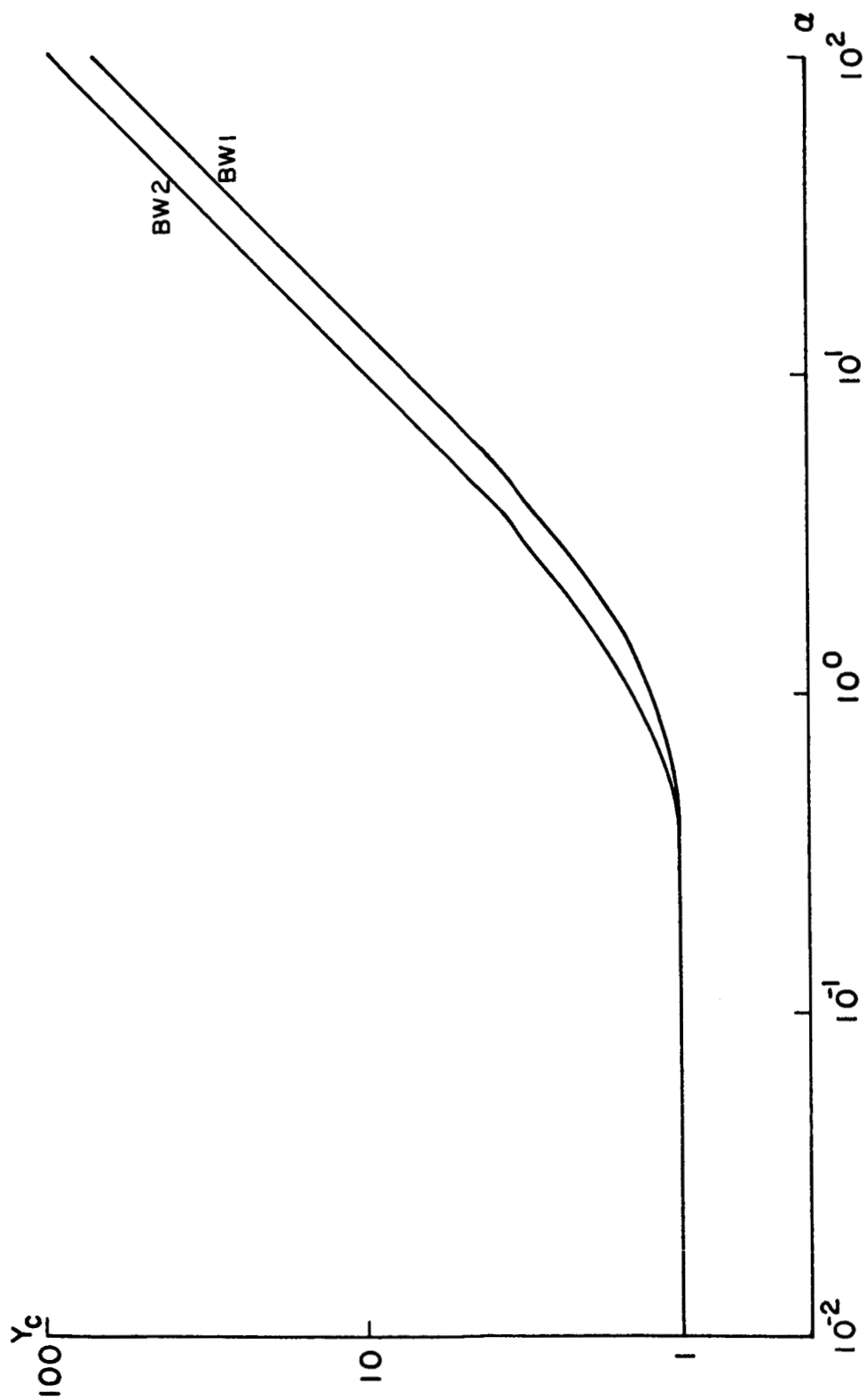
In the quasistatic theory, the line $Y = 1$ was not crossed by the symmetric dispersion curve and crossed only once for the anti-symmetric solution when $\alpha < 1$. When b was increased from 0.1 to 1.0, the point of intersection between the line $Y = 1$ and the dispersion curve was moved to the left on the graph. (Figures 5 and 6). We obtain different results when the complete dispersion relation is used. This can be expected since the crossing occurs for Z of the order of Y , i.e. when the quasistatic theory is not valid.

2.3 CUTOFF AND RESONANCES

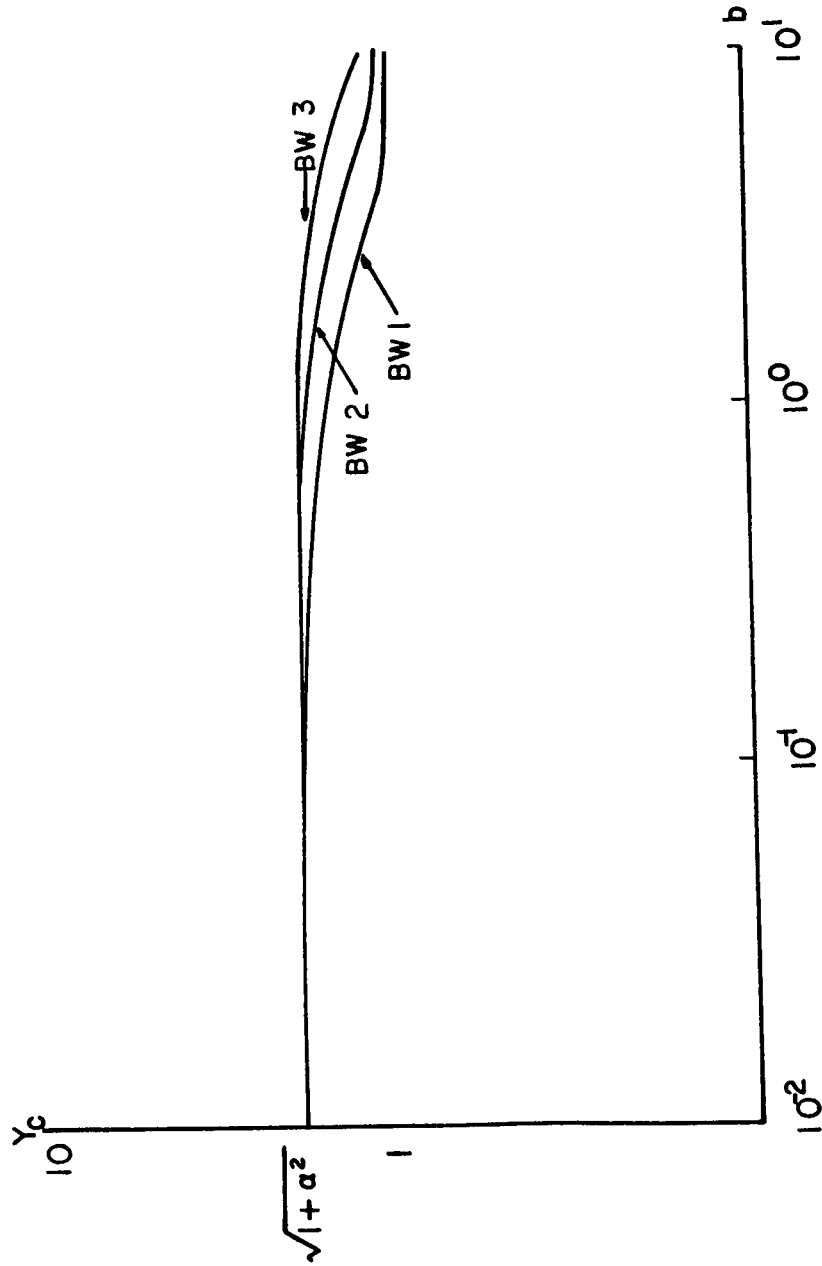
2.3.1 Cutoff

The cutoff is defined as the value of the frequency for which the propagation constant in the ξ direction outside the plasma vanishes. This condition will be satisfied either when Y and Z both go to zero or when $Y = Z$. The first case corresponds to the lowest order mode and to the low frequency body waves which have a cutoff of zero. For the second case the propagation constants inside the plasma can be written as

$$(42) \quad X_{1,2}^2 = \frac{Y^2}{S} \left[T(1 - S) \pm D\sqrt{T} \right]$$



CUTOFF FOR THE SYMMETRIC MODES VS α FOR $b=1.0$
FIGURE 13



CUTOFF FOR THE SYMMETRIC MODES VS b FOR $\alpha = 1.5$

FIGURE 14

and therefore

$$(43) \quad \beta_{1,2} = \mp Y^2 D \sqrt{T}$$

The dispersion relations reduce to

$$(44) \quad X_1 X_2 \coth X_1 b \coth X_2 b = 0 \quad (\text{Symmetric})$$

$$X_1 X_2 \tanh X_1 b \tanh X_2 b = 0 \quad (\text{Antisymmetric})$$

The condition $X_1 = 0$, $X_2 = 0$, $\coth X_1 b = 0$ or $\tanh X_1 b = 0$ are not consistent with the condition $Y = Z$ so that the dispersion relation can be further simplified to

$$\coth X_2 b = 0 \quad (\text{Symmetric})$$

$$\tanh X_2 b = 0 \quad (\text{Antisymmetric})$$

These two equations can be satisfied for $Y = Z$ only when

$$1 \leq Y \leq \sqrt{1 + \alpha^2}$$

1.) Symmetric case

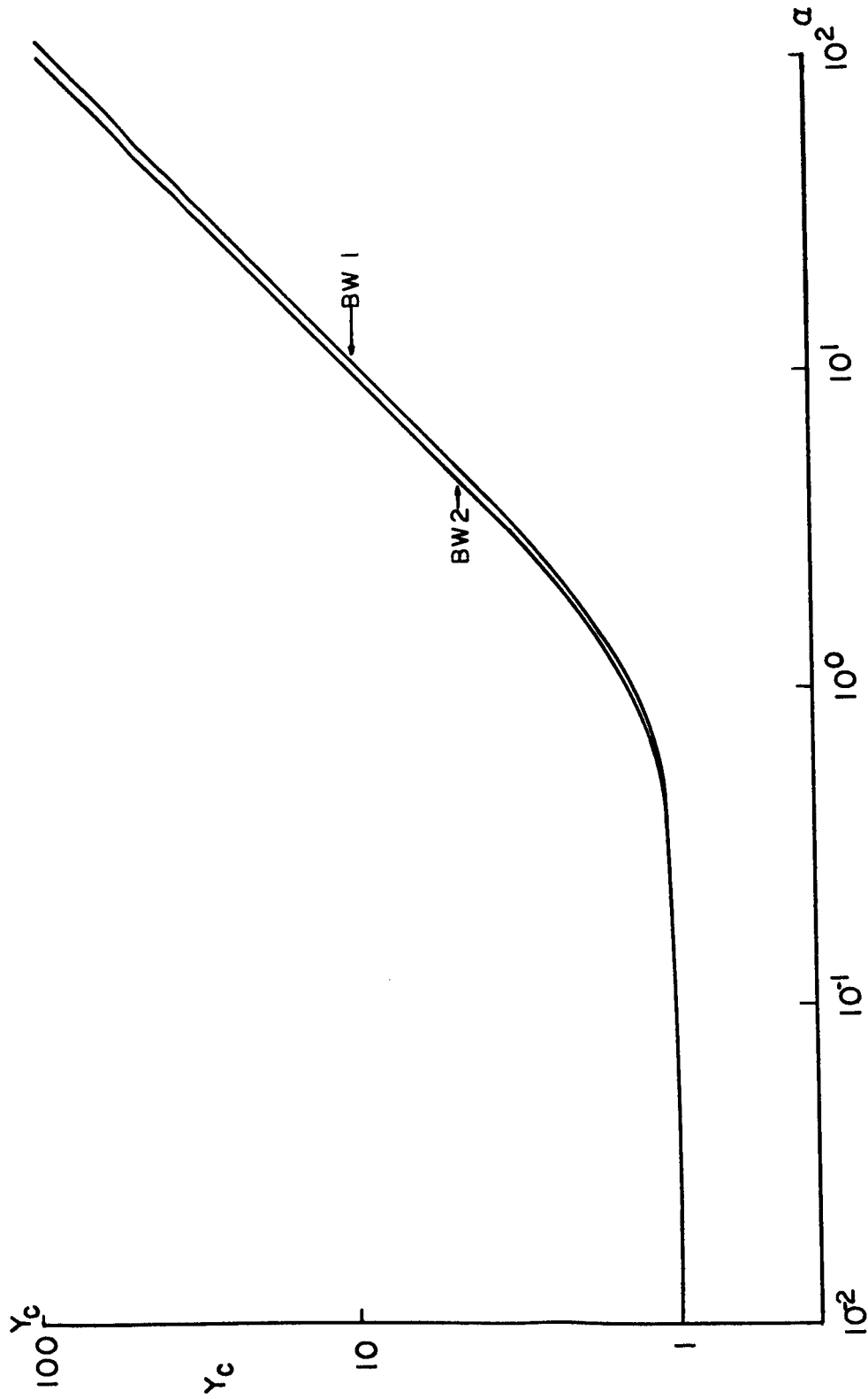
The cutoff frequency for the mode of order k is obtained from

$$X_2 b = i \left(-\frac{\pi}{2} + k\pi \right) \quad k \geq 1$$

which yields

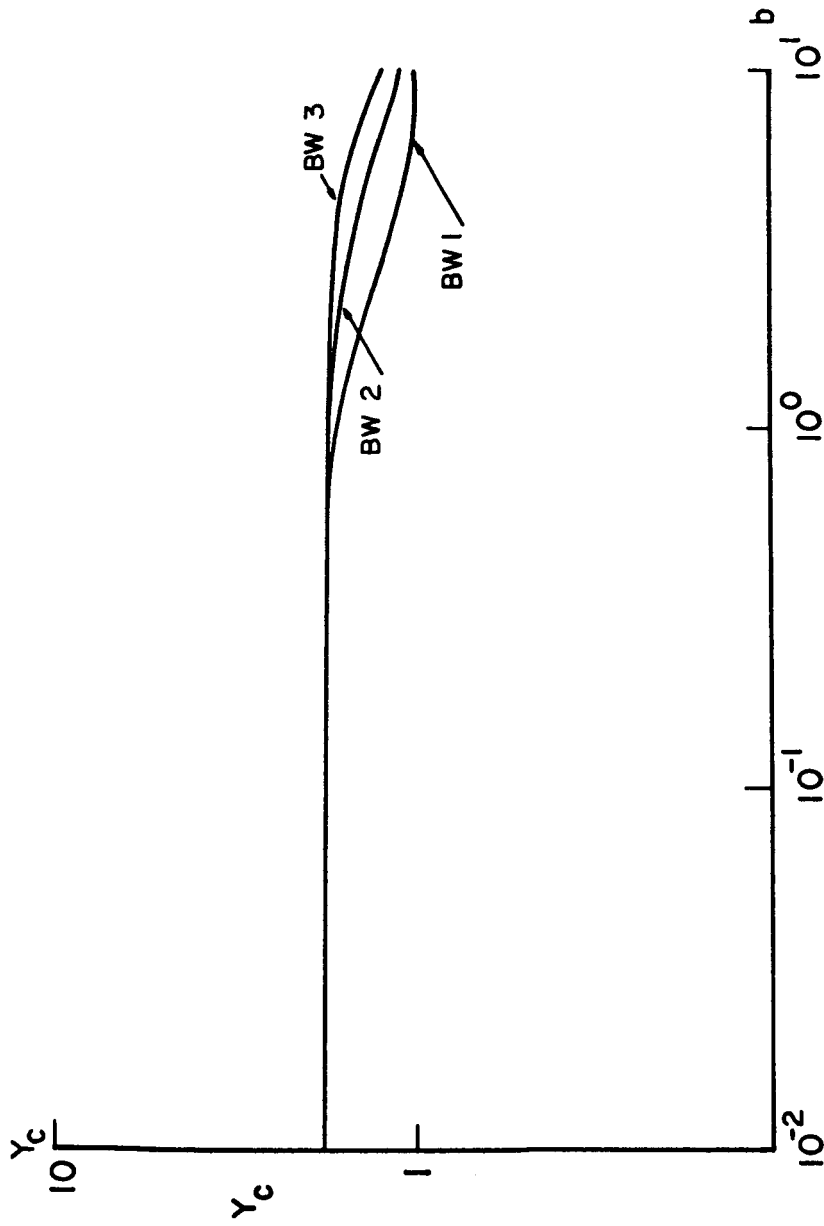
$$(45) \quad Y_{c,k} = \left[1 + \frac{\alpha^2 \left(-\frac{\pi}{2} + k\pi \right)^2}{b^2 + \left(-\frac{\pi}{2} + k\pi \right)^2} \right]^{\frac{1}{2}} \quad k \geq 1$$

Figures 13 and 14 show $Y_{c,k}$ plotted versus α and b for the lowest order modes. In the limit of large b (Figure 14), the cutoff frequency for a given magnetic field strength approaches ω_p ($Y_{c,k} = 1$). For thin slabs ($b \ll \pi$) the cutoff frequencies approach the upper hybrid frequency $\sqrt{1 + \alpha^2}$. Figure 13 shows the variation of $Y_{c,k}$ versus α for $b = 1$. Because of the relatively low value of b , the curves for the order 2 and higher are practically the same. The



CUTOFF FOR THE ANTISYMMETRIC MODES VS α FOR $b=1$

FIGURE 15



CUTOFF FOR THE ANTISYMMETRIC MODES VS b FOR $\alpha=1.5$

FIGURE 16

cutoff of the higher order modes occurs practically at the upper hybrid frequency.

In the case where α is greater than one ($\Omega > \omega_p$) there exists a value of slab thickness, b_k , given by

$$b_k = \frac{(2k-1)\pi}{2} \left[-1 + \frac{\alpha}{\sqrt{\alpha^2-1}} \right]^{1/2}$$

such that if b is greater than b_k , the cutoff frequency for the mode of order k is smaller than α . Figure 22 shows a case where this particular condition is satisfied for the mode of order one. As we can see, the wave is first forward and becomes backward as $Z \rightarrow \infty$. If b is sufficiently large, the dispersion curve does not cross $Y = \alpha$ and has only a positive slope.

2.) Antisymmetric case

The cutoff frequency for the mode of order k is obtained by writing

$$X_2 b = i k \pi \quad k \geq 1$$

which yields:

$$(46) \quad Y_{c,k} = \left[1 + \frac{\alpha^2 k^4 \pi^4}{k^2 \pi^2 + b^2} \right]^{1/2} \quad k \geq 1$$

Figures 15 and 16 show $Y_{c,k}$ plotted versus α and b for the lowest order modes. For a given value of α , as b increases, $Y_{c,k}$ tends to 1. The cutoff frequencies become smaller and approach the plasma frequency. When b is much smaller than π , the cutoff frequencies approach the upper hybrid frequency $\sqrt{1 + \alpha^2}$.

When α is greater than one, the cutoff frequency may be smaller than the cyclotron frequency if b is greater than b_k , where

$$b_k = k \pi \left[-1 + \frac{\alpha}{\sqrt{\alpha^2 - 1}} \right]^{\frac{1}{2}}$$

Under these circumstances, the wave of order k starts as a forward wave, crosses the line $Y = \alpha$ and becomes a backward wave as $Z \rightarrow \infty$. If b is much greater than b_k , the wave remains forward until it reaches the resonance for $\omega = \Omega$. It does not cross the line $Y = \alpha$.

2.3.2 Resonances

The resonances occur when the phase velocity vanishes for a finite value of the frequency. This is equivalent to saying that Z becomes very large for a finite value of Y . (See Footnote 1). Two cases have to be considered depending upon whether α is greater or smaller than one. The case $\alpha = 1$ is just a limiting case of the two previous ones. The symmetric and antisymmetric dispersion relations yield the same resonances. These are:

$$1.) \quad \alpha < 1$$

$$Y = \alpha \quad (\omega = \Omega)$$

$$Y = \sqrt{\frac{1 + \alpha^2}{2}} \quad \left(\omega = \sqrt{\frac{\omega_p^2 + \Omega^2}{2}} \right)$$

$$Y = 1 \quad (\omega = \omega_p)$$

$$2.) \quad \alpha > 1$$

$$Y = 1 \quad (\omega = \omega_p)$$

$$Y = \alpha \quad (\omega = \Omega)$$

¹It should be noted that as the resonance is approached, the theory breaks down and the thermal velocity of the electron should be included in the analysis to obtain the correct results. As obtained by Diamant, Granatstein and Schlesinger (1966) the phase velocity of the wave does not vanish but approaches the electron thermal velocity.

2.4 RESULTS OF THE ANALYSIS

2.4.1 Results from the General Dispersion Relation

The dispersion relation in the symmetric case is found from

Equation 20 to be

$$(48) \quad \begin{vmatrix} \cosh X_1 b + \frac{TX_0}{X_1} \sinh X_1 b & \cosh X_2 b + \frac{TX_0}{X_2} \sinh X_2 b \\ \beta_1 \left(\cosh X_1 b + \frac{X_0}{X_1} \sinh X_1 b \right) & \beta_2 \left(\cosh X_2 b + \frac{X_0}{X_2} \sinh X_2 b \right) \end{vmatrix} = 0$$

$$\text{where } X_{1,2}^2 = \frac{B \pm \sqrt{\Delta}}{2} \quad \text{and} \quad \Delta = B^2 - 4C$$

For the antisymmetric case we simply interchange the cosh and sinh functions.

Since we are interested in the trapped modes, we need only consider X_0 real and positive. Depending upon the sign of Δ and its absolute value, different cases may occur. These include:

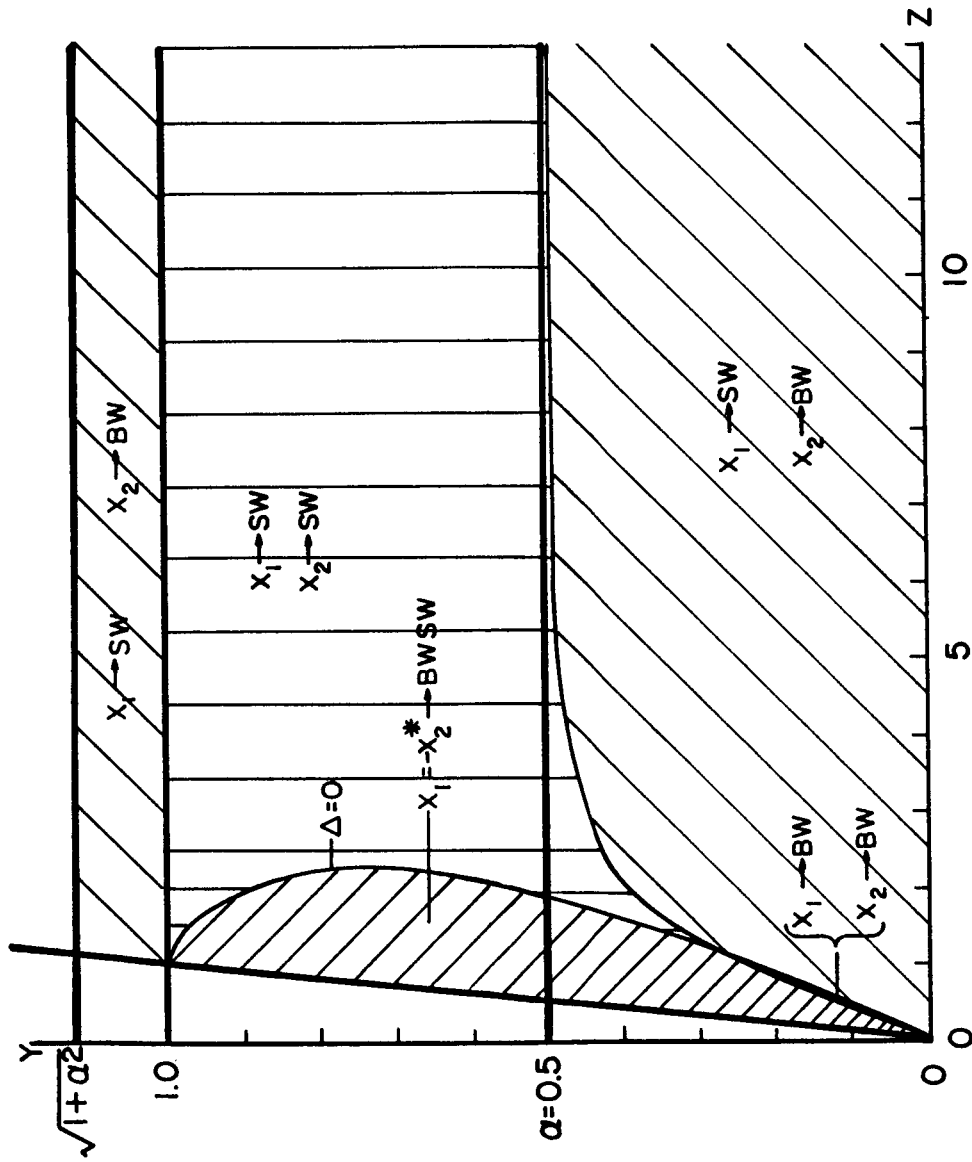
$$1.) \quad \Delta < 0$$

The two propagation constants X_1 and X_2 are complex and the wave is both oscillatory and damped. These are the body wave-surface waves mentioned in the introduction. Using the fact that $X_1 = -X_2^*$, where * means complex conjugate, it is easily shown that the dispersion relation is pure imaginary.

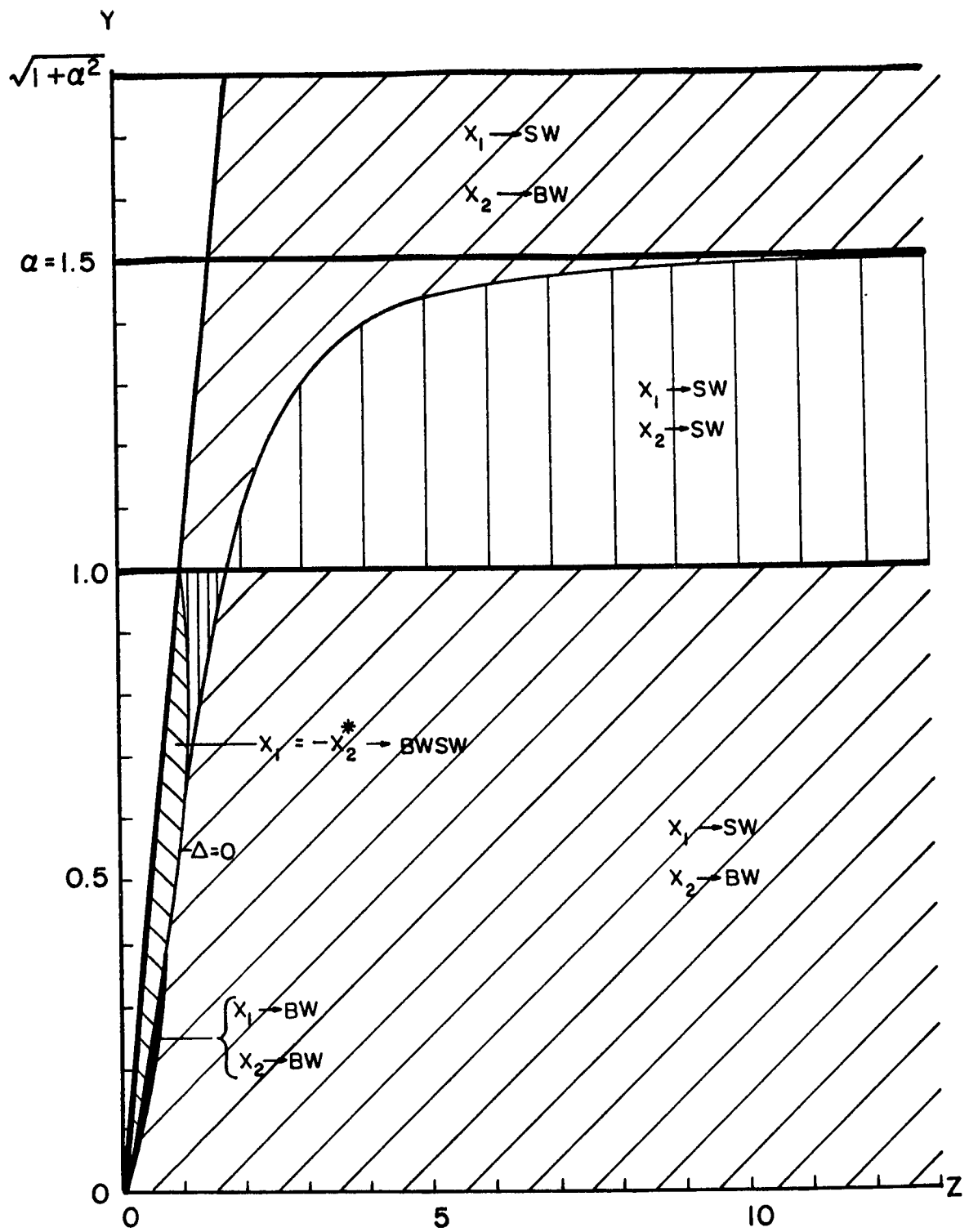
$$2.) \quad \Delta = 0$$

The dispersion equation is identically satisfied. However, the closed curve $\Delta = 0$ is not a solution because we do not have two independent solutions to match the boundary conditions.

$$3.) \quad \Delta > 0, \quad B \pm \sqrt{\Delta} > 0$$



REGIONS OF THE YZ PLANE WHERE DIFFERENT MODES PROPAGATE, $\alpha < 1$
FIGURE 17



REGIONS OF THE YZ PLANE WHERE DIFFERENT MODES
PROPAGATE, $\alpha > 1$
FIGURE 18

The quantities X_1^2 and X_2^2 are both positive implying the existence of two surface waves. The dispersion relation is real.

$$4.) \quad \Delta > 0, B + \sqrt{\Delta} > 0, \quad B - \sqrt{\Delta} < 0$$

For this case X_1 is real and X_2 is pure imaginary. The fields are then a superposition of one surface wave corresponding to X_1 and one body wave, corresponding to X_2 . The dispersion relation is real for the symmetric case and pure imaginary for the antisymmetric case.

$$5.) \quad \Delta > 0, B + \sqrt{\Delta} < 0, \quad B - \sqrt{\Delta} < 0$$

In this case, X_1 and X_2 are both imaginary. Two body waves propagate. The dispersion relation is real.

No other cases are possible in the region of the $Y Z$ plane where trapped modes propagate. It should be emphasized at this point that the different regions described above are different depending upon whether α is smaller or greater than one. The two possible configurations are shown on Figures 17 and 18.

When α becomes very small, the surface enclosed by the curve $\Delta = 0$ becomes large and in the limit of α going to zero, the surface occupies the whole frequency band $0 < Y < 1$. This seems to be in contradiction to the results obtained in the case where no magnetic field was present. However, it can be shown that the imaginary part of X_1 and X_2 becomes very small so that in the limit of zero magnetic field, $X_1 = X_2$ are both real. In other words, the body wave-surface wave becomes a pure surface wave. On the contrary, when α becomes large, the surface enclosed in the curve $\Delta = 0$ shrinks to zero and the wave becomes a pure body wave.

The fields have been obtained as a superposition of two waves, one corresponding to X_1 (wave 1) and one corresponding to X_2 (wave 2). In the limit of large Z/Y , X_1 has an asymptotic value of Z , and X_2 tends to the propagation constant obtained in the quasistatic approximation, except for $Y = \alpha$. We will show in section 2.5 that in general the magnitude of wave 2 is much greater than the magnitude of wave 1. Therefore, when we speak about the type of wave propagating we will refer to wave 2.

Three groups of solutions are obtained.

1.) The low frequency body wave

When α is smaller than one, these waves are forward and exist in the frequency band $0 < Y < \alpha$. Resonance is approached from the low frequency side and occurs for $Y = \alpha$. When α is greater than one, the waves exist in the frequency band $0 < Y < 1$. Resonance occurs for $Y = 1$. The effect of increasing the thickness of the slab is to shift the curves to the left, implying an increase in the group velocity of the wave for a given frequency. As the order of the mode is increased, the group velocity of the wave at a given frequency decreases. Resonance is approached for larger values of Z for the mode of order $n + 1$ than for the mode of order n .

2.) The high frequency body waves

When α is smaller than one, the waves propagate in the frequency gap $1 < Y < \sqrt{1 + \alpha^2}$ and approach the resonant frequency $Y = 1$ from the high frequency side. When b is small, the group velocity is very small and negative, implying backward waves. The resonance is approached for

very large values of Z . As b is increased, the cutoff frequency decreases until, when b is sufficiently large, the lowest order body wave starts as a forward wave. When b is increased further, higher order modes start as forward waves. They reach a maximum frequency and become backward as they approach resonance. The resonance is approached more rapidly when b becomes larger.

When α is greater than one, for small values of b , the waves are backward, existing in the frequency band $\alpha < Y < \sqrt{1 + \alpha^2}$. They approach the resonant frequency $Y = \alpha$ from the high frequency side. As b is increased, the cutoff frequency decreases. When b becomes sufficiently large, the body waves exist in the region $1 < Y < \sqrt{1 + \alpha^2}$. The waves start as forward waves, cross the line $Y = \alpha$, reach a maximum of frequency and become backward as they approach resonance from the high frequency side. If b is further increased, the body waves of lowest order remain forward and approach the resonance from the low frequency side. These results are different from the ones obtained in the quasistatic theory where a cutoff at $Y = \sqrt{1 + \alpha^2}$ was found. (The error in the quasistatic theory is not surprising, since this approximation is not valid near $Z = Y$.) The dispersion curves for higher order modes are situated above the preceding ones.

3.) The lowest order mode

The lowest order mode always starts as a forward body wave-surface wave. For large values of b in the symmetric case and for the antisymmetric case, the dispersion curve is very close to the $Y = Z$ line. This implies that the propagation constant in the ξ direction

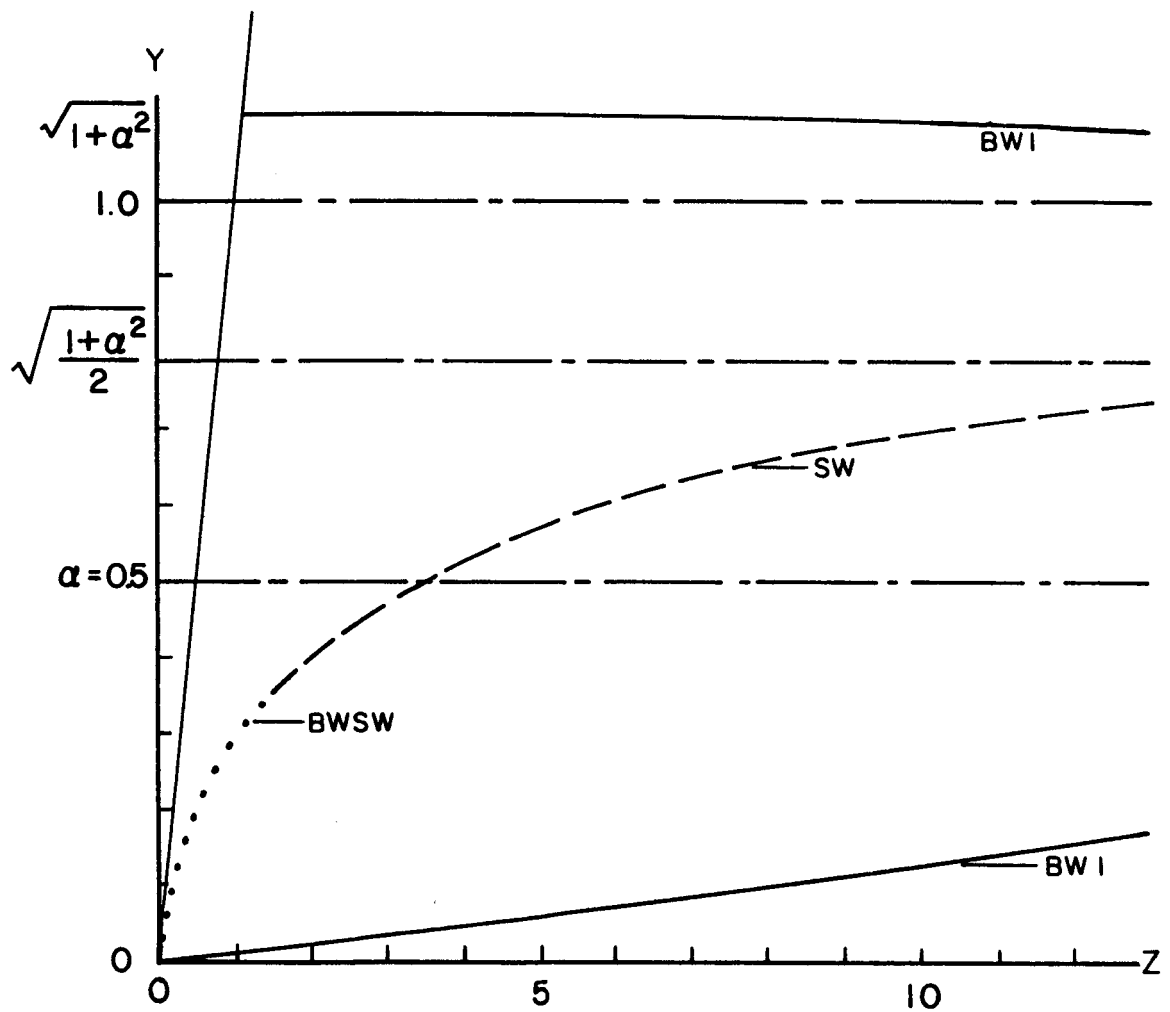
outside the plasma is almost equal to zero. In other words, the wave is very loosely guided by the slab. (Granatstein, Schlesinger and Vigant, 1963). When α is smaller than one, the body wave-surface wave becomes a surface wave as the dispersion curve crosses the curve $\Delta = 0$. The S W is forward in the symmetric case. For the antisymmetric case, the surface wave is forward when b is large, and backward when b is small. In both these cases, it approaches the resonance $Y = \sqrt{\frac{1 + \alpha^2}{2}}$. As Z increases, the fields are damped more rapidly as a function of distance from the interface so that the energy of the wave is concentrated at the interface. When α is larger than one, the B W S W becomes the B W O as it crosses the curve $\Delta = 0$. For the symmetric case, the wave is forward and approaches resonance at $Y = 1$. For the antisymmetric case, however, the wave is first forward and then backward when b is small. The resonance $Y = \alpha$ is then approached from the high frequency side. The wave is purely forward for large b and approaches the resonance $Y = \alpha$ from the low frequency side.

All of these features are exhibited when we chose the following set of values for α and b :

$\alpha = 0.5$	$b = 0.1$
$\alpha = 0.5$	$b = 1.$
$\alpha = 1.5$	$b = 0.1$
$\alpha = 1.5$	$b = 1.$

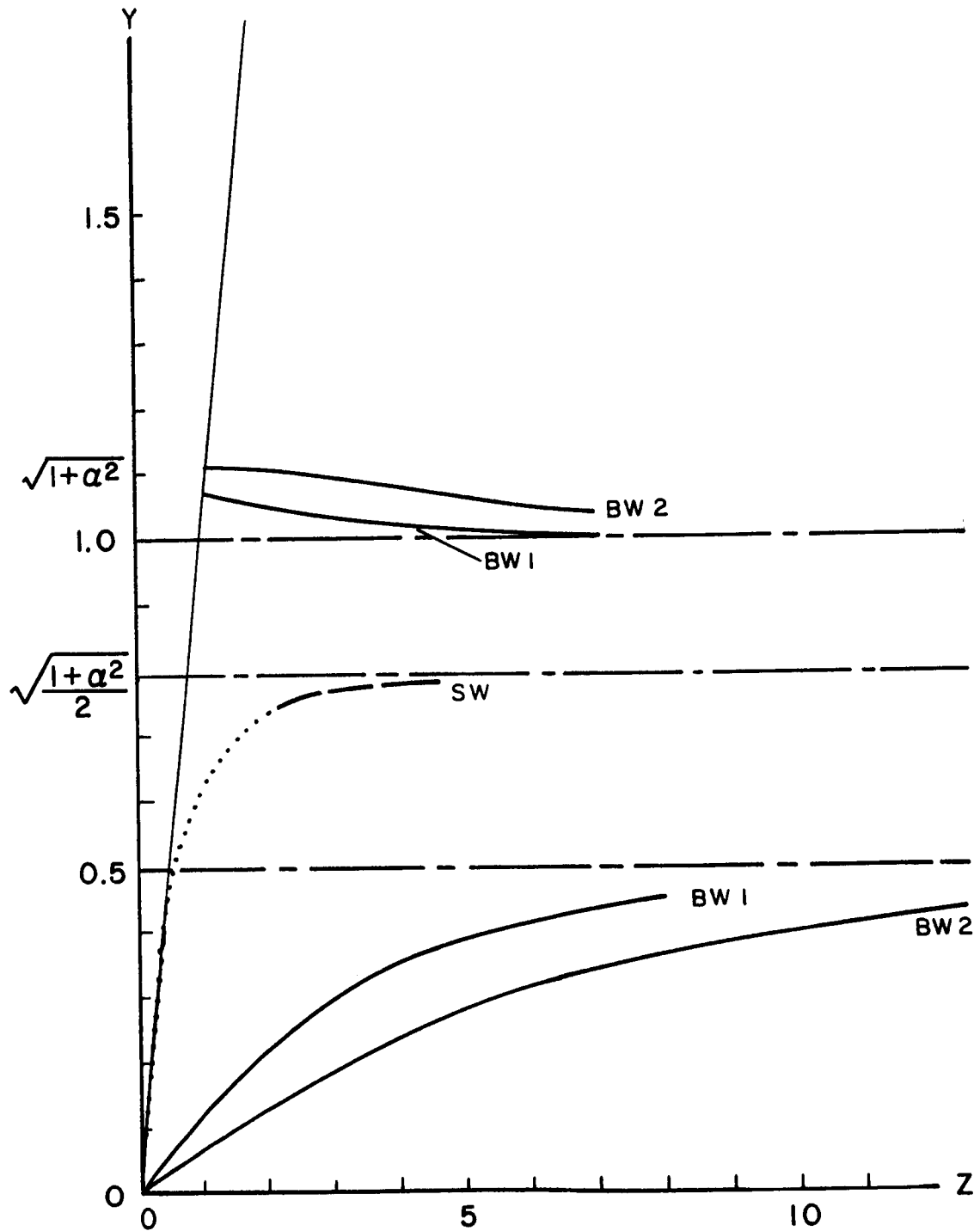
In addition, these values are representative of those found in laboratory experiments.

The dispersion curves obtained are quite different depending

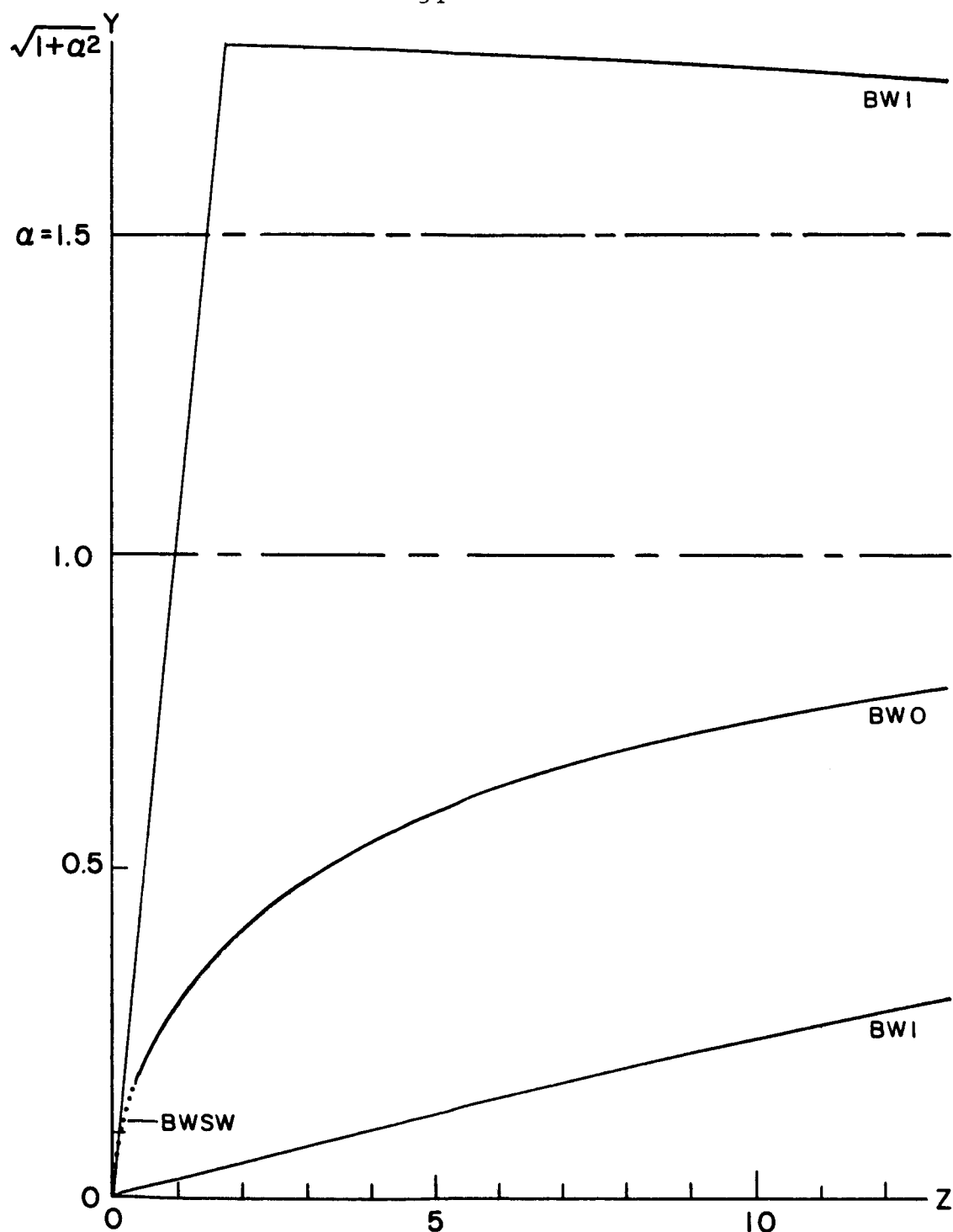


SOLUTION OF THE EXACT SYMMETRIC DISPERSION
RELATION FOR $\alpha = 0.5$ AND $b = 0.1$

FIGURE 19

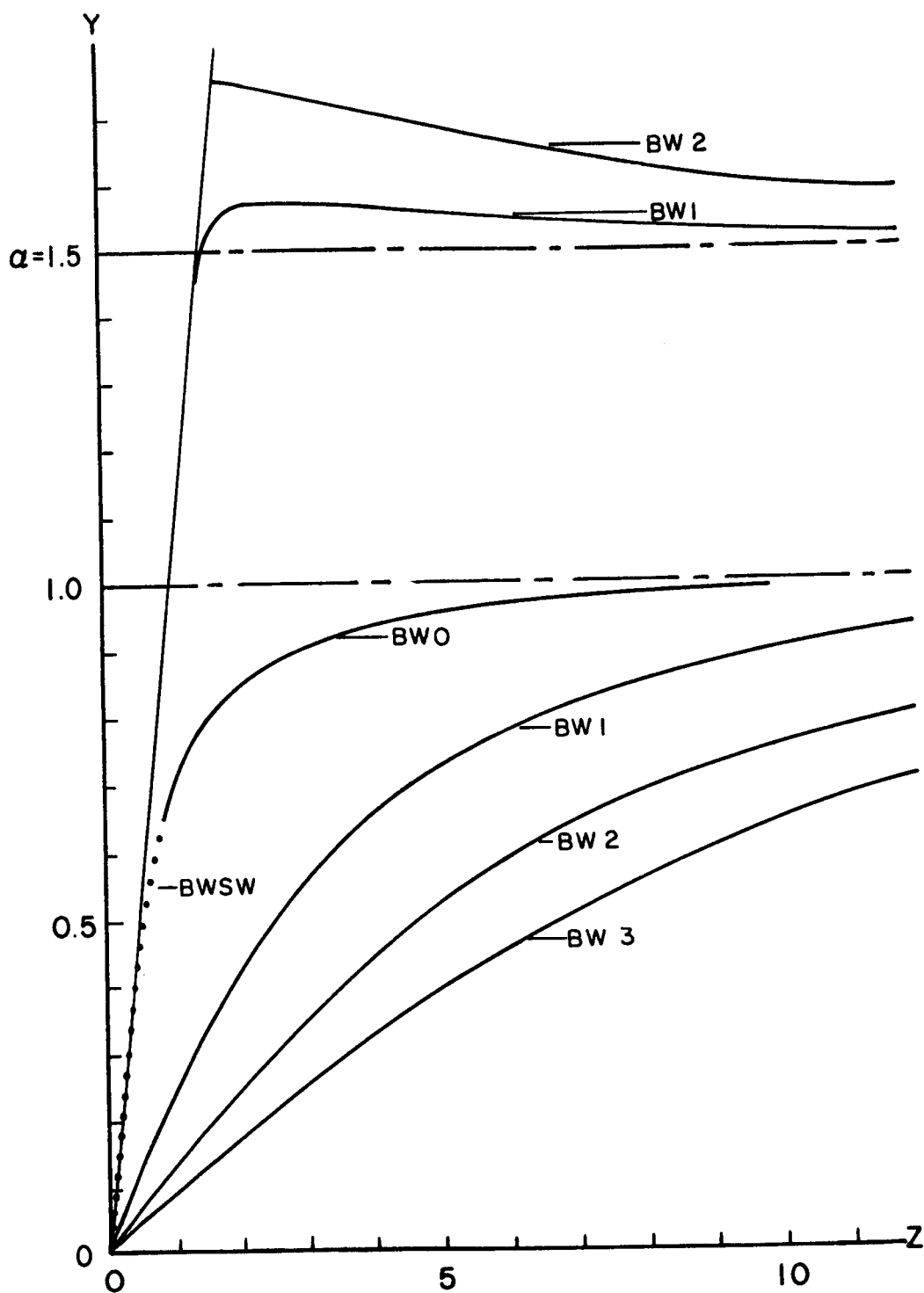


SOLUTION OF THE EXACT SYMMETRIC DISPERSION
RELATION FOR $\alpha=0.5$ AND $b=1.0$
FIGURE 20



SOLUTION OF THE EXACT SYMMETRIC DISPERSION
RELATION FOR $\alpha=1.5$ AND $b=0.1$

FIGURE 21



SOLUTION OF THE EXACT SYMMETRIC DISPERSION
RELATION FOR $\alpha = 1.5$ AND $b = 1.0$
FIGURE 22

upon whether the symmetric or the antisymmetric case is considered. Consequently, the two cases will be analyzed separately.

2.4.2 Symmetric Case

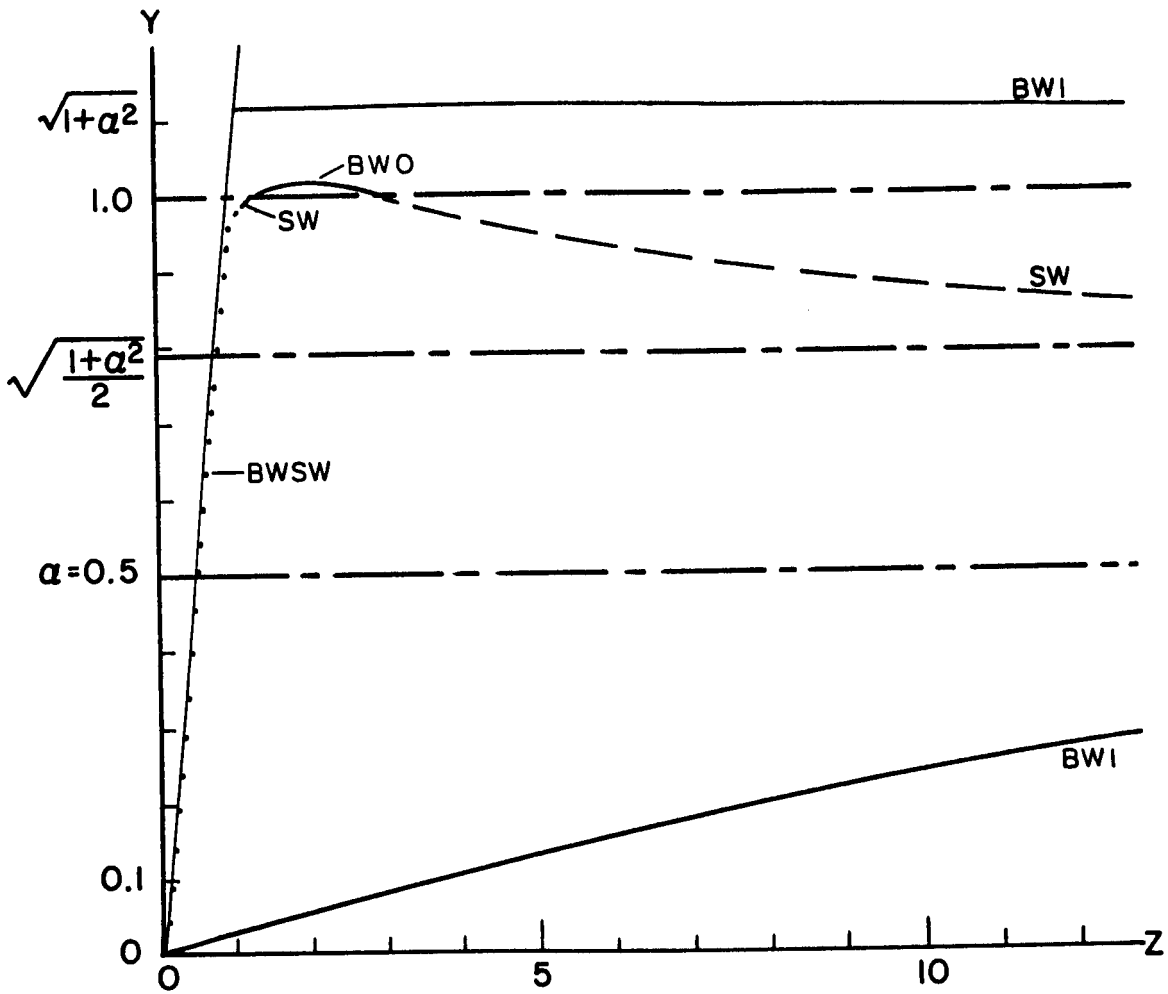
1.) $\alpha = 0.5$, $b = 0.1$ and 1.0

The behavior just described is shown on Figures 19 and 20. For the lowest order mode, it should be noted that for $b = 0.1$ the surface wave exists for values of Y smaller than α . When $b = 1.0$ the surface wave exists only in a very narrow gap of frequencies near resonance and its group velocity is very small. This is different from the result of quasistatic theory where the surface wave starts at $Y = \alpha$. When b is small, the low frequency body wave approaches resonance for very large values of Z .

When the order of the mode increases, the dispersion curve approaches resonant frequency very slowly. Yip and Seshadri (1966) introduce a different definition of resonance and obtain different resonances which depend on the order of the mode. We do not feel that their definition contributes new understanding and therefore consider it superfluous.

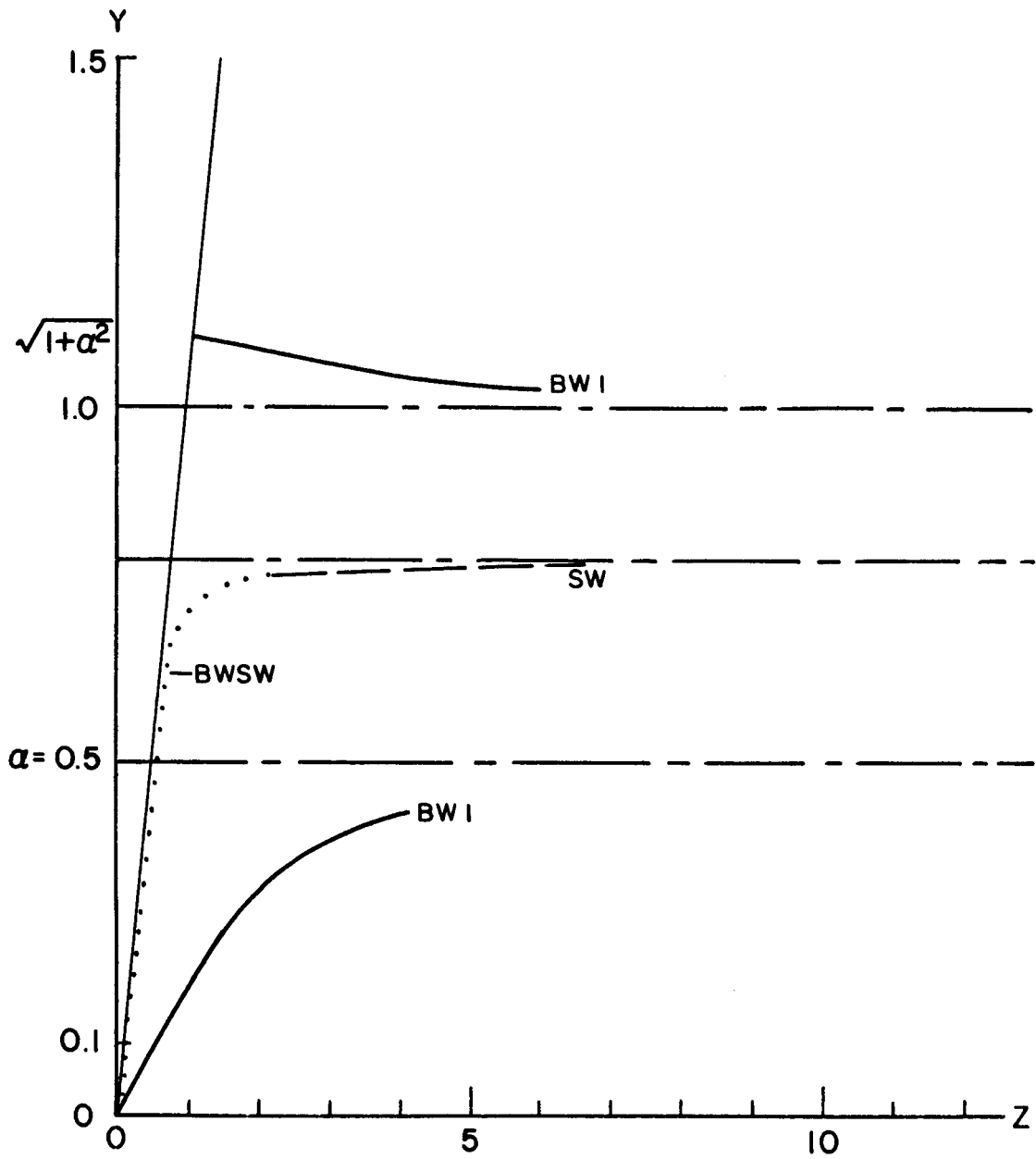
2.) $\alpha = 1.5$, $b = 0.1$ and 1.0

As shown on Figures 21 and 22, the B W O exists for very low frequencies when the slab thickness is small, ($b = 0.1$) but only close to the resonant frequency $Y = 1$ when b is large ($b = 1$). This again differs from the quasistatic theory which predicted the existence of B W O down to zero frequency. The relation $b > b_1$ is satisfied when $\alpha = 1.5$ and $b = 1$ so that the cutoff frequency for the high frequency



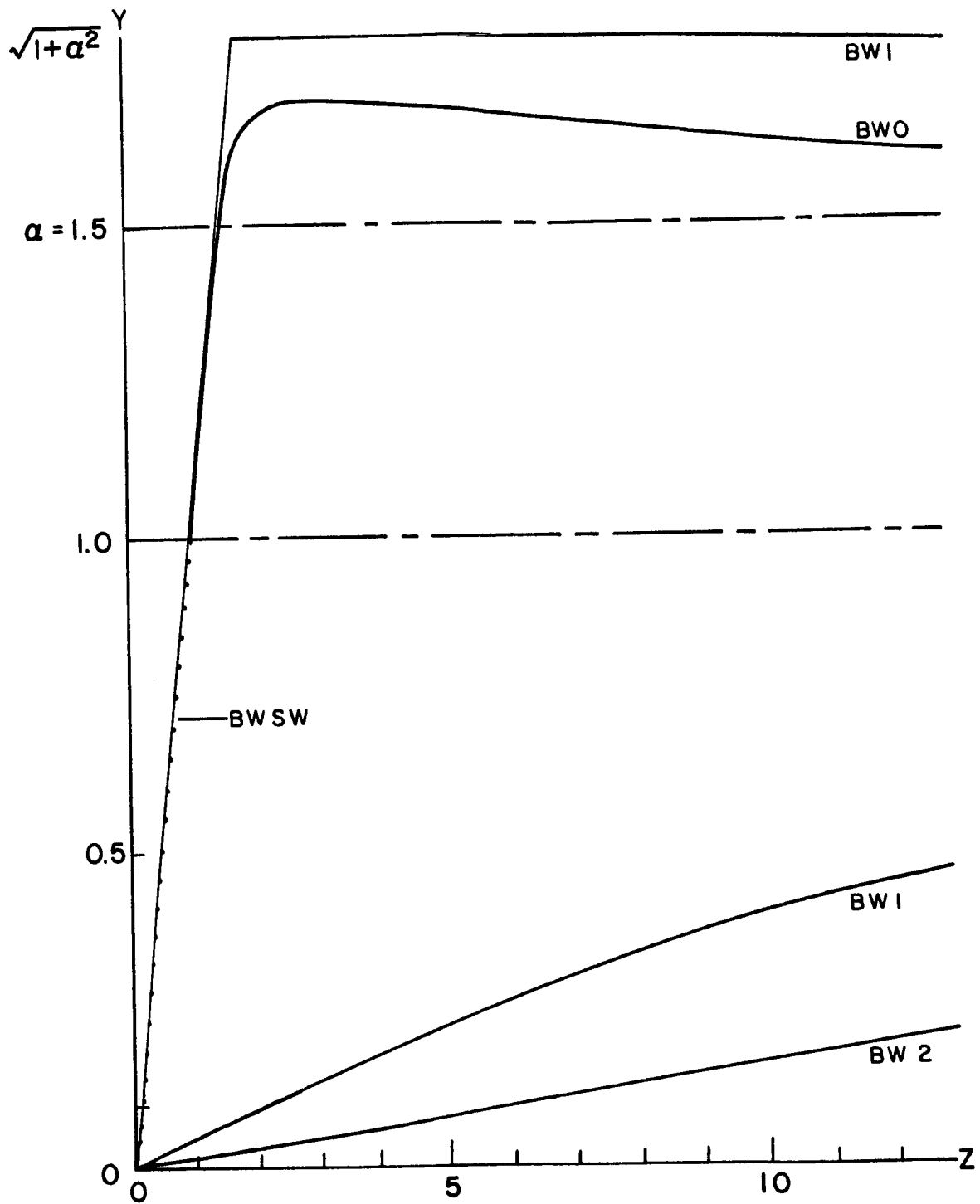
SOLUTION OF THE EXACT ANTISYMMETRIC DISPERSION
RELATION FOR $\alpha = 0.5$ AND $b = 0.1$

FIGURE 23



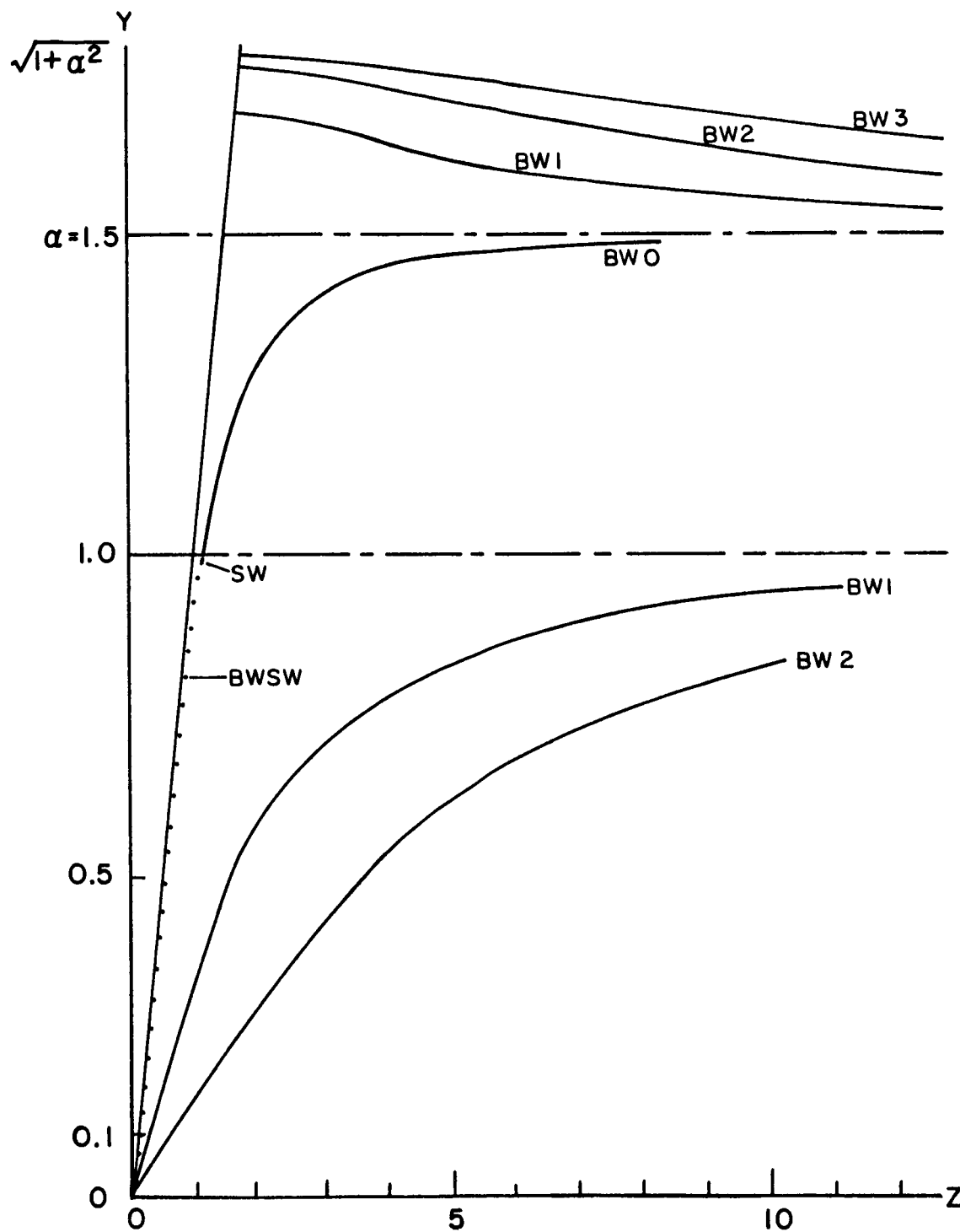
SOLUTION OF THE EXACT ANTISYMMETRIC DISPERSION
RELATION FOR $\alpha=0.5$ AND $b=1.0$

FIGURE 24



SOLUTION OF THE EXACT ANTISYMMETRIC DISPERSION
RELATION FOR $\alpha = 1.5$ AND $b = 0.1$

FIGURE 25



SOLUTION OF THE EXACT ANTISYMMETRIC DISPERSION
RELATION FOR $\alpha = 1.5$ AND $b = 1.0$

FIGURE 26

B W 1 is smaller than α . The forbidden gap, $1 < Y < \alpha$ predicted in the quasistatic analysis does not exist any more for large values of b .

2.4.3 Antisymmetric Case

1.) $\alpha = 0.5$, $b = 0.1$ and 1. Figures 23 and 24

When no magnetic field is present, we have seen that for thin slabs the surface wave was first forward and then backward. When the slab thickness becomes larger, the surface wave remained forward. For small values of the magnetic field, similar characteristics can be seen for the lowest order mode. When b is small (Figure 23) the lowest order mode starts as a B W S W. As it crosses the curve $\Delta = 0$ it becomes a forward surface wave until $Y = 1$. For Y larger than one the wave is a body wave which is first forward and then backward. The dispersion curve crosses again the line $Y = 1$ and corresponds to a backward surface wave. When $b = 1$ the wave is a forward surface wave with very low group velocity as soon as the dispersion curve crosses the loop $\Delta = 0$. It should be emphasized that the backward B W O of the quasistatic theory now exists only above the line $Y = 1$ in the exact theory. In the long wavelength region, the B W O has been replaced by a B W S W. Also, the S W was predicted in quasistatic theory to be a backward wave, while it is shown here that it remains forward for large value of b .

2.) $\alpha = 1.5$, $b = 0.1$ and 1. Figures 25 and 26

The forbidden gap $1 < Y < \alpha$, found in the quasistatic theory does not exist any more in the exact theory. For small values of b the dispersion curve of the B W O propagates close to the line $Y = Z$ and is therefore very loosely guided by the slab. This is shown on

Figure 25. When $b = 1$, the B W O remains forward as indicated by the Figure 26.

It should be noted at this point that there exists a very narrow gap of frequencies for which the lowest order mode is a pure surface wave. This region is situated below the line $Y = 1$ and above the curve $\Delta = 0$. Although this region is very small, it always exists when α is greater than one. For small values of b the dispersion curve is very close to the line $Y = Z$ so that the region where this forward surface wave exists is very small since the curve $\Delta = 0$ is tangent to the line $Y = 1$. However, as b increases, the gap of frequencies for which this surface wave exists becomes larger because the dispersion curve is not near the line $Y = Z$.

We can see a similarity between the behavior of the dispersion curves of the lowest order mode for α smaller and α greater than one. In both cases, these waves are first forward and then backward for thin slabs and purely forward when the thickness of the slab is increased. It should be emphasized, however, that the fields inside the slab have different characteristics depending upon whether the S W or the B W O is considered. For the surface wave, the fields vary as $\sinh$ and $\cosh$ of real numbers and in the limit of large Z the variation is of the form $\sinh \left[\frac{1 - \alpha^2}{1 + \alpha^2} \right] K_{\zeta} \xi$ or $\cosh \left[\frac{1 - \alpha^2}{1 + \alpha^2} \right] K_{\zeta} \xi$. The amplitude of all the fields is a maximum at the interface. However, for the B W O only the axial components of the field have their maximum at the interface. The other components have their maximum at the center. Furthermore, the

field variation is as sines and cosines of real numbers and in the limit of large Z_1 the field distribution is of the form $\sin \frac{\pi \xi}{2a}$ or $\cos \frac{\pi \xi}{2a}$.

Because of these significant differences between the surface wave and B W O, we do not feel that B W O should be classified as a surface wave as done by Granatstein and Schlesinger (1964).

2.5 FIELDS CHARACTERISTICS

This section will be devoted to the study of the relative amplitude of the different field components inside the plasma. All the components will be normalized by setting E_2 in Equations 20 and 23 equal to unity. We will first consider the limits of the coefficients of the hyperbolic functions in some particular cases for the symmetric solution. The E_ζ and H_ζ field components for the symmetric case will then be plotted, showing some of the essential features of the wave.

For the antisymmetric case, we do not need to consider again the limit of the coefficients of the hyperbolic functions, since they are the same as in the symmetric case. The coefficient of the cosh functions in the symmetric case are now the coefficients of the sinh functions and inversely. In order to relate the present study to the case of the plasma slab bounded on one side by a conducting screen, all the field components for the antisymmetric solution are plotted and analyzed.

2.5.1 Symmetric Case

It was found in section 2.1.2 that the field components inside the plasma could be written in the following form:

$$(49) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} E_1 \cosh K_{\xi_1} \xi + E_2 \cosh K_{\xi_2} \xi \\ \frac{-i E_1 \beta_1 \cosh K_{\xi_1} \xi - i E_2 \beta_2 \cosh K_{\xi_2} \xi}{D Y Z} \\ \frac{-i E_1 (TY^2 + X_1^2) \sinh K_{\xi_1} \xi - i E_2 (TY^2 + X_2^2) \sinh K_{\xi_2} \xi}{X_1 Z} \\ \frac{E_1 \beta_1 \sinh K_{\xi_1} \xi + E_2 \beta_2 \sinh K_{\xi_2} \xi}{D X_1 Z} \\ \frac{-E_1 \beta_1 \sinh K_{\xi_1} \xi - E_2 \beta_2 \sinh K_{\xi_2} \xi}{DY X_1} \\ \frac{-i E_1 TY \sinh K_{\xi_1} \xi - i E_2 TY \sinh K_{\xi_2} \xi}{X_1} \end{bmatrix} \times \exp (i K_{\zeta} \zeta - i \omega t)$$

where E_1 is given by:

$$(50) \quad \beta_1 \left(\cosh X_1 b + \frac{X_0}{X_1} \sinh X_1 b \right) E_1 + \beta_2 \left(\cosh X_2 b + \frac{X_0}{X_2} \sinh X_2 b \right) = 0$$

and the dispersion relation (Equation 20) must be satisfied. As indicated earlier, the fields inside the plasma are the superposition of two waves, designated as wave 1 (corresponding to E_1) and wave 2 (corresponding to E_2). We are now going to consider some particular limiting cases.

2.5.1.1 Zero Magnetic Field Limit

In the limit of B going to zero, X_1^2 and X_2^2 tends to the same limit

$$X^2 = Z^2 - Y^2 + 1$$

and the coefficients $\beta_{1,2}$ reduce to

$$(51) \quad \beta_{1,2} \rightarrow \mp D Z \sqrt{Y^2 - 1}$$

From Equation 50 we see that $E_1 = 1$ and the field components simplify to give

$$(52) \quad \begin{bmatrix} E_\zeta \\ H_\zeta \\ E_\xi \\ E_\eta \\ H_\xi \\ H_\eta \end{bmatrix} = \begin{bmatrix} 2 \cosh K_{\xi_2} \xi \\ 0 \\ -2i \frac{Z}{X_2} \sinh K_{\xi_2} \xi \\ 0 \\ 0 \\ -2i \frac{TY}{X_2} \sinh K_{\xi_2} \xi \end{bmatrix} \times \exp(i K_\zeta \zeta - i\omega t)$$

Only one wave is propagating in the ζ direction. Since only E_ζ , E_ξ and H_η survive, this wave is transverse magnetic.

2.5.1.2 Infinite Magnetic Field Limit

When the strength of the magnetic field becomes infinite, the propagation constants inside the plasma tend to the following limits:

$$X_1^2 = Z^2 - Y^2$$

$$X_2^2 = T (Z^2 - Y^2)$$

The coefficients $\beta_{1,2}$ yield

$$\beta_1 = -\frac{1}{Y^2} (Z^2 - Y^2)$$

$$\beta_2 \rightarrow \frac{Z^2 (Y^2 - 1)}{\alpha^2 (Z^2 - Y^2)}$$

With these values, Equation 50 indicates that E_1 is proportional to $\frac{1}{\alpha^2}$ so that E_1 is much smaller than 1. Consequently wave 1 becomes negligible as $\alpha \rightarrow \infty$.

The field components can now be written

$$(53) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} \cosh K_{\xi_2} \xi \\ 0 \\ -\frac{i E_2 T Z}{\sqrt{T (Z^2 - Y^2)}} \sinh K_{\xi_2} \xi \\ 0 \\ 0 \\ -\frac{i E_2 T Y}{\sqrt{T (Z^2 - Y^2)}} \sinh K_{\xi_2} \xi \end{bmatrix} \times \exp (i K_{\zeta} \zeta - i \omega t)$$

The wave is again transverse magnetic and only one wave propagates.

2.5.1.3 Large Z/Y Limit

The coefficients $\beta_{1,2}$ can be written as

$$(54) \quad \beta_{1,2} = \frac{\alpha^2 (Z^2 - Y^2)}{2 Y^2 (Y^2 - \alpha^2)} \left[1 \pm \sqrt{1 + \frac{4 Y^2 Z^2 (Y^2 - 1)}{\alpha^2 (Z^2 - Y^2)^2}} \right]$$

When Z is much larger than Y they reduce to:

$$\beta_1 = \frac{\alpha^2 Z^2}{Y^2 (Y^2 - \alpha^2)}$$

$$\beta_2 = \frac{-Y^2 + 1}{(Y^2 - \alpha^2)}$$

In order to study the field components, we consider four different cases. In all these cases, the amplitude of E_1 is much smaller than the amplitude of E_2 because the ratio β_2/β_1 is small.

1.) Limit of large Z/Y for the surface wave

The propagation constants are given by

$$X_1^2 = Z^2$$

$$X_2^2 = Z^2 T/S$$

As Z tends to infinity, X_1 and X_2 both tend to infinity and Y reaches the limit $\sqrt{\frac{1+\alpha^2}{2}}$ so that the field components are given by:

$$(55) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} \cosh K_{\xi_2} \xi \\ 0 \\ -i \frac{1-\alpha^2}{1+\alpha^2} \sinh K_{\xi_2} \xi \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \exp (i K_{\zeta} \zeta - i \omega t)$$

All the magnetic field components of the wave vanish. This explains why the results are in agreement with the quasistatic theory.

2.) Limit of large Z/Y when Y tends to $\alpha \neq 1$

As indicated in section 2.2.1, the propagation constants are now given by

$$\begin{aligned} X_1^2 &= Z^2 \\ X_2^2 &= Z^2 T/S - 2 T Y^2 \end{aligned}$$

In this case X_2^2 is finite so that the field components are

$$(56) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} \cosh K_{\xi_2} \xi \\ 0 \\ 0 \\ 0 \\ -\frac{(Y^2 - 1)}{\alpha X_2} \sinh K_{\xi_2} \xi \\ -i \frac{T Y}{X_2} \sinh K_{\xi_2} \xi \end{bmatrix} \times \exp (i K_{\zeta} \zeta - i \omega t)$$

The wave is T M; however, neither the condition $X_2 \gg Y$ nor the condition $\bar{H} = 0$ is satisfied. This is why the dispersion relation obtained in this case is not the same as the one for the quasistatic theory (Equations 31 and 32).

3.) Limit of large Z/Y when Y goes to 1

The propagation constants are given by

$$X_1^2 = Z^2$$

$$X_2^2 = Z^2 T/S$$

Taking the limits properly, we find X_2^2 finite and the only nonzero component of the field corresponds to E_ζ .

$$(57) \quad E_\zeta = \cosh K_{\xi_2} \xi \exp (i K_\zeta \zeta - i\omega t)$$

The wave is pure electric. The same result is obtained when we consider the infinite magnetic field limit (Equation 53) near resonance. As Y goes to 1 in Equation 53, only E_ζ remains different from zero. We simply obtain the ordinary longitudinal plasma oscillation.

4.) Low frequency body waves

We have seen that the phase velocity of the low frequency body wave was large so that the condition $Z \gg Y$ holds for these curves. Recall that X_1 is real and X_2 is pure imaginary for this region of the Y Z plane. Using Equations 14, 15, and 2, we find that $|E_1|$ is much smaller than 1.

2.5.1.4 Cutoff Limit

Cutoff occurs when the dispersion relation is satisfied for $X_0 = 0$. The propagation constants and the coefficients $\beta_{1,2}$ can be written

$$X_{1,2} = \frac{T}{S} Y^2 - TY^2 \pm \frac{DY^2}{S} \sqrt{T}$$

$$\beta_{1,2} = \mp DY^2 \sqrt{T}$$

Again X_2 is finite and the fields are simply given by

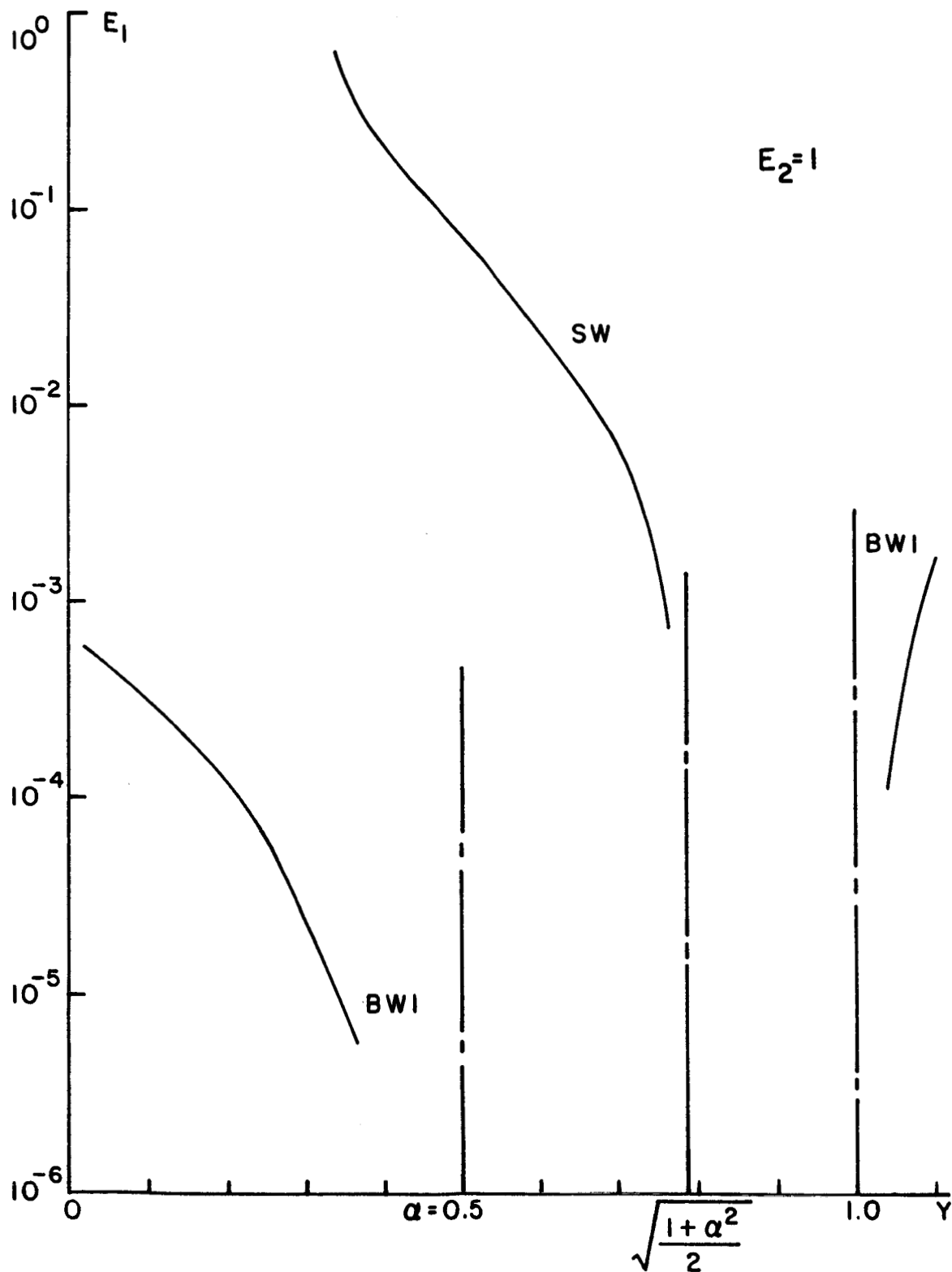
$$(58) \quad \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \\ E_{\xi} \\ E_{\eta} \\ H_{\xi} \\ H_{\eta} \end{bmatrix} = \begin{bmatrix} \cosh K_{\xi_2} \xi \\ -i \sqrt{T} \cosh K_{\xi_2} \xi \\ -i \frac{(TY^2 + X_2^2)}{X_2 Y} \sinh K_{\xi_2} \xi \\ \frac{\sqrt{Y^2 - 1}}{X_2} \sinh K_{\xi_2} \xi \\ -\frac{\sqrt{Y^2 - 1}}{X_2} \sinh K_{\xi_2} \xi \\ -\frac{TY}{X_2} \sinh K_{\xi_2} \xi \end{bmatrix} \times \exp(i K_{\zeta} \zeta - i\omega t)$$

The amplitude of E_1 vanishes as seen from Equation 20 since at cutoff $\cosh X_2 b$ vanishes. At the boundary of the slab, the wave is TEM as seen in Equation 58.

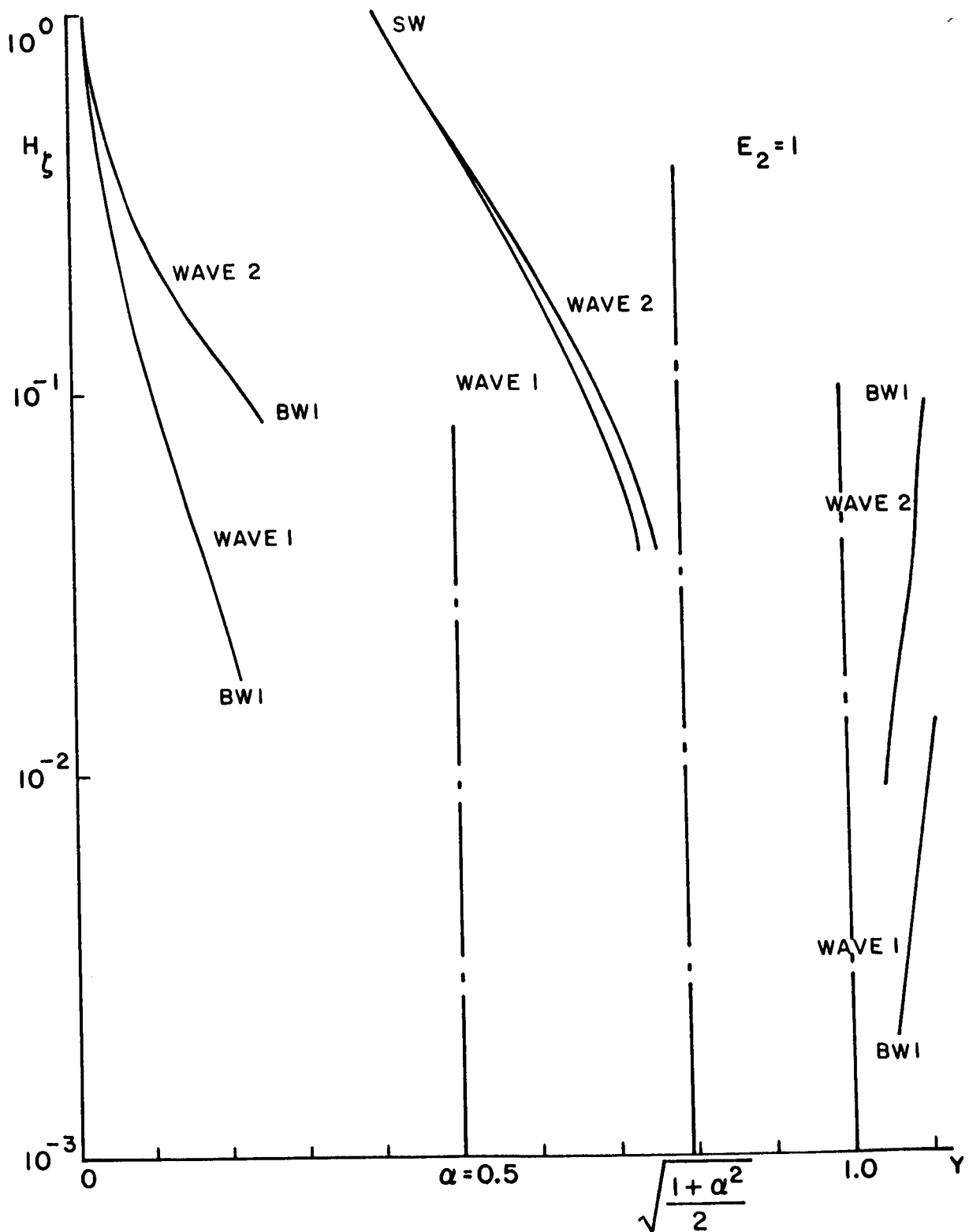
Except for the low frequency body wave case, where E_1 did not vanish but satisfies the relation $|E_1| \ll 1$, we have seen that for the above limits, the fields inside the plasma were given by the components of only one wave. We will see in section 2.6 that to very good accuracy the "one wave approximation" can in fact be used in the general case.

2.5.1.5 General Symmetric Case

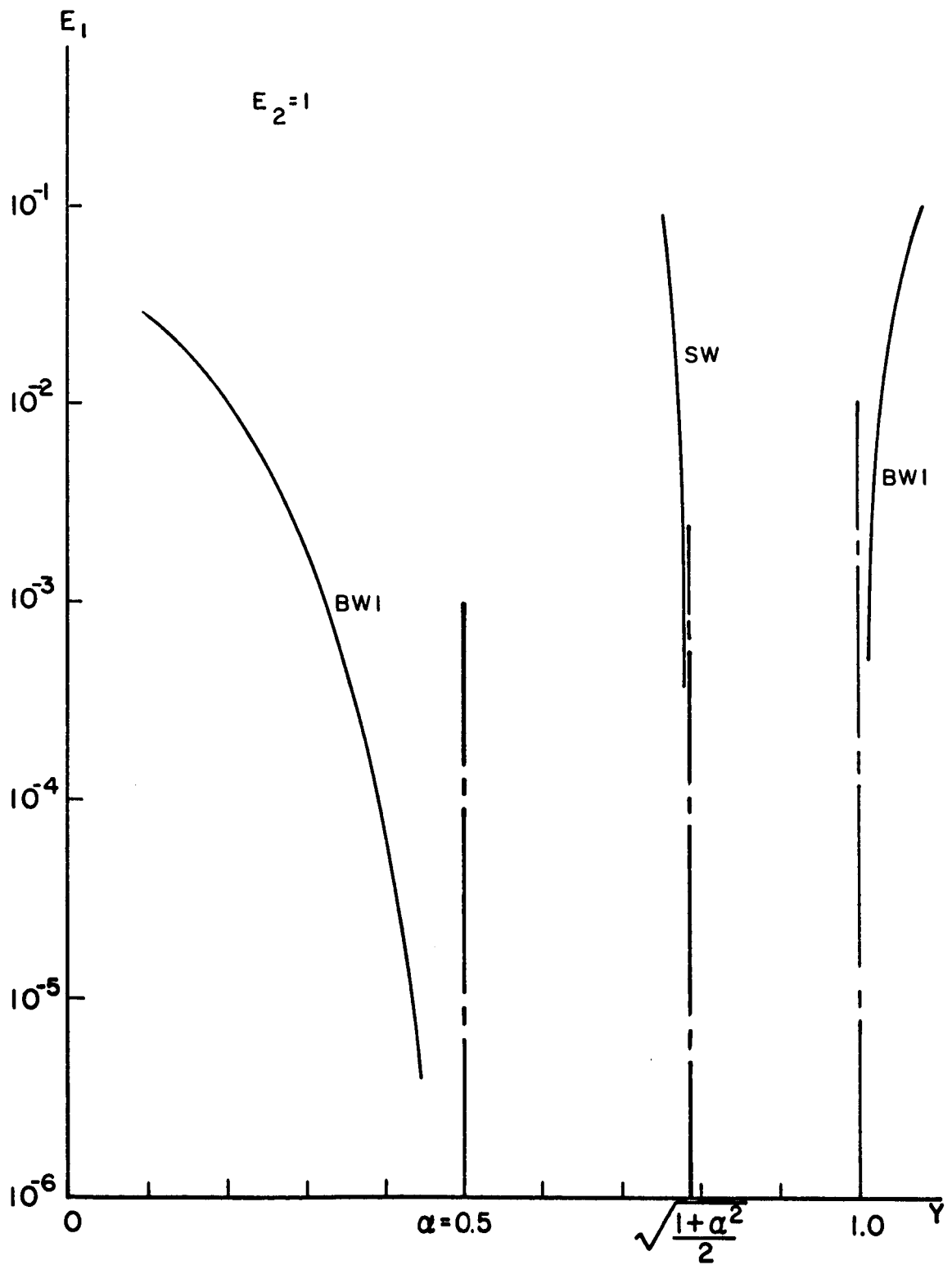
We consider the coefficients of the cosh functions in the expressions giving the longitudinal field components of the waves inside



E_1 VERSUS FREQUENCY FOR THE SYMMETRIC CASE,
WHEN $\alpha = 0.5$, $b = 0.1$
FIGURE 27

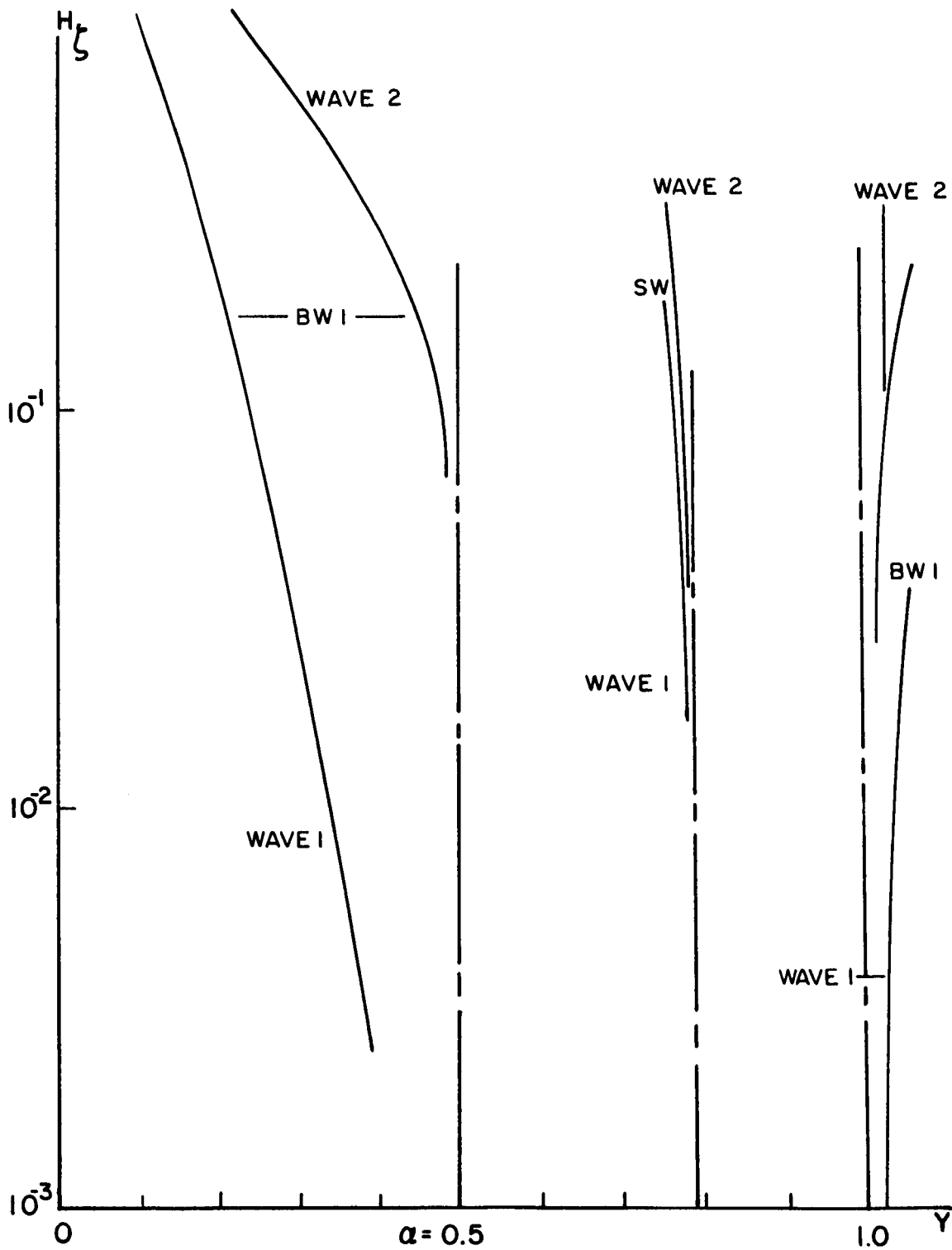


$H_{\zeta 1}$ AND $H_{\zeta 2}$ VERSUS FREQUENCY FOR THE SYMMETRIC CASE
WHEN $\alpha = 0.5$, $b = 0.1$
FIGURE 28



E_1 VS FREQUENCY FOR THE SYMMETRIC CASE WHEN
 $\alpha=0.5$ AND $b=1.0$

FIGURE 29



H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE SYMMETRIC CASE
WHEN $\alpha = 0.5$ AND $b = 1.0$

FIGURE 30

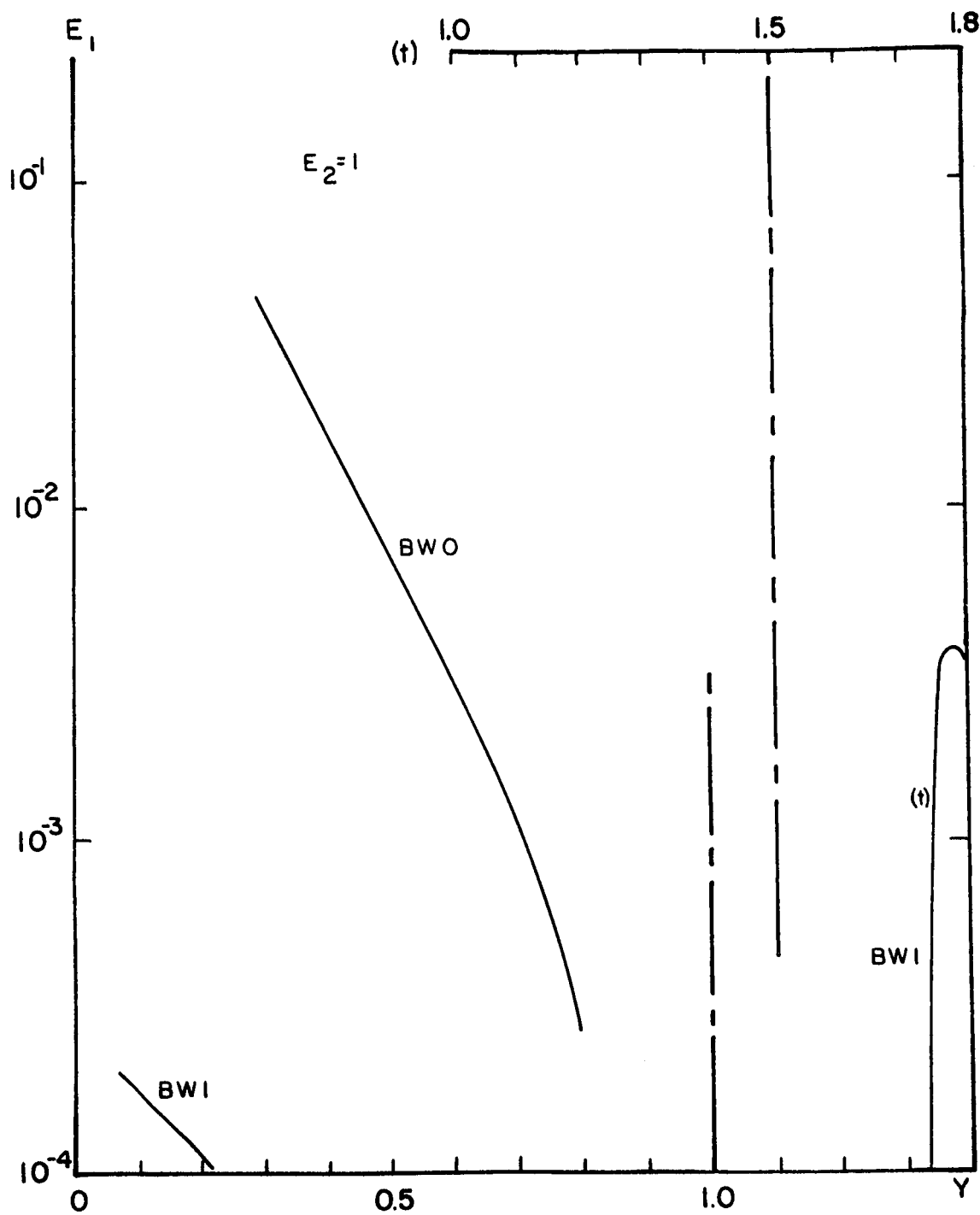
the plasma. These values are plotted for the four sets of values of the parameters α and b already used in this analysis: ($\alpha = 0.5$ and 1.5 , $b = 0.1$ and 1). Figures 27 to 34 show the variation of E_1 , H_{ζ_1} , and H_{ζ_2} for the lowest order modes, plotted versus frequency.

1.) $\alpha = 0.5$, $b = 0.1$ Figures 27 and 28

The high frequency body waves are essentially transverse magnetic since both H_{ζ_1} and H_{ζ_2} are much smaller than $E_2 = 1$. For the low frequency body wave, the wave is also transverse magnetic when Y is close to one. However, when the frequency decreases, the wave becomes hybrid and even transverse electric as $Y \rightarrow 0$. The surface wave is transverse magnetic in the quasistatic region as expected from section 2.5.1.3. When Y decreases, the amplitudes of H_{ζ_1} and H_{ζ_2} increase so as to become of the same order of magnitude as $E_2 = 1$. However, H_{ζ_1} and H_{ζ_2} are approximately equal but have opposite signs so that their sum is very small and the wave remains T M. This explains why the surface wave in the quasistatic theory is so close to the dispersion curve obtained with the complete set of Maxwell's equations. For all the body wave modes, $|E_1| \ll 1$ so that the characteristics of the waves are essentially given by wave 2.

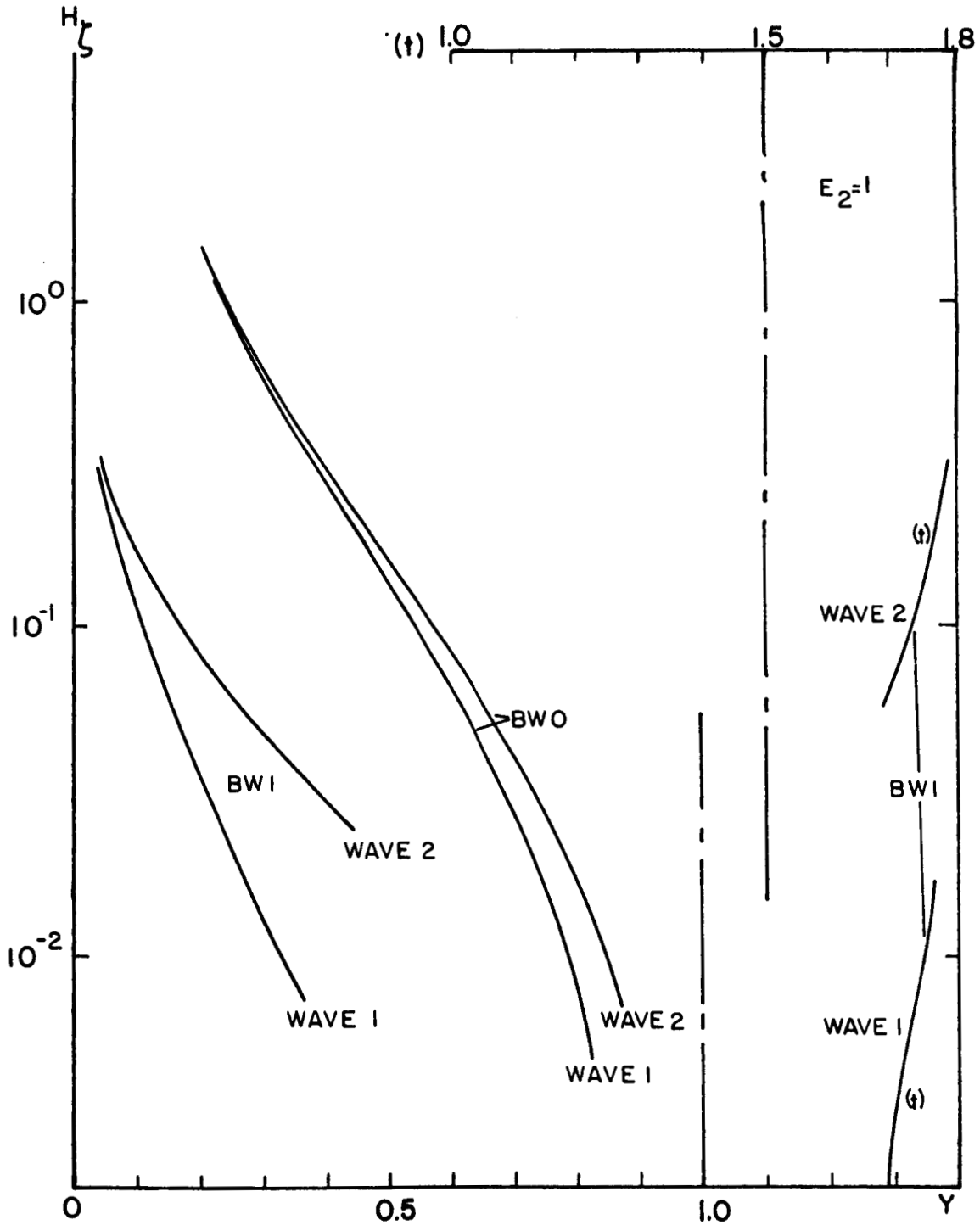
2.) $\alpha = 0.5$, $b = 1.0$ Figures 29 and 30

Again, for both the high frequency and the low frequency body waves, the wave is practically T M, except in the limit of Y going to zero in which case the low frequency body waves become T E. The effect of increasing b results in the low frequency body waves becoming hybrid for higher values of Y . For the surface wave, H_{ζ_1} and H_{ζ_2} have the



E_1 VS FREQUENCY FOR THE SYMMETRIC CASE WHEN
 $\alpha = 1.5$ AND $b = 0.1$

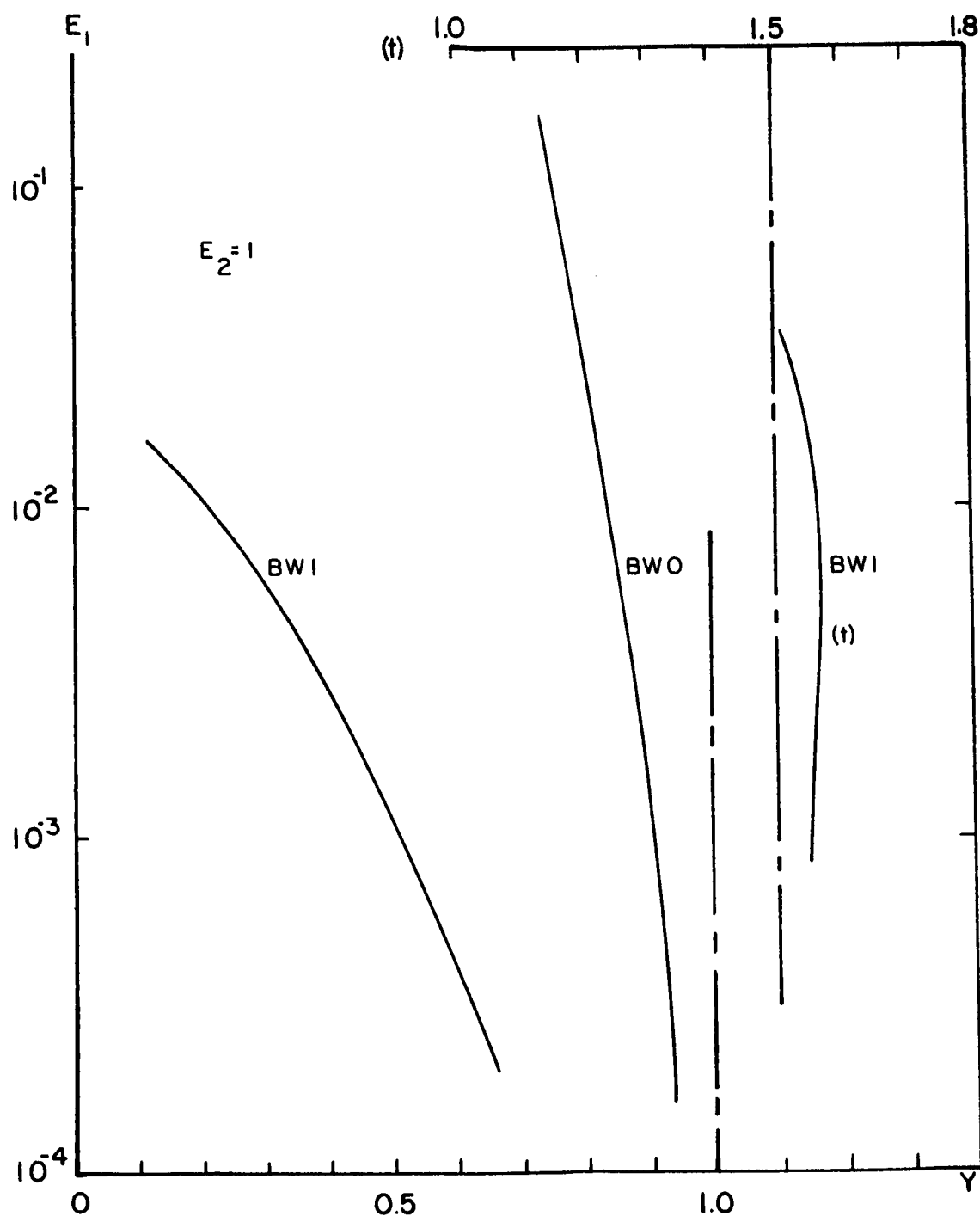
FIGURE 31



H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE SYMMETRIC CASE

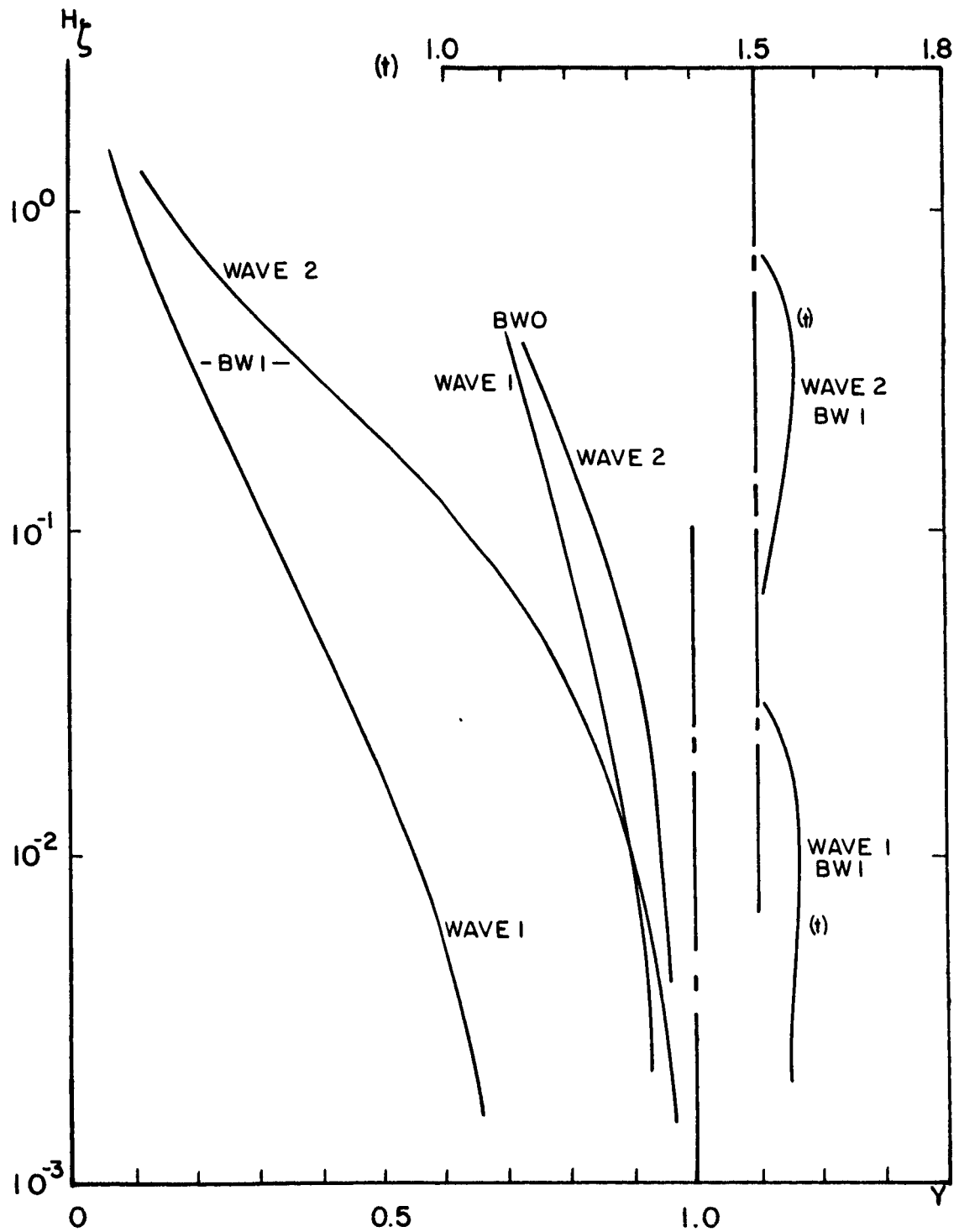
WHEN $\alpha=1.5$ AND $b=0.1$

FIGURE 32



E_1 VS FREQUENCY FOR THE SYMMETRIC CASE WHEN
 $\alpha=1.5$ AND $b=1.0$

FIGURE 33



H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE SYMMETRIC CASE
WHEN $\alpha=1.5$ AND $b=1.0$

FIGURE 34

same order of magnitude but have opposite sign so that the fields are transverse magnetic. The value of $|E_z|$ is much smaller than 1 for all the modes.

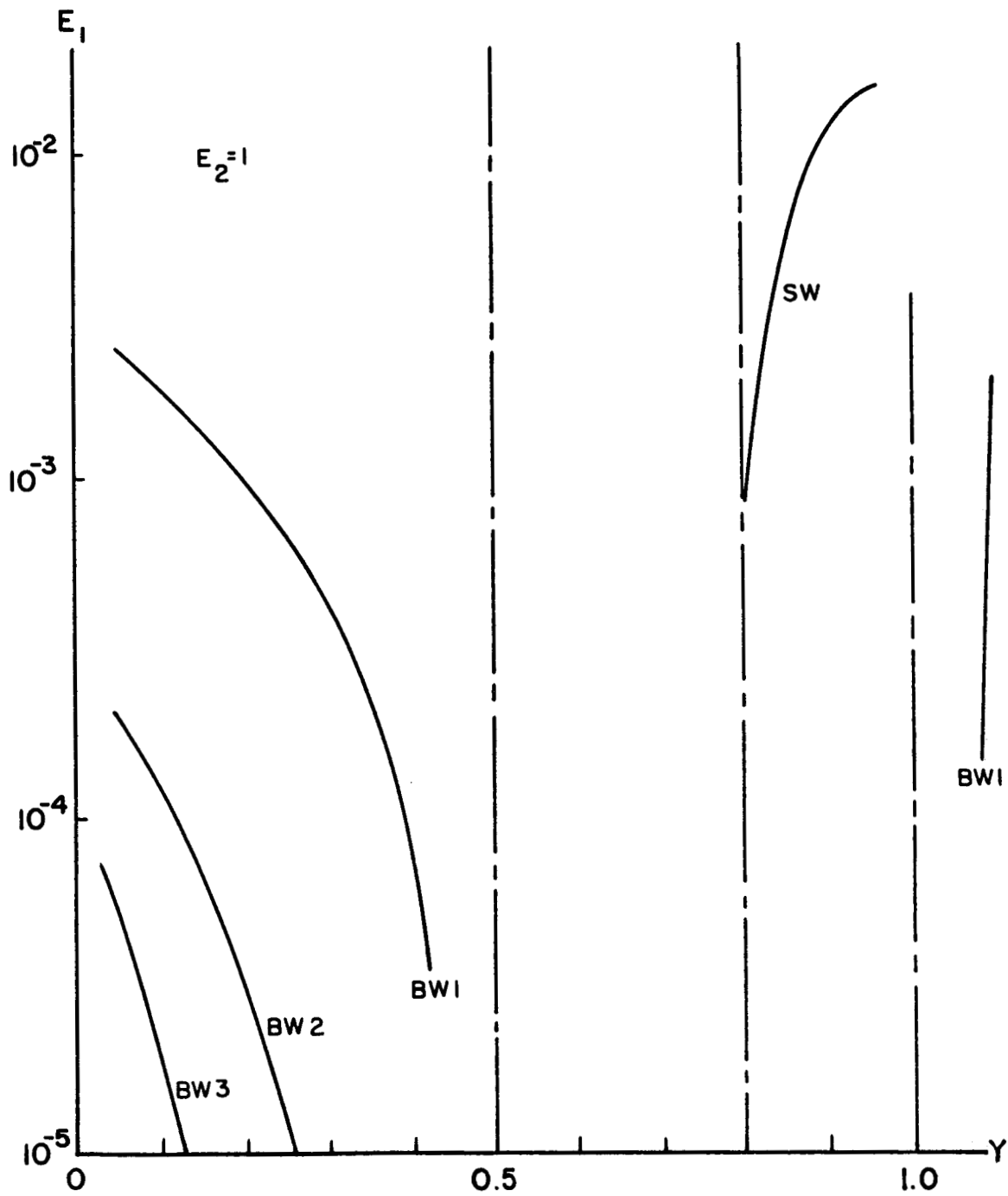
3.) $\alpha = 1.5$, $b = 0.1$ and 1. Figures 31 to 34

The same characteristics already discussed are seen on Figures 31 to 34. The coefficient $|E_z|$ is much smaller than 1 for both $b = 0.1$ and 1. The effect of increasing the slab thickness is to increase the relative magnitude of the wave 1 for a given frequency. The high frequency body waves are essentially transverse magnetic, except for a very narrow gap of frequency when $b = 1$. (Figure 34). The B W O is practically transverse magnetic because H_{ζ_1} and H_{ζ_2} have approximately the same magnitude but opposite signs. The low frequency body wave is transverse magnetic and becomes hybrid when Y decreases. As Y approaches zero, the wave becomes T E. The effect of increasing b is that the wave becomes hybrid for larger values of Y .

2.5.2 General Antisymmetric Case

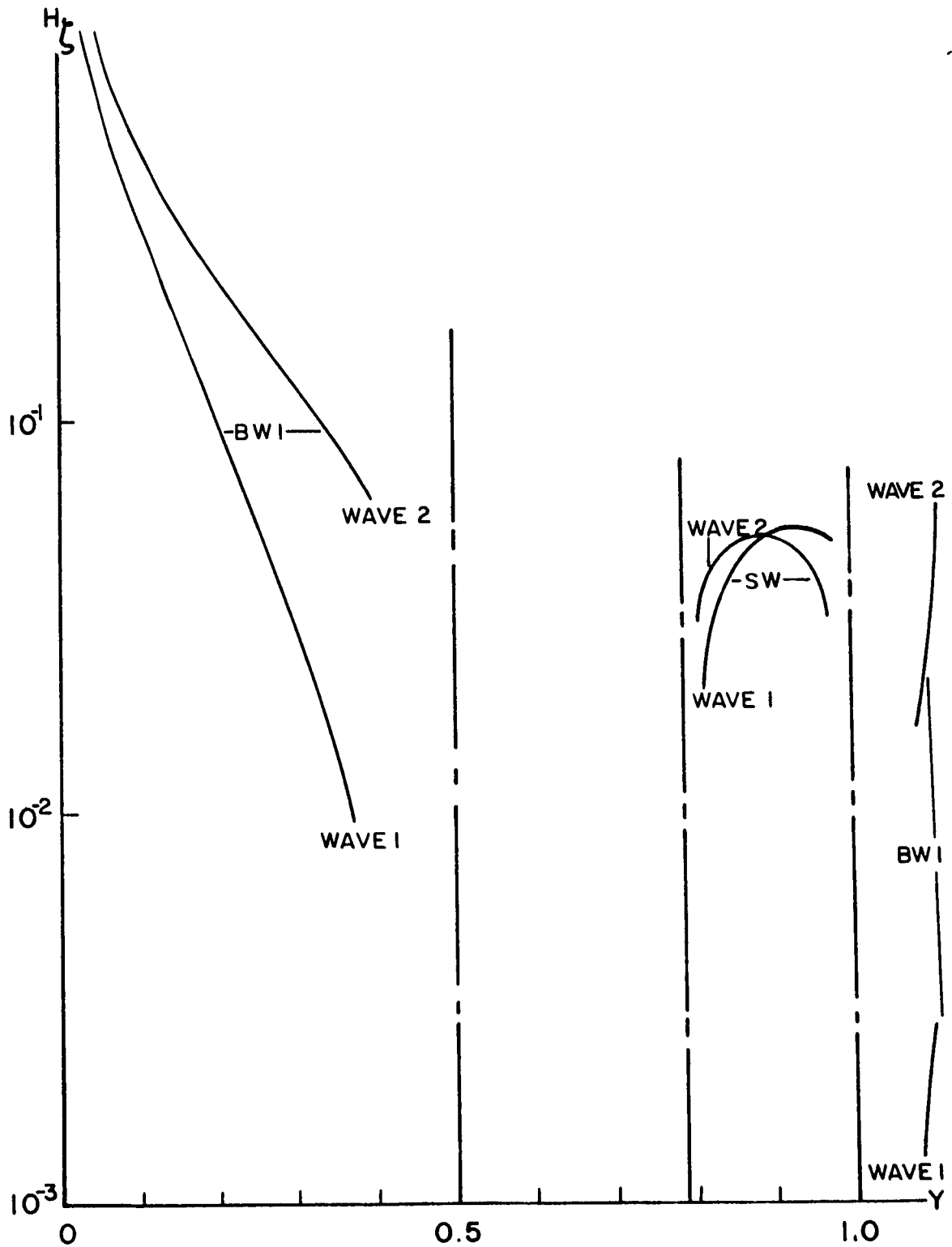
In this section we examine the behavior of all the field components. The values of the field components normalized with respect to $E_z = 1$ are plotted on Figures 35 to 58 versus the normalized frequency $Y = \frac{\omega}{\omega_p}$. Only the coefficients of the sinh and cosh functions are plotted. When the order of the mode increases, the intensity of the fields decreases very rapidly and it is not possible to plot all the curves on the same graph. Consequently, only the lowest order modes are shown.

We consider the same four sets of values of the parameter α and b and discuss the combinations of these parameters separately.



E_1 VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN
 $\alpha=0.5$ AND $b=0.1$

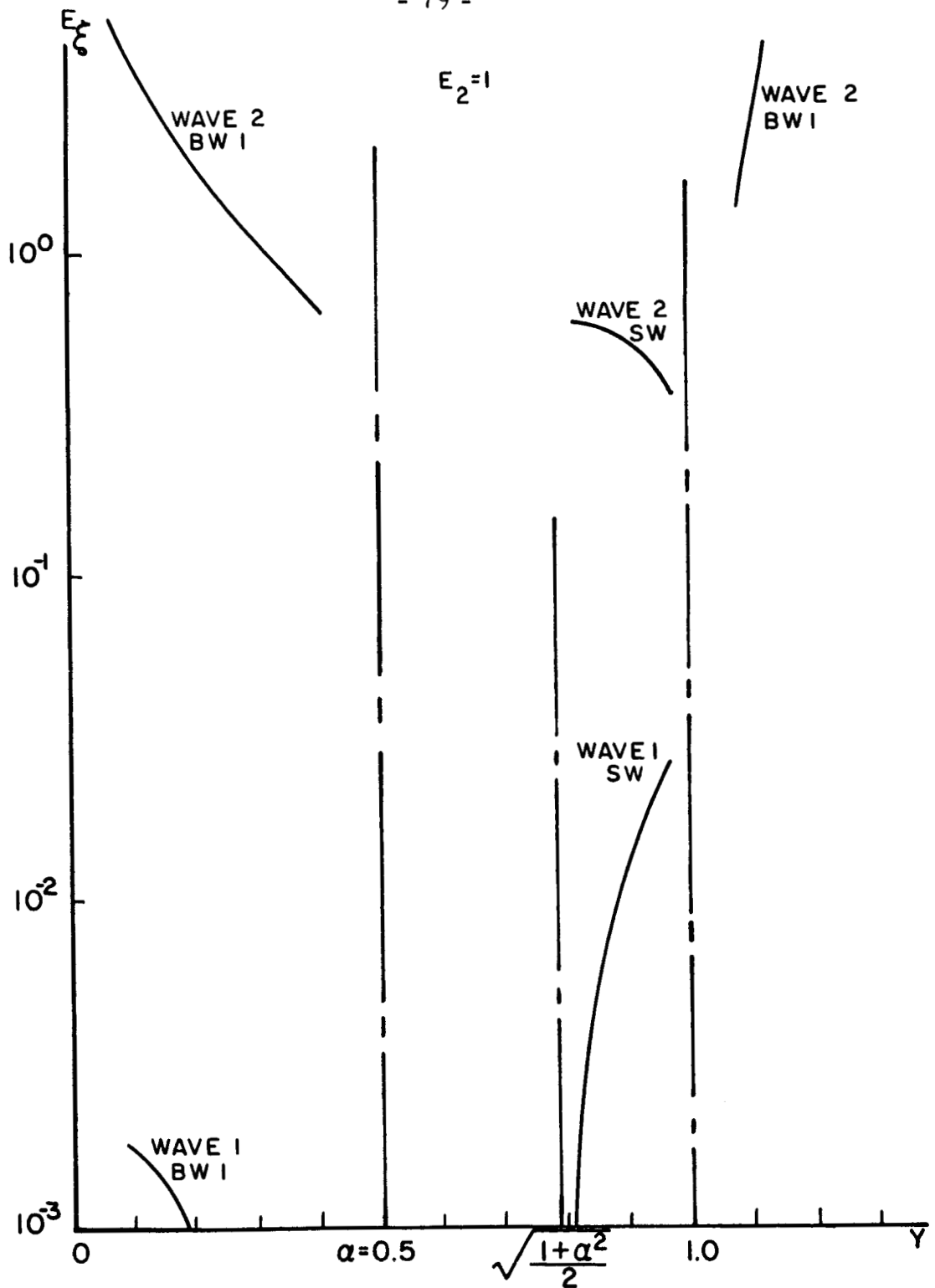
FIGURE 35



H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE

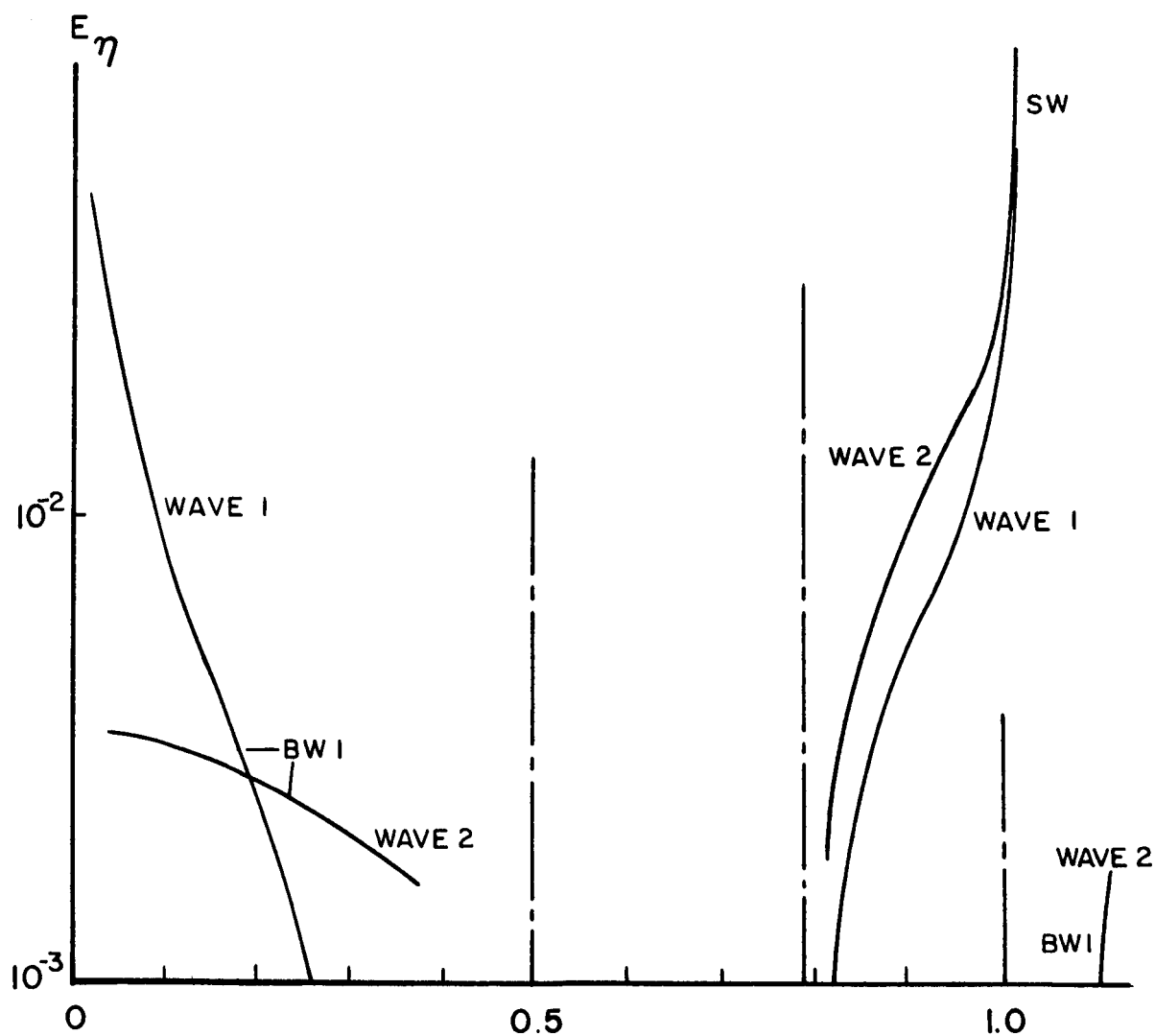
WHEN $\alpha=0.5$ AND $b=0.1$

FIGURE 36



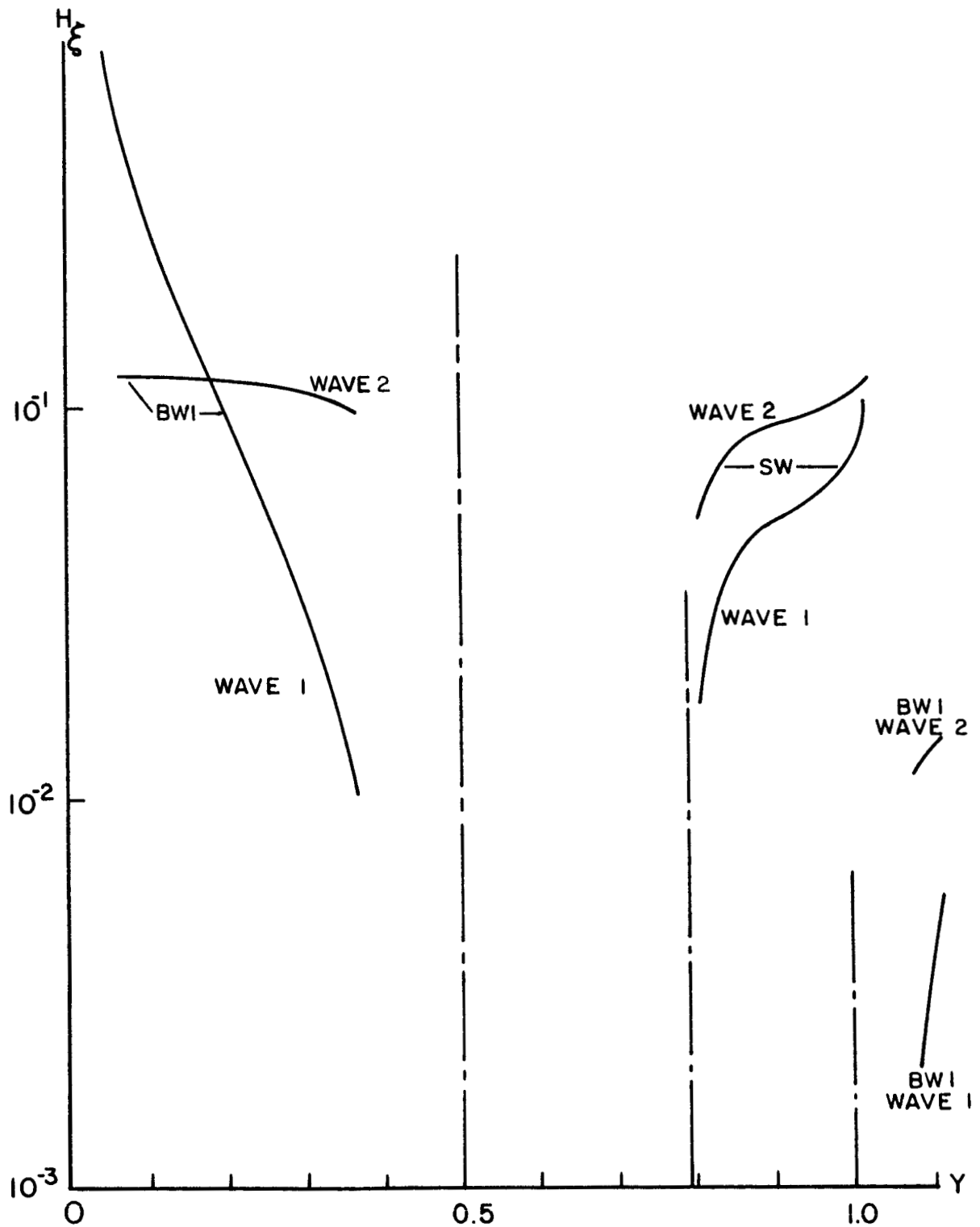
E_{ξ_1} AND E_{ξ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha = 0.5$ AND $b = 0.1$

FIGURE 37



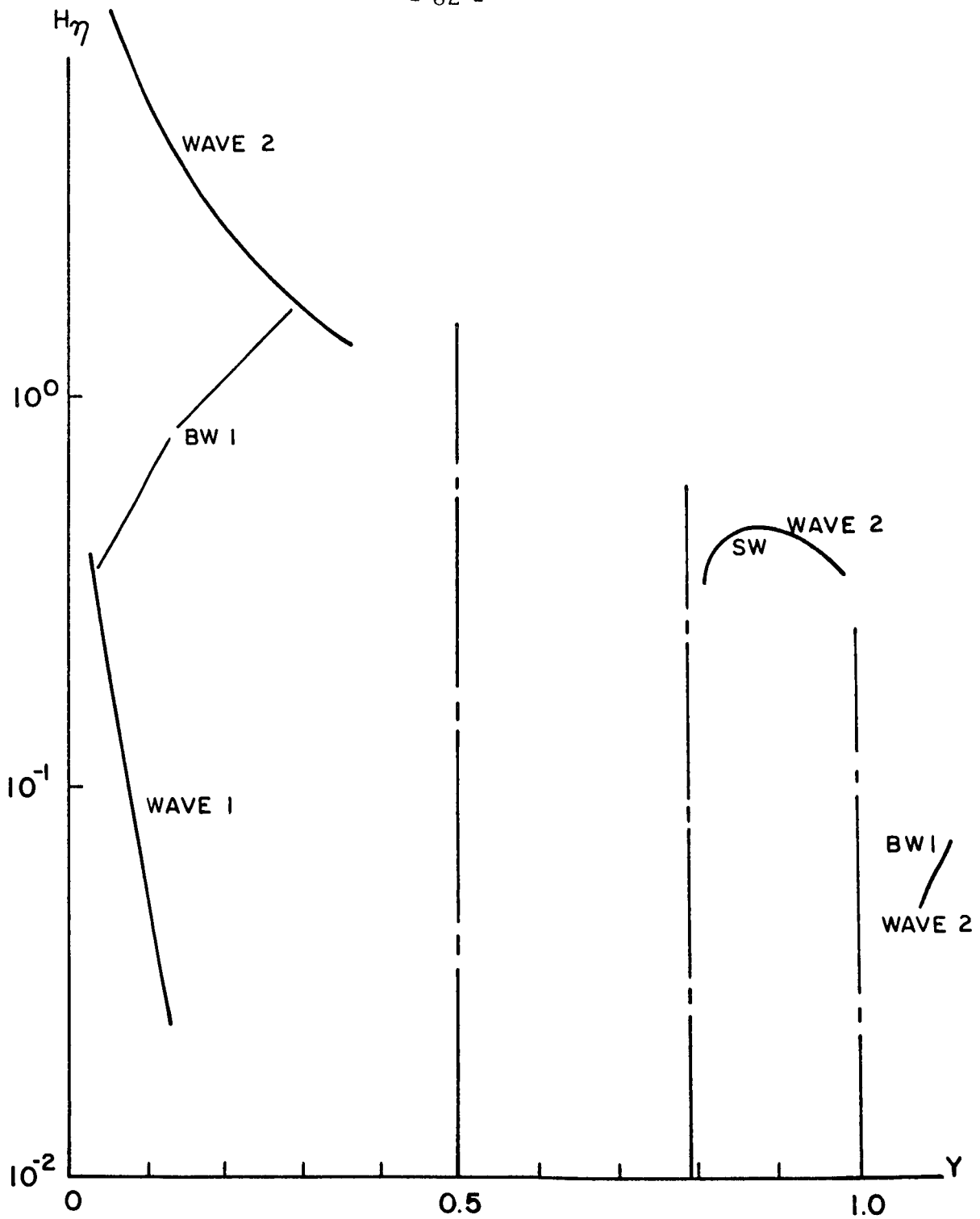
E_{η_1} AND E_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC
CASE WHEN $\alpha=0.5$ AND $b=0.1$

FIGURE 38



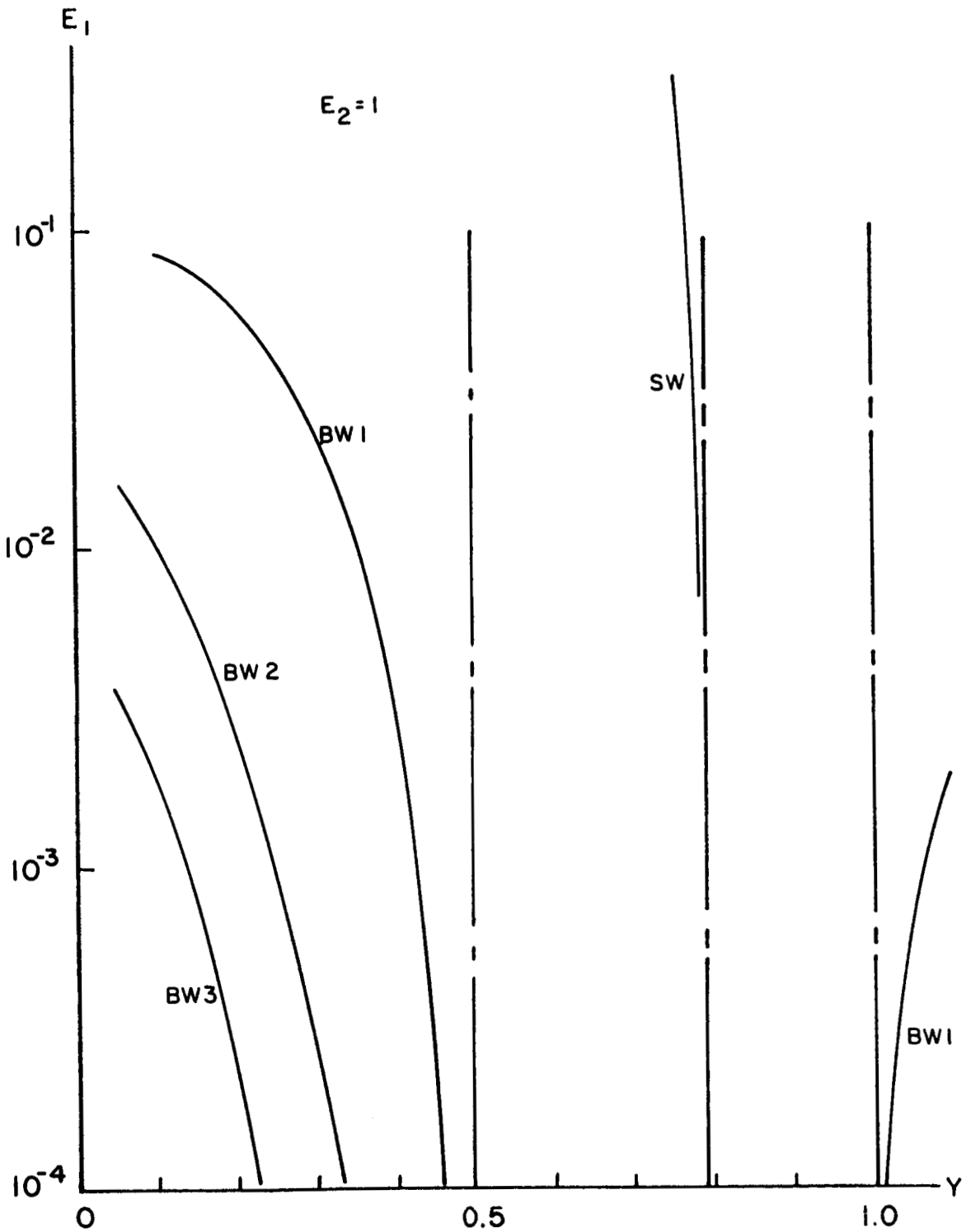
$H_{\xi 1}$ AND $H_{\xi 2}$ VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=0.5$ AND $b=0.1$

FIGURE 39



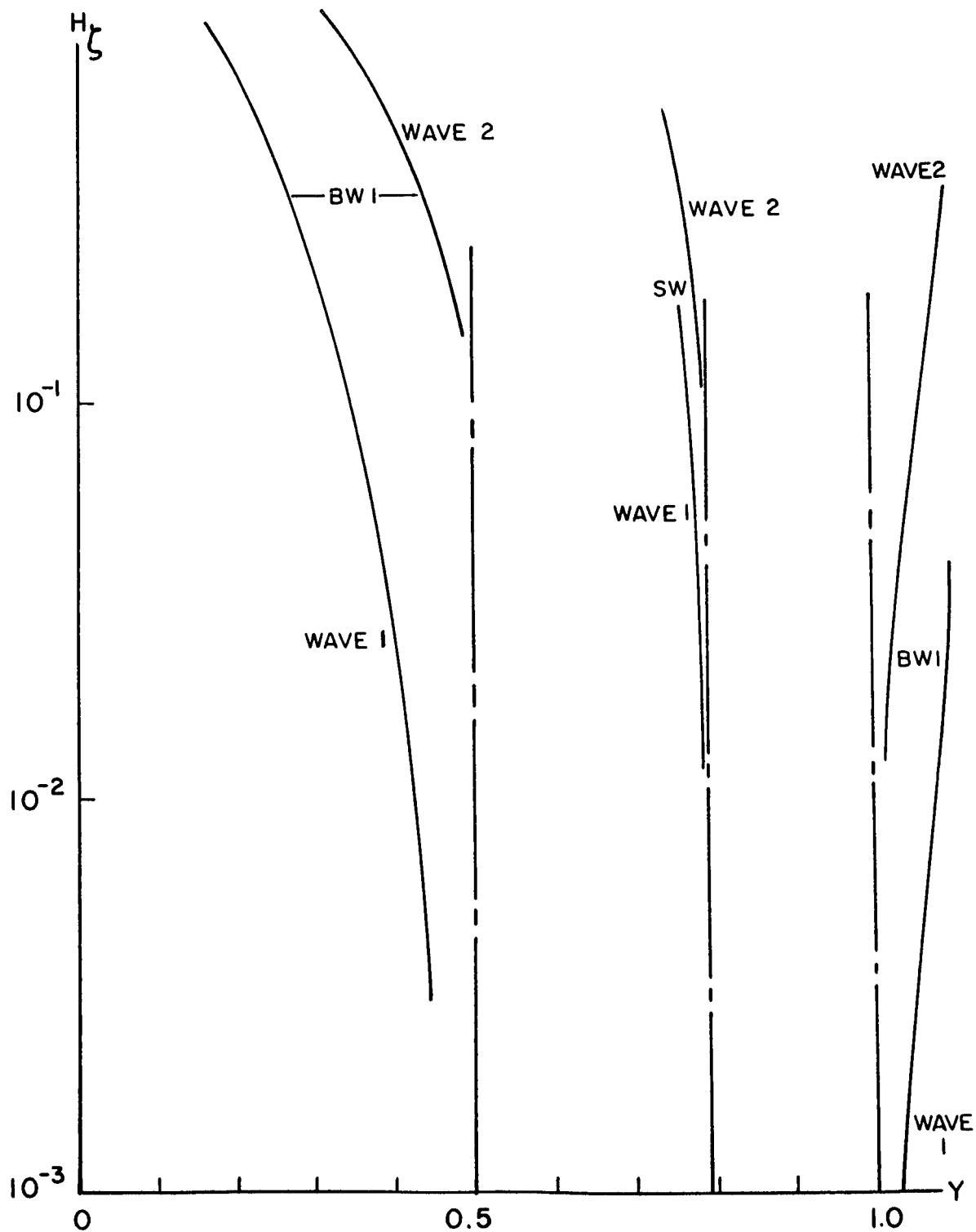
$H_{\eta 1}$ AND $H_{\eta 2}$ VS FREQUENCY FOR THE ANTISYMMETRIC
CASE WHEN $\alpha = 0.5$ AND $b = 0.1$

FIGURE 40



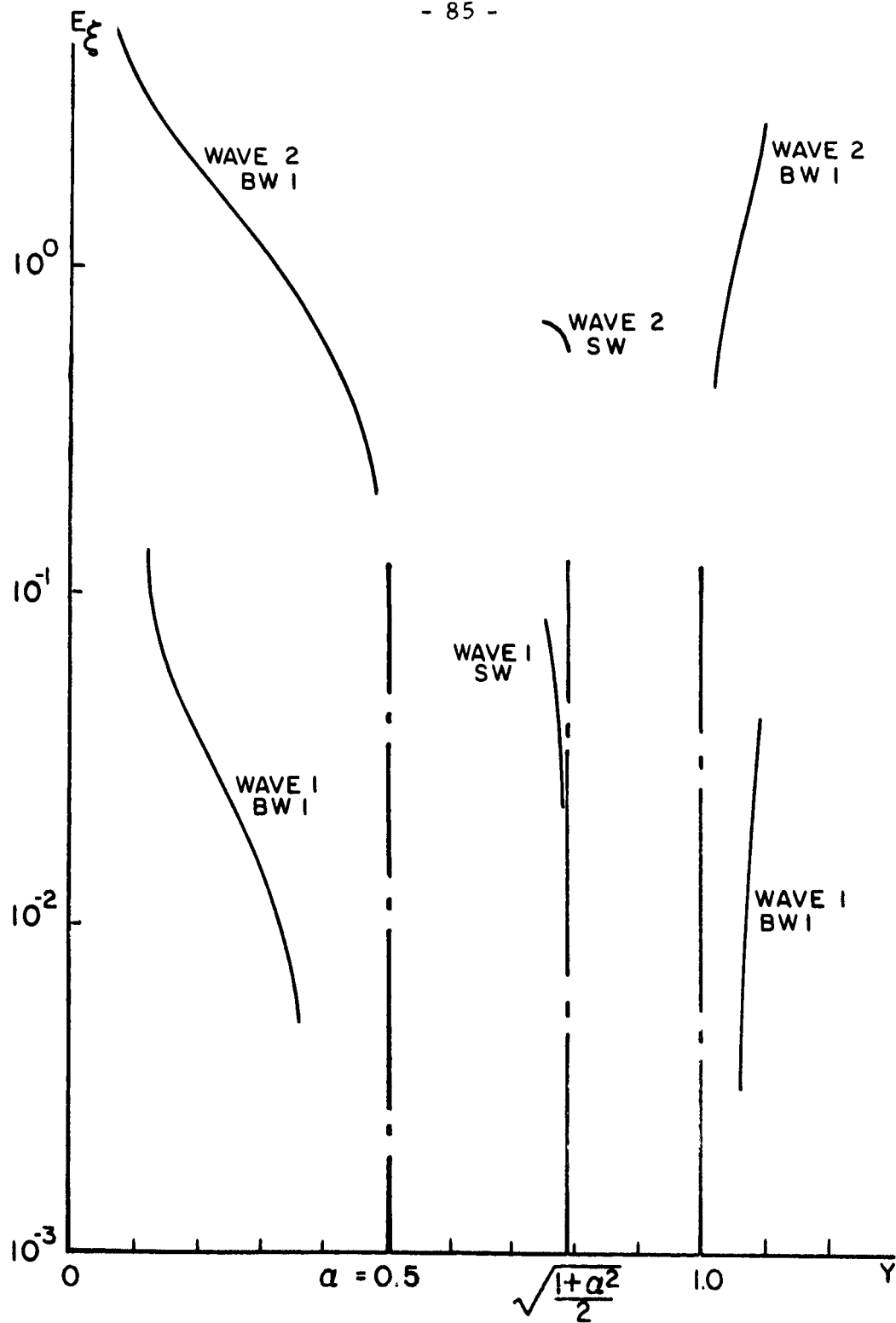
E_1 VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN
 $\alpha=0.5$ AND $b=1.0$

FIGURE 41



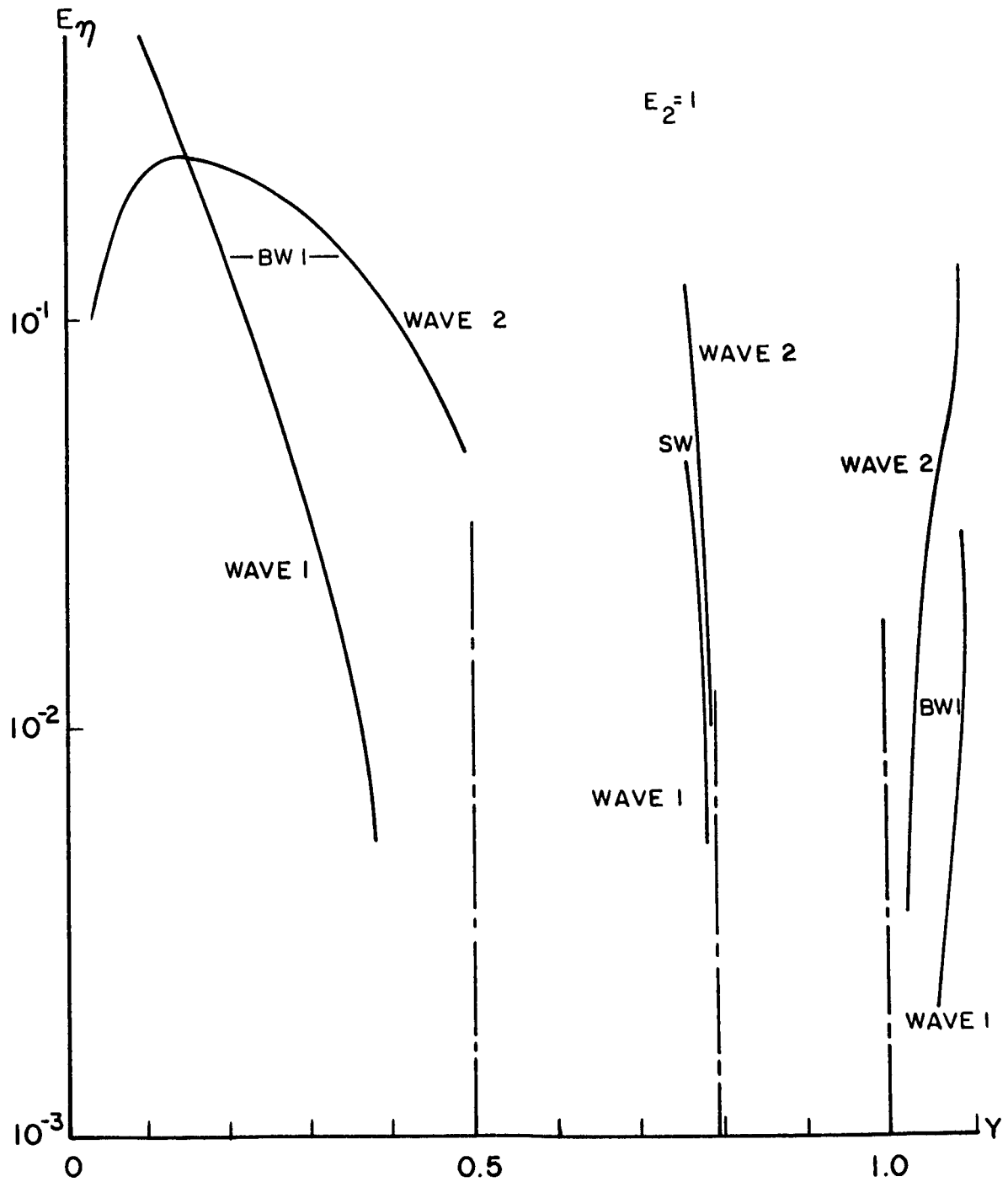
H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=0.5$ AND $b=1.0$

FIGURE 42



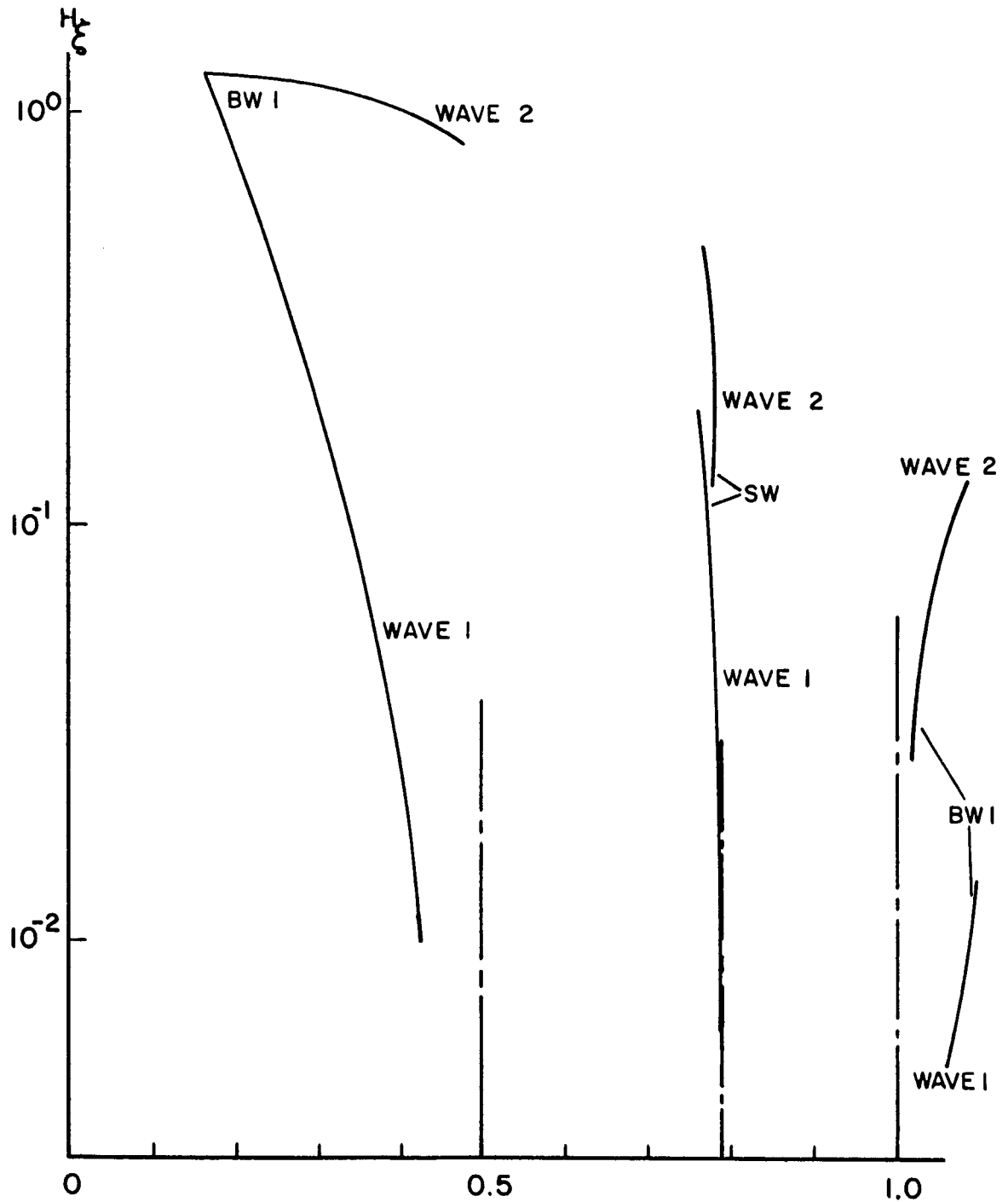
E_{ξ_1} AND E_{ξ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha = 0.5$ AND $b = 1.0$

FIGURE 43



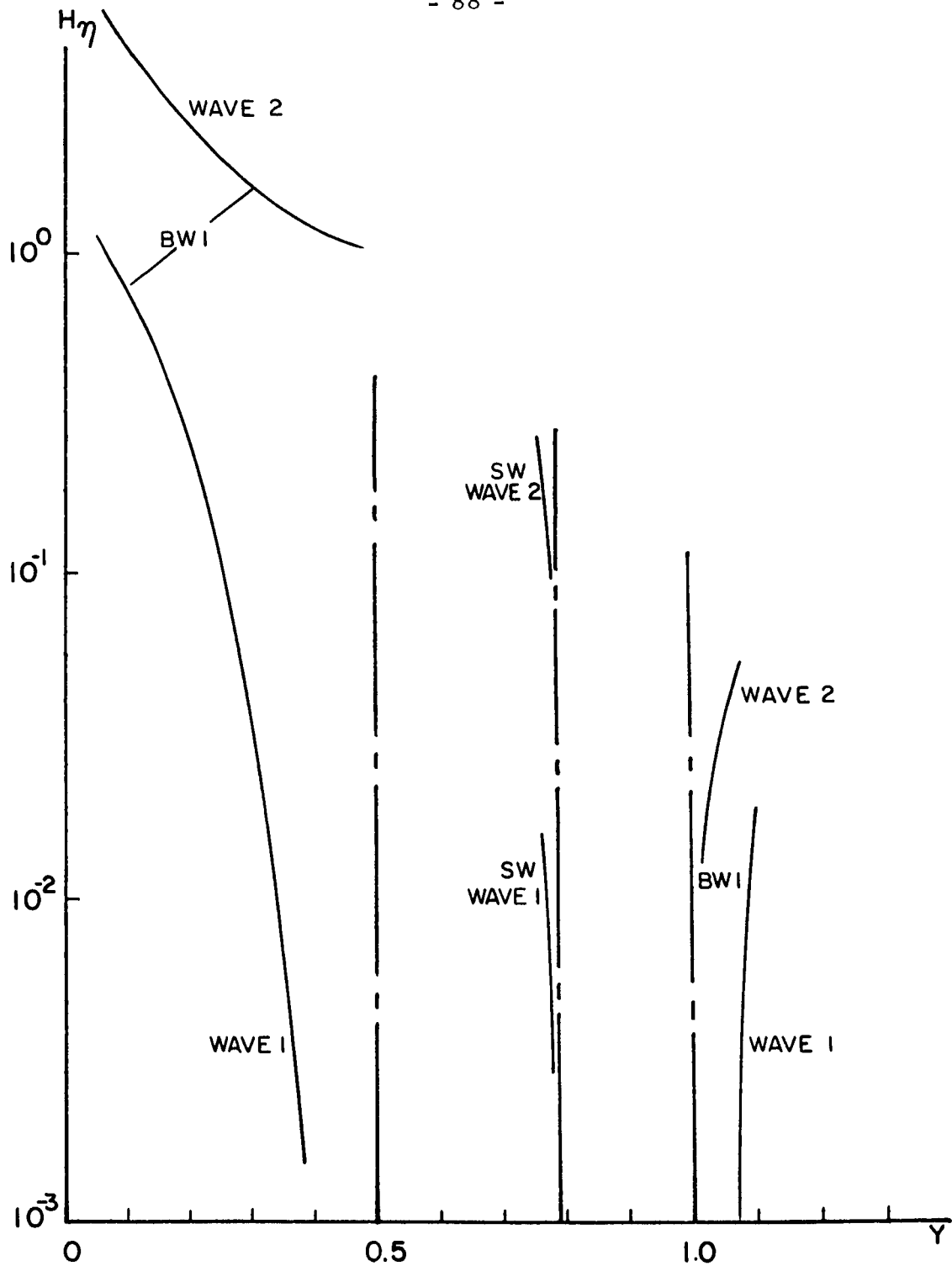
E_{η_1} AND E_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha = 0.5$ AND $b = 1.0$

FIGURE 44



$H_{\xi 1}$ AND $H_{\xi 2}$ VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=0.5$ AND $b=1.0$

FIGURE 45



H_{η_1} AND H_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC
CASE WHEN $\alpha = 0.5$ AND $b = 1.0$

FIGURE 46

1.) $\alpha = 0.5, b = 0.1$. Figures 35 to 40

Since the phase velocity of the low frequency body wave is very small, we find $|E_1| \ll 1$ as shown on Figure 35. When the frequency increases and approaches $Y = \alpha$, $|E_1|$ goes to zero. It is seen that both the surface wave and the high frequency B W 1 satisfy $|E_1| \ll 1$. The curve corresponding to B W 0 and to B W S W is not shown because of the lack of accuracy with the computer in this region.

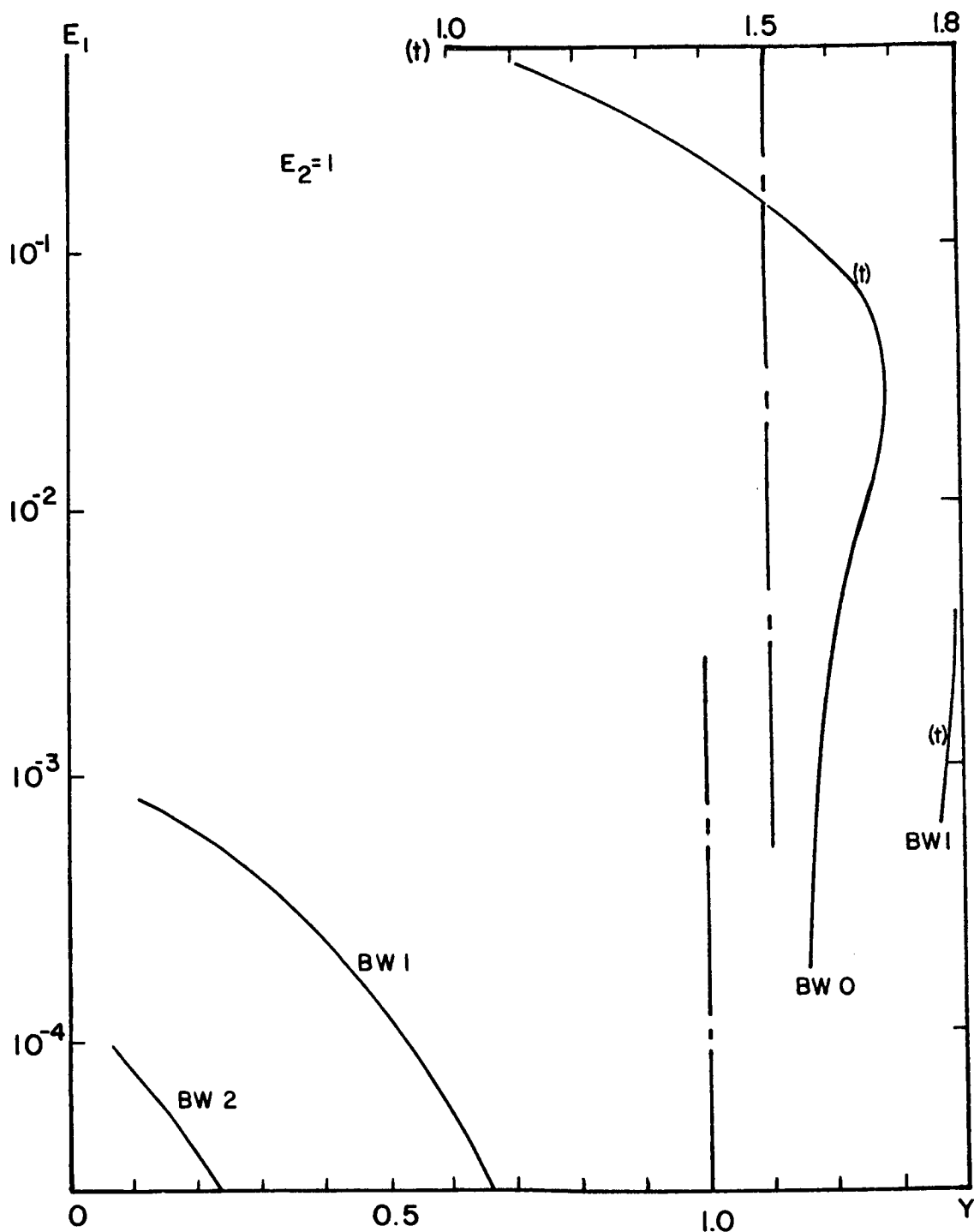
When we compare the amplitude of the other field components, we can make the following remarks:

For the high frequency body wave, and for the surface wave, only the E_ξ and the E_ζ component of wave 2 are important. The wave is essentially transverse magnetic.

For the low frequency body wave, when Y approaches α , the components of wave 1 become negligible (large Z/Y limit) and only E_ζ , H_ξ and H_η of wave 2 have a finite value (Equation 56). When Y decreases and becomes approximately 0.2, the two waves are present and H_ζ and E_ξ dominate all the other components. The wave then becomes transverse electric.

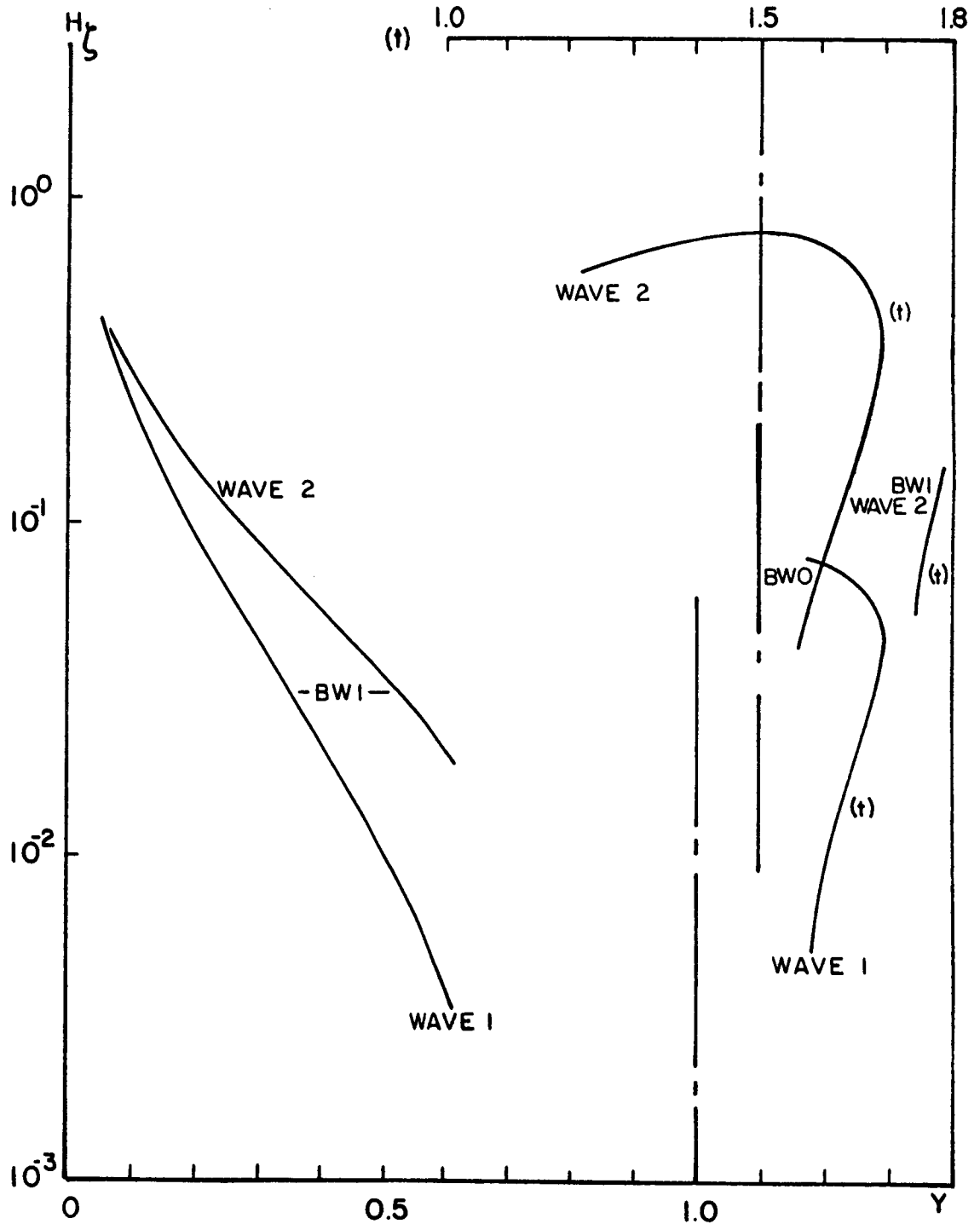
2.) $\alpha = 0.5, b = 1$. Figures 41 to 46

It is seen on Figure 41 that $|E_1|$ is much smaller than one for the low frequency and high frequency body waves. Since at a given frequency the phase velocity of the low frequency body wave increases with the increase of the slab thickness, $|E_1|$ is larger for these modes for thick slabs than for thin slabs. For the surface wave, the amplitude of E_1 increases as the frequency decreases from resonance. It



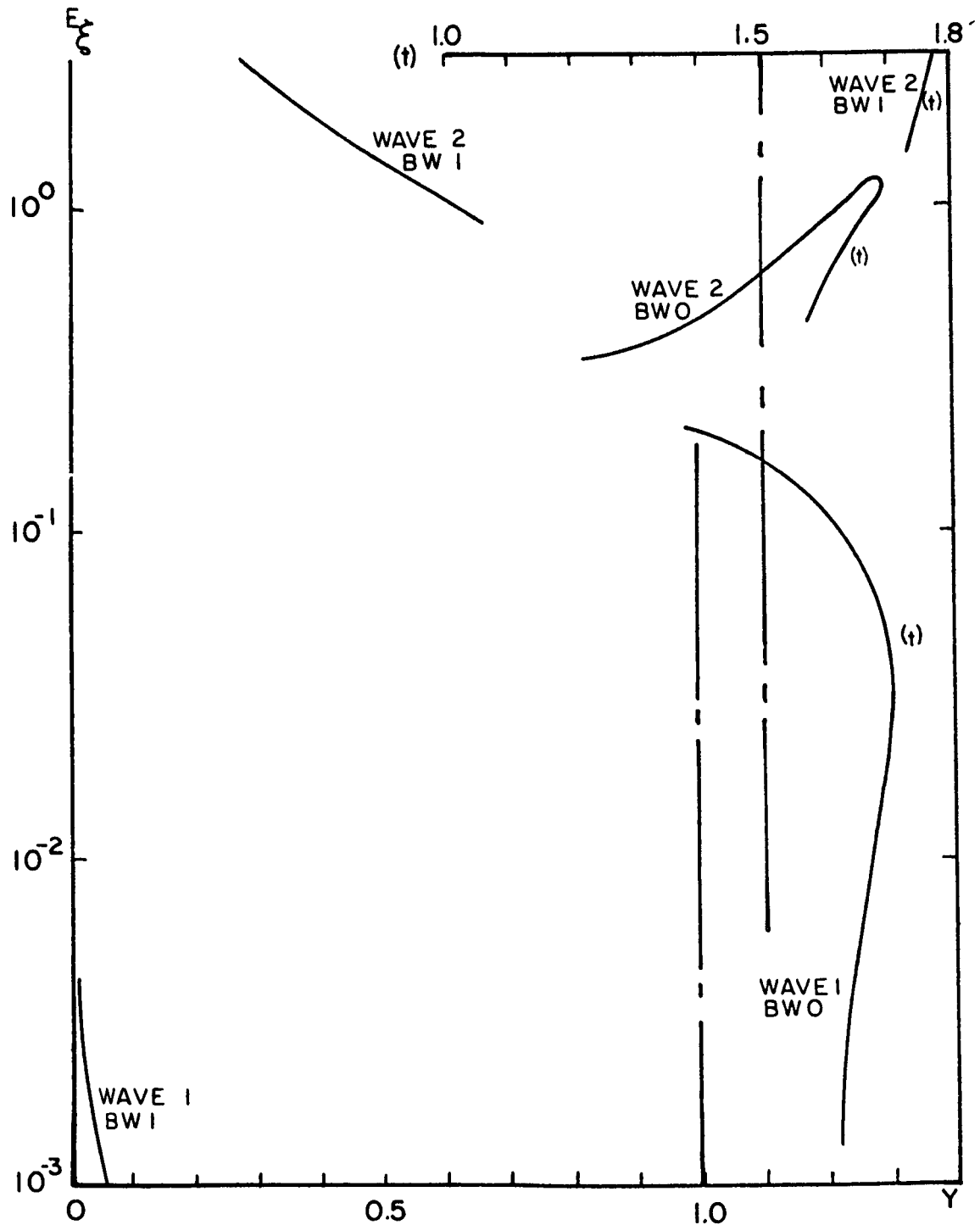
E_1 VS FREQUENCY FOR THE ANTISYMMETRIC CASE
WHEN $\alpha=1.5$ AND $b=0.1$

FIGURE 47



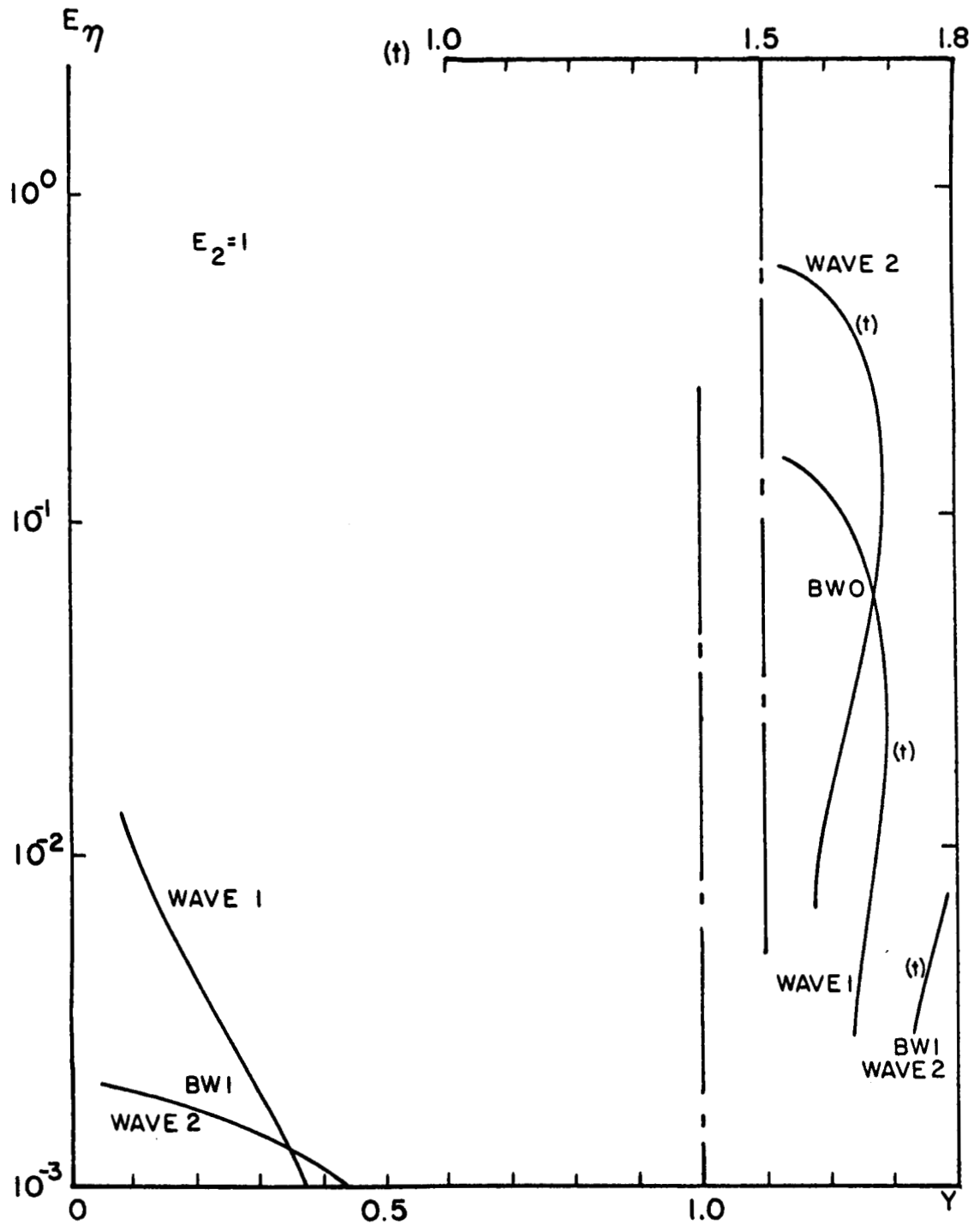
H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha = 1.5$ AND $b = 0.1$

FIGURE 48



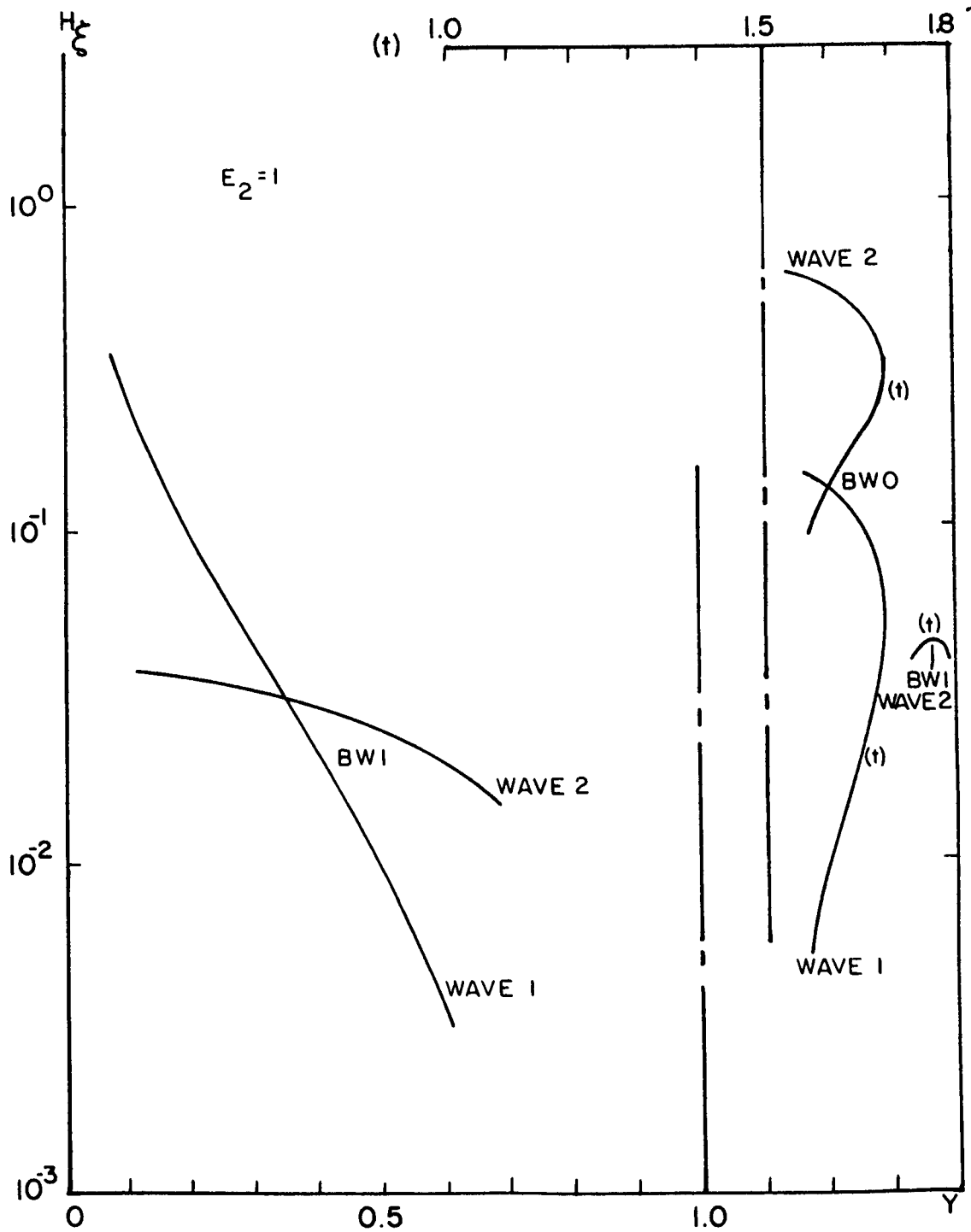
E_{ξ_1} AND E_{ξ_2} VS FREQUENCY FOR THE ANTISYMMETRIC
CASE WHEN $\alpha=1.5$ AND $b=0.1$

FIGURE 49



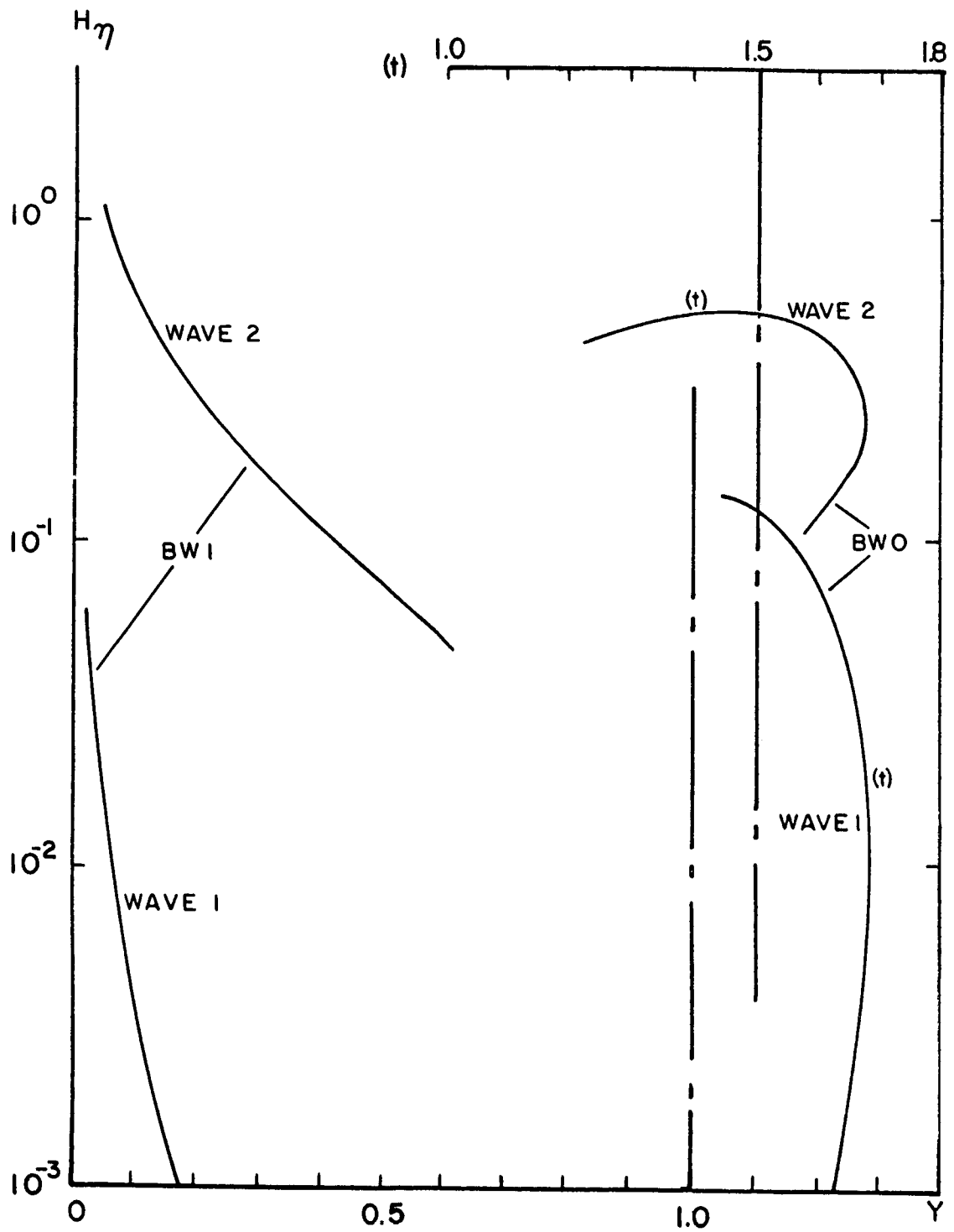
E_{η_1} AND E_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=1.5$ AND $b=0.1$

FIGURE 50



H_{ξ_1} AND H_{ξ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha = 1.5$ AND $b = 0.1$

FIGURE 51



H_{η_1} AND H_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=1.5$ AND $b=0.1$

FIGURE 52

becomes of the order unity when the surface wave becomes the B W S W.

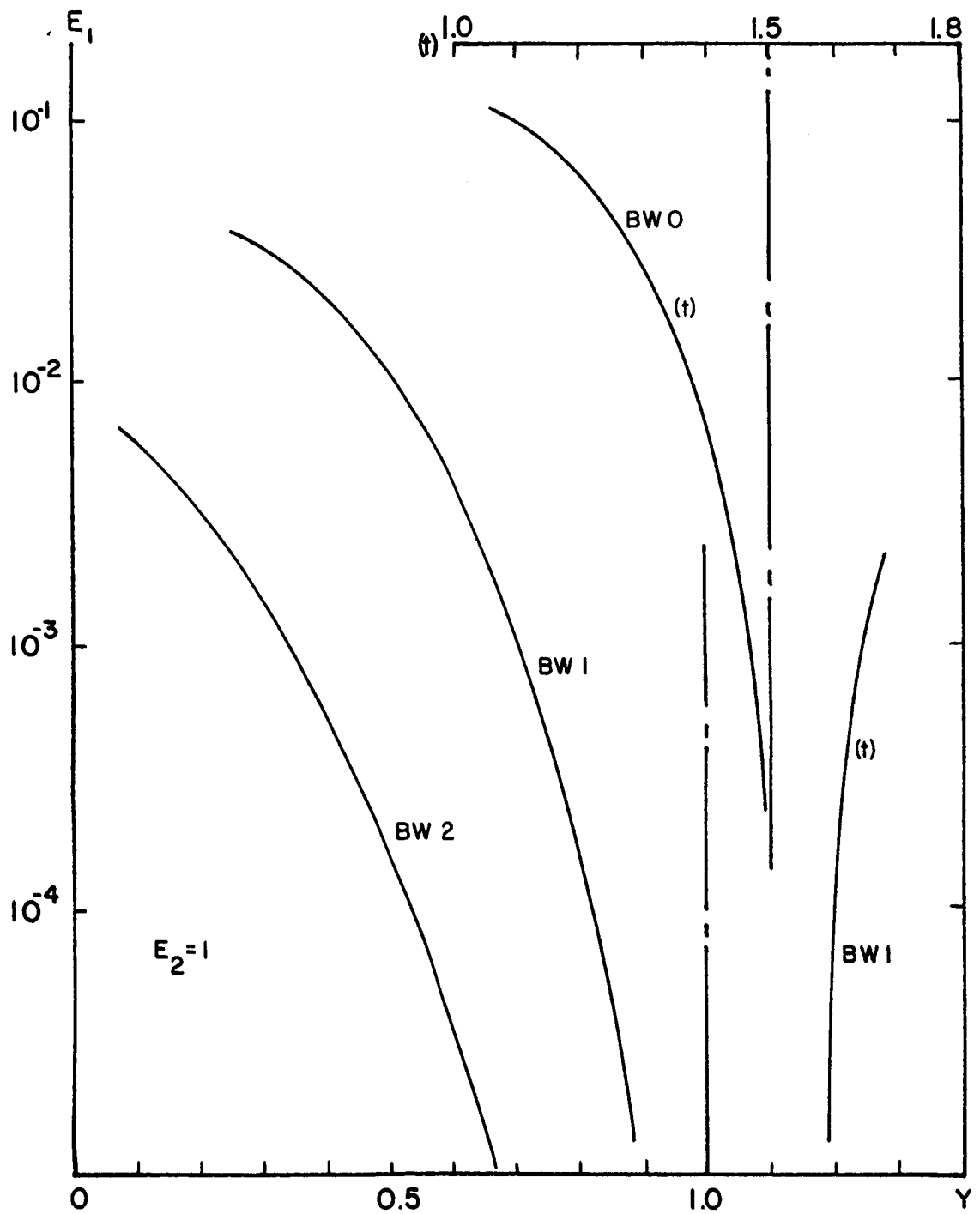
All the field components have the same variation as in the preceding case. However, now the wave is transverse magnetic only for several very narrow frequency gaps, namely $0.4 < Y < 0.5$; $0.74 < Y < 0.79$; and $1 < Y < 1.08$. In the other regions the wave is hybrid or becomes transverse electric at very low frequency. When b increases, the strength of the field components increases except for E_ξ which shows very little change in magnitude.

3.) $\alpha = 1.5$, $b = 0.1$. Figures 47 to 52

For all the low frequency body waves, the phase velocity is very small so that the relation $|E_1| \ll 1$ holds. The amplitude of E_1 for high frequency body waves is also much smaller than 1. For B W O, however, in the frequency gap $1 < Y < 1.6$, the amplitude of E_1 is approximately the same as $E_2 = 1$. As the dispersion curve approaches the asymptote $Y = \alpha = 1.5$, the amplitude of E_1 becomes much smaller than E_2 . These results are shown on Figure 47.

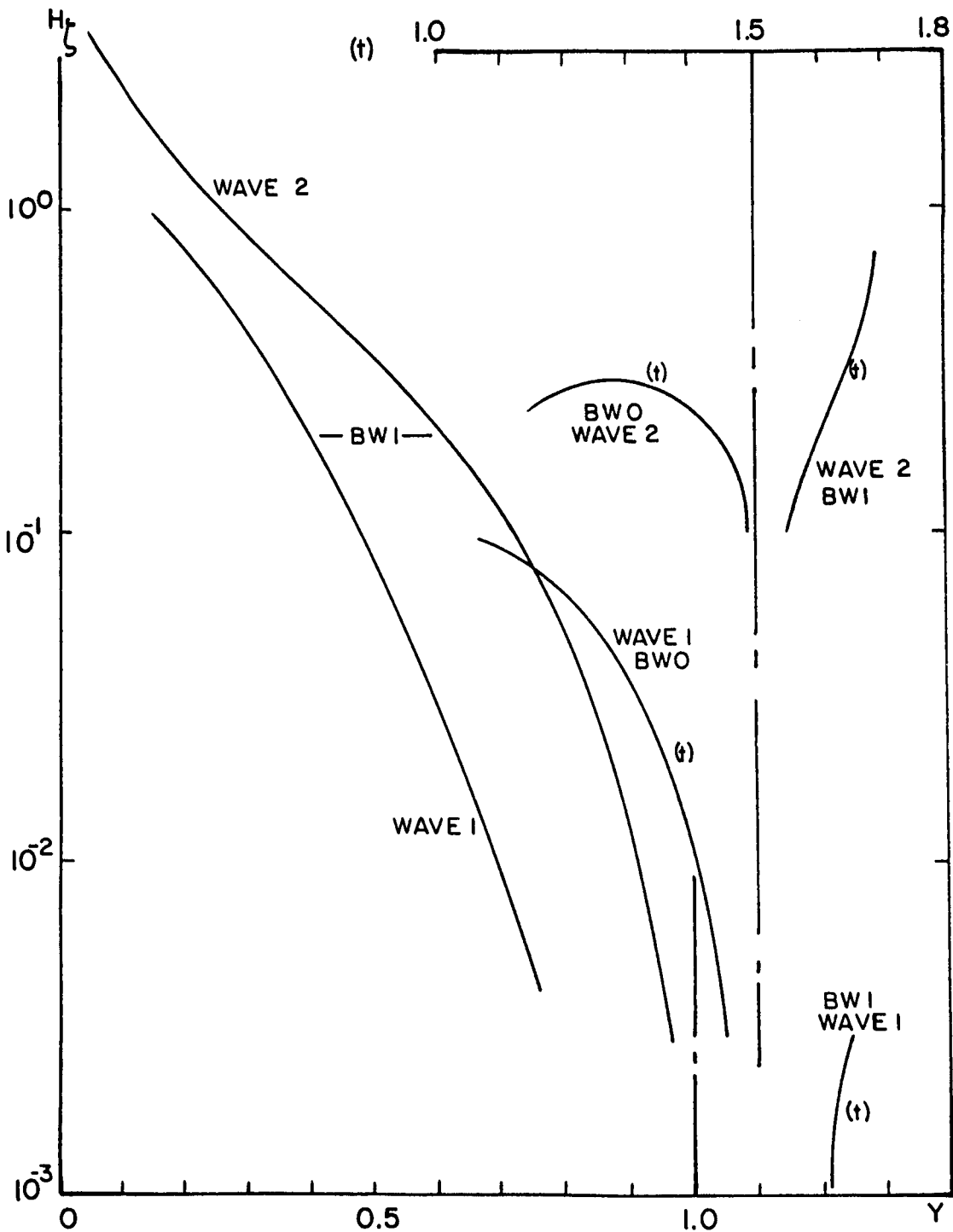
If we compare the amplitude of the fields corresponding to the high frequency body waves, we see that only E_ζ and E_ξ of wave 2 are important so that the waves are practically transverse magnetic.

For B W O, all the field components of wave 1 are negligible compared to wave 2 in the frequency gap $1.4 < Y < 1.6$. In addition, the amplitude of the field components of wave 2 are of the same order of magnitude. However, when the dispersion curve approaches the asymptote, only E_ζ , H_ξ and H_η of wave 2 take non zero values, as expected from the limits obtained in section 2.5.1.3 It should be



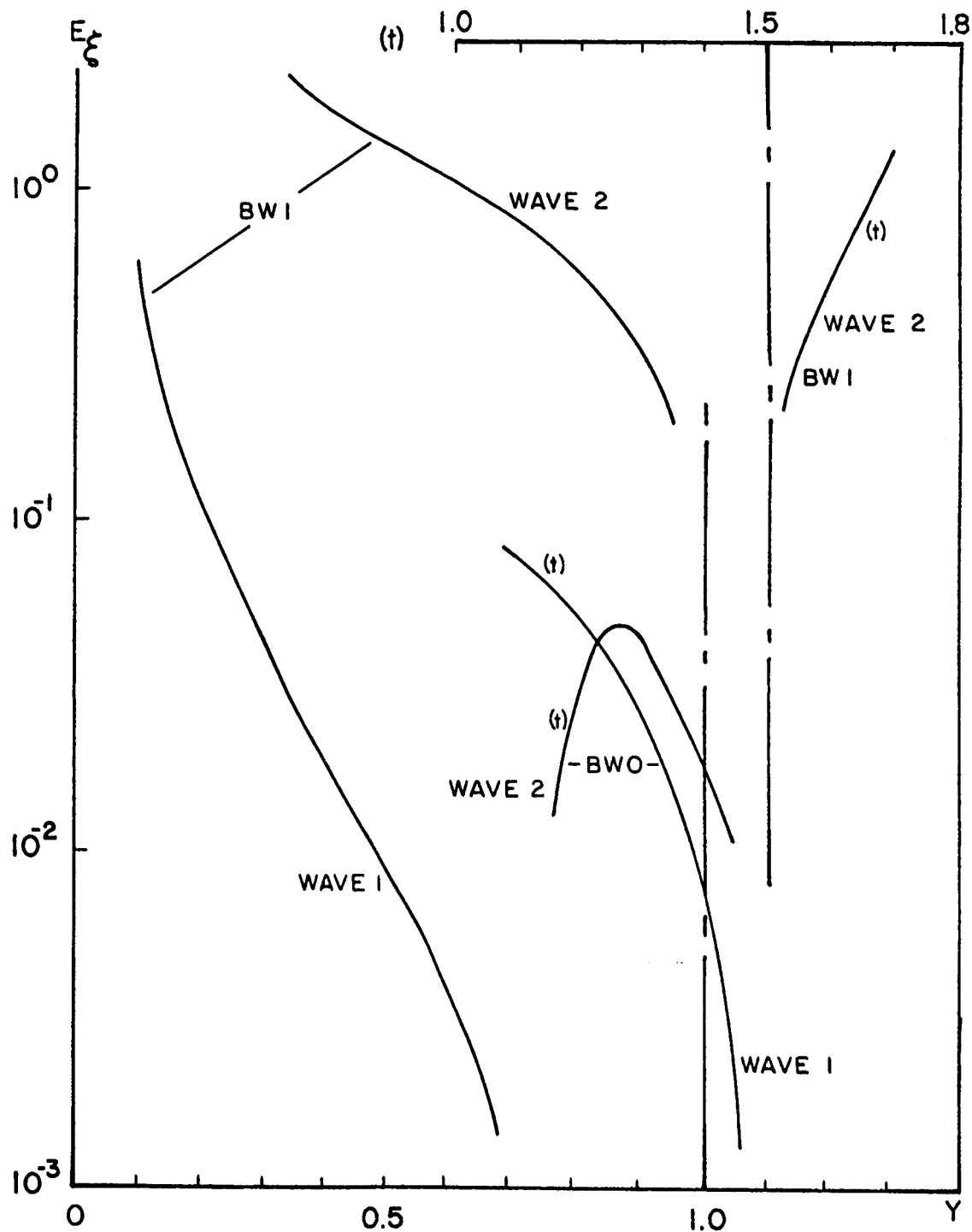
E_1 VS FREQUENCY FOR THE ANTISYMMETRIC CASE
WHEN $\alpha = 1.5$ AND $b = 1.0$

FIGURE 53

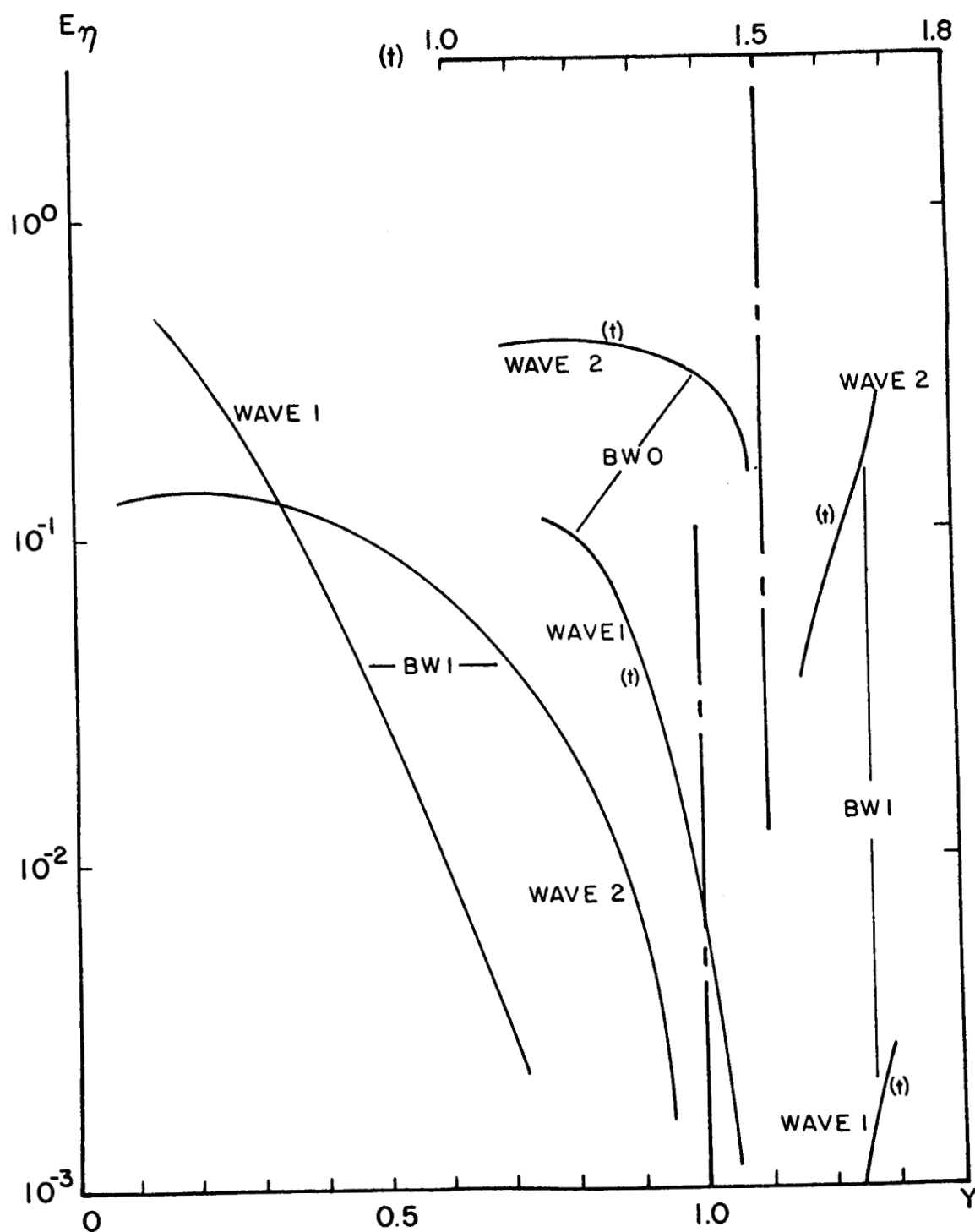


H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=1.5$ AND $b=1.0$

FIGURE 54

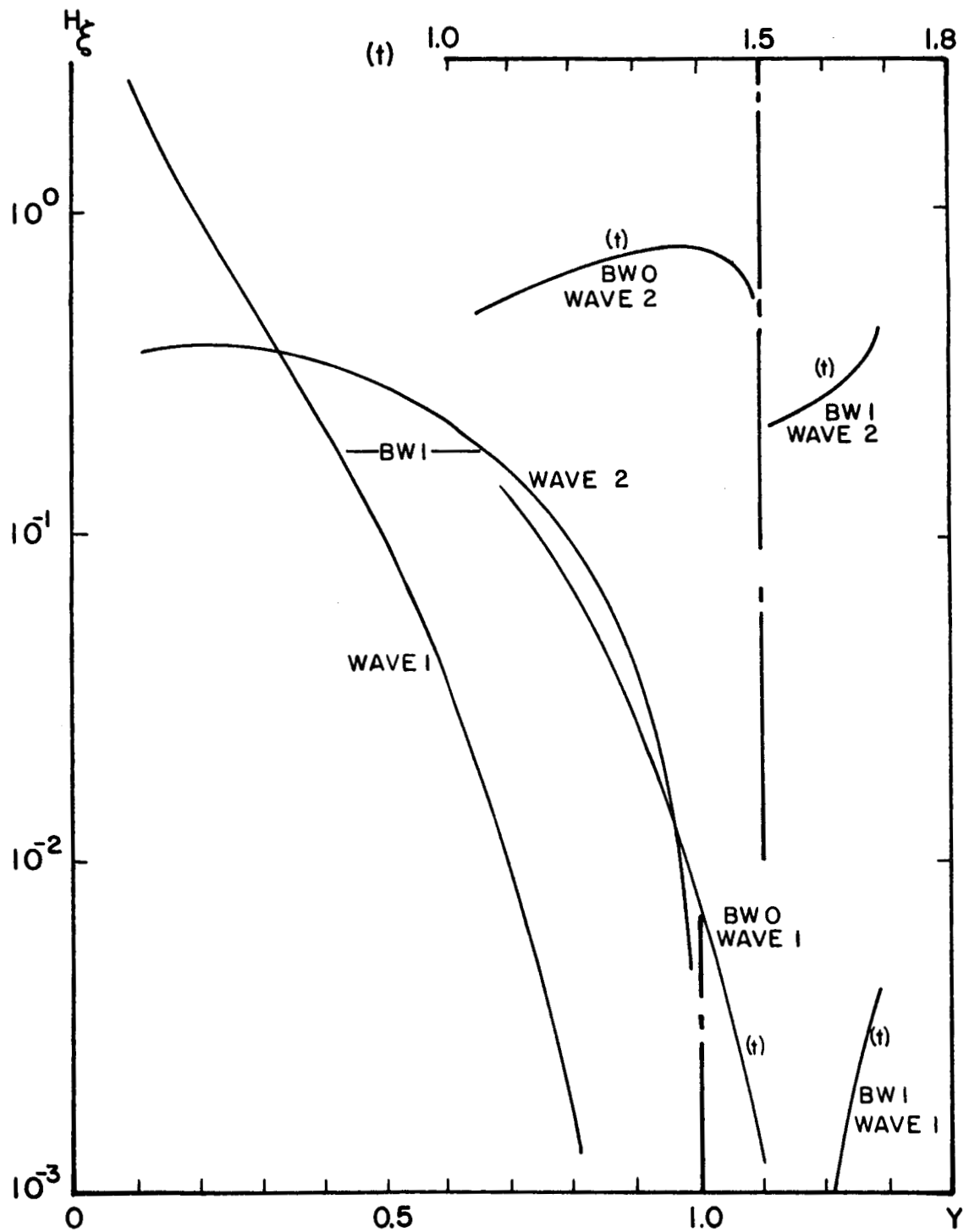


E_{ξ_1} AND E_{ξ_2} VS FREQUENCY FOR THE ANTISYMMETRIC
 CASE WHEN $\alpha=1.5$ AND $b=1.0$
 FIGURE 55



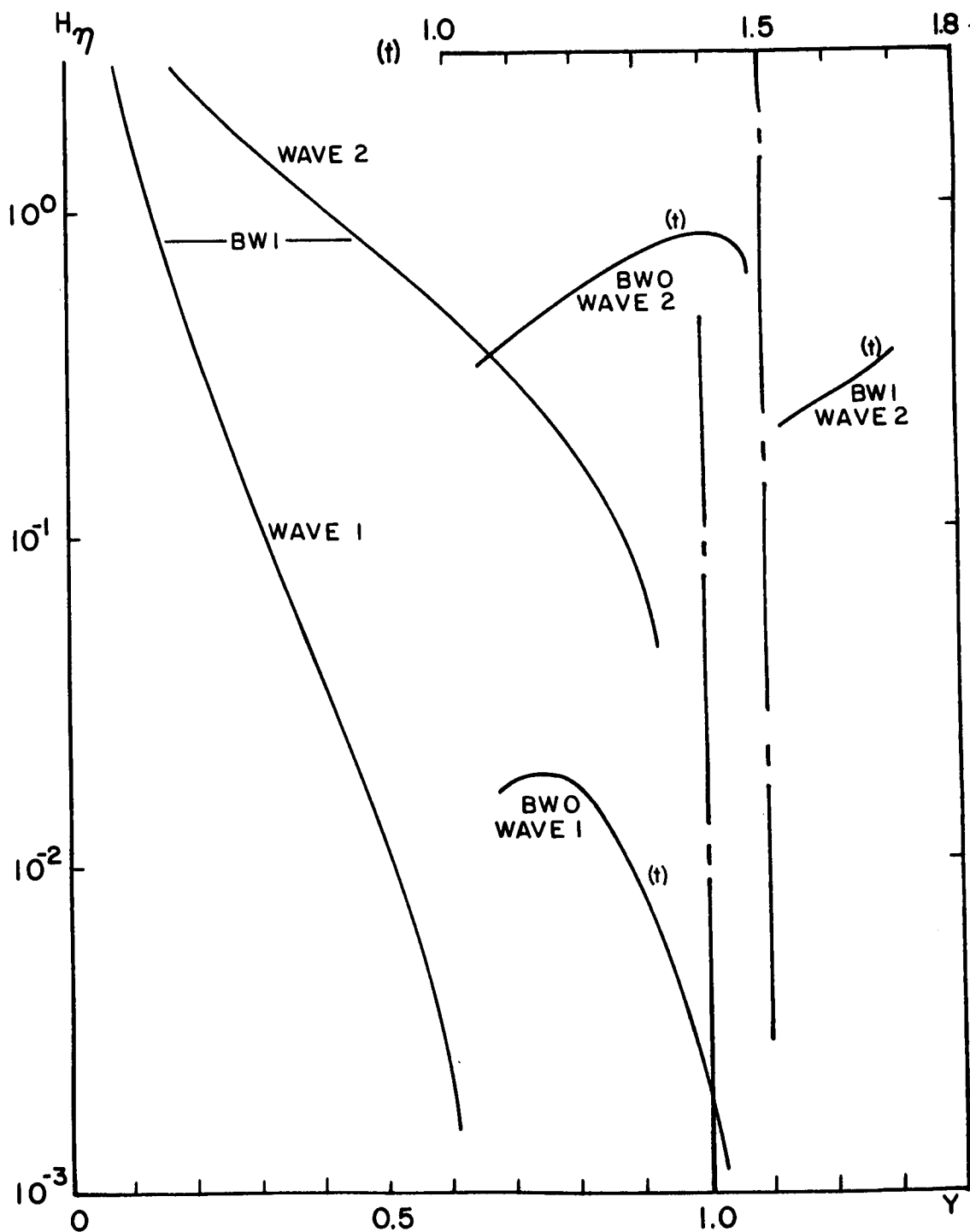
E_{η_1} AND E_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=1.5$ AND $b=1.0$

FIGURE 56



H_{ξ_1} AND H_{ξ_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $a=1.5$ AND $b=1.0$

FIGURE 57



H_{η_1} AND H_{η_2} VS FREQUENCY FOR THE ANTISYMMETRIC CASE WHEN $\alpha=1.5$ AND $b=1.0$

FIGURE 58

noted that the limiting values of H_{ξ_2} and H_{η_2} are much smaller than one so that the wave is almost pure electric.

Finally, the low frequency body wave curves indicate that for Y small, H_{ζ} goes to infinity so that the wave becomes transverse electric. When Y increases, H_{ζ} decreases and the two important components of the wave are E_{ζ} and E_{ξ} of wave 2. As Y approaches the resonant frequency $Y = 1$, only E_{ζ} of the wave 2 is important. Again, this was obtained in section 2.5.1.3.

4.) $\alpha = 1.5$, $b = 1$. Figures 53 to 58

For all the waves, the amplitude of E_1 is much smaller than one in the region of frequency considered (not for the B W S W). For the low frequency body wave, the phase velocity of the wave at a given frequency is larger for $b = 1$ than for $b = 0.1$ so that the amplitude of E_1 is larger than in the preceding case. The high frequency body wave is now hybrid with all the components of wave 2 present and of the same order of magnitude. The components of wave 1 are much smaller. For B W O, only E_{ξ_2} is small compared to the other components of the wave 2. Wave one is still negligible.

For the low frequency body wave, two regions have to be considered. Near the resonant frequency, only E_{ζ_2} is different from zero. (See Equation 57) Near $Y = 0$, H_{ζ} , E_{ξ} and H_{η} become the predominant components of wave 2. However, near $Y = 0$, some of the field components of wave 1 are also important, specifically for E_{η} and H_{ξ} wave 1 becomes the dominant wave.

Comparing this case with the preceding one, we see that the

components of the fields at a given frequency are larger than for $b = 0.1$, except E_ξ and E_ζ of wave 2 which still have the same order of magnitude.

2.5.3 Plasma Slab Bounded on One Side by a Conducting Screen

Seshadri and Owyang (1966) solved the problem of wave propagation along a magnetoplasma slab bounded on one side by a conducting screen. If we place a conducting screen in the plane $\xi = 0$ in the present study, the values of the parameters α and b chosen here are the same as the ones chosen by the above authors. If a comparison is made between their dispersion curves and our antisymmetric dispersion curves, it is found that they are almost identical. To explain this result, we have to consider the values of the fields in the plane $\xi = 0$. The boundary conditions on the conducting plane are

$$E_\zeta (\xi = 0) = 0$$

$$E_\eta (\xi = 0) = 0$$

$$H_\xi (\xi = 0) = 0$$

For our antisymmetrical solution E_ζ varies as a sinh function which vanishes automatically when $\xi = 0$. For thin slabs, $b = 0.1$, it can be seen on Figures 38, 39, 50 and 51 that except for very low frequency E_η and H_ξ are very small for $\xi = 0$ so that the boundary condition is almost satisfied. Therefore, the dispersion curves are practically the same.

When $b = 1.0$, however, E_η and H_ξ are not very small any more for the lowest order mode and for the low frequency B W 1. Some differences exist, therefore, for these two dispersion curves. For the higher order modes the boundary conditions are again almost satisfied

so that the dispersion curves are practically the same for the two problems.

2.6 THE "ONE WAVE" APPROXIMATION

We have seen in section 2.1 that the dispersion relation could be written in the form

$$(59) \begin{bmatrix} F_1 & F_2 \\ G_1 & G_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = 0$$

where, for the symmetric case

$$F_{1,2} = \cosh X_{1,2} b + \frac{TX_0}{X_{1,2}} \sinh X_{1,2} b$$

$$G_{1,2} = \beta_{1,2} \left[\cosh X_{1,2} b + \frac{X_0}{X_{1,2}} \sinh X_{1,2} b \right]$$

and for the antisymmetric case

$$F_{1,2} = \sinh X_{1,2} b + \frac{TX_0}{X_{1,2}} \cosh X_{1,2} b$$

$$G_{1,2} = \beta_{1,2} \left[\sinh X_{1,2} b + \frac{X_0}{X_{1,2}} \cosh X_{1,2} b \right]$$

Equation 59 yields

$$F_1 E_1 + F_2 E_2 = 0$$

and

$$G_1 E_1 + G_2 E_2 = 0$$

We have seen in section 2.5 that the amplitude of E_1 is much smaller than the amplitude of E_2 for almost all values of frequency in the case where a finite magnetic field is present. This is equivalent to saying

$$(60) \quad G_2 \ll G_1$$

and

$$F_2 \ll F_1$$

At cutoff G_2 and F_2 are both zero. The dispersion relation can therefore be reduced to

$$F_2 = 0$$

In the general case G_2 is small compared to G_1 , as implied by Equation 60, but different from zero. The zeros of the dispersion relation are approximately given by

$$F_2 = 0$$

In the limit of Z/Y going to infinity, $|E_1|$ vanishes so that $F_2 = 0$ is the true dispersion relation. Also, for $B = 0$ or for B going to infinity we have seen that the exact dispersion relation is

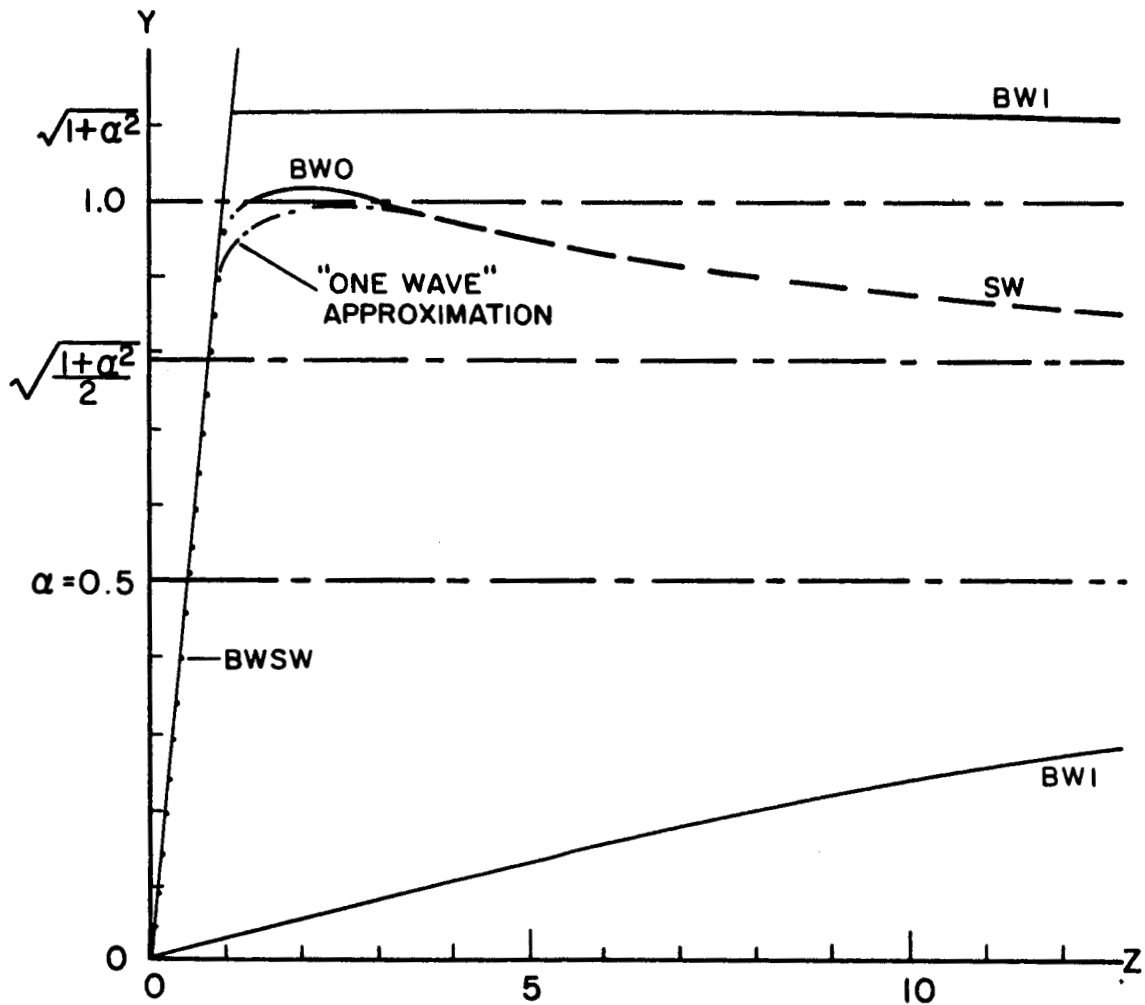
$$F_2 = 0$$

In order to include all the regions of the $Y Z$ plane, we have to consider the $B W S W$ region inside the loop $\Delta = 0$. The approximate dispersion relation can be rearranged a little to be written in the form.

$$(61) \quad X_2 \coth X_2 b + TX_0 = 0 \quad (\text{symmetric case})$$

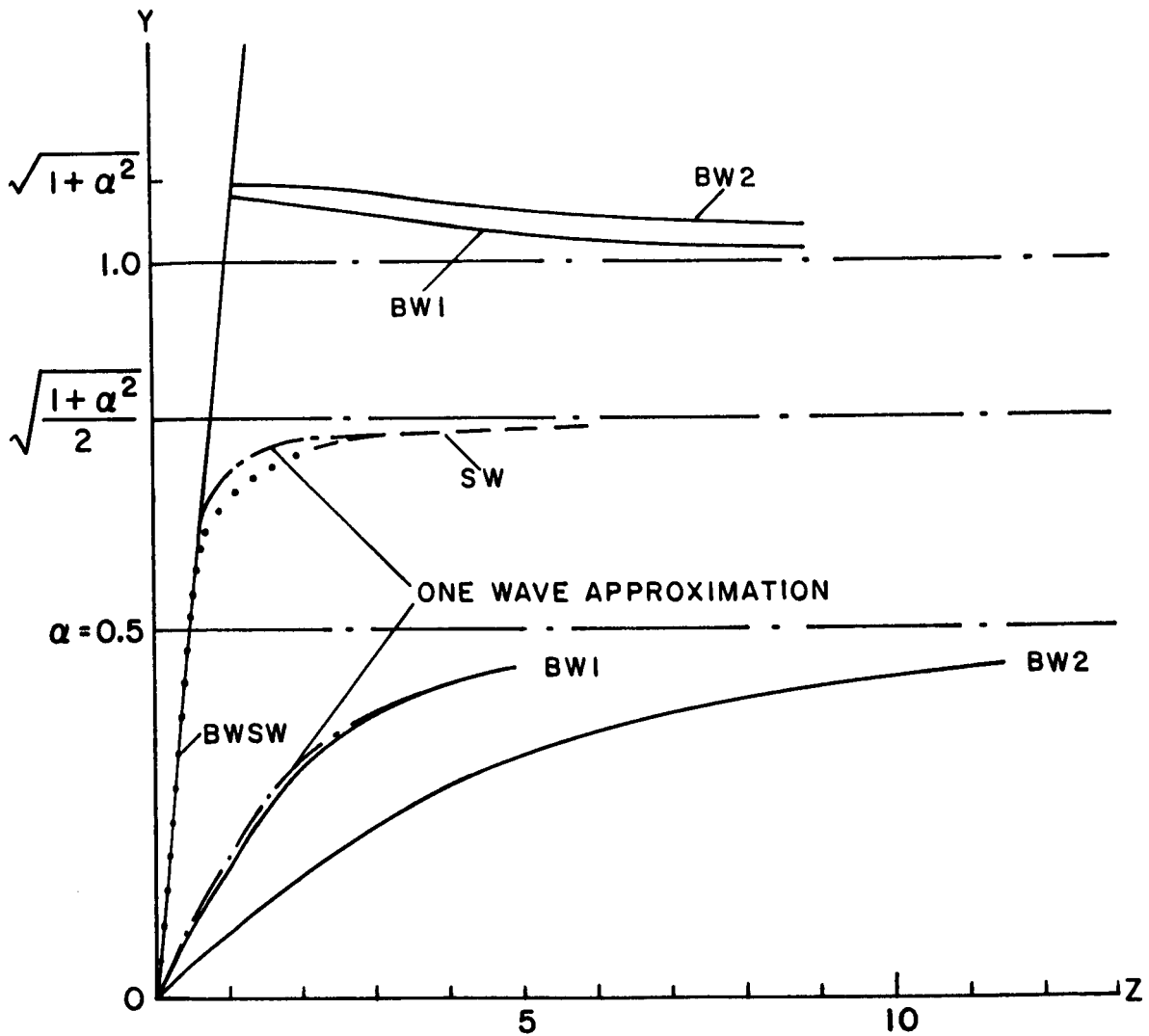
$$(62) \quad X_2 \tanh X_2 b + TX_0 = 0 \quad (\text{antisymmetric case})$$

Inside the loop and near the line $Y = Z$, the imaginary part of Equations 61 and 62 is very small so that the exact symmetric dispersion relation is practically satisfied when the real part of Equation 61 is zero and the exact antisymmetric dispersion relation is practically satisfied

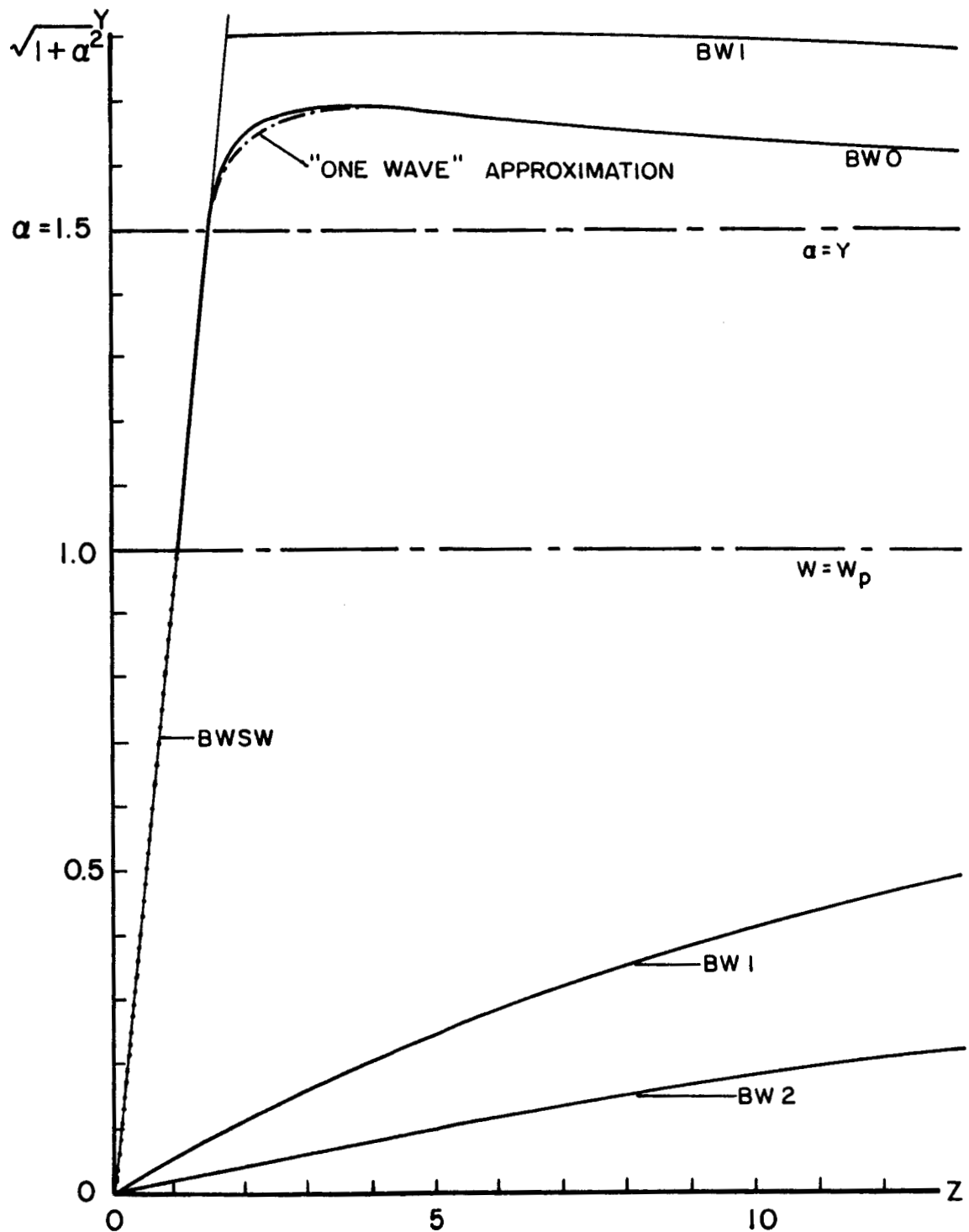


ANTISYMMETRIC CASE — EXACT AND APPROXIMATE
SOLUTION FOR $\alpha = 0.5$ AND $b = 0.1$

FIGURE 59

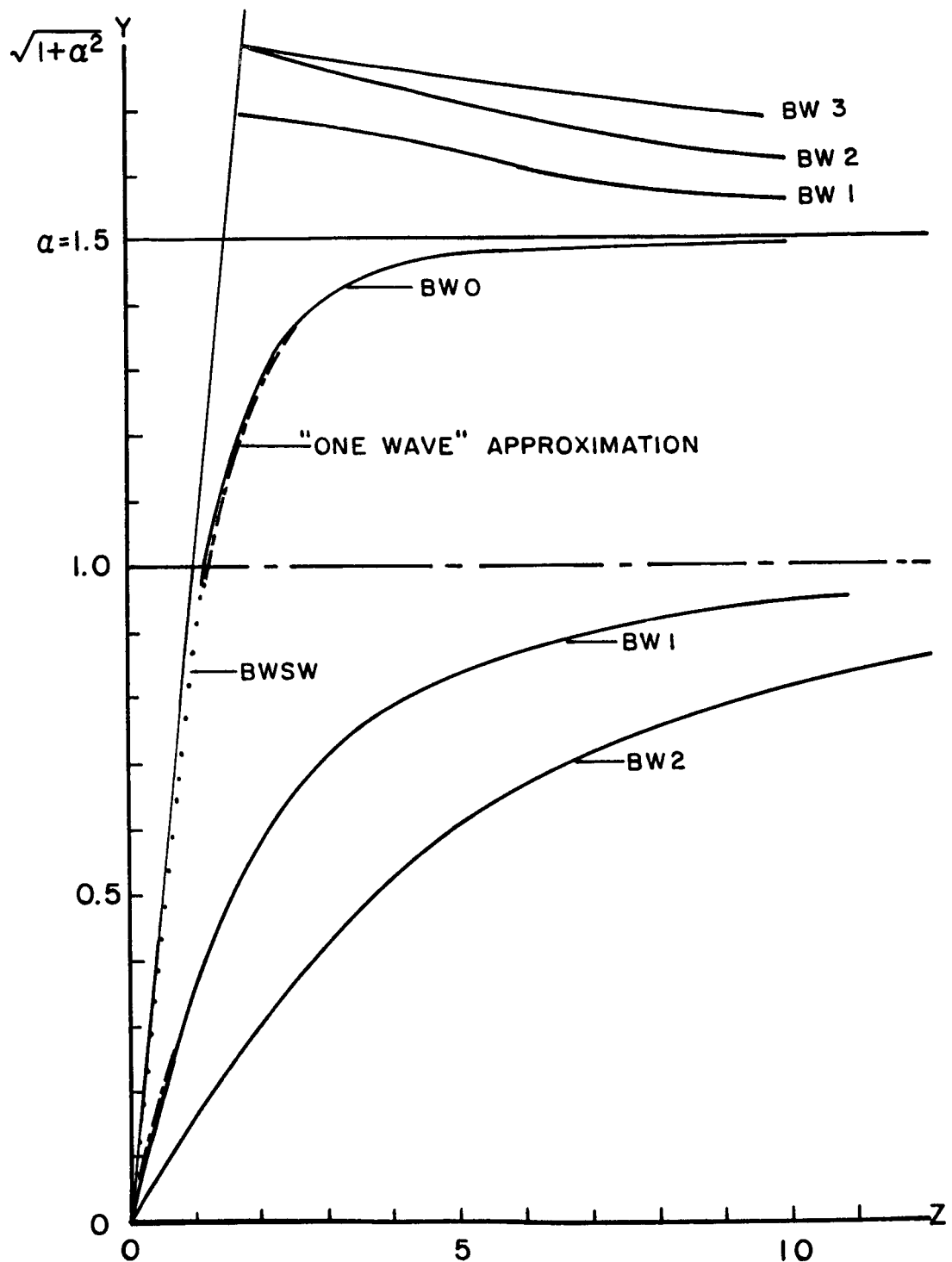


ANTISYMMETRIC CASE. EXACT AND APPROXIMATE
SOLUTION FOR $\alpha = 0.5$ AND $b = 1.0$
FIGURE 60



ANTISYMMETRIC CASE — EXACT AND APPROXIMATE
SOLUTION FOR $\alpha = 1.5$ AND $b = 0.1$

FIGURE 61



ANTISYMMETRIC CASE EXACT AND APPROXIMATE
SOLUTION FOR $\alpha = 1.5$ AND $b = 1.0$

FIGURE 62

when the real part of Equation 62 is zero. Since Equations 61 and 62 are real for the B W or for the S W, the approximate dispersion equation can be written for all cases as

$$(63) \quad \text{Re} (X_2 \coth X_2 b + TX_0) = 0 \quad (\text{symmetric})$$

$$(64) \quad \text{Re} (X_2 \tanh X_2 b + TX_0) = 0 \quad (\text{antisymmetric})$$

where Re means the real part of the enclosed function.

The approximate dispersion curves have been drawn on the same graph as the ones obtained with the exact theory. No differences could be found in the symmetric case between the exact theory and the "one wave" approximation. For the antisymmetric case, except for very low frequency, where the effect of the ions should be included in order to obtain valid results, the "one wave" approximation gives the same dispersion curves for the low frequency body waves as the one found with the exact theory. For the high frequency body waves, errors of no greater than one percent are found because at cutoff and for large Z the "one wave" approximation becomes the exact theory. For the lowest order mode, we expect some differences when the amplitude of E_1 is not much smaller than the amplitude of E_2 . This is shown on Figure 59 near $Y = 1$; on Figure 60 when the curve crosses the line $\Delta = 0$; on Figure 61 for Y near the light line, and on Figure 62 for the frequency gap $1 < Y < \alpha$. The approximate dispersion curve differs at most by five percent from the exact dispersion curves.

Although only four values of the parameters are presented, it was verified with the computer that the approximation was valid with an error of no more than five percent for much larger or much smaller

values of α and b . Since the approximation becomes the exact theory for $\alpha = 0$ and α going to infinity it is clear that the cases considered are in fact the most pessimistic.

In order to study the validity of the "one wave" approximation as a function of b , we need to rewrite the exact expression of the dispersion relation in the following form:

$$(65) \quad \frac{F_1 G_2}{G_1} - F_2 = 0$$

For small values of b , the phase velocity of the low frequency body wave is large so that $\frac{\beta_2}{\beta_1}$ is small. The term $\frac{F_1 G_2}{G_1}$ is, therefore, negligible and the approximate dispersion relation, $F_2 = 0$ is valid.

For large b , the argument that $|E_1|$ is small is not sufficient justification for using the "one wave" approximation because $|E_1 \cosh X_1 b|$ is large even though $|E_1|$ is small. The reason can be seen by examining Equation 65. The quantity $|E_1|$ becomes much smaller than one because G_1 is large. However, F_1 is of the same order of magnitude as G_1 . Nonetheless, the approximate dispersion relation $F_2 = 0$ still holds because when $F_2 = 0$, G_2 is also almost zero.

The same reasoning can be made for the high frequency waves. For small values of b the "one wave" approximation becomes exact at cutoff and for large values of Z so that the approximate dispersion curves differ at most by one or two percent from the exact dispersion curves. As b increases, $|E_1|$ becomes much smaller than one because G_1 is large. The approximation is still good because G_2 and F_2 are zero for almost the same points in the $Y Z$ plane.

Finally, for the lowest order mode, when b is large the dispersion curve becomes so close to the $Y = Z$ line and then to the line of resonance that the approximation cannot yield errors of more than one or two percent.

3 PLASMA COLUMN

3.1 DERIVATION OF THE DISPERSION RELATION

3.1.1 The Model

The physical model used to solve the problem of the plasma column is the same as the one used in the slab case. A cylindrical coordinate system (r, θ, ζ) is introduced. The axis of the column is the ζ axis along which the homogeneous magnetic field is directed. We consider an infinitely long column of plasma of radius a . A monochromatic wave propagates in the direction of the magnetic field so that the ζ, t , and θ dependence can be taken to be of the form:

$$e^{i K_{\zeta} \zeta - i \omega t + i m \theta}$$

The dimensions are normalized in the same way as for the slab configuration, now with

$$x = \frac{r \omega_p}{c}, \quad b = \frac{a \omega_p}{c}$$

3.1.2. The General Dispersion Relation

Maxwell's equations can be written in the following form:

$$(65) \quad \begin{bmatrix} (SY^2 - Z^2) & -iY^2 D \\ iY^2 D & (SY^2 - Z^2) \end{bmatrix} \begin{bmatrix} E_r \\ E_{\theta} \end{bmatrix} = \begin{bmatrix} iZ \frac{\partial}{\partial x} & -\frac{mY}{x} \\ -\frac{mZ}{x} & -iY \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} E_{\zeta} \\ H_{\zeta} \end{bmatrix}$$

$$(66) \quad \begin{bmatrix} H_r \\ H_{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -Z/Y & \frac{m}{Yx} \\ Z/Y & 0 & \frac{i}{Y} \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} E_r \\ E_{\theta} \\ E_{\zeta} \end{bmatrix}$$

and the axial components of the fields inside the plasma are given by:

$$(67) \quad \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial E_{\zeta}}{\partial x} \right) + \left(TY^2 - \frac{TZ^2}{S} - \frac{m^2}{x^2} \right) E_{\zeta} = -i Y \frac{ZD}{S} H_{\zeta}$$

$$(68) \quad \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial H_{\zeta}}{\partial x} \right) - \left(Y^2 \frac{(D^2 - S^2)}{S} + Z^2 + \frac{m^2}{x^2} \right) H_{\zeta} = i Y Z \frac{DT}{S} E_{\zeta}$$

The solution is easily found to be

$$(69) \quad E_{\zeta} = \sum_{i=1}^2 E_i I_m(X_i x)$$

$$(70) \quad H_{\zeta} = \sum_{i=1}^2 - \frac{i E_i \beta_i}{D Y Z} I_m(X_i x)$$

where $I_m(x)$ is the modified Bessel function of the first kind of order m , and where the rest of the notation is the same as used in Chapter 2.

The transverse components of the fields are found from Equations 65 and 66 and can be written

$$(71) \quad E_{\theta} = \left[\sum_{i=1}^2 \frac{E_i m}{x} I_m(X_i x) \left[-Z(SY^2 - Z^2) + \frac{\beta_i Y^2}{Z} \right] \right. \\ \left. + \sum_{i=1}^2 E_i X_i I'_m(X_i x) \left[ZDY^2 - \frac{\beta_i(SY^2 - Z^2)}{DZ} \right] \right] \\ \times \frac{1}{(SY^2 - Z^2)^2 - D^2 Y^4}$$

$$(72) \quad E_r = \left[\sum_{i=1}^2 \frac{E_i}{x} \frac{I_m(X_i x)}{I_m(X_i x)} \left[\frac{\beta_i (SY^2 - Z^2)}{D Z} - D Z Y^2 \right] + \sum_{i=1}^2 E_i X_i \frac{I'_m(X_i x)}{I_m(X_i x)} \left[Z (SY^2 - Z^2) - \frac{\beta_i Y^2}{Z} \right] \right] \times \frac{1}{(SY^2 - Z^2)^2 - D^2 Y^4}$$

$$(73) \quad H = \left[\sum_{i=1}^2 \frac{E_i}{D Y x} \frac{I_m(X_i x)}{I_m(X_i x)} \left[\beta_i (SY^2 - Z^2) - D^2 Y^2 Z^2 \right] + \sum_{i=1}^2 \frac{E_i X_i}{Y} \frac{I'_m(X_i x)}{I_m(X_i x)} \left[Z^2 (SY^2 - Z^2) - \beta_i Y^2 \right] + (SY^2 - Z^2)^2 - D^2 Y^4 \right] \times \frac{1}{(SY^2 - Z^2)^2 - D^2 Y^4}$$

$$(74) \quad H_r = \left[\sum_{i=1}^2 \frac{E_i}{Y x} \frac{I_m(X_i x)}{I_m(X_i x)} \left[Z^2 (SY^2 - Z^2) - \beta_i Y^2 + (SY^2 - Z^2)^2 - D^2 Y^4 \right] - \sum_{i=1}^2 \frac{E_i X_i}{D Y} \frac{I'_m(X_i x)}{I_m(X_i x)} \left[D Y^2 Z^2 - \beta_i (SY^2 - Z^2) \right] \right] \times \frac{1}{(SY^2 - Z^2)^2 - D^2 Y^4}$$

where I'_m is the derivative with respect to the total argument, and the coefficient $\exp (i K_\zeta - i\omega t + im\theta)$ has been omitted.

These expressions are very complicated. However, using an analogy with the slab problem (see Appendix A) we find the following identities:

$$(75) \quad \frac{Z^2 (SY^2 - Z^2) - \beta_i Y^2 + (SY^2 - Z^2)^2 - D^2 Y^4}{(SY^2 - Z^2)^2 - D^2 Y^4} \equiv - \frac{TY^2}{X_i^2}$$

$$(76) \quad \frac{D^2 Y^2 Z^2 - \beta_i (SY^2 - Z^2)}{(SY^2 - Z^2)^2 - D^2 Y^4} \equiv \frac{\beta_i}{X_i^2}$$

$$(77) \quad \frac{Z^2 (SY^2 - Z^2) - \beta_i Y^2}{(SY^2 - Z^2)^2 - D^2 Y^4} \equiv - \frac{TY^2 + X_i^2}{X_i^2}$$

The complexity of the dispersion relation without these simplifying identities has discouraged previous investigators from giving a complete analysis of the higher order modes. Also, the properties of the fields are much more in evidence in the simplified form. The field components inside the plasma can now be rewritten as:

$$(78) \quad E_\zeta = \sum_{i=1}^2 E_i I_m (X_i x)$$

$$(79) \quad H_\zeta = \sum_{i=1}^2 -i \frac{E_i \beta_i}{D Y Z} I_m (X_i x)$$

$$(80) \quad E_\theta = \sum_{i=1}^2 \frac{E_i m I_m (X_i x)}{x Z X_i^2} (TY^2 + X_i^2) + \frac{E_i \beta_i}{D X_i Z} I'_m (X_i x)$$

$$(81) \quad E_r = \sum_{i=1}^2 - \frac{E_i \beta_i m I_m (X_i x)}{D X_i^2 Z x} - \frac{i E_i (TY^2 + X_i^2)}{Z X_i} I'_m (X_i x)$$

$$(82) \quad H_\theta = \sum_{i=1}^2 - \frac{i E_i \beta_i m I_m (X_i x)}{D Y X_i^2 x} - i E_i \frac{TY}{X_i} I'_m (X_i x)$$

$$(83) \quad H_r = \sum_{i=1}^2 - \frac{E_i TY}{X_i^2} \frac{m I_m (X_i x)}{x} - \frac{E_i \beta_i}{D Y X_i} I'_m (X_i x)$$

while outside the slab the fields are simply given by:

$$E_\zeta = E_o K_m (X_o x)$$

$$H_\zeta = H_o K_m (X_o x)$$

$$E_r = - \frac{i Z}{X_o} E_o K'_m (X_o x) + \frac{m Y}{x X_o^2} H_o K_m (X_o x)$$

$$E_\theta = \frac{m Z E_o}{x X_o^2} K_m (X_o x) + \frac{i Y}{X_o} H_o K'_m (X_o x)$$

$$H_\theta = - \frac{i Y}{X_o} E_o K'_m (X_o x) + \frac{m Z}{x X_o^2} H_o K_m (X_o x)$$

$$H_r = - \frac{m Y}{x X_o^2} E_o K_m (X_o x) - \frac{i Z}{X_o} H_o K'_m (X_o x)$$

where $K_m (X)$ is the modified Bessel function of the second kind of order m . From the boundary conditions on the fields it is found that the relation

$$(84) \quad \begin{bmatrix} A_1 I_m (X_1 b) + B_1 I'_m (X_1 b) & A_2 I_m (X_2 b) + B_2 I'_m (X_2 b) \\ C_1 I_m (X_1 b) + D_1 I'_m (X_1 b) & C_2 I_m (X_2 b) + D_2 I'_m (X_2 b) \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = 0$$

must be satisfied, where

$$A_i = \frac{Y}{X_o} \frac{K'_m(X_o b)}{K_m(X_o b)} + \frac{1}{b X_o^2} \frac{\beta_i m}{D Y} - \frac{\beta_i m}{D Y X_i^2}$$

$$B_i = - \frac{T Y}{X_i}$$

$$C_i = \frac{m Z}{b X_o^2} + \frac{Y}{X_o} \frac{K'_m(X_o b)}{K_m(X_o b)} \frac{\beta_i}{D Y Z} - \frac{m (T Y^2 + X_i^2)}{b Z X_i^2}$$

$$D_i = - \frac{\beta_i}{D X_i Z}$$

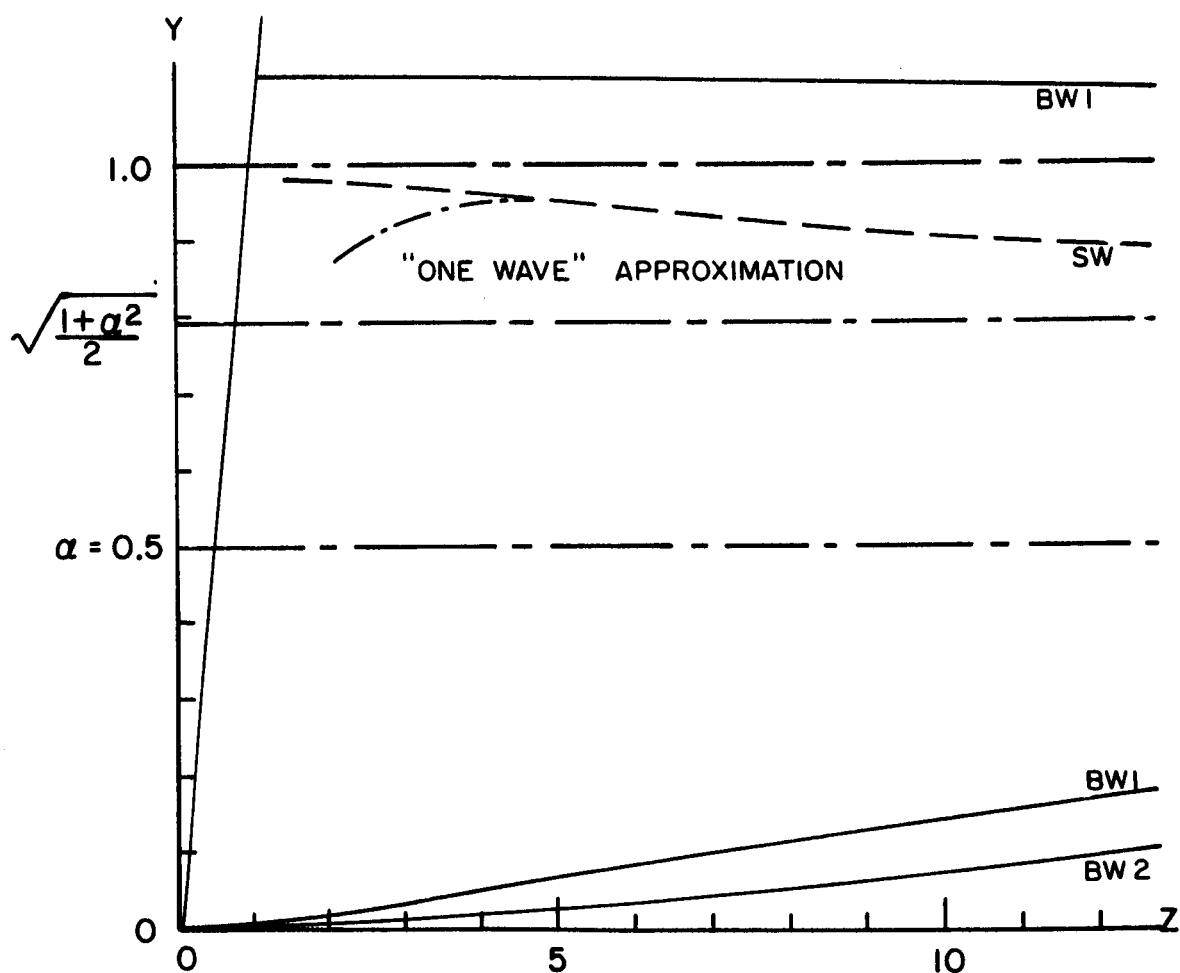
In order to obtain a nontrivial solution for the fields, the determinant of the 2×2 matrix must be set equal to zero. In contrast with previous expressions, the coefficients of the modified Bessel functions can be examined by a very simple analysis. As a consequence, we will be able to study the relative magnitude of E_1 and E_2 .

3.1.3 Symmetric Modes

When $m = 0$, the dispersion relation obtained corresponds to the circularly symmetric case

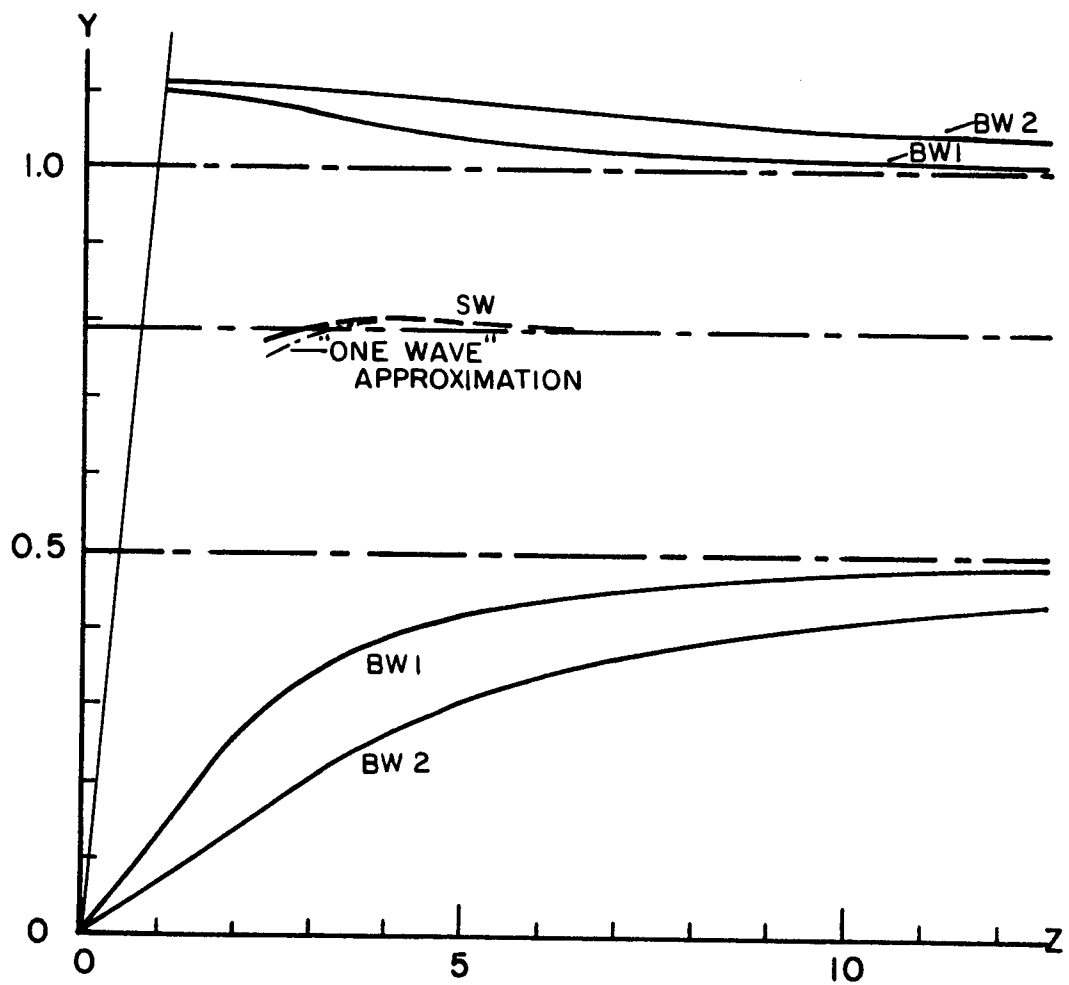
$$\begin{vmatrix} \frac{K'_o(X_o b)}{K_o(X_o b)} \frac{I_o(X_1 b)}{X_o} - \frac{T I'_o(X_1 b)}{X_1} & \frac{K'_o(X_o b)}{K_o(X_o b)} \frac{I_o(X_2 b)}{X_o} - \frac{T I'_o(X_2 b)}{X_2} \\ \beta_1 \left[\frac{K'_o(X_o b)}{K_o(X_o b)} \frac{I_o(X_1 b)}{X_o} - \frac{I'_o(X_1 b)}{X_1} \right] & \beta_2 \left[\frac{K'_o(X_o b)}{K_o(X_o b)} \frac{I_o(X_2 b)}{X_o} - \frac{I'_o(X_2 b)}{X_2} \right] \end{vmatrix} = 0$$

This dispersion relation has been analyzed by many authors (Formato and Gilardini, 1962; Camus, 1965; Yip and Seshadri, 1967). Therefore, we will not give the corresponding dispersion curves.



DIPOLE MODE. EXACT AND APPROXIMATE SOLUTION FOR
 $\alpha=0.5$ AND $b=0.1$

FIGURE 63



DIPOLE MODE. EXACT AND APPROXIMATE SOLUTION FOR
 $\alpha=0.5$ AND $b=1.0$
FIGURE 64

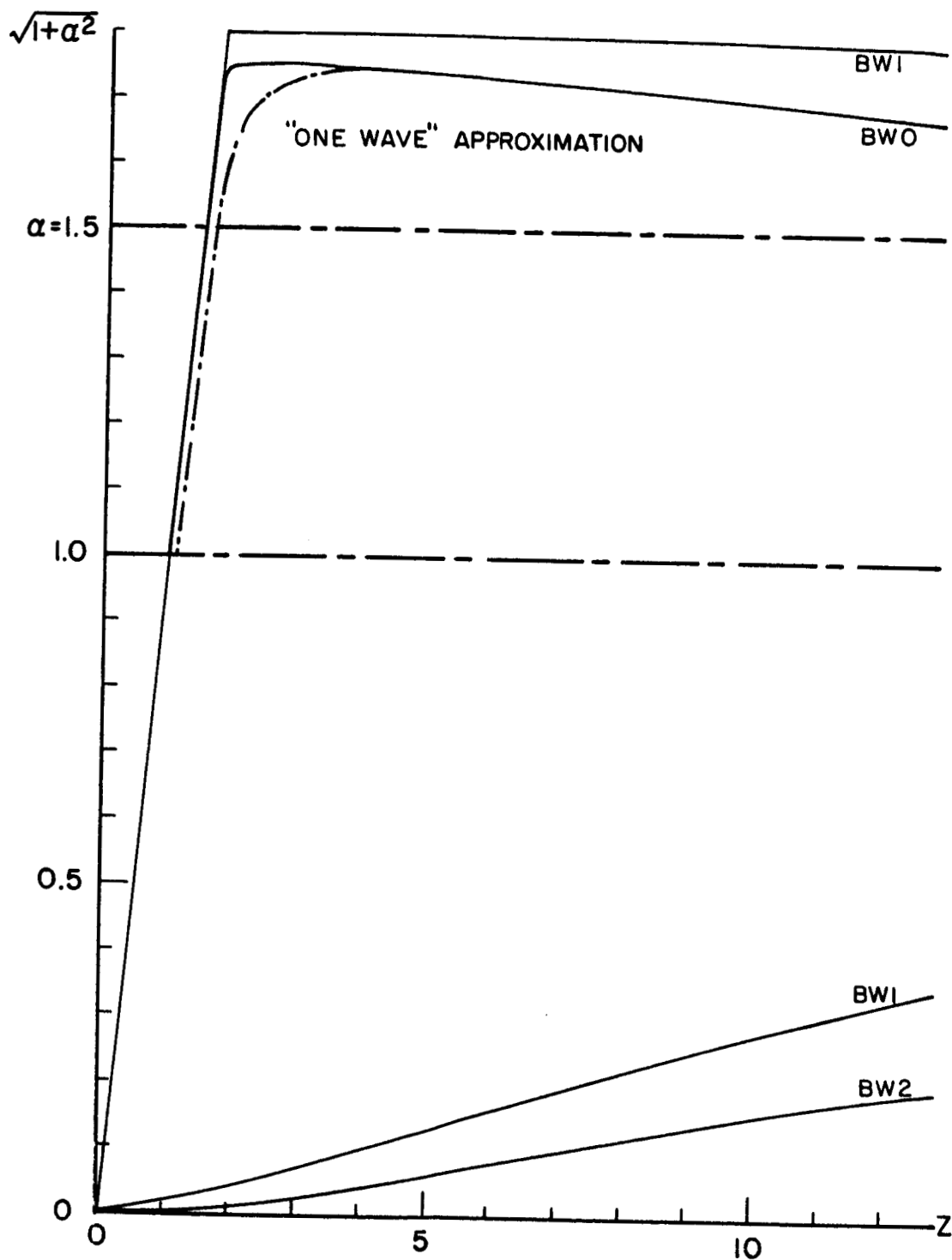
3.2 RESULTS OF THE ANALYSIS FOR THE DIPOLE MODE

Since X_1 and X_2 are not functions of geometry, the same wave type regions shown in Figures 17 and 18 are again obtained. Consequently, the general discussion of section 2.4.1 applies to the cylindrical geometry. However, the dispersion curves for the dipole mode are not identical with those from the antisymmetric case of the slab geometry. We will consider here the differences between the slab in the antisymmetric case and the dipole mode in the rod geometry.

The dispersion curves have not been calculated inside the loop $\Delta = 0$ because no subroutine was available on the computer for the Bessel functions of complex argument. However, the lowest order mode starts as a B W S W. The same values of the parameters α and b used in the slab geometry are again used for the cylinder.

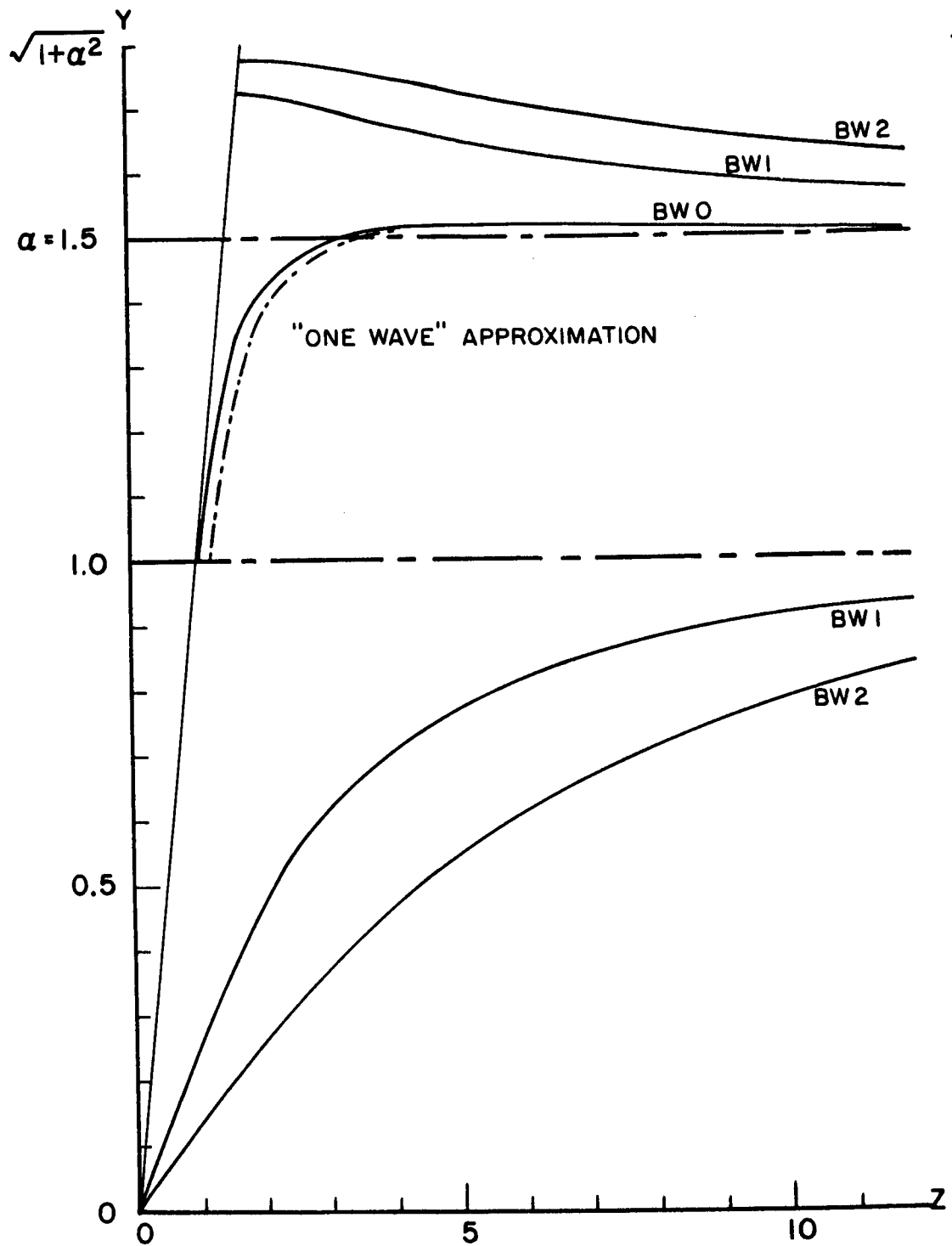
1.) $\alpha = 0.5$, $b = 0.1$ and 1. Figures 63 and 64

For a given frequency, the low frequency body waves have a group velocity which is smaller for the rod geometry than for the slab case. The lowest order mode remains below the line $Y = 1$ and propagates as a backward surface wave outside the loop $\Delta = 0$ when b is small. When no magnetic field is present, Granatstein, Schlesinger and Vigant (1963) showed that the dipole mode could not propagate above the line $Y = \frac{1}{\sqrt{2}}$. However, it should be noted that even for $\alpha < 1$, the magnetic field can become sufficiently large that the surface wave propagates for frequencies higher than the lower hybrid frequency and reaches the asymptote $Y = \sqrt{\frac{1+\alpha^2}{2}}$ from the high frequency side. This is seen to be the case for $\alpha = 0.5$. When $b = 1$



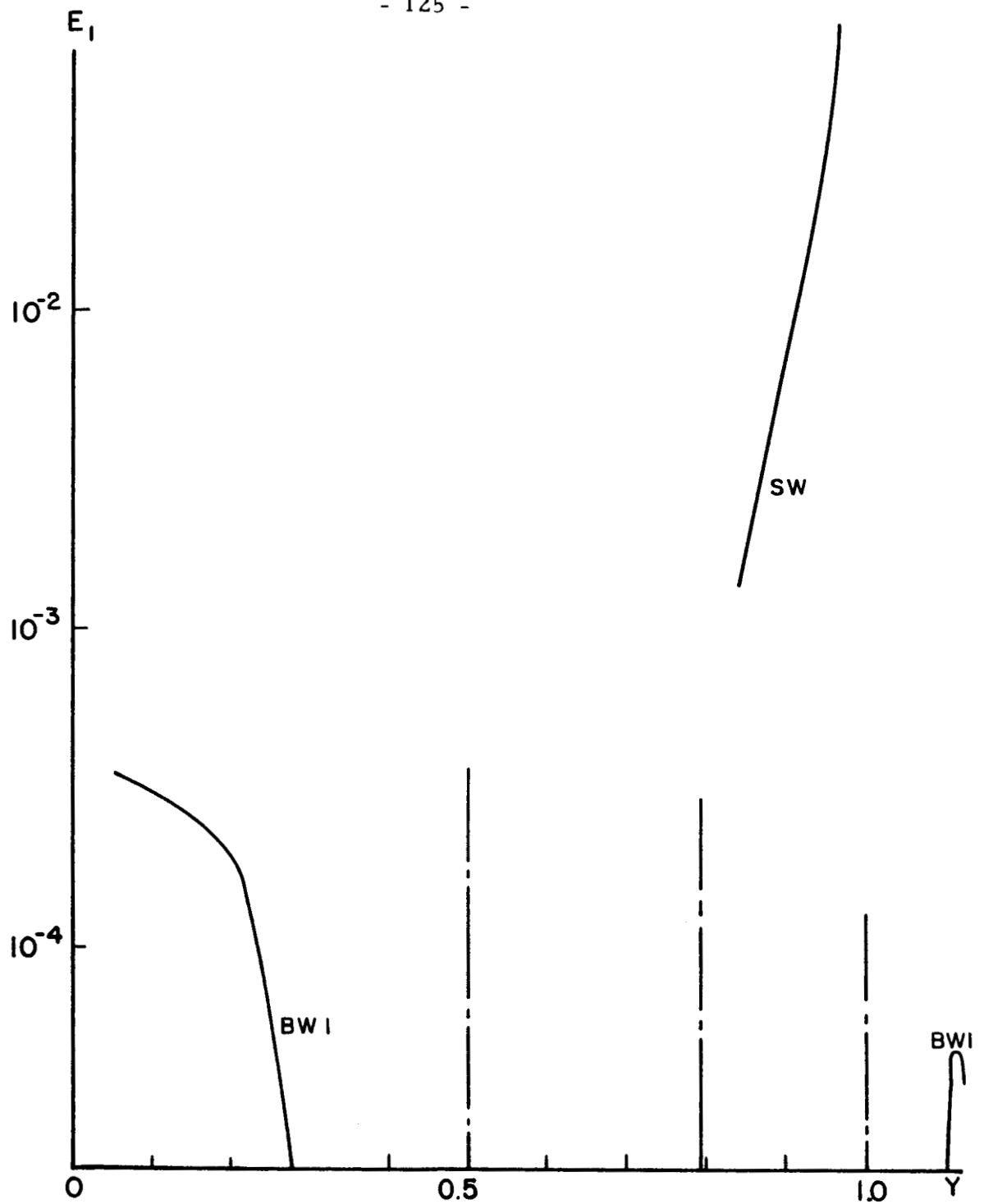
DIPOLE MODE. EXACT AND APPROXIMATE SOLUTION FOR
 $\alpha=1.5$ AND $b=0.1$

FIGURE 65



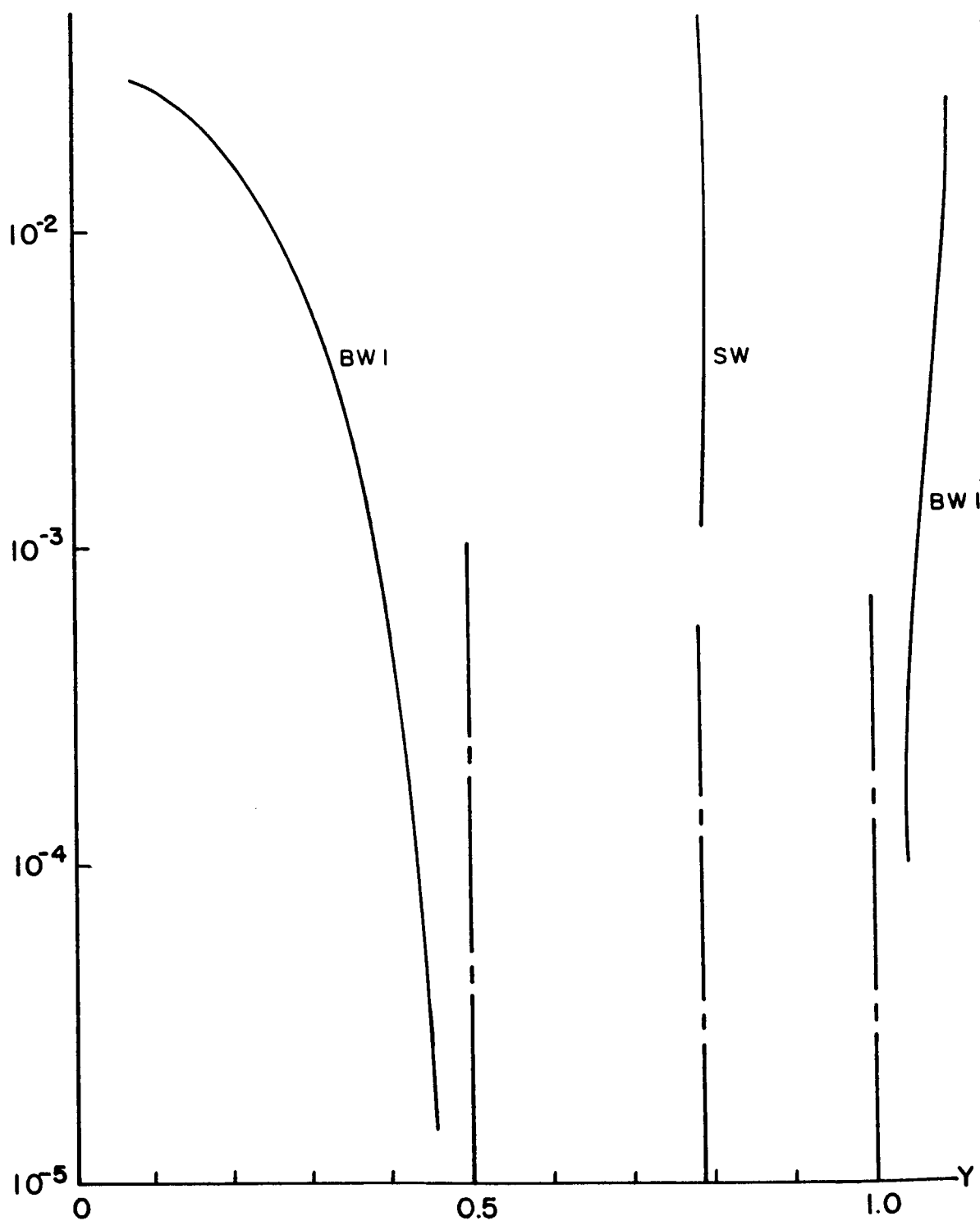
DIPOLE MODE. EXACT AND APPROXIMATE SOLUTION FOR
 $\alpha=1.5$ AND $b=1.0$

FIGURE 66

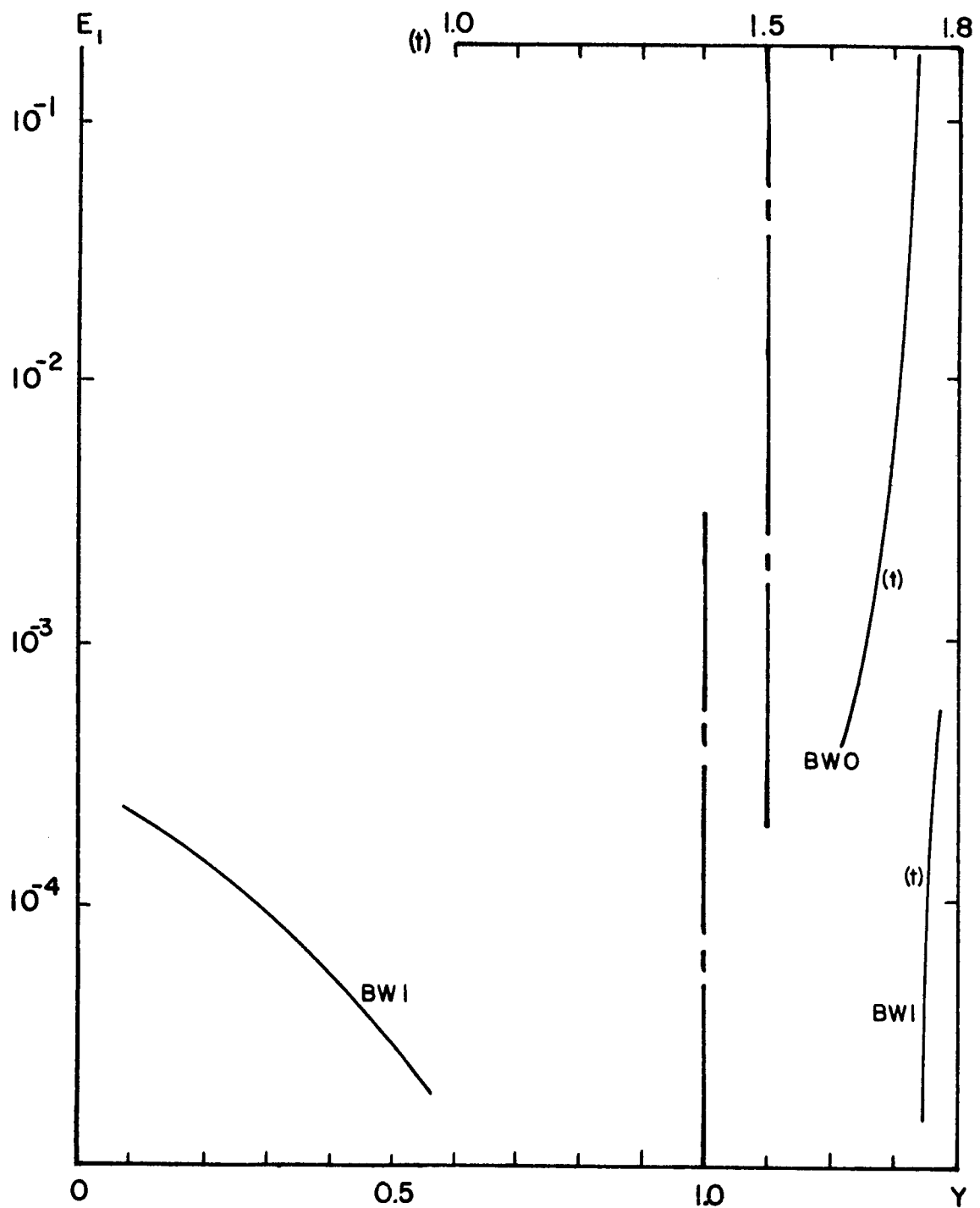


E_1 VS FREQUENCY FOR THE DIPOLE MODE WHEN
 $\alpha=0.5$ AND $b=0.1$

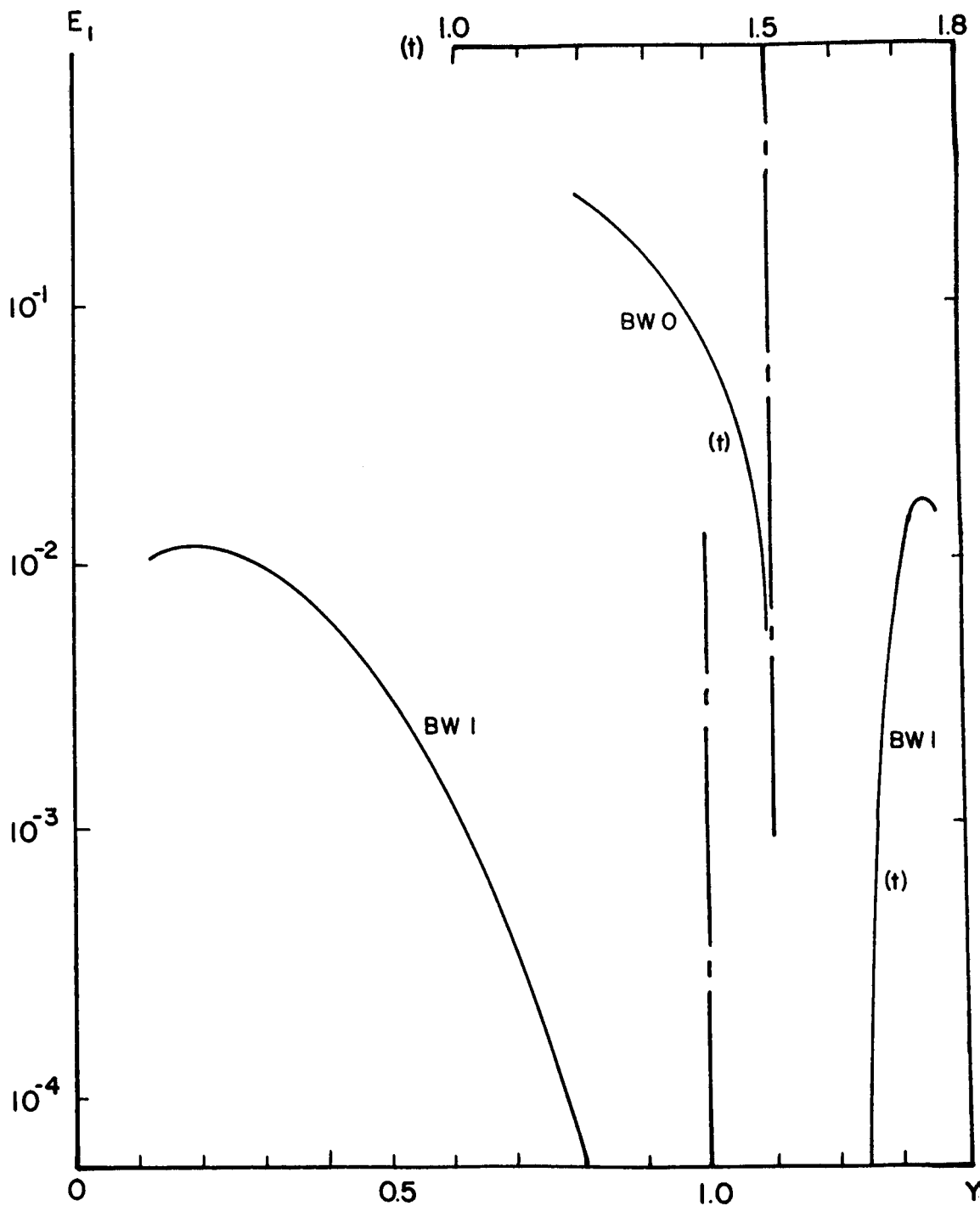
FIGURE 67



E_1 VS FREQUENCY FOR THE DIPOLE MODE WHEN $\alpha=0.5$
AND $b=1.0$
FIGURE 68



E_1 VS FREQUENCY FOR THE DIPOLE MODE WHEN
 $\alpha = 1.5$ AND $b = 0.1$
 FIGURE 69



E_1 VS FREQUENCY FOR THE DIPOLE MODE WHEN
 $\alpha = 1.5$ AND $b = 1.0$

FIGURE 70

this mode starts as a forward surface wave as its dispersion curve crosses the curve $\Delta = 0$, reaches a maximum frequency and becomes backward to approach the resonance $Y = \sqrt{\frac{1 + \alpha^2}{2}}$. In this last case, the group velocity of the wave is very small. Note that B W O does not exist at all in either case, which is different from the slab results.

2.) $\alpha = 1.5$, $b = 0.1$ and 1. Figure 65 and 66

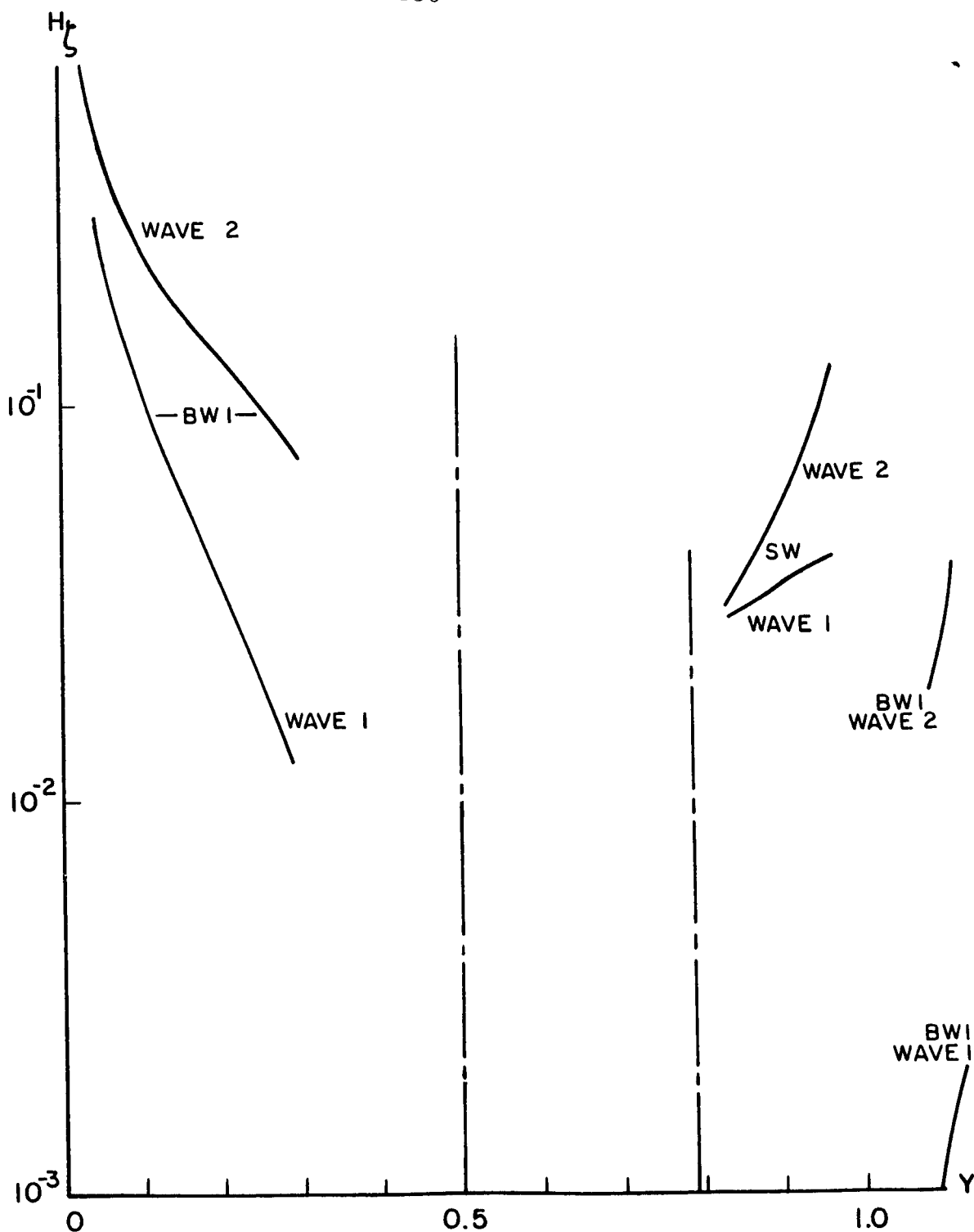
Again the low frequency body waves have a smaller group velocity for a given frequency for the dipole mode than for the antisymmetric case in the slab geometry. The forward surface wave region between the loop $\Delta = 0$ and the line $Y = 1$ still exists; however, it is much smaller than in the slab geometry because in the lowest order mode the dispersion curve is practically along the $Y = Z$ line.

3.3 FIELDS CHARACTERISTICS: "ONE WAVE" APPROXIMATION

3.3.1 Fields Characteristics

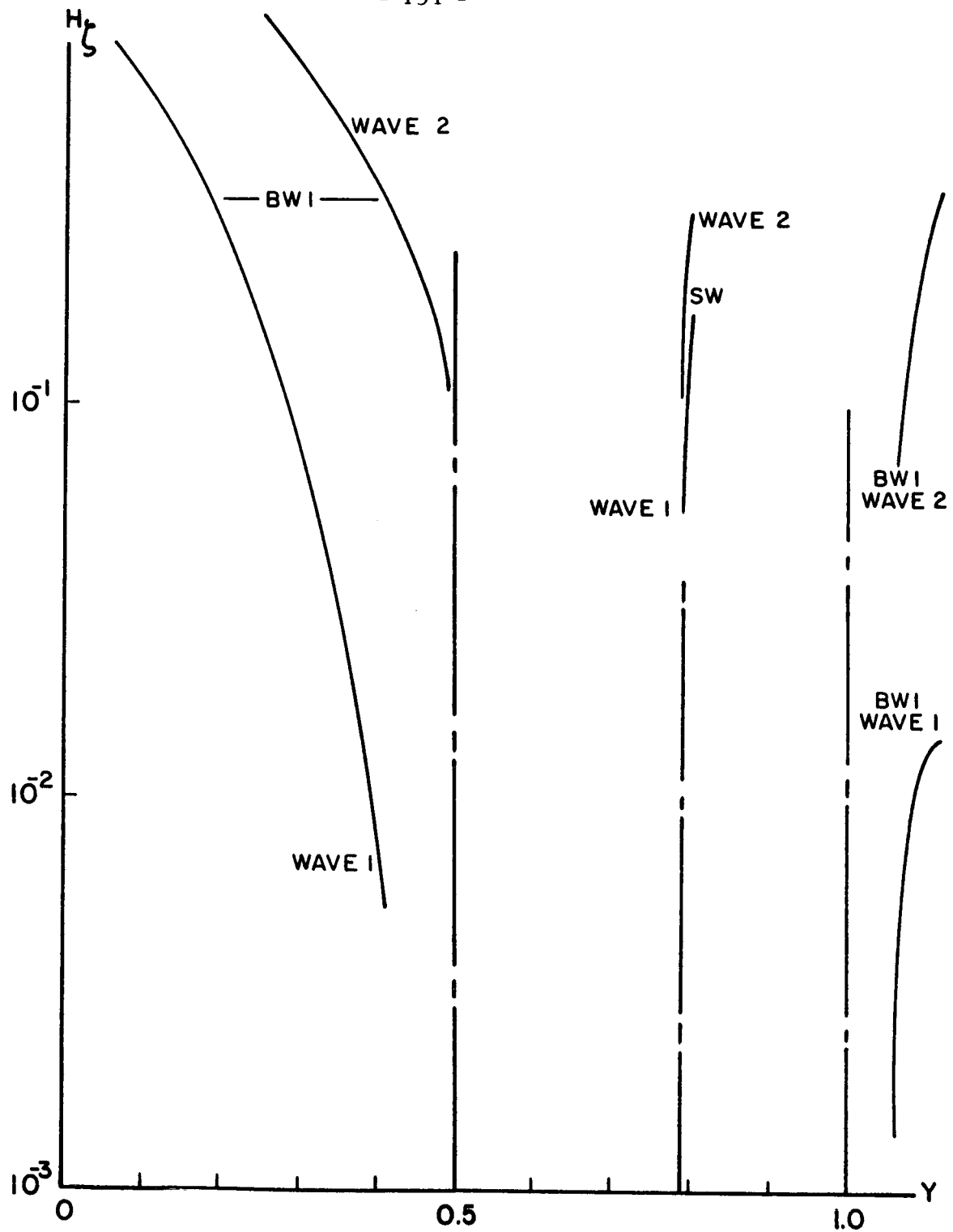
We consider the coefficients of the Bessel functions in the expression giving the longitudinal field components of the wave inside the plasma. All the components are normalized by setting E_2 in Equation 84 equal to unity.

Figures 67 to 70 show $|E_1|$ plotted versus frequency for the four sets of parameters chosen in this analysis. It is seen that for the low frequency and high frequency body waves the amplitude of E_1 is much smaller than one as in the slab geometry. However, for the lowest order mode, it becomes approximately equal to one for larger values of frequency than for the slab problem. Consequently, when we consider the "one wave" approximation, we may expect more discrepancy for the lowest order



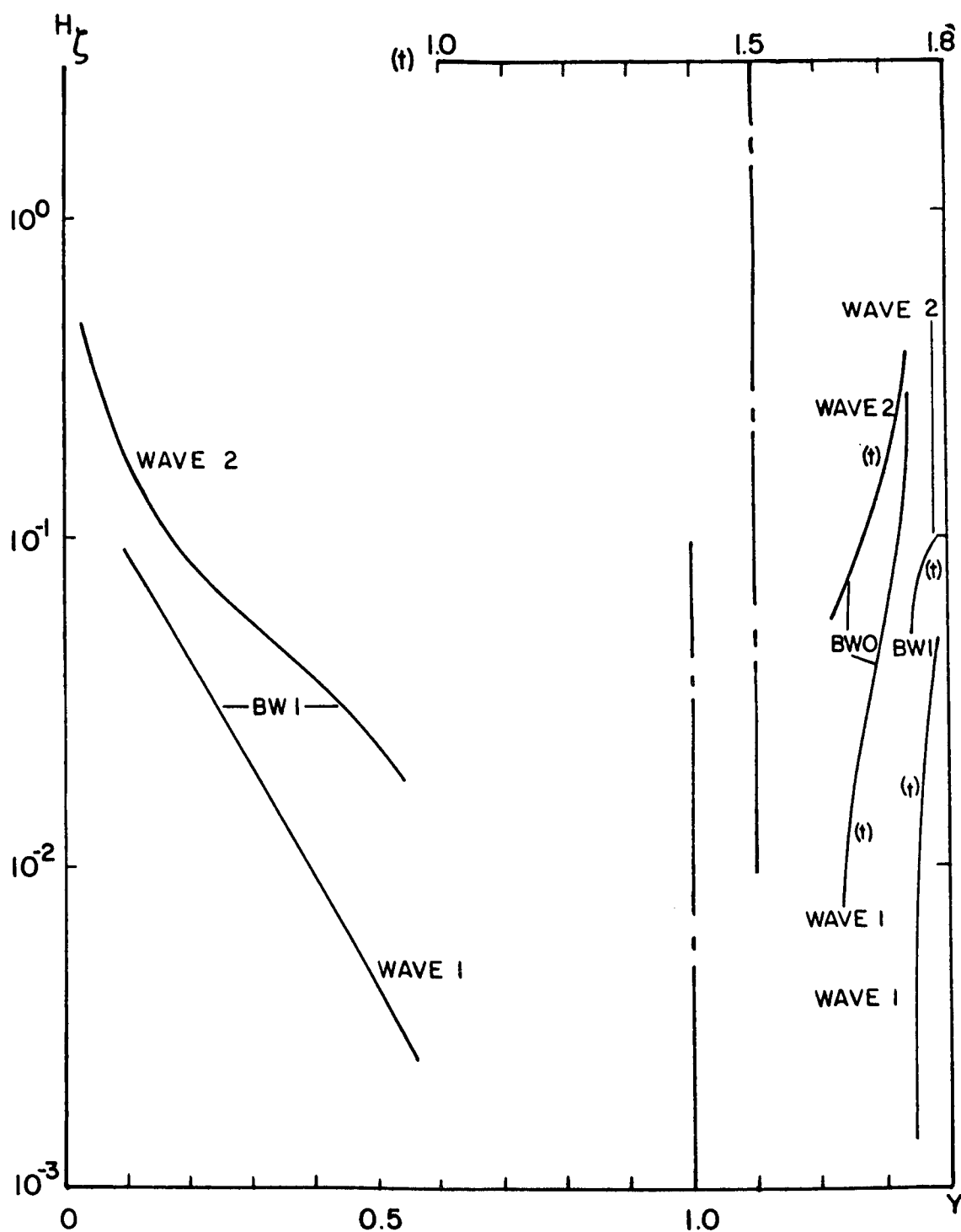
$H_{\zeta 1}$ AND $H_{\zeta 2}$ VS FREQUENCY FOR THE DIPOLE MODE
WHEN $\alpha=0.5$ AND $b=0.1$

FIGURE 71



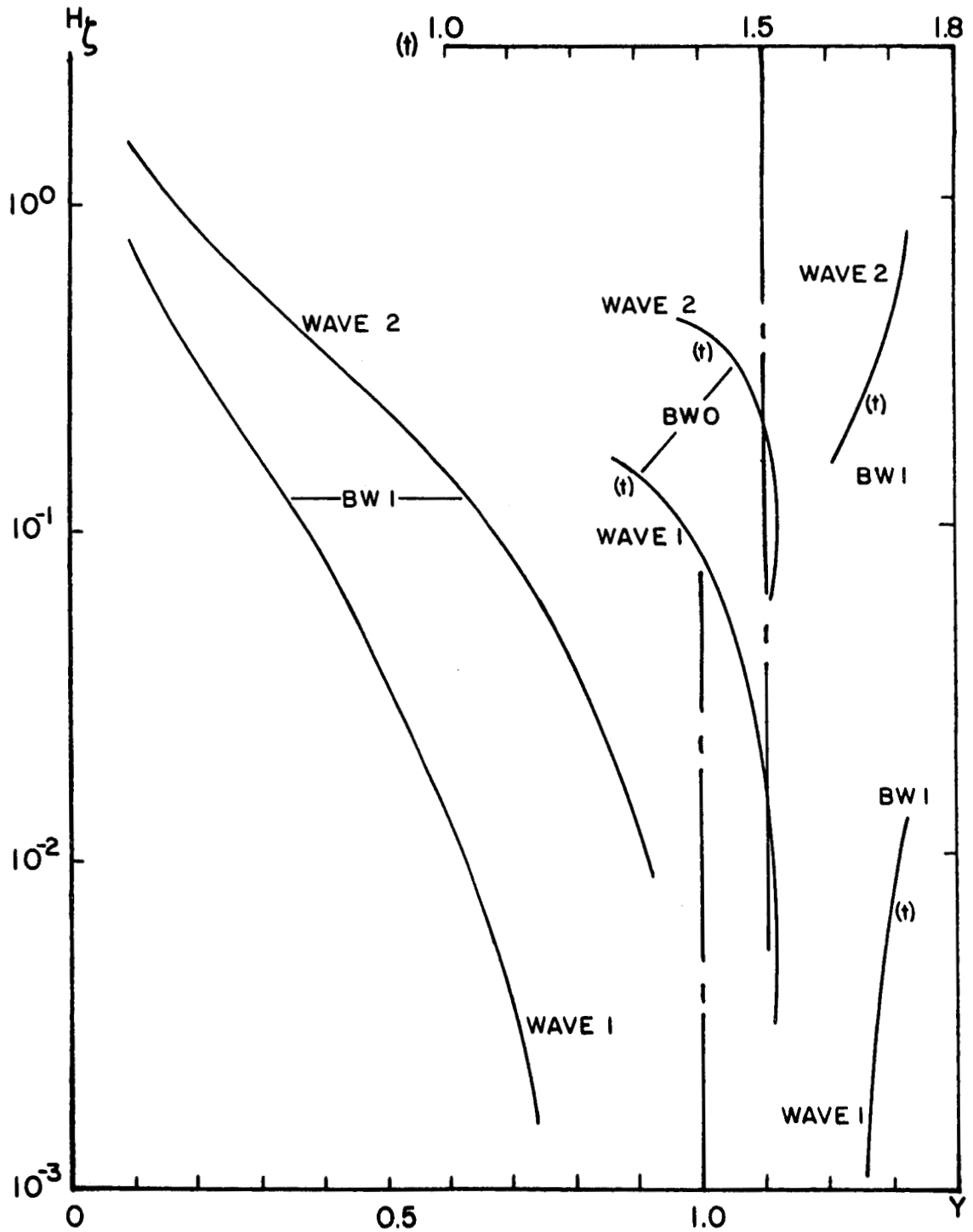
$H_{\zeta 1}$ AND $H_{\zeta 2}$ VS FREQUENCY FOR THE DIPOLE MODE
WHEN $\alpha=0.5$ AND $b=1.0$

FIGURE 72



H_{ζ_1} AND H_{ζ_2} VS FREQUENCY FOR THE DIPOLE MODE
WHEN $\alpha=1.5$ AND $b=0.1$

FIGURE 73



$H_{\zeta 1}$ AND $H_{\zeta 2}$ VS FREQUENCY FOR THE DIPOLE MODE
WHEN $\alpha=1.5$ AND $b=1.0$

FIGURE 74

dipolar mode than in the antisymmetric case of the slab geometry.

On Figures 71 to 74 the coefficients of the Bessel functions for the longitudinal components of the magnetic field corresponding to wave 1 and wave 2 are plotted versus frequency. The amplitude of the magnetic field corresponding to wave 1 is much smaller than the one corresponding to wave 2 except for S W when $\alpha = 0.5$ (Figures 71 and 72). For very low frequency, the amplitudes of both H_{ζ_1} and H_{ζ_2} become large so that the low frequency body waves become transverse electric. Near resonance the waves are transverse magnetic. For the high frequency body waves, near cutoff the wave is hybrid for large values of α and for small values of α when b is large. The wave is practically transverse magnetic when $\alpha = 0.5$ and $b = 0.1$ (Figure 71).

3.3.2 "One Wave" Approximation

3.3.2.1 Symmetric Case

The dispersion relation for the symmetric case was obtained in section 3.1.3. and can be written in the form:

$$\begin{vmatrix} K_1 & K_2 \\ L_1 & L_2 \end{vmatrix} = 0$$

The dispersion relation can be reduced to $K_2 = 0$ in the limit of zero or infinite magnetic field, cutoff and large Z/Y . Because of the great similarity in the coefficients for this mode in the slab and rod cases, it is not surprising to find that the "one wave" approximation obtained by writing $K_2 = 0$ is still valid.

3.3.2.2 Dipole Mode

In the limit of infinite magnetic field and for cutoff and large Z/Y , E_1 becomes negligible or even vanishes. However, in the limit of zero magnetic field the amplitude of E_1 does not become equal one, so that we cannot consider only one wave to be present. For small values of the magnetic field, we therefore expect some discrepancies for the lowest order mode.

The approximate dispersion curves are shown together with the exact dispersion curves on Figures 63 to 66. Errors of not more than one percent are found for the body waves, except in the very low frequency regions ($Y < 0.1$). For the lowest order mode, however, it is seen that when Z is smaller than four, and $\alpha = 0.5$, errors of the order 10 percent may be found. This is not surprising since in this region E_1 has approximately the same amplitude as E_2 . As the intensity of the magnetic field increases, the "one wave" approximation becomes more and more valid.

4 CONCLUSION

A study of the trapped modes in magnetoplasma slabs and rods was presented. The full set of Maxwell's equations was used and an homogeneous distribution of electrons was assumed. We applied the results of the cold plasma theory and neglected the effect, due to ions and to collisions. Both of these effects become important only in the very low frequency region. Trivelpiece and Gould (1959) found that the effect of collisions on the dispersion curves is small when the collision frequency is small compared to the wave frequency. However, introducing collisions will allow for the damping of the waves as they propagate. This is not included in the present theory.

The exact dispersion relation using the cold plasma theory was obtained and solved showing the existence of body waves, surface waves and body wave-surface waves. By studying the different regions of propagation for these modes, it was found that the surface waves exist as forward waves in a very narrow frequency band even when a strong magnetic field is applied ($\Omega > \omega_p$). However, a careful study of the fields inside the plasma shows, fundamental differences for the lowest order mode when the electron cyclotron frequency is greater or smaller than the electron plasma frequency. It was shown that near resonance the waves were transverse magnetic and that as the wavelength increases, the waves become hybrid or even transverse electric as the wavelength goes to infinity. For the cutoff of the high frequency body waves, the waves become TEM modes at the slab boundary.

By studying the relative amplitude of the two waves propagating inside the plasma, we were able to show that the "one wave" approximation could give very accurate results, except in the very low frequency limit. Using analogies with the slab problem, we were able to simplify the dispersion relation for the plasma rod and express the field components relatively simply as a function of wavelength and frequency. The "one wave" approximation is valid in the circularly symmetric case and for the low frequency and high frequency body waves in the dipole mode. However, for the lowest order mode, in the long wave length limit, errors of 10 percent are found when the strength of the applied magnetic field is relatively small.

Several extensions of this investigation are suggested. First, because of the similarities which exist between the dielectric tensor for the plasma and the permeability tensor for ferrites, our analysis can be modified easily to study wave propagation in ferrites.

Second, the inclusion of ions and of electron thermal velocity will modify the behavior of the dispersion curves for both small K_z and large K_z . These modifications can be achieved by use of a fluid theory including a pressure term, or, if possible, by a kinetic theory analysis. Some theoretical investigations have already been performed along these lines (Diament, Granatstein and Schlesinger, 1966; Sarkar and Lonngren, 1967).

Third, it would be desirable to include the effect of non-uniformity of the electron density inside the slab and rod. Although some work has been done on this problem also (de Santis, 1965), only the quasistatic approximation was used.

Finally, the effect of plasma drifting and relative electron-ion drifts should be examined to explore the possibility of trapped mode instabilities.

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APPENDIX A

In this section we shall indicate how the identities (Equations 75 to 77) were obtained:

We rewrite Maxwell's equations in the following form:

$$- \frac{m}{x} H_{\zeta} + Z H_{\theta} = SY E_r - i DY E_{\theta}$$

$$- Z H_r - i \frac{\partial H_{\zeta}}{\partial x} = i DY E_r + SY E_{\theta}$$

$$\frac{1}{x} \frac{\partial}{\partial x} (x H_{\theta}) - \frac{i m H_r}{x} = - i Y T E_{\zeta}$$

$$- \frac{m E_{\zeta}}{x} + Z E_{\theta} = - Y H_r$$

$$Z E_r + i \frac{\partial E_{\zeta}}{\partial x} = Y H_{\theta}$$

$$\frac{1}{x} \frac{\partial}{\partial x} (x E_{\theta}) - \frac{i m E_r}{x} = i Y H_{\zeta}$$

When $m = 0$, the coefficients of the fields in these equations are found to be the same as the coefficients of the corresponding fields for the slab geometry (Equations 4 to 9). Therefore, for the $m = 0$ mode, we expect to obtain the coefficients of the Bessel functions and the coefficients of the sinh and cosh functions to be the same. We then verify by substitution and obtain the identities given in section 3.1.2. Noting that when m is different from zero the coefficient of I'_m in Equation 72 and the coefficient of I_m in Equation 71 have a common factor, we can also simplify the coefficients of I_m . In the same way we simplify all the coefficients of the modified

Bessel functions and of their derivatives; the fields can then be simplified to the expressions given by Equations 78 to 83.