APPLICATIONS OF HAMILTON-JACOBI THEORY

TO TRAJECTORY OPTIMIZATION

by

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INTRODUCTION AND SUMMARY

This study is concerned with some special methods of applying the Hamilton-Jacobi theory of calculus of variations to rocket trajectory optimization. The basic theory is presented in Chapter I and consists of necessary conditions of calculus of variations stated for the Mayer control problem, followed by the development of Hamilton-Jacobi theory for such problems. The procedure for utilizing the decomposition of the Hamiltonian into base and perturbing Hamiltonians is explained.

In the second chapter a simple Mayer control problem of the Zermelo navigational problem type is solved to illustrate the theory of Chapter I. Two methods of splitting the Hamiltonian are compared.

The main problem is treated in Chapter III and is that of minimizing the fuel of a low-thrust rocket rendezvousing a satellite in a circular orbit. The terms in the Hamiltonian involving thrust of the rocket engine are taken as the perturbing Hamiltonian. The partial differential equation for the base Hamiltonian is solved by a combination of separation of variables and Lagrange's method for linear equations. The procedure for completion of the problem is indicated but not carried out, the analysis becoming too complicated for a closed form solution. Transversality conditions are used to obtain enough conditions to determine the constants of integration. Some conclusions on the usefulness of the method are given in a final paragraph.

CHAPTER I

DEVELOPMENT OF HAMILTON JACOBI THEORY FOR MAYER CONTROL PROBLEMS

During recent years much interest has developed in trajectory analysis of space flight. Present day scientists working on space projects are faced with many problems which are extremely complex in nature. One of the major problems is the development of "optimal control theory". Generally the optimal control problems in trajectory analysis are analogous to the classical problems of Bolza and Mayer in calculus of variations. Extensions of the classical theory are made to include control variables.

CLASSICAL MAYER PROBLEM

The classical Mayer problem in calculus of variations is that of finding in a class of admissible arcs

$$y_i(t), \qquad t_o \leq t \leq t_1, \qquad i = 1,\ldots,n$$

satisfying differential equations

$$f_j(t,y,\dot{y}) = 0, \qquad j = 1,\ldots, m < n$$

and end conditions

$$J_k(t_1, y(t_1),t_o,y(t_o)) = 0, \qquad k = 1,\ldots, p \leq 2n+2$$

one which minimizes a function of the form

$$g(t_1, y(t_1), t_o, y(t_o)).$$

Here, in the arguments of the functions, we use y to denote the set $y_1, \ldots, y_n$ and $\dot{y}$ to denote the set $\dot{y}_1, \ldots, \dot{y}_n$. Similar notations will be used for other variables. The super dot indicates derivative with respect to t. Partial derivatives will usually be denoted by subscript variables and summation by the tensor analysis device of repeated indices.

MAYER PROBLEM INVOLVING CONTROL VARIABLES

The Mayer problem involving control variables as occur in trajectory problems may be expressed in the following form.

The problem is to find in a class of admissible arcs

$$y_i(t), \; u_j(t), \quad t_o \leq t \leq t_1, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m$$

satisfying differential equations

$$\dot{y}_i = f_i(t, y, u)$$

and end conditions

$$J_k(t_1, y(t_1), t_o, y(t_o)) = 0, \qquad k = 1, \ldots, p \leq 2n+2$$

one which will minimize a function of the form

$$g(t_1, y(t_1), t_o, y(t_o)).$$

In this study the admissible arcs will be those arcs whose elements (t, y) lie in a given n+1 dimensional open region R and whose control variables u lie in a region U. The derivatives $\dot{y}_i$ are unrestricted.

The end points $(t_1, y(t_1), t_o, y(t_o))$ of the admissible arcs lie in a 2n+2 dimensional open space S and y, $\dot{y}$, u are continuous functions of t. The given functions f_i, J_k, g are assumed to have continuous partial derivatives in their arguments to as high as second order.

FIRST NECESSARY CONDITION: MULTIPLIER RULE

The classical first necessary condition involving the Euler-Lagrange equations can be stated for the Mayer problem with control variables in the following form. (II)*

THEOREM 1. A normal admissible arc E is said to satisfy the multiplier rule if there exists a function

$$H(t,y,u,\lambda) = \lambda_i f_i, \qquad i = 1,2,\ldots,n$$

where λ's are functions of t not simultaneously zero and continuous along the arc E, such that the equations

$$(1) \qquad \dot{\lambda}_i = -H_{y_i}, \qquad \dot{y}_i = H_{\lambda_i}, \qquad H_{u_j} = 0, \qquad j = 1,\ldots,m$$

are satisfied, if the end point conditions $J_k = 0$, $k = 1,\ldots,p$ hold, and if the transversality matrix

$$\left\| \begin{matrix} H(t_o) + g_{t_o} & -H(t_1) + g_{t_1} & -\lambda_i(t_o) + g_{y_i}(t_o) & \lambda_i(t) + g_{y_i}(t_1) \\ J_{kt_o} & J_{kt_1} & J_{ky_i}(t_o) & J_{ky_i}(t_1) \end{matrix} \right\|$$

is of rank p. Every minimizing arc must satisfy the multiplier rule.

*Roman numerals in parentheses refer to the bibliography at the end of the thesis.

Solutions of equations (1) are called extremals, and equations (1) are called the canonical equations of extremals. They are the Euler-Lagrange equations for the problem, and the function H is analogous to the Hamiltonian of mechanics. For definition and discussion of normal admissible arcs see Bliss (I, p. 213). In trajectory problems the arcs are usually normal and will be assumed so here.

WEIERSTRASS CONDITION

The Weierstrass condition for the Mayer control problem can be stated as follows: (II)

THEOREM 2. Along the minimizing arc E, the inequality

$$H(t,y,\lambda,\overline{u}) \leq H(t,y,\lambda,u)$$

must hold at each element (t,y,λ,u) of E for every $\overline{u}$ in U. Thus $H(t,y,\lambda,u)$ is a maximum with respect to the control variables for a minimizing arc, for which reason this condition is often called the Maximum Principle.

ELIMINATION OF CONTROL VARIABLES

An arc along which the determinant $\left| H_{u_j u_h} \right| \neq 0$ is said to be non-singular. In the proofs to be considered it will be assumed that all arcs are non-singular. The equations $H_{u_j} = 0$ can then be solved for the control variables in terms of multipliers and state variables. The control variables can then be eliminated from the Hamiltonian. This will be supposed done, and the Hamiltonian will be written as

$$H^*(t,y,\lambda) = H\,(t,y,u(t,y,\lambda),\lambda).$$

It follows that the canonical equations of the extremals can be expressed in terms of H^*. For, if

$$H_{u_j} = 0, \qquad j = 1,\ldots,m$$

of the set of equations (1) can be solved for

$$u_j = u_j(t,y,\lambda),$$

then
$$H^*_{y_i} = H_{y_i} + H_{u_j} u_{jy_i},$$

$$H^*_{\lambda_i} = H_{\lambda_i} + H_{u_j} u_{j\lambda_i}.$$

Since $H_{u_j} = 0$, it follows that

$$H^*_{y_i} = H_{y_i} \quad \text{and} \quad H^*_{\lambda_i} = H_{\lambda_i}$$

or
$$\dot{y}_i = H_{\lambda_i} = H^*_{\lambda_i}$$

and
$$\dot{\lambda}_i = -H_{y_i} = -H^*_{y_i}.$$

Hereafter H^* will be denoted by H because of the equivalence of two Hamiltonians.

THE HAMILTON-JACOBI EQUATION

The Hamilton-Jacobi theory involves the formulation of the Hamilton-Jacobi equation, which is a partial differential equation of first order. The importance of the theory in calculus of variations is based on the relation between the Hamilton-Jacobi equation and the Euler-Lagrange equations, which is established by several theorems which follow.

The relation

$$S_t + H(t,y,S_y) = 0, \tag{2}$$

is called the Hamilton-Jacobi equation. The equation has dependent variable S and n+1 independent variables $t, y_1, \ldots, y_n$. The complete solution of (2) will have n+1 arbitrary constants. However, one is additive and is of no importance here, so we shall consider a solution with n independent constants, no one of which is additive, to be a complete solution.

THEOREM 3. Let the Hamilton Jacobi equation (2) have the solution $S = S(t, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_m)$ depending on m ($\leq n$) parameters $\alpha_1, \ldots, \alpha_m$. Then each derivative S_{α_j} is a first integral of the canonical Euler equations system

$$\dot{y}_i = H_{\lambda_i}, \qquad \dot{\lambda}_i = -H_{y_i};$$

that is, S_{α_j} = constant along an extremal.

JACOBI'S THEOREM

THEOREM 4. Let $S(t, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n)$ be a complete integral of the Hamilton Jacobi equation (2), that is, a solution depending on n-parameters $\alpha_1, \ldots, \alpha_n$ and having the n by n determinant $\left|S_{\alpha_i y_h}\right| \neq 0$. Also let $\beta_1, \ldots, \beta_n$ be n arbitrary constants. Then the functions

$$y_i = y_i(t, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n), \qquad i = 1, \ldots, n \tag{3}$$

defined by the relations $S_{\alpha_i} = \beta_i$, together with the functions $\lambda_i = S_{y_i}$, constitute a general solution of the canonical system

$$\dot{y}_i = H_{\lambda_i}, \qquad \dot{\lambda}_i = -H_{y_i}, \qquad i = 1,\ldots,n.$$

For proofs of theorems 3 and 4 see (III, p. 90).

HAMILTON JACOBI PERTURBATION THEORY

In celestial mechanics the path of a planet is disturbed by the presence of other heavenly bodies. This disturbing force is proportional to the mass of the disturbing body, which is very small compared to the mass of the sun. The Hamiltonian is expressed as a sum of two parts. The one which corresponds to the motion of the planet without the disturbing influence is called the base Hamiltonian, and the one corresponding to the disturbing factor is called the perturbing Hamiltonian.

The low thrust rocket problems in trajectory analysis can be treated on the basis of perturbation theory. The thrust of the engine is considered as the disturbing factor. This helps to avoid the complexity of the non-linear differential equations in the solution of the problem.

The following theorem shows how to obtain a complete integral of order n of the Hamilton-Jacobi equation for the base Hamiltonian in case it involves fewer than n λ's (VII, p. 29).

THEOREM 5. Let $H(t,y_1,\ldots,y_n,\lambda_1,\ldots,\lambda_n)$ be the Hamiltonian for a dynamical system. Let $H_o = H_o(t,y_1,\ldots,y_n,\lambda_1,\ldots,\lambda_k)$, where $k < n$, be the base Hamiltonian and let $S\ (t,y_1,\ldots,y_n,\alpha_1,\ldots,\alpha_k)$ be a solution of the Hamilton-Jacobi equation for H_o depending on k independent parameters $(\alpha_1,\ldots,\alpha_k)$ with $S_{y_i\alpha_j} \neq 0$, $i,j = 1,2,\ldots,k$. Then

$$S^* = S(t, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_k) + \sum_{i=k+1}^{n} \alpha_i y_i,$$

where $(\alpha_{k+1}, \ldots, \alpha_n)$ are independent parameters, is a complete solution of order n for the base Hamilton-Jacobi equation.

From Theorem 4 it follows that

$$(4) \qquad \beta_i = S^*_{\alpha_i}, \qquad \lambda_i = S^*_{y_i}, \qquad i = 1, \ldots, n.$$

We solve these equations for y's and λ's in terms of α's and β's, thus $y_i = y_i(\alpha, \beta, t)$ and $\lambda_i = \lambda_i(\alpha, \beta, t)$, and substitute these values in the perturbing Hamiltonian, say H_1. Now H_1 is expressed in α's and β's as variables.

On considering S to be a generating function for a canonical transformation with α's and β's as new variables, it follows that the new Hamiltonian is $S_t + H$, (III, p. 79). But

$$S_t + H = S_t + H_o + H_1, \text{ and } S_t + H_o = 0$$

when S is a complete integral of the Hamilton-Jacobi equation for the base solution. Hence the H_1 is the Hamiltonian for the total problem in terms of the variables α_i, β_i; and the canonical equations for extremals in these coordinates are

$$\dot{\alpha}_i = H_{1\beta_i}, \qquad \dot{\beta}_i = -H_{1\alpha_i}.$$

The solution of these equations gives the extremals for the problem with 2n constants of integration (VII, p. 27; VIII, p. 137). By the use of the set of equations (4) we can express the trajectory in terms of y's and t. This theory can be extended to splitting the Hamiltonian into more than two parts.

CHAPTER II

SOLUTION OF A SIMPLE MAYER CONTROL PROBLEM BY HAMILTON JACOBI THEORY

The main purpose of this chapter is to illustrate the solving of calculus of variations control problems by the use of the Hamilton Jacobi theory developed in the previous chapter. The simple problem treated here is to some extent analogous to the major problem of Chapter III. Two methods of developing the solution are given. This problem may be viewed as a particular case of Zermelo's problem for navigating ships. The two dimensional rectangular system of coordinates is used.

STATEMENT OF THE PROBLEM

Consider a body moving on a level fluid surface which has velocity components px and qy along the cartesian directions, the body having velocity relative to the fluid of constant magnitude V generated by the thrust of the engine. It is required to minimize the time in going from one given point to another.

The equations of motion are

$$\dot{x} = px + V \cos \alpha,$$

$$\dot{y} = qy + V \sin \alpha,$$

where V, p and q are constants and the control variable α is the angle between the x-axis and the direction of the thrust.

THE HAMILTONIAN

The Hamiltonian for this system of equations is

$$H = \lambda_1(px + V\cos\alpha) + \lambda_2(qy + V\sin\alpha),$$

where λ_1 and λ_2 are multipliers and are functions of t, not simultaneously zero.

By theorem 2 of Chapter I, for final time to be minimum, H must be maximum. Hence

$$H_\alpha = 0 \quad \text{and} \quad H_{\alpha\alpha} \leq 0,$$

therefore
$$H_\alpha = -\lambda_1 V\sin\alpha + \lambda_2 V\cos\alpha = 0,$$

or $\tan\alpha = \dfrac{\lambda_2}{\lambda_1}$ and $\sin\alpha = \dfrac{\lambda_2}{\pm\sqrt{(\lambda_1^2 + \lambda_2^2)}}$, $\cos\alpha = \dfrac{\lambda_1}{\pm\sqrt{(\lambda_1^2 + \lambda_2^2)}}$.

Also
$$H_{\alpha\alpha} = -\lambda_1 V\cos\alpha - \lambda_2 V\sin\alpha \leq 0,$$

and inequality holds if and only if the positive sign is considered before the radical in the values of $\sin\alpha$ and $\cos\alpha$.

When α is eliminated, the Hamiltonian assumes the following form

$$H = \lambda_1 px + \lambda_2 qy + V\sqrt{(\lambda_1^2 + \lambda_2^2)},$$

$$= H_1 + H_0$$

where $H_0 = V\sqrt{(\lambda_1^2 + \lambda_2^2)}$ and $H_1 = \lambda_1 px + \lambda_2 qy$.

SOLUTION OF BASE HAMILTONIAN

Taking H_o as base Hamiltonian, the Hamilton-Jacobi equation for H_o is

$$S_t + V\sqrt{(S_x^2 + S_y^2)} = 0.$$

A method of separation of variables can be used. (VI, Chapter XII) Assume that the solution of the above equation is

$$S = S_1(t) + S_2(x) + S_3(y),$$

hence $$dS_1/dt + V\sqrt{((dS_2/dx)^2 + (dS_3/dy)^2)} = 0.$$

Let $dS_2/dx = \alpha_1$ and $dS_3/dy = \alpha_2$, where α_1 and α_2 are parametric constants. This is possible since x and y are not involved in the equation explicitly.

Now $$dS_1/dt = -V\sqrt{(\alpha_1^2 + \alpha_2^2)},$$

and hence $$S_1 = -Vt\sqrt{(\alpha_1^2 + \alpha_2^2)}.$$

Therefore $$S = \alpha_1 x + \alpha_2 y - Vt\sqrt{(\alpha_1^2 + \alpha_2^2)}.$$

By Jacobi's theorem (Theorem 4, Chapter I) the arbitrary β's and multipliers λ's are given by $\beta_i = S_{\alpha_i}$ and $\lambda_i = S_{y_i}$.

Therefore

(1)

$$\beta_1 = x - V\alpha_1 t/\sqrt{(\alpha_1^2 + \alpha_2^2)},$$

$$\beta_2 = y - V\alpha_2 t/\sqrt{(\alpha_1^2 + \alpha_2^2)},$$

$$\lambda_1 = \alpha_1, \qquad \lambda_2 = \alpha_2.$$

SOLUTION OF PERTURBING HAMILTONIAN

Using the above set of equations, we can eliminate x, y, λ_1 and λ_2 and express H_1 in new variables:

$$H_1 = p\alpha_1\beta_1 + q\alpha_2\beta_2 + Vt\ (p\alpha_1^2 + q\alpha_2^2)/\sqrt{(\alpha_1^2 + \alpha_2^2)}.$$

The canonical equations for H_1 are

$$\dot{\beta}_1 = p\beta_1 + Vt\ (p\alpha_1^3 + (2p - q)\ \alpha_1\alpha_2^2)/(\alpha_1^2 + \alpha_2^2)^{3/2},$$

$$\dot{\beta}_2 = q\beta_2 + Vt\ ((2q - p)\ \alpha_1^2\ \alpha_2 + q\alpha_2^3)/(\alpha_1^2 + \alpha_2^2)^{3/2},$$

$$\dot{\alpha}_1 = -p\alpha_1,$$

$$\dot{\alpha}_2 = -q\alpha_2.$$

The last two equations of the above set give

$$(2) \qquad \alpha_1 = c_1 e^{-pt}, \qquad\qquad \alpha_2 = c_2 e^{-qt},$$

where c_1 and c_2 are constants of integration, and the first two equations reduce to

$$\dot{\beta}_1 - p\beta_1 = V\ \frac{c_1 e^{-pt}(pc_1^2\ e^{-2pt} + (2p - q)c_2^2 e^{-2qt})}{(c_1^2\ e^{-2pt} + c_2^2\ e^{-2qt})^{3/2}}\ t = F_1(t),$$

$$\dot{\beta}_2 - q\beta_2 = V\ \frac{c_2 e^{-qt}((2q - p)c_1^2 e^{-2pt} + qc_2^2 e^{-2qt})}{(c_1^2\ e^{-2pt} + c_2^2\ e^{-2qt})^{3/2}}\ t = F_2(t),$$

which are linear differential equations and have the solutions

$$\beta_1 = e^{pt}\left(\int \bar{e}^{pt} F_1(t)\, dt + c_3\right),$$

$$\beta_2 = e^{qt}\left(\int \bar{e}^{qt} F_2(t)\, dt + c_4\right).$$

By the use of the equations in set (1) and (2) we can express

(3) $$x = e^{pt}\left(\int \bar{e}^{pt} F_1(t)\, dt + c_3\right) + V\, c_1 \bar{e}^{pt} t / \sqrt{(c_1^2\, \bar{e}^{2pt} + c_2^2\, \bar{e}^{2qt})}$$

$$y = e^{qt}\left(\int \bar{e}^{qt} F_2(t)\, dt + c_4\right) + V\, c_2 \bar{e}^{qt} t / \sqrt{(c_1^2\, \bar{e}^{2pt} + c_2^2\, \bar{e}^{2pt})},$$

which are the parametric equations of the trajectory. Also from (1) and (2) we can evaluate the multipliers in terms of t

(4) $$\lambda_1 = c_1\, \bar{e}^{pt}, \qquad \lambda_2 = c_2\, \bar{e}^{qt}.$$

The end point values of x and y can be used to evaluate the constants of integration in the solution of the problem. If the initial and final points of the trajectory nove on some curves or surfaces, then the transversality conditions can be used for evaluation of constants of integration.

A PARTICULAR CASE OF THE PROBLEM

When the velocity of the fluid is proportional to the distance from the origin, then $p = q$.

In this case we get the multipliers from equations (4) as

$$\lambda_1 = c_1\, \bar{e}^{pt}, \qquad \lambda_2 = c_2\, \bar{e}^{pt},$$

and the equations of the trajectory are deduced from the equations (3) to be

$$x = c_3\, e^{pt} - V\, c_1 / p\sqrt{(c_1^2 + c_2^2)},$$

$$y = c_4\, e^{pt} - V\, c_2 / p\sqrt{(c_1^2 + c_1^2)}.$$

It is easily verified that the above equations satisfy the Euler-Lagrange equations obtained from the Hamiltonian in its original form. Assuming that an optimum trajectory exists, it is given by these equations with the constants determined by the initial and end conditions. Elimination of t gives the geometrical trajectory as the straight line

$$c_4 x - c_3 y = V\,(c_2 c_3 - c_1 c_4) / \sqrt{(c_1^2 + c_2^2)}.$$

A SECOND METHOD OF SOLUTION

The second method of solving the problem (similar to that used in Chapter III) involves expressing the Hamiltonian H as $H_0 + H_1$ where

$$H_0 = \lambda_1 px + \lambda_2 qy \quad \text{and} \quad H_1 = V\sqrt{(\lambda_1^2 + \lambda_2^2)}.$$

Taking H_0 as base Hamiltonian, the Hamilton-Jacobi equation is

$$S_t + px\, S_x + qy\, S_y = 0.$$

Let us assume that the solution of the above equation is

$$S = S_1(t) + S_2(x,y),$$

hence

$$dS_1/dt + px\,\partial S_2/\partial x + qy\,\partial S_2/\partial y = 0.$$

Since t is not involved explicitly, we can put $dS_1/dt = -\alpha_1$ in the equation; hence

(5) $$px\, \partial S_2/\partial x + qy\, \partial S_2/\partial y = \alpha_1 .$$

Its subsidiary equations are (VI, Chapter XII)

$$\frac{dx}{px} = \frac{dy}{qy} = \frac{dS_2}{\alpha_1} .$$

Considering the first two terms of the subsidiary equations and integrating them, we get

$$\frac{x^q}{y^p} = a,$$

where a is an arbitrary constant.

The last two terms in the subsidiary equations give

$$S_2 - \frac{\alpha_1}{q} \log y = b,$$

where b is an arbitrary constant.

The general solution of equation (5) is

$$\varphi \left(S_2 - \frac{\alpha_1}{q} \log y, \frac{x^q}{y^p}\right) = 0,$$

where φ is an arbitrary function.

One such solution is

$$\left(S_2 - \frac{\alpha_1}{q} \log y\right) - \alpha_2 \frac{x^q}{y^p} = 0,$$

where α_2 is a parametric constant, which gives

$$S_2 = \frac{\alpha_1}{q} \log y + \alpha_2 \frac{x^q}{y^p} .$$

Hence a complete integral of equation (5) is

$$S = -\alpha_1 t + \frac{\alpha_1}{q} \log y + \alpha_2 \frac{x^q}{y^p} .$$

By Jacobi's theorem, arbitrary β's and multipliers λ's are evaluated as follows

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{1}{q} \log y - t, \qquad \text{also} \quad y = e^{(\beta_1 + t)q};$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \frac{x^q}{y^p} , \qquad \text{also} \quad x = \beta_2^{1/q} e^{(\beta_1 + t)p};$$

$$\lambda_1 = \frac{\partial S}{\partial y} = q\alpha_2 x^{q-1} y^{-p} \qquad \text{or} \quad \lambda_1 = q\alpha_2 \beta_2^{1-1/q} e^{-(\beta_1 + t)p};$$

$$\lambda_2 = \frac{\partial S}{\partial y} = \frac{\alpha_1}{q} y^{-1} - p\alpha_2 x^q y^{-(p + 1)}$$

$$\text{or} \qquad \lambda_2 = \left(\frac{\alpha_1}{q} - p\alpha_2 \beta_2 \right) e^{-(\beta_1 + t)q}.$$

SOLUTION OF THE PERTURBING HAMILTONIAN

The perturbing Hamiltonian H_1 can be expressed in series by using the binomial expansion. We consider only the first two terms, assuming λ_1 to be large relative to λ_2. Hence,

$$H_1 = V (\lambda_1 + \lambda_2^2 /2\lambda_1).$$

Eliminating λ_1 and λ_2 and getting H_1 in terms of new variables α's and β's give

$$H_1 = VA\ (q\alpha_2 B + 2^{-1}q^{-3}\alpha_2^{-1}B^{-1}\ (\alpha_1 - qp\alpha_2\beta_2)^2),$$

where $$A = e^{-(\beta_1 + t)p} \qquad \text{and} \qquad B = \beta_2^{1-1/q}.$$

The canonical equations are

$$\dot{\alpha}_1 = -pVA\ (q\alpha_2 B + 2^{-1}q^{-3}\alpha_2^{-1}B^{-1}\ (\alpha_1 - pq\alpha_2\beta_2)^2),$$

$$\dot{\alpha}_2 = VA\ (q\alpha_2 B\ \beta_2^{-1} + 2^{-1}q^{-3}\alpha_2^{-1}B^{-1}\ \beta^{-1}\ (\alpha_1 - pq\alpha_2\beta_2)$$
$$+ q^{-2}pB^{-1}\ (\alpha_1 - pq\alpha_2\beta_2)\ ),$$

$$\dot{\beta}_1 = -Vq^{-3}\alpha_2^{-1}B^{-1}\ (\alpha_1 - qp\alpha_2\beta_2)A,$$

$$\dot{\beta}_2 = -VA\ (qB - q^{-2}pB^{-1}\ \beta\alpha_2^{-1}\ (\alpha_1 - qp\alpha_2\beta_2)$$
$$- 2^{-1}q^{-3}\alpha_2^{-2}B^{-1}\ (\alpha_1 - qp\alpha_2\beta_2)^2).$$

The above four differential equations determine the optimum trajectory in terms of the new variables α_i, β_i and four constants of integration, which would be evaluated by use of the end point conditions. The completion of the problem by this method is much more complicated than by the first method, so we do not carry the work further here. This problem illustrates that the choice of method of splitting the Hamiltonian is important. Other considerations, such as the magnitude of the thrust may also influence the choice of how to split the Hamiltonian.

The problem of the next chapter, like the one in this chapter, has a Hamiltonian consisting of rational terms plus a term involving the square root of a quadratic in λ's. The thrust occurs only in the radical part. Since our rendezvous problem involves low thrust, it is desirable to choose this part as the perturbing Hamiltonian. Hence the procedure will be similar to that of the second method in this chapter.

CHAPTER III

RENDEZVOUS OF A ROCKET TO A SATELLITE MOVING IN CIRCULAR ORBIT

A rocket moving under the law of gravity and the thrust of the engine is to rendezvous with a satellite moving in a circular orbit about the earth. The variable angle of the thrust, called the control variable, is a function of time. It is desired to find the equations of the path of the rocket that will require the least amount of fuel. Investigation of this problem was suggested by William E. Miner who has developed extensive applications of perturbation theory to rocket trajectory problems, (V).

The method used here is that of Hamilton-Jacobi theory for perturbing planetary motions. The Hamiltonian is formulated by multipliers and the first order differential equiations of motion of the rocket.

ASSUMPTIONS

The following assumptions are made involving the physical conditions of the problem.

The path of the rocket is assumed to be in a plane, and hence a two dimensional polar coordinate system is used, with origin as the center of the earth.

The rocket is considered as a particle of variable mass.

Air resistance is assumed to be negligible.

Earth is assumed to be a perfect sphere and not rotating with respect to the coordinate system.

Thrust magnitude is considered proportional to the rate of flow of mass and is assumed to be constant.

Change in thrust direction is assumed to be instantaneous.

The circular path of the satellite is coplanar to the path of the rocket.

NOTATIONS

t	Independent time variable
t_o	Initial time
t_f	Final time
r	Variable radius vector
θ	Variable angle made by radius vector with the initial line of the polar coordinates system used
u	Velocity along the radius vector
v	Velocity along the perpendicular to the radius vector
m	Mass of the rocket
c	Rate of flow of mass
T	Constant thrust magnitude of the engine
α	Variable control angle made by the thrust with the radius vector
λ_i	Lagrange multipliers in Hamiltonian
k	Gravitational constant
R_1	Radius of the earth

R_2 Radius of the circular orbit of the satellite with center at the origin

ω Uniform angular velocity of the satellite

θ_o Angular position of the satellite at time t_o

EQUATIONS OF MOTION

From figure 2, the equations of motion can be expressed as (IV, Chapter IV)

$$\ddot{r} - r\dot{\theta}^2 = -k/r^2 + (T/m)\cos\alpha,$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = (T/m)\sin\alpha,$$

$$\dot{m} = -c.$$

The radial and tangential velocities u and v are given by

$$u = \dot{r}$$

$$v = r\dot{\theta}\ .$$

Hence we can express the above equations of motion as first order differential equations in the following form,

(1)
$$\dot{u} = v^2/r - k/r^2 + (T/m)\cos\alpha,$$

$$\dot{v} = -uv/r + (T/m)\sin\alpha,$$

$$\dot{r} = u,$$

$$\dot{\theta} = v/r,$$

$$\dot{m} = -c.$$

GEOMETRY OF THE PROBLEM

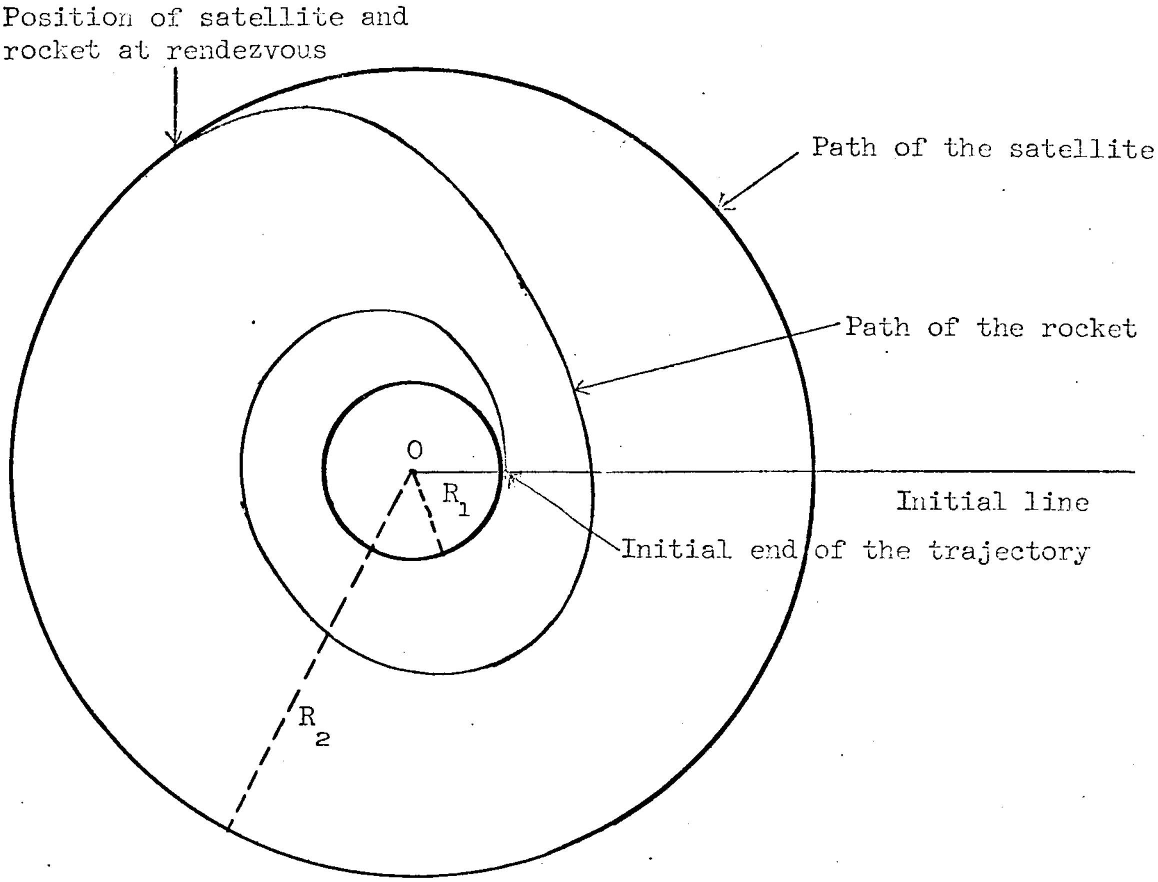

Fig. 1

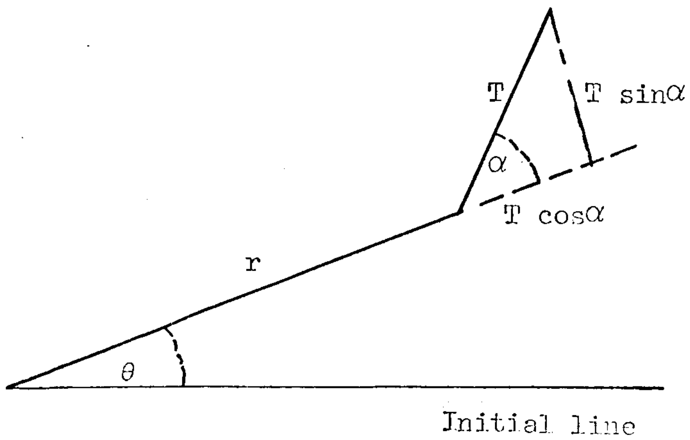

Fig. 2

The initial and final end conditions of the trajectory can be expressed in the following form

(2) $J_1 = t_o = 0,$

$J_2 = u(t_o) = 0,$

$J_3 = v(t_o) = 0,$

$J_4 = r(t_o) - R_1 = 0,$

$J_5 = \theta(t_o) = 0,$

$J_6 = u(t_f) = 0,$

$J_7 = v(t_f) - R_2\omega = 0,$

$J_8 = r(t_f) - R_2 = 0,$

$J_9 = \theta(t_f) - \omega t_f - \theta_o = 0$

$J_{10} = m(t_f) - m_f = 0,$

where R_1, R_2 and m_f are constants.

Minimizing the mass of the rocket at the time t_o is equivalent to minimizing the consumption of the fuel and also equivalent to the minimization of the time of the transit of the rocket. This can be expressed as

$$J_o = m(t_o).$$

FIRST TRANSFORMATION OF THE VARIABLES

The first set of transformed coordinates are obtained from the Lagrangian L, for the unit mass and the two body problem without thrust. Thus

$$L = (1/2)(\dot{r}^2 + r^2\dot{\theta}^2) + k/r$$

and we define

$$u = \partial L/\partial \dot{r} \qquad \text{and} \qquad w = \partial L/\partial \dot{\theta}$$

Hence, $$u = \dot{r} \qquad \text{and} \qquad w = r^2\dot{\theta} = rv.$$

The equations of motion in transformed variables are

(3) $$\dot{u} = ((w^2/r^3) - k/r^2) + (T/m)\cos\alpha,$$

$$\dot{w} = r(T/m)\sin\alpha,$$

$$\dot{r} = u,$$

$$\dot{\theta} = w/r^2,$$

$$\dot{m} = -c.$$

ELIMINATION OF CONTROL VARIABLE BY WEIERSTRASS CONDITION

The Hamiltonian for set of equations (3) is

$$H = \lambda_1\,(w^2/r^3 - k/r^2 + (T/m)\cos\alpha) + \lambda_2\,(r(T/m)\sin\alpha) + \lambda_3 u + \lambda_4 w/r^2 - c\lambda_5$$

where λ_1, λ_2, λ_3, λ_4 and λ_5 are multipliers and are functions of t, not simultaneously zero.

From the Weierstrass condition for maximum H,

$$H_\alpha = 0 \qquad \text{and} \qquad H_{\alpha\alpha} \le 0,$$

hence $$H_\alpha = -\lambda_1(T/m)\sin\alpha + \lambda_2 r(T/m)\cos\alpha = 0,$$

or $\tan\alpha = r\lambda_2/\lambda_1$ and $\sin\alpha = r\lambda_2/\pm\sqrt{(\lambda_1^2 + r^2\lambda_2^2)}$, $\cos\alpha = \lambda_1/\pm\sqrt{(\lambda_1^2 + r^2\lambda_2^2)}$.

Also $$H_{\alpha\alpha} = -\lambda_1(T/m)\cos\alpha - \lambda_2 r(T/m)\sin\alpha \le 0$$

and this holds if and only if the positive sign is considered before the radical in the values of $\sin\alpha$ and $\cos\alpha$.

Elimination of α gives

$$H = \lambda_1(w^2/r^3 - k/r^2) + \lambda_3 u + \lambda_4 w/r^2 - c\lambda_5 + (T/m)\sqrt{(\lambda_1^2 + r^2\lambda_2^2)},$$

$$= H_0 + H_1,$$

where $H_0 = \lambda_1(w^2/r^3 - k/r^2) + \lambda_3 u + \lambda_4 w/r^2 - c\lambda_5$ and $H_1 = (T/m)\sqrt{(\lambda_1^2 + r^2\lambda_2^2)}$.

Considering H_0 as the base Hamiltonian, the Hamilton-Jacobi equation for H_0 is

(4) $$\partial S/\partial t + (w^2/r^3 - k/r^2)\partial S/\partial u + u\,\partial S/\partial r + (w/r^2)\partial S/\partial\theta - c\,\partial S/\partial m = 0.$$

Let its solution be

$$S = S_1(t) + S_2(\theta) + S_3(m) + S_4(u,r);$$

then the Hamilton-Jacobi equation assumes the form

$$dS_1/dt + (w^2/r^3 - k/r^2)\partial S_4/\partial u + u\,\partial S_4/\partial r - (w/r^2)dS_2/d\theta - c\,dS_3/dm = 0,$$

which does not involve t, θ and m explicitly.

Hence $\quad dS_1/dt = \alpha_1, \quad dS_2/d\theta = \alpha_2, \quad dS_3/dm = \alpha_3,$

where $\alpha_1, \alpha_2, \alpha_3$ are parametric constants.

The Hamilton-Jacobi equation can now be written as

$$(5) \qquad (w^2/r^3 - k/r^2)\,\partial S_4/\partial u + u\,\partial S_4/\partial r = c\,\alpha_3 - \alpha_1 - \alpha_2 w/r^2,$$

which is in the form of Lagrange's linear equation (VI, Chapter XII), and its subsidiary equations are

$$(6) \qquad \frac{du}{w^2/r^3 - k/r^2} = \frac{dr}{u} = \frac{dS_4}{c\alpha_3 - \alpha_1 - \alpha_2 w/r^2}\,.$$

Considering the first subsidiary equation, we have

$$u^2 - 2k/r + w^2/r^2 = -a^2,$$

which we write as

$$f = -a^2,$$

where $-a^2$ is a constant of integration with the sign chosen so as to give a periodic shape to the trajectory.

On substituting for u from above in the last subsidiary equation, we have

$$dr/\sqrt{(-a^2r^2 + 2kr - w^2)} = r\,dS_4/(c\alpha_3 - \alpha_1)r^2 - \alpha_2 w,$$

the ambiguous sign of the radical being absorbed in the arbitrary constants $\alpha_1, \alpha_2, \alpha_3$.

On integrating, we have

$$S_4 - ((c\alpha_3 - \alpha_1)\,(-\sqrt{(-a^2r^2 + 2kr - w^2)}/a^2 + (k/a^3)\sin^{-1}(a^2r-k)/\sqrt{(k^2-w^2a^2)})$$

$$- \alpha_2 \sin^{-1}(kr - w^2)/r\sqrt{(k^2 - w^2a^2)} = b,$$

where b is a constant of integration;

$$\text{or } S_4 - (\frac{c\alpha_3 - \alpha_1}{-w^2+2kr-u^2r^2}\,(-ur + \frac{kr}{\sqrt{(-w^2+2kr-u^2r^2)}} \sin^{-1}\frac{-u^2r^2+kr-w^2}{\sqrt{(kr-w^2)^2+u^2r^2w^2)}})$$

$$- \alpha_2 \sin^{-1}\frac{kr - w^2}{\sqrt{((kr-w^2)^2 + u^2r^2w^2)}}\,) = b.$$

If we define

$$X = -w^2 + 2kr - u^2r^2,$$

$$Y = \sqrt{((kr - w^2)^2 + u^2r^2w^2)},$$

we can simplify the above equation to

$$S_4 - (((c\alpha_3 - \alpha_1)/X)(-ur + (kr/\sqrt{X})\sin^{-1} X/Y) - \alpha_2 \sin^{-1}(kr-w^2)/Y) = b,$$

$$\text{or } S_4 - g = b.$$

The general solution of (6) will be

$$\varphi\,(f,\ S_4 - g) = 0,$$

where φ is an arbitrary function.

It follows that

$$S_4 = g + \alpha_4 f + \alpha_6,$$

where α_4 and α_6 are parametric constants, is a solution and may be taken as a complete integral of equation (5).

Hence an integral of Hamilton Jacobi equation (4) is

$$S = \alpha_1 t + \alpha_2 \theta + \alpha_3 m + (((c\alpha_3 - \alpha_1)/X)\ (-ur + (kr/\sqrt{X})\ \sin^{-1} X/Y)$$

$$- \alpha_2 \sin^{-1} (kr - w^2)/Y) - \alpha_4 X/r^2,$$

the additive constant α_6 being dropped as explained on page 7. The complete integral of Hamilton-Jacobi equation (4) by Theorem 5 Chapter I, is

$$S^* = S + \alpha_5 w.$$

By Jacobi's theorem we have $\partial S^*/\partial\alpha_i = \beta_i$ with arbitrary β_i and hence we get

(7) $$\beta_1 = t - (1/X)\ (-ur + (kr/\sqrt{X})\ \sin^{-1}(X - kr)/Y),$$

$$\beta_2 = \theta - \sin^{-1} (kr - w^2)/Y,$$

$$\beta_3 = m + (c/X)(-ur + (kr/\sqrt{X})\ \sin^{-1} (X - kr)/Y),$$

$$\beta_4 = -X/r^2,$$

$$\beta_5 = w.$$

Also by Jacobi's theorem, the multipliers λ_1, λ_2, λ_3, λ_4, λ_5 are the partial derivatives of S^* with respect to u, w, r, θ, m, respectively. On letting Z denote $k^2 + \beta_4 \beta_5^2$, we find the results, after some simplification, to be as follows:

$$\begin{aligned}(8)\quad \lambda_1 &= (2/\beta_4)(c\alpha_3 - \alpha_1)(\beta_1 - t)u + ((c\alpha_3 - \alpha_1)/\beta_4^2 r^2)(u(\beta_4(\beta_1 - t)r^2 + ur) \\ &\quad + \beta_4 r) - (k/r\beta_4)(rk^2 - 2\beta_5^2\beta_4 - k\beta_5^2)/Z + \alpha_2\beta_5(kr - \beta_5^2)/rZ + 2\alpha_4 u,\end{aligned}$$

$$\begin{aligned}\lambda_2 &= 2\beta_5(c\alpha_3 - \alpha_1)(\beta_1 - t)/r^2\beta_4 + (c\alpha_3 - \alpha_1)\beta_5(r\beta_4(\beta_1 - t) + u)/\beta_4^2 r^3 \\ &\quad - k(2\beta_5 r^2 Z + (-u^2 + kr - \beta_5^2)(r^2\beta_4 + \beta_5^2)\beta_5)/ur^2\sqrt{(\beta_4^2 r^2 - \beta_4(r^2 - 1)(u^2 + \beta_5^2)Z} \\ &\quad + \alpha_2(\beta_5 Z r^2 + (kr - \beta_5^2)(\beta_4 r^2 + \beta_5^2)\beta_5)/ur^3\beta_5 Z + 2\alpha_4\beta_5/r^2 + \alpha_5,\end{aligned}$$

$$\lambda_4 = \alpha_2,$$

$$\lambda_5 = \alpha_3,$$

λ_3 can be computed the same way as other λ's, but we do not need it for our H_1.

From these sets of equations (7) and (8), as explained in first chapter, we need to solve for λ_1, λ_2 and r in terms of β's, α's and t. We then substitute these values in the perturbing Hamiltonian $H_1 = \sqrt{(\lambda_1^2 + r^2\lambda_2^2)}$, to obtain an expression in α's, β's and t, say $H_1(\alpha, \beta, t)$. The canonical equations for this Hamiltonian H_1 are

$$\dot{\beta}_i = -\partial H_1/\partial\alpha_i \qquad \dot{\alpha}_i = \partial H_1/\partial\beta_i .$$

The solution of these equations gives the final trajectory of the rocket in terms of α's and β's and ten constants of integration. It would be difficult, if not impossible, to get the solution in closed form. We conclude our study of the problem without attempting approximation procedures for its completion.

The constants of integration are determined by the initial and the final end conditions of the trajectory and the transversality conditions at the final point. By Theorem 1 of Chapter I, the transversality matrix in terms of the original variables is

$$
\left\|\begin{array}{cccccc}
H(t_o) & -H(t_f) & -\lambda_1(t_o) & -\lambda_2(t_o) & -\lambda_3(t_o) & -\lambda_4(t_o) \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
-\lambda_5(t_o)+1 & \lambda_1(t_f) & \lambda_2(t_f) & \lambda_3(t_f) & \lambda_4(t_f) & \lambda_5(t_f) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right\|
$$

This matrix has eleven rows and twelve columns and must be of rank ten.

This gives the following conditions,

$$\lambda_5(t_o) = 1,$$

$$H(t_f) = \omega\lambda_4(t_f).$$

The ten initial and final end conditions and the above two conditions are enough to determine the ten constants of integration and the initial and final times.

CONCLUSIONS

Although the method given here for obtaining a complete integral of the base Hamilton-Jacobi equation for the rocket trajectory problem is simple, involving only a linear partial differential equation, it leads to complicated involvement of u and r in equations (7) and (8). This makes it very difficult to obtain the Hamiltonian H_1 as a function of α_i, β_i, and t. Also it would seem difficult to devise a computer programming method that would be any improvement over other known methods. Further study of canonical transformation theory would seem desirable in an effort to obtain a coordinate system in which H_1 would be simpler.

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