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OPTIMUM RELAXATION TIME FOR A MAXWELL CORE DURING FORCED VIBRATION OF A ROCKET ASSEMBLY

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When the core of a case-bonded viscoelastic assembly is made of a Maxwell solid, an optimum relaxation time is found which minimizes the displacement amplitude and the bond stress response at resonance. For a Voigt solid the displacement amplitude and the bond stress response at resonance decreases with retardation time, but no optimum time exists in the same sense.

In a previous paper concerned with the forced vibration response of a case-bonded viscoelastic cylinder[1]*, the authors presented numerical results which indicated the existence of an optimum relaxation time τ for a Maxwell solid. At the optimum relaxation time τ the bond stress amplitude response is minimized. It was observed on the basis of some numerical calculations that the optimum relaxation time τ decreases with increasing values of the resonant frequency ω.

It is the purpose of this brief note to prove the existence of an optimum relaxation time τ for an assembly consisting of a solid cylinder bonded to a thin casing, if the cylinder is made of a material which is a Maxwell solid. It is also demonstrated that a Voigt solid has no optimum retardation time.

The present analysis starts with the law of conservation of energy for the system. Neglecting thermodynamic effects the law of conservation of energy may be written

$$P_{ext} = \frac{d}{dt}(KE) + \int_V \sigma_{ij}\dot{\epsilon}_{ij}\, dV \tag{1}$$

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*Superscript numbers in squared brackets designate references listed in the bibliography.

where σ_{ij} are the components of stress and $\dot{\epsilon}_{ij}$ are the components of strain rate; t is the parameter time. The term P_{ext} is the rate of work of the external forces, and KE is the kinetic energy of the system. The integral represents the rate of work done by the internal stresses during deformation.

There is no change in KE over one cycle of sinusoidal vibration. Hence the integral of Equation (1) from $t = 0$ to $t = 2\pi/\omega$ may be written

$$\int_0^{2\pi/\omega} P_{ext}\, dt = \int_0^{2\pi/\omega} \int_V \sigma_{ij}\, \dot{\epsilon}_{ij}\, dV\, dt \tag{2}$$

In the particular problem under consideration, which is the same as that previously studied[1], the rate of work of the external force may be written

$$P_{ext} = \int_{-\pi}^{+\pi} -pa \frac{\partial}{\partial t} u(a,\theta,t)\, d\theta \tag{3}$$

where p is the normal surface traction applied to the outer surface of the casing, u is the radial displacement of particles under load; a is the radius of the common surface between the cylinder and its casing.

It is convenient to separate the stress and strain-rate components into deviatoric components S_{ij}, e_{ij} and its mean normal components σ, ϵ, respectively. If this is done the product $\sigma_{ij}\dot{\epsilon}_{ij}$ can be written

$$\sigma_{ij}\, \dot{\epsilon}_{ij} = S_{ij}\dot{e}_{ij} + 3\sigma\dot{\epsilon} \tag{4}$$

Should the assumption again be made[1] that the cylinder is elastic in dilatation

$$\sigma = K\epsilon \tag{5}$$

where K is the bulk modulus of elasticity. The integral on the right side of Equation (2) then becomes

$$\int_0^{2\pi/\omega} \int_V \sigma_{ij} \dot{\epsilon}_{ij} \, dV \, dt = \int_0^{2\pi/\omega} \int_V S_{ij} \dot{e}_{ij} \, dV \, dt \tag{6}$$

Equation (6) shows that all vibratory energy dissipation is due to distortion and none to volume change.

Let us write the pressure load $p(\theta,t)$ as a phasor

$$p = \mathrm{Re}\,(P_o e^{j\omega t}) \tag{7}$$

where P_o is the complex amplitude and ω the real frequency. Then the radial displacement $u(r,\theta,t)$ can be expressed

$$u = \mathrm{Re}\left(\frac{U_o}{P_o}(j\omega, r)\, P_o\, e^{j\omega t}\right) \tag{8}$$

where the complex displacement transfer function $U_o(j\omega,r)/P_o$ is given explicitly by Equation (37a) of Reference [1], namely

$$\frac{U_o}{P_o} = \frac{a\, J_1(\alpha r)}{\left[\rho a \omega^2 + 2G - \frac{hE}{a(1-\nu^2)}\right] J_1(z) - \left(K + \frac{4}{3} G\right) z\, J_o(z)} \tag{9}$$

where E, ν are respectively Young's modulus and Poisson's ratio for the casing material ρ is the mass density of the casing, per unit area of the middle surface and h is the wall thickness of the casing; G is the shear modulus of the core. The terms $J_n(x)$ is Bessel's function of the order n and

$$\alpha = \omega \sqrt{\gamma/(K + \tfrac{4}{3} G)} \tag{10}$$

$$z = \alpha a$$

Substituting Equations (7), (8) into Equation (3) one finds that

$$P_{ext} = -2\pi a \, Re \, (P_o e^{j\omega t}) \, Re \left(j\omega \frac{U_o}{P_o} (j\omega, a) \, P_o e^{j\omega t} \right) \tag{11}$$

Then as a consequence of the lemma

$$\int_0^{2\pi/\omega} Re(c e^{j\omega t}) \, Re \, (d e^{j\omega t}) dt = \frac{\pi}{\omega} \, Re \, (c\bar{d}) \tag{12}$$

it is found by substituting Equation (9) into the left side of Equation (2), that the expression for the work of the external forces in one cycle of vibration becomes

$$\int_0^{2\pi/\omega} P_{ext} \, dt = 2\pi^2 a \left| P_o \right|^2 I_m \left[\frac{U_o}{P_o} (j\omega, a) \right] \tag{13}$$

In order to evaluate the double integral on the right side of Equation (6) one writes the principal deviatoric strain components e_{rr}, $e_{\theta\theta}$ and e_{zz} as phasors,

$$e_{rr} = Re \, (E_1 e^{j\omega t}) \, , \quad e_{\theta\theta} = Re \, (E_2 e^{j\omega t}) \, , \quad e_{zz} = Re \, (E_3 e^{j\omega t}) \tag{14}$$

where E_1, E_2, and E_3 are the complex deviatoric strain amplitudes.

The principal deviatoric stress components S_{rr}, $S_{\theta\theta}$ and S_{zz} may be obtained by using Equation (14) and the definition of complex shear modulus $G_c(j\omega)$,

$$S_{rr} = \mathrm{Re}(2G_cE_1e^{j\omega t}), \quad S_{\theta\theta} = \mathrm{Re}(2G_cE_2e^{j\omega t}), \quad S_{zz} = \mathrm{Re}(2G_cE_3e^{j\omega t}) \tag{15}$$

Substituting Equations (14), (15) into Equation (6) and using the lemma of Equation (12) one obtains for the work done by the internal stresses in one cycle of vibration the expression

$$\int_0^{2\pi/\omega} \int_V \sigma_{ij}\varepsilon_{ij}\, dV\, dt = 2\pi\, I_m\left[G_c(j\omega)\right] \int_A (|E_1|^2 + |E_2|^2 + |E_3|^2)\, dA \tag{16}$$

Equation (2) may now be applied by using the results of Equations (13) and (16). A relationship is obtained between U_o/P_o and G_c involving the strain amplitudes, namely

$$I_m\left[\frac{U_o}{P_o}(j\omega, a)\right] = \frac{I_m\left[G_c(j\omega)\right]}{\pi a\, |P_o|^2} \int_A (|E_1|^2 + |E_2|^2 + |E_3|^2)\, dA \tag{17}$$

The complex, deviatoric strain amplitudes E_1, E_2 and E_3 are related to the displacement transfer function $\frac{U_o}{P_o}(j\omega, a)$ through the strain-displacement equations of Reference [2], namely

$$E_1 = \frac{U_o}{P_o}(j\omega, a)\left[\frac{2}{3}\alpha\, J_0(\alpha r) - \frac{J_1(\alpha r)}{r}\right] P_o \,/\, J_1(z) \tag{18a}$$

$$E_2 = \frac{U_o}{P_o}(j\omega, a)\left[-\frac{\alpha}{3} J_0(\alpha r) + \frac{J_1(\alpha r)}{r}\right] P_o \,/ J_1(z) \tag{18b}$$

$$E_3 = \frac{U_o}{P_o}(j\omega, a)\left[-\frac{\alpha}{3} J_0(\alpha r)\right] P_o \,/\, J_1(z) \tag{18c}$$

The area integral appearing on the right side of Equation (17) can now be expressed as

$$\int_A (|E_1|^2 + |E_2|^2 + |E_3|^2)dA = 2\pi |P_o|^2 \left| \frac{U_o(j\omega, a)}{P_o} \right|^2 m(z) \tag{19}$$

The factor

$$m(z) = \frac{e^{-2j\angle z}}{|J_1(z)|^2} \int_0^z \left\{ \left| \frac{2}{3} J_o(x) - \frac{J_1(x)}{x} \right|^2 + \left| \frac{J_1(x)}{x} - \frac{1}{3} J_o(x) \right|^2 + \left| \frac{1}{3} J_o(x) \right|^2 \right\} x dx \tag{20}$$

is a non-negative real valued function of z involving integrals of Bessel functions.

When Equation (19) is inserted into Equation (17) one obtains a single expression in $\frac{U_o(j\omega, a)}{P_o}$ and $G_c(j\omega)$ alone, namely

$$I_m\left[\frac{U_o(j\omega, a)}{P_o} \right] = \frac{2}{a} m(z) \left| \frac{U_o(j\omega, a)}{P_o} \right|^2 I_m\left[G_c(j\omega) \right] \tag{21}$$

For the particular values of the parameters used in Reference [1], $m(z)$ is practically insensitive to τ over a wide range of frequencies and time constants, for both Voigt and Maxwell solids. The term $m(z)$, however, does vary significantly with frequency.

In lightly damped systems the resonant frequencies are approximately equal to the respective resonant frequencies of the corresponding perfectly elastic system. Within such lightly damped systems, therefore, it follows that U_o/P_o is approximately pure imaginary at resonance. The right hand side of Equation (21) is then non-negative, for positive ω, in which case U_o must lead P_o by approximately 90 degrees. Therefore, it is reasonable to substitute $|U_o/P_o|$ for $I_m(U_o/P_o)$ and write Equation (21) as

$$\left|\frac{U_o(j\omega,a)}{P_o}\right| = \left(\frac{a}{2m}\right) \Big/ I_m\left[G_c(j\omega)\right] \tag{22}$$

The coefficient $a/2m$ of Equation (22) is practically independent of viscoelastic time constant, for the parameters considered.

In the case of Voigt and Maxwell solids, the complex shear modulus G_c depends on the time parameter τ only through the product $\omega\tau$, so that

$$G_c = G_o\, g(j\omega\tau) \tag{23}$$

where

$$g = 1 + j\omega\tau \quad , \quad \text{Voigt Solid} \tag{24a}$$

$$g = j\omega\tau \,/\, (1 + j\omega\tau) \quad , \quad \text{Maxwell Solid} \tag{24b}$$

and the static modulus of rigidity G_o is real.

The g loci are plotted in Figure 1 as a function of $\omega\tau$. It is apparent that $\text{Im}\ g$ is unbounded for a Voigt solid, but has a maximum at $\omega\tau = 1$ for a Maxwell solid. Hence $1/\text{Im}\ g$ is a monotonically decreasing expression for a Voigt solid, but first decreases, then increases for a Maxwell solid when $1/\text{Im}\ g$ is regarded as a function of τ for a fixed positive value of ω.

Thus, there exists an optimum relaxation time τ, for a Maxwell solid, which minimizes the displacement amplitude at resonance. Moreover the optimum τ is simply the reciprocal of the resonant frequency ω, i.e.

$$\tau_{opt} = 1/\omega \qquad \text{Maxwell solid} \tag{25}$$

For a Voigt solid, the displacement amplitude at resonance decreases monotonically with increasing numerical values of τ. Thus there is no optimum retardation time for a Voigt solid, in the sense developed above.

At the same values of $\omega\tau$ and G_o the Voigt solid always provides greater vibration attenuation than the Maxwell solid, since the former has a greater value of $|\mathrm{Im}\ G_c|$.

Let us now examine the amplitude of the radial bond stress $\sigma_{rr}(a,\theta,t)$ at resonance. The appropriate stress transfer function is $\frac{\Sigma_o^{(rr)}}{P_o}(j\omega,a)$ at the lowest circumferential wave number and is given by Equation (39) of Reference [1] as

$$\frac{\Sigma_o^{(rr)}(j\omega,a)}{P_o(j\omega)} = \frac{z\,J_o(z) - 2(C_2/C_1)^2\,J_1(z)}{\left\{\frac{\rho}{h\gamma}\left(\frac{h}{a}\right)z^2 + \left(\frac{C_2}{C_1}\right)^2\left[2 - \frac{E}{G_c\,(1-\nu^2)}\left(\frac{h}{a}\right)\right]\right\} J_1(a) - z\,J_o(z)} \tag{26}$$

where the dimensionless frequency

$$z = \omega a/C_1 \tag{27}$$

and the terms

$$C_1 = \sqrt{\left(K + \frac{4}{3}G_c\right)/\gamma} \quad ; \quad C_2 = \sqrt{G_c/\gamma} \tag{28}$$

It may be related to the bond displacement transfer function $\frac{U_o(j\omega,a)}{P_o}$ by means of Equation (9) thereby providing the equality

$$\frac{\Sigma_o^{(rr)}}{P_o}(j\omega,a) = d\left[\frac{U_o(j\omega,a)}{P_o}\right] \tag{29}$$

where the parameter d is defined by the relation

$$d = \frac{\gamma}{a}\left(C_1^2\,z\,J_o(z)\,/\,J_1(z) - 2\,C_2^2\right) \tag{30}$$

For the numerical values used in Reference [1], namely

$$\rho/h\gamma = 1.96 \quad ; \quad E/G_o(1-\nu^2) = 22,500,$$
$$C_1^{(o)}/C_2^{(o)} = 30.35, \quad ; \quad h/a = 0.1, \tag{31}$$

the parameter d is practically insensitive to τ over a wide range of frequencies ω for both Voigt and Maxwell materials.

Substitution of Equation (29) into Equation (22) yields the following result

$$\left| \frac{\Sigma_o^{(rr)}(j\omega,a)}{P_o} \right| = \left(\frac{ad}{2m}\right) \Big/ I_m\left[G_c(j\omega)\right], \text{ at resonance} \tag{32}$$

The coefficient ad/2m which appears in Equation (31) is practically independent of the viscoelastic time constant τ. Thus the same conclusions regarding optimum values of τ previously deduced for $\left| U_o/P_o \right|$ also hold with respect to $\left| \Sigma_o^{(rr)}/P_o \right|$. In particular τ given by Equation (25) also minimizes $\left| \Sigma_o^{(rr)}/P_o \right|$ at resonance for a Maxwell solid.

In order to study numerically the degree of approximation inherent in Equation (25), a digital computer program was developed to search for the optimum value of τ by iteration. The program was applied to the Maxwell cylinder assembly defined by Equation (31). The program works in the following way.

First the parameter $\tau C_2^{(o)}/a$ is fixed and the parameter $\omega a/C_1^{(o)}$ is varied to determine a maximum value of $\left| \Sigma_o^{(rr)}/P_o \right|$. This locates the resonant frequency and peak amplitude as a function of relaxation time. Next $\tau C_2^{(o)}/a$ is varied and the resonant peak amplitudes so generated are examined for a minimum peak amplitude. An optimum $\tau C_2^{(o)}/a$ and corresponding resonant $\omega a/C_1^{(o)}$ are also determined during this process.

The calculation procedure described above is performed for the first three resonant frequencies given in Reference [1]. Results are listed in Table I, below.

Table I. OPTIMUM RELAXATION TIME FOR MAXWELL CORE MATERIAL

	First	Second	Third
$\tau C_2^{(o)}/a$	0.012061	0.006540	0.004257
$\omega a/C_1^{(o)}$	2.729058	5.032956	7.735903
$\underset{\tau C_2^{(o)}/a}{\min}\left[\underset{\omega a/C_1^{(o)}}{\max}\left\lvert\frac{\Sigma_o^{(rr)}}{P_o}(j\omega,a)\right\rvert\right]$	501.268	242.630	179.112
$\omega\tau$	0.998985	0.999003	0.999337

It is apparent from the Table that the optimum relaxation time can be computed quite accurately from Equation (25), for the particular example studied. The large stress amplitude ratios for this configuration attest to the low degree of damping in the system, and therefore to the validity of the assumption that $\frac{U_o}{P_o}(j\omega,a)$ is pure imaginary.

A comparison of the above $\omega a/C_1^{(o)}$ values with those given in Figures (2), (3), (4) of Reference [2] for the corresponding all-elastic assembly, reveals that even at optimum damping, damping has negligible effect on resonant frequency.

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REFERENCES

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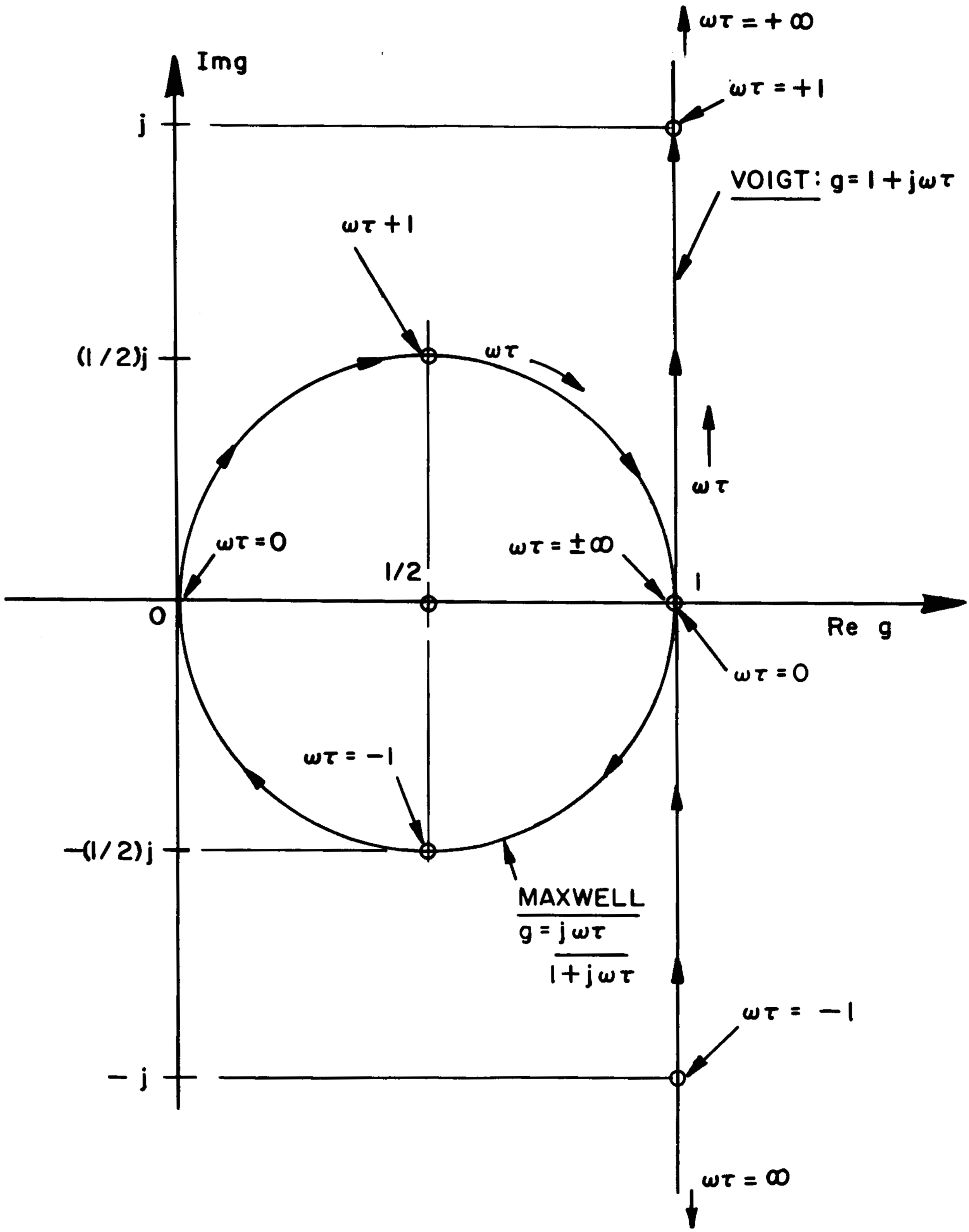

FIGURE 1. FREQUENCY LOCI OF COMPLEX MODULI FOR VOIGT AND MAXWELL SOLIDS. $g = G_e(j\omega)/G_o$