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EFFECTS OF CORRELATED NOISE WITH APPLICATIONS TO APOLLO TRACKING PROBLEMS

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by B. Kruger

Goddard Space Flight Center

Greenbelt, Md.

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ABSTRACT

The uncertainty associated with a least-squares fitted polynomial is a function of the standard deviation of the data noise, the correlation of the data noise, the degrees of freedom of the polynomial, and the number of data points. The standard deviation and the rms standard deviation of the polynomial are therefore better measures of data quality and the information contained in the data than the commonly used residuals from a least-square fitted polynomial.

The standard deviation η_i and the rms standard deviation $\bar{\eta}$ of least-squares fitted polynomials are analyzed in this paper. The general equations derived are demonstrated on several Apollo tracking problems:

> It is shown that $\bar{\eta}$ does not improve for higher sampling rates of angular data than 2 or 5 per second, for narrow and wide bandwidth setting of the angular servoloops, respectively. (Apollo GO, NO-GO decision.)
>
> The effect of negative correlation of measurement errors is analyzed. The effect of phase noise in range rate data is reduced by a factor $\sqrt{N}$ due to the negative correlation of -1/2 between adjacent measurement points.
>
> The effect of random-walk phase noise on the range rate data is shown to be proportional to the two-way propagation time of the signal.

The maximum error in a least-squares fitted polynomial is also analyzed.

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EFFECTS OF CORRELATED NOISE WITH APPLICATIONS TO APOLLO TRACKING PROBLEMS

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B. Kruger

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INTRODUCTION

A frequently used method for evaluation of tracking data is to fit a polynomial $\bar{y}_i$

$$\bar{y}_i = a_0 + a_1\, i + a_2\, i^2 + \cdots + a_{k-1}\, i^{k-1} \tag{1}$$

of degree $k - 1$, i.e., with k degrees of freedom, to the N measured data points y_i in the least-squares sense. The residuals v_i

$$v_i = y_i - \bar{y}_i \tag{2}$$

are then formed and the variance σ_ϵ^2

$$\sigma_\epsilon^2 = \frac{\sum_{i=1}^{N} v_i^2}{N - k} \tag{3}$$

is used as a measure of the quality of the data.

The basic shortcoming of this method is that σ_ϵ does not reflect the amount of information contained in the data. Due to the noise on the data y_i the constants a_ν in $\bar{y}_i$ can only be determined with limited accuracy; a standard deviation σ_{a_ν} can be associated with each a_ν. Taking the correlation between the coefficients into account, we can find the standard deviation η_i for the polynomial $\bar{y}_i$ at point i. It is suggested that η_i is a better measure for the data quality than σ_ϵ. The variance σ_ϵ^2 does not reflect the correlation of the data noise nor the number of data points available. On the other hand, η_i is a function of the data noise correlation and the number of data points available. In addition, η_i reflects the increased uncertainty due to an increase in k.

The standard deviation η_i of the polynomial $\bar{y}_i$ and especially the root mean square (rms) $\bar{\eta}$ of η_i, as defined in the section on data with uncorrelated noise, are therefore better and more accurate measures of data quality and the information contained in the data than σ_ϵ alone.

In this paper the relations between the standard deviations of the polynomial are derived for correlated and uncorrelated data noise. It is shown that η_i and $\overline{\eta}$ are always proportional to σ_ϵ and that the proportional factors are functions of the data noise correlation, the number of data points, and the number of degrees of freedom of the polynomial fit.

The general equations derived are demonstrated on several Apollo tracking problems.

THE LEAST-SQUARES FIT

Let us assume that we have N observed data points y_i at equally spaced intervals. The intervals are normalized to length 1 without loss of generality. A polynomial

$$\overline{y}_i = a_0 + a_1 i + a_2 i^2 + \cdots + a_{k-1} i^{k-1} \tag{4}$$

is fitted to the data by the least-squares method. The residuals v_i are given by

$$v_i = y_i - \overline{y}_i \; . \tag{5}$$

The sum of the residuals squared is minimized by varying the coefficients a_ν of $\overline{y}_i$. From

$$\frac{\partial}{\partial a_\nu} \sum_{i=1}^{N} v_i^2 = 0 \tag{6}$$

we obtain

$$\begin{aligned}
& a_0 \Sigma i^0 + a_1 \Sigma i + \cdots + a_{k-1} \Sigma i^{k-1} = \Sigma i^0 y_i \\
& a_0 \Sigma i + a_1 \Sigma i^2 + \cdots + a_{k-1} \Sigma i^k = \Sigma i \, y_i \\
& \quad \vdots \\
& a_0 \Sigma i^{k-1} + a_1 \Sigma i^k + \cdots + a_{k-1} \Sigma i^{2k-2} = i^{k-1} y_i \; ,
\end{aligned} \tag{7}$$

where all sums are taken from 1 to N. From this set of linear equations the coefficients a_ν may be solved for and inserted in Equation 4. It is shown in Appendix A that the result may conveniently be written in determinant form:

$$\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{j=N} \begin{vmatrix} 0 & 1 & j & j^2 & \cdots & j^{k-1} \\ 1 & A_0 & A_1 & A_2 & \cdots & A_{k-1} \\ i & A_1 & A_2 & A_3 & \cdots & A_k \\ i^2 & A_2 & A_3 & A_4 & \cdots & A_{k+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ i^{k-1} & A_{k-1} & A_k & A_{k+1} & \cdots & A_{2k-2} \end{vmatrix} y_j , \tag{8}$$

where

$$A_\nu = \sum_{i=1}^{N} i^\nu \tag{9}$$

and

$$|A| = \begin{vmatrix} A_0 & A_1 & \cdots & A_{k-1} \\ A_1 & A_2 & \cdots & A_k \\ \vdots & \vdots & & \vdots \\ A_{k-1} & A_k & \cdots & A_{2k-2} \end{vmatrix} .$$

For brevity we introduce the notation* $\begin{vmatrix} 0 & j \\ i & A \end{vmatrix}$ for the larger determinant so that

$$\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} y_j , \tag{10}$$

which is the basic equation that will be used for the analysis of the standard deviation η_i of the polynomial $\bar{y}_i$.

*Note that $\begin{vmatrix} o & j \\ i & A \end{vmatrix}$ is not a linear operator.

DATA WITH UNCORRELATED NOISE

Basic Relations and Definitions

The measurements y_i may be written

$$y_i = Y_i + \epsilon_i \;, \tag{11}$$

where

Y_i = the true value

ϵ_i = measurement error.

The ϵ_i are assumed to be stationary stochastic variables with zero mean and standard deviation σ_ϵ. The error $\Delta\overline{y}_i$ due to the measurement errors ϵ_i is obtained by substituting Equation 11 in Equation 10:

$$\Delta\overline{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j \;. \tag{12}$$

For uncorrelated noise we have

$$[\sigma(C_1\epsilon_1 + C_2\epsilon_2 + \ldots + C_N\epsilon_N)]^2 = (C_1^2 + C_2^2 + \ldots C_N^2)\,\sigma_\epsilon^2 \;,$$

and hence

$$\sigma^2_{\Delta\overline{y}_i} = \eta_i^2 = \frac{\sigma_\epsilon^2}{|A|^2} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix}^2 , \tag{13}$$

where

η_i = standard deviation of the least-squares fitted polynomial $\overline{y}_i$ at point i.

In Appendix B, Equation 3, it is shown that

$$\sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix}^2 = - |A| \begin{vmatrix} 0 & i \\ i & A \end{vmatrix} , \tag{B3}$$

and thus

$$\eta_i^2 = - \frac{\begin{vmatrix} 0 & i \\ i & A \end{vmatrix}}{|A|} \sigma_\epsilon^2 \;. \tag{14}$$

It is suitable to normalize the measuring interval to unit length by dividing i by N. Furthermore, η_i is symmetric around $i/N = 1/2$. A new variable u is therefore introduced:

$$\frac{i}{N} = \frac{1 + u}{2} \ , \tag{15}$$

and it is shown in Appendix C that, with the approximation

$$A_\nu = \frac{N^{\nu+1}}{\nu + 1} \ , \tag{16}$$

Equation 14 may be written

$$\frac{\eta_i^2}{\sigma_\epsilon^2} N = P_0^2(u) + 3P_1^2(u) + \ . \ . \ . \ (2k + 1)\ P_k^2(u) \ , \tag{17}$$

where P_ν are the Legendre polynomials. The error introduced by the approximation 16 is of magnitude N^{-2} (see Appendix C) and vanishes, therefore, for large N.

In Figure 1 the normalized standard deviation η_{ni} for $\bar{y}_i$,

$$\eta_{ni} = \frac{\eta_i}{\sigma_\epsilon} \sqrt{N} \ , \tag{18}$$

is shown graphically for values of k from 1 through 6. We note that, for $i/N = 0$ or $i/N = 1$,

$$\eta_{ni} \Big|_{i=\substack{0\\N}} = k \ . \tag{19}$$

This result is true for all k, which may be shown by expanding Equation 14 in powers of i:

$$\eta_{ni}^2 = N \frac{\eta_i^2}{\sigma_\epsilon^2} = \frac{N}{|A|} [\Delta_{11} - (\Delta_{12} + \Delta_{21})\ i$$

$$+ (\Delta_{13} + \Delta_{22} + \Delta_{31})\ i^2 + \ . \ . \ . \ ] \ , \tag{20}$$

where Δ_{11}, Δ_{12}, etc., are the minors of $|A|$. If $i = 0$ or $i = N$,

$$\eta_{ni}^2 \Big|_{i=\substack{0\\N}} = N \frac{\Delta_{11}}{|A|} \ .$$

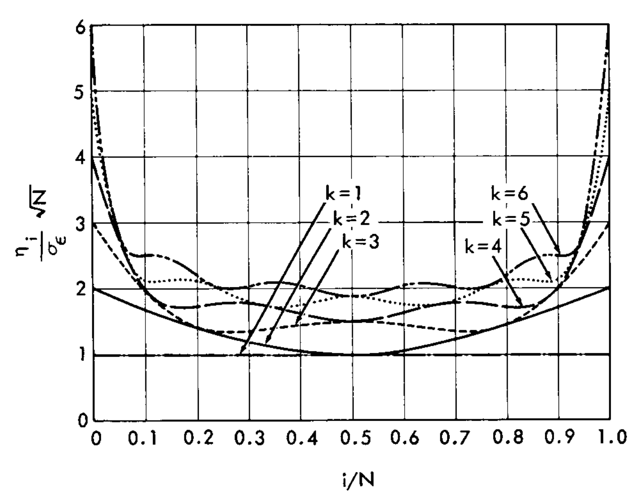

Figure 1—The standard deviation η_i for least-square fitted polynomials $\bar{y}_i$ of degree k − 1. The standard deviation of the data noise is σ_ϵ. N = number of data points.

It is shown in Appendix D, Equation D11, that

$$\Delta_{11} = \frac{k^2}{N} \, |A| \; ,$$

and thus

$$\eta_{ni} \Big|_{i=\substack{0\\N}} = k \; .$$

Sometimes it is necessary to use $\bar{y}_i$ for prediction outside the interval of observation. An example is the prediction of yearly oscillator drift based on one or two months of actual observation. Figure 2 shows the rapid increase of η_i outside the interval of observation.

The Root Mean Square Standard Deviation

The standard deviation η_i for the least-squares fitted polynomial $\bar{y}_i$ varies with i, as can be seen from Figure 1. It is therefore desirable to define an average standard deviation. We choose the root mean square (rms) standard deviation $\bar{\eta}$ defined by

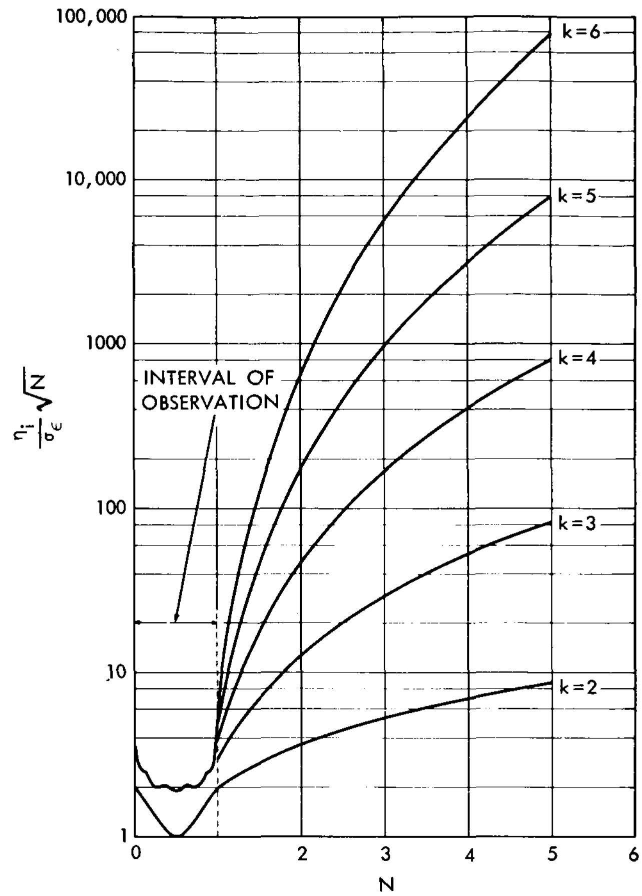

Figure 2—The standard deviation η_i of the least-squares fitted polynomial $\bar{y}_i$ increases rapidly, if $\bar{y}_i$ is used for prediction outside the interval of N observed data points.

$$\bar{\eta}^2 = \frac{1}{N} \sum_{i=1}^{N} \eta_i^2 \; , \tag{21}$$

or, using Equation 14,

$$\bar{\eta}^2 = -\frac{\sigma_\epsilon^2}{N|A|} \sum_{i=1}^{N} \begin{vmatrix} 0 & i \\ i & A \end{vmatrix} \; . \tag{22}$$

From Appendix B, Equation B4, we obtain

$$\bar{\eta} = \sqrt{\frac{k}{N}} \, \sigma_\epsilon \; . \tag{23}$$

This equation demonstrates a simple relationship between $\bar{\eta}$ and σ_ϵ. For $k = 1$ we obtain the well-known equation for the standard deviation of a time invariant quantity which is measured N times.

Because $N \geq k$ we have

$$\bar{\eta} \leq \sigma_\epsilon \tag{24}$$

For the normalized rms standard deviation

$$\bar{\eta}_n = \frac{\bar{\eta}}{\sigma_\epsilon}\sqrt{N}$$

we obtain

$$\bar{\eta}_n^2 = \frac{1}{N}\sum \eta_{ni}^2 = k$$

or

$$\bar{\eta}_n = \sqrt{k} \; . \tag{25}$$

Note that the Equations 23 and 25 are exact; and no assumptions have been made about the size of N. Figure 3 shows the relation between $\bar{\eta}$ and η_i for k = 3 and k = 6.

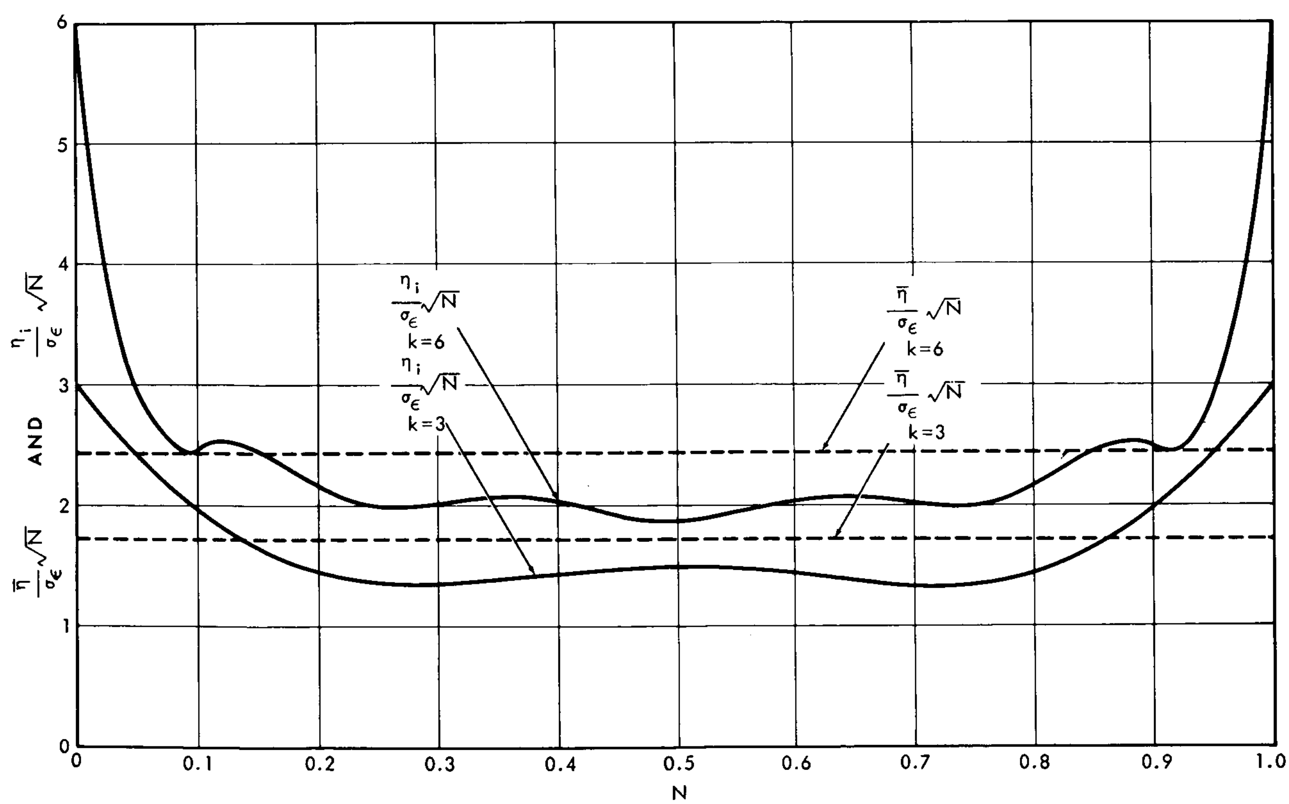

Figure 3—The relationship between the normalized standard deviation $(\bar{\eta}_i/\sigma_\epsilon)\sqrt{N}$ and the normalized rms standard deviation $(\bar{\eta}/\sigma_\epsilon)\sqrt{N}$ for least-squares fitted polynomials with k degrees of freedom.

DATA WITH CORRELATED NOISE

Basic Relations

If the correlation coefficient between the noise of measurements y_ℓ and $y_{\ell+m}$ is ρ_m, then we have the basic relation for a stationary process:

$$[\sigma(C_1\epsilon_1 + C_2\epsilon_2 + \cdots C_N\epsilon_N)]^2 = \Bigg[\sum_1^N C_i^2 + 2\rho_1 \sum_1^{N-1} C_i\, C_{i+1} + \cdots$$
$$+ 2\rho_v \sum_1^{N-v} C_i\, C_{i+v} + \cdots + 2\rho_{N-1}\, C_1\, C_N\Bigg]\, \sigma_\epsilon^2 \,,$$

where σ_ϵ is the standard deviation of errors in the measured values y_i. From Equation 12 for the error $\Delta\bar{y}_i$ in the least-squares fitted polynomial $\bar{y}_i$,

$$\Delta\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j \,, \tag{12}$$

we thus find the standard deviation η_i for $\bar{y}_i$ to be

$$\eta_i^2 = \frac{1}{|A|^2} \Bigg[\sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix}^2 + 2\rho_1 \sum_{j=1}^{N-1} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \cdot \begin{vmatrix} 0 & j+1 \\ i & A \end{vmatrix} + \cdots$$
$$+ 2\rho_\nu \sum_{j=1}^{N-\nu} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \cdot \begin{vmatrix} 0 & j+\nu \\ i & A \end{vmatrix} + \cdots 2\rho_{N-1} \begin{vmatrix} 0 & 1 \\ i & A \end{vmatrix} \cdot \begin{vmatrix} 0 & N \\ i & A \end{vmatrix}\Bigg] \sigma_\epsilon^2 \,. \tag{26}$$

For the root mean square standard deviation $\bar{\eta}$

$$\bar{\eta}^2 = \frac{1}{N} \sum_{i=1}^{N} \eta_i^2 \,,$$

we obtain

$$\bar{\eta}^2 = \frac{1}{|A|^2} \Bigg[\sum_{i=1}^{N} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix}^2 + 2\rho_1 \sum_{i=1}^{N} \sum_{j=1}^{N-1} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \cdot \begin{vmatrix} 0 & j+1 \\ i & A \end{vmatrix} + \cdots$$
$$+ 2\rho_\nu \sum_{i=1}^{N} \sum_{j=1}^{N-\nu} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \cdot \begin{vmatrix} 0 & j+\nu \\ i & A \end{vmatrix} + \cdots + \rho_{N-1} \sum_{i=1}^{N} \begin{vmatrix} 0 & 1 \\ i & A \end{vmatrix} \cdot \begin{vmatrix} 0 & N \\ i & A \end{vmatrix}\Bigg] \sigma_\epsilon^2 \,. \tag{27}$$

In Appendix D it is shown that this equation may be simplified to

$$\bar{\eta}^2 = \frac{k}{N}\left[1 + 2\sum_{i=1}^{N-1} \rho_i - \frac{2k}{N}\sum_{i=1}^{N-1} i\rho_i + \frac{g(k)}{N^3}\sum_{i=1}^{N-1} i^3\rho_i - \cdots\right]\sigma_\epsilon^2\ , \tag{28}$$

where $g(k)$ is a function of k alone. The normalized root mean square standard deviation

$$\bar{\eta}_n = \frac{\bar{\eta}}{\sigma_\epsilon}\sqrt{N}$$

is then

$$\bar{\eta}_n^2 = k\left[1 + 2\sum_{i=1}^{N-1} \rho_i - \frac{2k}{N}\sum_{i=1}^{N-1} i\rho_i + \frac{g(k)}{N^3}\sum_{i=1}^{N-1} i^3\rho_i - \cdots\right]. \tag{29}$$

This is a powerful equation; the use of it will be demonstrated in the following subsections.

Positive Correlation from System Transfer Functions

The transfer functions of a system can in most cases be described by an equivalent electronic circuit. For instance, the transfer function of an antenna servo is a low-pass filter. If white noise is applied to a low-pass filter, the noise at the output terminals will be positively correlated. In this section the effects of positive correlation from maximum-flat (Butterworth) filters will be analyzed. The autocorrelation functions for the noise at the filter output are summarized in Reference 1. Figure 4 shows the autocorrelation function for a single-pole, a double-pole, and an "ideal" (infinitely many poles) filter as well as the corresponding electronics circuits. Maximum-flat filters are chosen because they commonly occur, they describe most systems adequately, and their mathematical treatment is relatively simple.

The autocorrelation function $R(\tau)$ of the output noise (for white input noise) for a single-pole filter (see References 1 or 2) is

$$R(\tau) = e^{-\omega_c|\tau|}, \tag{30}$$

where ω_c = 3-db angular cutoff frequency of the filter. If the sampling interval is h, then

$$\rho_1 = e^{-\beta_1}$$

and

$$\rho_i = e^{-i\beta_1} = (\rho_1)^i, \tag{31}$$

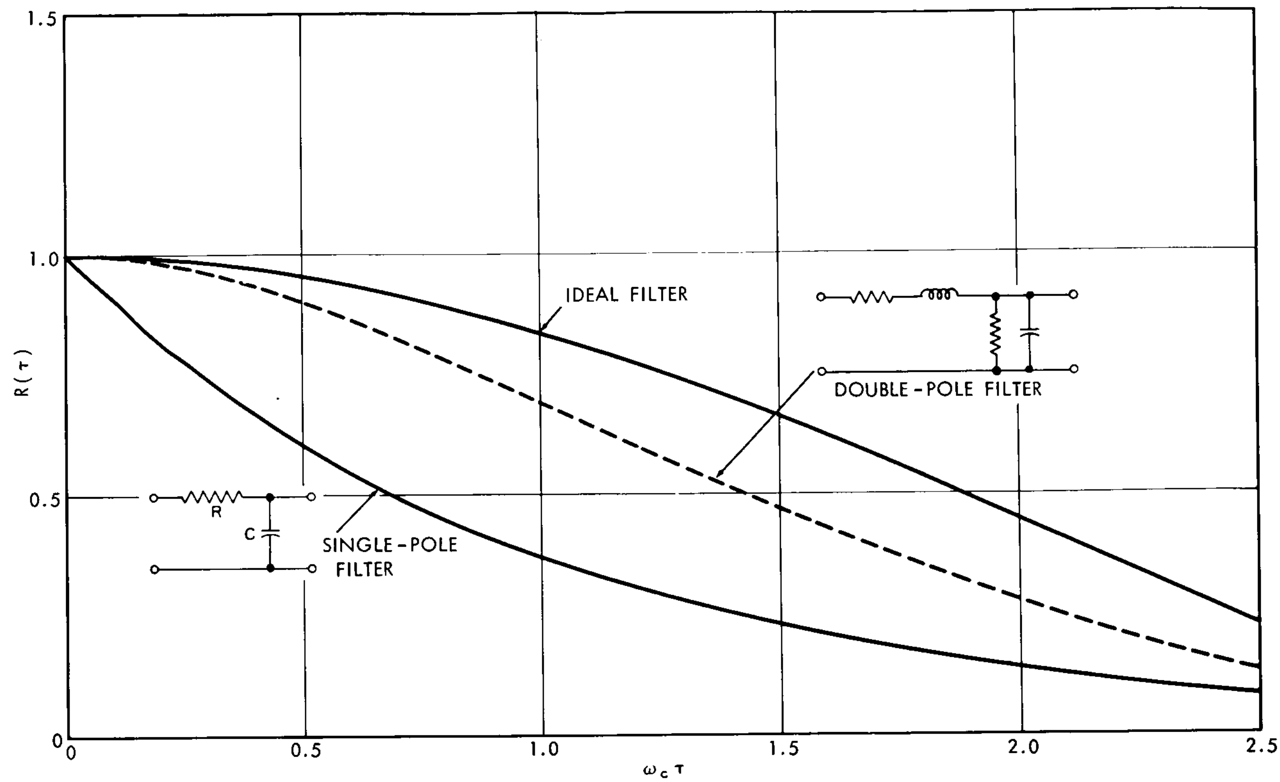

Figure 4—Electronic single- and double-pole filters. Normalized autocorrelation functions $R(\tau)$ of white noise passing through maximum-flat filters.

when $\beta_1 = \omega_c h$, and thus

$$\sum_{i=1}^{N-1} \rho_i = \sum_{i=1}^{N-1} \rho_1^i = \frac{\rho_1 (1 - \rho_1^{N-1})}{1 - \rho_1} . \tag{32}$$

But $\beta_1^{N-1} \ll 1$ for large N, so that

$$\sum_{i=1}^{N-1} \rho_i = \frac{\rho_1}{1 - \rho_1} . \tag{33}$$

For the sum $\Sigma i\rho_i$ we obtain

$$\sum_{i=1}^{N-1} i\rho_i = \sum_{i=1}^{N-1} i e^{-i\beta_1} = \frac{d}{d\beta_1} \sum_{i=1}^{N-1} - e^{-i\beta_1} . \tag{34}$$

For large N,

$$\sum_{i=1}^{N-1} i\rho_i = \frac{\rho_1}{(1 - \rho_1)^2} \quad . \tag{35}$$

The sums $\Sigma i^3 \rho_i$, etc., may be evaluated in a similar fashion.

Inserting these results in Equation 29 yields

$$\bar{\eta}_n^2 = k \left[1 + \frac{2\rho_1}{1 - \rho_1} - \frac{2k}{N} \frac{\rho_1}{(1 - \rho_1)^2} \cdots \right] , \tag{36}$$

and for large N

$$\bar{\eta}_n^2 = k \frac{1 + \rho_1}{1 - \rho_1} \tag{37}$$

or

$$\bar{\eta} = \sqrt{\frac{1 + \rho_1}{1 - \rho_1}} \, \bar{\eta}_o \ , \tag{38}$$

where $\rho_1 = e^{-\omega_c h}$ and $\bar{\eta}_o$ stands for the rms standard deviation of the fitted polynomial $\bar{y}_i$ for uncorrelated data noise.

We see from Equation 38 that the effect of positive correlation $e^{-i\omega_c h}$ is to multiply the rms standard deviation obtained for uncorrelated data by the factor

$$\sqrt{\frac{1 + \rho_1}{1 - \rho_1}} \ .$$

A maximum-flat (Butterworth) two-pole filter has the normalized autocorrelation function (Reference 1).

$$R(\tau) = \frac{\sin \left(\omega_c |\tau| \cos \frac{\pi}{4} + \frac{\pi}{4} \right) e^{-\omega_c |\tau| \cos \pi/4}}{\cos \frac{\pi}{4}} \tag{39}$$

and thus

$$\rho_i = (\sin i\beta_2 + \cos i\beta_2) \, e^{-i\beta_2} \ , \tag{40}$$

where $\beta_2 = \omega_c h/\sqrt{2}$. Performing the summation we obtain for large N

$$\sum_1^\infty \rho_i = \frac{\sin\beta_2 + \cos\beta_2 - e^{-\beta_2}}{e^{\beta_2} + e^{-\beta_2} - 2\cos\beta_2} ,$$

so that

$$1 + 2\sum_1^\infty \rho_i = \frac{\sin h\,\beta_2 + \sin\beta_2}{\cos h\,\beta_2 - \cos\beta_2} . \tag{41}$$

Thus

$$\bar{\eta}_n = \sqrt{\frac{\sin h\frac{\omega_c h}{\sqrt{2}} + \sin\frac{\omega_c h}{\sqrt{2}}}{\cos h\frac{\omega_c h}{\sqrt{2}} - \cos\frac{\omega_c h}{\sqrt{2}}}}\sqrt{k} \tag{42}$$

for a two-pole filter and for large N.

For an ideal filter, i.e., one with infinitely many poles, we obtain from References 1 and 2

$$R(\tau) = \frac{\sin\omega_c\tau}{\omega_c\tau} \tag{43}$$

and hence

$$\rho_i = \frac{\sin i\beta_1}{i\beta_1} , \tag{44}$$

where $\beta_1 = \omega_c \cdot h$. For large N we obtain from Reference 3, page 96

$$\sum_{i=1}^N \rho_i \approx \sum_1^\infty \rho_i = \frac{\pi - \beta_1}{2\beta_1} . \tag{45}$$

Also

$$\frac{1}{N}\sum_1^N i\rho_i = \frac{1}{N\beta_1}\sum_1^N \sin i\beta_1 = \frac{1}{N\beta_1}\, I_m \left\{\frac{1 - e^{jN\beta_1}}{1 - e^{j\beta_1}}\, e^{j\beta_1}\right\} , \tag{46}$$

where $j = \sqrt{-1}$. Thus

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} i\rho_i = 0 ,$$

provided that $1 - e^{j\beta_1} \neq 0$. We thus obtain from Equation 29

$$\overline{\eta}_n^2 = k \frac{\pi}{\omega_c h} \tag{47}$$

for an ideal low-pass filter and large N.

The Effect of Sampling Rate on Positively Correlated Data

An important problem is to find the maximum meaningful sampling rate if the total time of observation T is given and the data is positively correlated by a maximum-flat filter as described in the preceding subsection.

For a single-pole filter, Equation 38 may be written

$$\overline{\eta}^2 = \frac{k}{N} \frac{1 + \rho_1}{1 - \rho_1} \sigma_\epsilon^2 . \tag{48}$$

The total time of observation is T and the sampling interval is therefore

$$h = \frac{T}{N} \tag{49}$$

and thus

$$\rho_1 = e^{-\omega_c \frac{T}{N}}$$

or

$$\rho_1 = 1 - \left(\omega_c \frac{T}{N}\right) + \frac{1}{2!} \left(\omega_c \frac{T}{N}\right)^2 - \frac{1}{3!} \left(\omega_c \frac{T}{N}\right)^3 + \cdots . \tag{50}$$

Insertion in Equation 39 yields

$$\overline{\eta}^2 = k \frac{2}{\omega_c T} \left[1 + \frac{\left(\omega_c \frac{T}{N}\right)^2}{12} - \frac{\left(\omega_c \frac{T}{N}\right)^4}{180} + \cdots \right] \sigma_e^2 . \tag{51}$$

From this equation we see that $\bar{\eta}$ does not improve appreciably if $(\omega_c T/N) < 1$. In other words, little is gained by using a larger number of samples than N_{max}:

$$N_{max} = \omega_c T \ , \tag{52}$$

or a sampling rate N/T larger than

$$\frac{N_{max}}{T} = \omega_c \ . \tag{53}$$

Equation 37 may also be written

$$\bar{\eta}^2 = \frac{k}{N} \frac{1 + e^{-\omega_c \frac{T}{N}}}{1 - e^{-\omega_c \frac{T}{N}}} \sigma_\epsilon^2 \tag{54}$$

and, for $N \to \infty$,

$$\bar{\eta}^2 = \frac{2k}{\omega_c T} \sigma_\epsilon^2 \ , \tag{55}$$

where

σ_e = standard deviation of the noise of the data

k = degree of freedom of the least-squares fitted polynomial $\bar{y}_i$

ω_c = 3-db angular cutoff frequency of the single pole filter

T = total time of observation.

Figure 5 shows a comparison between $\bar{\eta}$ for correlated and uncorrelated noise with $T = 60$ sec and $\omega_c = 3$ rad/sec. This figure clearly demonstrates that little is gained by exceeding N_{max}. It is of interest to note that $\rho_1 = 1/e \approx 0.37$ for $N = N_{max}$.

From Equation 23 we see that the same $\bar{\eta}$ is obtained for correlated data with $N = \infty$ and uncorrelated data with $N = (1/2) N_{max}$.

For a double-pole filter we obtain from Equations 28 and 41

$$\bar{\eta}^2 = \frac{k}{N} \cdot \frac{\sinh \frac{\omega_c T}{\sqrt{2}\, N} + \sin \frac{\omega_c T}{\sqrt{2}\, N}}{\cosh \frac{\omega_c T}{\sqrt{2}\, N} - \cos \frac{\omega_c T}{\sqrt{2}\, N}} \sigma_\epsilon^2 \tag{56}$$

and, for $N \to \infty$,

$$\overline{\eta}^2 = \frac{2\sqrt{2}\,k}{\omega_c T}\,\sigma_\epsilon^2 \ . \tag{57}$$

For the ideal low-pass filter we obtain, from Equation 46 for $N \to \infty$,

$$\overline{\eta}^2 = \frac{\pi k}{\omega_c T}\,\sigma_\epsilon^2 \ . \tag{58}$$

Figure 6 shows the comparison between $\overline{\eta}$ for uncorrelated noise and noise correlated by a single-pole filter, a double-pole filter, and an ideal filter. It is seen that the difference between a double-pole and an ideal filter is very small. It is therefore sufficient to calculate with a double-pole filter for practical purposes.

Figure 7 and 8 show $\overline{\eta}$ for single- and double-pole filters with T = 60 sec and ω_c as parameter.

<u>Example</u>: What is the maximum meaningful sampling rate for angular data for Apollo tracking ships? The transfer function of the angle servos is approximated by a two-pole filter.

Expanding Equation 56 into a series we find

$$\overline{\eta}^2 = \frac{2\sqrt{2}}{\omega_c T}\left[1 + \frac{1}{180}\left(\frac{\omega_c T}{\sqrt{2}\,N}\right)^4 + \cdots\right] \ . \tag{59}$$

Somewhat arbitrarily we defined the maximum meaningful sampling rate as the rate for which $\overline{\eta}$ is within 5 percent of its value for $N = \infty$. Thus

$$\frac{1}{2}\cdot\frac{1}{180}\left(\frac{\omega_c T}{\sqrt{2}\,N_{max}}\right)^4 = 0.05 \tag{60}$$

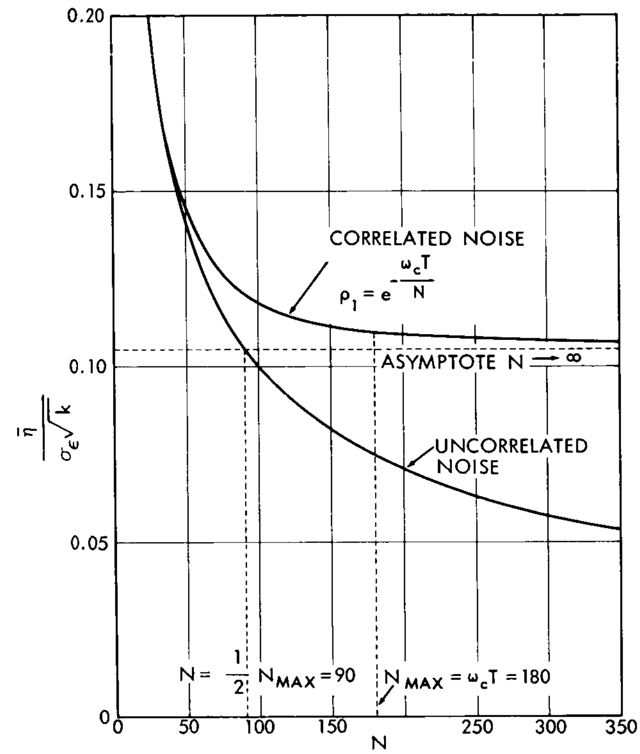

Figure 5—RMS standard deviation $\overline{\eta}$ of least-squares polynomial fits for uncorrelated and correlated noise. The time T of observation is 60 sec, and ω_c = 3 rad/sec. The number of samples N is varied.

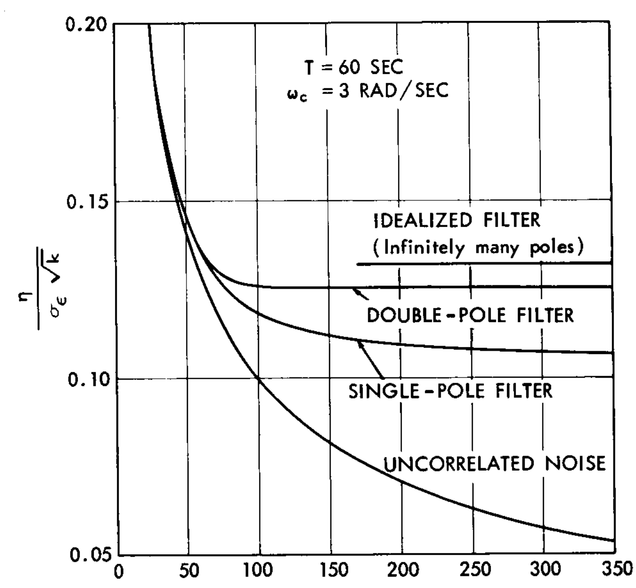

Figure 6—Comparison of the effect of different filters on the rms standard deviation $\overline{\eta}$ of least-squares fitted polynomials. White noise is applied to the filters.

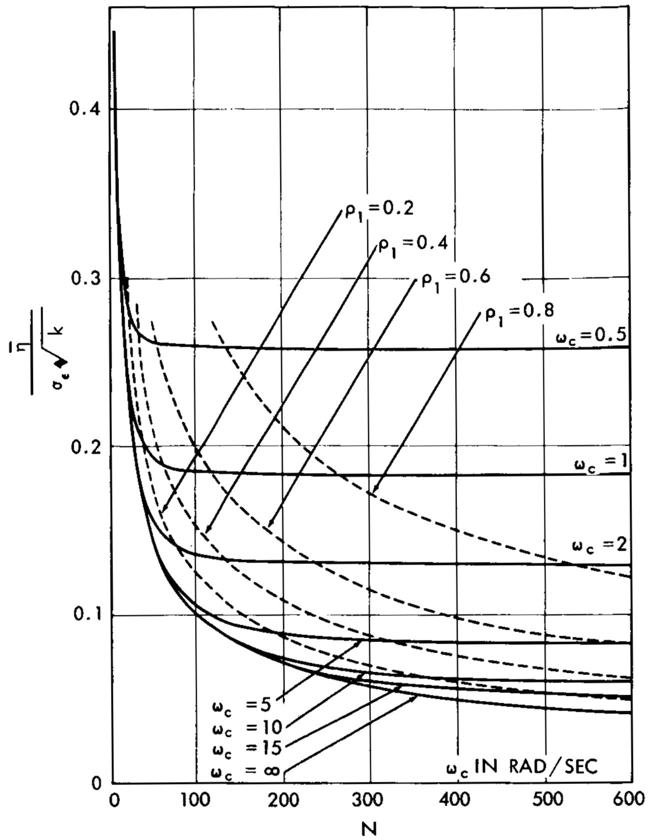

Figure 7—The effect of filter bandwidth of a single-pole filter on the rms standard deviation $\bar{\eta}$ of least-squares fitted polynomials. White noise is applied to the filter, and the time of observation T = 60 sec.

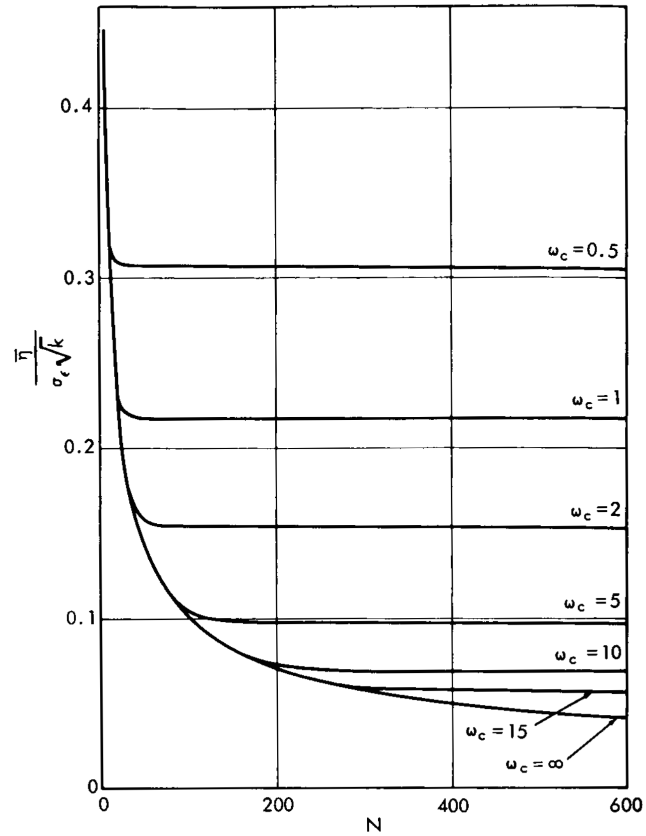

Figure 8—The effect of filter bandwidth of a double-pole filter on the rms standard deviation $\bar{\eta}$ of least-squares fitted polynomials. White noise is applied to the filter, and the time of observation T = 60 sec.

and

$$\left(\frac{N}{T}\right)_{max} = \frac{\omega_c}{3^{1/2}\, 2^{3/4}} . \tag{61}$$

The Angle Servo Bandwidth may be switched to either 1 Hz or 2.5 Hz (with gyro loop closed) according to Reference 4. We thus obtain from Equation 61 the maximum meaningful sampling rates as 2 and 5 samples per second respectively.

Negative Correlation Range Rate Data

A typical case of data with negatively correlated noise is range rate measurements in the non-destructive Doppler count mode (Reference 5). The Doppler frequency shift is continuously integrated by means of a counter, and the counter is read out at equal intervals T without destroying the information. If the information in the counter is Z_i, then the range rate $\dot{R}$ is

$$\dot{R}_1 = \frac{\Delta R_1}{T} = c_1 (Z_2 - Z_1) \qquad \dot{R}_2 = \frac{\Delta R_2}{T} = c_1 (Z_3 - Z_2) \qquad \dot{R}_3 = \frac{\Delta R_3}{T} = c_1 (Z_4 - Z_3) \; . \; . \; . \tag{62}$$

etc., where c_1 is a constant. The dominating errors in Z_i are of two types: a random, zero-mean, and uncorrelated noise with constant standard deviation σ_N, and a zero-mean random-walk noise with standard deviation σ_{RW} which is proportional to the root of time t:

$$\sigma_{RW} = c_2 \sqrt{t} \ ,$$

where c_2 is a constant.

We will first consider the noise with standard deviation σ_N. As the noise is uncorrelated we obtain

$$\begin{aligned} \sigma_{\dot{R}1}^{\ 1} &= c_1^2 \, \sigma_{z_1}^2 + c_1^2 \, \sigma_{z_2}^2 = 2c_1^2 \, \sigma_N^{\ 2} = \sigma_{\dot{R}N}^2 \\ \sigma_{\dot{R}2}^{\ 2} &\doteq c_1^2 \, \sigma_{z_2}^2 + c_1^2 \, \sigma_{z_3}^2 = 2c_1^2 \, \sigma_N^2 = \sigma_{\dot{R}N}^2 \ , \end{aligned} \tag{63}$$

where

$\sigma_{\dot{R}N}$ = the standard deviation of the range rate noise due to σ_N.

The errors in $\dot{R}_1$ and $\dot{R}_2$ are correlated because both contain the same error from Z_2. For this reason all adjacent measurements are correlated. Non-adjacent measurements, e.g., $\dot{R}_1$ and $\dot{R}_3$, are not correlated because they have no error in common.

From Equation 62 we obtain

$$\begin{aligned} [\sigma(\dot{R}_1 + \dot{R}_2)]^2 &= c_1^2 \, [\sigma(Z_3 - Z_1)]^2 = 2c_1^2 \sigma_N^2 = \sigma_{\dot{R}N}^2 \\ [\sigma(\dot{R}_1 + \cdots + \dot{R}_N)]^2 &= c_1^2 \, [\sigma(Z_N - Z_1)]^2 = 2c_1^2 \sigma_N^2 = \sigma_{\dot{R}N}^2 \ . \end{aligned} \tag{64}$$

But we also have

$$[\sigma(\dot{R}_1 + \cdots + \dot{R}_N)]^2 = N\sigma_{\dot{R}N}^2 + 2(N-1) \, \rho_1 \, \sigma_{\dot{R}N}^2 \ , \tag{65}$$

where ρ_1 is the correlation coefficient between adjacent measurements. The results from Equations 64 and 65 have to be identical for all N:

$$N + 2(N - 1) \, \rho_1 = 1 \ . \tag{66}$$

Hence

$$\rho_1 = -\frac{1}{2} \quad . \tag{67}$$

By putting $\dot{R}_i = y_i$ we obtain the least-square polynomial $\bar{y}_i$ from Equation 10:

$$\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} y_i \quad , \tag{10}$$

and the normalized rms standard deviation $\bar{\eta}_n$ of the polynomial is for large N, from Equation 29,

$$\bar{\eta}_n = k \left[1 - 1 - \frac{2k}{N} \left(-\frac{1}{2}\right)\right]$$

or

$$\bar{\eta}_n = \frac{k^2}{N} \quad , \tag{68}$$

which also may be written

$$\bar{\eta} = \frac{k}{N} \sigma_{\dot{R}N} \quad . \tag{69}$$

It is interesting to compare this result for negatively correlated noise with Equation 23 for uncorrelated noise:

$$\bar{\eta} = \sqrt{\frac{k}{N}} \sigma_\epsilon \quad . \tag{23}$$

We may thus write

$$\bar{\eta}\Big|_{\rho=-\frac{1}{2}} = \sqrt{\frac{k}{N}}\, \bar{\eta}\Big|_{\rho=0} \quad . \tag{70}$$

The negative correlation $\rho_1 = -1/2$ thus reduces the effect of the noise on the rms standand deviation of $\bar{y}_i$ by a factor $\sqrt{k/N}$.

The range rate data also gets an error contribution from the random-walk phase noise. This contribution is for a coherent range rate system proportional to the square root of the propagation time T_p of the electromagnetic wave going from the tracking station to the spacecraft and back again. The standard deviation $\sigma_{\dot{R}RW}$ of the error in the $\dot{R}$ measurements, caused by random walk, is thus

$$\sigma_{\dot{R}RW} = c_2 \sqrt{T_p} \ . \tag{71}$$

The normalized autocorrelation function for the range rate is shown in Figure 9. If the propagation time T_p is $p + x$ sampling intervals, where $0 < x < 1$, then the correlation coefficients are given by

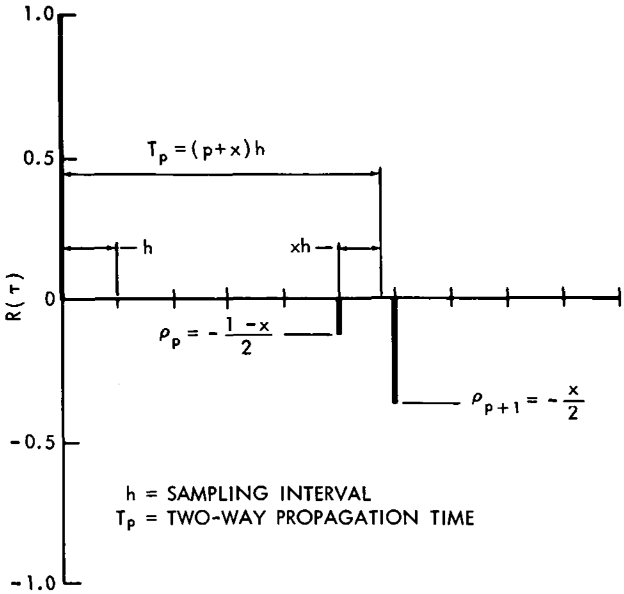

Figure 9—Normalized autocorrelation function $R(\tau)$ for range rate measurements (two-way Doppler) contaminated with random-walk phase noise.

$$\rho_p = -\frac{1-x}{2}$$

$$\rho_{p+1} = -\frac{x}{2} \ , \tag{72}$$

and all other $\rho = 0$, as shown in Appendix E. Hence

$$1 + 2\Sigma\rho_i = 0$$

and

$$-2\Sigma i\rho_i = p(1-x) + (p+1)x = T_p \ , \tag{73}$$

From Equation 28 we thus obtain

$$\bar{\eta}^2 = \frac{k^2}{N^2} T_p \sigma^2_{\dot{R}RW} \ .$$

With $\sigma_{\dot{R}RW} = c_2\sqrt{T_p}$, we obtain

$$\bar{\eta} = \frac{k}{N} c_2 T_p \quad . \tag{74}$$

The rms standard deviation of the least-squares fitted polynomial is thus proportional to the propagation time T_p. The effects of the random-walk error are therefore negligible for near-earth missions but may be dominating for "Deep Space" missions and galactic probes.

Negatively Correlated Data; Exponential Autocorrelation Function

Assume that we have a normalized autocorrelation function

$$R(\tau) = - e^{-\omega_c|\tau|} . \tag{75}$$

If the sampling interval is h, then

$$\rho_1 = - e^{-\beta_1}$$

and

$$\rho_i = - e^{-i\beta_1} ,$$

where

$$\beta_1 = \omega_c h , \tag{76}$$

and thus

$$\sum_1^{N-1} \rho_i = - \sum_1^{N-1} e^{-i\beta_1} = \frac{-|\rho_1|}{1 - |\rho_1|} \tag{77}$$

for large N in accordance with Equation 32. For the sum $\Sigma i\rho_i$ we obtain, from Equation 35 with $N \to \infty$,

$$\sum_1^{\infty} i\rho_i = - \frac{|\rho_1|}{(1 - |\rho_1|)^2} \quad . \tag{78}$$

Insertion in Equation 29 yields, for large N,

$$\bar{\eta} = \sqrt{\frac{1 - 3|\rho_1|}{1 - |\rho_1|}} \; \bar{\eta}_o \quad , \tag{79}$$

where $\bar{\eta}_o$ is the rms standard deviation of $\bar{y}_i$ for uncorrelated noise. The maximum amplitude for $|\rho_1|$ is

$$|\rho_1|_{max} = \frac{1}{3} \; . \tag{80}$$

For this value we obtain from Equation 29:

$$\bar{\eta}_n^2 = k \left[1 - 1 + \frac{2k}{N}\,\frac{3}{4}\right] \tag{81}$$

or

$$\bar{\eta} = \sqrt{\frac{3}{2}\,\frac{k}{N}}\;\sigma_\epsilon \; . \tag{82}$$

Except for a constant multiplier, this equation is identical with Equation 69 for noise, with $\rho_1 = -1/2$ and all other $\rho = 0$.

RELATED TOPICS

Determination of the Standard Deviation of Noise of Time-Varying Data

Let Y be a time-varying quantity which is observed at equally spaced time intervals. Without loss of generality, we normalize the time interval to 1, and Y will have the value Y_i at time i. During the observation an error ϵ_i is introduced so that the observed value y_i is

$$y_i = Y_i + \epsilon_i \; . \tag{83}$$

The error ϵ_i is assumed to be a zero-mean stochastic variable with standard deviation σ_ϵ. By definition,

$$\sigma_\epsilon^2 = \frac{1}{N} \sum_{1}^{N} \epsilon_i^2 \; . \tag{84}$$

when a least-squares polynomial $\bar{y}_i$ of order $k - 1$ is fitted to the data, the residuals v_i can be determined:

$$v_i = y_i - \bar{y}_i \tag{85}$$

and thus

$$\epsilon_i = v_i - (Y_i - \bar{y}_i) \; . \tag{86}$$

The conditions for a least-squares fit, Equation 6, may also be written

$$
\begin{aligned}
&\sum_{1}^{N} v_i = 0 \\
&\sum_{1}^{N} i\, v_i = 0 \\
&\quad \vdots \\
&\sum_{1}^{N} i^{k-1}\, v_i = 0 \quad .
\end{aligned}
\tag{87}
$$

From the summation of Equation 86 we obtain

$$
\sum_{1}^{N} \epsilon_i = \sum_{1}^{N} v_i - \sum_{1}^{N} (Y_i - \bar{y}_i)
\tag{88}
$$

or, using Equation 87,

$$
\sum_{1}^{N} \epsilon_i = - \sum_{1}^{N} (Y_i - \bar{y}_i)
\tag{89}
$$

and, by squaring Equation 86 before the summation,

$$
\sum_{1}^{N} \epsilon_i^2 = \sum_{1}^{N} v_i^2 - 2 \sum_{1}^{N} v_i (Y_i - \bar{y}_i) + \sum_{1}^{N} (Y_i - \bar{y}_i)^2 \quad .
\tag{90}
$$

If we assume that Y_i is generated by a polynomial

$$
Y_i = \alpha_0 + \alpha_1 i + \cdot \cdot \cdot + \alpha_{k-1} i^{k-1}
\tag{91}
$$

of degree $k - 1$ or lower, then

$$
Y_i - \bar{y}_i = (\alpha_0 - a_0) + (\alpha_1 - a_1) i + \cdot \cdot \cdot + (\alpha_{k-1} - a_{k-1}) i^{k-1}
\tag{92}
$$

and hence, from Equation 87,

$$
\sum_{1}^{N} v_i (Y_i - \bar{y}_i) = 0 \quad ,
$$

and Equation 90 reduces to

$$\sum_{1}^{N} \epsilon_i^2 = \sum_{1}^{N} v_i^2 + \sum_{1}^{N} (Y_i - \overline{y}_i)^2 \ . \tag{93}$$

In Appendix F it is shown that

$$\sum_{1}^{N} (Y_i - \overline{y}_i)^2 = \frac{k}{N} \sum_{1}^{N} \epsilon_i^2 \ , \tag{94}$$

if the ϵ_i are uncorrelated.

Hence

$$\sigma_\epsilon^2 = \frac{1}{N} \sum_{1}^{N} \epsilon_i^2 = \frac{1}{N-k} \sum_{1}^{N} v_i^2 \ , \tag{95}$$

if the ϵ_i are uncorrelated and if Y_i is a polynomial of equal or lower degree than $\overline{y}_i$.

The Maximum Error in a Polynomial Fit

An interesting question is: What error function ϵ_i produces the largest error in $\overline{y}_i$ if the errors are subject to the side condition

$$\sum_{1}^{N} \epsilon_i^2 = N\,\overline{\epsilon}^2 \ , \tag{96}$$

where $\overline{\epsilon}$ is a constant?

Equation 12 may be written

$$\Delta \overline{y}_i = -\frac{1}{|A|} \sum_{1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j = -\frac{1}{|A|} \sum_{1}^{N} c_j \, \epsilon_j \ . \tag{97}$$

Using the method of Lagrange's multipliers, we obtain for the maximization of $\Delta\overline{y}_i$

$$\frac{\partial}{\partial \epsilon_j} \left[\sum_{1}^{N} c_j \, \epsilon_j + \lambda \sum_{1}^{N} \epsilon_j^2 \right] = 0 \ . \tag{98}$$

Hence

$$c_1 + 2\lambda\,\epsilon_1 = 0$$
$$c_2 + 2\lambda\,\epsilon_2 = 0 \tag{99}$$
$$\vdots$$
$$c_j + 2\lambda\,\epsilon_j = 0$$

and

$$\epsilon_j = \frac{c_j}{c_1}\,\epsilon_1 \ . \tag{100}$$

Thus, from Equation 96,

$$N\,\overline{\epsilon}^2 = \epsilon_1^2\left[1 + \left(\frac{c_2}{c_1}\right)^2 + \cdots\right] = \frac{\epsilon_1^2}{c_1^2}\sum_1^N c_j^2$$

or

$$\epsilon_1 = \pm\frac{c_1\sqrt{N}\,\overline{\epsilon}}{\sqrt{\Sigma\,c_j^2}}$$

and

$$\epsilon_j = \pm\frac{c_j\sqrt{N}\,\overline{\epsilon}}{\sqrt{\Sigma\,c_j^2}} \ . \tag{101}$$

From Equation 97,

$$\Delta\overline{y}_{i\,max} = \frac{\sqrt{N}\,\overline{\epsilon}}{|A|}\,\sqrt{\Sigma\,c_j^2} \ .$$

But

$$\sum_1^N c_j^2 = \sum_1^N \begin{vmatrix} 0 & j \\ i & A \end{vmatrix}^2 = -\,|A|\,\begin{vmatrix} 0 & i \\ i & A \end{vmatrix}$$

and, using Appendix C, Equation C10,

$$\Sigma\,c_j^2 = \frac{1}{N}\,|A|^2\,\left[P_0^2(u) + 3\,P_1^2(u) + \cdots + (2k-1)\,P_{k-1}^2(u)\right] \ .$$

Thus

$$\Delta\bar{y}_{i\,max} = \bar{\epsilon}\sqrt{P_0^2(u) + 3P_1^2(u) + \cdots + (2k-1)\,P_{k-1}^2(u)}\ , \tag{102}$$

where

$$i = N\,\frac{u+1}{2}\ .$$

Comparing this result with Equation 17, we see that with $\bar{\epsilon} = \sigma_\epsilon$ the maximum error $\Delta\bar{y}_{i\,max}$ is $\sqrt{N}$ times larger than η_i for uncorrelated noise.

The error function is given by Equation 101, which also may be written

$$\epsilon_j = \frac{N\bar{\epsilon}}{|A|\sqrt{P_0^2(u) + 3P_1^2(u) + \cdots + (2k-1)\,P_{k-1}^2(u)}}\begin{vmatrix} 0 & j \\ i & A \end{vmatrix}\ , \tag{103}$$

where i is the point at which $\Delta\bar{y}_i$ is maximized.

GENERALIZATIONS

The polynomial in Equation 4 may be written

$$\bar{y}_i = \sum_{j=1}^{k} (i)^{j-1}\, a_{j-1}\ , \tag{104}$$

where k is the number of degrees of freedom. Equation 104 may be generalized to

$$\bar{y}_i = \sum_{j=1}^{k} Z_{ij}\, a_j\ , \tag{105}$$

where Z_{ij} are function of i and j. In this form Equation 105 includes linearized nonlinear systems such as described in Reference 6, Appendix B. The sums of the residuals is

$$s = \sum_{i=1}^{N}\left(\sum_{j=1}^{k} Z_{ij}\, a_i - y_i\right)^2\ . \tag{106}$$

where the summation in i is taken over the N measured values y_i.

The conditions for a minimum are

$$\frac{\partial s}{\partial a_\nu} = 2\sum_{i=1}^{N}\left(\sum_{j=1}^{k} Z_{ij}\ a_j - y_i\right) Z_{i\nu} = 0$$

or

$$\sum_{i=1}^{N}\sum_{j=1}^{k} Z_{ij}\ Z_{i\nu}\ a_j = \sum_{i=1}^{N} y_i\ Z_{i\nu}\ , \tag{107}$$

where ν takes the integer values from 1 to k. The sums are interchangeable, and Equation 107 may be written

$$\sum_{j=1}^{k} c_{\nu j}\ a_j = \sum_{i=1}^{N} y_i\ Z_{i\nu}\ , \tag{108}$$

where

$$c_{\nu j} = c_{j\nu} = \sum_{i=1}^{N} Z_{ij}\ Z_{i\nu}\ . \tag{109}$$

Solving for a_j yields

$$a_j = \frac{1}{|C|}\begin{vmatrix} C_{11} & C_{12} & \cdots & \Sigma y_i\ Z_{i1} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & \Sigma y_i\ Z_{i2} & \cdots & C_{2k} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ C_{k1} & C_{k2} & \cdots & \Sigma y_i\ Z_{ik} & \cdots & C_{kk} \end{vmatrix}\ , \tag{110}$$

where

$$|C| = \begin{vmatrix} C_{11} & \cdots & C_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ C_{k1} & \cdots & C_{kk} \end{vmatrix}\ .$$

Insertion in Equation 105 yields

$$\bar{y}_i = \frac{1}{|C|}\sum_{j=1}^{k} Z_{ij} \sum_{i=1}^{N} \begin{vmatrix} C_{11} & C_{12} & \cdots & Z_{i1} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & Z_{i2} & \cdots & C_{2k} \\ \vdots & \vdots & & \vdots & & \\ C_{k1} & C_{k2} & \cdots & Z_{ik} & \cdots & C_{kk} \end{vmatrix} y_i$$

or

$$\bar{y}_i = -\frac{1}{|C|}\sum_{r=1}^{N} \begin{vmatrix} 0 & Z_{i1} & Z_{i2} & \cdots & Z_{ik} \\ Z_{r1} & C_{11} & C_{12} & \cdots & C_{1k} \\ Z_{r2} & C_{21} & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_{rk} & C_{k1} & C_{k2} & \cdots & C_{kk} \end{vmatrix} y_r \, , \tag{111}$$

where the summation index r is identical with the previously used summation index i. In Equation 111 we recognize the generalized form of Equation 8. In analogy to Equation 10, the brief notation

$$\bar{y}_i = -\frac{1}{|C|}\sum_{r=1}^{N} \begin{vmatrix} 0 & Z_i \\ Z_r & C \end{vmatrix} y_r \tag{112}$$

is introduced.

For uncorrelated noise with standard deviation σ_ϵ, we obtain the standard deviation η_i of the estimate $\bar{y}_i$:

$$\eta_i^2 = \frac{\sigma_\epsilon^2}{|C|^2}\sum_{r=1}^{N} \begin{vmatrix} 0 & Z_i \\ Z_r & C \end{vmatrix}^2 = \frac{-1}{|C|}\begin{vmatrix} 0 & Z_i \\ Z_i & C \end{vmatrix} \sigma_\epsilon^2 \, , \tag{113}$$

which is the generalized form of Equation 14.

For the rms standard deviation $\bar{\eta}$, defined by

$$\bar{\eta}^2 = \frac{1}{N}\sum_{i=1}^{N} \eta_i^2 \, ,$$

we obtain

$$\overline{\eta}^2 = -\frac{\sigma_\epsilon^2}{N|C|}\sum_i \begin{vmatrix} 0 & Z_i \\ Z_i & C \end{vmatrix}$$

or

$$\overline{\eta}^2 = \frac{k}{N}\sigma_\epsilon^2 . \tag{114}$$

This equation is identical with Equation 23, which thus holds true for the generalized case.

We have found that Equations 8, 14, and 23 hold true for the generalized case. In particular, they hold true for polynomials fitted to unequally spaced data. It should be pointed out that no approximations have been made in the derivation of these equations; they are exact.

The generalization of the equations for correlated data is unfortunately not possible, one of the reasons being that ρ_i is not defined for unequally spaced data.

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National Aeronautics and Space Administration
Greenbelt, Maryland, January 31, 1967
311-02-89-01-51

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Appendix A

Derivation of $\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} y_j$

From the Equations 7, where all summations are from 1 to N

$$
\begin{aligned}
& a_0 \Sigma j^0 + a_1 \Sigma j^1 + \cdots + a_{k-1} \Sigma j^{k-1} = \Sigma j^0 y_j \\
& a_0 \Sigma j^1 + a_1 \Sigma j^2 + \cdots + a_{k-1} \Sigma j^k = \Sigma j y_j \\
& \vdots \\
& a_0 \Sigma j^{k-1} + a_1 \Sigma j^k + \cdots + a_{k-1} \Sigma j^{2k-2} = \Sigma j^{k-1} y_j \quad ,
\end{aligned}
$$

we can solve for the coefficients a_ν. With

$$A_\nu = \sum_{j=1}^{N} j^\nu$$

we obtain

$$a_\nu = \frac{1}{|A|} \begin{vmatrix} A_0 & A_1 & \cdots & A_{\nu-1} & \Sigma y_j & A_{\nu+1} & \cdots & A_{k-1} \\ A_1 & A_2 & \cdots & A_\nu & \Sigma j y_i & A_{\nu+2} & \cdots & A_k \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{k-1} & A_k & \cdots & A_{k+\nu-2} & \Sigma j^{k+\nu -} y_j & A_{k+\nu} & \cdots & A_{2k-2} \end{vmatrix} \tag{A1}$$

where

$$|A| = \begin{vmatrix} A_0 & \cdots & A_{k-1} \\ \vdots & & \vdots \\ A_{k-1} & \cdots & A_{2k-2} \end{vmatrix} \quad . \tag{A2}$$

Thus

$$\bar{y}_i = \frac{1}{|A|}\left\{ \begin{vmatrix} \Sigma y_j & A_1 \cdots A_{k-1} \\ \Sigma j y_j & A_2 \cdots A_k \\ \vdots & \vdots \quad \vdots \\ \Sigma j^{k-1} y_j & A_k \cdots A_{2k-2} \end{vmatrix} + \begin{vmatrix} A_0 & \Sigma y_j & A_2 \cdots A_{k-1} \\ A_1 & \Sigma j y_j & A_3 \cdots A_k \\ \vdots & \vdots & \vdots \quad \vdots \\ A_{k-1} & \Sigma j^{k-1} y_j & A_{k+1} \cdots A_{2k-2} \end{vmatrix} i + \cdots \right.$$

$$\left. + \begin{vmatrix} A_0 \cdots A_{k-2} & \Sigma y_j \\ A_1 \cdots A_{k-1} & \Sigma j y_j \\ \vdots \quad \vdots & \vdots \\ A_{k-1} \cdots A_{2k-3} & \Sigma j^{k-1} y_j \end{vmatrix} i^{k-1} \right\}, \tag{A3}$$

but

$$\begin{vmatrix} A_0 & \Sigma y_j & \cdots & A_{k-1} \\ A_1 & \Sigma j y_j & \cdots & A_k \\ \vdots & \vdots & & \vdots \\ A_{k-1} & \Sigma j^{k-1} y_j & \cdots & A_{2k-2} \end{vmatrix} = - \begin{vmatrix} \Sigma y_j & A_0 \cdots A_{k-1} \\ \Sigma j y_j & A_1 \cdots A_k \\ \vdots & \vdots \quad \vdots \\ \Sigma y^{k-1} y_j & A_{k-1} \cdots A_{2k-2} \end{vmatrix}. \tag{A4}$$

By rewriting all the determinants with the sums in the first column we obtain

$$\bar{y}_i = \frac{-1}{|A|} \begin{vmatrix} 0 & 1 & i \cdots i^{k-1} \\ \Sigma y_j & A_0 & A_1 \cdots A_{k-1} \\ \Sigma j y_j & A_1 & A_2 \cdots A_k \\ \vdots & \vdots & \vdots \quad \vdots \\ \Sigma j^{k-1} y_j & A_{k-1} & A_k \cdots A_{2k-2} \end{vmatrix}. \tag{A5}$$

Expanding the determinant after the first column reveals that the summation may be moved outside the determinant. y_j is a common factor in the first column and may also be moved outside the determinant. Thus

$$\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & 1 & i \cdots i^{k-1} \\ 1 & A_0 & A_1 \cdots A_{k-1} \\ j & A_1 & A_2 \cdots A_k \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ j^{k-1} & A_2 & A_3 \cdots A_{2k-2} \end{vmatrix} \cdot y_j \tag{A6}$$

or, with shorter notations,

$$\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & i \\ j & A \end{vmatrix} \cdot y_j \quad . \tag{A7}$$

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Appendix B

The Summation of $\sum_{j=1}^{N} \begin{vmatrix} 0 & i \\ j & A \end{vmatrix}^2$

Writing out the determinant in more detail, we have

$$
\sum_{j=1}^{N} \begin{vmatrix} 0 & i \\ j & A \end{vmatrix}^2 = \sum_{j=1}^{N} \begin{vmatrix} 0 & 1 & i & i^2 \cdots \\ 1 & A_0 & A_1 & A_2 \cdots \\ j^1 & A_1 & A_2 & A_3 \cdots \\ j^2 & A_2 & A_3 & A_4 \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}^2 = \sum_{j=1}^{N} \begin{vmatrix} 0 & 1 \quad i \cdots \\ \begin{vmatrix} 0 & 1 \quad i \cdots \\ 1 & \\ j & A \\ \cdot & \\ \cdot & \\ \cdot & \end{vmatrix} & A_0 \quad A_1 \cdots \\ \begin{vmatrix} 0 & 1 \quad i \cdots \\ j & \\ j^2 & A \\ \cdot & \\ \cdot & \\ \cdot & \end{vmatrix} & A_1 \quad A_2 \cdots \\ \begin{vmatrix} 0 & 1 \quad i \cdots \\ j^2 & \\ j^3 & A \\ \cdot & \\ \cdot & \\ \cdot & \end{vmatrix} & A_2 \quad A_3 \cdots \\ \vdots & \vdots \quad \vdots \end{vmatrix} \tag{B1}
$$

from the rule that

$$
Z \begin{vmatrix} b_0 & b_1 & b_2 \cdots \\ b_1 & b_2 & b_3 \cdots \\ b_2 & b_3 & b_4 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} Z\,b_0 & b_1 & b_2 \cdots \\ Z\,b_1 & b_2 & b_3 \cdots \\ Z\,b_2 & b_3 & b_4 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} .
$$

If the summation variable only occurs in a row or a column, the summation over the whole determinant may be replaced by summation of the elements of the row or column, thus:

$$\sum_{j=1}^{N} \begin{vmatrix} 1 & b_0 & b_1 & \cdots \\ j & b_1 & b_2 & \cdots \\ j^2 & b_2 & b_3 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} = \begin{vmatrix} \Sigma j^0 & b_0 & b_1 & \cdots \\ \Sigma j & b_1 & b_2 & \cdots \\ \Sigma j^2 & b_2 & b_3 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} .$$

This is easily verified by expansion into minors.

By applying this rule twice and observing that by definition

$$\sum_{j=1}^{N} j^{\nu} = A_{\nu} ,$$

we obtain

$$\sum_{j=1}^{N} \begin{vmatrix} 0 & i \\ j & A \end{vmatrix}^2 = \begin{vmatrix} 0 & 1 & i & \cdots \\ \begin{vmatrix} 0 & 1 & i & \cdots \\ A_0 & A_0 & A_1 & \cdots \\ A_1 & A_1 & A_2 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} & A_0 & A_1 & \cdots \\ \begin{vmatrix} 0 & 1 & i & \cdots \\ A_1 & A_0 & A_1 & \cdots \\ A_2 & A_1 & A_2 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} & A_1 & A_2 & \cdots \\ \begin{vmatrix} 0 & 1 & i & \cdots \\ A_2 & A_0 & A_1 & \cdots \\ A_3 & A_1 & A_2 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} & A_2 & A_3 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} . \tag{B2}$$

But

$$\begin{vmatrix} 0 & 1 & i & \cdots \\ A_0 & A_0 & A_1 & \cdots \\ A_1 & A_1 & A_2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} = -|A| \; ,$$

$$\begin{vmatrix} 0 & 1 & i & \cdots \\ A_1 & A_0 & A_1 & \cdots \\ A_2 & A_1 & A_2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} = \begin{vmatrix} A_1 & A_0 & A_2 & \cdots \\ A_2 & A_1 & A_3 & \cdots \\ A_3 & A_2 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} i = -|A| \, i \; ,$$

and so on. Thus

$$\sum_{j=1}^{N} \begin{vmatrix} 0 & i \\ j & A \end{vmatrix}^2 = \begin{vmatrix} 0 & 1 & i & \cdots \\ -|A| & A_0 & A_1 & \cdots \\ -|A| \, i & A_1 & A_2 & \cdots \\ -|A| \, i^2 & A_2 & A_3 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} = -|A| \cdot \begin{vmatrix} 0 & i \\ i & A \end{vmatrix} \; . \tag{B3}$$

Summation over i yields

$$\sum_{i=1}^{N} \begin{vmatrix} 0 & 1 & i & i^2 & \cdots \\ 1 & A_0 & A_1 & A_2 & \cdots \\ i & A_i & A_2 & A_3 & \cdots \\ i^2 & A_2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} =$$

$$- \sum_{i=1}^{N} \begin{vmatrix} 1 & A_1 & A_2 & \cdots \\ i & A_2 & A_3 & \cdots \\ i^2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} + \sum_{i=1}^{N} \begin{vmatrix} i & A_0 & A_2 & \cdots \\ i^2 & A_1 & A_3 & \cdots \\ i^3 & A_2 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} - \sum_{i=1}^{N} \begin{vmatrix} i^2 & A_0 & A_1 & \cdots \\ i^3 & A_1 & A_2 & \cdots \\ i^4 & A_2 & A_3 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} + \cdots$$

$$
= - \begin{vmatrix} A_0 & A_1 & A_2 & \cdots \\ A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} + \begin{vmatrix} A_1 & A_0 & A_2 & \cdots \\ A_2 & A_1 & A_3 & \cdots \\ A_3 & A_2 & A_4 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} - \begin{vmatrix} A_2 & A_0 & A_1 & \cdots \\ A_3 & A_1 & A_2 & \cdots \\ A_4 & A_2 & A_3 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} + \cdots
$$

$$
= - k \, |A| \, , \tag{B4}
$$

where k is the degree of freedom of the least-squares fitted polynomial. Thus

$$
\sum_{i=1}^{i=N} \sum_{j=1}^{j=N} \begin{vmatrix} 0 & i \\ j & A \end{vmatrix}^2 = k \, |A|^2 \ . \tag{B5}
$$

Appendix C

Evaluation of $\dfrac{\begin{vmatrix} 0 & i \\ i & A \end{vmatrix}}{|A|}$ for Large N.

Equation 9 defines A_ν:

$$A_\nu = \sum_{i=1}^{N} i^\nu \; . \tag{9}$$

It is shown in Reference 3, page 4, that

$$A_\nu = \sum_{1}^{N} i^\nu = N^{\nu+1} \left(\frac{1}{\nu+1} + \frac{1}{2N} + \frac{\nu}{12\,N^2} + \cdots \right) \tag{C1}$$

and, for large N,

$$A_\nu \approx \frac{N^{\nu+1}}{\nu+1} \; . \tag{C2}$$

This is the same approximation as

$$A_\nu \approx \int_0^N x^\nu \, dx = \frac{N^{\nu+1}}{\nu+1} \; . \tag{C3}$$

The error introduced in $|A|$ by this approximation is of magnitude N^{-2}. By subtracting rows and columns in a suitable fashion, the N^ν term may be eliminated:

$$\begin{array}{r} \\ \\ \\ \\ \text{row p} \\ \text{row p + 1} \\ \\ \\ \\ \end{array} \begin{vmatrix} A_0 & & A_{\ell-1} & A_\ell & \cdots \\ \cdot & & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \\ A_{p-1} & \cdots & A_{\ell+p-2} & A_{\ell+p-1} & \cdots \\ A_p & \cdots & A_{\ell+p-1} & A_{\ell+p} & \cdots \\ \cdot & & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \end{vmatrix} .$$

Subtract $p/2$ times row p from row $p+1$. Thereafter, subtract $\ell/2$ times column ℓ from column $\ell + 1$. The element of row $p+1$ and column $\ell + 1$ of $|A|$ is then

$$A_{\ell+p} - \frac{p}{2} A_{\ell+p-1} - \frac{\ell}{2}\left(A_{\ell+p-1} - \frac{p}{2} A_{\ell+p-2}\right)$$

$$= \frac{N^{\ell+p+1}}{\ell+p+1} + \frac{N^{\ell+p}}{2} + \cdot\cdot\cdot\cdot - \frac{p}{2}\left(\frac{N^{\ell+p}}{\ell+p} + \cdot\cdot\cdot\right) - \frac{\ell}{2}\left(\frac{N^{\ell+p}}{\ell+p} + \cdot\cdot\cdot\right)$$

$$= \frac{N^{\ell+p+1}}{\ell+p+1} + \text{terms of order } N^{\ell+p-1} \text{ and lower.}$$

The error introduced by the approximation is thus of order N^{-2}. With the above approximation,

$$\begin{vmatrix} 0 & i \\ i & A \end{vmatrix} = \begin{vmatrix} 0 & 1 & i & \cdots & i^{k-1} \\ 1 & N & \frac{N^2}{2} & \cdots & \frac{N^k}{k} \\ i & \frac{N}{2} & \frac{N^3}{3} & \cdots & \frac{N^{k+1}}{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ i^{k-1} & \frac{N^k}{k} & \frac{N^{k+1}}{k+1} & \cdots & \frac{N^{2k-1}}{2k-1} \end{vmatrix} \approx N^{k^2+1} \begin{vmatrix} 0 & 1 & x & \cdots & x^{k-1} \\ 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ x & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{k-1} & \frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix}, \qquad \text{(C4)}$$

where $x = i/N$. For more rapid evaluation of the determinant, we make the substitution

$$x = \frac{u+1}{2}$$

and observe that

$$\begin{vmatrix} 0 & 1 & \left(\frac{u+1}{2}\right) & \cdots & \left(\frac{u+1}{2}\right)^{k-1} \\ 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ \left(\frac{u+1}{2}\right) & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \left(\frac{u+1}{2}\right)^2 & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \left(\frac{u+1}{2}\right)^{k-1} & \frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} = \frac{1}{2^{k^2-k}} \begin{vmatrix} 0 & 1 & u & u^2 & \cdots & u^{k-1} \\ 1 & 1 & 0 & \frac{1}{3} & \cdots & \\ u & 0 & \frac{1}{3} & 0 & \cdots & \\ u^2 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \\ u^{k-1} & & & & & \end{vmatrix}. \qquad \text{(C5)}$$

This result may be verified by subtracting rows and columns in a suitable fashion. For $|A|$ we have

$$|A| = \begin{vmatrix} N & \frac{N^2}{2} & \cdots & \frac{N^k}{k} \\ \frac{N^2}{2} & \frac{N^3}{3} & \cdots & \frac{N^{k+1}}{k+1} \\ \vdots & \vdots & & \vdots \\ \frac{N^k}{k} & \frac{N^{k+1}}{k+1} & \cdots & \frac{N^{2k-1}}{2k-1} \end{vmatrix} = N^{k^2} \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} . \tag{C6}$$

From Reference 7, pages 429 to 431, we obtain

$$|A| = N^{k^2} \frac{[2!\ 3!\ \cdots\ (k-1)!]^3}{k!\ (k+1)!\ \cdots\ (2k-1)!} . \tag{C7}$$

Combining Equations C4, C5 and C7, we obtain

$$\frac{\begin{vmatrix} 0 & i \\ i & A \end{vmatrix}}{|A|} = \frac{k!\ (k+1)!\ \cdots\ (2k-1)!}{N \cdot 2^{k(k-1)}\ [2!\ 3!\ \cdots\ (k-1)!]^3} \begin{vmatrix} 0 & 1 & u & \cdots & u^{k-1} \\ 1 & 1 & 0 & \cdots & \\ u & 1 & \frac{1}{3} & \cdots & \\ \vdots & \vdots & \vdots & & \\ u^{k-1} & & & & \end{vmatrix} . \tag{C8}$$

By subtracting the second column from the other columns we obtain

$$\begin{vmatrix} 0 & 1 & u & u^2 & \cdots \\ 1 & 1 & 0 & \frac{1}{3} & \cdots \\ u & 0 & \frac{1}{3} & 0 & \cdots \\ u^2 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} = \begin{vmatrix} -1 & 1 & u & u^2 - \frac{1}{3} & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ u & 0 & \frac{1}{3} & 0 & \cdots \\ u^2 - \frac{1}{3} & \frac{1}{3} & 0 & \frac{4}{45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} ,$$

and the order of the determinant is reduced by 1.

Repeating the process, we obtain

$$\frac{\begin{vmatrix} 0 & i \\ i & A \end{vmatrix}}{|A|} = -\frac{1}{N}\left[1 + 3u^2 + 5\left(\frac{3u^2 - 1}{2}\right)^2 + 7\left(\frac{5u^3 - 3u}{2}\right)^2 \right.$$

$$\left. + 9\left(\frac{35u^4 - 30u^3 + 3}{8}\right)^2 + 11\left(\frac{63u^5 - 70u^3 + 15u}{8}\right)^2 + \cdots\right] . \tag{C9}$$

The polynomials in the squared parenthesis are the Legendre polynomials; thus

$$\frac{\begin{vmatrix} 0 & i \\ i & A \end{vmatrix}}{|A|} = -\frac{1}{N}\left[P_0^2(u) + 3P_1^2(u) + \cdots + (2k-1)P_{k-1}^2(u)\right] , \tag{C10}$$

where

$$i = N\frac{u+1}{2} .$$

Appendix D

Evaluation of $\bar{\eta}$ for Correlated Data

From Equation 27,

$$\bar{\eta}^2 = \frac{1}{N|A|^2}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\begin{vmatrix}0 & j\\ i & A\end{vmatrix}^2 + \cdots + 2\rho_\nu \sum_{i=1}^{N}\sum_{j=1}^{N-\nu}\begin{vmatrix}0 & j\\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & j+\nu\\ i & A\end{vmatrix} + \cdots\right].$$

With the substitution $\ell = j + \nu$ we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{N-\nu}\begin{vmatrix}0 & j\\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & j+\nu\\ i & A\end{vmatrix} = \sum_{i=1}^{N}\sum_{\ell=\nu+1}^{N}\begin{vmatrix}0 & \ell-\nu\\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & \ell\\ i & A\end{vmatrix} =$$

$$\sum_{i=1}^{N}\sum_{\ell=1}^{N}\begin{vmatrix}0 & \ell-\nu\\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & \ell\\ i & A\end{vmatrix} - \sum_{i=1}^{N}\sum_{\ell=1}^{\nu}\begin{vmatrix}0 & \ell-\nu\\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & \ell\\ i & A\end{vmatrix} \qquad \text{(D1)}$$

$$\sum_{i=1}^{N}\sum_{\ell=1}^{N}\begin{vmatrix}0 & \ell-\nu\\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & \ell\\ i & A\end{vmatrix} = \sum_{i=1}^{N}\sum_{\ell=1}^{N}\begin{vmatrix} 0 & 1 & \ell-\nu & \cdots \\ \begin{vmatrix}0 & 1 & \ell & \cdots\\ 1 & & & \\ i & A & & \\ \cdot & & & \\ \cdot & & & \end{vmatrix} & A_0 & A_1 & \cdots \\ \begin{vmatrix}0 & 1 & \ell & \cdots\\ i & & & \\ i^2 & A & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{vmatrix} & A_1 & A_2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

$$= - |A| \sum_{\ell=1}^{N} \begin{vmatrix} 0 & 1 & \ell-\nu & (\ell-\nu)^2 \cdots \\ 1 & & & \\ \ell & & & \\ \ell^2 & & A & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{vmatrix}$$

$$= - |A| \sum_{\ell=1}^{N} \left\{ \begin{vmatrix} 0 & 1 & \ell & \ell^2 \cdots \\ 1 & & & \\ \ell & & A & \\ \ell^2 & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{vmatrix} + \begin{vmatrix} 0 & 0 & -\nu & -2\nu\ell+\nu^2 \cdots \\ 1 & A_0 & A_1 & A_2 \\ \ell & A_1 & A_2 & A_3 \\ \ell^2 & A_2 & A_3 & A_4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \right\} . \qquad \text{(D2)}$$

The summation of the first determinant is given by Equation B4. The second determinant is expanded in minors after the first row:

$$\sum_{\ell=1}^{N} \begin{vmatrix} 0 & 0 & -\nu & -2\nu\ell+\nu^2 \cdots \\ 1 & A_0 & A_1 & A_2 \\ \ell & A_1 & A_2 & A_3 \\ \ell^2 & A_2 & A_3 & A_4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = -\nu \sum_{\ell=1}^{N} \begin{vmatrix} 1 & A_0 & A_2 \cdots \\ \ell & A_1 & A_3 \cdots \\ \ell^2 & A_2 & A_4 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$+ 2\nu \sum_{\ell=1}^{N} \begin{vmatrix} \ell & A_0 & A_1 & \cdots \\ \ell^2 & A_1 & A_2 & \cdots \\ \ell^3 & A_2 & A_3 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix} - \nu^2 \sum_{\ell=1}^{N} \begin{vmatrix} 1 & A_0 & A_1 \cdots \\ \ell & A_1 & A_2 \cdots \\ \ell^2 & A_2 & A_3 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \cdots$$

$$= -\nu \begin{vmatrix} A_0 & A_0 & A_2 \cdots \\ A_1 & A_1 & A_3 \cdots \\ A_2 & A_2 & A_4 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + 2\nu \begin{vmatrix} A_1 & A_0 & A_1 \cdots \\ A_2 & A_1 & A_2 \cdots \\ A_3 & A_1 & A_3 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} - \nu^2 \sum \begin{vmatrix} A_0 & A_0 & A_1 \cdots \\ A_1 & A_1 & A_2 \cdots \\ A_2 & A_2 & A_3 \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \cdots = 0 , \qquad \text{(D3)}$$

because two columns are identical in each determinant. In the same manner we may verify that all the other minors are zero. Hence

$$\sum_{i=1}^{N}\sum_{\ell=1}^{N}\begin{vmatrix}0 & \ell-\nu \\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & \ell \\ i & A\end{vmatrix} = -|A|\sum_{\ell=1}^{N}\begin{vmatrix}0 & \ell \\ \ell & A\end{vmatrix} = k\,|A|^2 . \tag{D4}$$

For the second term in Equation D1 we obtain

$$\sum_{i=1}^{N}\sum_{\ell=1}^{\nu}\begin{vmatrix}0 & \ell-\nu \\ i & A\end{vmatrix}\cdot\begin{vmatrix}0 & \ell \\ i & A\end{vmatrix} = -|A|\sum_{\ell=1}^{\nu}\begin{vmatrix}0 & \ell-\nu \\ \ell & A\end{vmatrix} . \tag{D5}$$

Expansion in minors of A yields

$$\sum_{\ell=1}^{\nu}\begin{vmatrix}0 & \ell-\nu \\ \ell & A\end{vmatrix} = \sum_{\ell=1}^{\nu}\left\{-\Delta_{11} + (2\ell-\nu)\Delta_{12} - [(\ell-\nu)^2+\ell^2]\,\Delta_{13} - \ell(\ell-\nu)\,\Delta_{22} + \cdots\right\}$$

$$= -\nu\left(\Delta_{11} - \Delta_{12} + \frac{1}{3}\,\Delta_{13} + \cdots\right) + \frac{\nu^3}{6}\,(\Delta_{22} - 4\Delta_{13} + \cdots) + \cdots , \tag{D6}$$

observing that $\Delta_{ij} = \Delta_{ji}$, where Δ_{ij} are the minors of $|A|$. With the approximation $A_\nu = N^{\nu+1}/\nu + 1$, we find in the same manner as in Equation C6 that

Δ_{11} is of order k^2-1 in N

Δ_{12}, Δ_{21} are of order k^2-2 in N

$\Delta_{13}, \Delta_{22}, \Delta_{31}$ are of order k^2-3 in N .

Thus, taking the dominating terms only,

$$\sum_{i=1}^{N}\sum_{\ell=1}^{\nu}\begin{vmatrix}0 & \ell-\nu \\ \ell & A\end{vmatrix}\begin{vmatrix}0 & \ell \\ \ell & A\end{vmatrix} = \nu\Delta_{11}\,|A| - \frac{\nu^3}{6}(\Delta_{22} - 4\Delta_{13})\,|A| + \cdots . \tag{D7}$$

With the approximation $A_\nu = N^{\nu+1}/\nu + 1$ we have

$$\Delta_{11} = N^{k^2-1} \begin{vmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+1} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{k+2} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots & \frac{1}{k+3} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \frac{1}{k+3} & \cdots & \frac{1}{2k-1} \end{vmatrix} . \tag{D8}$$

The rows are multiplied by factors so that 1's are obtained in the last column:

$$\begin{vmatrix} \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \vdots & & \vdots \\ \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} = \frac{k!}{(2k-1)!} \begin{vmatrix} \frac{k+1}{3} & \cdots & \frac{k+1}{k} & 1 \\ \vdots & & \vdots & \vdots \\ \frac{2k-1}{k+1} & \cdots & \frac{2k-1}{2k-2} & 1 \end{vmatrix}$$

and, by subtracting the bottom row from the other rows,

$$\begin{vmatrix} \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \vdots & & \vdots \\ \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} = \frac{k!}{(2k-1)!} \begin{vmatrix} \frac{(k-2)^2}{3(k+1)} & \frac{(k-2)(k-3)}{4(k+2)} & \cdots & \frac{k-2}{k(2k-2)} & 0 \\ \frac{(k-2)(k-3)}{4(k+1)} & \frac{(k-3)^2}{5(k+2)} & \cdots & \frac{k-3}{(k+1)(2k-2)} & 0 \\ \vdots & & & \vdots & \\ \frac{2k-2}{k(k+1)} & \cdots & \cdots & \frac{1}{(2k-3)(2k-2)} & 0 \\ \frac{2k-1}{k+1} & \cdots & \cdots & \frac{2k-1}{2k-2} & 1 \end{vmatrix} .$$

Taking out common factors,

$$\begin{vmatrix} \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \vdots & & \vdots \\ \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} = \frac{[(k-2)!k!]^2}{(2k-2)!(2k-1)!} \begin{vmatrix} \frac{1}{3} & \cdots & \frac{1}{k} \\ \vdots & & \vdots \\ \frac{1}{k} & \cdots & \frac{1}{2k-3} \end{vmatrix} . \tag{D9}$$

Successive application of this recurrence formula yields (see also Reference 8),

$$\begin{vmatrix} \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \vdots & & \vdots \\ \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} = \frac{[2!3!\cdots(k-2)!]^3 (k-1)(k!)^2}{k!(k+1)!\cdots(2k-1)!} \tag{D10}$$

or, using Equations C7, D8, and D10,

$$\Delta_{11} = \frac{k^2}{N} |A| . \tag{D11}$$

With the same procedure it may be shown that

$$\Delta_{22} - 4\Delta_{13} = \frac{g(k)}{N^3} |A| ,$$

where $g(k)$ is a function of k alone. We thus obtain, in decreasing degree of N,

$$\bar{\eta} = \frac{1}{N|A|^2} \left[k|A|^2 + 2 \sum_{1}^{N-1} \rho_\nu \left(k|A|^2 - \nu \frac{k^2}{N} |A|^2 - \nu^3 \frac{g(k)}{N^3} |A|^2 + \ldots \right) \right] \sigma_\epsilon^2 \tag{D12}$$

or

$$\bar{\eta} = \frac{k\sigma_\epsilon^2}{N} \left[1 + 2 \sum_{1}^{N-1} \rho_\nu - \frac{2k}{N} \sum_{1}^{N-1} \nu \rho_\nu - \frac{g(k)}{N^3} \sum_{1}^{N-1} \nu^3 \rho_\nu + \cdots \right] . \tag{D13}$$

Appendix E

The Autocorrelation Function of Range Rate

It is shown in Reference 8 that the Doppler count $\Delta\phi$ may be written

$$\Delta\phi = \phi(t+h+T_p) - \phi(t+h) - [\phi(t+T_p) - \phi(t)] , \tag{E1}$$

where

$\Delta\phi$ = phase difference, measured by Doppler count

t = reference time

h = sampling interval

T_p = two-way propagation time

$\phi(t)$ = phase transmitted at time t.

We consider the case where the phase is contaminated by a random-walk phase noise ϕ_{RW}. The standard deviation $\sigma_{\phi_{RW}}$ of the noise accumulated during a time interval ΔT is

$$\sigma^2_{\phi_{RW}} = E[\phi_{RW}(t+\Delta T) - \phi_{RW}(t)]^2 = k_1^2 \Delta T , \tag{E2}$$

where k_1 is a constant.

We consider the case where the two-way propagation time is larger than the sampling interval h:

$$T_p > h .$$

This situation is shown in Figure E1. The standard deviation $\sigma_{\Delta\phi_{RW}}$ of the Doppler count is then, according to Equation E1,

$$\sigma^2_{\Delta\phi_{RW}} = E[\phi_{RW}(t+h+T_p) - \phi_{RW}(t+T_p)]^2 - E[\phi_{RW}(t+h) - \phi_{RW}(t)]^2$$

and, using Equation E2,

$$\sigma^2_{\Delta\phi_{RW}} = 2k_1^2 h . \tag{E3}$$

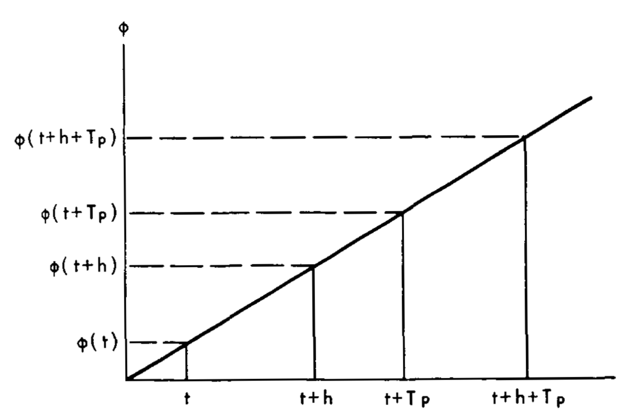

Figure E1—Phase vs. time for $h < T_p$.

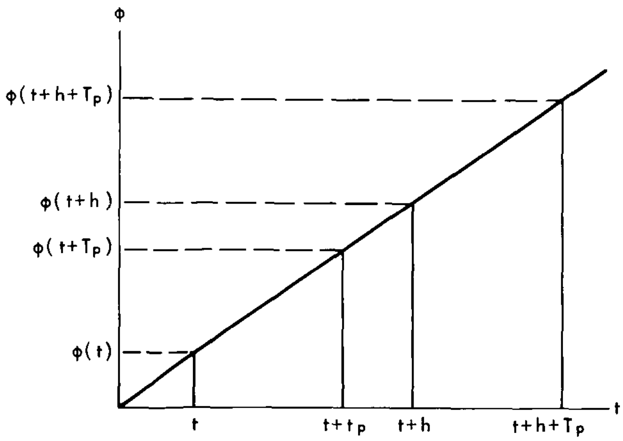

Figure E2—Phase vs. time for $T_p \leq h$.

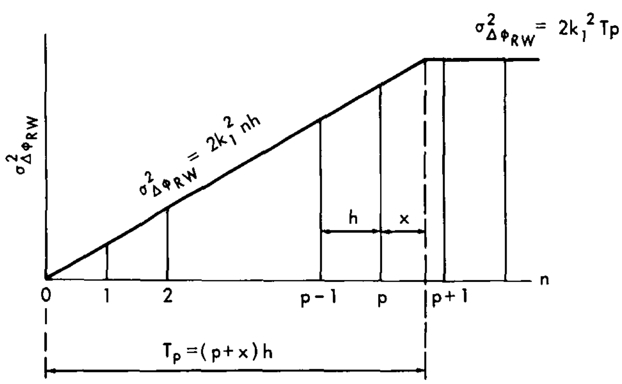

Figure E3—Standard deviation of phase noise due to random walk.

If, on the other hand, $T_p \leq h$, we obtain from Figure E2:

$$\sigma^2_{\Delta\phi_{RW}} = E[\phi_{RW}(t+h+T_p) - \phi_{RW}(t+h)]^2$$

$$+ E[\phi_{RW}(t+T_p) - \phi_{RW}(t)]^2$$

or

$$\sigma^2_{\Delta\phi_{RW}} = 2k_1^2 T_p \ . \tag{E4}$$

The standard deviation $\sigma_{\Delta\phi_{R_W}}$ for the Doppler count $\Delta\phi(nh)$ over n sampling intervals is obtained from Equations E3 and E4 and is shown in Figure E3 as a function of n. The two-way propagation delay is

$$T_p = (p+x)h \ , \tag{E5}$$

where

p = integer

$0 \leq x < 1$

h = sampling interval.

For the time interval $2h$, we obtain from Equation E3:

$$\sigma^2_{\Delta\phi_{RW}} = E\{\Delta\phi_{RW}(2h)\}^2 = 4k_1^2 h \ . \tag{E6}$$

But we can also consider the interval $2h$ as two intervals $h+h$. If ρ_1 is the correlation coefficient between these intervals, then

$$E\{\Delta\phi_{RW}(2h)\}^2 = E\{\Delta\phi_{RW}(h)\}^2 (1+1+2\rho_1) = 4k_1^2(1+\rho_1)h \ . \tag{E7}$$

Comparing Equations E6 and E7, we find:

$$\rho_1 = 0 \ . \tag{E8}$$

In the same manner

$$\rho_2 = 0$$
$$\vdots$$
$$\rho_{p-1} = 0 \ , \tag{E9}$$

where

ρ_ν = correlation coefficient between samples which are ν sampling intervals apart.

Consider now $p + 1$ sampling intervals. By summing $p + 1$ intervals, we obtain

$$E\{\Delta\phi_{RW} \, [(p+1)h]\}^2 = 2\,k_1^2\,(p + 1 + 2\rho_p)\,h \ . \tag{E10}$$

From Figure 12 and Equation E5, we also obtain

$$E\{\Delta\phi_{RW} \, [(p+1)h]\}^2 = 2\,k_1^2\,(p + x)\,h \tag{E11}$$

and hence

$$\rho_p = -\frac{1-x}{2} \ . \tag{E12}$$

For $p + 2$ sampling intervals, we obtain in the same manner

$$2\,k_1^2\,(p + 2 + 4\rho_p + 2\rho_{p+1})\,h = 2\,k_1^2\,(p + x)\,h$$

and thus

$$\rho_{p+1} = -\frac{x}{2} \ . \tag{E13}$$

For $p + 3$ sampling intervals, we obtain

$$2\,k_1^2\,(p + 3 + 6\rho_p + 4\rho_{p+1} + 2\rho_{p+2})\,h = 2\,k_1^2\,(p + x)\,h$$

or

$$\rho_{p+2} = 0 \ . \tag{E14}$$

In the same manner,

$$\rho_n = 0 \text{ for } n \geq p + 2 . \tag{E15}$$

If ΔR is the change in range during the sampling interval h, we may write the (average) range rate $\dot{R}$:

$$\dot{R} = \frac{\Delta R}{h} ,$$

where ΔR is proportional to $\Delta\phi$. The correlation coefficients or normalized autocorrelation functions for $\dot{R}$ are thus the same as for $\Delta\phi$. See also References 9 and 10.

Appendix F

The Evaluation of $\sum_{1}^{N} (Y_j - \bar{y}_j)^2$

From Equations 10 and 83,

$$\bar{y}_i = \frac{-1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} y_j$$

$$y_j = Y_j + \epsilon_j ,$$

we obtain

$$\bar{y}_i = -\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} (Y_j + \epsilon_j) . \qquad \text{(F1)}$$

It may be shown that

$$-\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} Y_j = Y_i ,$$

if Y_i is of equal or lower degree than $\bar{y}_i$. This result is self-evident because $\bar{y}_i = Y_i$ if all $\epsilon_i = 0$, or may be derived by evaluating the sum in the same manner as in Appendix A. We thus obtain

$$Y_i - \bar{y}_i = \frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j = \frac{1}{|A|} \begin{vmatrix} 0 & \Sigma j \epsilon_j \\ i & A \end{vmatrix} \qquad \text{(F2)}$$

and

$$\sum_{i=1}^{N} (Y_i - \bar{y}_i)^2 = \frac{1}{|A|^2} \sum_{i=1}^{N} \begin{vmatrix} 0 & \Sigma j \epsilon_j \\ i & A \end{vmatrix}^2 , \qquad \text{(F3)}$$

where

$$\begin{vmatrix} 0 & \Sigma j \epsilon_j \\ i & A \end{vmatrix} = \begin{vmatrix} 0 & \Sigma \epsilon_j & \Sigma j \epsilon_j & \Sigma j^2 \epsilon_j \cdots \\ 1 & A_0 & A_1 & A_2 & \cdots \\ i & A_1 & A_2 & A_3 & \cdots \\ i^2 & A_2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} .$$

Summing over i gives

$$\sum_{i=1}^{N} (Y_i - \bar{y}_i)^2 = - \frac{1}{|A|} \begin{vmatrix} 0 & \Sigma j \epsilon_j \\ \Sigma j \epsilon_j & A \end{vmatrix}$$

$$= - \frac{1}{|A|} \left[- \Sigma \epsilon_j \cdot \begin{vmatrix} \Sigma \epsilon_j & A_1 & \cdots \\ \Sigma j \epsilon_j & A_2 & \cdots \\ \vdots \end{vmatrix} + \Sigma j \epsilon_j \begin{vmatrix} \Sigma \epsilon_j & A_0 & A_2 & \cdots \\ \Sigma j \epsilon_j & A_1 & A_3 & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} - \cdots \right] . \tag{F4}$$

Multiplying the sums into the determinants results in elements of the form $\Sigma j^p \epsilon_j \Sigma j^q \epsilon_j$. If the data is uncorrelated, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} i^p \cdot j^q \epsilon_i \epsilon_j = 0 , \quad i \neq j , \tag{F5}$$

and thus, with $r = p + q$,

$$\sum_{j=1}^{N} j^p \epsilon_j \sum_{j=1}^{N} j^q \epsilon_j = \sum_{j=1}^{N} j^r \epsilon_j^2 \tag{F6}$$

for large N. Furthermore,

$$\sum_{j=1}^{N} j^r \epsilon_j^2 = \frac{1}{N} \sum_{j=1}^{N} j^r \sum_{j=1}^{N} \epsilon_j^2 = \frac{A_r}{N} \sum_{j=1}^{N} \epsilon_j^2 , \tag{F7}$$

which may be shown by induction by going from N to $N + 1$ and is obviously true for $N = 1$.

Equation F4 is thus reduced to

$$\sum_{i=1}^{N} (Y_i - \bar{y}_i)^2 = -\frac{\Sigma \epsilon_j^2}{N|A|} \left[- \begin{vmatrix} A_0 & A_1 & \cdots \\ A_1 & A_2 & \cdots \\ \vdots & \vdots & \\ \end{vmatrix} + \begin{vmatrix} A_1 & A_0 & \cdots \\ A_2 & A_1 & \cdots \\ \vdots & \vdots & \\ \end{vmatrix} - \cdots \right]$$

and hence

$$\sum_{i=1}^{N} (Y_i - \bar{y}_i)^2 = \frac{k}{N} \sum_{j=1}^{N} \epsilon_j^2 \ . \tag{F8}$$

In the same manner,

$$\rho_n = 0 \text{ for } n \geq p + 2 \text{ .} \tag{F15}$$

If ΔR is the change in range during the sampling interval h, we may write the (average) range rate $\dot{R}$:

$$\dot{R} = \frac{\Delta R}{h} ,$$

where ΔR is proportional to $\Delta\phi$. The correlation coefficients or normalized autocorrelation functions for $\dot{R}$ are thus the same as for $\Delta\phi$. See also References 9 and 10.

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