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EXPLORING IN AEROSPACE ROCKETRY
3. CALCULATION OF ROCKET VERTICAL-FLIGHT PERFORMANCE

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Presented to Lewis Aerospace Explorers
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3. CALCULATION OF ROCKET VERTICAL-FLIGHT PERFORMANCE

John C. Evvard*

In calculating the altitude potential of a rocket, one must take into account the forces produced by both the thrust of the engine and the gravitational pull of the Earth. A simplified approach can be developed for estimating peak altitude performance of model rocket vehicles. The principles involved, however, are basic and are applicable to all rocket-powered vehicles. The method of calculating vertical-flight performance is to use Newton's law to compute acceleration. Then, velocity and vertical distance, or altitude, are computed from acceleration. (Symbols used in these calculations are defined in the appendix.)

CALCULATIONS

According to Newton's law of motion, a mass M exerts a force (in weight units) of value M on its support. If the support is removed, this mass will fall freely with an acceleration of 32.2 feet per second per second. That is, the vertical speed will increase by 32.2 feet per second for each second of free fall. Imagine that this mass M is resting on a frictionless table top. A force of M (weight units) in a horizontal direction will produce an acceleration g of 32.2 feet per second per second in the horizontal direction. If the force is increased or decreased, the acceleration will be correspondingly increased or decreased. If the force in weight units is designated W and the acceleration is a, then this proportionality is expressed as

$$\frac{W}{a} = \frac{M}{g} \tag{1}$$

or

$$Wg = Ma$$

Physicists do not like to keep writing g in the equation. They distinguish between force and weight. Hence, they define the force F as Wg.

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$$F = Wg = Ma$$

Hence,

$$F = Ma \tag{2}$$

This is the equation we will use. Because this equation is independent of Earth's gravity, it is equally valid everywhere in the universe.

A thrusting rocket has at least two forces acting on it: (1) the force F_R due to the motor, and (2) the force $F_W = -Mg$ due to the weight of the rocket. The force F is the sum of F_R and F_W

$$F = F_R + F_W$$

or

$$F = F_R - Mg \tag{3}$$

Acceleration

From equations (2) and (3) the acceleration is given as

$$a = \frac{F_R}{M} - g$$

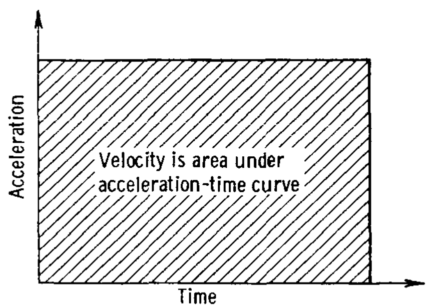
For convenience, the subscript on the symbol for the force due to the rocket motor will now be dropped so that

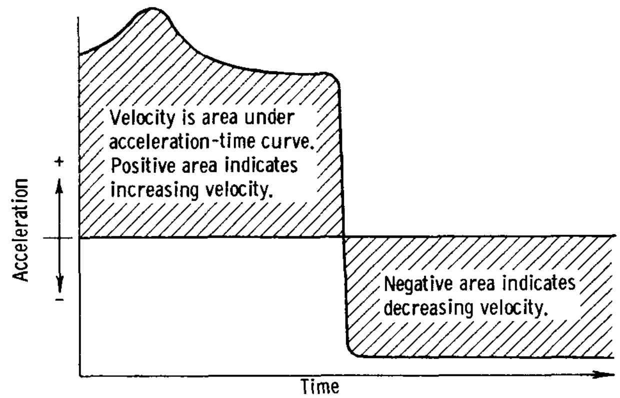
$$a = \frac{F}{M} - g \tag{4}$$

The acceleration is thus made up of two acceleration terms. The first, F/M, is due to the thrust-to-mass ratio. This would be the acceleration if there were no gravity. The second acceleration is that of gravity. This term reflects the so-called gravity loss. Equation (4) is general for vertical flight if instantaneous values of thrust and mass are inserted.

Velocity

If the acceleration is constant, then the velocity is clearly the acceleration multiplied by the time. This quantity is the area under the acceleration-time curve. If the



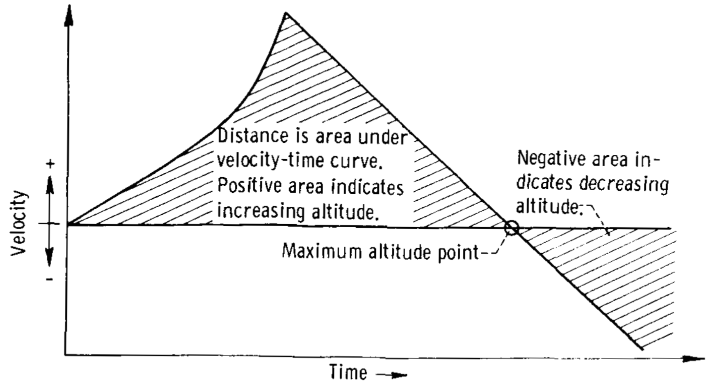


acceleration is not constant, then each increment in velocity is the local average acceleration multiplied by the time increment. The total velocity is the sum of the incremental velocity changes. This is, in fact, the area under the acceleration-time curve. Note that the areas generated by the curve can be positive or negative. A positive area denotes increasing velocity. A negative area denotes decreasing velocity. The velocity is zero initially and also when the positive and negative areas are equal. The velocity at any time is the area generated by the acceleration up to that time. Positive velocity means that the rocket is rising. Negative velocity means that the rocket is falling.

Flight Altitude

In a similar manner, each increment of vertical distance traveled Y by the rocket (flight altitude) is the local average velocity multiplied by the time increment. Thus, distance or altitude is the area under the velocity-time curve. The maximum altitude occurs when the velocity is zero (when the positive and negative areas under the acceleration-time curve are equal).

Note that the equations and graphical solution are general if instantaneous thrust and mass are employed in calculating the acceleration as a function of time. For example, the second-stage motor thrust and the combined weight of all remaining stages would be



used just after first-stage burnout. The thrust and duration of thrust are given in model rocket catalogs. Remember to use consistent units. If thrust is in pounds, multiply by 32.2 to get F. Use M in pounds, time in seconds. The value of g is 32.2 feet per second per second. If thrust is in ounces, the mass should be in ounces, but 32.2 is still the multiplication factor to obtain F. Note that maximum thrust and average thrust are quite different for most model rocket motors.

APPROXIMATE ANALYTIC SOLUTIONS

The propellant weight for model rockets will likely be small compared to the launch weight. Hence, the mass can be nearly constant. Also, an average thrust might be employed instead of instantaneous thrust. Hence, acceleration is constant. The following equations result for a single-stage rocket. These equations can be obtained from the area plots already discussed.

During powered flight

$$a = \left(\frac{F}{M} - g\right) \tag{5}$$

$$V_a = \left(\frac{F}{M} - g\right) t_a \tag{6}$$

$$y_a = \left(\frac{F}{M} - g\right) \frac{t_a^2}{2} \tag{7}$$

The time t_a is limited to the thrust duration of powered flight. The initial velocity was taken as zero. For coasting flight, time t_c is measured from burnout. The height increase during coasting flight is y_c.

For coasting flight

$$a = -g \tag{8}$$

$$V = -gt_c + V_a \tag{9}$$

But at peak altitude $V = 0$, so

$$t_c = \frac{V_a}{g} \tag{10}$$

$$y_c = -\frac{gt_c^2}{2} + V_a t_c \tag{11}$$

Inserting t_c from equation (10) into equation (11) gives

$$y_c = \frac{V_a^2}{2g} \tag{12}$$

The total height is then

$$Y = y_a + y_c$$

or

$$Y = \left(\frac{F}{M} - g\right)\frac{t_a^2}{2} + \left(\frac{F}{M} - g\right)^2 \frac{t_a^2}{2g}$$

This then reduces to

$$Y = \frac{F^2}{M^2}\frac{t_a^2}{2g} - \frac{F}{M}\frac{t_a^2}{2} \tag{13}$$

Let T be the total impulse as given in rocket motor tables. This is force in pounds multiplied by time in seconds. Then

$$F \approx \frac{T}{t_a} g \tag{14}$$

Substituting into equation (13),

$$Y = \frac{Tg}{2M}\left(\frac{T}{M} - t_a\right) \tag{15}$$

The t_a term in equation (15) results from the gravity loss. This subtraction from the flight altitude can be minimized by (a) choosing motors with high total impulse T, (b) designing rockets with low mass M, and (c) choosing motors with very short burning time (minimizing t_a).

Tabulated values of motor characteristics are now required. The model rocket catalogs generally list such motors. These have been found to have an average specific impulse of about 82.8 seconds. That is, the motors generate 82.8 pounds of thrust for each pound per second of propellant flow rate. The jet velocity of these motors is then $82.8 \times g = 82.8 \times 32.2 = 2666$ feet per second. Other useful motor characteristics are listed in table 3-I. The quantities T and t are the total impulse and burning time included in equation (15). The quantity m is the propellant weight. This should be small compared to the rocket weight if the assumptions of equation (15) are to hold. Division of propellant weight by burning time gives the average propellant flow rate, or burning rate, $\dot{m}$. The term gt_a is the velocity loss during powered flight due to gravity,

TABLE 3-I. - MODEL ROCKET MOTOR CHARACTERISTICS

Motor	Total impulse, T, lb-sec	Burning time, t_a, sec	Propellant weight, m, lb	Average propellant burning rate, $\dot{m}$, lb/sec	Velocity loss, gt_a, ft/sec	Distance loss, $gt_a^2/2$, ft
$\frac{1}{4}$ A.8	0.17	0.17	0.00211	0.0124	5.49	0.466
$\frac{1}{2}$ A.8	.35	.40	.00422	.01055	12.89	2.58
A.8	.70	.90	.00844	.00938	29.0	13.04
B.8	1.15	1.40	.0139	.00992	45.1	31.50
B3	1.15	.35	.0139	.0397	11.27	1.97
C.8	1.50	2.00	.0181	.00905	64.4	64.4

while $gt_a^2/2$ is the altitude loss due to gravity during powered flight (see eqs. (6) and (7)). The velocity loss during powered flight, of course, leads to an additional altitude loss during coasting flight.

Sample problem:

Each of three different rockets is to be fired with three separate motors. The loaded weights, or masses, M of the rockets are 0.15, 0.25, and 0.5 pound, respectively. The three motors to be used are the A.8, the B3, and the C.8. Use equation (15) to calculate the expected altitude for each of the rockets. (The values of T and t_a for each of the motors are given in table 3-I.) Note that the B3 engine outperforms the C.8 engine on the heavy rocket in spite of the smaller total impulse. This is due to the gravity-loss term.

Motor	T/M	t_a	T/M - t_a	Y, ft
0.15-Pound rocket				
A.8	4.67	0.9	3.77	283.2
B3	7.67	.35	7.32	903.5
C.8	10.00	2.00	8.00	1288
0.25-Pound rocket				
A.8	2.80	0.9	1.9	85.6
B3	4.60	.35	4.25	314.7
C.8	6.00	2.00	4.00	386.4
0.5-Pound rocket				
A.8	1.40	0.9	0.5	11.2
B3	2.30	.35	1.95	72.2
C.8	3.00	2.00	1.00	48.3

SIMPLE THEORY FOR MULTISTAGE ROCKETS

Let subscripts 1, 2, 3, . . ., and n refer to conditions of the first, second, third, . . ., and n^{th} stages during thrusting flight. For example, t_2 is the time increment during second stage firing, y_2 is the distance, or altitude, increase during second-stage firing, V_2 is the velocity increase due to the second stage, etc. The general equations (constant mass) for the n^{th} stage are

$$a_n = \left(\frac{F_n}{M_n} - g\right) \tag{16}$$

$$V_n = \left(\frac{F_n}{M_n} - g\right)t_n \tag{17}$$

Hence, the total velocity of the rocket after n stages have fired is

$$V = V_1 + V_2 + V_3 + \ldots V_n \tag{18}$$

Hence,

$$V = \left(\frac{F_1}{M_1} - g\right)t_1 + \left(\frac{F_2}{M_2} - g\right)t_2 + \ldots \left(\frac{F_n}{M_n} - g\right)t_n \tag{19}$$

$$y_n = \left(\frac{F_n}{M_n} - g\right)\frac{t_n^2}{2} + t_n\left[\left(\frac{F_1}{M_1} - g\right)t_1 + \left(\frac{F_2}{M_2} - g\right)t_2 + \ldots \left(\frac{F_{n-1}}{M_{n-1}} - g\right)t_{n-1}\right] \tag{20}$$

The second term of equation (20) is the velocity of the rocket just prior to n^{th} stage firing multiplied by the n^{th} stage firing time, and y_n is the altitude increase during n^{th} stage firing. The total altitude will then be

$$Y = y_1 + y_2 + y_3 \ldots y_n + y_c \tag{21}$$

or

$$Y = \left(\frac{F_1}{M_1} - g\right)\frac{t_1^2}{2} + \left[\left(\frac{F_2}{M_2} - g\right)\frac{t_2^2}{2} + \left(\frac{F_1}{M_1} - g\right)t_1t_2\right]$$

$$+ \left[\left(\frac{F_3}{M_3} - g\right)\frac{t_3^2}{2} + \left(\frac{F_2}{M_2} - g\right)t_2t_3 + \left(\frac{F_1}{M_1} - g\right)t_1t_3\right] + \ldots + \frac{V^2}{2g} \tag{22}$$

In equation (22) it is assumed that there is no time delay between stage firings. Note from equation (14) that

$$F_n = \frac{T_n g}{t_n}$$

Hence

$$V_n = \left(\frac{T_n}{M_n} - t_n\right)g \tag{17a}$$

$$V = \left(\frac{T_1}{M_1} - t_1\right)g + \left(\frac{T_2}{M_2} - t_2\right)g + \ldots \left(\frac{T_n}{M_n} - t_n\right)g \tag{18a}$$

$$y_n = \left(\frac{T_n}{M_n} - t_n\right)\frac{gt_n}{2} + t_n g\left[\left(\frac{T_1}{M_1} - t_1\right) + \left(\frac{T_2}{M_2} - t_2\right) + \ldots \left(\frac{T_{n-1}}{M_{n-1}} - t_{n-1}\right)\right] \tag{20a}$$

or

$$y_n = \frac{V_n t_n}{2} + t_n(V_1 + V_2 + V_3 + \ldots V_{n-1}) \tag{20b}$$

$$Y = y_1 + y_2 + \ldots + y_n + \frac{V^2}{2g} \tag{22a}$$

or

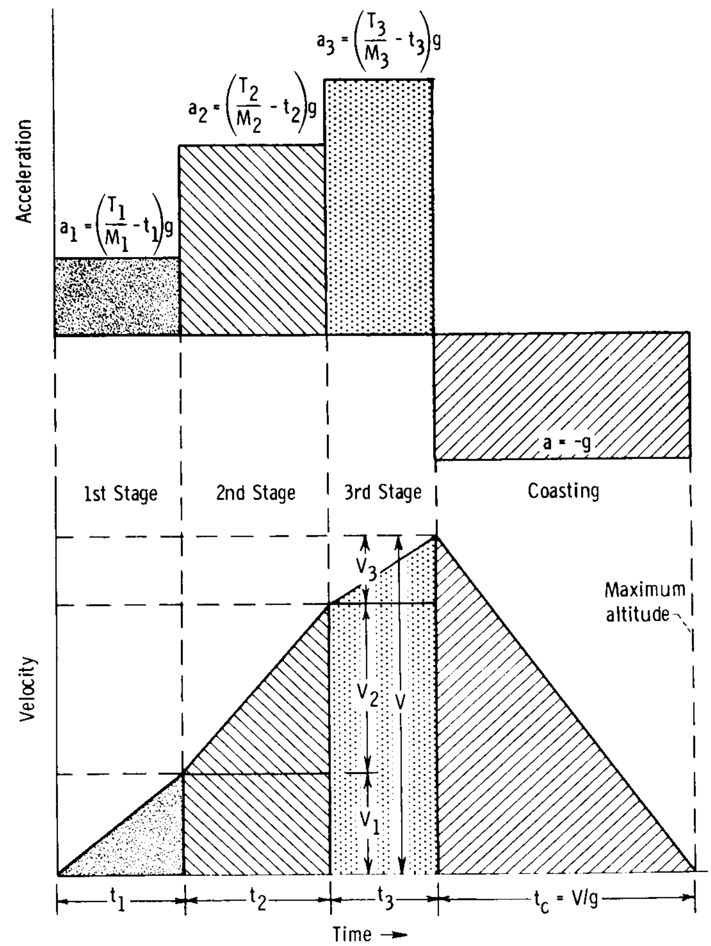
$$Y = \left(\frac{T_1}{M_1} - t_1\right)\frac{gt_1}{2} + \left[\left(\frac{T_2}{M_2} - t_2\right)\frac{gt_2}{2} + \left(\frac{T_1}{M_1} - t_1\right)gt_2\right]$$

$$+ \left[\left(\frac{T_3}{M_3} - t_3\right)\frac{gt_3}{2} + \left(\frac{T_2}{M_2} - t_2\right)gt_3 + \left(\frac{T_1}{M_1} - t_1\right)gt_3\right] + \ldots + \frac{V^2}{2g} \tag{22b}$$

or

$$Y = \frac{V_1 t_1}{2} + \left(\frac{V_2}{2} + V_1\right)t_2 + \left(\frac{V_3}{2} + V_2 + V_1\right)t_3 + \ldots + \frac{V^2}{2g} \tag{22c}$$

These mathematical derivations may be confusing. The final results, however, are almost self-evident from sketches of acceleration and velocity against time. Acceleration is constant for each stage and for coasting flight. The area under the acceleration-time curve gives the velocity. The velocity increase for each stage is then the area of the rectangle given by the product of acceleration and time. For example, the second-stage velocity increase from the sketch is a_2t_2 or $V_2 = \left[(T_2/M_2) - t_2\right]gt_2$. In a similar manner the terms of equation (22c) may be recognized as the various shaded areas of



the lower part of the sketch. For example, the term $V_1t_1/2$ of equation (22c) is the first triangular area of the velocity-time curve. The second term is the area of the rectangle plus the triangle over the time interval t_2, etc.

Sample problem:

A rocket is to be fired with a B3 motor for its first stage and an A.8 motor for its second stage. The launch weight of the rocket is 0.3 pound, and the second-stage weight is 0.15 pound. What altitude is the rocket expected to attain? (In the following calculations the subscript number denotes the stage.)

From the problem:

$$M_1 = 0.3 \text{ lb} \qquad M_2 = 0.15 \text{ lb}$$

From table 3-I:

$$T_1 = 1.15 \text{ lb-sec} \qquad T_2 = 0.7 \text{ lb-sec}$$

$$t_1 = 0.35 \text{ sec} \qquad t_2 = 0.9 \text{ sec}$$

From equation (17a):

$$V_1 = \left(\frac{1.15}{0.3} - 0.35\right)32.2 \qquad V_2 = \left(\frac{0.7}{0.15} - 0.9\right)32.2$$

$$V_1 = 112.1 \text{ ft/sec} \qquad V_2 = 121.4 \text{ ft/sec}$$

From equation (18):

$$V = 112.1 + 121.4$$

$$V = 233.5 \text{ ft/sec}$$

From equation (20b):

$$y_1 = \frac{112.1 \times 0.35}{2} \qquad y_2 = \frac{121.4 \times 0.9}{2} + (0.9 \times 112.1)$$

$$y_1 = 19.6 \text{ ft} \qquad y_2 = 155.5 \text{ ft}$$

Finally, from equation (22a):

$$Y = 19.6 + 155.5 + \frac{(233.5)^2}{64.4}$$

$$Y = 1021.7 \text{ ft}$$

These equations have neglected the change in mass associated with propellant ejection. Hence, the actual performance would be higher than the calculated value. On the other hand, wind resistance, which would decrease performance, has also been neglected. The actual performance would also change if the thrust were not constant with time. Most model rocket motors give a peak in thrust soon after ignition. High initial thrust leads to improved performance.

EXACT EQUATIONS FOR CONSTANT-THRUST ROCKET VEHICLES

At any point in time, neglecting drag,

$$a = \frac{F}{M} - g = \frac{I_{sp} g \dot{m}}{M} - g \tag{23}$$

where I_{sp} is the specific impulse ($\approx$82.8 sec for the model rocket motors that we have examined), $\dot{m}$ is the average propellant burning rate in pounds per second, and M is the instantaneous weight of the vehicle in pounds. Over the period of acceleration or motor thrust duration, this equation yields the following expression for velocity at burnout:

$$V = 2.3\, I_{sp} g \log \frac{M_i}{M_f} - gt \tag{24}$$

where M_i is the initial total mass of the vehicle, M_f is the final mass of the vehicle at burnout, and t is the burning time of the rocket motor. This is the same as equation (16) of the previous chapter except for the second (or gravity-loss) term. The powered-flight altitude is then given by the equation

$$y = I_{sp} g \frac{M_i}{\dot{m}} \left(1 - \frac{M_f}{M_i} - 2.3 \frac{M_f}{M_i} \log \frac{M_i}{M_f}\right) - \frac{gt^2}{2} \tag{25}$$

The maximum altitude (or altitude after coasting) is then

$$Y = y + y_c$$

or

$$Y = y + \frac{V^2}{2g} \tag{26}$$

Aerodynamic drag has been ignored in the relations presented herein. This drag force, which would be included in equation (3), generally has the form

$$F_D = \frac{1}{2} \rho V^2 C_D A$$

where ρ is the air density, and V is the instantaneous speed of the rocket. The drag coefficient C_D is related to the geometry of the rocket and the quality of flow (laminar, turbulent, etc.) over the surface of the rocket. The quantity A is a reference area to indicate rocket size. The theory and prediction charts for rocket performance with aerodynamic drag are presented in reference 1.

APPENDIX - SYMBOLS

A	area
a	acceleration
C_D	aerodynamic drag coefficient
F	force
F_D	force due to aerodynamic drag
F_R	force due to rocket motor
F_W	force due to weight of rocket
g	acceleration due to Earth's gravity
I_{sp}	specific impulse
M	mass of rocket
M_f	final mass of rocket
M_i	initial mass of rocket
$M_1, M_2, M_3, \ldots, M_n$	mass of rocket during respective firing of first, second, third, . . ., n^{th} stage
m	weight of propellant
$\dot{m}$	average burning rate of propellant
T	total impulse (force multiplied by time)
$T_1, T_2, T_3, \ldots, T_n$	total-impulse increase associated with firing of first, second, third, . . ., n^{th} stage, respectively
t	time
t_a	time duration of acceleration (for single-stage rocket)
t_c	time duration of coasting flight ($V > 0$)
$t_1, t_2, t_3, \ldots, t_n$	incremental time increase during firing of first, second, third, . . ., n^{th} stage
V	velocity of rocket
$V_1, V_2, V_3, \ldots, V_n$	incremental velocity increase associated with firing of first, second, third, . . ., n^{th} stage
W	force in weight units
Y	flight altitude ($V \geq 0$)

y	incremental altitude increase
y_c	altitude increase associated with coasting flight
$y_1, y_2, y_3, \ldots, y_n$	incremental altitude increase during firing of first, second, third, . . ., n^{th} stage
ρ	air density

REFERENCE

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