GENERATION AND DESCRIPTION OF A CLASS OF RANDOM PROCESSES *

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ABSTRACT

The generation of a possibly non-stationary random process having a specified autocorrelation function is examined. If the class of autocorrelation functions under consideration is suitably restricted, a finite linear system to be excited by white noise may be determined which yields at its output a random process having the specified autocorrelation function. The determination of this linear system provides a "spectral factorization" for the random process. Simple criteria are thus obtained which serve to identify autocorrelation functions in the class. The positive definite character of such functions may be tested by a straightforward process related to the factorization technique. Moreover, conditions may be stated whereby the factorization and the results derived from it are valid globally.

1. INTRODUCTION

Many problems of signal processing, especially filtering and prediction of random signals, have benefited from the use of the "shaping filter" technique. In general, this technique demands that a given random process be generated as the output of a system (called a shaping filter) whose input is white noise. The usual applications require that the random process be specified only by its second-order statistics, that is, by its spectral density if the process is wide-sense stationary, or more generally by its autocorrelation function if the process may be stationary or non-stationary. The term "factorization" (or "spectral factorization") is given to the process of determining a shaping filter from a given autocorrelation function. The shaping filter is usually required to consist of a finite number of lumped elements. This requirement facilitates not only mathematical analysis, but also the generation of the random process by analogue simulation or other menas. The simplest formulation of the factorization problem imposes additional assumptions requiring that both the given random process and the white noise input process be real and scalar valued and have zero mena.

The problem briefly described above has been the subject of many investigations which have met with varying degrees of success. Bode

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and Shannon [1] were among the first to apply spectral factorization to the smoothing and prediction theory of stationary random processes. In their case the spectral density of the random process, a rational function, was symmetrically factored into two parts, one of which was chosen to be the (non-unique) transfer function of the shaping filter. Darlington [2] investigated the non-stationary case, and demonstrated the existence of a spectral factorization when the type of time variations was suitably defined and restricted. At about the same time, Batkov [3] proposed a recursive algebraic solution to the factorization problem which seems to be invalid except in special cases. The most recent work on this problem was performed by Kalman [4], Stear [5], and Anderson [6]. Kalman essentially reformulated the problem in state variable terms. The results of Stear and Anderson, although derived by different methods, are similar and appear to provide a first step in demonstrating the existence of a factorization for the general non-stationary case. It is, in fact, the work of Kalman and Anderson which is most closely related to the approach taken in the present paper.

2. FORMULATION

It will be assumed that the shaping filter may be described in the following way:

$$\begin{aligned} \dot{x}(t) &= \beta(t)u(t) \\ y(t) &= \alpha^t(t)x(t) \end{aligned} \tag{1}$$

In this equation, $x(t)$, $\beta(t)$, and $\alpha(t)$ are real valued n-vectors,* and the input $u(t)$ is a scalar, zero-mean, white-noise process, i.e., $E\{u(t)u(\tau)\} = \delta(t-\tau)$. The output autocorrelation function is denoted by $r(t,\tau) = E\{y(t)y(\tau)\}$. The system is assumed to be causal.

The form chosen for the shaping filter is quite general. Some previous investigations have represented the filter by a single n^{th} order differential equation, which implies a stringent observability requirement on the present form. Although the absence of a feedback matrix in equation (1) may make this form unsuitable for practical simulation, the theory of equivalent systems [7] is sufficiently developed to indicate when the above system has an equivalent but practical realization.

The autocorrelation function of the process $y(t)$ may easily be expressed by means of (1). Assume that at some initial time t_0 the state vector is a random variable $x(t_0)$. Let

$$M_0 = E\{x(t_0)x^t(t_0)\} \tag{2a}$$

and

$$M(t) = M_0 + \int_{t_0}^{t} \beta(\lambda)\beta^t(\lambda)d\lambda. \tag{2b}$$

* The superscript t will be used to denote matrix transpose.

Then $M(t)$ is the covariance matrix of the state vector $x(t)$ and

$$r(t,\tau) = \begin{cases} \alpha^t(t)M(t)\alpha(\tau) & (t<\tau) \\ \alpha^t(t)M(\tau)\alpha(\tau) & (t>\tau). \end{cases} \tag{3}$$

The matrix $M(t)$ defined above has the following properties: both $M(t)$ and its derivative $\dot{M}(t)$ are symmetric, positive semidefinite matrices, and the rank of $\dot{M}(t)$ is at most unity. Moreover, any matrix with these properties can be expressed as in equation (2). Such matrices will be called <u>admissable</u>.

From equations (2) and (3), $r(t,\tau)$ must satisfy the following conditions:

A1. $r(t,\tau)$ is symmetric; i.e., $r(t,\tau) = r(\tau,t)$.

A2. $r(t,\tau)$ is separable; i.e., there exist (column) vectors $\gamma(t)$ and $\phi(t)$ such that

$$r(t,\tau) = \begin{cases} \phi^t(t)\gamma(\tau) & \text{for } t > \tau \\ \gamma^t(t)\phi(\tau) & \text{for } t < \tau. \end{cases}$$

The vectors $\gamma(t)$ and $\phi(t)$ are assumed to be included in the given data.

A3. $r(t,\tau)$, by virtue of being an autocorrelation function, must be non-negative definite [8]; i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i r(t_i,t_j) a_j \geq 0.$$

for all a_i, t_i and finite n.

The last requirement is not only physically reasonable, but also it has been established directly that functions of the form of equation (3), with $M(t)$ admissable, must satisfy condition A3. Hence, if any function $r(t,\tau)$ can be rewritten as in equation (3) with $M(t)$ admissable, then this function is non-negative definite. But once $r(t,\tau)$ is expressed in this form, the factorization problem is solved, since $\beta(t)$ can be determined readily from $\dot{M}(t)$.

The following definition and theorem summarize the discussion above. (The theorem was originally stated by Kalman [4] in slightly different terms.)

<u>Definition 1</u>. A function $r(t,\tau)$ satisfying A1 and A2 <u>admits a factorization</u> if there exists a random process $y(t)$ such that

$$r(t,\tau) = E\{y(t)y(\tau)\}$$

where $y(t)$ is generated by a shaping filter.

<u>Theorem 1</u>. A function $r(t,\tau)$ satisfying A1 and A2 admits a factorization if and only if there is a vector $\alpha(t)$ and an admissable matrix $M(t)$ such that

$$r(t,\tau) = \alpha^t(t)M[\min(t,\tau)]\alpha(\tau),$$

in which case $r(t,\tau)$ satisfies A3.

3. THE FACTORIZATION PROBLEM

This section will summarize and extend a factorization technique which has recently been developed. Assumptions A1-A3 will be in effect, and the following assumptions will also be required:

A4. The vectors ϕ and γ have a sufficient number of continuous derivatives.*

A5. The sets $\{\phi_i\}$ and $\{\gamma_i\}$ for $i = 1 \ldots n$ are each comprised of linearly independent functions over the appropriate interval of interest.

The last assumption is not restrictive.

In order to factor $r(t,\tau)$, let

$$\phi^t(t)\gamma(\tau) = \alpha^t(t)M(\tau)\alpha(\tau) \qquad t > \tau.$$

Because of A5, we may equate $\alpha(t) = \phi(t)$ to within an unimportant constant linear transformation which will be taken as the identity transformation for convenience. Then

$$\gamma = M\phi \tag{4}$$

is the basic equation which must be solved for an admissable matrix $M(t)$. Under certain conditions this equation may be converted into the following matrix Ricatti differential equation:**

$$\dot{M} = \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})(\gamma^{(k+1)} - M\phi^{(k+1)})^t}{\delta_k^2} \tag{5}$$

where the scalar quantity δ_k^2 is defined as

$$\delta_k^2 = \phi^{(k)t}\gamma^{(k+1)} - \phi^{(k+1)t}\gamma^{(k)}.\ ^{***}$$

In deriving equation (5) it was assumed that:

A6. For some $k < n$, $\delta_k^2(t) \neq 0$ for all t, and $\delta_i^2(t) \equiv 0$ for $0 \leq i < k$.

Let M_0, a positive-semidefinite matrix, denote an initial value of $M(t)$. Also, let $\Phi_k = [\phi, \phi^{(1)}, \ldots, \phi^{(k)}]$ and $\Gamma_k = [\gamma, \gamma^{(1)}, \ldots, \gamma^{(k)}]$.

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* The argument "t" will be omitted when it is clear from the context that no confusion will arise.

** Although derived independently, the method of transforming equation (4) into the Ricatti equation (5) is essentially that employed by Anderson [6], and will therefore be omitted here.

*** The k-th derivative of a function ϕ will be denoted $\phi^{(k)}$.

Then as a consequence of the derivation of equation (5), M_0 must satisfy

$$\Gamma_k(t_0) = M_0\Phi_k(t_0) \tag{6}$$

The standard existence theorem for ordinary differential equations [9] shows that equation (5) has a unique solution M(t) in a neighborhood of t_0, at which point $M(t_0) = M_0$. The authors have shown that if $\delta_i^2 \equiv 0$ for i < k as assumed, then $\delta_k^2 \geq 0$. The hypothesis of this statement is necessary. For example, if $r(t,\tau) = e^{-t}e^{\tau}$ for $t > \tau$, then $\delta_0^2 = 2$, whereas $\delta_1^2 = -2$. Of course, equation (5) is valid only as long as $\delta_k^2 > 0$.

With $\delta_k^2 > 0$, it is obvious from the form of equation (5) that the solution M(t) is admissable. It must now be established that equation (4) is satisfied in the region where M(t) is defined. Anderson has shown this result by exhibiting a linear differential equation which is satisfied by the vector quantity $[\gamma(t) - M(t)\phi(t)]$. If the initial condition for the linear differential equation is zero as implied by equation (6), then the solution, $[\gamma - M\phi]$ is everywhere zero, and the factorization problem has a local solution.

Thus far the existence of an initial matrix has only been postulated. That its existence is not obvious is apparent from the following consideration. In order for equation (6) to be valid, it is necessary that Γ_k and Φ_k be consistent in the sense that rank $(\Gamma_k) \leq$ rank (Φ_k). For if this rank condition is violated, then there is no M_0 satisfying $\Gamma_k = M_0\Phi_k$. In fact, it is demonstrated below that rank (Γ_k) = rank (Φ_k) = k + 1.

Consider a matrix R_k defined as $R_k = \Phi_k{}^t\Gamma_k$. We will show that R_k is nonsingular. Let $Y_k = \mathrm{col}[y^{(0)}, y^{(1)}, \ldots y^{(k)}]$*. Then following Loeve [8], $R_k = E\{Y_kY_k{}^t\}$ and is positive semidefinite. Assume that R_k is singular. Then the random variables $y^{(i)}(t)$ are linearly dependent; i.e., there exist continuous scalars $a_i(t)$ such that

$$\sum_{i=0}^{k} a_i y^{(i)} = 0$$

with probability one. Differentiation of this linear constraint yields an expression of the form

$$y^{(k+1)} = \sum_{i=0}^{k} b_i y^{(i)}$$

provided that $a_k \neq 0$. Therefore $y^{(k+1)}$ exists, and $R_{k+1} = E\{Y_{k+1}Y_{k+1}^t\} = \Phi_{k+1}^t\Gamma_{k+1}$ is a symmetric matrix. In particular, $\phi^{(k)t}\gamma^{(k+1)} = \phi^{(k+1)t}\gamma^{(k)}$; i.e., $\delta_k^2 = 0$, which contradicts assumption A6. Therefore the rank of R_k must be k+1 and R_k is positive definite.**

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* The derivatives $y^{(i)}(t)$ are to be interpreted in the mean square sense [8]. The existence of these derivatives is guaranteed if $\delta_i^2 \equiv 0$ for i = 0 ... k-1. The proof of this statement is simple but will not be included here.

** A more detailed analysis would show that R_k may be singular, but the points of singularity cannot be dense on any interval.

With R_k positive definite it is easy to show a method of constructing M_0. Let V and U be matrices consisting of n rows and n-k-1 columns, and let $U = M_0 V$. Then

$$\begin{bmatrix} \Phi_k^t \\ \cdots \\ V^t \end{bmatrix} M_0 [\Phi_k \vdots V] = \begin{bmatrix} \Phi_k^t \\ \cdots \\ V^t \end{bmatrix} [\Gamma_k \vdots U] = \begin{bmatrix} R_k & \vdots & \Phi_k^t U \\ \cdots & \vdots & \cdots \\ V^t \Gamma_k & \vdots & V^t U \end{bmatrix} \tag{7}$$

In general, non-unique matrices V and U may be found which satisfy $\Phi_k^t U = 0$, $V^t \Gamma_k = 0$, and $V^t U = I$. The matrix $[\Phi_k \vdots V]$ is then non-singular, and equation (7) may be inverted to yield a matrix M_0 which is symmetric, positive definite, and satisfies $\Gamma_k = M_0 \Phi_k$.

4. IDENTIFICATION OF AUTOCORRELATION FUNCTIONS

In the previous section it was shown how a class of functions $r(t,\tau)$ defined by a certain set of assumptions admits spectral factorization. With the exception of A3, these assumptions constitute a prescription for determining analytically whether a given $r(t,\tau) = \phi^t(t)\gamma(\tau)$ is a member of the class. It is assumption A3 which presents the problem because the definition of the non-negative definite property is better suited to determining that a function is not non-negative definite, than determining that it is. It would therefore be desirable to be able to identify functions which admit factorization without explicitly relying on the definition of the non-negative-definite property in A3, and without substantially increasing the complexity of the operations required by the remaining assumptions. The previous discussions indicate a way of explicitly avoiding A3 without further restricting the class of functions $r(t,\tau)$. Assumption A3 was used for only two purposes: first, for showing that $\delta_k^2(t) \geq 0$, and second, for showing that R_k is positive definite. Now, by eliminating A3 and modifying A6 to read:

A6' For some $k < n$, $\delta_k^2(t) > 0$ for all t, and $\delta_i^2(t) \equiv 0$ for all $0 \leq i < k$. Also, $R_k(t)$ is a non-negative definite matrix which may be singular at points not dense in any interval.

an equivalent set of assumptions is achieved. Note that A6' requires only the calculation of derivatives and determinants. Theorem 1 shows that A3 is implied by the new set of assumptions. These assumptions obviously comprise a set of conditions which are sufficient to determine whether a function $r(t,\tau)$ admits a factorization. These assumptions are necessary in the limited sense that if $\delta_k^2 < 0$ or if R_k is non-definite, then $r(t,\tau)$ is not non-negative definite and hence does not admit factorization.

It is possible that $r(t,\tau)$ may be such that $\delta_k^2 = 0$ at some points or that $\delta_i^2 \equiv 0$ for all $i = 0, 1, \ldots$. For functions of this type, the Ricatti equation (5) may not exist, or if it does, will contain singularities. This case has been investigated in detail with the result that A6' may be relaxed considerably.

The following examples illustrate how the new assumptions may be used to determine whether a given function admits factorization.

Example 1. $r(t,\tau) = -e^t e^{-\tau}$ for $t > \tau$. Let $\phi(t) = -e^t$ and $\gamma(t) = e^{-\tau}$. Then $\delta_0^2 = 2 > 0$, so that $k = 0$ and $R_0 = r(t,t) = -1 < 0$. Hence $r(t,\tau)$ does not admit factorization.

Example 2 (Kalman). Let $r(t,\tau) = f[\min(t,\tau)]$. If $f(t)$ is positive and increasing, then by setting $\phi(t) = 1$ and $\gamma(t) = f(t)$ we have $\delta_0^2 = f(t) > 0$ and $R_0(t) = r(t,t) = f(t) > 0$, which satisfies A6'. Hence $r(t,\tau)$ is non-negative definite.

5. SOLUTION DEFINED IN THE FUTURE

The above set of assumptions has been shown to be sufficient only locally because the factorization technique depends on the existence of a solution of a non-linear differential equation. It is well known that solutions of such equations may possess a finite escape time; i.e., become unbounded at a finite time after t_0. This behavior is clearly undesirable, especially where simulation is involved. One would hope to be able to avoid a finite escape time by requiring that $r(t,t)$ be bounded on every finite interval. The following example shows that this requirement does not insure boundedness of $M(t)$.

Example 3. Let $r(t,\tau) = \begin{cases} -\max(t,\tau) & \text{for } t,\tau < 0 \\ 0 & \text{for } t > 0 \text{ or } \tau > 0. \end{cases}$

Then for $t > \tau$, $\phi(t) = t$ and $\gamma(t) = -1$ provided that $t < 0$ and $\tau < 0$.

Equation (4) becomes $-1 = Mt$, which has the solution $M = -1/t$. This "solution" escapes at $t = 0$. However, $r(t,\tau)$ is non-negative definite, since $\delta_0^2 = 1 > 0$ and $R_k(t) = r(t,t) = -t > 0$ for $t < 0$. When $t > 0$ or $\tau > 0$ then $\gamma = \phi = 0$, but M which is admissable, cannot decrease and remains infinite for $t > 0$.

The reason that the unbounded behavior of $M(t)$ in the above example went undetected in $r(t,\tau)$ is that at $t = 0$ the shaping filter degenerated so that the escape of the state variable could not be observed at the filter output. The important concept here is observability. System (1) is said to be completely observable [10], if for any t there exists a finite $t' > t$ such that the functions $\{\phi_i(t)\}$ are linearly independent over the interval $[t,t']$.

The following theorem is relevant to the discussion of finite escape time but the proof is omitted here.

Theorem 2. If system (1) is completely observable and if $r(t,t)$ is bounded on every finite interval, then $M(t)$, the solution to equation (5), is bounded on every finite interval.

The requirement of complete observability in the above theorem is only sufficient to insure the boundedness of $M(t)$. Complete observability is not a necessary condition since a shaping filter may be non-observable and yet the covariance matrix, $M(t)$, of the state vector may be bounded.

Since determination of the shaping filter is the object of the factorization problem it might seem logically inconsistent to require, a priori, that the filter be completely observable. However, the observability property only requires knowledge of the "output part" of the filter, here specified by the given vector $\phi(t)$.

6. CONCLUSION

The follwoing example summarizes several points presented here.

<u>Example 4</u>. $r(t,\tau) = \tau/2 - \tau^2/6t$ for $t > \tau$. By choosing

$$\phi(t) = \begin{bmatrix} 1 \\ -1/t \end{bmatrix} \quad \text{and} \quad \gamma(t) = \begin{bmatrix} t/2 \\ t^2/6 \end{bmatrix}$$

one may calculate $\delta_o^2 \equiv 0$, $\delta_1^2 = 1/t^2 > 0$, and

$$R_1(t) = \begin{bmatrix} t/3 & 1/6 \\ 1/6 & 1/3t \end{bmatrix} \quad \text{which is positive definite only for } t > 0.$$

Hence $r(t,\tau)$ admits a factorization only for $t,\tau > 0$. Moreover, since the functions $-1/t$ and 1 are linearly independent over any positive interval, the factorization is global.

By choosing $t = 1$ as the initial time and

$$M_o = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

as the initial condition, one obtains the matrix

$$M(t) = \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix}$$

as the solution to the matrix Ricatti equation and

$$\beta(t) = \pm\begin{bmatrix} 1 \\ t \end{bmatrix}$$

for the coefficients of the shaping filter.

If the hypotheses of Theorem 2 are added to assumptions A1...A6', excluding A3, the result is a set of criteria which defines a large class of functions known to be capable of global factorization. These criteria may be applied in a striaghtforward fashion to an arbitrary function and moreover, any function satisfying these criteria must also satisfy the non-negative definite property, and must therefore be an autocorrelation function.

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