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SUMMARY

This paper provides a listing of techniques used to determine the eigenvalue bounds of a matrix defined over either the real or complex fields. Theorems concerning the condition of eigenvalues as a function of the related matrix are stated. Known theorems which determine the bounds are derived. Closed-form solutions are expressed in terms of (1) the matrix elements, (2) matrix norms, and (3) vectors and the eigenvalues of related matrices. Extensions of several results are made to infinite matrices. A comparison is made in terms of the relative size of the areas of eigenvalue inclusion for the various solutions. Examples in terms of eigenvalue bounds for particular matrices are given.

INTRODUCTION

In various applications of operator theory, it is often required to determine the spectrum $\sigma(A)$ of an operator A, that is, all scalars λ for which $A - \lambda$ has no inverse. For the n-dimensional operator, this problem is to determine those scalars λ for which there exists an associated nonzero vector x such that $Ax = \lambda x$. Solutions in this case can be assumed by requiring the vanishing of the determinant of the associated operator $A - \lambda I_n$ for the n-dimensional identity matrix I_n. Expansion of this determinant yields an nth degree polynomial, the roots of which are the eigenvalues of the matrix A. The roots of the general polynomial of degree n can be determined directly (that is, solvable by radicals (ref. 1)) if and only if $n \leqq 4$. However, various techniques do exist for determining upper and lower bounds for eigenvalues and very often this information is sufficient to solve various types of problems.

This paper is concerned with listing known techniques which determine eigenvalue bounds and with comparing their relative accuracies. These bounds are expressed in terms of (1) the elements of the matrix itself, (2) matrix norms, and (3) vectors and

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eigenvalues of related matrices. Extensions of several results are made to infinite matrices.

SYMBOLS

$a, g, j, k, m, n, r, s, t$; $M, N; \alpha, \beta, \gamma, \delta, \epsilon, \sigma$	real numbers
$a_{ij}, g_{ij}, t_{ij}, b_{kk}$	elements of a matrix, occupying ith or kth column and jth or kth row
$\|A\|$	matrix norm of A
$A_{ij}, A, B, B_\epsilon, G$; H, T, U, D	matrices defined over complex field
$\|A_{ij}\|$	equal to $\sup_{\substack{x \in \Omega_j \\ x \neq 0}} \frac{\|A_{ij}x\|_{\Omega_i}}{\|x\|_{\Omega_j}}$
$\|A_{ij}x\|_{\Omega_i}$	vector norm evaluated in subspace Ω_i
$\mathrm{diag}(a_1, \ldots, a_n)$	diagonal matrix with $a_1, \ldots, a_n$ down the main diagonal
$\left.\frac{d^k}{(d\lambda)^k}\right\|_{\lambda=0}$	kth derivative evaluated at $\lambda = 0$
e	column vector, all of whose components are 1
i	element for which its square is equal to -1
I_n	identity matrix of order n
J_i	ith Jordan block corresponding to λ_i
l_1	set of all vectors $x = x_1, x_2, \ldots$ for which it is true that $\sum_{i=1}^{\infty} \|x_i\| < \infty$
p	polynomial
P	permutation matrix
P_k	numerically equal to $\sum_{\substack{j=1 \\ j \neq k}}^{n} \|a_{kj}\|$

$q(x) = \frac{(x,A_x)}{(x,x)}$	or Rayleigh's quotient
$\|x\|_{\Omega_j}$	vector norm evaluated in subspace Ω_j
X,h,x,y,z,ν	vectors
$X_i,x_i,x_j,y_i,y_j,z_i,z_j,Z_{ii}$	components of vectors or vectors themselves
ϵ	error
λ	eigenvalue
λ^A	eigenvalue of matrix A
Λ	Jordan canonical or normal form of matrix
$\sigma(A)$	spectrum of A; i.e., the set of all complex numbers λ for which $A - \lambda$ has no inverse
$[\sigma(A)]^{-1}$	set whose elements are inverse of elements of $\sigma(A)$
$\psi(\lambda)$	equal to $\det(A - \lambda I)$
Ω	subspace
det	determinant
inf	infimum or greatest lower bound
max	maximum value
min	minimum value
sup	supremum or least upper bound
$\| \ \|$	modulus or absolute value
$\| \ \|$ (double bars) $\|\ \|$	equal to $< , >^{1/2}$ for a vector

$< , >$ complex scalar product

$\in$ belongs to a set

Subscripts:

i,j,k,l,n integers

max maximum

min minimum

Superscripts:

A,B,B_{ϵ},H matrices

T matrix transpose

α,k real numbers

$*$ conjugate transpose

$-$ complex conjugate

WELL-KNOWN THEOREMS FOR EIGENVALUES

Some well-known results concerning the eigenvalues of particular types of matrices are given in table I. Other results which are less well known than those in table I, but yet of some importance are:

(1) If A is a positive real matrix, that is, $a_{ij} > 0$, then there exists a real, positive eigenvalue which is simple and such that its absolute value is greater than that of any other eigenvalue. (See ref. 8.)

(2) If A is a nonnegative irreducible real matrix, that is, $a_{ij} \geqq 0$, then there exists a real positive eigenvalue. (See ref. 9.)

(3) If there exists a k such that A^k is a positive real matrix, then there exists an eigenvalue of A such that it is real, and its absolute value is greater than any other eigenvalue. If, in addition, k is an odd integer, then this eigenvalue is positive. (See ref. 10.)

TABLE I.- EIGENVALUE THEOREMS

Condition of A	Notation	Condition of λ	References
Nonzero operator	A	$\sigma(A)$ is a nonempty, closed, and bounded subset of complex numbers	2
		$\sigma(p(A)) = p(\sigma(A))$ where p is a polynomial	3
A^{-1} exists		$\sigma(A^{-1}) = \|\sigma(A)\|^{-1}$	3
Hermitian	$A = A^*$	All λ are real	4, 5, 6
Real symmetric	$A = A^T$	All λ are real	4, 5
Skew hermitian	$A = -A^*$	All λ are imaginary	4
Real skew symmetric	$A = -A^T$	All λ are imaginary	4
Isometry	$A^*A = I$	$\|\lambda_j\| = 1$	4, 6
Orthogonal	$A^TA = I$	$\|\lambda_j\| = 1$	4
Triangular, that is, $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$ or A^T		$\lambda_j = a_{jj}$	5
Permutation, that is, $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$	$A = P$	$\lambda_j = \cos\left(\frac{2\pi j}{n}\right) + i \sin\left(\frac{2\pi j}{n}\right)$ for $j = 0, 1, \ldots, n - 1$	7

THEOREMS FOR EIGENVALUE BOUNDS

The bounds for eigenvalues may be determined by various techniques. In general, these techniques express the bounds in terms of (1) the elements of the matrix itself, (2) matrix norms, and (3) vectors and eigenvalues of related matrices. Although the eigenvalues may be approximated by considering the roots of the characteristic equations, the necessary procedures (Newton's method, Graffe's method, etc.) require a "first guess" of the roots combined with successive iterations. These relations do not lend themselves to closed-form solutions of eigenvalue limits. In this paper, only those types of relations listed are investigated.

A.- Bounds by Matrix Elements

An important relationship giving the eigenvalue bounds in terms of the matrix elements and matrix order is provided by the following theorem. (See ref. 11.)

Theorem A1.- Let A be a complex matrix of order n. Define

$$G = \frac{1}{2}(A + A^*)$$

$$T = \frac{1}{2}(A - A^*)$$

Let

$$a = \max|a_{ij}|$$

$$g = \max|g_{ij}|$$

$$t = \max|t_{ij}|$$

$$\lambda = \alpha + i\beta$$

Then

$$|\lambda| \leqq na$$

$$|\alpha| \leqq ng$$

$$|\beta| \leqq nt$$

Proof: Let $Ax = \lambda x$ so that

$$\langle x,Ax\rangle = \langle x,\lambda x\rangle = \lambda\langle x,x\rangle$$

and

$$\langle Ax,x\rangle = \langle \lambda x,x\rangle$$

or

$$\langle x,A^*x\rangle = \bar{\lambda}\langle x,x\rangle$$

Then

$$<x,Ax> + <x,A^*x> = (\alpha + i\beta)<x,x> + (\alpha - i\beta)<x,x>$$

or

$$<x,(A + A^*)x> = 2\alpha<x,x>$$

$$<x,Gx> = \alpha<x,x>$$

Likewise

$$<x,Ax> - <x,A^*x> = 2i\beta<x,x>$$

or

$$<x,Tx> = i\beta<x,x>$$

$$-i<x,Tx> = \beta<x,x>$$

By the Cauchy-Schwarz inequality,

$$\left|\lambda<x,x>\right| = |\lambda|\left|<x,x>\right| = \left|<x,Ax>\right| \leqq \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||x_i||x_j| \leqq a\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i||x_j|\right)$$

$$\leqq \frac{a}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(|x_i|^2 + |x_j|^2\right) = \frac{a}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i|^2 + \frac{a}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_j|^2$$

$$= \frac{na}{2} + \frac{na}{2} = na$$

where the x terms are normalized so that $<x,x> = 1$. Thus $|\lambda| \leqq na$. Proceeding in a similar manner since

$$\alpha<x,x> = <x,Gx>$$

yields

$$|\alpha| \leqq ng$$

Likewise, since $\beta<x,x> = -i<x,Tx>$,

$$|\beta| \leqq nt$$

Bendixson (ref. 11) found a bound for the imaginary part for a real matrix A.

Theorem A2.- Let A be a real matrix of order n,

$$T = \frac{1}{2}\left(A - A^T\right)$$

$$\lambda = \alpha + i\beta$$

Then

$$|\beta| \leqq t\,\frac{\sqrt{n(n - 1)}}{2}$$

Proof: Since $Ax = \lambda x$ for $x = y + iz$,

$$A(y + iz) = (\alpha + i\beta)(y + iz) = (\alpha y - \beta z) + i(\alpha z + \beta y)$$

Equating real and imaginary parts yields

$$Ay = \alpha y - \beta z$$

$$Az = \alpha z + \beta y$$

so that

$$<y,Az> = <y,\alpha z> + <y,\beta y>$$

$$-<z,Ay> = -<z,\alpha y> + >z,\beta z>$$

and by adding

$$<y,Az> - <z,Ay> = \beta\left(<y,y> + <z,z>\right)$$

Now

$$<y,Az> - <z,Ay> = <y,Az> - <A^*z,y> = <y,Az> - <y,A^Tz> = <y,\left(A - A^T\right)z>$$

$$= \beta(<y,y> + <z,z>) = \beta\left(|y|^2 + |z|^2\right)$$

or by definition of T,

$$\beta\left(<y,y> + <z,z>\right) = 2<y,Tz>$$

Therefore

$$\frac{\beta}{2}\left(|y|^2 + |z|^2\right) = <y,Tz>$$

Since $T = -T^T$, $t_{ij} = -t_{ji}$ and $t_{1i} = 0$. Thus

$$<y,Tz> = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}y_iz_j = \sum_{i=1}^{n}\sum_{j=i+1}^{n} t_{ij}\left(y_iz_j - y_jz_i\right)$$

$$\leqq \sum_{i=1}^{n}\sum_{j=i+1}^{n} \left|t_{ij}\right|\left|y_iz_j - z_iy_j\right| \leqq t\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left|y_iz_j - z_iy_j\right|$$

where $t = \max\left|t_{ij}\right|$ and squaring the preceding equation yields

$$\beta^2\left(\|y\|^2 + \|z\|^2\right)^2 \leqq 4t^2\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left|y_iz_j - z_iy_j\right|\right)^2 \tag{1}$$

where $\|y\|^2 = \langle y,y \rangle$. By the arithmetic-geometric mean inequality, for real numbers r_i,

$$\left(r_1 + \cdots + r_m\right)^2 \leqq m\left(r_1^2 + \cdots + r_m^2\right)$$

There are n^2 elements in the matrix; the diagonals do not appear in this sum since $t_{ii} = 0$. For every two elements of the matrix, one combination is used in the summation. Thus, there are $\frac{n^2 - n}{2}$ or $\frac{n(n-1)}{2}$ combinations. Thus by the arithmetic-geometric mean inequality,

$$\left(\sum\sum \left|y_i z_j - z_i y_j\right|\right)^2 \leqq \frac{n(n-1)}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left|y_i z_j - z_i y_j\right|^2$$

Consider now

$$\left(|y|^2 + |z|^2\right)^2 - \left(|y|^2 - |z|^2\right)^2 = 4|y|^2|z|^2$$

By Lagrange's identity,

$$|y|^2|z|^2 = \langle y,z \rangle^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(y_i z_j - z_i y_j\right)^2$$

Thus

$$\left(|y|^2 + |z|^2\right)^2 \geqq 4 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(y_i z_j - z_i y_j\right)^2$$

Substituting this result into equation (1) yields

$$4\beta^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(y_i z_j - z_i y_j\right)^2 \leqq \beta^2\left(|y|^2 + |z|^2\right)^2 \leqq 4t^2\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} \left|y_i z_j - z_i y_j\right|\right)^2$$

$$\leqq 4t^2\left[\frac{n(n-1)}{2}\right] \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left|y_i z_j - y_j z_i\right|^2$$

Thus $\beta^2 \leqq t^2 \frac{n(n-1)}{2}$.

The importance of these two theorems lies in their ability to determine an upper bound for the real and imaginary components separately. However, the following theorem proven by Lévy-Hadamard-Gerschgorin (ref. 12) gives an even more basic result and has since been used as a cornerstone for many more theorems of eigenvalue bounds.

Theorem A3.- Every eigenvalue of a matrix is contained in at least one of the n disks whose centers are a_{ii} and whose radii are

$$r_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ik}| \qquad (i = 1, \ldots, n)$$

Proof: Let B be a matrix of order n. The system of equations $Bx = 0$ has a nontrivial solution if and only if $\det B = 0$. Let x_k be the dominant component of $x = (x_1, \ldots, x_n)$, that is, $|x_k| \geqq |x_i|$ for all i. Then, the kth equation is

$$b_{kk}x_k = -\sum_{\substack{m=1 \\ m \neq k}}^{n} b_{km}x_m$$

or

$$|b_{kk}||x_k| \leqq \sum_{\substack{m=1 \\ m \neq k}}^{n} |b_{km}||x_m| \leqq |x_k| \sum_{m \neq k}^{n} |b_{km}|$$

and thus

$$|b_{kk}| \leqq \sum_{\substack{m=1 \\ m \neq k}}^{n} |b_{km}|$$

Let $B = A - \lambda I$, where λ is such that $\det(A - \lambda I) = 0$, the "eigenvalue problem." Therefore

$$|\lambda - a_{kk}| \leqq \sum_{\substack{m=1 \\ m \neq k}}^{n} |a_{km}|$$

This theorem can be generalized to countably infinite dimensional operators which have a summable matrix representation; that is,

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij}| < \infty$$

Corollary A3: If A is a summable matrix whose eigenvectors $x = (x_1, x_2, \ldots)$ are in l_1 and $x = (x_1, x_2, \ldots)$ is in l_1 means that

$$\sum_{i=1}^{\infty} |x_i| < \infty$$

the results of theorem A3 hold.

Proof: For all eigenvectors x of A, x in l_1 implies that there exists a component, say $\bar{x}_k$ of x, for which $\bar{x}_k$ is a dominant component (that is, $|x_k| \geqq |x_i|$ for all i). The kth equation is then

$$b_{kk}x_k = -\sum_{\substack{m=1 \\ m \neq k}}^{\infty} b_{km}x_m$$

where $b_{ij} = a_{ij}$ for $i \neq j$ and $b_{ii} = a_{ii} - \lambda$ so that, as before, $|b_{kk}| \leqq \sum_{\substack{m=1 \\ m \neq k}} |b_{km}| < \infty$.

Thus all eigenvalues are bounded by

$$|\lambda - a_{kk}| \leqq \sum_{\substack{m=1 \\ m \neq k}}^{\infty} |a_{km}| < \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty$$

<u>Theorem A4</u>.- The following theorem is of interest with respect to the preceding corollary. For a summable matrix, an eigenvector is in l_1 if and only if it has a finite dominant component.

Proof: From $Ax = \lambda x$,

$$\left| \sum_{j=1}^{\infty} a_{ij}x_j \right| = |\lambda| \, |x_i| \qquad (i = 1, 2, \ldots)$$

so that

$$|\lambda| \sum_{i=1}^{\infty} |x_i| = \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{ij}x_j \right| \leqq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| \, |x_j|$$

Therefore

$$\sum_{j=1}^{\infty} |x_j| \leqq \frac{1}{|\lambda|} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| \, |x_j| \leqq \frac{|x_k|}{\lambda} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty$$

if x_k is a dominant component and $\lambda \neq 0$. Thus, $\sum_{i=1}^{\infty} |x_i| < \infty$ and $x \in l_1$.

Clearly, for $x \in l_1$, x has a dominant component

Corollary A4 (Frobenius): An almost immediate consequence of theorem 4 is the well-known "Theorem of Frobenius."

$$|\lambda|_{max} \leqq \max \sum_{m=1}^{n} |a_{km}|$$

$$|\lambda|_{min} \geqq \min\left(|a_{kk}| - \sum_{\substack{m=1 \\ k\neq m}}^{n} |a_{km}|\right)$$

Proof:

$$|\lambda - a_{kk}| \geqq |\lambda| - |a_{kk}|$$

so that from the preceding inequalities

$$|\lambda| \leqq |a_{kk}| + \sum_{\substack{m=1 \\ m\neq k}}^{n} |a_{km}| = \sum_{m=1}^{n} |a_{km}|$$

Also

$$|\lambda - a_{kk}| \geqq |a_{kk}| - |\lambda|$$

so that

$$|\lambda| \geqq |a_{kk}| - \sum_{\substack{m=1 \\ m\neq k}}^{n} |a_{km}|$$

Also, since $\det A = \det A^T$,

$$\sum_{\substack{m=1 \\ m\neq k}}^{n} |a_{km}|$$

may be replaced in theorem A3 and its corollary A3 by

$$\sum_{\substack{m=1 \\ m\neq k}}^{n} |a_{mk}|$$

Thus, the centers of the circles containing the eigenvalues will remain unchanged even though their radii will be changed.

Theorem A5.- As a further refinement of corollary A3, Alfred Brauer (ref. 13) was able to restrict the regions containing the eigenvalues by means of the "ovals of Cassini" in this theorem. Each eigenvalue of A lies in at least one of the $\frac{n(n-1)}{2}$ ovals of Cassini

$$|\lambda - a_{kk}||\lambda - a_{ll}| \leqq \left(\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|\right)\left(\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{lj}|\right)$$

and in at least one of the ovals

$$|\lambda - a_{kk}||\lambda - a_{ll}| \leqq \left(\sum_{\substack{i=1 \\ i\neq k}}^{n} |a_{ik}|\right)\left(\sum_{\substack{j=1 \\ j\neq l}}^{n} |a_{lj}|\right)$$

Proof: For $x = (x_1, x_2, \ldots x_n)$

$$(\lambda - a_{ii})x_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j \qquad (i = 1, 2, \ldots, n)$$

Let $|x_k| \geqq |x_l| \geqq |x_j|$ for $j \neq k$, $j \neq l$. Then

$$|\lambda - a_{kk}||x_k| \leqq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}||x_j| \leqq \left(\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|\right)|x_l|$$

and

$$|\lambda - a_{ll}||x_l| \leqq \sum_{\substack{j=1 \\ j\neq l}}^{n} |a_{lj}||x_j| \leqq \left(\sum_{\substack{j=1 \\ j\neq l}}^{n} |a_{lj}|\right)|x_k|$$

Therefore

$$|\lambda - a_{kk}||\lambda - a_{ll}||x_k||x_l| \leqq \left(\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|\right)\left(\sum_{\substack{j=1 \\ j\neq l}}^{n} |a_{lj}|\right)|x_k||x_l|$$

so that

$$\left|\lambda - a_{kk}\right|\left|\lambda - a_{ll}\right| \leqq \left(\sum_{\substack{j=1\\j\neq k}}^{n}\left|a_{kj}\right|\right)\left(\sum_{\substack{j=1\\j\neq l}}^{n}\left|a_{lj}\right|\right)$$

which proves the theorem.

Similarly, it may be shown that all eigenvalues of A are contained in at least one of the ovals

$$\left|\lambda - a_{kk}\right|\left|\lambda - a_{ll}\right| \leqq \left(\sum_{\substack{i=1\\i\neq k}}^{n}\left|a_{ik}\right|\right)\left(\sum_{\substack{i=1\\i\neq l}}^{n}\left|a_{ik}\right|\right)$$

There are n elements a_{ii} which, in part, form the ovals. The number of distinct subsets with two elements that can be chosen from this set of n elements is

$$\frac{n!}{2!\,(n - 2)!} = \frac{n(n - 1)(n - 2)!}{2(n - 2)!} = \frac{n(n - 1)}{2}$$

Thus, there are $\frac{n(n - 1)}{2}$ ovals.

Theorem A6.- Another inequality (ref. 14) giving the regions in which the eigenvalues are contained is presented in the following theorem. For the matrix $A = \left(a_{ij}\right)_n$,

$$\left|\lambda - a_{ii}\right| \leqq \left(\sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right|\right)^{\alpha}\left(\sum_{\substack{k=1\\k\neq i}}^{n}\left|a_{ki}\right|\right)^{1-\alpha}$$

for $0 \leqq \alpha \leqq 1$.

Proof: As was shown in theorem A3 and corollary A3, for the determinant of $A - \lambda I$ to vanish, the following inequalities must be satisfied:

$$\left|\lambda - a_{ii}\right| \leqq \sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right|$$

$$\left|\lambda - a_{ii}\right| \leqq \sum_{\substack{k=1\\i\neq k}}^{n}\left|a_{ki}\right|$$

Thus,

$$|\lambda - a_{ii}| = \left(|\lambda - a_{ii}|^{\alpha}\right)\left(|\lambda - a_{ii}|^{1-\alpha}\right) \leqq \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\right)^{\alpha}\left(\sum_{\substack{k=1 \\ i\neq k}}^{n} |a_{ki}|\right)^{1-\alpha}$$

whenever $0 \leqq \alpha \leqq 1$.

Corollary A6(a): Two simple corollaries to this theorem are presented,

$$|\lambda^{A}|_{max} \leqq \left[|a_{ii}| + \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\right)^{\alpha}\left(\sum_{\substack{k=1 \\ i\neq k}}^{n} |a_{ki}|\right)^{1-\alpha}\right]$$

$$|\lambda^{A}|_{min} \geqq \left[|a_{ii}| - \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\right)^{\alpha}\left(\sum_{\substack{k=1 \\ i\neq k}}^{n} |a_{ki}|\right)^{1-\alpha}\right]$$

$$|\lambda^{A}|_{max} \leqq \left[|a_{ii}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\right]^{\alpha}\left[|a_{ii}| + \sum_{\substack{k=1 \\ i\neq k}}^{n} |a_{ki}|\right]^{1-\alpha}$$

$$|\lambda^{A}|_{min} \geqq \left[|a_{ii}| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\right]^{\alpha}\left[|a_{ii}| + \sum_{\substack{k=1 \\ i\neq k}}^{n} |a_{ki}|\right]^{1-\alpha}$$

All for $0 \leqq \alpha \leqq 1$. The corollaries and theorem hold likewise for $\alpha = 1 - \beta$, $1 - \alpha = \beta$ where $0 \leqq \beta \leqq 1$.

Corollary A6(b): Corollary A6(b) is a direct consequence of theorem A5. For each α, $0 \leqq \alpha \leqq 1$, every eigenvalue of A lies in at least one of the $\frac{n(n-1)}{2}$ ovals,

$$|z - a_{ii}|\,|z - a_{jj}| \leqq \left[\left(\sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|\right)\left(\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|\right)\right]^{1-\alpha}\left[\left(\sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ji}|\right)\left(\sum_{\substack{k=1 \\ j\neq k}}^{n} |a_{jk}|\right)\right]^{\alpha}$$

For $\alpha = 0$ or $\alpha = 1$, this relation reduces to theorem A5.

Theorem A7.- As a further extension of theorem A5, the largest eigenvalue may be bounded from the results of reference 13. Each eigenvalue λ satisfies

$$|\lambda| \leq \frac{1}{2} \max_{\substack{k,j=1,2,\ldots,n \\ k\neq j}} \left\{ |a_{kk}| + |a_{jj}| + \left[\left(|a_{kk}| - |a_{jj}| \right)^2 + 4P_k P_j \right]^{1/2} \right\} = M$$

where

$$P_k = \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|$$

Proof: Assume that $|a_{rr}| \geqq |a_{ss}|$.

(1) If $|\lambda| \leqq |a_{rr}|$, then

$$\begin{aligned} |\lambda| &\leqq \frac{1}{2}\left(|a_{rr}| + |a_{ss}|\right) + \frac{1}{2}\left(|a_{rr}| - |a_{ss}|\right) \\ &\leqq \frac{1}{2}\left\{ |a_{rr}| + |a_{ss}| + \left[\left(|a_{rr}| - |a_{ss}| \right)^2 + 4P_r P_s \right]^{1/2} \right\} \leqq M \end{aligned}$$

since

$$P_r \geqq 0$$

$$P_s \geqq 0$$

(2) If $|\lambda| > |a_{rr}| \geqq |a_{ss}|$, then

$$0 < |\lambda| - |a_{rr}| \leqq |\lambda - a_{rr}|$$

$$0 < |\lambda| - |a_{ss}| \leqq |\lambda - a_{ss}|$$

From the corollary A6(b),

$$\left(|\lambda| - |a_{rr}|\right)\left(|\lambda| - |a_{ss}|\right) \leqq |\lambda - a_{rr}|\,|\lambda - a_{ss}| \leqq P_r P_s$$

or

$$|\lambda|^2 - \left(|a_{rr}| + |a_{ss}|\right)|\lambda| + |a_{rr}a_{ss}| - P_r P_s \leqq 0$$

and

$$\underbrace{\left\{|\lambda| - \frac{1}{2}\left[|a_{rr}| + |a_{ss}| + \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_rP_s}\right]\right\}}_{\text{Part (1)}}$$

$$\times \underbrace{\left\{|\lambda| - \frac{1}{2}\left[|a_{rr}| + |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_rP_s}\right]\right\}}_{\text{Part (2)}} \leqq 0$$

Thus, either Part (1) $\geqq 0$ and Part (2) $\leqq 0$ or Part (1) $\leqq 0$ and Part (2) $\geqq 0$. However, since

$$\text{Part (1)} = |\lambda| + \frac{-|a_{rr}| - |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_rP_s}}{2}$$

$$\leqq |\lambda| + \frac{-|a_{rr}| - |a_{ss}| + \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_rP_s}}{2} = \text{Part (2)}$$

then it must be true that Part (1) $\leqq 0 \leqq$ Part (2). Thus from part (1), it follows that

$$|\lambda| \leqq \frac{1}{2}\left\{|a_{rr}| + |a_{ss}| + \left[\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_rP_s\right]^{1/2}\right\} \leqq M$$

In addition, if a third condition is satisfied, namely,

$$|a_{kk}a_{jj}| > P_kP_j$$

then a similar type of lower bound for the modulus of the eigenvalues of A can be formulated (ref. 13).

Theorem A8.- If

$$|a_{kk}a_{jj}| > P_kP_j \qquad (k, j = 1, 2, \ldots, n)$$

then

$$|\lambda| \geqq \min_{k,j=1,2,\ldots,n} \left\{ |a_{kk}| + |a_{jj}| - \left[\left(|a_{kk}| - |a_{jj}| \right)^2 + 4P_k P_j \right]^{1/2} \right\} = m > 0$$

Proof: As was shown in the proof of theorem A7

$$|\lambda| - \frac{1}{2}\left[|a_{rr}| + |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}| \right) + 4P_r P_s} \right] \geqq 0$$

or

$$|\lambda| \geqq \frac{1}{2}\left[|a_{rr}| + |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}| \right) + 4P_r P_s} \right] \geqq m$$

Assume that m is attained where $k = \gamma$, $j = \delta$, so that

$$m = \left\{ |a_{\gamma\gamma}| + |a_{\delta\delta}| - \left[\left(|a_{\gamma\gamma}| - |a_{\delta\delta}| \right)^2 + 4P_\gamma P_\delta \right]^{1/2} \right\}$$

$$= \left[|a_{\gamma\gamma}| + |a_{\delta\delta}| - \left(|a_{\gamma\gamma}|^2 - 2|a_{\gamma\gamma}a_{\delta\delta}| + |a_{\delta\delta}|^2 + 4P_\gamma P_\delta \right)^{1/2} \right]$$

$$> \left[|a_{\gamma\gamma}| + |a_{\delta\delta}| - \left(|a_{\gamma\gamma}|^2 - 2|a_{\gamma\gamma}a_{\delta\delta}| + |a_{\delta\delta}|^2 + 4|a_{\gamma\gamma}a_{\delta\delta}| \right)^{1/2} \right]$$

$$= \left\{ |a_{\gamma\gamma}| + |a_{\delta\delta}| - \left[\left(|a_{\gamma\gamma}| + |a_{\delta\delta}| \right)^2 \right]^{1/2} \right\} = 0$$

Note that all the previous theorems have given bounds only for the modulus of the eigenvalues. However, for a particular case, more definite information may be implied from exact information regarding the values of the elementary symmetric functions of $\lambda_1, \lambda_2, \ldots, \lambda_n$. Several important results concerning these functions are given by the following theorem.

Theorem A9.- For an arbitrary matrix $A = (a_{ij})_n$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

$$\prod_{i=1}^{n} \lambda_i = \det A$$

and

$$\sum \lambda_i^{\,2} \leqq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{\,2}$$

if A is real.

Proof: Let

$$\psi(\lambda) = \det(A - \lambda I)$$

By a Maclaurin's series expansion of $\psi(x)$, the coefficient of λ^{k-1} is

$$\left.\frac{d^{k-1}\psi(\lambda)}{(d\lambda)^{k-1}}\right|_{\lambda=0} = (k - 1)!\left(a_{11} + \ . \ . \ . + a_{nn}\right)$$

Also by the fundamental theorem of algebra,

$$\psi(\lambda) = \left(\lambda_1 - \lambda\right)\left(\lambda_2 - \lambda\right) \ . \ . \ . \left(\lambda_n - \lambda\right)$$

where the $\lambda_1, \ . \ . \ ., \lambda_n$ are the eigenvalues of A. Then

$$\left.\frac{d^{k-1}\psi(\lambda)}{(d\lambda)^{k-1}}\right|_{\lambda=0} = (k - 1)!\left(\lambda_1 + \ . \ . \ . \ \lambda_n\right)$$

Thus

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} \equiv \text{Trace of } A$$

Also for $\psi = \det(A - \lambda I)$

$$\psi(0) = \left.\frac{d^0\psi(\lambda)}{(d\lambda)^0}\right|_{\lambda=0} = \det A$$

and for $\psi(\lambda) = \prod_{i=1}^{n} \lambda_i - \lambda$

$$\psi(0) = \prod_{i=1}^{n} \lambda_i$$

Thus $\Pi \lambda_i = \det A$. Likewise, the other elementary symmetric functions are the corresponding coefficients of the characteristic equation.

Since the multiplicity of λ_i in A is the same as the multiplicity of λ_i^k in A^k, then

$$\text{Trace of } A^k = \sum_{i=1}^{n} \lambda_i^k$$

(Note, if all $\lambda = 0$, then $\psi(\lambda) = \lambda^n = \det(A - \lambda I) = \det A = 0$.) Let $k = 2$ so that

$$\text{Trace } A^2 = \lambda_1^2 + \ldots + \lambda_n^2$$

Also

$$\text{Trace } A^2 = \sum_{i=1}^{n} a_{ii}^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ i \neq k}}^{n} a_{ik} a_{ki}$$

$$\text{Trace } A^T A = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ki}^2 = \sum_{i=1}^{n} a_{ii}^2 + \sum_{\substack{k=1 \\ k \neq i}}^{n} \sum_{i=1}^{n} a_{ik}^2$$

Since $a_{ik}^2 + a_{ki}^2 \geqq 2a_{ik}a_{ki}$, then

$$\text{Trace } A^2 \leqq \text{Trace } A^T A$$

Thus

$$\sum_{k=1}^{n} \lambda_k^2 \leqq \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2$$

B.- Bounds by Matrix Norms

In this section, the eigenvalue bounds are determined in terms of matrix norms. A matrix norm $||A||$ of a square matrix A is any bounded real-valued function for which the following matrix norm properties are true:

(1) $||A|| > 0$ whenever $A \neq 0$

(2) $||\alpha A|| = |\alpha| \, ||A||$ where α is a scalar

(3) $||A + B|| \leqq ||A|| + ||B||$

(4) $||AB|| \leqq ||A|| \, ||B||$

(Property (3) implies that the norm is a continuous function of A; that is, $\left| \|A\| - \|A_n\| \right| < \epsilon$ for any ϵ, whenever $\|A - A_n\|$ is sufficiently small.)

The following is a listing of several possible norms for an arbitrary matrix $A = \left(a_{ij}\right)_n$:

$$\|A\|_E = \sqrt{\text{Trace } (A^*A)} = \text{Square root of sum of squares of } A \quad \text{(Euclidean)}$$

$$\|A\|_e = \text{Maximal row (column) sum of } \left(|a_{ij}|\right)_n$$

$$\|A\|_{e^T} = \text{Maximal row (column) sum of } \left(|a_{ij}|\right)_n^*$$

$$\|A\|_g = \left\|G^{-1}AG\right\|_e$$

where

G	any nonsingular matrix
$\|A\|$	any matrix norm

$G_e = g$

$$e = \begin{Bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}$$

$$\|A\|_S = \sum_{i,j=1}^{n} |a_{ij}|$$

$\|A\|_I = \max\limits_{\|x\|=1} \|Ax\|$ which is the induced norm

A relation between the eigenvalue bounds and the value of powers of the matrix is given in reference 15.

Theorem B1.- All eigenvalues λ^A of the matrix A are contained within the unit circle if and only if

$$\lim_{n \to \infty} A^n = 0$$

Proof (1): Assume that all eigenvalues of A are contained within the unit circle. Then choose an arbitrary $\epsilon > 0$ so that

$$|\lambda^A|_{max} + \epsilon < 1$$

It is now desirable to find a matrix norm with

$$\|A\| \leqq |\lambda^A|_{max} + \epsilon < 1$$

for by property (4) of matrix norms, it follows that

$$\|A^n\| < \|A\|^n$$

If $\|A\| < 1$, then

$$\lim_{n\to\infty} \|A\|^n = 0$$

This relation implies that

$$\lim_{n\to\infty} \|A^n\| = 0$$

By the contrapositive of property (1) of matrix norms and norm continuity

$$\lim_{n\to\infty} \|A^n\| = 0$$

implies that

$$\lim_{n\to\infty} A^n = 0$$

so that the sufficiency portion would be proven.

A desired matrix actually does exist. Define $\|A\|_g$ to be the maximal row sum of absolute values of $G^{-1}AG$.

Let Λ be the Jordan canonical or normal form of A so that $A = T^{-1}\Lambda T$. Then Λ is an upper triangular matrix. Let $P = \mathrm{diag}(\delta^{-n}, \delta^{1-n}, \delta^{2-n}, \ldots)$ where $\delta > 0$. Then

$$P^{-1}\Lambda P = \begin{bmatrix} \lambda_i & \delta & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_i & \delta & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_i & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_j \end{bmatrix}$$

Thus $\|P^{-1}\Lambda P\|_e \leqq |\lambda^A|_{max} + \epsilon$ since the value of its maximal row sum may be made sufficiently close to $|\lambda^A|_{max}$ by choosing δ small enough. Transforming the e norm by P, that is, Pe = g, yields

$$\|\Lambda\|_g = \|P^{-1}\Lambda P\|_e \leqq |\lambda^A|_{max} + \epsilon < 1$$

Therefore, by defining the norm $\|A\| \equiv \|\Lambda\|_g$, the sufficiency portion is proven.

Proof (2): Let $\lim_{n\to\infty} A^n = 0$. Now

$$A^n = \left(T^{-1}\Lambda T\right)^n = \left(T^{-1}\Lambda T\right)_1\left(T^{-1}\Lambda T\right)_2 \ . \ . \ . \ \left(T^{-1}\Lambda T\right)_n = T^{-1}\Lambda^n T$$

Therefore, $\lim_{n\to\infty} A^n = 0$ implies

$$\lim_{n\to\infty} T^{-1}\Lambda^n T = 0$$

or

$$0 = T^{-1}\left(\lim_{n\to\infty} \Lambda^n\right)T$$

Thus

$$\lim_{n\to\infty} \Lambda^n = 0$$

Then each element ${}_na_{ij}$ of the Jordan matrix Λ^n must be such that

$$\lim_{n\to\infty}\left[{}_na_{ij}\right] = 0$$

for all i,j.

Let Λ be partitioned into block diagonal form where each block corresponds to a distinct eigenvalue of A. That is,

$$\Lambda = \begin{bmatrix} J_1 & & & & 0 \\ & J_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & J_k \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & \cdot & & \\ & & \cdot & & \\ & & & \cdot & 1 \\ 0 & & & & \lambda_i \end{bmatrix}$$

Then

$$\Lambda^n = \begin{bmatrix} J_1^n & & & & 0 \\ & J_2^n & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & J_k^n \end{bmatrix}$$

$$J_i^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda^{n-1} & \cdots & \binom{n}{j}\lambda^{n-j} \\ 0 & \lambda_i^n & \cdots & \binom{n}{j+1}\lambda^{n+1-j} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda^n \end{bmatrix}$$

where

$$\binom{n}{j} = \frac{n!}{j!\,(n - j)!}$$

Thus, in particular,

$$\lim_{n \to \infty} \lambda_i^n = 0$$

so that $|\lambda_i| < 1$ for all i. The importance of this relation is obvious when iterative (numerical) techniques defining the matrix A as the error in the approximate solution are considered.

Theorem B2.- One of the most significant and generalized results is given by the following theorem (ref. 15). For an arbitrary matrix A, the largest possible eigenvalue modulus is $|\lambda^A|_{max} \leqq \|A\|$ for any matrix norm of A.

Proof: Let $\|A\| = \alpha$, a real scalar. Also define $B_\epsilon = \frac{A}{(\alpha + \epsilon)}$ where ϵ is positive.

Consider

$$\|B_\epsilon\| = \left\|\frac{A}{(\alpha + \epsilon)}\right\| = \frac{1}{(\alpha + \epsilon)}\|A\| = \frac{\alpha}{\alpha + \epsilon} < 1$$

for all $\epsilon > 0$, that is, $\|B_\epsilon\| < 1$. By the proof of theorem B1 (proof (1)), $\|B_\epsilon\| < 1$ implies that $\lim_{n \to \infty} B_\epsilon^n = 0$, which, by the result of theorem B1, implies that for all eigenvalues of B_ϵ, $\lambda_i^{B_\epsilon}$, it is true that $\left|\lambda_i^{B_\epsilon}\right| < 1$. If λ_i^A is any eigenvalue of A, there will exist a corresponding eigenvalue of B such that

$$\lambda_i^{B_\epsilon} = \frac{\lambda_i^A}{\alpha + \epsilon}$$

From this relation, since $\left|\lambda_i^{B_\epsilon}\right| < 1$ for all eigenvalues of B_ϵ, then

$$\left|\frac{\lambda_i^A}{\alpha + \epsilon}\right| < 1$$

or

$$\left|\lambda_i^A\right| < \alpha + \epsilon = \|A\| + \epsilon$$

Therefore $\left|\lambda_i^A\right|_{max} \leqq \|A\|$ since the relation is true for all $\epsilon > 0$. By using this result, a more precise bound is established. (See ref. 2.)

Theorem B3.-

$$|\lambda^A|_{max} = \lim_{n \to \infty} \|A^n\|^{1/n}$$

Proof: From previous theorems,

$$|\lambda^{A^n}|_{max} \leqq \|A^n\|$$

and

$$\lambda^{A^n} = (\lambda^A)^n$$

Thus for all n

$$|\lambda^A|_{max} \leqq \|A^n\|^{1/n} \tag{2}$$

so that

$$|\lambda^A|_{max} \leqq \inf_n \|A^n\|^{1/n}$$

and thus

$$|\lambda^A|_{max} \leqq \lim_{n\to\infty} \inf_n \|A^n\|^{1/n}$$

By theorem B1, $|\lambda^B|_{max} < 1$ implies $\lim_{n\to\infty} \|B^n\| = 0$, so for all $n > N$, $\|B^n\| < 1$ and $\|B^n\|^{1/n} < 1$. Thus,

$$\sup_{n>N} \|B^n\|^{1/2} \leqq 1$$

and

$$\lim_{n\to\infty} \sup_n \|B^n\|^{1/n} \leqq 1$$

Then for arbitrary A and $\epsilon > 0$, let $C = \left(|\lambda^A|_{max} + \epsilon\right)^{-1} A$ so that $|\lambda_C|_{max} < 1$ and

$$\lim_{n\to\infty} \sup_n \|C^n\|^{1/n} \leqq 1$$

Therefore

$$\lim_{n\to\infty} \sup_n \|A^n\|^{1/n} \leqq |\lambda^A|_{max} \tag{3}$$

Combining inequalities (2) and (3) yields

$$|\lambda^A|_{max} \leqq \lim_{n\to\infty} \inf_n \|A^n\|^{1/n} \leqq \lim_{n\to\infty} \sup_n \|A^n\|^{1/n} \leqq |\lambda^A|_{max}$$

Theorem B4.- If A is Hermitian, then $\|A\|_I = \sup\{|\lambda| : \lambda \epsilon\, \sigma(A)\}$. The proof of this theorem is given in reference 3.

If the matrix is partitioned such that each diagonal submatrix is square, then eigenvalue bounds may be determined by procedures similar to those used in the section "A.- Bounds by Matrix Elements."

Let A be any matrix order n, which is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{N1} & & \cdots & A_{NN} \end{bmatrix}$$

where the diagonal submatrices A_{ii} are square of order n_i. Define the matrix norm by

$$\left\|A_{ij}\right\| \equiv \sup_{\substack{x \in \Omega_j \\ x \neq 0}} \frac{\left\|A_{ij}x\right\|_{\Omega_i}}{\left\|x\right\|_{\Omega_j}}$$

for an arbitrary vector norm over the subspace Ω_k. If the diagonal submatrices A_{ii} are nonsingular and if

$$\left(\left\|A_{jj}^{-1}\right\|\right)^{-1} > \sum_{\substack{k=1 \\ k \neq j}}^{N} \left\|A_{jk}\right\| \qquad (1 \leqq j \leqq N)$$

then the matrix A is said to be "block strictly diagonally dominant." (See ref. 16.)

Theorem B5.- For every partitioning of the matrix A, each eigenvalue λ^A satisfies

$$\left[\left\|\left(A_{j,j} - \lambda I_j\right)^{-1}\right\|\right]^{-1} \leqq \sum_{\substack{k=1 \\ k \neq j}}^{N} \left\|A_{j,k}\right\|$$

whenever the $\left(A_{jj} - \lambda I_j\right)^{-1}$ exist.

Proof: Assume that $A - \lambda I$ is singular. Then there exists a nonzero partitioned vector $X = \begin{bmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{bmatrix}$ such that

$$(A - \lambda I)X = 0$$

Consider $A - \lambda I$ in its partitioned form; this relation implies

$$\sum_{\substack{j=1 \\ i \neq j}}^{N} A_{ij} X_j = -(A_{ii} - \lambda I_i) X_i \tag{4}$$

Let X_r be the largest component of X, that is,

$$\|X_r\| \geqq \|X_j\| \qquad (1 \leqq j \leqq N)$$

Divide X by $\|X_r\|$. Then from equation (4)

$$\left\| \sum_{\substack{j=1 \\ j \neq r}}^{N} A_{rj} X_j \right\| = \left\| (A_{rr} - \lambda I_r) X_r \right\| \tag{5}$$

From the Cauchy-Schwarz inequality, the left-hand side of equation (5) is such that

$$\left\| (A_{rr} - \lambda I_r) X_r \right\| \leqq \sum_{\substack{j=1 \\ j \neq r}}^{n} \|A_{rj}\| \, \|X_j\| \leqq \sum_{\substack{j=1 \\ j \neq r}}^{n} \|A_{rj}\| \tag{6}$$

since

$$1 = \|X_r\| \geqq \|X_j\|$$

by the division of X.

Let $Z_{rr} = (A_{rr} - \lambda I_r) X_r$. Then

$$\left\| (A_{rr} - \lambda I_r) X_r \right\| = \frac{\left\| (A_{rr} - \lambda I_r) X_r \right\|}{\|X_r\|} = \frac{\|Z_{rr}\|}{\left\| (A_{rr} - \lambda I_r)^{-1} Z_{rr} \right\|} \geqq \left[\left\| (A_{rr} - \lambda I_r)^{-1} \right\| \right]^{-1} \tag{7}$$

since

$$\frac{\left\| (A_{rr} - \lambda I_r)^{-1} X_r \right\|}{\|X_r\|} \leqq \sup_{X} \frac{\left\| (A_{rr} - \lambda I_r)^{-1} X \right\|}{\|X\|} = \left\| (A_{rr} - \lambda I_r)^{-1} \right\|$$

The first part of the inequality follows from the fact that

$$\inf \frac{\|Bx\|}{\|x\|} = \inf \frac{\|y\|}{\|B^{-1}y\|} = \frac{1}{\sup \dfrac{\|B^{-1}y\|}{\|y\|}} = \frac{1}{\|B^{-1}\|} = \|B^{-1}\|^{-1}$$

From this equality and continuity of the norm, if B is singular, the definition $\|B^{-1}\|^{-1} = 0$ is obtained. Then from equations (5), (6), and (7)

$$\left[\left\|\left(A_{rr} - \lambda I_r\right)^{-1}\right\|\right]^{-1} \leqq \sum_{\substack{j=1 \\ j \neq r}}^{n} \|A_{rj}\|$$

If, in theorem A5, $|\lambda - a_{ii}|$ is replaced by the general form $\left[\left\|\left(A_{ii} - \lambda I_i\right)^{-1}\right\|\right]^{-1}$ and

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$$

is replaced by

$$\sum_{\substack{l=1 \\ l \neq i}}^{n} \|A_{il}\|$$

then an identical proof (ref. 16) will give the following corollary.

Corollary B5: All eigenvalues of A, λ^A, lie in the union of the $\frac{N(N - 1)}{2}$ point sets defined by

$$\left[\left\|\left(A_{ii} - \lambda I_i\right)^{-1}\right\|^{-1} \left\|\left(A_{jj} - \lambda I_j\right)^{-1}\right\|\right]^{-1} \leqq \left(\sum_{\substack{l=1 \\ l \neq i}}^{n} \|A_{i,l}\|\right)\left(\sum_{\substack{l=1 \\ l \neq j}}^{n} \|A_{j,l}\|\right)$$

where

$$1 \leqq i$$

$$j \leqq N$$

$$i \neq j$$

In a similar manner, if these substitutions are made in theorem A6, and an identical proof is used, the result will be the following corollary. (See ref. 16.)

Corollary B5: For any α with $0 \leqq \alpha \leqq 1$, each eigenvalue of A satisfies

$$\left[\left\|\left(A_{jj} - \lambda I_j\right)^{-1}\right\|\right]^{-1} \leqq \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \|A_{jk}\|\right)^{\alpha} \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \|A_{kj}\|\right)^{\alpha - 1}$$

for at least one j, $1 \leqq j \leqq N$.

C.- Bounds by Vectors and Related Matrices

This section determines eigenvalue bounds in terms of vectors or in terms of the eigenvalues of related matrices. Most of the following proofs depend upon the quadratic form of a matrix combined with simple geometric inequalities.

Bendixson proved the following result for a real matrix $A = (a_{ij})_n$; it was extended by Hirsch (ref. 11) to the complex case.

Theorem C1.- If $\lambda^A = \alpha + i\beta$, and $\lambda_{max}^{\frac{1}{2}(A+A^*)}$ and $\lambda_{min}^{\frac{1}{2}(A+A^*)}$ are the largest and smallest eigenvalues of $\frac{1}{2}(A + A^*)$, then

$$\lambda_{max}^{\frac{1}{2}(A+A^*)} \geqq \alpha \geqq \lambda_{min}^{\frac{1}{2}(A+A^*)}$$

Proof: Let H be an arbitrary Hermitian matrix and U be the unitary transformation such that U^*HU is a diagonal matrix. If the equality

$$<x,Hx> = \sigma<x,x>$$

is satisfied by a nontrivial x, then

$$\lambda_{max}^{H} \geqq \sigma \geqq \lambda_{min}^{H}$$

For

$$<x,Hx> = <Uy,HUy> = <y,U^*HUy> = \sum \lambda_i^H y_i^2$$

If $<x,Hx> = \sigma<x,x>$, then

$$\sigma<x,x> = \sigma<Uy,Uy> = \sigma<y,U^*Uy> = \sigma \sum_{i=1}^{n} |y|_i^2$$

Thus

$$\lambda_{max}^{H} y_1\bar{y}_1 + \ldots + \lambda_{min}^{H} y_n\bar{y}_n = \sigma \sum_{i=1}^{n} y_i\bar{y}_i$$

or

$$\lambda_{max}^{H}\left(\sum_{i=1}^{n} y_i\bar{y}_i\right) \geqq \sigma\left(\sum_{i=1}^{n} y_i\bar{y}_i\right) \geqq \lambda_{min}^{H}\left(\sum_{i=1}^{n} y_i\bar{y}_i\right)$$

Therefore

$$\lambda^H_{max} \geqq \sigma \geqq \lambda^H_{min}$$

As was shown in theorem A1 when $\lambda^A = \alpha + i\beta$ and x is an eigenvector corresponding to λ^A

$$\frac{1}{2}\sum\left(a_{ij} + \bar{a}_{ji}\right)\bar{x}_i x_j = \alpha<x,x>$$

so that

$$\lambda_{max}^{\frac{1}{2}(A+A^*)} \geqq \alpha \geqq \lambda_{min}^{\frac{1}{2}(A+A^*)}$$

Corollary C1: Since $\frac{A - A^*}{2i}$ is also Hermitian and since

$$\beta = \frac{1}{2i}\sum\left(a_{ij} - \bar{a}_{ji}\right)\bar{x}_i x_j$$

then

$$\lambda_{max}^{\frac{1}{2i}(A-A^*)} \geqq \beta \geqq \lambda_{min}^{\frac{1}{2i}(A-A^*)}$$

Just as in the proof of theorem C1, related vectors may be used to define eigenvalues and their bounds.

If λ_i is an eigenvalue with a corresponding eigenvector x_i for the complex matrix A, $Ax_i = \lambda x_i$; thus

$$<x_i,Ax_i> = <x_i,\lambda x_i> = \lambda<x_i,x_i>$$

or

$$\lambda_i = \frac{<x_i,Ax_i>}{<x_i,x_i>}$$

In the more general form, this quotient

$$q(x) \equiv \frac{<x,Ax>}{<x,x>}$$

for arbitrary vector x is called the Rayleigh's quotient.

If A is Hermitian, there exists a unitary matrix U such that U^*AU is a diagonal matrix and $U^*U = I$. Then

$$<x,Ax> = <Uy,AUy> = <y,U^*AUy> = <y,(\text{diag } \lambda)y>$$

Also if $<x,x> = 1$,

$$1 = <x,x> = <x,U^*Ux> = <Ux,Ux> = <y,y>$$

Thus the values assumed by $<x,Ax>$ on $<x,x> = 1$ are equal to the values assumed by $<y,(\text{diag } \lambda_i)y>$ on $<y,y> = 1$. However,

$$<y,(\text{diag } \lambda_i)y> = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$$

so that

$$<y,(\text{diag } \lambda_i)y> \geqq \lambda_{min} <y,y> = \lambda_{min}$$

and

$$<y,(\text{diag } \lambda_i)y> \leqq \lambda_{max} <y,y> = \lambda_{max}$$

Thus

$$\lambda_{min}^{A} = \min_{y} \frac{<y,(\text{diag } \lambda_i)y>}{<y,y>} = \min_{x} \frac{<x,Ax>}{<x,x>}$$

and

$$\lambda_{max}^{A} = \max_{y} \frac{<y,(\text{diag } \lambda_i)y>}{<y,y>} = \max_{x} \frac{<x,Ax>}{<x,x>}$$

Theorem C2.- A relation which gives eigenvalue bounds of the matrix A in terms of eigenvalues of the related matrix A^*A is

$$\lambda_{min}^{A^*A} \leqq \left|\lambda_i^{A}\right|^2 \leqq \lambda_{max}^{A^*A}$$

Proof: Let x_i be an eigenvector corresponding to the eigenvalue λ_i of A so that $Ax_i = \lambda_i x_i$ and

$$<Ax_i,Ax_i> = <\lambda_i x_i,\lambda_i x_i>$$

or

$$<x_i,A^*Ax_i> = \lambda_i <\lambda_i x_i,x_i> = \lambda_i \bar{\lambda}_i <x_i,x_i>$$

Thus $\left|\lambda_i\right|^2 = \dfrac{<x_i,A^*Ax_i>}{<x_i,x_i>}$, and by the same reasoning as that of theorem C1

$$\lambda_{min}^{A^*A} \leqq \left|\lambda_i^{A}\right|^2 \leqq \lambda_{max}^{A^*A}$$

Corollary C2(a):

$$\lambda_{min}^{A^*A} \leqq \left|\lambda_{min}^{A}\right|^2 \leqq \left|\lambda_{max}^{A}\right|^2 \leqq \lambda_{max}^{A^*A}$$

Corollary C2(b): If the matrix A is real, then

$$\lambda_{min}^{A^TA} \leqq \left|\lambda_{min}^{A}\right|^2 \leqq \left|\lambda_{max}^{A}\right|^2 \leqq \lambda_{max}^{A^TA}$$

The largest eigenvalue cannot only be bounded by considering related vectors but, in fact, can be approximated as closely as desired. This result is due to Collatz. (See ref. 12.)

Theorem C3.- For a matrix A of order k, with k distinct modulus eigenvalues and for an arbitrary $\epsilon > 0$, there exists an $N > 0$ such that

$$\left| \frac{\|A^n \nu\|}{\|A^{n-1}\nu\|} - \left|\lambda^A\right|_{max} \right| < \epsilon \qquad (n > N)$$

Proof: Let $\nu = \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_n y_n$ where $y_1, y_2, \ldots, y_n$ are the linearly independent eigenvectors, and let $x_i \beta_i y_i$ so that $\nu = x_1 + x_2 + \ldots + x_n$. Assume that the eigenvalues of A are ordered such that

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$$

Then

$$A^n \nu = \lambda_1^n \left[x_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n x_2 + \ldots + \left(\frac{\lambda_n}{\lambda_1}\right)^n x_n \right]$$

As n approaches ∞, then $\left(\frac{\lambda_i}{\lambda_1}\right)$ approaches 0 for all i. Thus,

$$\lim_{n\to\infty} \frac{\|A^n \nu\|}{\|A^{n-1}\nu\|} = \frac{\|\lambda_1^n x_1\|}{\|\lambda_1^{n-1} x_1\|} = |\lambda_1|$$

Several theorems which give eigenvalue bounds in terms of eigenvalues of related matrices were shown by Wittmeyer. (See ref. 17.) Some of these theorems are given below.

Theorem C4.-

$$\lambda_{max}^{(AB)^*(AB)} \leqq \lambda_{max}^{A^*A} \lambda_{max}^{B^*B}$$

Proof: Let

$$h = Bx$$

$$z = Ah$$

so that $z = ABx$. Then

$$<z,z> = <h,A^*Ah> = <y,(\text{diag}\ \lambda_i)y> = \sum \lambda_i y_i \bar{y}_i \leqq \left|\lambda_{max}^{A^*A}\right| <y,y>$$

where

$$<y,y> = <y,U^*Uy> = <Uy,Uy> = <h,h>$$

so that

$$|z| \leqq \left(\lambda_{max}^{A^*A}\right)^{1/2} |h|$$

and

$$|h| \leqq \left(\lambda_{max}^{B^*B}\right)^{1/2} |x|$$

Thus,

$$|z| \leqq \left(\lambda_{max}^{A^*A}\right)^{1/2} \left(\lambda_{max}^{B^*B}\right)^{1/2} |x|$$

Let x be the eigenvector of $(AB)^*(AB)$ corresponding to $\lambda_{max}^{(AB)^*(AB)}$; that is,

$$(AB)^*(AB)x_1 = \lambda_{max}^{(AB)^*(AB)} x_1$$

Let $z_1 = ABx_1$. Then

$$|z_1|^2 = (z_1,z_1) = (ABx_1,ABx_1) = \left[x_1,(AB)^*(AB)x_1\right] = \lambda_{max}^{(AB)^*(AB)} |x_1|^2$$

Thus, let $z = z_1$ so that

$$|x_1| \left[\lambda_{max}^{(AB)^*(AB)}\right]^{1/2} = |z_1| \leqq |x_1| \left(\lambda_{max}^{A^*A}\right)^{1/2} \left(\lambda_{max}^{B^*B}\right)^{1/2}$$

Corollary C4: It follows from theorem C2 and its corollary that

$$\left|\lambda_{max}^{AB}\right| \leqq \left[\lambda_{max}^{(AB)^*(AB)}\right]^{1/2}$$

and also

$$\left|\lambda_{max}^{AB}\right| \leqq \left(\lambda_{max}^{A^*A}\right)^{1/2} \left(\lambda_{max}^{B^*B}\right)^{1/2}$$

In a manner similar to that of theorem C4,

$$|h| \geqq \left(\lambda_{min}^{B^*B}\right)^{1/2} |x|$$

$$|z| \geqq \left(\lambda_{min}^{A^*A}\right)^{1/2} |h|$$

so that

$$|z| \geqq \left(\lambda_{min}^{A^*A}\right)^{1/2} \left(\lambda_{min}^{B^*B}\right)^{1/2} |x|$$

Let x be such that $(AB)^*(AB)x_1 = \lambda_{min}^{(AB)^*(AB)} x_1$ so that for $z_1 = ABx_1$ results in

$$|z_1| = \lambda_{min}^{(AB)^*(AB)} |x_1|$$

Theorem C5.- Letting $z = z_1$ in the preceding equation yields

$$|x_1| \left[\lambda_{min}^{(AB)^*(AB)}\right]^{1/2} = |z_1| \geqq |x_1| \left(\lambda_{min}^{A^*A}\right)^{1/2} \left(\lambda_{min}^{B^*B}\right)^{1/2}$$

and thus proves

$$\lambda_{min}^{(AB)^*(AB)} \geqq \lambda_{min}^{A^*A} \lambda_{min}^{B^*B}$$

Corollary C5: In a similar manner the relationships

$$\left|\lambda_{min}^{AB}\right| \geqq \left[\lambda_{min}^{(AB)^*(AB)}\right]^{1/2}$$

and

$$\left|\lambda_{min}^{AB}\right| \geqq \left(\lambda_{min}^{A^*A}\right)^{1/2} \left(\lambda_{min}^{B^*B}\right)^{1/2}$$

are proved.

Theorem C6.- Another theorem which follows from a somewhat different geometric consideration if A and B are normal matrices is

$$\left[\lambda_{max}^{(A+B)^*(A+B)}\right]^{1/2} \geqq \left(\lambda_{max}^{A^*A}\right)^{1/2} + \left(\lambda_{max}^{B^*B}\right)^{1/2}$$

Proof: Consider

$$\lambda_{max}^{(A+B)^*(A+B)} = \max_x \frac{<x,(A+B)^*(A+B)x>}{<x,x>}$$

$$= \max_x \left(\frac{<x,A^*Ax>}{<x,x>} + \frac{<x,B^*Bx>}{<x,x>} + \frac{<x,B^*Ax>}{<x,x>} + \frac{<x,A^*Bx>}{<x,x>} \right)$$

$$\leqq \max_x \frac{<x,A^*Ax>}{<x,x>} + \max_x \frac{<x,B^*Bx>}{<x,x>} + \max_x \frac{<x,B^*Ax>}{<x,x>} + \max_x \frac{<x,A^*Bx>}{<x,x>}$$

$$= \lambda_{max}^{A^*A} + \lambda_{max}^{B^*B} + \lambda_{max}^{B^*A} + \lambda_{max}^{A^*B}$$

By the corollary to theorem C4 is

$$\lambda_{max}^{A^*A} + \lambda_{max}^{B^*B} + \lambda_{max}^{B^*A} + \lambda_{max}^{A^*B} \leqq \lambda_{max}^{A^*A} + \lambda_{max}^{B^*B} + \left(\lambda_{max}^{BB^*}\lambda_{max}^{A^*A}\right)^{1/2} + \left(\lambda_{max}^{AA^*}\lambda_{max}^{B^*B}\right)^{1/2}$$

$$= \lambda_{max}^{A^*A} + \lambda_{max}^{B^*B} + 2\left(\lambda_{max}^{A^*A}\lambda_{max}^{B^*B}\right)^{1/2} = \left[\left(\lambda_{max}^{A^*A}\right)^{1/2} + \left(\lambda_{max}^{B^*B}\right)^{1/2}\right]^2$$

Thus

$$\left[\lambda_{max}^{(A+B)^*(A+B)}\right]^{1/2} \leqq \left(\lambda_{max}^{A^*A}\right)^{1/2} + \left(\lambda_{max}^{B^*B}\right)^{1/2}$$

Corollary C6:

$$\left|\lambda_{max}^{A+B}\right| \leqq \left(\lambda_{max}^{A^*A}\right)^{1/2} + \left(\lambda_{max}^{B^*B}\right)^{1/2}$$

Theorems C4 to C6 with their respective corollaries may be repeated as corollaries for the special case where A and B are real matrices and the transposed conjugate is replaced by the transpose.

COMPARISON AND COMPUTATION

From the results of the preceding sections, it is seen that there are many ways to compute the bounds for eigenvalues. However, certain theorems give more precise eigenvalue bounds in all cases than others. Comparison is now made between the inclusion regions of eigenvalues for several theorems of the section "A.- Bounds by Matrix Elements."

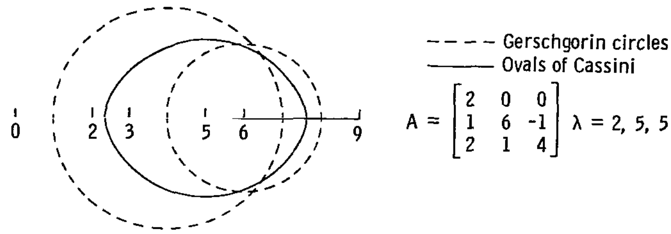


Figure 1.- Comparison of inclusion region for matrix A using Gerschgorin circles (theorem A3) and ovals of Cassini (theorem A5).

Since

$$\max_i \sum_{\substack{j=1 \\ j\neq 1}}^{n} |a_{ij}| \leqq n\left|\max a_{ij}\right|$$

Corollary A3 gives a smaller region than theorem A1 does in all cases.

Theorem A5 (ovals of Cassini) give a smaller region than theorem A3 since every point of the oval lies in at least one of the two circles which form it. An example of this condition is seen in figure 1 and is proven as follows:

If z is contained in the ovals of Cassini, then

$$|z - a_{kk}| \leqq \left(\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|\right) \frac{\left(\sum_{\substack{j\neq l \\ j=1}}^{n} |a_{lj}|\right)}{|z - a_{ll}|}$$

for some k,l. If $|z - a_{ll}| > \sum_{\substack{j=1 \\ j\neq l}}^{n} |a_{lj}|$,

$$\frac{\sum |a_{lj}|}{|z - a_{ll}|} < 1$$

and

$$|z - a_{kk}| \leqq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|$$

Thus, every point within the oval lies within at least one of the circles which form it.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 10 & 2 \\ 3 & 2 & 30 \end{bmatrix}$$

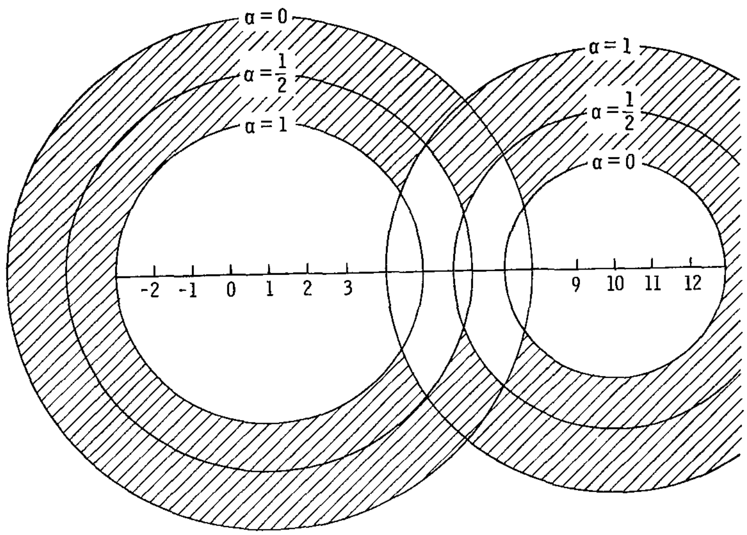


Figure 2.- Regions which can be excluded from containing eigenvalues by using the results of theorem A6 for matrix A.

Theorem A6 gives a smaller inclusion region for all $0 \leqq \alpha \leqq 1$ than theorem A3. The area which can be excluded by using values of $\alpha = 0, 1/2, 1$ is shown as the shaded area in figure 2 for the matrix A whose approximate eigenvalues are 30.55, 10.07, and 0.38 (theorem A6).

If

$$|z - a_{ii}| \leqq \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right)^{\alpha} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ki}| \right)^{1-\alpha}$$

then choose the larger of the two summations,

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ and } \sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ki}|$$

Suppose it is

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

then

$$\left(\sum_{\substack{k=1\\k\neq i}}^{n}\left|a_{ki}\right|\right)^{1-\alpha} \leqq \left(\sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right|\right)^{1-\alpha}$$

Hence,

$$\left|z - a_{ii}\right| \leqq \left(\sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right|\right)^{\alpha}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right|\right)^{1-\alpha} = \sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}\right|$$

Hence, the region of inclusion for eigenvalues for theorem A6 (and hence its corollary) is contained within the region of inclusion for eigenvalues for corollary A3. Figure 2 contains the region of the first two eigenvalues only. (See ref. 14.) The region obtained by using theorem A3 corresponds to $\alpha = 1$.

Theorem B5 reduces to theorem A3 in the special case where each submatrix is a single element of the matrix. For an example of a case in which theorem B5 gives a better result than theorem A3, consider the partitioned matrix (ref. 16)

$$A = \left[\begin{array}{cc|cc} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ \hline -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

with eigenvalues $\lambda = 1, 3, 5, 7$ and where the vector norm is taken as

$$\|x\| \equiv \left(\sum_{i=1}^{2}\left|x_i\right|^2\right)^{1/2}$$

for $x = (x_1, x_2)$ and

$$\|A_{ij}\| \equiv \sup_{x} \frac{\|A_{ij}x\|}{\|x\|}$$

Clearly

$$\|A_{12}\| = \|A_{21}\| = 1$$

For $\|A_{11}\|$ and $\|A_{22}\|$ consider

$$\left\|\left(A_{ii} - \lambda I\right)^{-1}\right\|^{-1} = \inf_{x} \frac{\left\|\left(A_{ii} - \lambda I\right)x\right\|}{\|x\|} = \inf_{x} \frac{\left\|\begin{matrix} 4 - \lambda & -2 \\ -2 & 4 - \lambda \end{matrix}\right\|}{\|x\|} = \inf_{x} \frac{\left\|\begin{bmatrix} (4 - \lambda)x_1 - 2x_2 \\ -2x_1 + (4 - \lambda)x_2 \end{bmatrix}\right\|}{\left(x_1^2 + x_2^2\right)^{1/2}}$$

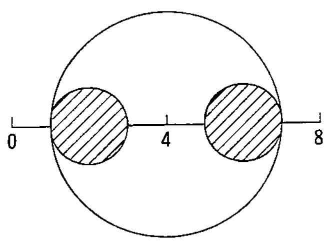
$$= \inf_{x} \left\{\frac{\left[(4 - \lambda)x_1 - 2x_2\right]^2 + \left[-2x_1 + (4 - \lambda)x_2\right]^2}{x_1^2 + x_2^2}\right\}^{1/2}$$

$$= \inf_{x} \left[(4 - \lambda)^2 + 4 - 8(4 - \lambda)\frac{x_1 x_2}{x_1^2 + x_2^2}\right]^{1/2}$$

Now $\frac{\partial}{\partial x_1} \frac{x_1 x_2}{x_1^2 + x_2^2} = 0$ and $\frac{\partial}{\partial x_2} \frac{x_1 x_2}{x_1^2 + x_2^2} = 0$ for $x_1 = x_2$ and $x_1 = -x_2$.

For $\lambda < 4$, the infimum occurs at $x_1 = x_2$. For $\lambda > 4$, the infimum occurs at $x_1 = -x_2$. At $x_1 = x_2$, the theorem requires $|\lambda - 2| \leqq 1$ and at $x_1 = -x_2$, it requires $|\lambda - 6| \leqq 1$.

A comparison between theorem B5 (whose inclusion region is shaded) and theorem A3 is given in figure 3.



$$A = \left[\begin{array}{cc|cc} -4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ \hline -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{array}\right]$$

Figure 3.- Comparison of inclusion regions for matrix A by using theorem B5 (hatched area) and theorem A3.

APPLICATIONS

Questions relating to the convergence of series and sequences of matrices arise in many situations. The eigenvalue bounds of the related matrices can give sufficiency conditions for convergence. For example, consider the system of linear equations $Ax = y$ where A is a $n \times n$ nonsingular matrix of coefficients and x and y are n dimensional vectors. (See ref. 18.) Let G be an approximate inverse of A so that the approximate solution is $z = Gy$. It can be shown by induction that for any integer k

$$x = \sum_{m=0}^{k} (I - GA)^m z + (I - GA)^{k+1} x = z + \sum_{m=1}^{k} D^m z + D^{k+1} x$$

Denote the error in z by $\epsilon = x - z$ and let $D = I - GA$. Thus

$$x - z = \epsilon = \sum_{m=1}^{k} D^m z + D^{k+1} x$$

If $\left|\lambda_{max}^D\right| < 1$, the $\lim\limits_{k \to \infty} D^k = 0$. Thus

$$\epsilon = \sum_{m=1}^{\infty} D^m z$$

places a bound on the error ϵ. As another example, consider the equation $Ax = m$ where A is nonsingular. (See ref. 19.) If x represents the solution and x_k is the kth approximation, let $\nu_k = x - x_k$ and $y_k = m - Ax_k = A\nu_k$. Then $\lim\limits_{k \to \infty} \nu_k = 0$ if and only if $\lim\limits_{k \to \infty} y_k = 0$ and in either case $\lim\limits_{k \to \infty} x_k = x$. To determine an iteration on the set of x_k terms, let $A = A_1 + A_2$ with A_1 nonsingular. Define the kth iteration by $x_k = A_1^{-1} m - A_1^{-1} A_2 x_{k-1}$. Since $A_1 \nu_{k+1} = -A_2 \nu_k$, for convergence of the iteration, it is necessary and sufficient that all eigenvalues of $A_1 - A_2$ be contained in the unit circle of the complex plane.

Iterative schemes may be established to give close approximations to all the eigenvalues of a matrix. A survey of these techniques together with comparative accuracy and computation time is given by White. (See ref. 20.)

CONCLUDING REMARKS

A listing of techniques which determine the eigenvalue bounds of a matrix defined over either the real or complex fields is presented. The condition, modulus, or numerical value of eigenvalues as a function of the corresponding matrices are listed without proofs for several well-known types of matrices. Other known theorems which determine the bounds have been proven in detail. These results have been expressed in terms of (1) the matrix elements, (2) matrix norms, and (3) vectors and the eigenvalues of related matrices. Also extensions of several results have been made to countably infinite matrices.

A comparison has been made in terms of the relative size of the areas of eigenvalue inclusion for several solutions. This comparison has shown that some solutions give better results in all cases than other solutions. Examples in terms of eigenvalue bounds for particular matrices have been given.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 13, 1967,
126-62-02-03-23.

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