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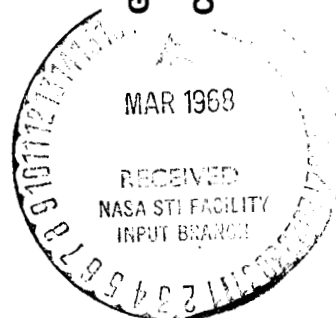
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ABSTRACT

The time dependent and path dependent processes occurring prior to fracture in the regions close to the crack tip have been considered for an axially symmetrical geometry. To account for nonlinear behavior of glasslike high linear polymers within the localized zones of high stress and to describe the rheological response of the bulk material, a model is developed, which postulates that the crack is surrounded by a Dugdale-type thin plastic zone, while the less stressed matrix is assumed to behave as a linear viscoelastic solid.

According to the present model the yield point itself is a function of time, as suggested by Crochet. This implies that a faster loading corresponds to a higher yield stress and allows for quantitative description of the initial stages of delayed fracture, i. e., the process of formation of the craze.

Several cases of creep failure, under different histories of loading, are discussed. In particular, a universal equation $p_0/p_G = [K(0)/K(t_*)]^{1/2}$ for creep failure under step-load condition is derived. It relates the tensile strength p_0 to the time of application of the load and material characteristics: p_G is the Griffith stress, while $K(t)$ encompasses the rheological properties of the solid.

This equation is shown to be true both for the 3- and 2-dimensional problems, if only the ratio p_o/Y_o is small (Y_o denotes initial yield stress). It resembles considerably the Williams equation derived earlier for delayed fracture caused by a spherical void under hydrostatic tension.

In the limit, both Griffith theory as well as its modification by Irwin and Orowan can be derived from the present analysis.

DELAYED FRACTURE IN VISCOELASTIC-PLASTIC SOLIDS

1. Introduction

It is realized that the presence of a crack in a glass-like linear high polymer induces non-uniform deformations, such that the stresses and strains within the zones close to the crack tip are markedly different from the corresponding quantities in the 'matrix' surrounding the crack. The experimental evidence, gathered by Berry [2], Cessna and Sternstein [4], Kambour [9], shows that fracture in high polymers is preceded by a considerable irreversible deformation, localized along the crack plane and constituting an "extension" of the crack. This phenomenon has been observed for amorphous thermoplastics by Kambour [9]. The term "craze" has been coined for the regions where the physical properties of highly deformed material have been changed. It is there, where most of the energy required per unit area of the new surface is invested: according to rough estimates of Kambour in fracturing of glassy polymers (like PMMA) 1.5% of energy available is used to overcome intermolecular attraction, about 16% is lost as work on plastic deformations and the rest is used on irreversible viscoelastic displacements inside the craze. It was further shown [9], that formation of a craze and its final size is not uniquely determined by the load or deformation that acts at any instant on the bulk sample.

It appears thus that neither purely elastic nor elastic-plastic considerations would be adequate to describe the path-dependent growth of plastic regions and the rheological response of an apparently glassy matrix. Until recently there were no analytical treatments available

(within the framework of continuum mechanics), which would attempt to describe the time-dependent processes along with the path-dependent plastic deformation at the crack tip occurring prior to fracture. The earlier works by Williams [21], Schapery and Williams [18], Knauss [10] have tried to describe the delayed fracture in glass-like polymers applying the theory of linear viscoelastic solids. In some of these works, Williams [21], the singularity in stresses at the crack tip have been intuitively avoided by introducing a Neuber-like characteristic size, over which the stresses were averaged. However, this quantity, being somewhat ambiguously defined, entered in all the pertinent final results (the material ahead of crack tip was considered to act as the set of independent columns, to each of which a critical strain criterion was successively applied to describe the spreading crack).

On the other hand, the energy considerations based upon the first law of thermodynamics, as adapted to fracture problems by Schapery [17], were carried out by the use of the exact stress distribution in a linear viscoelastic body containing a spherical void subject to hydrostatic tension: this geometry being chosen, Williams [22], to avoid analytical difficulties in attacking the problem of a real crack.

The fracture behavior of polymers has been studied both experimentally and analytically under different histories of loading and for several non-uniaxial stress states, by Smith [19] and Knauss [10]. These experiments have shown that the critical stress (or critical deformation) does depend on the rate at which strains are applied and on the path taken through the space of external loadings.

Recently Cessna and Sternstein [4] have proposed a rheological model for the quantitative description of viscoelastic plastic behavior of glassy polymers prior to fracture. The suggested model consists of a Maxwell element joined in series with a Voigt element, in which the dash-pot is non-linear; it obeys the Eyring hyperbolic-sine flow law. This model does incorporate the minimum number of required features for the expected response of the solid, namely, immediate elastic deformation, viscous flow and retarded elastomeric response, which is stress-sensitive and which displays a stress-activated yielding, i. e., yielding at the high stresses in the critical zones at the crack tip, but no yielding at lower stresses in the bulk material. They [4] determined experimentally five rheological constants characteristic for the model, adjusting it in this way to describe PMMA. Their failure data seems to agree well with the theoretical predictions, based upon an approximate analysis of stresses and strains ahead of the crack. Although the use of the discrete rheological constants, instead of the corresponding spectrum, may appear insufficient in other applications, Cessna and Sternstein's approach illustrates well the problems encountered in the viscoelastic-plastic analysis. It gives also an excellent insight into fracture mechanics of glassy polymers.

In connection with the present study the observation by Cessna and Sternstein [4] is noteworthy namely that the parameter related to the size of the plastic region at the moment of failure initiation appeared to be a material constant, almost insensitive to the initial stress concentration factor and the strain rate (within the range 10^{-2} to 10^{-4} sec^{-1}). This implies, as will be evident in the next chapters, that the critical opening displacement, postulated in the present paper, does

exist*. To support this statement we shall also refer to the earlier experimental results, Smith [19], Low [11], Knauss [10] and Mueller and Knauss [13], who in addition related the critical strain to the temperature reduced strain-rates applied during a constant strain rate failure test.

Once the critical displacement w^* or critical strain ϵ^* is defined, we shall proceed to formulate the condition for incipient fracture by requiring that the displacement at the end of the crack shall be equal to its critical value. This is a sufficient condition for fracture, c.f., Murrell [12], Rice [16], Wnuk [23], and it has been used by a number of authors: Williams [21], Goodier and Field [6] and Olesiak and Wnuk [14].

The present paper offers an attempt to solve the problem of stress, strain and displacement distributions around a penny-shaped crack, under an assumption that the Dugdale type model of the crack is adequate to allow for the quantitative description of the plasticity effects occurring at the crack tip, while the matrix itself will be assumed to behave as a linear viscoelastic solid (no further specifications as to what rheological model we have in mind is necessary). In that way the stresses, strains and displacement fields around the crack are to be related to the scheme of loading, time-dependent response of bulk

* Some investigators do not like the critical strain criterion for fracture, as for instance W. S. Blackburn (private communication). They argue that it is reasonable to have it as a result of the theory rather than as a starting point.

material and to the plastic yielding present in the critical zones, where the solid behaves, according to the model used, as a non-linear medium.

Furthermore, in the more general case considered here, the yield stress itself is time dependent through the equation suggested by Crochet [5]:

$$Y(t) = A + B e^{-C\chi} \quad (1.1)$$

where A, B, C are material constants, while for the function χ , describing the history of loading, some suitable representation is chosen, as for instance in [5]:

$$\chi = [(\epsilon_{ij}^V - \epsilon_{ij}^E)(\epsilon_{ij}^V - \epsilon_{ij}^E)]^{\frac{1}{2}} \quad (1.2)$$

Here the superscripts "V" and "E" denote viscoelastic and purely elastic components of the strains. Equation (1.1) asserts that Y is a monotonically decreasing function of χ . This implies, [5], that a faster loading corresponds to a higher yield stress, as required by experimental evidence, Heller and Stoll, Freudenthal [8].

Equation (1.2) indicates also that the initial response of the material is of the elastic-perfectly plastic type, as initially $\chi=0$ and we have yielding along the critical zone, whose length is determined by the magnitude of the applied pressure and the initial value of the yield stress $Y(0) = A + B$. Next, the decrease of the yield point, under constant load insufficient to cause the immediate fracture, will result in lengthening and widening of the critical zones. Increase in size of the

plastic zones will be accompanied by the growth of displacements at the end of the crack. This stage of delayed fracture may be compared with the process of formation of the craze (Kambour [9]).

The practical purpose of the present analysis is to predict the time to fracture initiation by relating the material characteristics, scheme of loading, the path taken in the space of external loads, as well as the temperature and the strain-rates. All these quantities together will determine the creep failure.

In the limit of vanishing viscoelastic-plastic deformation at the flaw tip, the critical zone and the bulk material approach a homogeneous elastic domain, with a high gradient of stress near the flaw tip (so called 'non-relaxed' crack). Under such a condition, the present theory yields Griffith's result. At the other extreme, where a plastic region of constant size constitutes the critical zone, the present analysis reduces to Irwin-Orowan's modification of the Griffith theory.

Somewhat in between these two extremes falls the case considered by Williams, i. e. , that of a crack in a viscoelastic material under plane strain or plane stress conditions [21] and that of a spherical void under hydrostatic tension [22]. In both cases Williams considered material to be linearly viscoelastic, which allowed him to predict the delayed fracture, but obviously he was not able to take into account any plastic deformation. A similar representation, too, can be derived from the present theory, if plastic effects are assumed to be small.

2. Stress And Strain Distribution. Time-Dependent Yield Point.

The problem of a growing crack in a viscoelastic medium or that of a crack of constant length but subjected to certain variation in time loading cannot be treated by means of standard method of solution to linear viscoelastic stress analysis, i. e., by the use of the correspondence principle. Both the above mentioned cases reduce to a mixed boundary value problem for which the regions where the different types of boundary conditions are prescribed, are time-dependent. This does not allow to evaluate the required integral transforms with respect to time.

For one important case, however, when all the time-dependent quantities are monotonically increasing with time, the distribution of stresses and strains around a crack can be determined by means of an "extended correspondence principle", as proposed by Graham [7].

In the model developed here we shall consider a penny-shaped crack to be surrounded by a Dugdale-type, thin plastic zone, while the bulk material will be assumed to behave as a linear viscoelastic solid, governed by the constitutive equations:

$$\begin{aligned} s_{ij} &= \int_{-\infty}^t G_1(t-\tau) \frac{\partial e_{ij}(\tau)}{\partial \tau} d\tau \\ s &= \int_{-\infty}^t G_2(t-\tau) \frac{\partial e(\tau)}{\partial \tau} d\tau \end{aligned} \tag{2.1}$$

where $G_1(t)$ and $G_2(t)$ are the relaxation moduli in shear and isotropic

compression respectively, s_{ij} and e_{ij} denote the deviatoric parts of stress and strain tensors, while $\delta_{ij}s$ and $\delta_{ij}e$ are the hydrostatic parts of these tensors. It can be readily shown that the Graham [7] formula for the displacements in the crack plane ($z=0$)

$$u_z(\rho, t) = \frac{2}{\pi} K(o) \int_{\rho}^{a(t)} \frac{dv}{(v^2 - \rho^2)^{\frac{1}{2}}} \int_0^v \frac{s p(s, t) ds}{(v^2 - s^2)^{\frac{1}{2}}} +$$

$$+ \frac{2}{\pi} \int_0^t K(\tau) \operatorname{Re} \left\{ \int_{\rho}^{a(t-\tau)} \frac{dv}{(v^2 - \rho^2)^{\frac{1}{2}}} \int_0^v \frac{\rho p(s, t-\tau) ds}{(v^2 - s^2)^{\frac{1}{2}}} \right\} d\tau \quad (2.2)$$

can be written simply as

$$u_z(\rho, t) = u_z^o(\rho, t) + \int_0^t \frac{\dot{K}(\tau)}{K(o)} u_z^o(\rho, t-\tau) d\tau \quad (2.3)$$

Here $u_z^o(t)$ is the elastic solution to the same boundary value problem $p(\rho, t)$ is the pressure applied at the crack surface (including sections of the constant yield stress), ρ is the radial dimensionless coordinate $\rho = r/a$ (the other coordinates being z and θ), $a(t)$ is an outside radius of the plastic zone; it is larger than, or in the limit, equal to the length of the crack $l(t)$, and finally the function $K(t)$ is defined as [7]:

$$K(t) = \mathfrak{L}^{-1} \left[\frac{2(2G_1^*(s) + G_2^*(s))}{s^2(G_1^*(s) + 2G_2^*(s)G_1^*(s))} ; s \rightarrow t \right] \quad (2.4)$$

where $G_1^*(s)$ and $G_2^*(s)$ are the Laplace transforms of the relaxation moduli and \mathcal{L}^{-1} denotes the inverse Laplace transform. Formula (2.3) is valid for any linear viscoelastic solid with properties characterized by the function (2.4). Formulae of the same type are shown to be true for all components of displacement and strain tensors, while the stresses in the viscoelastic matrix are the same they would have been in an elastic body.

Under the additional simplifying assumption that the Poisson ratio is constant, the moduli G_1 and G_2 can be simply related: $G_1 = (1-2\nu)G_2/(1+\nu)$ and equation (2.4) reduces to, [7]:

$$K(t) = \mathcal{L}^{-1} \left[\frac{2(1-\nu)}{s^2 G_1^*(s)} ; s \rightarrow t \right] \quad (2.5)$$

or

$$K(t) = 2(1-\nu) D(t) \quad (2.6)$$

where $D(t)$ is the creep compliance.

Our model of a crack is different from those introduced earlier by Goodier and Field [6] for the 2-dimensional problem and Olesiak and Wnuk [14] for the 3-dimensional problem, in two major aspects: first we allow the yield stress to vary with time $Y = Y(t)$, according to equation (1.1). This function is a priori unknown and it will be determined after the solution to a corresponding mixed boundary value problem for a linear viscoelastic half-space is found. Only after all the components of the tensors ϵ_{ij}^v and ϵ_{ij} are known, may the invariant χ be constructed,

as required by equations (1.1) and (1.2). The second difference lies in the assumed nature of the matrix surrounding the crack: it is sensitive to the time-dependent processes in contrast to the previous solutions.

According to Crochet [5], the yield stress, which initially equals Y_0 , decreases with time, allowing more and more yielding to occur at the tip of the yielded zone. The rate of this increase is determined by the strain intensity function $\chi = [(\epsilon_{ij} - \epsilon_{ij}^0)(\epsilon_{ij} - \epsilon_{ij}^0)]^{\frac{1}{2}}$, ϵ_{ij} denoting the viscoelastic strains evaluated at the most strained point belonging to the elastic domain. In our case it is the point at the end of the plastic zone $r = a$.

After instantaneous yield, which determines the initial length of the plastic zone, the yielding will proceed: we imagine this process can be approximated by a stair-step like function, as shown in Figure 1, while the length of plastic zone increases at the same time. (In fact, the true distribution of stresses σ_z will look more like the continuous curve in Figure 1b). This distribution, in turn, will be represented by a single straight line, drawn at certain medium height, i. e., at the average between the initial yields stress Y_0 and the instantaneous yield stress $Y(t)$ evaluated at the point $r = a$.

$$\langle Y(t) \rangle = \frac{1}{2} [Y_0 + Y(t)] \quad (2.7)$$

of course, the average $\langle Y \rangle$ is a function of time, so far unknown.

The distribution of stresses around the crack in a viscoelastic matrix, according to the extended correspondence principle, is the same as it would be in the elastic solid. Therefore, using the results of Olesiak and Wnuk [14], we have

$$\text{for } 0 \leq \rho < m, \quad \sigma_z = -p(t)$$

$$\sigma_r = 0$$

$$\sigma_\theta = -(1+2\nu) P(t)$$

$$\text{for } m \leq \rho \leq 1, \quad \sigma_z = \langle Y \rangle$$

$$\sigma_r = \kappa p(t) \left(\frac{m}{\rho}\right)^2 - \kappa \langle Y \rangle \left[1 - \left(\frac{m}{\rho}\right)^2\right] + \langle Y \rangle$$

$$\sigma_\theta = 2\nu \langle Y \rangle - \kappa p(t) \left(\frac{m}{\rho}\right)^2 + \kappa \langle Y \rangle \left[1 - \left(\frac{m}{\rho}\right)^2\right] \quad (2.8)$$

$$\text{and for } \rho \geq 1, \quad \sigma_z = \frac{2\langle Y \rangle}{\pi} \left[(1+\lambda) \sin^{-1} \left(\frac{1-m^2}{\rho^2-m^2} \right)^{\frac{1}{2}} - \lambda \sin^{-1} \left(\frac{1}{\rho} \right) \right]$$

$$\sigma_r = \frac{2\langle Y \rangle}{\pi} \left\{ [(1+\lambda) (1 + \kappa \left(\frac{m}{\rho}\right)^2) - \kappa] \sin^{-1} \left(\frac{1-m^2}{\rho^2-m^2} \right)^{\frac{1}{2}} + \right.$$

$$\left. \kappa \lambda \sin^{-1} \left[(\rho(1+\lambda))^{-1} \left(\frac{\rho^2-1}{\rho^2-m^2} \right)^{\frac{1}{2}} \right] - \lambda \sin^{-1} \left(\frac{1}{\rho} \right) \right\}$$

$$\sigma_\theta = \frac{2\langle Y \rangle}{\pi} \left\{ [(1+\lambda) (2\nu - \kappa \left(\frac{m}{\rho}\right)^2) + \kappa] \sin^{-1} \left(\frac{1-m^2}{\rho^2-m^2} \right)^{\frac{1}{2}} - \right.$$

$$\left. -2\nu \lambda \sin^{-1} \left(\frac{1}{\rho} \right) - \kappa \lambda \sin^{-1} \left[(\rho(1+\lambda))^{-1} \left(\frac{\rho^2-1}{\rho^2-m^2} \right)^{\frac{1}{2}} \right] \right\}.$$

This is true for case 1, i. e., loading $p = p(t)$ applied on the surface of the crack, while for the case 2, i. e., tensile stress applied at infinity we obtain:

$$\text{for } 0 \leq \rho < m, \quad \sigma_z = 0$$

$$\sigma_r = p(t) (\kappa - 1)$$

$$\sigma_\theta = p(t) (2\nu + \kappa)$$

for $m \leq \rho \leq 1$, $\sigma_z = \langle Y \rangle$

$$\sigma_r = \langle Y \rangle [(1-\lambda)(1-\kappa) + \kappa (\frac{m}{\rho})^2] \quad (2.9)$$

$$\sigma_\theta = \langle Y \rangle [(1-\lambda)(2\nu + \kappa) - \kappa (\frac{m}{\rho})^2]$$

and for $\rho \geq 1$, $\sigma_z = \frac{2\langle Y \rangle}{\pi} \left[\frac{\pi}{2}\lambda - \lambda \sin^{-1}(\frac{1}{\rho}) + \sin^{-1}\left(\frac{1-m^2}{\rho^2-m^2}\right)^{\frac{1}{2}} \right]$

$$\sigma_r = \frac{2\langle Y \rangle}{\pi} \left\{ [1-\kappa + \kappa (\frac{m}{\rho})^2] \sin^{-1}\left(\frac{1-m^2}{\rho^2-m^2}\right)^{\frac{1}{2}} - (1-\kappa)\lambda \sin^{-1}(\frac{1}{\rho}) \right\}$$

$$\sigma_\theta = \frac{2\langle Y \rangle}{\pi} \left\{ [2\nu + \kappa - \kappa (\frac{m}{\rho})^2] \sin^{-1}\left(\frac{1-m^2}{\rho^2-m^2}\right)^{\frac{1}{2}} - (2\nu + \kappa)\lambda \sin^{-1}(\frac{1}{\rho}) \right\}.$$

The ratio $\lambda = P/\langle Y \rangle$ is here a function of time, as both p and $\langle Y \rangle$ can vary with time. Thus it is seen that the stresses will also vary with time. Other notations: $\kappa = (1-2\nu)(1+\nu)/2$, $m = \ell/a$, with " ℓ " as the crack length and " a " as the outside radius of the plastic zone. All stresses are given on three different sections: along the crack $0 \leq \rho < m$, inside the plastic zone $m \leq \rho \leq 1$ and outside of it $\rho \geq 1$. It can readily be observed that the stresses pass through a discontinuity at the point $\rho = m$.

The dimensionless length of plastic zone m is related to the dimensionless load, λ , as required by the finiteness condition, [14] :

$$m = \begin{cases} \left(\frac{(1+2\lambda)^{\frac{1}{2}}}{1+\lambda} \right) & (1) \\ (1-\lambda^2)^{-\frac{1}{2}} & (2) \end{cases} \quad (2.10)$$

The numbers in brackets behind the formulae designate the corresponding case of loading: we shall use this type of parallel notation throughout the paper.

At the point $\rho=1$, i. e., at the end of the plastic zone, we have

$$\begin{aligned}
 (1) \quad \sigma_z &= \langle Y \rangle & (2) \quad \sigma_z &= \langle Y \rangle \\
 \sigma_r &= \langle Y \rangle \left[1 + \kappa \frac{\lambda + \lambda^2}{(1+\lambda)^2} \right] & \sigma_r &= \langle Y \rangle [1 + (\kappa-1)\lambda - \kappa\lambda^2] \\
 \sigma_\theta &= \langle Y \rangle \left[2\nu - \kappa \frac{\lambda + \lambda^2}{(1+\lambda)^2} \right] & \sigma_\theta &= \langle Y \rangle [2\nu - (2\nu + \kappa)\lambda + \kappa\lambda^2]
 \end{aligned} \tag{2.11}$$

and the elastic strain field:

$$\begin{aligned}
 (1) \quad \epsilon_z^o &= \frac{\langle Y \rangle}{E} 2\kappa & (2) \quad \epsilon_z^o &= \frac{\langle Y \rangle}{E} [2\kappa + (1-2\kappa)\lambda] \\
 \epsilon_r^o &= \frac{\langle Y \rangle}{E} 2\kappa \left[(1+\nu) \frac{\lambda + \lambda^2}{2(1+\lambda)^2} + 1 \right] & \epsilon_r^o &= \frac{\langle Y \rangle}{E} \left\{ 2\kappa + [-\nu + \kappa(\nu-1)]\lambda - \right. \\
 & & & \left. - \kappa(1+\nu)\lambda^2 \right\} \\
 \epsilon_\theta^o &= - \frac{\langle Y \rangle}{E} (1+\nu) 2\kappa \frac{\lambda + \lambda^2}{(1+\lambda)^2} & \epsilon_\theta^o &= \frac{\langle Y \rangle}{E} \left\{ -1[\nu + (1+\nu)\kappa] + \kappa(1+\nu)\lambda^2 \right\}
 \end{aligned} \tag{2.12}$$

The further analysis, leading to the derivation of the function $\langle Y(t) \rangle$, will be carried out under the assumption that the ratio of applied load to yield stress is small ($\lambda \ll 1$). Then the above equations reduce to

$$\begin{aligned}
 \sigma_z &= \langle Y \rangle & \epsilon_z^o &= \frac{\langle Y \rangle}{E} 2\kappa \\
 \sigma_r &= \langle Y \rangle & \epsilon_r^o &= \frac{\langle Y \rangle}{E} 2\kappa \\
 \sigma_\theta &= 2\nu \langle Y \rangle & \epsilon_\theta^o &= 0
 \end{aligned} \tag{2.13}$$

These values of stresses and strains at the point $\rho = 1$ are the same for both cases of loading (the terms $O(\lambda)$ and higher have been neglected).

The viscoelastic strains at the tip of plastic zone are given by

$$\begin{aligned} \epsilon_r = \epsilon_z &= \frac{\langle Y(t) \rangle}{E} 2\kappa + \frac{2\kappa}{E} \int_0^t \frac{\dot{K}(\tau)}{K(0)} \langle Y(t-\tau) \rangle d\tau \\ \epsilon_\theta &= 0 \end{aligned} \quad (2.14)$$

Combining (2.14) and (2.13) we can now construct the strain intensity function χ . First, we observe that $\chi = [(\epsilon_{ij} - \epsilon_{ij}^0)(\epsilon_{ij} - \epsilon_{ij}^0)]^{\frac{1}{2}}$ can be rewritten as

$$\chi = [(\epsilon_{ij} - \epsilon_{ij}^0)(\epsilon_{ij} - \epsilon_{ij}^0) + \frac{1}{3}(\epsilon_{ii} - \epsilon_{ii}^0)^2]^{\frac{1}{2}} \quad (2.15)$$

or upon substituting

$$\epsilon_\theta - \epsilon_\theta^0 = 0$$

$$\begin{aligned} (\epsilon_{ij} - \epsilon_{ij}^0)(\epsilon_{ij} - \epsilon_{ij}^0) &= (\epsilon_z - \epsilon_z^0)^2 + (\epsilon_r - \epsilon_r^0)^2 - \\ &\quad - (\epsilon_r - \epsilon_r^0)(\epsilon_z - \epsilon_z^0) \end{aligned} \quad (2.16)$$

$$\frac{1}{3}(\epsilon_{ii} - \epsilon_{ii}^0)^2 = \frac{1}{3}[(\epsilon_z - \epsilon_z^0) + (\epsilon_r - \epsilon_r^0)]^2$$

we have

$$\chi = [(\epsilon_z - \epsilon_z^0)^2 + (\epsilon_r - \epsilon_r^0)^2]^{\frac{1}{2}} \quad (2.17)$$

and finally, taking into account (2.14) and (2.13), we obtain

$$\chi = \frac{2\sqrt{2}\kappa}{E} \int_0^t \frac{\dot{K}(\tau)}{K(0)} \langle Y(t-\tau) \rangle d\tau \quad (2.18)$$

Next, we proceed to determine the function $\langle Y(t) \rangle$ itself.

Recalling the Crochet equation $Y(t) = A + B \exp(-C\chi)$ where A, B, C are material constants, and using the definition of the average yield stress $2\langle Y \rangle = Y_0 + Y(t)$, we derive an integral equation for the unknown function $Y(t)$, namely,

$$Y(t) = A + B \exp \left\{ - \frac{2\kappa C}{E} \left[(A+B) \left(\frac{K(t)}{K(0)} - 1 \right) + \int_0^t \frac{\dot{K}(\tau)}{K(0)} Y(t-\tau) d\tau \right] \right\} \quad (2.19)$$

By subsequent two-fold differentiation we reduce this formula to an equivalent differential equation:

$$\frac{2}{\alpha} \frac{-\ddot{Y}(Y-A) + \dot{Y}^2}{(Y-A)^2} = 2(A+B) \frac{\ddot{K}(t)}{K(0)} + \frac{\dot{K}(t)}{K(0)} \dot{Y} \quad (2.20)$$

or, in a more compact form

$$-\ddot{y}y + \dot{y}^2 = y^2 [P(t)\dot{y} + Q(t)] \quad (2.21)$$

where

$$y(t) = Y(t) - A, \quad \alpha = 2 \cdot 2\kappa \frac{C}{E} \quad (2.22)$$

$$P(t) = \frac{\alpha}{2} \frac{\dot{K}(t)}{K(0)}, \quad Q(t) = \alpha(A+B) \frac{\ddot{K}(t)}{K(0)}$$

To illustrate the possible behavior of the function $Y(t)$ we shall consider the particular case of a Maxwell solid for which $P = \alpha/\tau_0$, $Q = 0$.

Equation (2.21) reduces then to

$$\ddot{y}y - \dot{y}^2 = - \frac{\alpha}{2\tau_0} y^2 \dot{y}, \quad (2.23)$$

similarly for a 3-element solid (a spring E_1 connected in series with a Voigt element E_2, τ_2) we have $P = \frac{\alpha}{2} b e^{-t/\tau_2}$, $Q = [-\alpha b (A+B)/\tau_2] e^{-t/\tau_2}$; thus equation (2.21) becomes

$$e^{t/\tau_2} \frac{d}{dt} \left(\frac{\dot{Y}}{Y} \right) = -\frac{\alpha b}{2} \dot{Y} + \frac{\alpha b}{\tau_2} (A+B) \quad (2.24)$$

where $b = E_1/E_2\tau_2$, τ_0 and τ_2 denote the relaxation times for Maxwell and Voigt elements, respectively. The first of the last two equations can be readily integrated:

$$Y(t) = (2A+B) \left\{ 1 - \frac{B}{2(A+B)} \exp \left[-\frac{\alpha}{2\tau_0} (2A+B)t \right] \right\}^{-1} - (A+B) \quad (2.25)$$

In obtaining this result we have reduced equation (2.23) to a first order differential equation, observing that $(\ddot{Y}Y - \dot{Y}^2)/Y^2 = \frac{d}{dt} (\dot{Y}/Y)$ and eliminating two integration constants by use of the initial conditions:

$$Y(0) = A+B \quad (2.26)$$

$$Y(0) = -\alpha B(A+B) \frac{\dot{K}(0)}{K(0)}$$

It is seen that according to equation (2.23) the initial yield stress $Y(0)$ equals $A+B$, while for long times, it tends to the value $Y(\infty) = A$. Insertion of (2.25) into the relation defining the average yield stress gives

$$\langle Y(t) \rangle = Y_0 / \Psi(t), \quad \Psi(t) = 1 + \beta - \beta e^{-ct/\tau_0} \quad (2.27)$$

where

$$\beta = \frac{B}{2A + B} \quad (2.28)$$

$$c = 2\kappa C(2A + B)/E$$

It is noteworthy that for an incompressible material $c = 0$ and therefore it turns out that the stress $\langle Y \rangle$, according to the present model, remains constant anyway.

We shall now discuss briefly the effect of a time-dependent yield stress (as obtained for the simplest case of Maxwell solid), on the stress distribution as well as strains and displacements in the crack plane.

The stresses at the tip of the plastic zone, $\rho = 1$, decrease with time after load has been applied, in the same way as $\langle Y(t) \rangle$ diminishes. This is shown in Figure 2. It is obvious that the rate of "plastic relaxation" strongly depends upon the numerical values of the constant A , B , C , appearing in the Crochet equation. To have a rough estimate we assume here the numbers suggested by the experiments of Heller, Stoll and Freudenthal [8] and concerning the filled elastomers (combination of polyurethane rubber with potassium chloride) which have been used for modeling the mechanical properties of solid rocket propellants. For $A = 100$ psi, $B = 25$ psi and $C = 400$ we have $B/A = 0.25$, $\beta = 0.111$ and $c \approx 400 Y(0)/E$ which can vary between 1 and 40, depending on the ratio of initial yield stress to Young's modulus. (The factor $2(1+\nu)(1-2\nu)(1+B/2A) \approx 1$ for $\nu = 0.3$ and $B/A = 0.25$). Although the diagrams in Figure 2 show that the effect of coefficient c on the rate of plastic relaxation is considerable, it will be demonstrated that it has very little effect on the time dependence of the displacement at the crack tip.

This, in turn, will result in very little sensitivity of the shape of delayed fracture curve to the changes in numerical value of parameter c . Indeed, the diagrams in Figure 3 drawn for $c = 1$ and $c = 10$ differ only slightly. They were obtained under step load condition and they illustrate the growth of displacements at the crack tip. The lengthening of the plastic zone, as shown in Figure 4, depends on the parameter c in a more distinct way, while the curve governing the delayed fracture is practically insensitive to changes in c : the curve for $c = 10$ has been omitted in Figure 5, as it was almost identical with that obtained for $c = 1$ (curve 2 in Figure 5).

In all these diagrams, shown in Figure 3, 4, and 5, the fine lines illustrate the corresponding results which follow under the assumption of constant yield stress $\langle Y(t) \rangle = Y_0$. Although the differences are not significant, it should be pointed out, that they may become more pronounced for larger ratios B/A , i. e., when the mechanical properties of a material are more strongly rate-dependent.

To derive the equations, according to which the curves in Figure 3, 4, and 5 were drawn, we consider first the displacements at the crack tip. These, according to the results of Olesiak and Wnuk [14], and for small $\lambda = p(t)/\langle Y \rangle$, are given by the relation common for both schemes of loading:

$$u_z^0(l, t) = w^0(t) = \frac{4(1-\nu^2)\ell}{\pi E_g} \langle Y \rangle \frac{\lambda^2}{2} + O(\lambda^4). \quad (2.29)$$

Thus, for step load $p(t) = p_0 H(t)$, time-dependent yield stress $\langle Y \rangle = Y_0 / \Psi(t)$ and the constant crack length $\ell = \ell_0$ we have

$$w^0(t) = \frac{2(1-\nu^2) \ell_0 p_0^2}{\pi E_g Y_0} H(t) \Psi(t) \quad (2.30)$$

and the viscoelastic counter-part, according to the extended correspondence principle (2.3) will take the form (for $t \geq 0$):

$$w(t) = w(0) \left[\Psi(t) + \int_0^t \frac{K(\tau)}{K(0)} \Psi(t - \tau) d\tau \right] \quad (2.31)$$

where $W(0) = 2(1-\nu^2) \ell_0 p_0^2 / \pi E_g Y_0$, E_g denotes the glassy Young modulus and Y_0 is the initial yield stress. Substituting $\Psi(t)$ from (2.27) as well as choosing $\dot{K}(t)/K(0) = 1/\tau_0$ (Maxwell element) and integrating, we obtain:

$$\frac{w(t)}{w(0)} = 1 + \beta \left(1 - \frac{1}{c}\right) + (1 + \beta) \frac{t}{\tau_0} - \left(1 - \frac{1}{c}\right) \beta e^{-ct/\tau_0} \quad (2.32)$$

This relation is illustrated in Figure 3. Similarly, the length of the plastic zone, given by expansion of equation (2.8) for small λ as

$$a \simeq \ell_0 \left[1 + \frac{\lambda^2}{2}\right] = \ell_0 \left[1 + \frac{\lambda_0^2}{2} \Psi^2(t) H(t)\right], \quad \lambda_0 = p_0/Y_0 \quad (2.33)$$

will increase with time, as shown in Figure 4 (in preparing the plots $\lambda_0 = p_0/Y_0$ was chosen as $1/4$).

The delayed fracture occurs for time $t = t_*$, at which moment the displacement $w(t)$ attains its critical value w^* ; thus equation (2.30) governs the delayed fracture, upon substitution $t = t_*$, $w(t) = w^*$. Furthermore, it is easy to show that the ratio

$$\frac{w^*}{w(o)} = \frac{\pi E_g Y_o w^*}{2(1-\nu_o^2) l_o p_o^2} \quad (2.34)$$

equals the square of the Griffith critical stress divided by the square of the applied load. The fracture energy γ appearing in the Griffith formula is allowed here to possess a broader meaning: it equals the plastic energy dissipation, $\gamma = Y_o w^*$ as shown by Wnuk [23] if $\lambda \ll 1$, for all those cases where this energy prevails over the work of cohesion forces (for ductile materials the plastic energy dissipation is about 1000 times greater than the work against cohesion forces). Only in the limit ($Y \rightarrow E$), when plasticity effects disappear or become negligible, will the quantity γ regain its original sense of surface energy. In this case, according to a simple Polanyi formula [15], γ is given again as a product of stress (E) and critical displacement w^* . Taking this into account we have for delayed fracture

$$\frac{p_o}{p_G} = \left[1 + \beta \left(1 - \frac{1}{c} \right) + \left(1 + \beta \right) \frac{t_*}{\tau_o} - \left(1 - \frac{1}{c} \right) \beta e^{-ct_*/\tau_o} \right]^{-\frac{1}{2}} \quad (2.35)$$

This relation is illustrated by curve 2 in Figure 5.

All the above formulae simplify significantly when the yield stress $\langle Y \rangle$ is assumed to be constant and equal to Y_o . In this case $\beta = 0$, $\Psi(t) \equiv 1$ and we have

$$\begin{aligned} w(t) &= w(o) \left[1 + t/\tau_o \right] \\ a(t) &= a(o) = l_o \left[1 + \lambda_o^2/2 \right] \\ \frac{p_o}{p_a} &= \left(1 + t_*/\tau_o \right)^{\frac{1}{2}} \end{aligned} \quad (2.36)$$

These equations have served to draw the fine lines in the diagrams in Figures 3, 4 and 5.

Comparing curves 1 and 2 in Figure 5 we notice that the one obtained at $\langle Y \rangle = \text{const.}$ (curve 1) and given by the third of equations (2.36), does not predict the time to fracture t_* in a conservative way. In fact, the decrease in $\langle Y \rangle$ accelerates the time processes at the crack tip. Therefore, the predicted fracture will occur earlier than it would under the assumption of a constant yield stress. In order to have the predictions on the safe side and retain the possibly the simplest form of the equation governing the delayed fracture, it is suggested that the lower value of yield stress is taken as an estimate for $\langle Y \rangle$. Thus for $\langle Y \rangle = \text{const.} = Y(\infty)$, we obtain

$$\frac{p_o}{p_G} = \left[\frac{Y(\infty)}{Y(o)} \right]^{\frac{1}{\beta}} (1 + t_*/\tau_o)^{-\frac{1}{\beta}} \quad (2.37)$$

instead of equation (2.36). This gives a lower bound to the delayed fracture curve, as shown in Figure 5 (curve 3 was obtained for $Y(\infty)/Y(o) = 100/125$).

Finally a simple numerical example is solved: for given

$$p_o/p_G = 1/2, \quad p_o/Y_o = 1/4, \quad c = 1, \quad \beta = 0.111$$

we find from (2.33) time to failure $t_* = 2.7\tau_o$. For this value of t_* it follows from (2.25) that the yield stress decreases from Y_o to $0.9045 Y_o$ at the point of fracture, while the plastic zone increases from $1.03125 \rho_o$ to $1.03806 \rho_o$.

It shows in this specific case that the 10% decrease in yield stress influences the other results only insignificantly, particularly those which are of practical importance. The length of the plastic zone at the point

of fracture is only about 0.7% larger than the initial length. Also times to fracture evaluated from the simple equation (third in (2.34), $t_{*}=3\tau_0$) and this resulting from (2.33) are not markedly different. For a different polymer and larger times to failure, those differences could be more pronounced.

The present result, however, does encourage us to make one further step and assume $\langle Y \rangle = \text{const}$. This will simplify the following analysis considerably and will enable us to deduce a formula valid for large sizes of plastically deformed zones around the penny-shaped crack. The case of plane strain and plane stress will also be briefly discussed.

3. Delayed Fracture. Viscoelastic-Plastic Deformation At The Crack Tip

The starting point for the derivation of an equation governing the delayed fracture is the knowledge of displacements in the crack plane. In particular, the displacements at the crack tip according to the Dugdale model, as developed by Olesiak and Wnuk [14] for a penny-shaped crack, are given by

$$u_z^0(l, t) = w^0(t) = \frac{4(1-\nu^2)l_0 \langle Y \rangle}{\pi E_g} \begin{cases} 1 + \lambda - (1 + 2\lambda)^{\frac{1}{2}} & (1) \\ 1 - (1 - \lambda^2)^{\frac{1}{2}} & (2) \end{cases} \quad (3.1)$$

This, substituted in (2.3), gives the displacement at the tip of the crack in the viscoelastic-plastic solid, i. e.,

$$w(t) = \frac{2Y_0^2 w^*}{P_G \Psi(t)} \begin{cases} 1 + \lambda(t) + [1 + 2\lambda(t)]^{\frac{1}{2}} + \int_0^{t_*} \frac{\dot{K}(\tau)}{K(0)} [1 + \lambda(t-\tau) + (1 + 2\lambda(t-\tau))^{\frac{1}{2}}] d\tau & (1) \\ 1 - (1 - \lambda^2(t))^{\frac{1}{2}} + \int_0^{t_*} \frac{\dot{K}(\tau)}{K(0)} [1 - (1 - \lambda^2(t-\tau))^{\frac{1}{2}}] d\tau & (2) \end{cases} \quad (3.2)$$

where

$$p_G = [\pi E_g \gamma/2(1-\nu^2)\ell_0]^{\frac{1}{2}}, \quad \gamma = Y_0 w^* \quad (3.3)$$

$$\Psi(t) = \frac{Y_0}{\langle Y \rangle}, \quad \lambda = p(t)/\langle Y(t) \rangle$$

This equation is generally true, i. e., it holds for any rheological response of the solid ($K(t)$), time-dependent yield stress ($\Psi(t)$), and an arbitrary loading increasing monotonically with time, $p(t)$. Assuming further a constant yield stress $\langle Y \rangle = Y_0$ ($\Psi \equiv 1$) and evaluating $w(t)$ at the time $t = t_*$, we obtain the equation governing delayed fracture:

$$\phi_0 = \begin{cases} 1 + \lambda(t_*) - (1 + 2\lambda(t_*))^{-\frac{1}{2}} + \int_0^{t_*} \frac{K(\tau)}{K(0)} [1 + \lambda(t_* - \tau) - (1 + 2\lambda(t_* - \tau))^{-\frac{1}{2}}] d\tau & (1) \\ 1 - (1 - \lambda^2(t_*))^{-\frac{1}{2}} + \int_0^{t_*} \frac{K(\tau)}{K(0)} [1 - (1 - \lambda^2(t_* - \tau))^{-\frac{1}{2}}] d\tau & (2) \end{cases} \quad (3.4)$$

Here $\phi_0 = \ell_*/\ell_0$ or $\phi_0 = p_G^2/2Y_0^2$, while $\lambda = p(t)/Y_0$. It is therefore seen that the parameter ϕ_0 can have at least two interpretations:

(1) ratio of characteristic length $\ell_* = \pi E_g w^*/4(1-\nu^2)Y_0$ which may be viewed as a material property and the initial crack length ℓ_0 ; (2) one-half of the square of the ratio of the Griffith stress and the yield stress.

For the step load $p(t) = p_0 H(t)$ we obtain from (3.4):

$$\phi_0 \frac{K(0)}{K(t_*)} = \begin{cases} 1 + \lambda_0 - \sqrt{1 + 2\lambda_0} & (1) \\ 1 - \sqrt{1 - \lambda_0^2} & (2) \end{cases} \quad (3.5)$$

where $\lambda_0 = p_0/Y_0$. This can be solved with respect to λ_0 , yielding the required relation for the tensile strength as a function of the time of load application:

$$\lambda_o = \phi_o \frac{K(o)}{K(t_*)} + \left[2\phi_o \frac{K(o)}{K(t_*)} \right]^{\frac{1}{2}} \quad (1) \quad (3.6)$$

$$\lambda_o = \begin{cases} \left\{ \phi_o \frac{K(o)}{K(t_*)} \left[2 - \phi_o \frac{K(o)}{K(t_*)} \right] \right\}^{\frac{1}{2}} & \text{for } \phi_o \leq 1 \\ \left\{ \frac{K(o)}{K(t_*)} \left[2 - \phi_o \frac{K(o)}{K(t_*)} \right] \right\}^{\frac{1}{2}} & \text{for } \phi_o \geq 1 \end{cases} \quad (2) \quad (3.7)$$

for case 2. These equations are true, let us recall, for a viscoelastic-plastic solid, axially symmetric geometry and the step load applied either directly on the crack surface (case 1) or at infinity (case 2), while the yield stress is assumed constant and equal to its initial value. Here we do not impose any restrictions as to the size of plastic zone around the crack.

It is easily seen that the critical load required for instantaneous fracture, $t_* = 0$, equals

$$\lambda_o(t_*=0) = \phi_o + 2\phi_o \quad (1)$$

$$\lambda_o(t_*=0) = \begin{cases} [\phi_o(2 - \phi_o)]^{\frac{1}{2}} & \text{for } \phi_o \leq 1 \\ 1 & \text{for } \phi_o \geq 1 \end{cases} \quad (2) \quad (3.8)$$

which is identical with the result of the elastic-plastic analysis, according to Olesiak and Wnuk [14]. At loads smaller than $\lambda_o(t_* = 0)$ fracture is also possible, but a certain finite time t_* is now required before instability can take place. To illustrate the above general result, i. e., equation (3.7), we shall substitute $K(t_*)/K(o) = 1 + t_*/\tau_o$ as it follows for a

Maxwell element and $K(t_*)/K(0) = 1 + \frac{E_1}{E_2} (1 - 3^{-t_*/\tau_2})$ for the matrix behaving like a standard linear solid.

The delayed fracture curves, obtained in this way for both cases of loading, are shown in Figure 6 (case 1) and Figure 7 (case 2). It is seen that the minimum load, at which fracture still occurs, is zero for the Maxwell model, while for the 3-parameter model it is

$$\lambda_{\min} = \lambda_0(t_* = \infty) = \begin{cases} \frac{\phi_0}{1+E_1/E_2} + \left(\frac{2\phi_0}{1+E_1/E_2} \right)^{\frac{1}{2}} \\ \left\{ \frac{\phi_0}{1+E_1/E_2} \left[2 - \frac{\phi_0}{1+E_1/E_2} \right] \right\}^{\frac{1}{2}} \end{cases} \quad (3.9)$$

These equations determine the horizontal asymptotic lines in the diagrams in Figure 6b and 7b. Below λ_{\min} there will be no fracture.

The parameter $\phi_0 = l_*/l_0$ illustrates the influence of the initial crack length upon the critical load. This can be observed in both Figure 6 and Figure 7: the higher the value of ϕ_0 (smaller initial crack length) results in shifting the creep failure curve upwards, which corresponds to larger magnitudes of the critical load at the same time to break.

Particularly interesting is the curve obtained for $\phi_0 = 1$ in the case 2, as shown in Figure 7. This curve is valid for initial crack length $l_0 = l_*$ and any length smaller than that. It means the $\phi_0 = 1$ curve is true also for a flawless solid, without any crack present, or with microcracks only. Hence, we can conclude that for very small crack lengths (of order l_* or smaller) the critical load becomes independent of crack length as required by the second of equations (3.7) in contradiction to the Griffith

theory. The breaking stress in this case has a finite value, which for the two particular rheological models, can be read out straight from the $\phi_0 = 1$ curves in Figure 7. The result seems to support Berry's [2] conclusion about the existence of "inherent flaw", which is characteristic for a given polymer and which either exists in an apparently flawless solid or perhaps can be generated upon application of load.

It seems also that the present result fits well with earlier experimental data on tensile strength of glasses, Taylor [20] and Baker and Preston [1], provided the proper rheological model is chosen. Agreement is observed over several decades of time between the present theoretical predictions, equation (3.7), and the measured tensile strength of rubber (polybutyl methacrylate), Bueche [3].

For larger initial crack lengths, i. e., $l_0 \gg l_*$, or equivalently, when the applied load constitutes a small ratio of the yield stress ($\lambda \ll 1$), all the above equations simplify significantly. Equation (3.4) becomes now

$$2 \phi_0 = \lambda_0^2(t_*) + \int_0^{t_*} \frac{K(\tau)}{K(0)} \lambda_0^2(t_* - \tau) d\tau \quad (3.10)$$

which is common for both schemes of loading. Hence, for any given history of loading $p(t) = p_0 \varphi(t)$, the creep failure is determined by

$$\frac{p_0}{p_G} = \left[\varphi^2(t_*) + \int_0^{t_*} \frac{K(\tau)}{K(0)} \varphi^2(t_* - \tau) d\tau \right]^{-\frac{1}{2}} \quad (3.11)$$

The minimum load, below which there will be no fracture is obtained from equation (3.11) as

$$p_{\min} = p_G [\varphi^2(\infty) + \lim_{t_* \rightarrow \infty} \int_0^{t_*} \frac{\dot{K}(\tau)}{K(0)} \varphi^2(t_* - \tau) d\tau]^{-\frac{1}{2}} \quad (3.12)$$

Let us consider an example of application of equation (3.11):
for a loading linearly increasing with time $p(t) = p_0 \gamma t$, and for the
matrix represented by the Maxwell element $\dot{K}(\tau)/K(0) = 1/\tau_0$, we have

$$\frac{p(t_*)}{p_G} = (1 + t_*/3\tau_0)^{-\frac{1}{2}} \quad (3.13)$$

while for the same history of loading and the 3-parameter element
 $\dot{K}(\tau)/K(0) = E_1 e^{-t/\tau_2}/E_2 \tau_2$ equation (3.11) yields

$$\frac{p(t_*)}{p_G} = \left\{ 1 + \frac{E_1}{E_2} \left[1 + 2 \left(\frac{\tau_2}{t_*} \right)^2 (1 - e^{-t_*/\tau_2}) - 2 \left(\frac{\tau_2}{t_*} \right) \right] \right\}. \quad (3.14)$$

It is easily verified that if the time to fracture is assumed zero, then the
required opening pressure equals the Griffith critical load. In fact,
for small t_* both the above equations reduce to

$$p(t_*) \simeq p_G [1 - t_*/6\tau_0] \quad (3.15)$$

$$p(t_*) \simeq p_G \left[1 - \frac{E_1}{E_2} t_*/6\tau_2 \right]$$

respectively.

The examples of application given here are merely illustrative,
as for any loading $\varphi(t)$ monotonically increasing with time and any
rheological characteristic $K(t)$ the equation (3.11), upon carrying out
the prescribed integration, will supply the basic relation between the

magnitude of the load, the way it is applied and the time to failure, i. e., it will generate all the data necessary to predict delayed fracture in polymers for which the ratio of the Griffith stress to the yield point is small.

It is noteworthy that equation (3.11) assumes a particularly simple form for a step loading $p(t) = p_o H(t)$. It becomes then

$$\frac{p_o}{p_G} = \left[1 + \int_0^{t_*} \frac{\dot{K}(\tau)}{K(o)} d\tau \right]^{-\frac{1}{2}} \quad (3.16)$$

or even shorter

$$\frac{p_o}{p_G} = \left[\frac{K(o)}{K(t_*)} \right]^{\frac{1}{2}} \quad (3.17)$$

This equation yields $p_o = p_G$ for $t_* = 0$ as expected. It is interesting to note that the minimum load (below which there will be no fracture) follows immediately from equation (3.16) upon recalling equation (2.6) and allowing t_* to tend to infinity. We have

$$p_{\min} = p_G \left[\frac{E_e}{E_g} \right]^{\frac{1}{2}} \quad (3.17a)$$

where E_e and E_g denote the rubbery and glassy moduli, respectively. In spite of the simplicity of the equation (3.17) and (3.17a) they do have a rather general range of application. As a matter of fact, we are going to show that the same equation can be obtained for 2-dimensional states.

To begin with it should be noted that equation (3.11) is true not only for a geometry considered here (penny-shaped), but also for 2-dimensional case of either plane stress or plane strain. Indeed,

assuming as a point of departure the results of Goodier and Field [6], based on the Dugdale-type model, and following the same line of argument as that at the beginning of this chapter, we arrive at

$$\frac{\pi^2}{4} \phi_0 = \ln \left\{ \sec \frac{\pi}{2} \lambda(t_*) \right\} + \int_0^{t_*} \frac{K(\tau)}{K(0)} \ln \left\{ \sec \frac{\pi}{2} \lambda(t_* - \tau) \right\} d\tau \quad (3.18)$$

which is an analogue to equation (3.4). It holds for both plane stress and plane strain, provided that the substitution for $\phi_0 = p_G/2Y^2$ is done correctly, i. e., one should not overlook that the critical stress p_G used to define ϕ_0 , has different values in each of the two cases. In an explicit form we have $\phi_0 = \pi w^* E_g / 4 \lambda_0 Y_0$ for plane stress and $\phi_0 = \pi w^* E_g / 4(1-\nu^2) \lambda_0 Y_0$ for plane strain.

Next, upon assuming a step load condition $p(t) = p_0 H(t)$, equation (3.18) yields

$$\lambda_0 = \frac{2}{\pi} \cos^{-1} \exp \left[- \phi_0 \frac{K(0)}{K(t_*)} \right] \quad (3.19)$$

This is equivalent to equation (3.7) obtained previously. Finally, letting $\phi_0 \rightarrow 0$ i. e., excluding small cracks, we obtain as before equation (3.9).

This is done by expanding $\ln \sec \frac{\pi}{2} \lambda(t)$ into a power series:

$$\ln \sec \frac{\pi}{2} \lambda(t) = \frac{\lambda^2(t)}{2} \left(\frac{\pi}{2} \right)^2 + \dots$$

and observing that for any given history of loading $\lambda / \sqrt{2\phi_0} = \left(\frac{p_0}{p_G} \right) \varphi(t)$.

To make this last step possible we have again used the relation

$\gamma = Y_0 w^*$, which is true for small λ . Thus, equations (3.14) and (3.15), obtained for a load linearly increasing with time, will also be true for the

2-dimensional case. The same can be said about equation (3.17), governing creep failure under the step load condition. It may be concluded then, that equation (3.17), being the simplest of all equations derived here, retains its rather general meaning. Therefore, we shall call this equation universal, and the resulting curve, which relates tensile strength to the time of application of the load, according to equation (3.17), will be hereafter referred to as a universal curve for delayed fracture.

We shall use equation (3.17) to predict fracture in solithane* which is one of the polymers tested in our laboratories [13]. The plot done according to equation (3.17) and using the temperature reduced creep data from [13], is shown in Figure 8 (curve 1). Curve 2, plotted in the same figure, was obtained for the same material and using Williams [22] formula, which was derived for an entirely different geometry (spherical void), different scheme of loading (hydrostatic tension) and by a different procedure (energy criterion for fracture). The Williams formula, written in the way consistent with our notation, has the form

$$\frac{P_o}{P_G} = \left[\frac{K(o)}{2K(t_*) - K(o)} \right]^{\frac{1}{2}}$$

The resemblance of both curves is striking.

* Equivoluminal composition of solithane 113.

Finally, the plots in Figure 9 and Figure 10 respectively show the delayed fracture curves for the 3-dimensional case (case 2) and the 2-dimensional case (plane strain and plane stress, load applied at infinity). No assumptions have been made as to the size of the plastic zone. The set of curves in Figure 9 was obtained from equation (3.9), while that in Figure 10 is governed by equation (3.17), upon substituting for $K(o)/K(t_*)$ the reduced creep data for solithane. The dotted lines on both figures show the delayed fracture curves, according to the universal equation (3.15), which holds for small ϕ_o . There are three universal curves on each diagram, since now the vertical axis has a different scale: λ_o instead of p_o/p_G ; the relation being $\lambda_o = \frac{p_o}{p_G} 2\phi_o$. Indeed, the agreement between solid and dotted curves is satisfactory only for small ϕ_o , as expected.

4. Outline Of A Problem For Further Investigation.

Before concluding this report some comments are to be made. Up to this point, it has been tacitly assumed that the critical displacement w^* remains constant throughout all ranges of the applied strain-rate, and this constant value was assumed to be equal w_g^* i. e., the "glassy displacement" appearing to be critical at the instantaneous fracture (very large strain rates). This is only approximately true. In general, the quantity w^* is shown to be a certain function of the strain rate. This information is supplied by the experiments, as for instance those of Knauss [10] and Mueller and Knauss [13]. The diagram in Figure 11 shows an example of the relation between the critical strain and the strain

rate, as measured by Mueller and Knauss [13] for solithane at a temperature of 20°C (all these data can be reduced to any other temperature if the proper value for temperature shift α_T is inserted).

In order to incorporate these data into the present theory of creep failure we shall substitute the function $w^* = w^*(\epsilon)$, known from the experiment, into eq. (3.2) from which the universal relation (3.17) has been derived. In particular, under step load condition and for small λ , equation (3.2) reduces to

$$w(t) = w(o) K(t)/K(o) \quad , \quad w(o) = 2(1-\nu^2) \ell_o p_o^2 / \pi E_g Y_o \quad (4.1)$$

which upon substitution $t = t_*$, $w(t) = w^*$ gives condition for fracture. However, before we proceed further, we shall have to rewrite equation (4.1) in terms of strains at the crack tip rather than displacements, as the experimental data are given in strains. Therefore, we define the strain at the end of the crack

$$\epsilon(t) = \frac{w(t) - d}{d} \quad (4.2)$$

Here d denotes certain hypothetical initial width of an infinitesimal element placed between two opposite elastic-plastic interfaces. This quantity was introduced on several occasions for instance by Goodier and Field [16] and Olesiak and Wnuk [14], although, to the authors knowledge, no numerical evaluations were ever given. Fortunately, as we are going to show, the length d (sometimes called "gauge width") can be eliminated from the final result (we shall also attempt to give a simple estimate for it).

It is easy to show that the glassy critical displacement satisfies the relation

$$w_g^* = \left(\frac{p_G}{p_o} \right)^2 w(o) \quad (4.3)$$

Substituting (4.3) into (4.2) we have

$$\frac{w(o)}{d} = (1 + \epsilon_g^*) \left(\frac{p_o}{p_G} \right)^2 \quad (4.4)$$

This will permit to eliminate the ratio $w(o)/d$. On the other hand, if one insists on the direct estimate of the gauge width, it can also be obtained from (4.2):

$$d = \gamma \left\{ Y_o (1 + \epsilon_g^*) \right\} \quad (4.5)$$

where γ denotes fracture energy associated with instantaneous fracture:

$\gamma = Y_o w_g^*$ while Y_o is the initial yield point and ϵ_g^* is the breaking strain at very large rates.

Next, we do have to define some average strain rate, as under the step load condition, the strain at the crack tip is not constant, but it changes with time as follows from (4.1) and (4.2):

$$\dot{\epsilon}(t) = \frac{w(o)}{d} \frac{\dot{K}(t)}{K(o)} \quad (4.6)$$

To make it comparable with the constant strain data we average the function $\dot{\epsilon}(t)$ over the period of time (o, t_*) , as follows:

$$\langle \dot{\epsilon}(t) \rangle = \frac{1}{t_*} \int_0^{t_*} \dot{\epsilon}(t) dt = \frac{w(o)}{d} \left[\frac{K(t_*)}{K(o)} - 1 \right] \quad (4.7)$$

or, eliminating d by use of (4.4) we have

$$\langle \epsilon \rangle = \frac{1 + \dot{\epsilon}_g}{t_*} \left[\frac{K(t_*)}{K(o)} - 1 \right] \left(\frac{p_o}{p_G} \right)^2 \quad (4.8)$$

Hence, it is seen that the larger the applied load, the higher are the strain rates (the relation (4.8) is somewhat indirect as both t_* and $K(t_*)$ change with load. Nevertheless, the above conclusion is obvious).

Introducing the derived formulae into the equation $w^* = w(o)K(t_*)/K(o)$, which governs the delayed fracture under the step load condition and for small ratios p_o/Y_o , we obtain

$$\frac{p_o}{p_G} = \left[\frac{K(o)}{K(t_*)} \frac{1 + \epsilon_g^*}{1 + \epsilon^* [\langle \epsilon(t) \rangle]} \right]^{\frac{1}{2}} \quad (4.9)$$

where $\dot{\epsilon} [\langle \dot{\epsilon} \rangle]$ is the function determining the rate dependence of the critical strain and ϵ_g^* is the "glassy" breaking strain. This equation does incorporate two sets of data: the rheological characteristic $K(t)$, available for instance from creep tests, and those obtained in the failure tests performed under constant rate condition. Of course, it reduces to the universal equation (3.17), if one assumes $\epsilon^* [\langle \dot{\epsilon} \rangle] = \text{const.} = \epsilon_g^*$.

The specification of the function ϵ^* and evaluating the time to fracture initiation according to the equation (4.8) would yield more refined data pertinent to the creep failure. This may not be a straightforward process, as the average rate $\langle \dot{\epsilon} \rangle$ depends on a priori unknown t_* and most likely use of certain iteration procedure will be necessary. It is believed, however, that the problem deserves further attention.

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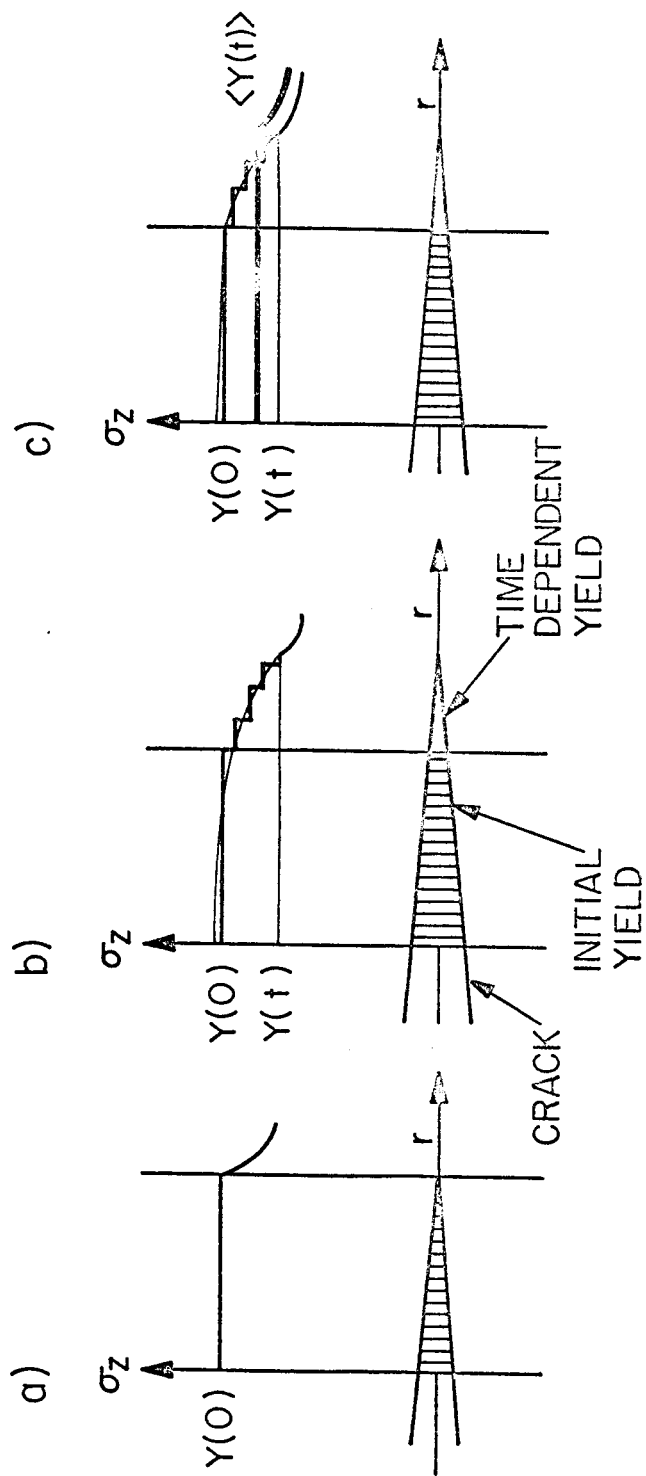


Fig. 1. Formation of the yielded zone and the distribution of stresses σ_z within this zone.

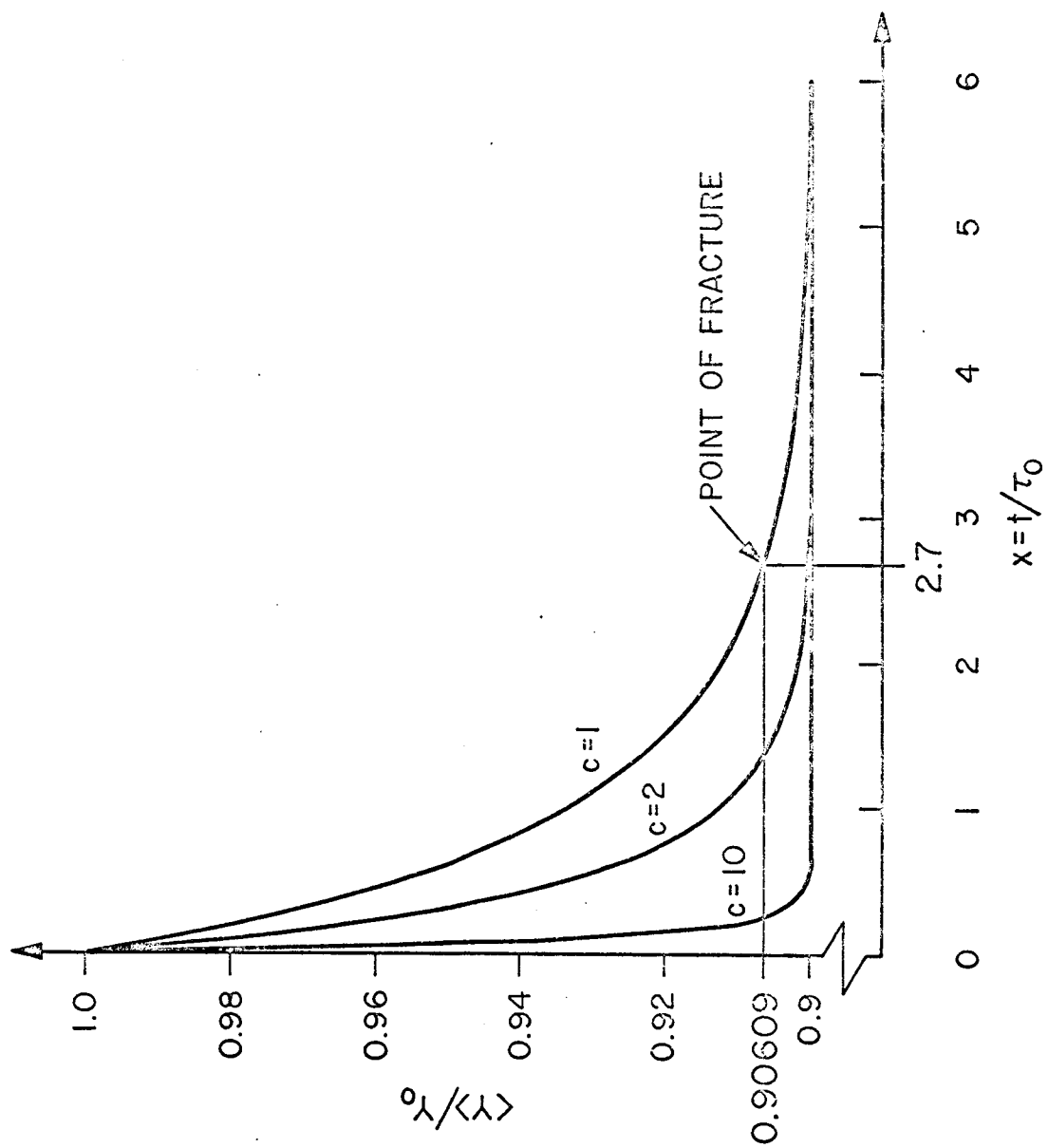


Fig. 2. Decrease of the yield stress prior to fracture (including sample calculation contained in text).

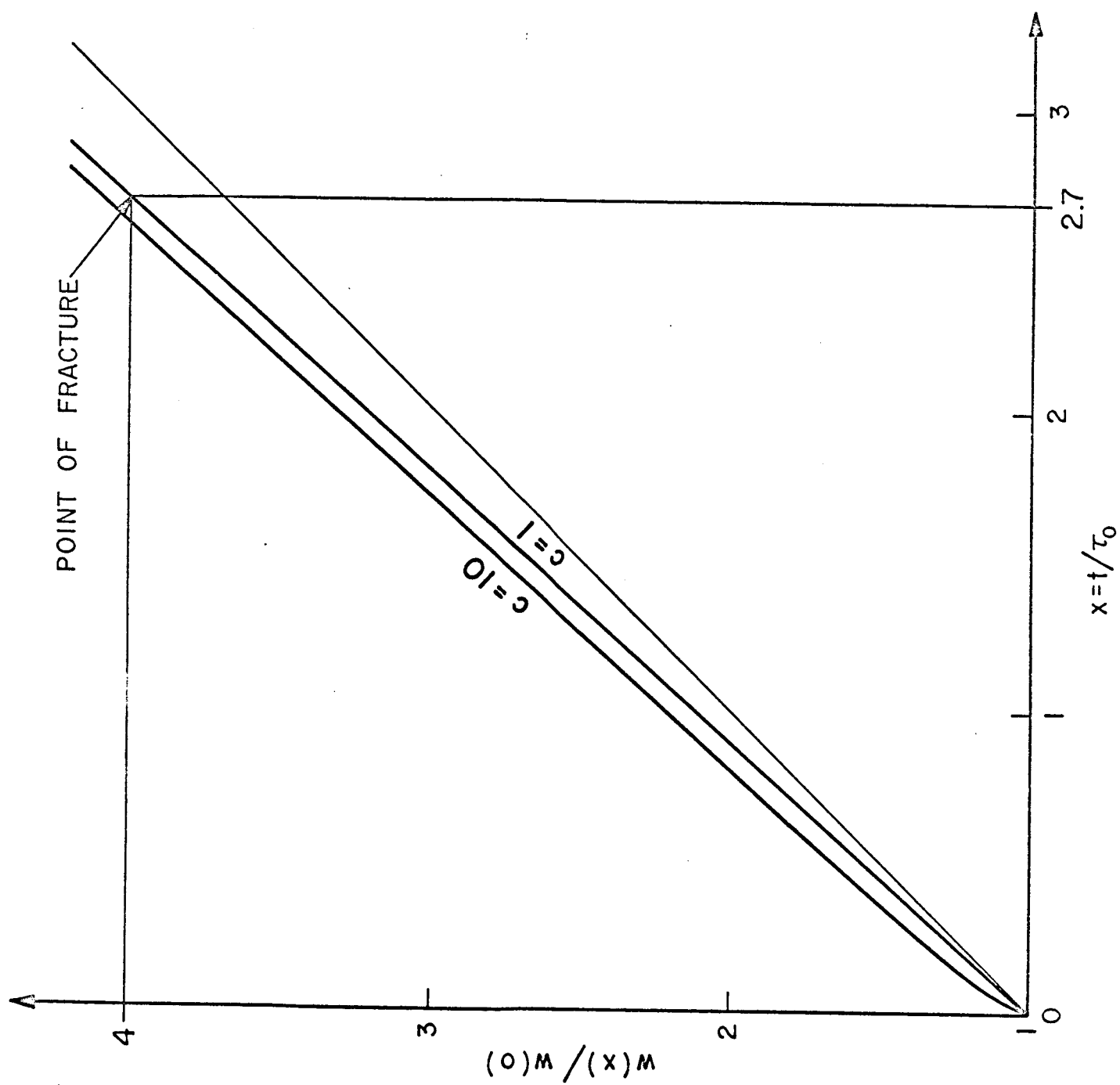


Fig. 3. Growth of displacement at the crack tip prior to fracture (solid line corresponds to time dependent yield stress, while fine line results when yield stress is assumed constant).

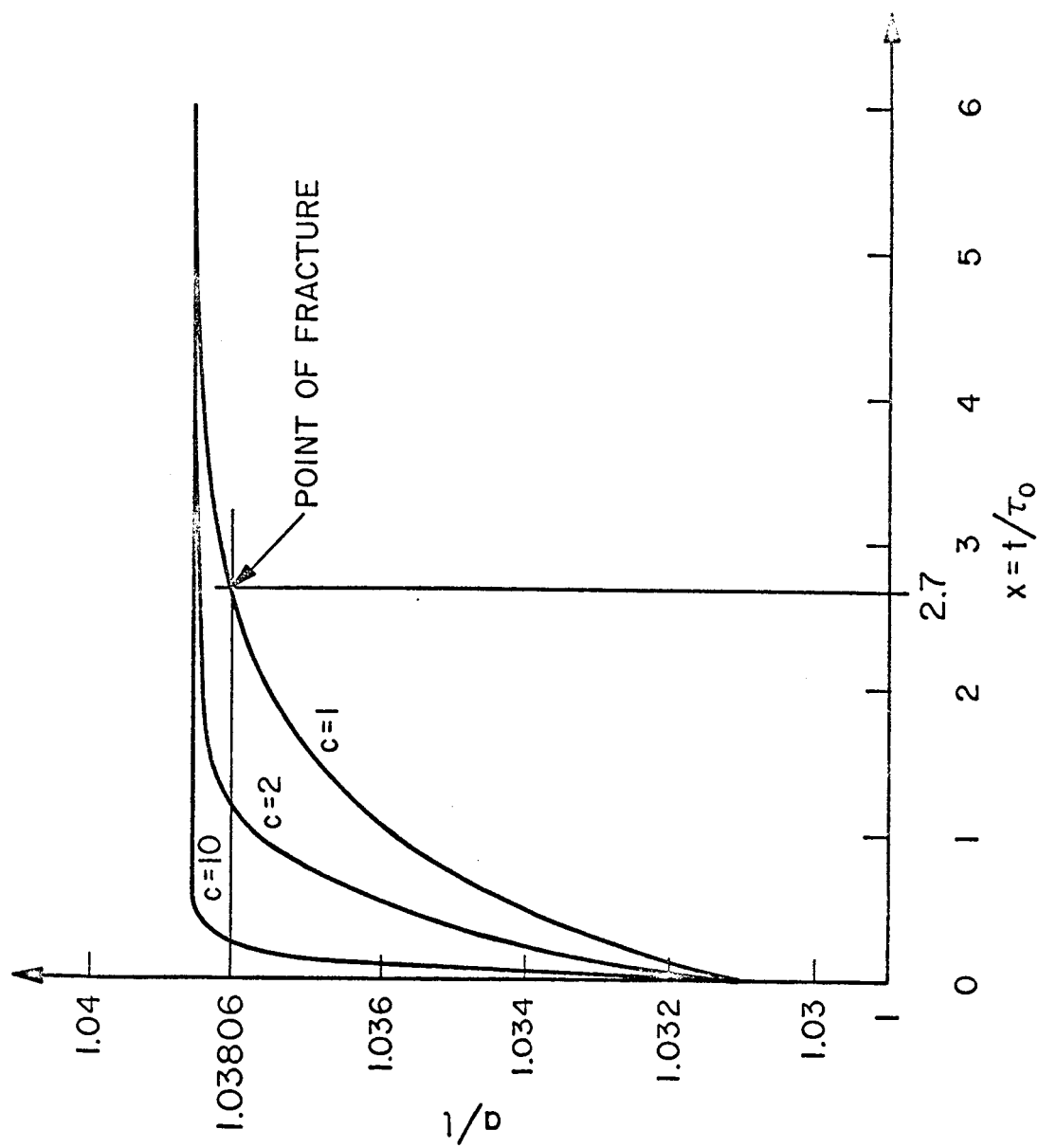


Fig. 4. Extension of plastic zone prior to fracture.
(including sample calculation contained in text):

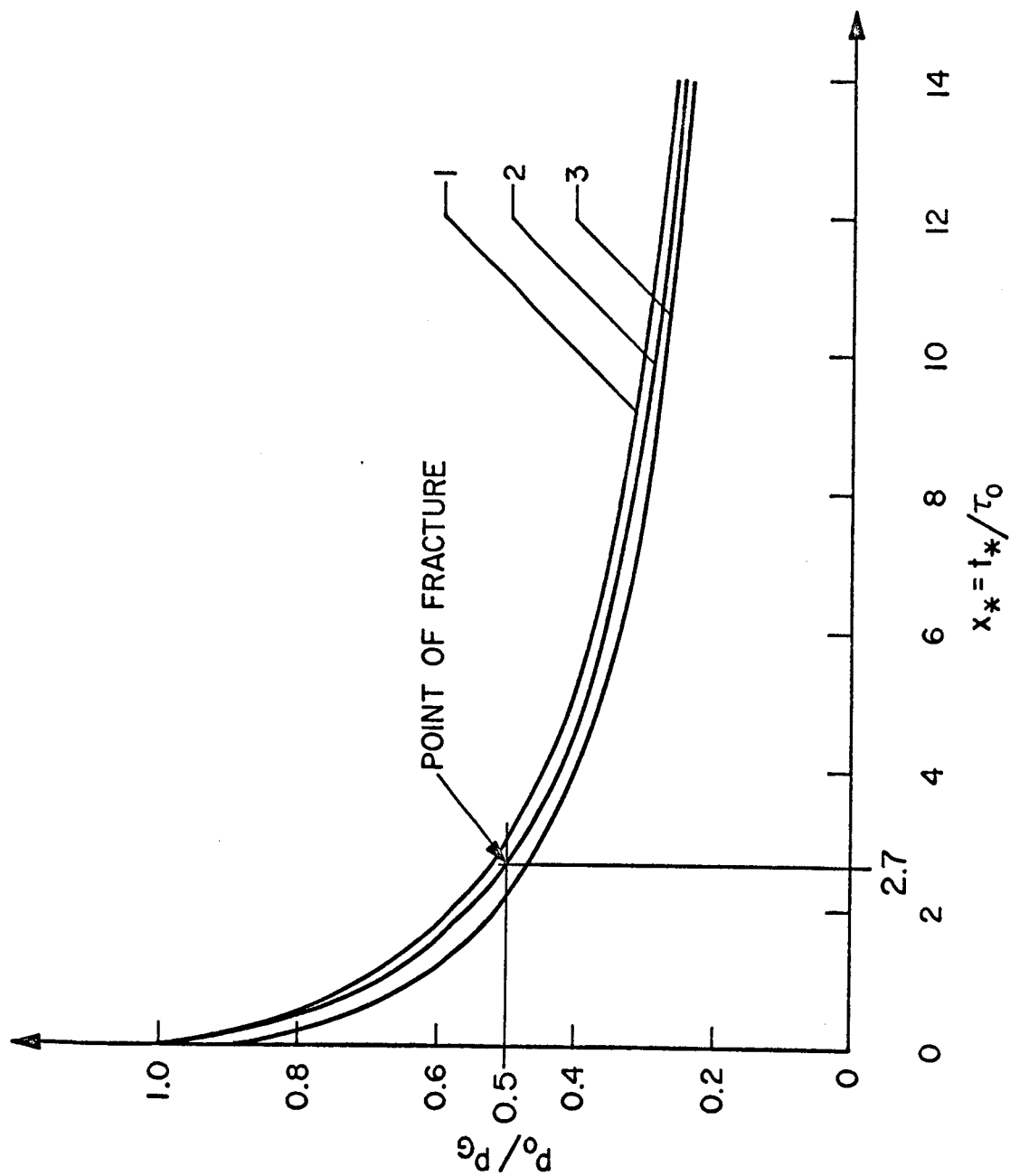


Fig. 5. Delayed fracture at a function load level
 1 and 3 obtained under an assumption of the constant yield
 stress: $\langle Y \rangle = Y_0$ curve 1
 $\langle Y \rangle = Y_\infty$ curve 3
 2 yield stress allowed to vary with time.

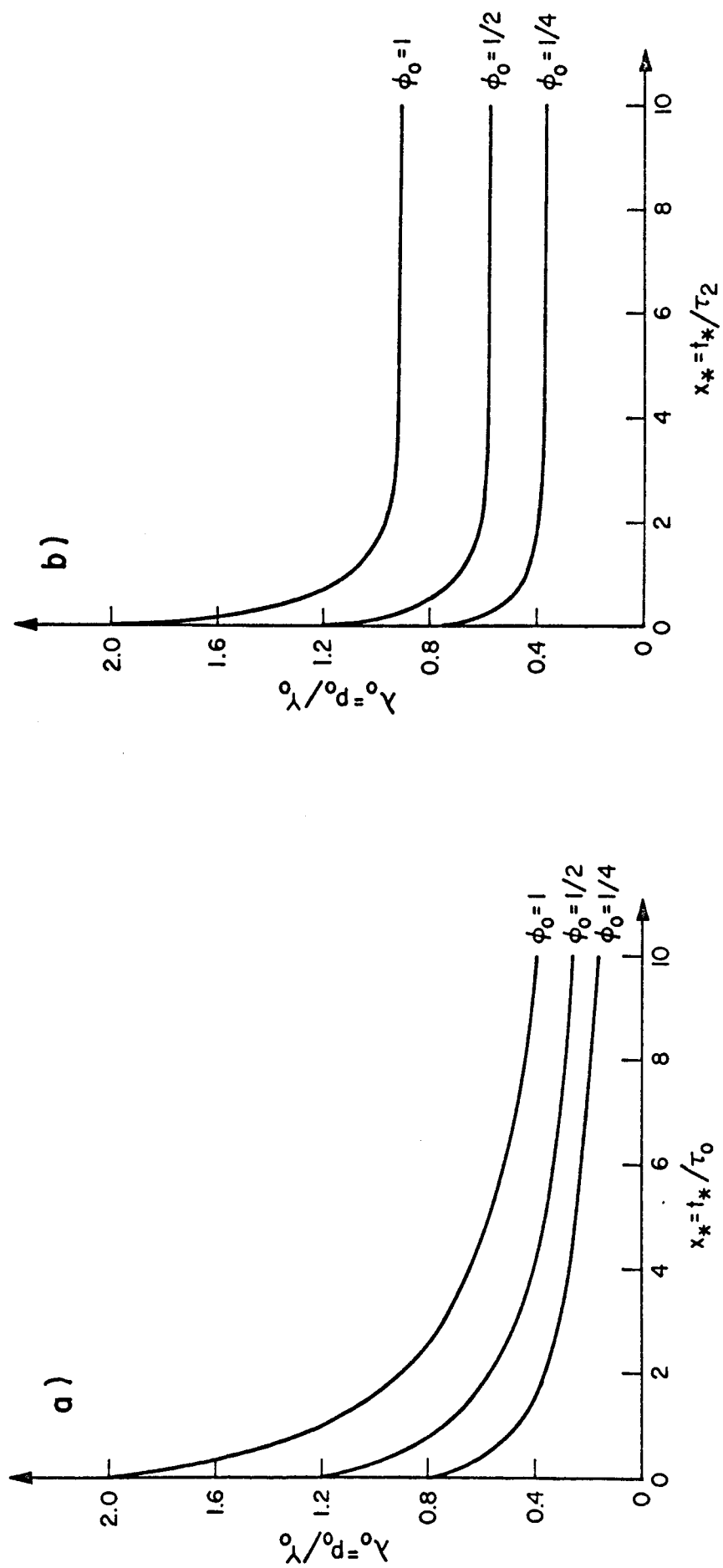


Fig. 6. Delayed fracture caused by a pressurized crack (load applied on crack surface):
a) Maxwell solid,
b) standard linear solid ($E_1/E_2 = 2$).

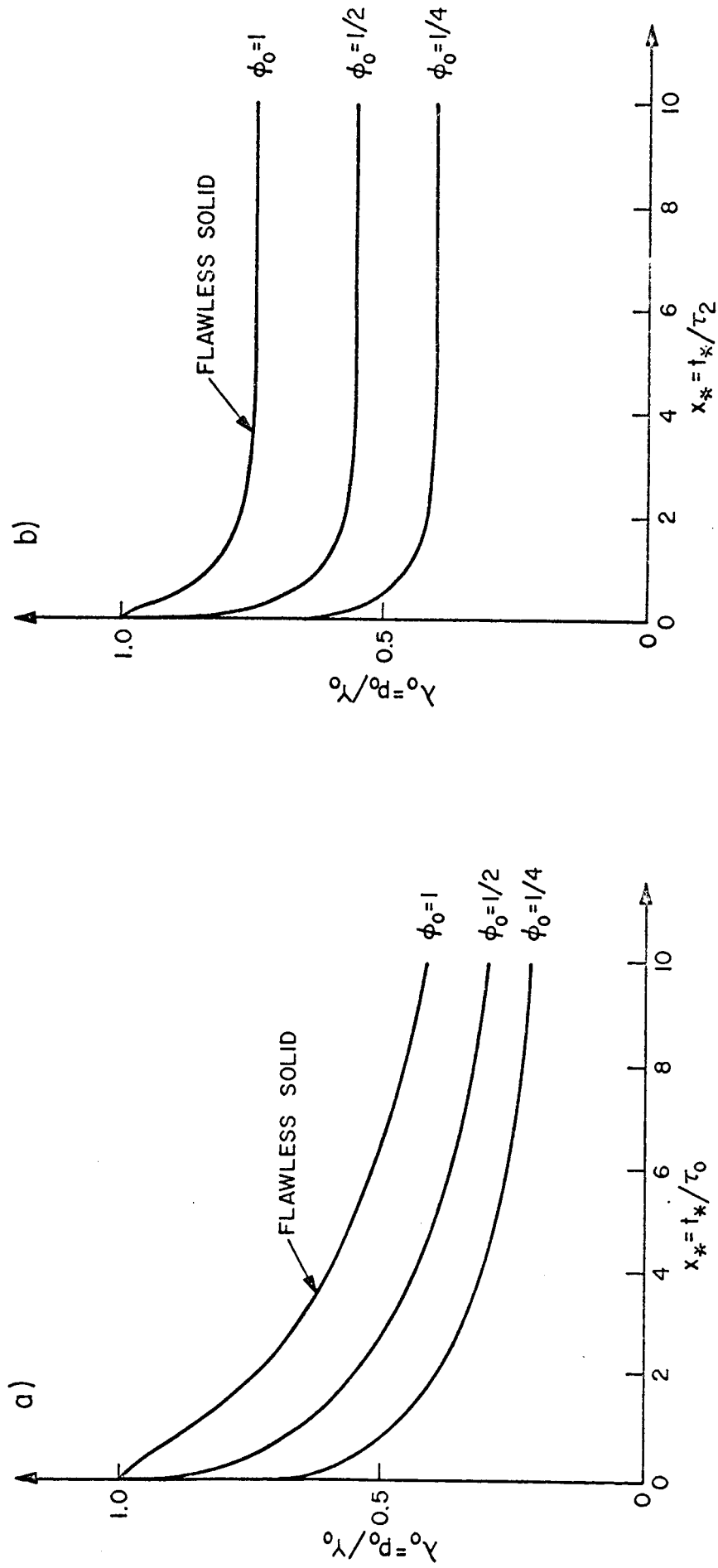


Fig. 7. Delayed fracture caused by a penny shaped crack (tensile stress applied at infinity):
a) Maxwell solid,
b) standard linear solid ($E_1/E_2 = 2$).

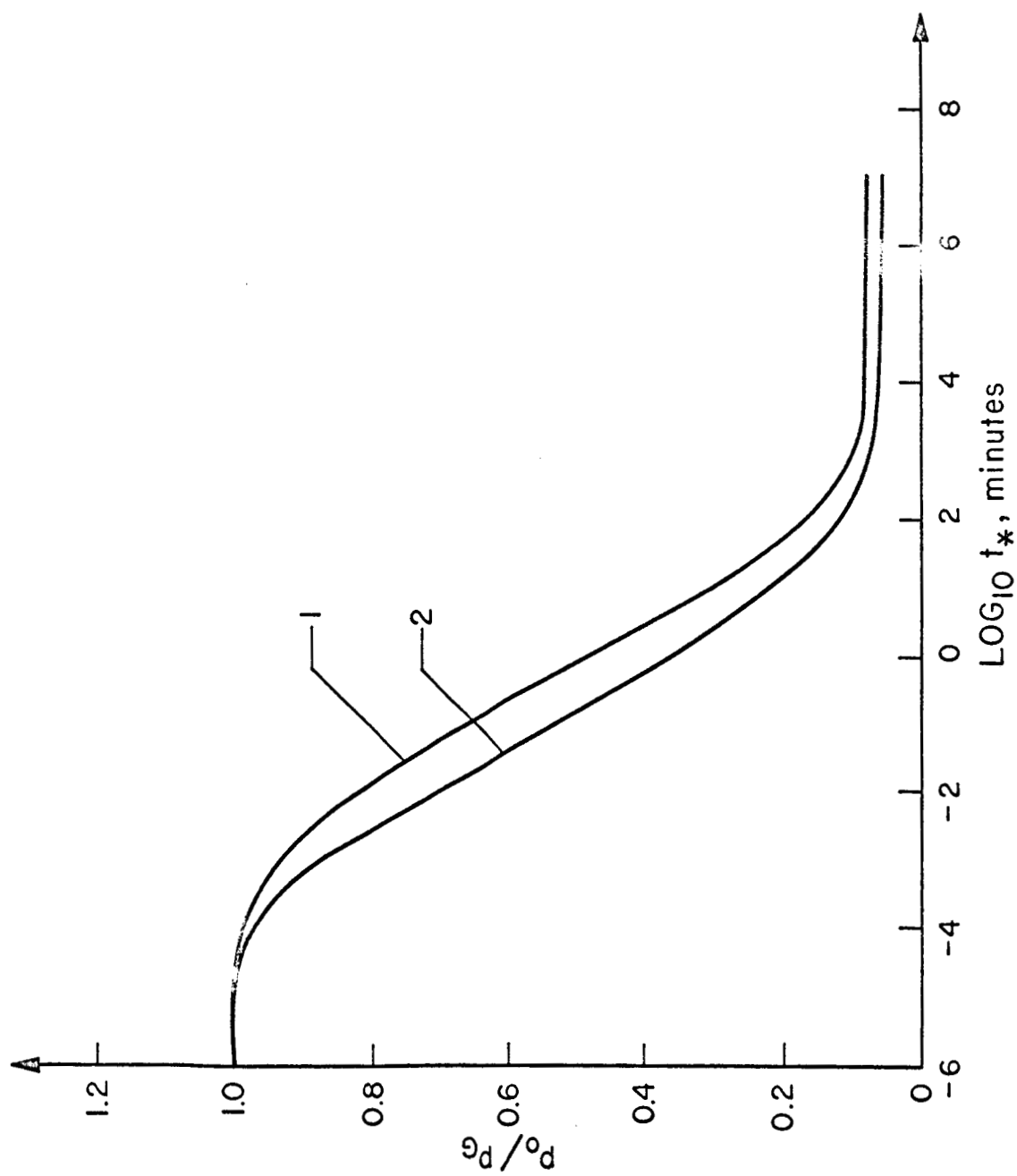


Fig. 8. Creep failure curves: referred to the glass transition temperature

1- present paper (universal curve)

2- Williams' result for a spherical void, Ref. [22]

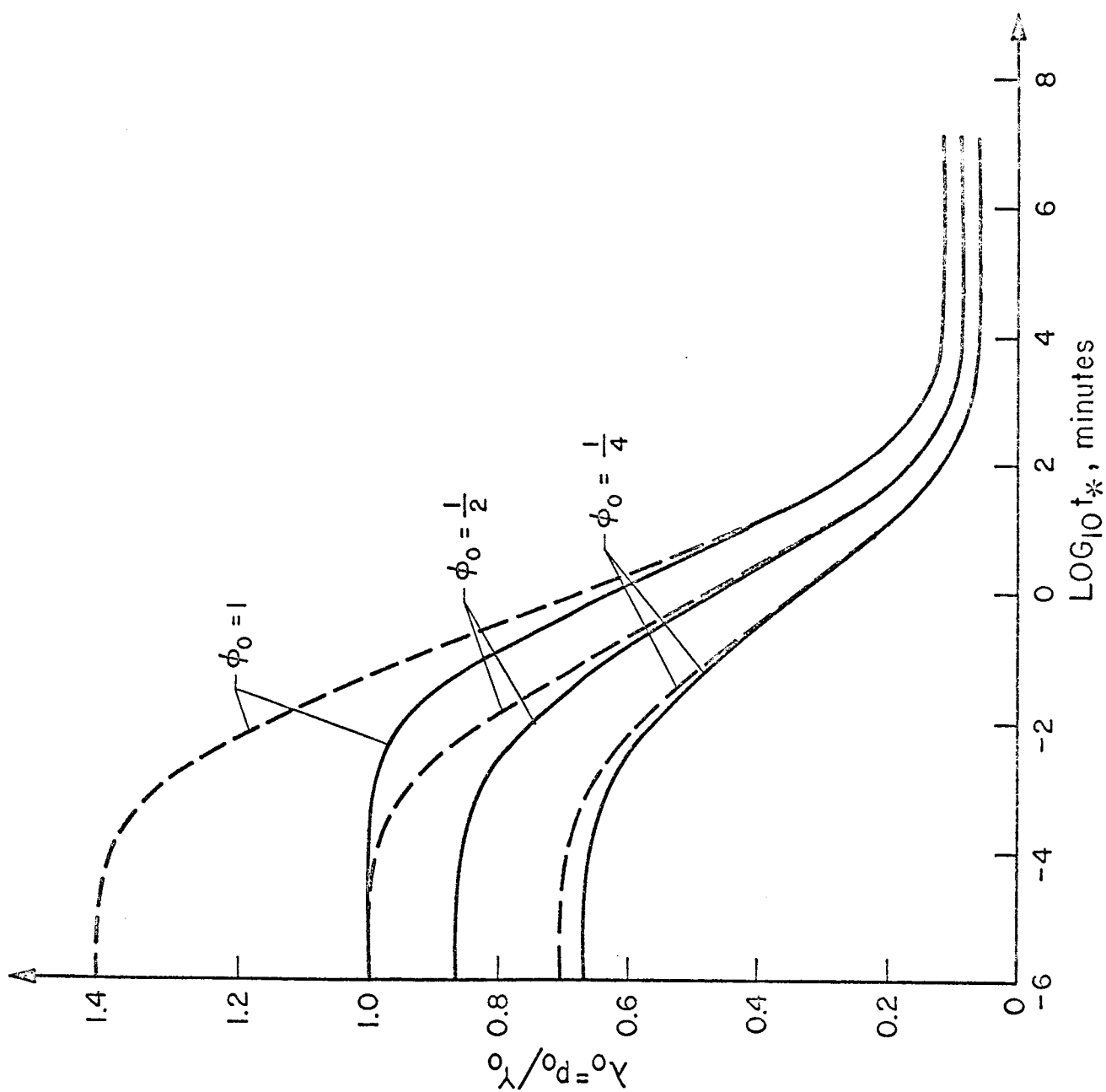


Fig. 9. Creep failure curves for a penny shaped crack (tensile stress applied at infinity); referred to the glass transition temperature
solid lines : plasticity effects at the crack tip taken into account,
dotted lines : plasticity effects neglected.

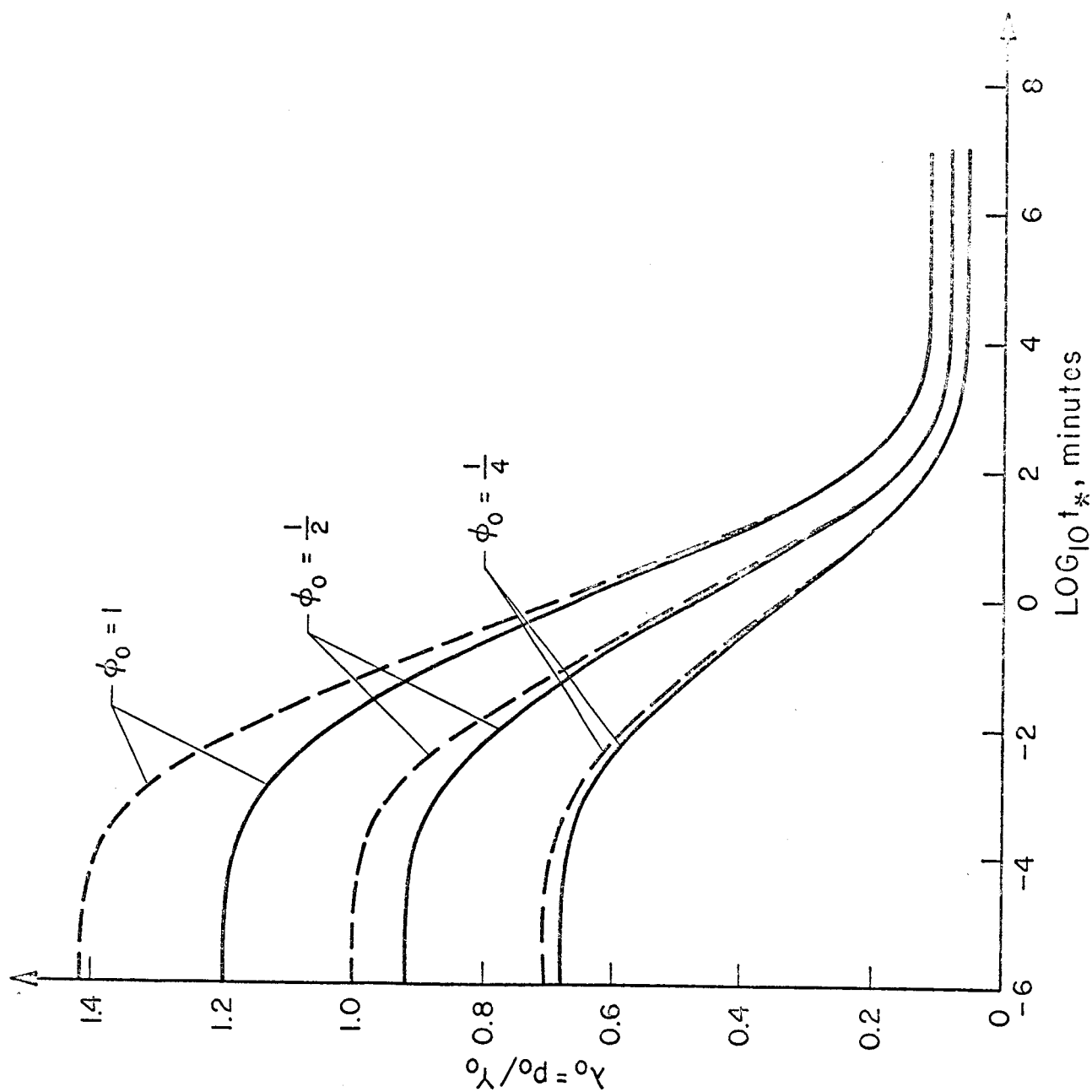


Fig.10 Creep failure curves for the two-dimensional case (plane stress and plane strain); referred to the glass transition temperature
solid lines : plasticity effects at the crack tip taken into account,
dotted lines: plasticity effects neglected.

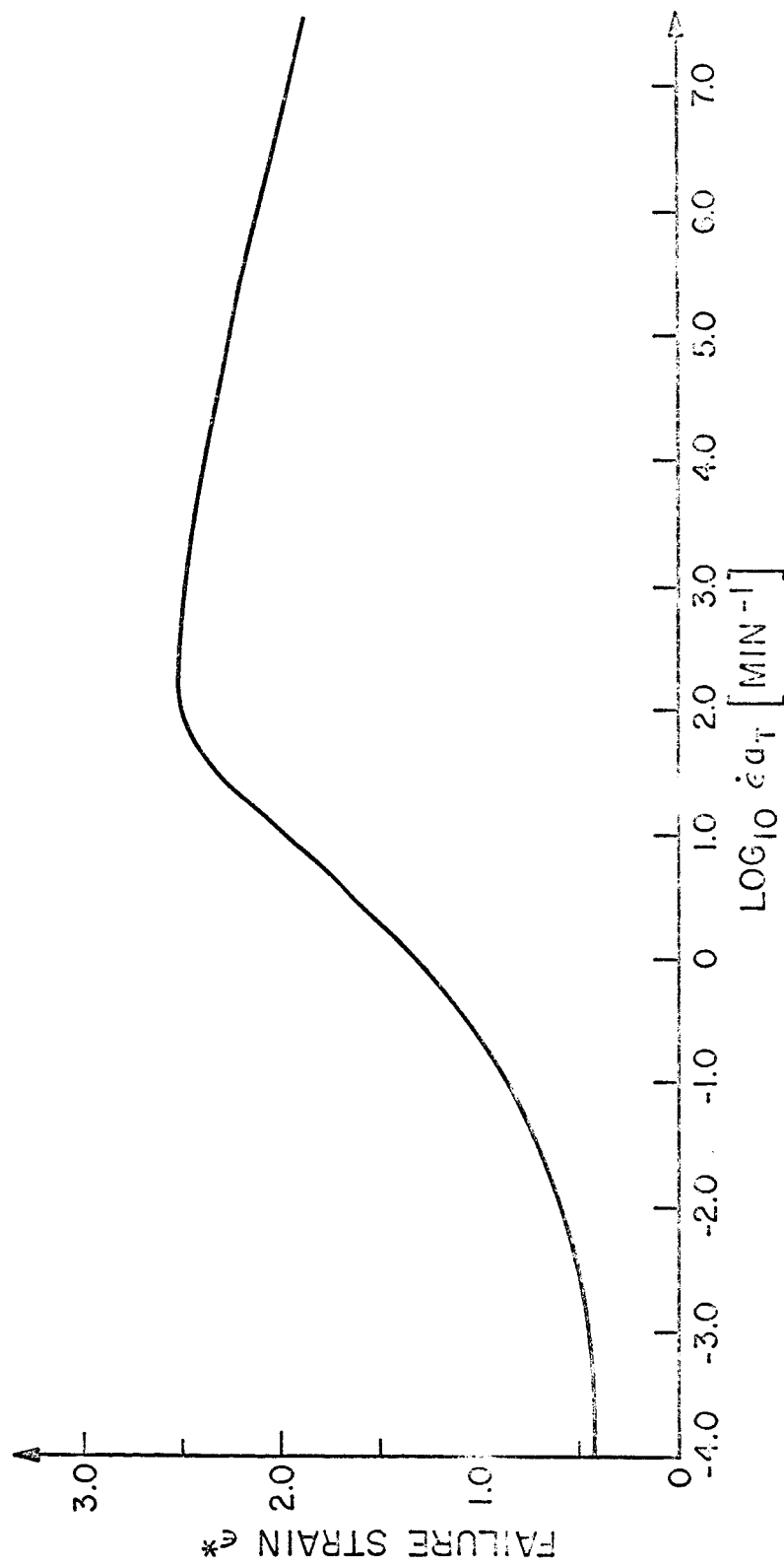


Fig. 11 Temperature-rate-dependent failure strains for unswollen solithane (reference temperature 20°C.).