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DYNAMIC STABILITY OF SPACE VEHICLES

Volume VII - The Dynamics of Liquids in Fixed and Moving Containers

by L. L. Fontenot

Prepared by
GENERAL DYNAMICS CORPORATION
San Diego, Calif.
for George C. Marshall Space Flight Center

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FOREWORD

This report is one of a series in the field of structural dynamics prepared under contract NAS 8-11486. The series of reports is intended to illustrate methods used to determine parameters required for the design and analysis of flight control systems of space vehicles. Below is a complete list of the reports of the series.

Volume I	Lateral Vibration Modes
Volume II	Determination of Longitudinal Vibration Modes
Volume III	Torsional Vibration Modes
Volume IV	Full Scale Testing for Flight Control Parameters
Volume V	Impedence Testing for Flight Control Parameters
Volume VI	Full Scale Dynamic Testing for Mode Determination
Volume VII	The Dynamics of Liquids in Fixed and Moving Containers
Volume VIII	Atmospheric Disturbances that Affect Flight Control Analysis
Volume IX	The Effect of Liftoff Dynamics on Launch Vehicle Stability and Control
Volume X	Exit Stability
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Volume XV	Shell Dynamics with Special Applications to Control Problems

The work was conducted under the direction of Clyde D. Baker and George F. McDonough, Aero Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The General Dynamics Convair Program was conducted under the direction of David R. Lukens.

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INTRODUCTION

For well over a century, scientists and engineers have evidenced an interest in the dynamics of solids containing liquid cavities. Apart from the classical contributions reported in Lamb's first edition of "Hydrodynamics" (1895), only recently has the problem of a partly filled cavity with free surface been explored. Numerous articles devoted to this problem have been published in various countries. This was stimulated by a great number of technical problems requiring the determination of the motion of a body with partly filled liquid cavity. Examples of these include anti-roll passive tank stabilization systems employed in ocean-going ships, dynamics of rockets, and seismic oscillations of structures under water pressure.

The combined efforts of workers in the field have given us a comprehensive theory branching out in several directions. To present this theory completely is beyond the scope of this study; therefore the discussion is not general but is restricted to those aspects concerning stability and control of liquid-rocket powered missiles and space vehicles.

That the effect of liquid propellant motions must be considered in the design of most liquid-rocket powered missiles and space vehicles is well known; for the most part, the problem is one of vehicle stability and control. Generally, the propellant motions interact with both the control system and vehicle dynamics, which also couple with each other. The natural frequencies of the oscillating propellants are usually much closer to the rigid body control frequencies than to the elastic body frequencies. If the natural frequencies of the propellants become too close to the control frequency of the vehicle or the natural frequency of the control sensor, the situation may become critical. Under these circumstances, the oscillating propellants exert large forces and moments on the vehicle, which may saturate the control system and ultimately lead to structural failure. Thus, the responses of forces and moments exerted by the vibrating propellants on the vehicle must be sufficiently well defined analytically so that their effects can be incorporated into analyses of the overall system dynamic behavior.

Some recent surveys [1, 2] catalogue the numerous papers available in the field; in the main, the studies cited therein treat general and specific problems of the motion of liquids in fixed and moving vessels. The methods used in these studies are varied, and the assumptions are based on approximations which are sometimes confusing and difficult to justify. We will not discuss these papers in detail; however, a perusal of them by the reader will disclose an apparent lack of agreement as to the exact analytic statement for the motion of liquids enclosed in moving containers. Moreover, the usual method of obtaining the forces exerted on a space vehicle by oscillating propellants is to compute the liquid reactions forces assuming that the motion of the propellant tanks does not depend on the dynamic state of the vehicle. These reaction forces then are treated as generalized forces acting on the vehicle in an arbitrary state of motion. This computation of events generally gives rise to "equivalent" mechanical models, which are then combined with similar representations for other dynamic elements of the vehicle to obtain the overall system dynamic models. Generally, in such an investigation the analyst uses a reference system moving with respect to inertial space. Invariably, the motions of the propellant tanks are linearized with respect to the tank rates, both translation and rotation. These rates, in turn, depend on the vehicle body rates. The arbitrary linearization of the equations of motion with respect to the body rates is questionable and can lead to serious difficulties. This and the fact that the propellant tanks have not been considered as an integral part of the vehicle make it difficult for the engineer to properly assess the attitude stability characteristics of liquid propellant space vehicles.

There appears to be a need for an analytical review that will clarify concept and introduce the engineer to the more advanced works and the research literature. The main purpose of this investigation is to satisfy this need. To avoid becoming a collection of formulae, many fundamental notions presented elsewhere are included in detail. An excellent collection of such formulae may be found in [3, 4, 5], to which the reader is referred.

The basic materials treated have been kept as modest as possible. A brief review of certain fundamental results from theoretical hydrodynamics necessary to describe the motion of a heavy liquid enclosed in a rigid vessel which is itself in motion is presented. An energy formulation of the system (vessel plus liquid) is written for six degrees of freedom. These concepts are then extended to the case of the planar motion of a liquid propellant vehicle having a single tank and engine. For simplicity the tank is taken to be a prismatic cylinder. The results apply trivially to more than one tank and engine. The planar equations of motion are then used to obtain the perturbation equations of motion. To this end, the vehicle motion is treated as a summation of perturbations from a known reference motion and motion in which vehicle body axes remain coincident with reference axes. The role of the liquid motions in the perturbation equations are then isolated and identified. The liquid in the propellant tank is replaced by a simple mechanical system (system of pendula plus discrete mass and moment of inertia), and planar perturbation equations for the entire vehicle are derived. The effects of the mechanical system motions in the perturbation equations are isolated and identified. The role of the liquid motions and mechanical system motions are compared to show that the mechanical system can duplicate the action of the liquid. The analysis is extended to tanks of arbitrary shape having rotational symmetry, such as commonly occurs in most vehicles. The role of the liquid motions and mechanical system motions are again compared. The oscillations of a heavy liquid in a fixed vessel are treated. In addition, the oscillations of systems of solid bodies with a liquid (systems with a liquid "member") are considered.

The review does not deal with the two directions in which intensive investigations are being currently conducted. These are non-linear oscillations and the problem of damping. A study of these questions involves difficulties of a fundamental nature. A number of algorithms pertaining to the theory of non-linear oscillations have been published, but all of them are unwieldly and, most important, no one has thus far managed to prove their convergence. Moreover, the very question of the existence of periodic solutions of resulting non-linear systems still remains open. Even more complicated is the problem of oscillation of a viscous liquid. The formulation of the problems comprises a great number of difficulties. The problems of the dynamics of a body with a liquid under conditions of weightlessness have become pertinent most recently, but only the first steps have so far been made in this direction and it is still premature to speak of results.

2/ FUNDAMENTAL RESULTS FROM THEORETICAL HYDRODYNAMICS

BASIC EQUATIONS FOR MOTION OF A HEAVY LIQUID ENCLOSED IN A PARTLY FILLED VESSEL WHICH IS ITSELF IN MOTION

Consider the motion of a frictionless liquid enclosed in a partly filled vessel which is itself in motion, Fig. 1. To describe the motion of the system, take a cartesian frame of reference fixed relatively to the container, say an origin o and three axes ox, oy, oz. Reference oxyz is orientated in such a manner that oz is measured positively along the outward directed normal to the undisturbed free surface. Thus the free surface, denoted by S(t), coincides with plane xoy (the plane z = o) when the container and liquid are at rest.

Let

$$z = \zeta\,(x,\ y,\ t) \tag{2.1}$$

be the equation of S(t) when it is displaced. Denote by Σ (t) the wetted surface of the vessel, and by τ (t) the variable volume enclosed by S(t) and Σ (t). Let Σ , τ , and S represent the corresponding values of Σ(t), τ (t) and S(t) in the undisturbed position. All surfaces are assumed to be piece wise smooth.

Suppose that at time t the vessel is coincident with inertial space and that it is moving relatively to inertial space with motion described by an observer in inertial space as a velocity $\bar{u}$ of o and an angular velocity $\bar{\omega}$. Then the position vector $\bar{r}$ of a particular liquid particle $P \in \tau(t)$ at time t is the same for an observer moving with the vessel as it is for an observer in inertial space.

The point P, if rigidly attached to the moving frame of reference oxyz, has the velocity

$$\bar{V} = \bar{u} + \bar{\omega} \times \bar{r}\ . \tag{2.2}$$

Thus, if P is fixed in inertial space instead of in oxyz, it will appear to an observer in oxyz to move with velocity $-\bar{V}$.

Denote by $\bar{q}(P,\ t)$, $\bar{v}(P,\ t)$ the velocities of the liquid particle at point $P(x,\ y,\ z) \in \tau(t)$ at time t as estimated by observers in inertial space and

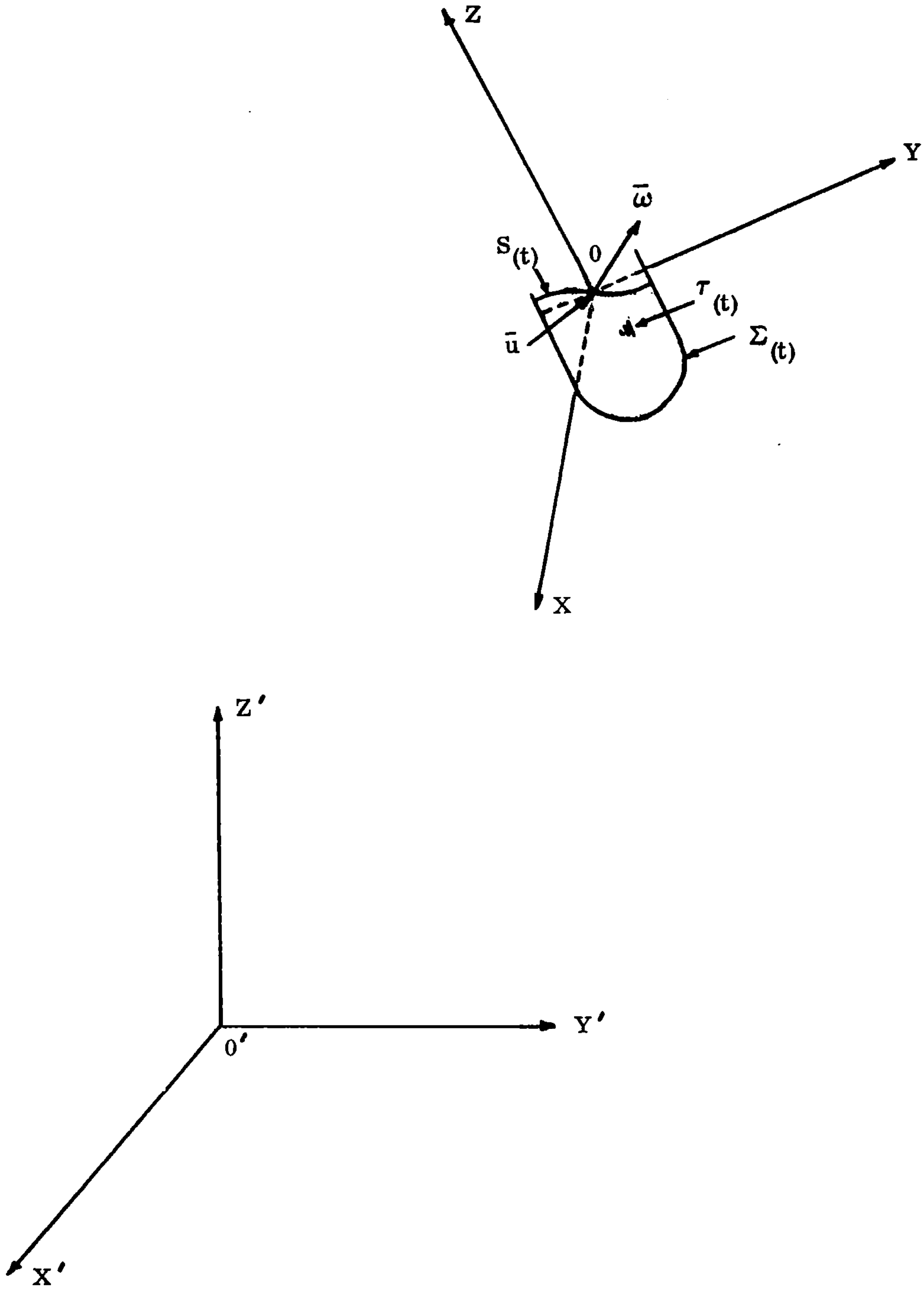


Figure 1. Partially filled vessel in motion.

oxyz respectively. Then

$$\bar{q} = \bar{V} + \bar{v}, \; \bar{v} = \frac{d\bar{r}}{dt}, \tag{2.3}$$

in which the position vector $\bar{r}$ is referred to the moving reference oxyz.

Assume the liquid to be homogeneous and incompressible throughout the motion. Moreover, neglect interfacial tension forces and capillary contact effects between liquid and boundary. Then, the motion of the liquid in $\tau(t)$, when referred to the moving frame of reference oxyz, is completely described by the following formulae:

Equation of motion

$$\frac{d\bar{q}}{dt} + \bar{\omega} \times \bar{q} = \frac{d\bar{v}}{dt} + 2\bar{\omega} \times \bar{v} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + \bar{a} = \bar{f} - \frac{1}{\rho}\nabla p, \; P\epsilon\,\tau(t),$$

$$\frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + [(\bar{q} - \bar{V})\nabla]\,\bar{q}, \; \frac{dv}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v}\,\nabla)\,\bar{v}, \; \bar{a} = \frac{d\bar{u}}{dt} + \bar{\omega} \times \bar{u}, \; P\epsilon\,\tau(t) \tag{2.4}$$

Equation of continuity

$$\nabla\bar{q} = \nabla\bar{v} = 0, \; P\epsilon\,\tau(t) \tag{2.5}$$

Boundary conditions (kinematical)

$$q_n - V_n = v_n = 0, \; P\epsilon\,\Sigma(t) \tag{2.6}$$

$$q_n - V_n = \zeta_t \cos(n, z), \; P\epsilon\,S(t)$$

Boundary conditions (physical)

$$p \perp \Sigma(t), \tag{2.7}$$

$$p(x, y, \zeta, t) = \text{const.}, \; P\epsilon\,S(t)$$

Forces and moments

$$-\bar{F}_1 = \rho \iiint_{\tau(t)} \frac{\partial \bar{q}}{\partial t}\, d\tau + \rho \iiint_{\tau(t)} (\bar{\omega} \times \bar{q})\, d\tau - \rho \iiint_{\tau(t)} \bar{f}\, d\tau + \rho \iint_{S(t)} \bar{q}\zeta_t \cos(n, z)\, ds \tag{2.8}$$

$$= \rho \iiint_{\tau(t)} \frac{\partial \bar{v}}{\partial t}\, d\tau + 2\rho \iiint_{\tau(t)} (\bar{\omega} \times \bar{v}) d\tau + \rho \iiint_{\tau(t)} (\dot{\bar{\omega}} \times \bar{r}) d\tau + \rho \iiint_{\tau(t)} [\bar{\omega} \times (\bar{\omega} \times \bar{r})] d\tau$$

$$- \rho \iiint_{\tau(t)} (\bar{f} - \bar{a}) d\tau + \rho \iint_{(t)} \bar{v}\, \zeta_t \cos(n, z) ds,$$

$$-L_1 = \rho \iiint_{\tau(t)} (\bar{r} \times \frac{\partial \bar{q}}{\partial t}) d\tau + \rho \iiint_{\tau(t)} [\bar{\omega}\, (\bar{r} \cdot \bar{q}) - \bar{q}(\bar{\omega} \bar{r})] d\tau - \rho \iiint_{\tau(t)} (\bar{r} \times \bar{f}) d\tau$$

$$+ \rho \iint_{S(t)} (\bar{r} \times \bar{q}) \zeta_t \cos(n, z) ds$$

$$= \rho \iiint_{\tau(t)} (\bar{r} \times \frac{\partial \bar{v}}{\partial t}) d\tau + 2\rho \iiint_{\tau(t)} [\bar{\omega} (\bar{r}\bar{v}) - \bar{v}(\bar{\omega} \bar{r})] d\tau + \rho \iiint_{\tau(t)} (\bar{\omega} \bar{r}) (\bar{r} \times \bar{\omega}) d\tau$$

$$+ \rho \iiint_{\tau(t)} [\dot{\bar{\omega}} (\bar{r}\bar{r}) - \bar{r}(\dot{\bar{\omega}} \bar{r})] d\tau - \rho \iiint_{\tau(t)} [\bar{r} \times (\bar{f} - \bar{a})] d\tau$$

$$+ \rho \iint_{S(t)} (\bar{r} \times \bar{v})\, \zeta_t \cos(n, z) ds$$

in which $\bar{f}$ is the vector of body forces (such as gravity) per unit mass; ρ is the mass density; p is the pressure intensity at point P(x, y, z) (independent of direction); $\bar{a}$ is the absolute acceleration of 0 as measured by an observer in inertial space.

Formula (1.4) is the second law of motion applied to a liquid particle of infinitesimal volume, and can also be obtained immediately from Euler's equation on application of the classical expression for rates of change of a vector viewed from inertial and moving space. Note that the equation of motion is expressed in either of two forms; one of which is in terms of the absolute velocity $\bar{q}$, and the other in terms of the relative velocity $\bar{v}$.

Formula (1.5) states that the net flow rate of liquid into any small volume must be zero. It likewise is expressed in either of two forms.

Formula (1.6) is the kinematic condition which must be satisfied at the solid boundary $\Sigma(t)$, namely, that the component of velocity normal to the boundary must be equal to the velocity component of the boundary normal to itself. Note that the component of relative velocity normal to $\Sigma(t)$ is zero, $v_n = 0$;

the component of the absolute velocity normal to $\Sigma(t)$ is equal to V_n, the velocity of the boundary normal to itself. This follows from the relationship (2.3).

Formula (2.6_2) is the kinematic condition which must be satisfied at the free surface, Lord Kelvin's condition, and is a corollary of (2.5).[2] $\zeta_t \left(\equiv \frac{\partial \zeta}{\partial t}\right)$ is the apparent velocity of movement of the free surface S(t) in the direction oz; i.e., ζ_t is the velocity of movement along the straight line x = const., y = const., of the point of intersection of the surface $z = \zeta$ with this straight line.

Formula (2.7_1) states that for a frictionless liquid in contact with a rigid boundary, the liquid thrust shall be normal to the boundary.

Formula (2.7_2) expresses the constancy of pressure at the free surface S(t).

Formula (2.8_1), expressed in either of two forms, is the computation of the force resulting from the action of the liquid motion on the vessel surface $\Sigma(t)$. It is obtained from an integration of formula (2.4) throughout $\tau(t)$ and application of the divergence theorem and formulae (2.5, 6, 7).

Formula (2.8_2) is the computation for the moments about the origin of the forces exerted by the liquid on the vessel surface $\Sigma(t)$.

There is an important corollary to formula (2.4) concerning the vorticity of liquid elements. Let the extraneous body forces be conservative,

$$\bar{f} = -\nabla \Omega \tag{2.9}$$

then, on introducing formula (2.9) into (2.4) and operating with ∇x on the resulting equation of motion, we get

$$\frac{d}{dt}(\nabla \times \bar{q}) + \bar{\omega} \times (\nabla \times \bar{q}) = [(\nabla \times \bar{q})\nabla]\bar{q}\,, \tag{2.10}$$

$$\frac{d}{dt}(\nabla \times \bar{q}) = \frac{\partial}{\partial t}(\nabla \times \bar{q}) + [(\bar{q} - \bar{V})\nabla](\nabla \times \bar{q})\,,$$

and (2.11)

$$\frac{d}{dt}(\nabla \times \bar{v}) + \bar{\omega} \times (\nabla \times \bar{v}) + 2\dot{\bar{\omega}} = 2\nabla(\bar{\omega}\bar{v}) - \bar{\omega} \times (\nabla \times \bar{v}) + [(\nabla \times \bar{v})\nabla]\bar{v}\,,$$

[2]See notes.

$$\frac{d}{dt}(\nabla \times \bar{v}) = \frac{\partial}{\partial t}(\nabla \times \bar{v}) + (\bar{v}\nabla)(\nabla \times \bar{v}) ,$$

in which $(\nabla \times \bar{q})$ is the absolute vorticity of liquid elements and $(\nabla \times \bar{v})$ the relative vorticity of liquid elements, that is, the vorticity as measured by an observer in inertial and moving space respectively. $(\nabla \times \bar{q})$ and $(\nabla \times \bar{v})$ are not independent. Indeed, operating on (2.3) with $\nabla \times$ we see

$$(\nabla \times \bar{q}) = 2\bar{\omega} + (\nabla \times \bar{v}) \tag{2.12}$$

Consider a liquid particle which has no vorticity $(\nabla \times \bar{q}) = 0$; from (2.10), it follows that

$$\frac{d(\nabla \times \bar{q})}{dt} = 0 ,$$

and therefore the particle never acquires vorticity. This implies that if

$$(\nabla \times \bar{q}) = 0 \tag{2.13}$$

At some time $t = t_0$, then it is zero for all time, and

$$(\nabla \times \bar{v}) = -2\bar{\omega} \tag{2.14}$$

using (2.12).

There is a significant corollary to formula (2.5) concerning the flux of the liquid. From the divergence theorem

$$\iint\limits_{\Sigma(t)+S(t)} v_n \, ds = \iiint\limits_{\tau(t)} \nabla \bar{v} \, d\tau ,$$

which, in light of formulae (2.5, 6), gives

$$\iint\limits_{S(t)} \zeta_t \cos(n, z) \, ds = 0 . \tag{2.15}$$

It is known that the most general solution of formula (2.5) can be expressed in either of the two forms [6]

$$\bar{q} = \bar{q}_0 + \bar{q}_1 , \; \bar{q}_0 = \nabla\phi , \; \nabla\bar{q}_1 = 0 , \tag{2.16}$$

$$\bar{v} = \bar{v}_0 + \bar{v}_1 , \; \bar{v}_0 = \nabla\psi , \; \nabla\bar{v}_1 = 0 .$$

Vectors $\bar{q}_0$ and $\bar{v}_0$ are the irrotational components of the absolute and relative velocity vectors respectively. $\bar{q}_1$, $\bar{v}_1$ are the corresponding vortical components.

Substituting (2.16) into formula (2.5), we obtain

$$\nabla \phi = \nabla \psi = 0 , \tag{2.17}$$

which states that the irrotational components of the absolute and relative velocity vectors must satisfy Laplace's equation.

When there is no vorticity

$$\bar{q}_1 = 0 , \tag{2.18}$$

$$(\nabla \times \bar{v}_1) = -2 \bar{\omega} ,$$

formulae (2, 10, 11) are satisfied identically and the absolute velocity is completely determined to within an arbitrary additive function of time by the velocity potential ϕ.

Formula (2.18_2) is satisfied identically if we take

$$\bar{v}_1 = -\bar{\omega} \times \bar{r} \tag{2.19}$$

Thus the relative velocity becomes

$$\bar{v} = \nabla \psi - \bar{\omega} \times \bar{r} \tag{2.20}$$

Introducing formulae (2.16_1) with $\bar{q}_1 = 0$ and (2.20) into (2.4) and integrating the resulting equation of motion, we get

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \Omega + \frac{1}{2} v^2 - \frac{1}{2} V^2 = D(t) \tag{2.21}$$

$$\frac{p}{\rho} + \frac{\partial \psi}{\partial t} + \Omega + \bar{a}\bar{r} + \frac{1}{2} v^2 - \frac{1}{2} (\bar{\omega} \times \bar{r})^2 = C(t)$$

where

$$v^2 = \nabla \psi - |\bar{\omega} \times \bar{r}|^2 = |\nabla \phi - \bar{V}|^2 , \; V^2 = |\bar{V}|^2 , \; \bar{f} = -\nabla \Omega ,$$

and D(t), C(t) are instantaneous constants, that is, functions of t only. Therefore at a given instant the constants have the same value throughout the liquid. Expression (2.21) is the unsteady pressure equation expressed in either of two forms.

Substituting formulae (2.16_1) with $\bar{q}_1 = 0$ and (2.20) into the boundary conditions (2.6), we obtain

$$\frac{\partial \phi}{\partial n} = \bar{u}\bar{n} + \omega_x [y \cos(n,z) - z \cos(n,y)] + \omega_y [z \cos(n,x) - x \cos(n,z)]$$

$$+ \omega_z [x \cos(n,y) - y \cos(n,x)], \quad P \epsilon \Sigma(t), \qquad (2.22)$$

$$\frac{\partial \phi}{\partial n} = \bar{u}\bar{n} + \omega_x [y \cos(n,z) - z \cos(n,y)] + \omega_y [z \cos(n,x) - x \cos(n,z)]$$

$$+ \omega_z [x \cos(n,y) - y \cos(n,x)] + \zeta_t \cos(n,z), \quad P \epsilon S(t),$$

$$\frac{\partial \psi}{\partial n} = \omega_x [y \cos(n,z) - z \cos(n,y)] + \omega_y [z \cos(n,x) - x \cos(n,z)]$$

$$+ \omega_z [x \cos(n,y) - y \cos(n,x)], \quad P \epsilon \Sigma(t),$$

$$\frac{\partial \psi}{\partial n} = \omega_x [y \cos(n,z) - z \cos(n,y)] + \omega_y [z \cos(n,x) - x \cos(n,z)]$$

$$+ \omega_z [x \cos(n,y) - y \cos(n,x)] + \zeta_t \cos(n,z), \quad P \epsilon S(t),$$

where cos (n, x), ... denote the cosines of the angles formed by the outward directed normal to the surface of contact at the point P.

Introduce a new potential φ such that

$$\phi = \bar{u}\bar{r} - \omega_x yz - \omega_z xy - \omega_y xz + \int_0^t \tfrac{1}{2} u^2 \, dt + \varphi, \qquad (2.23)$$

$$\psi = -\omega_x yz - \omega_y xz - \omega_z xy + \varphi,$$

$$\phi = \psi + \bar{u}\bar{r} + \int_0^t \tfrac{1}{2} u^2 \, dt.$$

then it follows from (2.17, 21, 22) that

$$\Delta \varphi = 0, \quad P \epsilon \tau(t), \qquad (2.24)$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \dot{\omega}_x yz - \dot{\omega}_y xz - \dot{\omega}_z xy + \bar{a}\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2} (\bar{\omega} \times \bar{r})^2 = C(t), \quad P \epsilon \tau(t), \qquad (2.25)$$

$$\frac{\partial \varphi}{\partial n} = 2\left[\omega_x y \cos(n, z) + \omega_y z \cos(n, x) + \omega_z x \cos(n, y)\right], \quad P \epsilon \Sigma(t) \tag{2.26}$$

$$2\left[\omega_x y \cos(n, z) + \omega_y z \cos(n, x) + \omega_z x \cos(n, y)\right] + \zeta_t \cos(n, z),$$

$$P \epsilon\ S(t)\ .$$

Since the pressure must be independent of position at the free surface we require

$$\frac{\partial \varphi}{\partial t} - \dot{\omega}_x yz - \dot{\omega}_y xz - \dot{\omega}_z xy + \bar{a}\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}(\bar{\omega} \times \bar{r})^2 = 0, \quad P \epsilon\ S(t), \ (z = \zeta), \tag{2.27}$$

which is the condition (2.7a). The forces and moments (2.8) become

$$-F_{1_x} = \rho \iiint\limits_{\tau(t)} \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint\limits_{\tau(t)} \frac{\partial v^2}{\partial x}\, d\tau - \rho \iiint\limits_{\tau(t)} f_x\, d\tau + \rho\, a_x \iiint\limits_{\tau(t)} d\tau \tag{2.28}$$

$$-\rho(\omega_y^2 + \omega_z^2) \iiint\limits_{\tau(t)} x\, d\tau + \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} y\, d\tau + \rho(\omega_z \omega_x - \dot{\omega}_y) \iiint\limits_{\tau(t)} z\, d\tau$$

$$-F_{1_y} = \rho \iiint\limits_{\tau(t)} \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint\limits_{\tau(t)} \frac{\partial v^2}{\partial y}\, d\tau - \rho \iiint\limits_{\tau(t)} f_y\, d\tau + \rho\, a_y \iiint\limits_{\tau(t)} d\tau$$

$$+\rho(\omega_x \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} x\, d\tau - \rho(\omega_x^2 + \omega_z^2) \iiint\limits_{\tau(t)} y\, d\tau + \rho(\omega_z \omega_y - \dot{\omega}_x) \iiint\limits_{\tau(t)} z\, d\tau$$

$$-F_{1_z} = \rho \iiint\limits_{\tau(t)} \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint\limits_{\tau(t)} \frac{\partial v^2}{\partial z}\, d\tau - \rho \iiint\limits_{\tau(t)} f_z\, d\tau + \rho\, a_z \iiint\limits_{\tau(t)} d\tau$$

$$+\rho(\omega_x \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} x\, d\tau + \rho(\omega_y \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} y\, d\tau - \rho(\omega_x^2 + \omega_y^2) \iiint\limits_{\tau(t)} z\, d\tau$$

$$-L_{1_x} = \rho \iiint\limits_{\tau(t)} \left[y \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) - z \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau + \tfrac{1}{2}\rho \iiint\limits_{\tau(t)} \left(y \frac{\partial v^2}{\partial z} - z \frac{\partial v^2}{\partial y}\right) d\tau$$

$$-\rho \iiint\limits_{\tau(t)} (y f_z - z f_y)\, d\tau + \rho\, a_z \iiint\limits_{\tau(t)} y\, d\tau - \rho\, a_y \iiint\limits_{\tau(t)} z\, d\tau$$

$$+\rho(\omega_x \omega_z - \dot{\omega}_z) \iiint\limits_{\tau(t)} x y\, d\tau - \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} x z\, d\tau$$

$$+ \rho(\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} y^2 \, d\tau + \rho(\omega_z^2 - \omega_y^2) \iiint_{\tau(t)} y z \, d\tau$$

$$- \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} z^2 \, d\tau$$

$$-L_{1_y} = \rho \iiint_{\tau(t)} [z \frac{\partial}{\partial x} (\frac{\partial \varphi}{\partial t}) - x \frac{\partial}{\partial z} (\frac{\partial \varphi}{\partial t})] d\tau + \tfrac{1}{2} \rho \iiint_{\tau(t)} (z \frac{\partial v^2}{\partial x} - x \frac{\partial v^2}{\partial z}) d\tau$$

$$- \rho \iiint_{\tau(t)} (z f_x - x f_z) d\tau - \rho a_z \iiint_{\tau(t)} x \, d\tau + \rho a_x \iiint_{\tau(t)} z \, d\tau - \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} x^2 \, d\tau$$

$$- \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} x y \, d\tau + \rho(\omega_x^2 - \omega_z^2) \iiint_{\tau(t)} x z \, d\tau + \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y z \, d\tau$$

$$+ \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} z^2 \, d\tau$$

$$-L_{1_z} = \rho \iiint_{\tau(t)} [x \frac{\partial}{\partial y} (\frac{\partial \varphi}{\partial t}) - y \frac{\partial}{\partial x} (\frac{\partial \varphi}{\partial t})] d\tau + \tfrac{1}{2} \rho \iiint_{\tau(t)} (x \frac{\partial v^2}{\partial y} - y \frac{\partial v^2}{\partial x}) d\tau$$

$$- \rho \iiint_{\tau(t)} (x f_y - y f_x) d\tau + \rho a_y \iiint_{\tau(t)} x \, d\tau - \rho a_x \iiint_{\tau(t)} y \, d\tau + \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x^2 \, d\tau$$

$$+ \rho (\omega_y^2 - \omega_x^2) \iiint_{\tau(t)} x y \, d\tau + \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} x z \, d\tau - \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y^2 \, d\tau$$

$$- \rho (\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} y z \, d\tau$$

Given $\bar{u}$, $\bar{\omega}$, certain initial conditions and the geometry of the vessel, we can in principle find φ. With φ known, the motion of the liquid and its important consequences are completely determined. It should be pointed out that without further knowledge of $\bar{\omega}$ and $\bar{u}$, except insofar as they are given functions of time, we cannot logically discard nonlinearities involving them.

The problem as stated is somewhat academic because in dynamic studies of aerospace vehicles $\bar{u}$ and $\bar{\omega}$ are not given functions of time which are independent of vehicle motion and orientation with respect to inertial space. In reality they depend on the body rates of the vehicle and consequently are not known until the total motion of the system is known.

SUMMARY OF FUNDAMENTAL RESULTS

The basic equations describing the motion of a liquid enclosed in a partly filled vessel which is itself in motion have been derived. Principal assumptions employed in the derivation were:

(1) frictionless liquid;
(2) homogeneous liquid;
(3) liquid incompressible throughout motion;
(4) interfacial tension forces and capillary contact effects between liquid and boundaries neglected;
(5) liquid motion irrotational as viewed by an observer in inertial space.

The pertinent formulae derived are:

(1) Velocities

$$\bar{q} = \bar{u} - \omega_x \nabla y z - \omega_y \nabla x z - \omega_z \nabla x y + \nabla\varphi \qquad (2.29)$$

$$\bar{v} = - \omega_x \nabla y z - \omega_y \nabla x z - \omega_z \nabla x y + \nabla\varphi$$

(2) Equation of motion

$$\frac{p}{\rho} + \frac{\partial\varphi}{\partial t} - \dot{\omega}_x y z - \dot{\omega}_y x z - \dot{\omega}_z x y + \bar{a}\,\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}(\bar{\omega} \times \bar{r})^2 = C(t) \qquad (2.30)$$

(3) Equation of continuity

$$\Delta\varphi = 0$$

(4) Boundary conditions (kinematical)

$$\frac{\partial\varphi}{\partial n} = \begin{cases} 2\,[\omega_x y \cos(n, z) + \omega_y z \cos(n, x) + \omega_z x \cos(n, y)], \; P \epsilon \Sigma(t), & (2.31) \\ 2\,[\omega_x y \cos(n, z) + \omega_y z \cos(n, x) + \omega_z x \cos(n, y)] + \zeta_t \cos(n, z), \; P \epsilon S(t), & \end{cases}$$

$$\iint\limits_{S(t)} \zeta_t \cos(n, z)\, ds = 0$$

(5) **Constancy of pressure at free surface**

$$\frac{\partial \varphi}{\partial t} - \dot{\omega}_x\, y\, z - \dot{\omega}_y\, x\, z - \dot{\omega}_z\, x\, y + \bar{a}\,\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}(\bar{\omega} \times \bar{r})^2 = 0\,, \quad z = \zeta \tag{2.32}$$

(6) **Forces and moments**

$$-F_{1_x} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial x}\, d\tau - \rho \iiint_{\tau(t)} f_x\, d\tau + \rho\, a_x \iiint_{\tau(t)} d\tau \tag{2.33}$$

$$-\rho(\omega_y^2 + \omega_z^2) \iiint_{\tau(t)} x\, d\tau + \rho(\omega_x\, \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y\, d\tau + \rho(\omega_z\, \omega_x - \dot{\omega}_y) \iiint_{\tau(t)} z\, d\tau$$

$$-F_{1_y} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\, d\tau - \rho \iiint_{\tau(t)} f_y\, d\tau + \rho\, a_y \iiint_{\tau(t)} d\tau$$

$$+\rho(\omega_x\, \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x\, d\tau - \rho(\omega_x^2 + \omega_z^2) \iiint_{\tau(t)} y\, d\tau + \rho(\omega_z\, \omega_y - \dot{\omega}_x) \iiint_{\tau(t)} z\, d\tau$$

$$-F_{1_z} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z}\, d\tau - \rho \iiint_{\tau(t)} f_z\, d\tau + \rho\, a_z \iiint_{\tau(t)} d\tau$$

$$+\rho(\omega_x\, \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} x\, d\tau + \rho(\omega_y\, \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} y\, d\tau - \rho(\omega_x^2 + \omega_y^2) \iiint_{\tau(t)} z\, d\tau$$

$$-L_{1_x} = \rho \iiint_{\tau(t)} \left[y \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) - z \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \left(y \frac{\partial v^2}{\partial z} - z \frac{\partial v^2}{\partial y}\right) d\tau$$

$$-\rho \iiint_{\tau(t)} (y\, f_z - z\, f_y)\, d\tau + \rho\, a_z \iiint_{\tau(t)} y\, d\tau - \rho\, a_y \iiint_{\tau(t)} z\, d\tau$$

$$+\rho(\omega_x\, \omega_z - \dot{\omega}_z) \iiint_{\tau(t)} x\, y\, d\tau - \rho(\omega_x\, \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x\, z\, d\tau$$

$$+\rho(\omega_y\, \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} y^2\, d\tau + \rho(\omega_z^2 - \omega_y^2) \iiint_{\tau(t)} y\, z\, d\tau$$

$$-\rho(\omega_y\, \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} z^2\, d\tau$$

$$-L_{1_y} = \rho \iiint_{\tau(t)} \left[z \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t}\right) - x \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau + \frac{1}{2}\rho \iiint_{\tau(t)} \left(z \frac{\partial v^2}{\partial x} - x \frac{\partial v^2}{\partial z} \right) d\tau$$

$$- \rho \iiint_{\tau(t)} (z f_x - x f_z) \, d\tau - \rho a_z \iiint_{\tau(t)} x \, d\tau + \rho a_x \iiint_{\tau(t)} z \, d\tau - \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} x^2 \, d\tau$$

$$- \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} x y \, d\tau + \rho(\omega_x^2 - \omega_z^2) \iiint_{\tau(t)} x z \, d\tau + \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y z \, d\tau$$

$$+ \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} z^2 \, d\tau$$

$$-L_{1_z} = \rho \iiint_{\tau(t)} \left[x \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right) - y \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau + \frac{1}{2}\rho \iiint_{\tau(t)} \left(x \frac{\partial v^2}{\partial y} - y \frac{\partial v^2}{\partial x} \right) d\tau$$

$$- \rho \iiint_{\tau(t)} (x f_y - y f_x) \, d\tau + \rho a_y \iiint_{\tau(t)} x \, d\tau - \rho a_x \iiint_{\tau(t)} y \, d\tau + \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x^2 \, d\tau$$

$$+ \rho (\omega_y^2 - \omega_x^2) \iiint_{\tau(t)} x y \, d\tau + \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} x z \, d\tau - \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y^2 \, d\tau$$

$$- \rho (\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} y z \, d\tau$$

Thus, with $\bar{u}$, $\bar{\omega}$, certain initial conditions and the geometry of the vessel given, φ can presumably be determined. Consequently the motion of the liquid in $\tau(t)$ is known. It should be pointed out that the determination of φ involves difficulties of a fundamental nature. A number of algorithms pertaining to the non-linear oscillations of a liquid enclosed in partly filled prismatic cylinders have been published but all of them are still very clumsy and, most important, no one has thus far succeeded in proving their convergence. Moreover, the very question of the existence of periodic solutions of non-linear systems still remains open.

STOKES' PROBLEM

Consider the classical problem of the motion of a heavy liquid enclosed in a completely filled vessel which is itself in motion.

Denote by τ the volume of the liquid cavity (the plane S is no longer a free surface). Let the volume τ be set in motion in any manner. Moreover, assume that this motion is known to us, i.e., the instantaneous translational and angular velocities of the vessel are known. The velocity potential of the absolute motion of the liquid must satisfy

$$\Delta \phi = 0$$

The boundary condition is

$$\frac{\partial \phi}{\partial n} = \bar{u}\,\bar{n} + \omega_x \left[y \cos (n, z) - z \cos (n, y)\right] + \omega_y \left[z \cos (n, x) - x \cos (n, z)\right]$$
$$+ \omega_z \left[x \cos (n, y) - y \cos (n, x)\right], \quad P \epsilon\, \Sigma + S \, . \tag{2.34}$$

This condition may be satisfied by writing

$$\phi = u_x \phi_x^* + u_y \phi_y^* + u_z \phi_z^* + \omega_x \varphi_x^* + \omega_y \varphi_y^* + \omega_z \varphi_z^* \, ,$$

where ϕ_x^* , . . . are harmonic and satisfy the boundary conditions

$$\frac{\partial \phi_x^*}{\partial n} = \cos (n, x) \, , \; \frac{\partial \phi_y^*}{\partial n} = \cos (n, y) \, , \; \frac{\partial \phi_z^*}{\partial n} = \cos (n, z) \, , \; P \epsilon\, \Sigma + S \, ,$$

$$\frac{\partial \varphi_x^*}{\partial n} = y \cos (n, z) - z \cos (n, y) \, , \; P \epsilon\, \Sigma + S \, , \tag{2.35}$$

$$\frac{\partial \varphi_y^*}{\partial n} = z \cos (n, x) - x \cos (n, z) \, , \; P \epsilon\, \Sigma + S \, ,$$

$$\frac{\partial \varphi_z^*}{\partial n} = x \cos (n, y) - y \cos (n, x) \, , \; P \epsilon\, \Sigma + S \, ,$$

so that ϕ_x^* , . . . depend solely on the geometry of τ and not on its motion. Such is the classical problem of Stokes. ϕ_x^*, ... are sometimes called Stokes potentials.

If for ϕ_x^*, ϕ_y^*, ϕ_z^*, φ_x^*, φ_y^*, φ_z^* we substitute x, y, z, $-zy+\varphi_x^*$, $-xz+\varphi_y^*$, $-xy+\varphi_z^*$ we get

$$\varphi = \bar{u}\,\bar{r} - \omega_x (yz - \varphi_x^*) - \omega_y (xz - \varphi_y^*) - \omega_z (xy - \varphi_z^*)\,, \tag{2.36}$$

where φ_x^*, ... are harmonic and satisfy conditions

$$\frac{\partial \varphi_x^*}{\partial n} = 2\, y \cos (n, z)\,, \quad P \in \Sigma + S\,, \tag{2.37}$$

$$\frac{\partial \varphi_y^*}{\partial n} = 2\, z \cos (n, x)\,, \quad P \in \Sigma + S\,,$$

$$\frac{\partial \varphi_z^*}{\partial n} = 2\, x \cos (n, y)\,, \quad P \in \Sigma + S\,.$$

Note that the differential system

$$\Delta \varphi_x^* = 0\,, \quad P \in \tau\,,$$

$$\frac{\partial \varphi_x^*}{\partial n} = 2\, y \cos (n, z)\,, \quad P \in \Sigma + S$$

is equivalent to the variational problem for the functional

$$G_x^* = \tfrac{1}{2} \iiint_\tau (\nabla \varphi_x^*)^2 \, d\tau - 2 \iint_{S+\Sigma} \varphi_x^* \, y \cos (n, z)\, ds\,.$$

Similarly,

$$G_y^* = \tfrac{1}{2} \iiint_\tau (\nabla \varphi_y^*)^2 \, d\tau - 2 \iint_{S+\Sigma} \varphi_y^* \, z \cos (n, x)\, ds\,,$$

$$G_z^* = \tfrac{1}{2} \iiint_\tau (\nabla \varphi_z^*)^2 \, d\tau - 2 \iint_{S+\Sigma} \varphi_z^* \, x \cos (n, y)\, ds\,.$$

These expressions may be combined into a single formula

$$G_i^* (\varphi^*) = \tfrac{1}{2} \iiint_\tau (\nabla \varphi^*)^2 \, d\tau - 2 \iint_{S+\Sigma} \varphi^* \, \eta_i \, ds \tag{2.38}$$

where

$$\eta_i = \begin{cases} y \cos(n, z) & \text{for } i = 1 , \\ z \cos(n, x) & \text{for } i = 2 , \\ x \cos(n, y) & \text{for } i = 3 . \end{cases}$$

Thus the problem of determining Stokes' potentials is equivalent to finding the extremum of functional (2.38). The method of Ritz may be used to solve this variational problem.

Consider the motion of the entire system (liquid plus solid). Denote by $\dot{T}$ the total kinetic energy of the system. Thus

$$T = T_1 + T_s ,$$

$$T_1 = \frac{1}{2}\iiint_{\tau(t)} \rho |q|^2 d\tau = \frac{1}{2}\iiint_{\tau(t)} \rho\, |\bar{V} + \bar{v}|^2\, d\tau \ , \quad T_s = \frac{1}{2} \iiint_{\tau^*} \rho\, |\bar{V}|^2\, d\tau^* ,$$

where T_1 is the kinetic energy of the liquid in $\tau(t)$ and T_s is the kinetic energy of the solid, i.e., vessel proper. τ^*, ρ^* represent the volume and mass density of the vessel proper,respectively. Let Π_1 be the potential energy of the liquid in $\tau(t)$,

$$\Pi_1 = \iiint_{\tau(t)} \rho\ \Omega\ d\tau$$

$$\bar{f} = -\nabla\ \Omega$$

then, the equations of motion for the complete system may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T}{\partial \bar{u}} = \bar{F}_{ext} + \iiint_{\tau^*} \rho^* \bar{f}\, d\tau^* + \iiint_{\tau(t)} \rho\ \bar{f}\ d\tau \qquad (2.39)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T}{\partial \bar{u}} = \bar{L}_{ext} + \iiint_{\tau^*} \rho^* (\bar{r} \times \bar{f})\, d\tau^* + \iiint_{\tau(t)} \rho\, (\bar{r} \times \bar{f})\ d\tau \ ,$$

$$\frac{d}{dt}\ (T_1 + \Pi_1) = -\iint_{S(t)+\Sigma(t)} p\ (V_n + v_n)\ ds = -\iint_{S(t)+\Sigma(t)} p\ V_n\ ds \ ,$$

$$p\,(x,\ y,\ \zeta,\ t) = 0\ , \quad P \in S(t)\ ,$$

$$\nabla \bar{v} = 0\ , \quad P \in \tau(t)$$

$$v_n = \begin{cases} 0\ , & P \in \Sigma(t)\ , \\ \zeta_t\ \cos\,(n,\ z)\ , & P \in S(t)\ , \end{cases}$$

in which $\iiint\limits_{\tau(t)} \rho \bar{f}\, d\tau$, $\iiint\limits_{\tau^*} \rho^* \bar{f}\, d\tau^*$ are the resultant of body forces per unit mass acting on the liquid in $\tau(t)$ and on the solid respectively, and $\bar{F}_{ext.}$, $\bar{L}_{ext}$ are externally applied forces and moments.

Formulae (2.39_1) and (2.39_2) are Kirchhoff's equations in vector form (Lagrange's equations referred to moving coordinates).

Formula (2.39_3) states that the rate of change of total energy of any portion of the liquid (assumed incompressible) as it moves about is equal to the rate of working of the pressures on the boundary. In deriving this expression it was assumed that $\frac{\partial \Omega}{\partial t} = 0$. The unsteady pressure equation may be derived from the formula. Moreover, condition (2.39_4) is a corollary of (2.39_3) for the irrotational motion of a liquid enclosed in a fixed vessel.

Formulae (2.39_1) and (2.39_2) can be written as

$$\frac{d}{dt}\left(\frac{\partial T_s}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T_s}{\partial \bar{u}} = \bar{F}_{ext} + \iiint\limits_{\tau^*} \rho^* \bar{f}\, d\tau^* + \iiint\limits_{\tau(t)} \rho \bar{f}\, d\tau - \frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{u}}\right) - \bar{\omega} \times \frac{\partial T_1}{\partial \bar{u}} ,$$

$$\frac{d}{dt}\left(\frac{\partial T_s}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T_s}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T_s}{\partial \bar{u}} = \bar{L}_{ext} + \iiint\limits_{\tau^*} \rho^* (\bar{r} \times \bar{f})\, d\tau^* + \iiint\limits_{\tau(t)} \rho (\bar{r} \times \bar{f})\, d\tau$$

$$- \frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{\omega}}\right) - \bar{\omega} \times \frac{\partial T_1}{\partial \bar{\omega}} - \bar{u} \times \frac{\partial T_1}{\partial \bar{u}} .$$

Now, if the liquid had been absent ($T_1 = 0$), the right side of these equations would have contained only $\bar{F}_{ext} + \iiint\limits_{\tau^*} \bar{f}\, d\tau^*$ and $\bar{L}_{ext} + \iiint\limits_{\tau^*} \rho^* (\bar{r} \times \bar{f})\, d\tau^*$. The action of the liquid pressures must therefore be represented by the remaining terms on the right. Thus the action of the liquid is represented by the force and moment

$$-\bar{F}_1 = \frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T_1}{\partial \bar{u}} - \iiint\limits_{\tau(t)} \rho \bar{f}\, d\tau ,$$

$$-\bar{L}_1 = \frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T_1}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T_1}{\partial \bar{u}} - \iiint\limits_{\tau(t)} \rho (\bar{r} \times \bar{f})\, d\tau ,$$

which can be verified from formulae (2.8). To be sure

$$\frac{\partial T_1}{\partial \bar{u}} = \iiint\limits_{\tau(t)} (\bar{u} + \bar{\omega} \times \bar{r} + \bar{v})\, \rho\, d\tau ,$$

$$\frac{\partial T_1}{\partial \bar{\omega}} = \iiint_{\tau(t)} (\bar{r} \times \bar{u} + \bar{r} \times (\bar{\omega} \times \bar{r}) + \bar{r} \times \bar{v}) \rho d\tau ,$$

$$\bar{\omega} \times \frac{\partial T_1}{\partial \bar{u}} = \iiint_{\tau(t)} (\bar{\omega} \times \bar{u} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + \bar{\omega} \times \bar{v}) \rho \, d\tau ,$$

$$\bar{\omega} \times \frac{\partial T_1}{\partial \bar{\omega}} = \iiint_{\tau(t)} [\bar{\omega} \times (\bar{r} \times \bar{u}) - (\bar{\omega}\,\bar{r})(\bar{\omega} \times \bar{r}) + \bar{\omega} \times (\bar{r} \times \bar{v})] \rho \, d\tau ,$$

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \bar{u}}\right) = \iiint_{\tau(t)} [\dot{\bar{u}} + \dot{\bar{\omega}} \times r + \bar{\omega} \times \bar{v} + \frac{\partial \bar{v}}{\partial t} + (\bar{v} \nabla) \bar{v}] d\tau , \quad [\frac{d}{dt}(\rho d\tau) = 0]$$

$$= \iiint_{\tau(t)} (\dot{\bar{u}} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \bar{v}) \rho \, d\tau + \iiint_{\tau(t)} \frac{\partial (\rho \bar{v})}{\partial t} d\tau + \iint_{S(t)} \rho \bar{v} \zeta_t \cos(n, z) \, ds ,$$

$$\rho [\frac{\partial \bar{v}}{\partial t} + (\bar{v} \nabla) \bar{v}] = \frac{\partial (\rho \bar{v})}{\partial t} + \nabla (\rho \bar{v} : \bar{v}) ,$$

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \bar{v}) = 0 ,$$

$$\iiint_{\tau(t)} \nabla (\rho \bar{v} : \bar{v}) d\tau = \iint_{S(t)+\Sigma(t)} \rho \bar{v} v_n \, ds = \iint_{S(t)} \rho \bar{v} \zeta_t \cos(n, z) \, ds ,$$

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \bar{\omega}}\right) = \iiint_{\tau(t)} [\bar{v} \times \bar{u} + \bar{r} \times \dot{\bar{u}} + \bar{v} \times (\bar{\omega} \times \bar{r}) + \bar{r} \times (\dot{\bar{\omega}} \times \bar{r}) + \bar{r} \times (\bar{\omega} \times \bar{v})] \rho \, d\tau$$

$$+ \iiint_{\tau(t)} [\bar{r} \times \frac{\partial (\rho \bar{v})}{\partial t}] d\tau + \iint_{S(t)} \rho (\bar{r} \times \bar{v}) \zeta_t \cos(n, z) \, ds ,$$

and therefore

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T_1}{\partial \bar{u}} = \iiint_{\tau(t)} \frac{\partial (\rho \bar{v})}{\partial t} d\tau + 2 \iiint_{\tau(t)} \rho (\bar{\omega} \times \bar{v}) d\tau + \iiint_{\tau(t)} \rho (\dot{\bar{\omega}} \times \bar{r}) d\tau$$

$$+ \iiint_{\tau(t)} \rho [\bar{\omega} \times (\bar{\omega} \times \bar{r})] d\tau + \iiint_{\tau(t)} \rho \bar{a} \, d\tau + \iint_{S(t)} \rho \bar{v} \zeta_t \cos(n, z) \, ds$$

$$= -\bar{F}_1 + \iiint_{\tau(t)} \rho \bar{f} \, d\tau ,$$

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T_1}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T_1}{\partial \bar{u}} = \iiint_{\tau(t)} \rho [\bar{r} \times \frac{\partial (\rho \bar{v})}{\partial t}] d\tau + 2 \iiint_{\tau(t)} \rho [\bar{\omega} (\bar{r}\,\bar{v}) - \bar{v} (\bar{\omega}\,\bar{r})] d\tau$$

$$+ \iiint_{\tau(t)} \rho(\bar{\omega} \cdot \bar{r})(\bar{r} \times \bar{\omega})\, d\tau + \iiint_{\tau(t)} \rho\,[\dot{\bar{\omega}}(\bar{r}\,\bar{r}) - \bar{r}\,(\dot{\bar{\omega}}\,\bar{r})]\, d\tau$$

$$+ \iiint_{\tau(t)} \rho(\bar{r} \times \bar{a})\, d\tau + \iint_{S(t)} \rho(\bar{r} \times \bar{v})\, \zeta_t \cos(n, z)\, ds$$

$$= -\bar{L}_1 + \iiint_{\tau(t)} \rho(\bar{r} \times \bar{f})\, d\tau$$

as adduced.

If the motion of the liquid is irrotational when measured in inertial space, then

$$\bar{v} = \nabla \psi - \bar{\omega} \times \bar{r},$$

$$= \nabla \varphi - \bar{\omega} \times \bar{r} - \omega_x \nabla y\, z - \omega_y \nabla x\, z - \omega_z \nabla x\, y,$$

using (2.23) and formulae (2.39) become

$$a_x \{\rho \iiint_{\tau(t)} d\tau + \iiint_{\tau^*} \rho^* d\tau^*\} - (\omega_y^2 + \omega_z^2)\{\rho \iiint_{\tau(t)} x\, d\tau + \iiint_{\tau^*} \rho^* x\, d\tau^*\} \tag{2.40}$$

$$+ (\omega_y\, \omega_x - \dot{\omega}_z)\{\rho \iiint_{\tau(t)} y\, d\tau + \iiint_{\tau^*} \rho^* y\, d\tau^*\} + (\omega_x\, \omega_z + \dot{\omega}_y)\{\rho \iiint_{\tau(t)} z\, d\tau + \iiint_{\tau^*} \rho^* z\, d\tau^*\}$$

$$- 2\,\dot{\omega}_y\, \rho \iiint_{\tau(t)} z\, d\tau - \rho \iiint_{\tau(t)} f_x\, d\tau - \iiint_{\tau^*} \rho^* f_x\, d\tau^* + \rho \iiint_{\tau(t)} \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right) d\tau$$

$$+ \tfrac{1}{2}\, \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial x}\, d\tau = F_{ext_x}$$

$$a_y \{\rho \iiint_{\tau(t)} d\tau + \iiint_{\tau^*} \rho^* d\tau^*\} + (\omega_x\, \omega_y + \dot{\omega}_z)\{\rho \iiint_{\tau(t)} x\, d\tau + \iiint_{\tau^*} \rho^* x\, d\tau^*\}$$

$$- (\omega_x^2 + \omega_z^2)\{\rho \iiint_{\tau(t)} y\, d\tau + \iiint_{\tau^*} \rho^* y\, d\tau^*\} + (\omega_y\, \omega_z - \dot{\omega}_x)\{\rho \iiint_{\tau(t)} z\, d\tau + \iiint_{\tau^*} \rho^* z\, d\tau^*\}$$

$$- 2\,\dot{\omega}_z\, \rho \iiint_{\tau(t)} x\, d\tau - \rho \iiint_{\tau(t)} f_y\, d\tau - \iiint_{\tau^*} \rho^* f_y\, d\tau^* + \rho \iiint_{\tau(t)} \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right) d\tau$$

$$+ \tfrac{1}{2}\, \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\, dv = F_{ext_y}$$

$$a_z \{\rho \iiint_{\tau(t)} d\tau + \iiint_{\tau^*} \rho^* \, d\tau^*\} + (\omega_x \, \omega_z - \dot{\omega}_y) \{\rho \iiint_{\tau(t)} x \, d\tau + \iiint_{\tau^*} \rho^* x \, d\tau^*\}$$

$$+ (\omega_y \, \omega_z + \dot{\omega}_x) \{\rho \iiint_{\tau(t)} y \, d\tau + \iiint_{\tau^*} \rho^* y \, d\tau^*\} - (\omega_x^2 + \omega_y^2) \{\rho \iiint_{\tau(t)} z \, d\tau + \iiint_{\tau^*} \rho^* z \, d\tau^*\}$$

$$- 2 \, \dot{\omega}_x \, \rho \iiint_{\tau(t)} y \, d\tau - \rho \iiint_{\tau(t)} f_z \, d\tau - \iiint_{\tau^*} \rho^* f_z \, d\tau^* + \rho \iiint_{\tau(t)} \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) d\tau$$

$$+ \tfrac{1}{2} \, \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} \, d\tau = F_{ext_z}$$

$$a_z \{\rho \iiint_{\tau(t)} y \, d\tau + \iiint_{\tau^*} \rho^* y \, d\tau^*\} - a_y \{\rho \iiint_{\tau(t)} z \, d\tau + \iiint_{\tau^*} \rho^* z \, d\tau^*\}$$

$$+ (\omega_x \, \omega_z - \dot{\omega}_y) \{\rho \iiint_{\tau(t)} x \, y \, d\tau + \iiint_{\tau^*} \rho^* x \, y \, d\tau^*\} - (\omega_x \, \omega_y + \dot{\omega}_z) \{\rho \iiint_{\tau(t)} x \, z \, d\tau + \iiint_{\tau^*} \rho^* x \, z \, d\tau^*\}$$

$$+ (\omega_y \, \omega_z + \dot{\omega}_x) \{\rho \iiint_{\tau(t)} y^2 \, d\tau + \iiint_{\tau^*} \rho^* y^2 \, d\tau^*\} + (\omega_z^2 - \omega_y^2) \{\rho \iiint_{\tau(t)} y \, z \, d\tau + \iiint_{\tau^*} \rho^* y \, z \, d\tau^*\}$$

$$- (\omega_y \, \omega_z - \dot{\omega}_x) \{\rho \iiint_{\tau(t)} z^2 \, d\tau + \iiint_{\tau^*} \rho^* z^2 \, d\tau^*\} + 2 \, \dot{\omega}_z \, \rho \iiint_{\tau(t)} x \, z \, d\tau - 2 \, \dot{\omega}_x \iiint_{\tau(t)} y^2 \, d\tau$$

$$- \rho \iiint_{\tau(t)} (y \, f_z - z \, f_y) \, d\tau - \iiint_{\tau^*} \rho^* (y \, f_z - z \, f_y) \, d\tau^* + \rho \iiint_{\tau(t)} \left[y \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) - z \, \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau$$

$$+ \tfrac{1}{2} \, \rho \iiint_{\tau(t)} \left(y \, \frac{\partial v^2}{\partial z} - z \, \frac{\partial v^2}{\partial y}\right) d\tau = L_{ext_x}$$

$$- a_z \{\rho \iiint_{\tau(t)} x \, d\tau + \iiint_{\tau^*} \rho^* x \, d\tau^*\} + a_x \{\rho \iiint_{\tau(t)} z \, d\tau + \iiint_{\tau^*} \rho^* z \, d\tau^*\}$$

$$- (\omega_x \, \omega_z - \dot{\omega}_y) \{\rho \iiint_{\tau(t)} x^2 \, d\tau + \iiint_{\tau^*} \rho^* x^2 \, d\tau^*\} - (\omega_y \, \omega_z + \dot{\omega}_x) \{\rho \iiint_{\tau(t)} x \, y \, d\tau + \iiint_{\tau^*} \rho^* x \, y \, d\tau^*\}$$

$$+ (\omega_x^2 - \omega_z^2) \{\rho \iiint_{\tau(t)} x \, z \, d\tau + \iiint_{\tau^*} \rho^* x \, z \, d\tau^*\} + (\omega_x \, \omega_y - \dot{\omega}_z) \{\rho \iiint_{\tau(t)} yz \, d\tau + \iiint_{\tau^*} \rho^* y \, z \, d\tau^*\}$$

$$+ (\omega_x \, \omega_z + \dot{\omega}_y) \{\rho \iiint_{\tau(t)} z^2 \, d\tau + \iiint_{\tau^*} \rho^* z^2 \, d\tau^*\} + 2 \, \dot{\omega}_x \, \rho \iiint_{\tau(t)} x \, y \, d\tau - 2 \, \dot{\omega}_y \, \rho \iiint_{\tau(t)} z^2 \, d\tau$$

$$- \rho \iiint_{\tau(t)} (z \, f_x - x \, f_z) \, d\tau - \iiint_{\tau^*} \rho^* (z \, f_x - x \, f_z) \, d\tau^* + \rho \iiint_{\tau(t)} \left[z \, \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t}\right) - x \, \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau$$

$$+ \frac{1}{2} \rho \iiint_{\tau(t)} (z \frac{\partial v^2}{\partial x} - x \cdot \frac{\partial v^2}{\partial z}) \, d\tau = L_{ext_y} \ ,$$

$$a_y \{\rho \iiint_{\tau(t)} x \, d\tau + \iiint_{\tau^*} \rho^* x \, d\tau^*\} - a_x \{\rho \iiint_{\tau(t)} y \, d\tau + \iiint_{\tau^*} \rho^* y \, d\tau^*\}$$

$$+ (\omega_x \omega_y + \dot{\omega}_z) \{\rho \iiint_{\tau(t)} x^2 \, d\tau + \iiint_{\tau^*} \rho^* x^2 \, d\tau^*\} + (\omega_y^2 - \omega_x^2) \{\rho \iiint_{\tau(t)} x \, y \, d\tau + \iiint_{\tau^*} \rho^* x \, y \, d\tau^*\}$$

$$+ (\omega_y \omega_z - \dot{\omega}_x) \{\rho \iiint_{\tau(t)} x \, z \, d\tau + \iiint_{\tau^*} \rho^* x \, z \, d\tau^*\} - (\omega_x \ \omega_y - \dot{\omega}_z) \{\rho \iiint_{\tau(t)} y^2 \, d\tau + \iiint_{\tau^*} \rho^* y^2 \, d\tau^*\}$$

$$- (\omega_x \omega_z + \dot{\omega}_y) \{\rho \iiint_{\tau(t)} y \, z \, d\tau + \iiint_{\tau^*} \rho^* y \, z \, d\tau^*\} - 2 \, \dot{\omega}_z \, \rho \iiint_{\tau(t)} x^2 \, d\tau + 2 \, \dot{\omega}_y \, \rho \iiint_{\tau(t)} y \, z \, d\tau$$

$$- \rho \iiint_{\tau(t)} (x \, f_y - y \, f_x) \, d\tau - \iiint_{\tau^*} \rho^* (x \, f_y - y \, f_x) \, d\tau^* + \rho \iiint_{\tau(t)} [x \frac{\partial}{\partial y} (\frac{\partial \varphi}{\partial t}) - y \frac{\partial}{\partial x} (\frac{\partial \varphi}{\partial t})] \, d\tau$$

$$\frac{1}{2} \rho \iiint_{\tau(t)} (x \frac{\partial v^2}{\partial y} - y \ \frac{\partial v^2}{\partial x}) \, d\tau = L_{ext\ z} \ ,$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \dot{\omega}_x \ y \ z - \dot{\omega}_y \, x \ z - \dot{\omega}_z \, x \ y + \bar{a} \ \bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2} (\bar{\omega} \times \bar{r})^2 = C(t), \ P \epsilon \ \tau(t),$$

$$\frac{\partial \varphi}{\partial t} - \dot{\omega}_x \ y \ z - \dot{\omega}_y \ x \ z - \dot{\omega}_z \ x \ y + \bar{a} \ \bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2} (\bar{\omega} \times \bar{r})^2 = 0, \ P \epsilon \ \dot{S}(t), \ z = \zeta,$$

$$\Delta \varphi = 0 \quad P \epsilon \ \tau(t)$$

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 2 \ [\omega_x \ y \cos (n, \ z) + \omega_y \ z \cos (n, \ x) + \omega_z \ x \cos (n, \ y)] \ , \ P \epsilon \ \Sigma(t) \ , \\ 2 \ [\omega_x \ y \cos (n, \ z) + \omega_y \ z \cos (n, \ x) + \omega_z \ x \cos (n, \ y)] + \zeta_t \cos (n, \ z) \ , \\ \quad P \epsilon \ S(t) \ . \end{cases}$$

Thus, with the external forces and moments, certain initial conditions and the geometry of the vessel given, the motion of the system can, in principle, be determined from (2.40).

3/ PLANAR EQUATIONS FOR THE MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

PLANAR EQUATIONS FOR THE MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

Consider the planar motion of the launch vehicle illustrated in Figure 2. The vehicle consists of an engine, a main body which is assumed to be rigid and a rigid tank partly filled with a frictionless liquid. For simplicity, the tank is assumed to be a prismatic cylinder, Figure 3.

To describe the motion of the system take three cartesian frames of reference $o' x' y' z'$ fixed relatively to the vehicle at a distance l below the "capped" free surface, o x y z fixed relatively to the container and $o_e x_e y_e z_e$ fixed relatively to the swivelling engine. Reference o x y z is oriented in such a manner that o z is measured positively along the outward directed normal to the undisturbed free surface. Thus the free surface, denoted by S(t), coincides with plane x o y (the plane $z = 0$) when the vehicle and liquid are at rest.

Let

$$z = \zeta (x, y, t)$$

be the equation of S(t) when it is displaced. Denote by $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$ the wetted surface of the vessel, and by $\tau(t)$ the variable volume enclosed by S(t) and $\Sigma(t)$. Let Σ, τ and S represent the values of $\Sigma(t)$, $\tau(t)$ and S(t) in the undisturbed position. In addition let C be the boundary curve of $\Sigma_1(t)$. All surfaces are presumed to be piecewise smooth.

Coordinate systems o x y z and $o_e x_e y_e z_e$ are related to $o' x' y' z'$ as follows:

$$z' = z + l \quad , \qquad z' = - l_1 - (z_e \cos\beta + y_e \sin\beta) \quad ,$$

$$y' = y \quad , \qquad y' = y_e \cos\beta - z_e \sin\beta \quad ,$$

$$x' = x \quad , \qquad x' = x_e$$

Suppose that at time t the vehicle is coincident with inertial space and that it is moving relatively to inertial space with motion described by an observer in inertial space as a velocity $\bar{u}$ of $0'$ and an angular velocity $\bar{\omega}$. Then the position

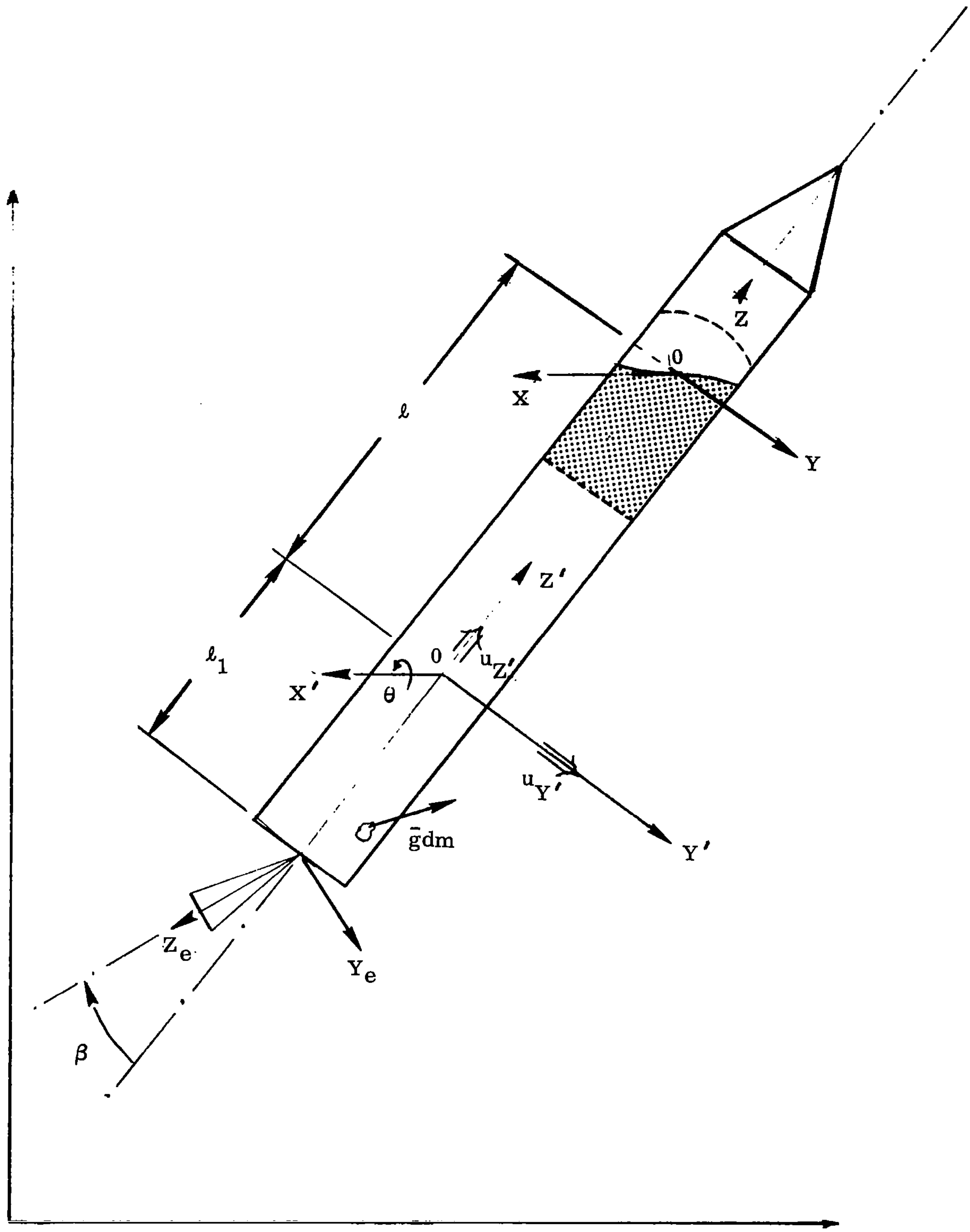


Figure 2. Launch vehicle experiencing planar motion.

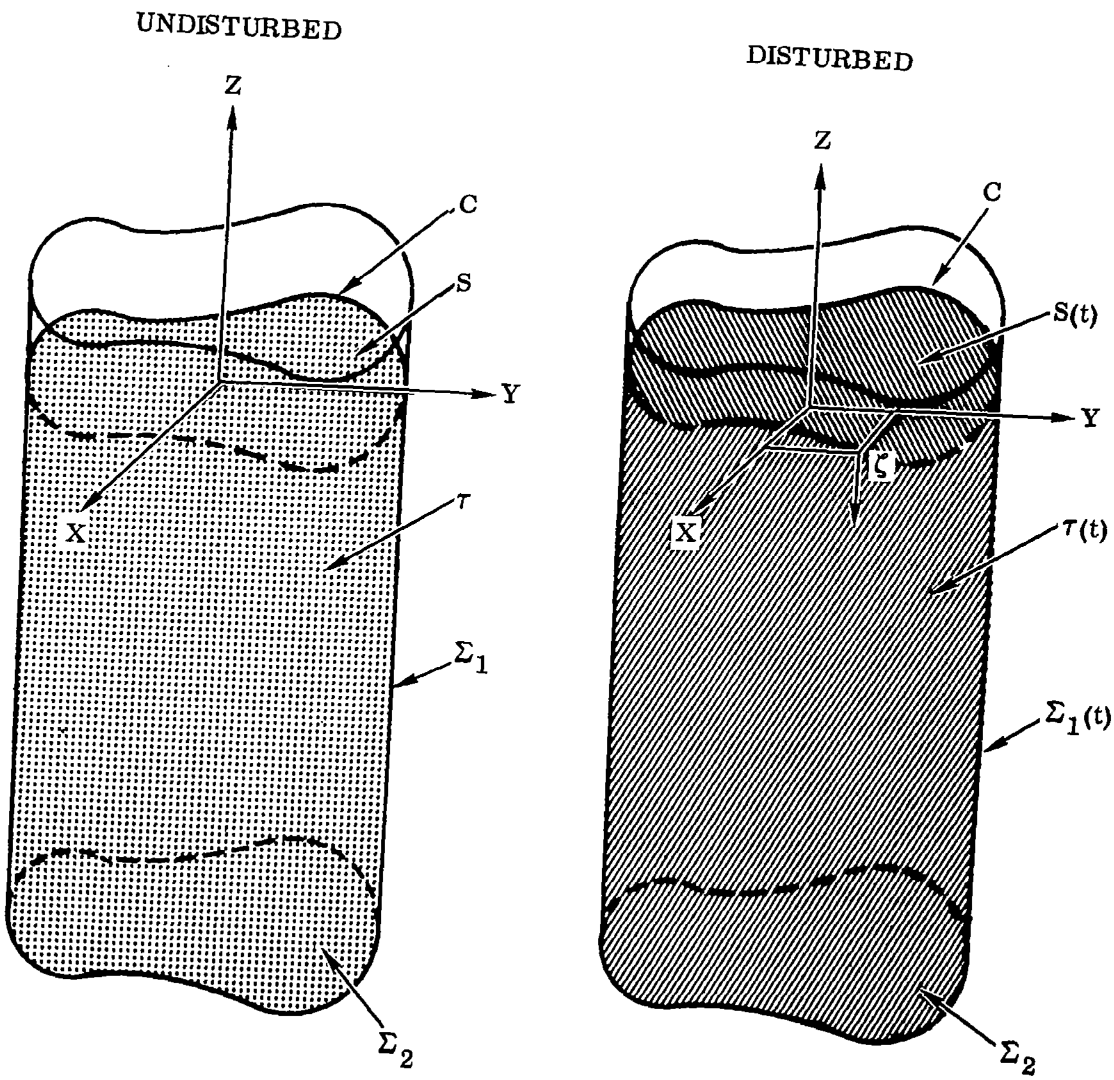


Figure 3. Prismatic cylindrical propellant tank.

vector $\bar{r}'$ of a particular point P on the vehicle at time t is the same for an observer moving with the vehicle as it is for an observer in inertial space.

The point P, if rigidly attached to the vehicle, has the velocity

$$\bar{V} = \{0, (u_y' - z'\dot{\theta}), (u_z' + y'\dot{\theta})\} .$$

Denote by $\bar{q}$ (P, t), $\bar{v}$ (P, t) the velocities of the liquid particle $P \epsilon \tau(t)$ at time t as estimated by observers in inertial space and o x y z respectively. Then

$$\bar{q} = \{0, (u_y' - (z + l)\dot{\theta}, (u_z' + y\dot{\theta})\} + \bar{v} ,$$

in which $\bar{q}$, $\bar{v}$ are referred to moving reference o x y z.

The velocity of a generic mass point on the engine is, referring to Figure 2,

$$\bar{V} = \{0, [u_y' + (\dot{\theta} - \dot{\beta})(z_e \cos\beta + y_e \sin\beta) + l_1\dot{\theta}], [u_z' - (\dot{\theta} - \dot{\beta})(z_e \sin\beta - y_e \cos\beta)]\} .$$

The total kinetic energy of the vehicle is therefore

$$T = T_{\text{main body}} + T_{\text{engine}} + T_{\text{liquid}} ,$$

where

$$T_{\text{main body}} = \tfrac{1}{2}(u_y^{2\prime} + u_z^{2\prime}) \iiint_{\tau^*} dm + \tfrac{1}{2}\dot{\theta}^2 \iiint_{\tau^*} (y'^2 + z'^2)\, dm + \dot{\theta}\{u_z' \iiint_{\tau^*} y'\, dm$$

$$- u_y' \iiint_{\tau^*} z'\, dm\} ,$$

$$T_{\text{engine}} = \tfrac{1}{2}(u_y^{2\prime} + u_z^{2\prime}) \iiint_{\tau_e} dm + \tfrac{1}{2}\dot{\theta}^2 \iiint_{\tau_e} [y_e^2 + (l_1 + z_e)^2]\, dm \qquad (3.1)$$

$$+ \dot{\theta}\{u_z' \iiint_{\tau_e} y_e\, dm + u_y' \iiint_{\tau_e} (l_1 + z_e)\, dm\} + \tfrac{1}{2}\dot{\beta}^2 \iiint_{\tau_e} (y_e^2 + z_e^2)\, dm$$

$$- \dot{\beta}\dot{\theta} \iiint_{\tau_e} (y_e^2 + z_e^2)\, dm - \dot{\beta}\{[(u_y' + l_1\dot{\theta})\cos\beta - u_z'\sin\beta] \iiint_{\tau_e} z_e\, dm$$

$$+ [(u_y' + l_1\dot{\theta})\sin\beta + u_z'\cos\beta] \iiint_{\tau_e} y_e\, dm\} + \dot{\theta}\{[(u_y' + l_1\dot{\theta})(\cos\beta - 1)$$

$$- u_z'\sin\beta] \iiint_{\tau_e} z_e\, dm + [(u_y' + l_1\dot{\theta})\sin\beta + u_z'(\cos\beta - 1)] \iiint_{\tau_e} y_e\, dm\} ,$$

$$T_{\text{liquid}} = \tfrac{1}{2}(u_y^{2} + u_z^{2}{}') \iiint_{\tau(t)} \rho \, d\tau + \tfrac{1}{2}\dot{\theta}^2 \iiint_{\tau(t)} [y^2 + (z+1)^2]\rho \, d\tau + \tfrac{1}{2}\iiint_{\tau(t)} v^2 \rho \, d\tau$$

$$+ \dot{\theta}\{u_z' \iiint_{\tau(t)} y \rho \, d\tau - u_y' \iiint_{\tau(t)} (1+z)\rho \, d\tau + u_y' \iiint_{\tau(t)} v_y \rho \, d\tau$$

$$+ u_z' \iiint_{\tau(t)} v_z \rho \, d\tau + \dot{\theta} \iiint_{\tau(t)} [y v_z - (z+1) v_y]\rho \, d\tau \; .$$

Assume the liquid to be homogeneous and incompressible throughout the motion. Neglect interfacial tension forces and capillary contact effects between liquid and tank walls. Moreover, assume that the liquid is not draining from the tank. Then, with the absolute velocity of liquid particles irrotational, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_y'}\right) - \dot{\theta}\frac{\partial T}{\partial u_z'} = F_{ext_y}' + \rho \iiint_{\tau(t)} g_y' \, d\tau + \iiint_{\tau^*} g_y' \, dm + \iiint_{\tau_e} g_y' \, dm \; ,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_z'}\right) + \dot{\theta}\frac{\partial T}{\partial u_y'} = F_{ext_z}' \quad \rho \iiint_{\tau(t)} g_z' \, d\tau + \iiint_{\tau^*} g_z' \, dm + \iiint_{\tau_e} g_z' \, dm \; , \qquad (3.2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + u_y'\frac{\partial T}{\partial u_z'} - u_z'\frac{\partial T}{\partial u_y'} = L_{ext_x}' + \rho \iiint_{\tau(t)} [y g_z' - (1+z) g_y'] \, d\tau$$

$$+ \iiint_{\tau^*} (y' g_z' - z' g_y') + \iiint_{\tau_e} [y_e \cos \dot{\beta} - z_e \sin \beta) g_z'$$

$$+ (l_1 + z_e \cos \beta + y_e \sin \beta) g_y'] \, dm \; ,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = \iiint_{\tau_e} (g_z' \sin \beta - g_y' \cos \beta) z_e \, dm - \iiint_{\tau_e} (g_z' \cos \beta + g_y' \sin \beta) y_e \, dm$$

$$+ Q_\beta \; ,$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \ddot{\theta} y (1+z) + a_y y + a_z(z+1) + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}\dot{\theta}^2 [y^2 + (z+1)^2] = C(t), \;\; P \epsilon \, \tau(t) \; ,$$

$$\frac{\partial \varphi}{\partial t} - \ddot{\theta} y (1+z) + a_y y + a_z (z+1) + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}\dot{\theta}^2 [y^2 + (z+1)^2] = 0, \;\; P \epsilon \, S(t), \;\; z = \zeta ,$$

$$\bar{v} = \nabla \varphi + (0, \, 0, \, -2y\dot{\theta}) \, , \;\; P \epsilon \, \tau(t) \; ,$$

$$\Delta \varphi = 0 \; , \;\; P \epsilon \, \tau(t)$$

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 2\,\dot\theta\, y \cos(n, z)\,, & P \in \Sigma_1(t),\ \Sigma_2(t)\,, \\ 2\,\dot\theta\, y \cos(n, z) + \zeta_t \cos(n, z)\,, & P \in S(t)\,, \end{cases}$$

in which

$$\bar g = (0\,,\ g_{y'}\,,\ g_{z'})$$

is the vector of body forces per unit mass, and

$$\bar g = -\nabla \Omega\ .$$

Q_β is the generalized force which, in general, is associated with the spring moment per unit angle of the hinge when locked, with the damping moment of the hinge, with the engine actuator moment, and with the engine actuator position required by the autopilot.

Formulae $(3.2_{1,2,3})$ express the dynamic equilibrium of the complete system. Formula (3.2_4) is the Euler-Lagrange equation which describes the motion of the engine. Formula (3.2_5) is the unsteady pressure equation which is obtained from a quadrature of the Euler-Lagrange equation for motion of the liquid subject to the above assumptions. Formula (3.2_6) expresses the constancy of pressure at the free surface of the liquid. Formula (3.2_7) is a corollary of the assumption of absolute irrotational motion.

Using (3.1, 2), we get

$$a_{y'}\{\iiint_{\tau^*} dm + \iiint_{\tau_e} dm + \rho\iiint_{\tau(t)} d\tau\} - \dot\theta^2\{\iiint_{\tau^*} y'\,dm + \iiint_{\tau_e} y_e\,dm + \rho\iiint_{\tau(t)} y\,d\tau\} \tag{3.3}$$

$$-\ddot\theta\{\iiint_{\tau^*} z'\,dm + \iiint_{\tau_e}(z_e + l_1)\,dm + \rho\iiint_{\tau(t)}(z + l)\,d\tau\} - \iiint_{\tau^*} g_{y'}\,dm - \rho\iiint_{\tau(t)} g_{y'}\,dm$$

$$-\iiint_{\tau_e} g_{y'}\,dm + \rho\iiint_{\tau(t)}\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial t}\right)d\tau + \tfrac{1}{2}\rho\iiint_{\tau(t)}\frac{\partial v^2}{\partial y}\,d\tau + [\ddot\theta(\cos\beta + 1) - 2\,\dot\beta\,\dot\theta\sin\beta$$

$$+(\dot\theta^2 + \dot\beta^2)\sin\beta - \ddot\beta\cos\beta]\iiint_{\tau_e} z_e\,dm + [\,\ddot\theta\sin\beta + 2\,\dot\beta\,\dot\theta\cos\beta - (\dot\theta^2 + \dot\beta^2)\cos\beta$$

$$+\dot\theta^2 - \ddot\beta\sin\beta]\iiint_{\tau_e} y_e\,dm + 2\,\ddot\theta\, l_1\iiint_{\tau_e} dm = F_{ext_{y'}}\ ,$$

$$a_{z'}\{\iiint_{\tau^*} dm + \iiint_{\tau_e} dm + \rho\iiint_{\tau(t)} d\tau\} - \dot\theta^2\{\iiint_{\tau^*} z'\,dm + \iiint_{\tau_e}(z_e + l_1)\,dm + \rho\iiint_{\tau(t)}(l + z)\,d\tau\}$$

$$+ \ddot{\theta} \{ \iiint_{\tau^*} y' \, dm + \iiint_{\tau_e} y_e \, dm + \rho \iiint_{\tau(t)} y \, d\tau \} - \iiint_{\tau^*} g_{z'} \, dm - \rho \iiint_{\tau(t)} g_{z'} \, d\tau - \iiint_{\tau_e} g_{z'} \, dm$$

$$- 2 \ddot{\theta} \rho \iiint_{\tau(t)} y \, d\tau + \rho \iiint_{\tau(t)} \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) d\tau + \frac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} \, d\tau + [\ddot{\theta} \sin \beta + 2 \dot{\beta} \dot{\theta} \cos \beta$$

$$- (\dot{\theta}^2 + \dot{\beta}^2) \cos \beta - \dot{\theta}^2 - \ddot{\beta} \sin \beta] \iiint_{\tau_e} z_e \, dm + [\theta (\cos \beta - 1) - 2 \dot{\beta} \dot{\theta} \sin \beta$$

$$+ (\dot{\theta}^2 + \dot{\beta}^2) \sin \beta - \ddot{\beta} \cos \beta] \iiint_{\tau_e} y_e \, dm + 2 l_1 \dot{\theta}^2 \iiint_{\tau_e} dm = F_{ext_{z'}} ,$$

$$a_{z'} \{ \iiint_{\tau^*} y' \, dm + \iiint_{\tau_e} y_e \, dm + \rho \iiint_{\tau(t)} y \, d\tau \} - a_{y'} \{ \iiint_{\tau^*} z' \, dm + \iiint_{\tau_e} (l_1 + z_e) \, dm$$

$$+ \rho \iiint_{\tau(t)} (1 + z) \, d\tau \} + \ddot{\theta} \{ \iiint_{\tau^*} (y'^2 + z'^2) \, dm + \iiint_{\tau_e} [y_e^2 + (l_1 + z_e)^2] \, dm$$

$$+ \rho \iiint_{\tau(t)} [y^2 + (1 + z)^2] \, d\tau \} - \iiint_{\tau^*} (y' g_{z'} - z' g_y) \, d\tau - \rho \iiint_{\tau(t)} [y \, g_{z'} - (1 + z) \, g_{y'}] \, d\tau$$

$$- \iiint_{\tau_e} [(y_e \cos \beta - z_e \sin \beta) \, g_{z'} + (l_1 + z_e \cos \beta + y_e \sin \beta) \, g_{y'}] \, dm$$

$$- 2 \rho \ddot{\theta} \iiint_{\tau(t)} y^2 \, d\tau + \rho \iiint_{\tau(t)} [y \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) - (z + 1) \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right)] \, d\tau + \frac{1}{2} \rho \iiint_{\tau(t)} [y \frac{\partial v^2}{\partial z}$$

$$- (z + 1) \frac{\partial v^2}{\partial y}] \, d\tau - \ddot{\beta} \iiint_{\tau_e} (y_e^2 + z_e^2) \, dm + \{ a_{y'} (\cos \beta + 1) - a_{z'} \sin \beta$$

$$+ l_1 [2 \ddot{\theta} (\cos \beta - 1) - 2 \dot{\beta} \dot{\theta} \sin \beta + \dot{\beta}^2 \sin \beta - \ddot{\beta} \cos \beta] \} \iiint_{\tau_e} z_e \, dm$$

$$+ \{ a_{y'} \sin \beta + a_{z'} (\cos \beta - 1) + l_1 [2 \ddot{\theta} \sin \beta + 2 \dot{\beta} \dot{\theta} \cos \beta - \dot{\beta}^2 \cos \beta$$

$$- \ddot{\beta} \sin \beta] \} \iiint_{\tau_e} y_e \, dm + 2 a_{y'} l_1 \iiint_{\tau_e} dm = L_{ext_{x'}} ,$$

$$\ddot{\beta} \iiint_{\tau_e} (y_e^2 + z_e^2) \, dm - \ddot{\theta} \iiint_{\tau_e} (y_e^2 + z_e^2) \, dm + [a_{z'} \sin \beta - a_{y'} \cos \beta - l_1 \ddot{\theta} \cos \beta$$

$$+ l_1 \dot{\theta}^2 \sin \beta] \iiint_{\tau_e} z_e \, dm - [a_{y'} \sin \beta + a_{z'} \cos \beta + l_1 \ddot{\theta} \sin \beta + l_1 \dot{\theta}^2 \cos \beta] \iiint_{\tau_e} y_e \, dm$$

$$- \iiint_{\tau_e} (g_{z'} \sin\beta - g_{y'} \cos\beta)\, z_e \; dm + \iiint_{\tau_e} (g_{z'} \cos\beta + g_{y'} \sin\beta)\, y_e \, dm = Q_\beta \, ,$$

$$\frac{p}{\rho} + \frac{\partial\varphi}{\partial t} - \ddot{\theta}\, y\,(1+z) + a_{y'}\, y + a_{z'}(z+1) + \tfrac{1}{2} v^2 + \Omega - \tfrac{1}{2}\dot{\theta}^2 [y^2 + (1+z)^2] = C(t), \; P \epsilon \tau(t),$$

$$\frac{\partial\varphi}{\partial t} - \ddot{\theta}\, y\,(1+z) + a_{y'}\, y + a_{z'}(1+z) + \tfrac{1}{2} v^2 + \Omega - \tfrac{1}{2}\dot{\theta}^2 [y^2 + (1+z)^2] = 0, \; P \epsilon S(t), \; z = \zeta \, ,$$

$$\bar{v} = \nabla\varphi + (0,\ 0,\ -2y\dot{\theta}), \; P \epsilon \tau(t) \, ,$$

$$\Delta\varphi = 0, \; P \epsilon \tau(t) \, ,$$

$$\frac{\partial\varphi}{\partial n} = 0, \; P \epsilon \Sigma_1(t) \, ,$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial z} = 2\dot{\theta}\, y, \; P \epsilon \Sigma_2(t), \; z = -h \, ,$$

$$\frac{\partial\varphi}{\partial n} = (2\dot{\theta}\, y + \zeta_t) \cos(n,\ z) \, , \; P \epsilon S(t)$$

$$\bar{g} = (0,\ g_{y'},\ g_{z'}) \, ,$$

$$\bar{g} = -\nabla\Omega \, ,$$

$$a_{y'} = \dot{u}_{y'} - \dot{\theta}\, u_{z'} \, , \; a_{z'} = \dot{u}_{z'} + \dot{\theta}\, u_{y'} \, .$$

We now make the following simplifications and identifications:

(1) The specific body force are independent of position,

$$\bar{g} = \text{const};$$

(2) Origin of reference system $o'x'y'z'$ is at center of mass of vehicle when engine is "locked-out" and liquid "frozen" solid,

$$\iiint_{\tau^*} y' \, dm + \iiint_{\tau_e} y_e \; dm + \rho \iiint_{\tau} y \, d\tau = 0 \, ,$$

$$\iiint_{\tau^*} z' \; dm + \iiint_{\tau_e} (l_1 + z_e) \, dm + \rho \iiint_{\tau} (1+z) \, d\tau = 0 \, ;$$

(3) Engine is symmetrical about its line of centers,

$$\iiint y_e \; dm = 0 \, ;$$

(4) Origin of coordinate system oxyz is such that

$$\iiint_{\tau} y\, d\tau = \iint_{S} dx\, dy \int_{-h}^{0} y\, dz = h \iint_{S} y\, dx\, dy = 0,$$

the tank is symmetrical about longitudinal axis of vehicle;

(5) Total moment of inertia of system about axis $o'x'$ is

$$I_{o'x'} = I_{o'x'}^{sb} + I_{o'x'}^{e} + I_{o'x'}^{l} ,$$

$$I_{o'x'}^{sb} = \iiint_{\tau^*} (y'^2 + z'^2)\, dm ,$$

$$I_{o'x'}^{e} = \iiint_{\tau_e} [y_e^2 + (l_1 + z_e)^2]\, dm ,$$

$$I_{o'x'}^{l} = \iiint_{\tau} [y^2 + (l + z)^2]\, d\tau ;$$

(6) Total mass of system is

$$M = M_{sb} + M_e + M_l ,$$

$$M_{sb} = \iiint_{\tau^*} dm ,$$

$$M_e = \iiint_{\tau_e} dm ,$$

$$L_l = \rho \iiint_{\tau} d\tau ;$$

(7) Moment of inertia of engine about axis $o_e x_e$ is

$$I_{o_e y_e}^{e} = \iiint_{\tau_e} (y_e^2 + z_e^2)\, dm ;$$

(8) First moment of mass of engine about axis $o_e y_e$ is

$$S_{o_e y_e}^{e} = \iiint_{\tau_e} z_e\, dm ;$$

(9) **Second moment of mass of liquid about axis oz is**

$$I_{Oz}^{1} = \rho \iiint_{\tau} y^2 \, d\tau \ ;$$

(10) **Various volume integrals associated with liquid are evaluated as**

$$\rho \iiint_{\tau(t)} d\tau = \rho \iiint_{\tau} d\tau + \rho \iint_{S} \zeta \, dx \, dy \ ,$$

$$\rho \iiint_{\tau(t)} y \, d\tau = \rho \iiint_{\tau} y \, d\tau + \rho \iint_{S} y \, \zeta \, dx \, dy \ ,$$

$$\rho \iiint_{\tau(t)} (1 + z) \, d\tau = \rho \iiint_{\tau} (1 + z) \, d\tau + \rho \iint_{S} (\zeta^2/2 + 1 \, \zeta) \, dx \, dy \ ,$$

$$\rho \iiint_{\tau(t)} [y^2 + (1 + z)^2] \, d\tau = \rho \iiint_{\tau} [y^2 + (1 + z)^2] \, d\tau + \rho \iint_{S} [1^2 + y^2) \zeta + 1 \, \zeta^2 + \zeta^3/3] \, dx \, dy;$$

(11) Potential Ω is

$$\Omega = - g_{y'} \, y - g_{z'} (z + 1) \ .$$

With these simplifications and definitions, formulae (3.3) may be written as follows:

Component of force equation along axis $o'y'$

$$M \, (a_{y'} - g_{y'}) + \rho (a_{y'} - g_{y'}) \iint_{S} \zeta \, dx \, dy - \rho \, \ddot{\theta} \iint_{S} (1 \, \zeta + \zeta^2/2) \, dx \, dy - \rho \, \dot{\theta}^2 \iint_{S} y \, \zeta \, dx \, dy$$

$$+ \rho \iiint_{\tau(t)} \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y} d\tau + [\ddot{\theta} (\cos \beta + 1) - 2 \, \dot{\beta} \, \dot{\theta} \sin \beta \tag{3.4}$$

$$+ (\dot{\theta}^2 + \dot{\beta}^2) \sin \beta - \ddot{\beta} \cos \beta] S_{o_e \mathring{y}_e} + 2 \, l_1 \, \ddot{\theta} \, M_e = F_{ext_{y'}}$$

Component of force equation along axis $o'z'$

$$M \, (a_{z'} - g_{z'}) + \rho (a_{z'} - g_{z'}) \iint_{S} \zeta \, dx \, dy - \rho \, \ddot{\theta} \iint_{S} y \, \zeta \, dx \, dy - \rho \, \dot{\theta}^2 \iint_{S} (1 \, \zeta + \zeta^2/2) \, dx \, dy$$

$$+ \rho \iiint_{\tau(t)} \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} d\tau - [\ddot{\theta} \sin \beta + 2 \, \dot{\beta} \, \dot{\theta} \cos \beta \tag{3.5}$$

$$- (\dot{\theta}^2 + \dot{\beta}^2) \cos\beta - \dot{\theta}^2 - \ddot{\beta} \sin\beta \,] S_{o_e^\bullet y_e} + 2\, l_1 M_e\, \dot{\theta}^2 = F_{ext_z'}$$

Moment equation about axis $o'x'$

$$(I_{o'x'} - 2\, I_{oz}^1)\, \ddot{\theta} + \rho\, (a_{z'} - g_{z'}) \iint\limits_S y\, \zeta\, dx\, dy - \rho\, (a_{y'} - g_{y'}) \iint\limits_S (1\, \zeta + \zeta^2/2)\, dx\, dy$$

$$+ \rho\, \ddot{\theta} \iint\limits_S [(1^2 - y^2)\, \zeta + \zeta^2 + \zeta^3/3]\, dx\, dy + \rho \iiint\limits_{\tau(t)} [y \frac{\partial}{\partial z} (\frac{\partial \varphi}{\partial t}) - (1 + z) \frac{\partial}{\partial y} (\frac{\partial \varphi}{\partial t})]\, d\tau$$

$$+ \tfrac{1}{2}\, \rho \iiint\limits_{\tau(t)} [y \frac{\partial v^2}{\partial z} - (1 + z) \frac{\partial v^2}{\partial y}]\, d\tau - I_{o_e^\circ x_e}\, \ddot{\beta} + \{(a_{y'} - g_{y'})(\cos\beta + 1) \tag{3.6}$$

$$- (a_{z'} - g_{z'}) \sin\beta + l_1\, [2\, \ddot{\theta}\, (\cos\beta - 1) - 2\, \dot{\beta}\, \dot{\theta} \sin\beta + \dot{\beta}^2 \sin\beta$$

$$- \ddot{\beta}\, \cos\beta \,]\, \}\, S_{o_e^\bullet y_e} + 2\, l_1\, (a_{y'} - g_{y'})\, M_e = L_{ext_x'}$$

Engine equation

$$I_{o_e^\circ x_e}\, (\ddot{\beta} - \ddot{\theta}) + [\, (a_{z'} - g_{z'}) \sin\beta - (a_{y'} - g_{y'}) \cos\beta - l_1\, \ddot{\theta}\, \cos\beta \tag{3.7}$$

$$+ l_1\, \dot{\theta}^2 \sin\beta \,]\, S_{o_e^\circ y_e} = Q_\beta$$

Constancy of pressure at free surface

$$\frac{\partial \varphi}{\partial t} - \ddot{\theta}\, y\, (1 + \zeta) + (a_{y'} - g_{y'})\, y + (a_{z'} - g_{z'})\, (1 + \zeta) + \tfrac{1}{2}\, v^2 \tag{3.8}$$

$$- \tfrac{1}{2}\, [\, y^2 + (1 + \zeta)^2]\, \dot{\theta}^2 = 0\,, \quad z = \zeta$$

Kinematics of liquid

$$\bar{v} = \nabla \varphi + (0,\ 0,\ - 2\, \dot{\theta}\, y)\,, \quad P \epsilon\, \tau(t) \tag{3.9}$$

$$\Delta \varphi = 0\,, \quad P \epsilon\, \tau(t)$$

$$\frac{\partial \varphi}{\partial n} = 0\,, \quad P \epsilon\, \Sigma_1(t)$$

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial z} = 2\, \dot{\theta}\, y, \quad P \epsilon\, \zeta(t), \quad z = -h\,,$$

$$\frac{\partial \varphi}{\partial n} = (2\, \dot{\theta}\, y + \zeta_t) \cos\, (n,\ z), \quad P \epsilon\, S(t)\,.$$

The solution to the Neumann problem (3.9), subject to the restriction $\iint\limits_S \zeta_t\, dx\, dy = 0$, may be taken in the form of a series

$$\varphi(x, y, z, t) = \sum_1^\infty (\alpha_i(t) \cosh k_i\, (z + h) + \gamma_i(t) \sinh k_i\, z)\, \varphi_i\, (x,\ y) + \alpha_o(t) \tag{3.10}$$

This function is harmonic as indicated by (3.9_2), and satisfies (3.9_3) if the infinitely many values k_i^2 ($i = 1, 2, \ldots$) are the values of k^2 (eigenvalues) for which the two-dimensional scalar Helmholtz equation

$$\Delta \varphi + k^2 \varphi = 0, \ P \epsilon S , \tag{3.11}$$

has a non-zero solution satisfying

$$\frac{\partial \varphi}{\partial n} = 0, \ P \epsilon \Sigma_1 \text{ (or C)} . \tag{3.12}$$

Functions φ_i ($i = 1, 2, \ldots$) are the corresponding solutions (eigenfunctions) of (3.11).

We point out, without proof, several important properties of the system of functions φ_i and the numbers k_i^2:

(1) Of the infinite number of numbers k_i^2, all are real and positive;

(2) The set of functions φ_i is orthogonal

$$(\varphi_i, \varphi_i) \equiv \iint_S \varphi_i \ \varphi_j \, dx \, dy = \begin{cases} 0, & i \neq j \\ \|\varphi_i\|^2, & i = j \end{cases} , \tag{3.13}$$

and can be normalized;

(3) Any function $\mu(x, y)$ which has continuous second-order derivatives in and on the boundary of S and which is orthogonal to a constant:

$$(\mu, 1) \equiv \iint \mu(x, y) \, dx \, dy = 0 ,$$

and which satisfies the boundary conditions, may be expressed as a uniformly convergent series of eigen functions

$$\mu(x, y) = \sum_1^\infty c_i \ \varphi_i(x, y) ; \tag{3.14}$$

(4) If function μ which is continuously twice-differentiable satisfies the condition (3.12), then the series (3.14) not only converges uniformly to μ, but the series obtained from it by termwise differentiation also converges in the mean to the corresponding derivative of μ;

(5) In addition to the condition for orthogonality, written above, the following equations hold:

$$\iint \nabla \varphi_i \nabla \varphi_i \; dx\,dy = \begin{cases} 0, & i \neq j \\ k_i^2 \; \|\varphi_i\|^2, & i = j \end{cases} . \tag{3.15}$$

We are now in a position to expand $2\,y\,\dot{\theta}$ and $\zeta\,(x,\ y,\ t)$ as series of the form

$$2\,\dot{\theta}\;y = 2\,\dot{\theta}\,\{\sum_1^{\infty} \eta_i\,(t)\,\varphi_i\,(x,\ y) + \eta_o\,(t)\}\ , \tag{3.16}$$

$$\zeta\,(x,\ y,\ t) = \sum_1^{\infty} \xi_i\,(t)\,\varphi_i\,(x,\ y) + \xi_o\,(t)\ .$$

Such expansions are possible, since for a fixed value of t functions $2\,\dot{\theta}\,(y - \eta_o)$ and $\zeta\,(x,\ y,\ t) - \xi_o\,(t)$ are developable in series of the form (3.14). The coefficients of these series depend, in general, on t. Choosing η_o and ξ_o so that $y - \eta_o$ and $\zeta - \xi_o$ are orthogonal to a constant (unity), we get

$$\eta_i\,(t) = \frac{(y,\ \varphi_i)}{\|\varphi_i\|^2}\ , \tag{3.17}$$

$$\eta_o\,(t) = \frac{(y,\ 1)}{A} = 0\ ,\quad (y,\ 1) = 0\ ,$$

and

$$\xi_i\,(t) = \frac{(\zeta,\ \varphi_i)}{\|\varphi_i\|^2}\ , \tag{3.18}$$

$$\xi_o\,(t) = \frac{(1,\ \zeta)}{A}\ ,$$

respectively, using the condition for orthogonality (3.13). Here A is the area of the cross section of the cylinder. Now since

$$\frac{\partial\,\zeta}{\partial\,n} = 0\ ,\quad P \in C\ ,$$

the series obtained from (3.16_2) by termwise differentiation converges in the mean to the corresponding derivative of ζ. Note that this implies a 90° contact angle. For contact angles other than this such an expansion as (3.16_2) is still possible with, of course, certain modifications.

For (3.5) to satisfy the boundary condition (3.9_4) it is necessary that

$$\gamma_i\,(t) = 2\,\dot{\theta}\;\frac{(y,\ \varphi_i)}{k_i\;\|\varphi_i\|^2\;\cosh k_i\,h}\ . \tag{3.19}$$

To satisfy the boundary condition (3.9$_5$) it is necessary for

$$\sum_1^\infty \{k_i \varphi_i \sinh k_i (\zeta + h) - \nabla\varphi_i \nabla\zeta \cosh k_i (\zeta + h)\} \alpha_i (t)$$

$$+ \sum_1^\infty \{k_i \varphi_i \cosh k_i \zeta - \nabla\varphi_i \nabla\zeta \sinh k_i \zeta\} \gamma_i (t) = + 2 \dot{\theta} y + \zeta_t$$

$$= \sum_1^\infty \gamma_i (t) k_i \cosh k_i h \varphi_i + \zeta_t$$

However

$$\psi_i = -\frac{1}{k_i^2} \Delta \psi_i = -\frac{1}{k_i^2} \nabla (\nabla \psi_i) ,$$

so that this expression can be written, on application of a known vector identity, in the form

$$\sum_1^\infty \alpha_i (t) \nabla \{\frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i\} + \sum_1^\infty \gamma_i (t) \nabla \{\frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \nabla \varphi_i\} \tag{3.20}$$

$$= - \zeta_t = - \sum_1^\infty \dot{\xi}_i (t) \varphi_i - \dot{\xi}_0 (t) ,$$

using (3.18). Multiplying both sides of this equality by φ_j and integrating over S, we get

$$- \|\varphi_i\|^2 \dot{\xi}_i = \sum_1^\infty \alpha_j (t) \iint_S \varphi_i \nabla \{\frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j\} d x d y$$

$$+ \sum_1 \gamma_j (t) \iint_S \varphi_i \nabla \{\frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j\} d x d y$$

but

$$\varphi_i \nabla \{\frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j\} = \nabla \{\varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j\}$$

$$- \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_i \nabla \varphi_j ,$$

$$\varphi_i \nabla \{\frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j\} = \nabla \{\varphi_i \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j\}$$

$$- \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_i \nabla \varphi_j ,$$

and moreover

$$\iint_S \nabla \{ \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \} \, dx\, dy = \int_C \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \frac{\partial \varphi_j}{\partial n} \, d L = 0 \ ,$$

$$\iint_S \nabla \{ \varphi_i \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \} \, dx\, dy = 0 \ ,$$

in accordance with the divergence theorem and boundary condition (3.12). Therefore, it follows

$$\dot{\xi}_i = \sum_1^{\infty} C_{ij} \alpha_j + 2 \dot{\theta} \mu_i \tag{3.21}$$

where

$$C_{ij} = \frac{1}{\|\varphi_i\|^2} \iint_S \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \nabla \varphi_i \, dx\, dy \ , \tag{3.22}$$

$$\mu_i = \frac{1}{\|\varphi_i\|^2} \sum_1^{\infty} \frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h} \iint_S \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \nabla \varphi_i \, dx\, dy \ .$$

If we integrate both sides of (3.20) over S we obtain the constraint expressing the constancy of volume. To be sure

$$\iint_S \nabla \{ \frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i \} \, dx\, dy = \int_C \frac{\sinh k_i (\zeta + h)}{k_i} \frac{\partial \varphi_i}{\partial n} \, d L = 0 \ ,$$

$$\iint_S \nabla \{ \frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \nabla \varphi_i \} \, dx\, dy = \int_C \frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \frac{\partial \varphi_i}{\partial n} \, d L = 0 \ ,$$

and

$$\sum_1^{\infty} \dot{\xi}_i \iint_S \varphi_i \, dx\, dy + \dot{\xi}_o \iint_S dx\, dy = A \dot{\xi}_o \ ,$$

so that

$$A \dot{f}_o = 0 \ .$$

But the volume is given by

$$\tau(t) = \iint_S dx\, dy \int_{-h}^{\zeta} dz = h \iint_S dx\, dy + \iint_S \zeta \; dx\, dy = \tau_0 + A\, \xi_0 ,$$

and

$$\dot{\tau}(t) = 0 = A\, \dot{\xi}_0$$

as adduced.[3]

Substituting the expansions for φ and ζ in expression (3.8), multiplying the resulting equation by φ_i and integrating over S, we get, after considerable manipulation and rearranging indices,

$$(a_z' - g_z')\|\varphi_i\|^2 \xi_i + \sum_1^{\infty} \dot{\alpha}_j \iint_S \varphi_i \varphi_j \cosh k_j(\zeta + h)\, dx\, dy + (a_y' - g_y')(y, \varphi_i) \tag{3.23}$$

$$+ \tfrac{1}{2}\sum_1^{\infty}\sum_1^{\infty} \alpha_i\, \alpha_l \iint_S \varphi_i\, [\nabla\varphi_j\, \nabla\varphi_l \cosh k_j(\zeta + h) \cosh k_l(\zeta + h) + k_j k_l \varphi_j \varphi_l \sinh k_j(\zeta + h)\, \cdot$$

$$\sinh k_l(\zeta + h)]\, dx\, dy + \{ 2\sum_1^{\infty} \frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h} \iint_S \varphi_i \varphi_j \sinh k_j \zeta \; dx\, dy$$

$$- \sum_1^{\infty} \xi_j \iint_S y\, \varphi_i \varphi_j\, dx\, dy - \xi_0\, (y, \varphi_i)\}\, \ddot{\theta} + 2\, \{\tfrac{1}{2} \sum_1^{\infty}\sum_1^{\infty} \alpha_l \frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h}\; \cdot$$

$$\iint_S \varphi_i\, [\nabla\varphi_j\, \nabla\varphi_l \sinh k_j \zeta \cosh k_l(\zeta + h) + k_j k_l \varphi_j \varphi_l \cosn k_j \xi \sinh k_l(\zeta + h)]\, dx\, dy$$

$$+ \tfrac{1}{2} \sum_1^{\infty}\sum_1^{\infty} \alpha_j \frac{(y, \varphi_l)}{k_l \|\varphi_l\|^2 \cosh k_l h} \iint_S \varphi_i\, [\nabla\varphi_j\, \nabla\varphi_l \cosh k_j(\zeta + h) \sinh k_l \zeta$$

$$+ k_j k_l \varphi_j \varphi_l \sinh k_j(\zeta + h) \cosh k_l \zeta\,]\, dx\, dy - 2\sum_1^{\infty} \alpha_j k_j \iint_S y\, \varphi_i \varphi_j\; \cdot$$

$$\sinh k_j(\zeta + h)\, dx\, dy\}\, \dot{\theta} - \tfrac{1}{2}\, \dot{\theta}^2 \{\iint_S \varphi_i\, [(1 + \zeta)^2 - 3\, y^2]\, dx\, dy$$

$$- 4 \sum_1^{\infty}\sum_1^{\infty} \frac{(y, \varphi_j)(y, \varphi_l)}{k_j k_l \|\varphi_l\|^2 \|\varphi_l\|^2 \cosh k_j h \cosh k_l h} \iint_S \varphi_i\, [\nabla\varphi_j\, \nabla\varphi_l \sinh k_j \zeta \sinh k_l \zeta$$

$$+ k_j k_l \varphi_j \varphi_l \cosh k_j \zeta \cosh k_l \zeta\,]\, dx\, dy + 8 \sum_1^{\infty} \frac{(y, \varphi_j)}{\|\varphi_j\|^2 \cosh k_j h} \iint_S \varphi_i \varphi_j \cosh k_j \zeta\, dx\, dy\} = 0$$

---

[3]See notes.

Likewise, substituting the expansions for φ and ζ in expression (3.8) and integrating the resulting equation over S, we obtain

$$A\,\{\dot{\alpha}_0 + (a_z' - g_z')(1+\xi_0)\} + \sum_1^\infty \dot{\alpha}_i \iint_S \varphi_i \cosh k_i(\zeta+h)\,dx\,dy \tag{3.24}$$

$$+ \tfrac{1}{2}\sum_1^\infty\sum_1^\infty \alpha_i\,\alpha_j \iint_S [\nabla\varphi_i\,\nabla\varphi_j \cosh k_i(\zeta+h)\cosh k_j(\zeta+h) + k_i\,k_j\,\varphi_i\,\varphi_j \sinh k_i(\zeta+h)\,\cdot$$

$$\sinh k_j(\zeta+h)\,dx\,dy + \{2\sum_1^\infty \frac{(y,\varphi_i)}{k_i\,\|\varphi_i\|^2\cosh k_i\,h}\iint_S \varphi_i \sinh k_i\,\zeta\,dx\,dy - 1\,A$$

$$-\sum_1^\infty (y,\varphi_i)\,\xi_i\}\,\ddot{\theta} + 2\,\{\tfrac{1}{2}\sum_1^\infty\sum_1^\infty \alpha_j \frac{(y,\varphi_i)}{k_i\,\|\varphi_i\|^2\cosh k_i\,h}\iint_S [\nabla\varphi_i\,\nabla\varphi_j \cosh k_j(\zeta+h)\sinh k_i\,\zeta$$

$$+ k_i\,k_j\,\varphi_i\,\varphi_j \sinh k_j(\zeta+h)\cosh k_i\,\zeta]\,dx\,dy + \tfrac{1}{2}\sum_1^\infty\sum_1^\infty \alpha_i \frac{(y,\varphi_j)}{k_j\,\|\varphi_j\|^2\cosh k_j\,h}\,\cdot$$

$$\iint_S [\nabla\varphi_i\,\nabla\varphi_j \cosh k_i(\zeta+h)\sinh k_j\,\zeta + k_i\,k_j\,\varphi_i\,\varphi_j \sinh k_i(\zeta+h)\cosh k_j\,\zeta]\,dx\,dy$$

$$-2\sum_1^\infty \alpha_i\,k_i \iint_S y\,\varphi_i \sinh k_i(\zeta+h)\,dx\,dy\}\,\dot{\theta} - \tfrac{1}{2}\,\{\iint_S [(1+\zeta)^2 - 3\,y^2]\,dx\,dy$$

$$-4\sum_1^\infty\sum_1^\infty \frac{(y,\varphi_i)\,(y,\varphi_j)}{k_i\,k_j\,\|\varphi_i\|^2\|\varphi_j\|^2\cosh k_i\,h\,\cosh k_j\,h}\iint_S [\nabla\varphi_i\,\nabla\varphi_j \sinh k_i\,\zeta\,\sinh k_j\,\zeta$$

$$+ k_i\,k_j\,\varphi_i\,\varphi_j \cosh k_i\,\zeta\,\cosh k_j\,\zeta]\,dx\,dy + 8\sum_1^\infty \frac{(y,\varphi_i)}{\|\varphi_i\|^2\cosh k_i\,h}\iint_S \varphi_i \cosh k_i\,\zeta\,dx\,dy\}\,\dot{\theta}^2 = 0\,.$$

Introducing the expansion for φ into (3.4, 5, 6) gives

$$M\,(a_y' - g_y') + \rho\,(a_y' - g_y')\iint_S \zeta\,dx\,dy - \rho\,\ddot{\theta}\,\{\iint_S (1\,\zeta + \zeta^2/2)\,dx\,dy \tag{3.25}$$

$$+ 2\sum_1^\infty \frac{(y,\varphi_i)}{k_i\,\|\varphi_i\|^2\cosh k_i\,h}\iint_S y\,\nabla\,[\frac{\cosh k_i\,\zeta - \cosh k_i\,h}{k_i}\,\nabla\varphi_i]\,dx\,dy\} - \rho\,\dot{\theta}^2\iint_S y\,\zeta\,dx\,dy$$

$$-\rho\sum_1^\infty \dot{\alpha}_i \iint_S y\,\nabla\,[\frac{\sinh k_i(\zeta+h)}{k_i}\,\nabla\varphi_i]\,dx\,dy + \tfrac{1}{2}\rho\iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\,d\tau + [\ddot{\theta}\,(\cos\beta + 1)$$

$$+ (\dot{\theta} - \dot{\beta})^2 \sin\beta - \ddot{\beta}\cos\beta]\,S^{e}_{o_{e}y_{e}} + 2\,l_1\,\ddot{\theta}\,M_e = F_{ext_y}'\,,$$

$$M\,(a_z' - g_z') + \rho\,(a_z' - g_z')\iint_S \zeta\,dx\,dy - \rho\,\ddot{\theta}\,\{\iint_S y\,\zeta\,dx\,dy \tag{3.26}$$

$$- 2 \sum_{1}^{\infty} \frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \iint_S (\sinh k_i \zeta + \sinh k_i h) \varphi_i \, dx \, dy - \rho \dot{\theta}^2 \iint_S (1 \zeta + \zeta^2/2) dx \, dy$$

$$+ \rho \sum_{1}^{\infty} \dot{\alpha}_i \iint_S [\cosh k_i (\zeta + h) - 1] \varphi_i \, dx \, dy + \tfrac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} d\tau - [(\ddot{\theta} - \ddot{\beta}) \sin \beta$$

$$- (\dot{\theta} - \dot{\beta})^2 \cos \beta - \dot{\theta}^2] S_{o_e y_e}^{\circ} + 2 \, l_1 \, M_e \, \dot{\theta}^2 = F_{ext_{z'}} ,$$

and

$$\{I_{o'x'} - 2 I_{o'z}^{1} + \rho \iint [(l^2 - y^2) \zeta + \zeta^2 + \zeta^3/3] dx \, dy + 2 \rho \sum_{1}^{\infty} \frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \cdot \tag{3.27}$$

$$\iint_S y \, \{(1 + \zeta) \nabla [\frac{\cosh k_i \zeta}{k_i} \nabla \varphi_i] + k_i (1 - h) \varphi_i \cosh k_i h + 2 \, (\sinh k_i \zeta$$

$$+ \sinh k_i h) \varphi_i \} dx \, dy\} \ddot{\theta} + \rho (a_{z'} - g_{z'}) \iint_S y \zeta \, dx \, dy - \rho (a_{y'} - g_{y'}) \iint_S (l \zeta + \zeta^2/2) \, dx \, dy$$

$$+ \rho \sum_{1}^{\infty} \dot{\alpha}_i \iint_S y \, \{(1 + \zeta) \nabla [\frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i] + 2 [\cosh k_i (\zeta + h) - 1] \varphi_i \} dx \, dy$$

$$+ \tfrac{1}{2} \rho \iiint [y \frac{\partial v^2}{\partial z} - (l + z) \frac{\partial v^2}{\partial y}] d\tau - I_{o_e x_e}^{\circ} \ddot{\beta} + \{(a_{y'} - g_{y'}) (\cos \beta + 1)$$

$$- (a_{z'} - g_{z'}) \sin \beta + l_1 [2 \ddot{\theta} (\cos \beta - 1) - 2 \dot{\beta} \dot{\theta} \sin \beta + \dot{\beta}^2 \sin \beta - \ddot{\beta} \cos \beta]\} S_{o_e y_e}^{\circ}$$

$$+ 2 \, l_1 \, (a_{y'} - g_{y'}) M_e = L_{ext_{x'}} ,$$

respectively.

Formulae (3.7, 9_1, 10, 19, 21, 22, 23, 24, 25, 26, 27) together with the appropriate control system, flight path data and aerodynamic data are sufficient to describe the motion of the system. The dependent variables occurring in these expressions are ξ_i, $\dot{\theta}$, u_y, u_z and β. Note that the body rates u_y, u_z, $\dot{\theta}$ are not generalized coordinates; therefore, to determine the true orientation of the vehicle, it is necessary to express these rates in terms of three independent coordinates such as Euler angles. However, stability analyses of closed loop vehicle attitude control systems treat vehicle rigid body motions as a summation of perturbations from a reference motion and motions in which vehicle body axes remain coincident with reference axes. Moreover, the perturbation quantities are presumed to be infinitesimals so that products of infinitesimals and their derivatives can be neglected. In this case the perturbation equations, when referred to vehicle fixed axes, can be integrated to yield the orientation of the perturbed state with respect to the reference state.

The perturbation equations of motion are obtained by taking the vectorial difference between the perturbed equations of motion and reference equations of motion. The resulting expressions are equated to the appropriate perturbed external forces and control system forces.

4/ PERTURBATION EQUATIONS FOR THE PLANAR MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

PERTURBATION EQUATIONS FOR THE PLANAR MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

For simplicity suppose the vehicle to be moving in the direction of constant acceleration

$$(0,\ 0,\ a_{z'}) = (0,\ 0,\ a_r)\ ,$$

under the action of the force field

$$(0,\ 0,\ g_{z'}) = (0,\ 0,\ -g_r)\ ,$$

the only external force being the thrust T directed along the longitudinal axis of the missile (Fig. 4). The free surface of the liquid forms a plane perpendicular to $a_{z'}$, i.e., $\zeta = 0$, and the engine is aligned with the longitudinal axis of the vehicle ($\beta = 0$). We shall call this the reference state.

A small disturbance from the reference state is effected by letting

$$a_{y'} = (a_{y'})_r + \delta a_{y'} = a_r \sin\theta + \delta a_{y'}\ , \tag{4.1}$$

$$a_{z'} = (a_{z'})_r + \delta a_{z'} = a_r \cos\theta + \delta a_{z'}\ ,$$

$$g_{y'} = -g_r \sin\theta\ ,$$

$$g_{z'} = -g_r \cos\theta\ ,$$

in formulae (3.7, 23, 24, 25, 26, 27), and considering $\delta a_{y'}$, $\delta a_{z'}$, θ, β, ζ, ... initially infinitesimal so that when product terms are neglected the resulting equations become linear. These equations are the perturbed equations of motion.

To obtain the perturbation equations, project the reference state equation of motion onto the instantaneous position of the body axes and subtract them from the perturbed equations of motion. Thus, we get perturbation force equation along axis $o'y'$

$$M\ \delta a_{y'} + \rho \sum_{1}^{\infty} (y,\ \varphi_i)\ \ddot{\xi}_i + 2\ (S_{o_e^\circ y_e} + l_1 M_e)\ \ddot{\theta} - S_{o_e^\circ y_e}\ \ddot{\beta} = \delta F_{ext_{y'}}\ , \tag{4.2}$$

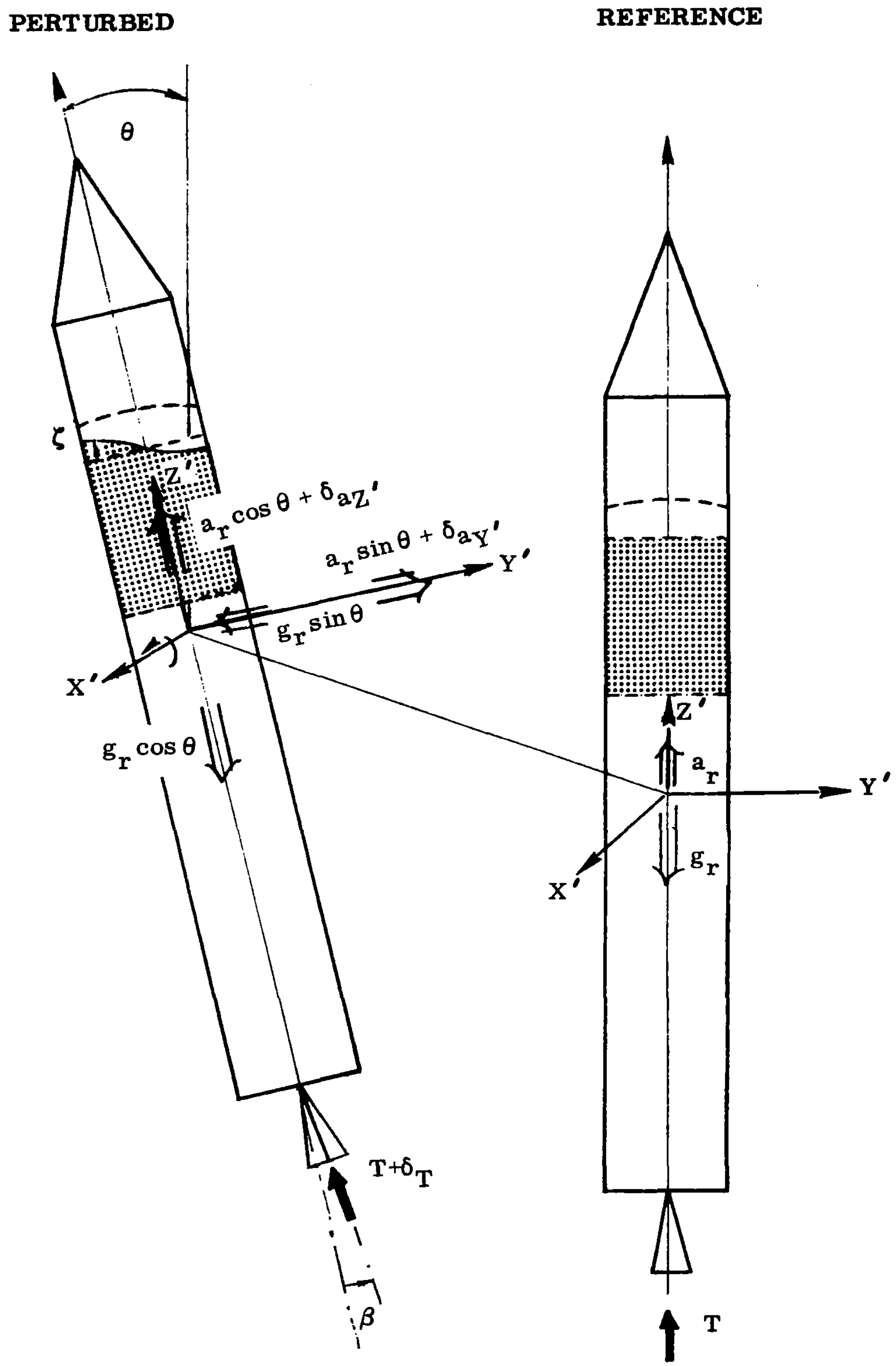


Figure 4. Yaw plane perturbation model for vehicle.

Perturbation force equation along axis $o'z'$

$$M\ \delta a_{z'} = \delta F_{ext_{z'}}\ , \tag{4.3}$$

Perturbation moment equation about axis $o'x'$

$$\{ I_{o'x'} - 4\, I_{o_z}^{1} + 8\,\rho \sum_{1}^{\infty} \frac{\cosh k_i h - 1}{k_i \sinh k_i h} \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2} \}\ \ddot\theta + \rho \sum_{1}^{\infty} [\frac{2\ (\cosh k_i h - 1)}{k_i \sinh k_i h} - 1](y, \varphi_i)\ddot\xi_i$$

$$+ \rho\,(a_r + g_r) \sum_{1}^{\infty} (y,\ \varphi_i)\ \xi_i + 2\,(S_{o_e y_e}^{\bullet} + l_1\ M_e)\ \delta a_{y'} - (I_{o_e y_e}^{\bullet} + l_1\ S_{o_e y_e}^{\bullet})\ \ddot\beta \tag{4.4}$$

$$= \delta L_{ext_{x'}}\ ,$$

Perturbation engine equation

$$I_{o_e x_e}^{\bullet}\ \ddot\beta - (I_{o_e y_e}^{\bullet} + l_1\ S_{o_e y_e}^{\bullet})\ \ddot\theta - S_{o_e y_e}^{\bullet}\ \delta a_{y'}\ + (a_r + g_r) S_{o_e y_e}^{\bullet}\ \beta = Q_\beta\ , \tag{4.5}$$

Constancy of pressure at liquid free surface

$$\frac{\rho \|\varphi_i\|^2}{k_i \tanh\ k_i h}\ \ddot\xi_i + \rho\,\|\varphi_i\|^2\,(a_r\ + g_r)\ \xi_i + \rho\,[\frac{2\ (\cosh k_i h - 1)}{k_i \sinh k_i h} - 1]\,(y,\ \varphi_i)\,\ddot\theta \tag{4.6}$$

$$+ \rho\,(y,\ \varphi_i)\ \delta a_{y'}\ = 0\ \ ,$$

in which $\delta F_{ext_{y'}}$, $\delta F_{ext_{z'}}$, $\delta L_{ext_{x'}}$, Q_β are perturbation forces and moments.

Also

$$\alpha_i\ = \frac{\dot\xi_i}{k_i\ \sinh k_i\ h} + 2\ \dot\theta\ \frac{(y,\ \varphi_i)}{k_i\,\|\varphi_i\|^2 \cosh k_i\ h}\ \frac{(\cosh k_i\ h\ -\ 1)}{\sinh k_i\ h}\ .$$

Denote by Y, Z the components of the displacement vector from the origin of the reference state to that of the perturbed state, measured along the instantaneous position of the body axes; then, to the same order of smallness, we have

$$\delta a_{y'} = \ddot Y \tag{4.7}$$

$$\delta a_{z'} = \ddot Z$$

With the definitions given previously, we see that the action of the liquid in formulae (4.2, 3, 4) is represented by the forces and moment

$$- \delta F_y^{1\prime} = M_1 \ddot{Y} + \rho \iint_S y \, \zeta_{tt} \, dx \, dy \tag{4.8}$$

$$- \delta F_z^{1\prime} = M_1 \ddot{Z}$$

$$- \delta L_x^{1\prime} = \rho \, \ddot{\theta} \iiint_\tau (\nabla \phi^*)^2 \, d\tau + \rho \iint_S \phi^* \zeta_{tt} \, dx \, dy + \rho \, (a_r + g_r) \iint_S y \, \zeta \, dx \, dy$$

in which

$$\phi^* = \varphi^* - y \, (1 + z) \ , \tag{4.9}$$

$$\varphi^* = 2 \sum_1^\infty \frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \left\{ \frac{\cosh k_i h - 1}{\sinh k_i h} \varphi_i \cosh k_i (z + h) + \varphi_i \sinh k_i z \right\} ,$$

$$\rho \iiint_\tau (\nabla \phi^*)^2 \, d\tau = \rho \iiint_\tau [y^2 + (1 + z)^2] \, d\tau - 4 \rho \iiint_\tau y^2 \, d\tau + \rho \iiint_\tau (\nabla \varphi^*)^2 \, d\tau ,$$

$$\rho \iiint_\tau (\nabla \varphi^*)^2 = 8 \rho \sum_1^\infty \frac{\cosh k_i h - 1}{k_i \sinh k_i h} \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} ,$$

$$\zeta = \sum_1^\infty \xi_i \varphi_i + \xi_0$$

and ϕ^*, φ^* satisfy

$$\Delta \phi^* = \Delta \varphi^* = 0, \ P \epsilon \tau \tag{4.10}$$

$$\frac{\partial \phi^*}{\partial n} = y \cos (n, z) - (1 + z) \cos (n, y), \ P \epsilon \Sigma, S,$$

$$\frac{\partial \varphi^*}{\partial n} = 2 y \cos (n, z) , \ P \epsilon \Sigma , S .$$

From (2.34) et. seq., we see that ϕ^*, φ^* are modified Stokes potentials which are determined solely from the geometry of the vessel, the free surface being "capped".

With the introduction of potential ψ, where

$$\Delta \psi = 0 \quad , \ P \epsilon \tau , \tag{4.11}$$

$$\frac{\partial \psi}{\partial n} = \begin{cases} 0 & , \ P \epsilon \Sigma , \\ \zeta_t & , \ P \epsilon S , \end{cases}$$

formula (4.6) may be written as

$$\rho \frac{\partial \psi}{\partial t} + \rho\, \ddot{\theta}\, \phi^* + \rho\, y\, \ddot{Y} + \rho\, (a_r + g_r)\, \zeta \;= 0 \;, \tag{4.12}$$

$$\psi \;=\; \sum_1^{\infty} \dot{\xi}_i(t)\, \frac{\cosh k_i (z+h)}{k_i \,\sinh k_i\, h}\, \varphi_i \;,$$

$$y \;=\; \sum_1^{\infty} \frac{(y,\, \varphi_i)}{\|\varphi_i\|^2}\, \varphi_i \;.$$

Thus, it follows from formulae (4.8, 9, 10, 11, 12) that the action of the liquid can be determined also from a computation of forces and moment (along $o'y'$, $o'z'$ and about $o'x'$ respectively) resulting from the solution of the linear problem shown in Fig. 5, Y, Z, θ, ζ being taken as infinitesimals.

Formula (4.9_3) represents the moment of inertia of the "capped" liquid about the center of rotation, and obviously does not equal the moment of inertia of the "frozen" liquid about that point. Indeed, if we denote by $I_0^{*\prime}$ the moment of inertia of the "capped" liquid about $0'$, and by $\widetilde{I}_0{}'$ that of the "frozen" liquid about the same point, then

$$\frac{I_0^{*\prime}}{\widetilde{I}_0{}'} = 1 - \frac{4\rho}{\widetilde{I}_0{}'} \iiint_{\tau} y^2\, d\tau + \frac{8\rho}{\widetilde{I}_0{}'} \sum_1^{\infty} \frac{\cosh k_i\, h - 1}{k_i \,\sinh k_i\, h}\, \frac{(y,\, \varphi_i)^2}{\|\varphi_i\|^2} \;, \tag{4.13}$$

$$\widetilde{I}_0{}' = \rho \iiint_{\tau} [y^2 + (1+z)^2]\, d\tau \;,$$

a ratio which can be shown to be less than unity. In particular,

$$\frac{I_{cg}^*}{\widetilde{I}_{cg}} \;=\; 1 - \frac{4\rho}{\widetilde{I}_{cg}} \iiint_{\tau} y^2\, d\tau + \frac{8\rho}{\widetilde{I}_{cg}} \sum_1^{\infty} \frac{\cosh k_i\, h - 1}{k_i \,\sinh k_i\, h}\, \frac{(y,\, \varphi_i)^2}{\|\varphi_i\|^2} \;, \tag{4.14}$$

$$\widetilde{I}_{cg} = \rho \iiint_{\tau} [\, y^2 + (z + h/2)^2]\, d\tau \;,$$

at the center of gravity of the liquid $(1 = h/2)$. Moreover,

$$I_0^{*\prime} = I_{cg}^* + M_l\, (1 - h/2)^2 \;, \tag{4.15}$$

$$\widetilde{I}_0{}' = \widetilde{I}_{cg} + M_l\, (1 - h/2)^2 \;.$$

The behavior of (4.13) may be illustrated effectively by considering a rectangular tank such as shown in Fig. 6. We have, for this example,

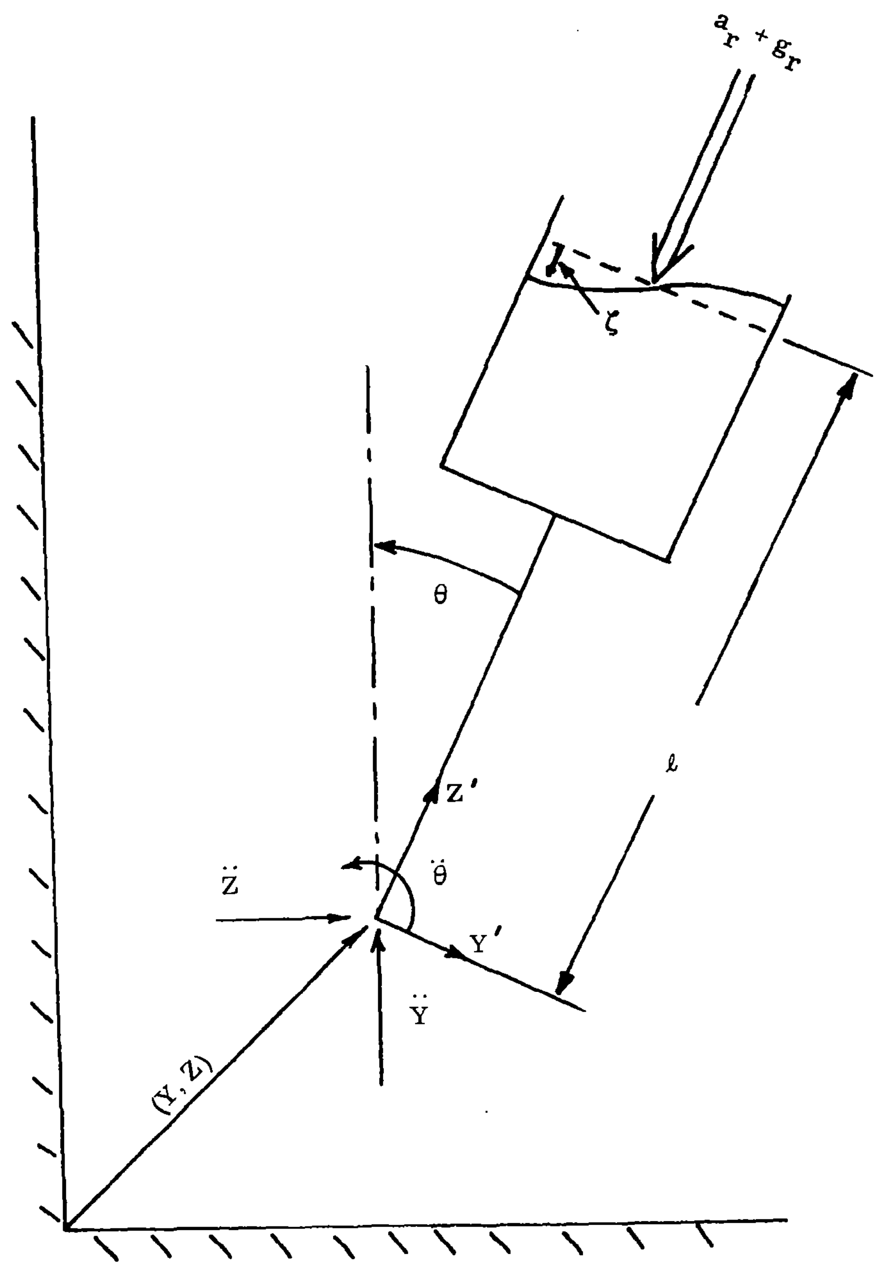


Figure 5. Linear model for simulating action of liquid in perturbation equations.

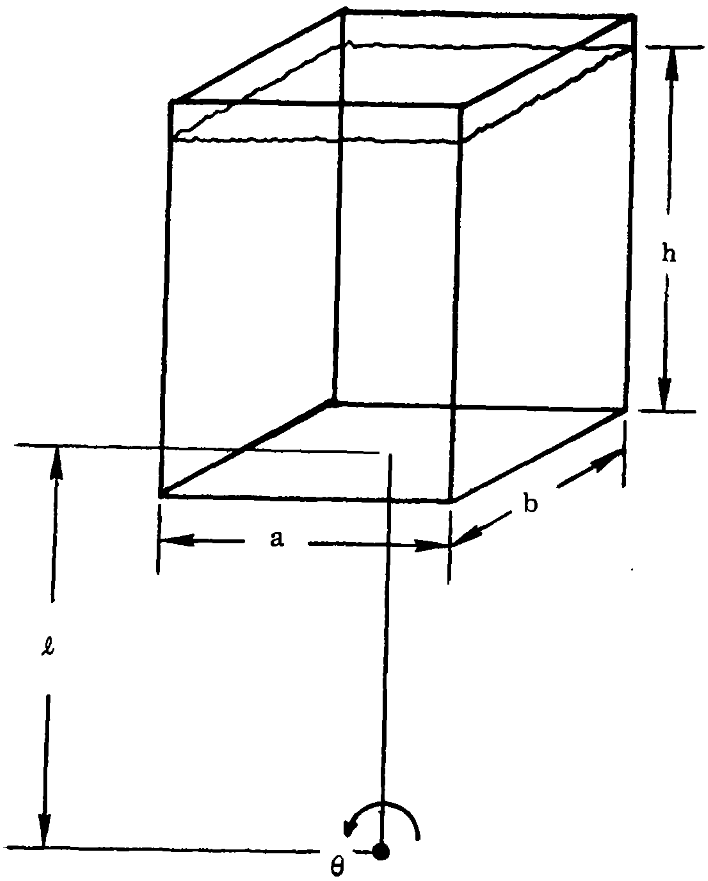


Figure 6. Liquid filled rectangular tank.

$$\frac{I_0^{*\prime}}{\tilde{I}_0{}'} = 1 - \frac{(1/3)}{\{(l/a)^2 - (l/a)(h/a) + 1/3\,(h/a)^2 + 1/12\}} + \frac{(64/\pi^5)\,(a/h)}{\{(l/a)^2 - (l/a)(h/a) + 1/3\,(h/a)^2 + 1/12\}} \cdot$$

$$\sum_{1,3,..} \{1/i^5 \tanh i\pi h/2a\} . \qquad (4.16)$$

This ratio is plotted in Fig. 7 as a function of l/h with a/h as parameter. Note that for any given a/h the "capped" moment of inertia of the liquid is less for rotation about the center of gravity of the liquid ($l = h/2$) than for any other position. Similar results may be obtained for other configurations.

To put formulae (4.8, 12) into symmetrical form, we introduce the transformation

$$\xi_1 = \frac{(y,\ \varphi_1)}{\|\varphi_1\|^2}\ q_1 \ , \qquad (4.17)$$

getting

$$- \delta F_y^{1\prime} = M_1\ \delta a_y{}' + \rho \sum_1^\infty \frac{(y,\ \varphi_1)^2}{\|\varphi_1\|^2}\ q_1 \ ,$$

$$- \delta F_z^{1\prime} = M_1\ \delta a_z{}' \ ,$$

$$- \delta L_x^{1\prime} = I_0^*\ \ddot{\theta} + \rho \sum_1^\infty \left[\frac{2\,(\cosh k_1 h - 1)}{k_1 \sinh k_1\ h} - 1\right] \frac{(y,\ \varphi_1)^2}{\|\varphi_1\|^2}\ \ddot{q}_1 \qquad (4.18)$$

$$+ \rho\,(a_r + g_r) \sum_1^\infty \frac{(y,\ \varphi_1)^2}{\|\varphi_1\|^2}\ q_1 \ ,$$

$$\frac{\rho\,(y,\ \varphi_1)^2}{k_1 \|\varphi_1\|^2 \tanh k_1\ h}\ \ddot{q}_1 + \rho\,(a_r + g_r) \frac{(y,\ \varphi_1)^2}{\|\varphi_1\|^2}\ q_1 + \rho \left[\frac{2\,(\cosh k_1\ h\ -\ 1)}{k_1 \sinh k_1\ h} - 1\right] \frac{(y,\ \varphi_1)^2}{\|\varphi_1\|^2}\ \ddot{\theta}$$

$$+ \rho\,\frac{(y,\ \varphi_1)^2}{\|\varphi_1\|^2}\ \delta a_y{}' = 0$$

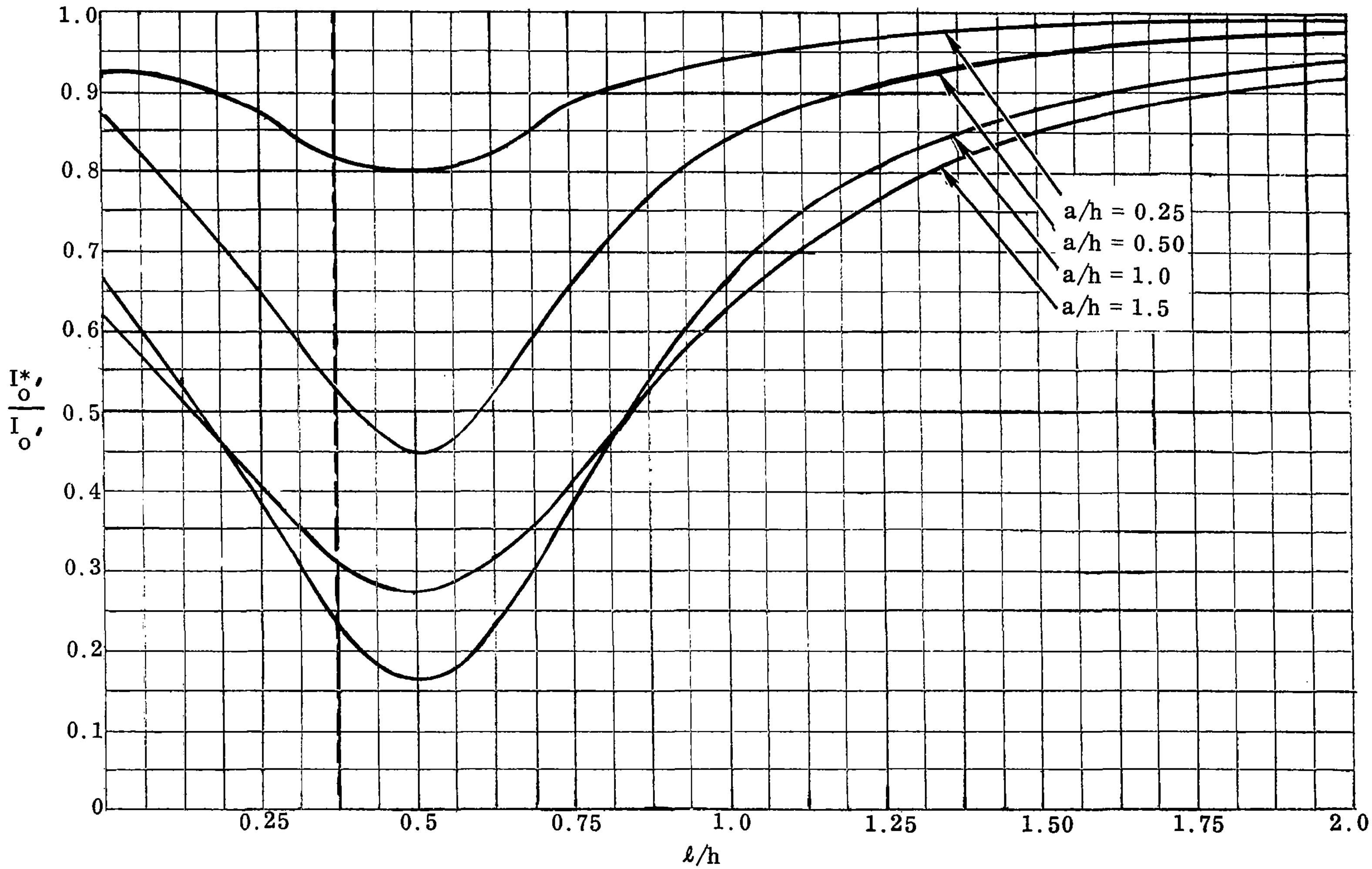


Figure 7. Ratio of "capped" moment of inertia to "frozen" moment of inertia for rectangular liquid filled tank.

5/ MECHANICAL ANALOG FOR REPRESENTING THE ACTION OF THE LIQUID IN THE PERTURBATION EQUATIONS OF MOTION

MECHANICAL ANALOG FOR REPRESENTING THE ACTION OF THE LIQUID IN THE PERTURBATION EQUATIONS OF MOTION

We enquire how to duplicate the action of the liquid, represented by formulae (4.18), with a mechanical system. Accordingly, consider the system of pendula attached to the vehicle shown in Fig. 8, where

M_{pi} ~ Mass of i th pendulum,

L_{si} ~ Distance from center of rotation to hinge point of i th pendulum,

L_{pi} ~ Length of i th pendulum arm,

I_O ~ Rigid moment of inertia,

M_O ~ Rigid mass,

L_O ~ Distance from center of rotation to point at which M_O is situated.

The velocity of the mass of the i th pendulum is simply

$$\bar{V}_i = \{ 0, [(u_y' - L_{si}\dot{\theta}) + (\dot{\theta} + \dot{q}_i) L_{pi} \cos q_i], [u_z' + L_{pi}(\dot{\theta} + \dot{q}_i)\sin q_i]\} \quad , \qquad (5.1)$$

and that of mass M_O is

$$\bar{V}_O = \{ 0, (u_y' - L_O\dot{\theta}), u_z'\} \; . \qquad (5.2)$$

It follows that the kinetic energy of the mechanical system is

$$T_{ms} = \tfrac{1}{2}(u_y'^2 + u_z'^2)\{M_O + \sum_1^N M_{pi}\} + \tfrac{1}{2}\dot{\theta}^2\{I_O + M_O L_O^2 + \sum_1^N M_{pi}(L_{si} - L_{pi})^2\} \qquad (5.3)$$

$$- \dot{\theta} u_y' \{M_O L_O + \sum_1^N M_{pi}(L_{si} - L_{pi})\} + \tfrac{1}{2}\sum_1^N M_{pi} L_{pi}^2 \dot{q}_i^2 + \dot{\theta}\sum_1^N M_{pi} L_{pi}^2 \dot{q}_i$$

$$+ \sum_1^N M_{pi} L_{pi}\dot{q}_i \{u_z' \sin q_i + (u_y' - L_{si}\dot{\theta})\cos q_i\}$$

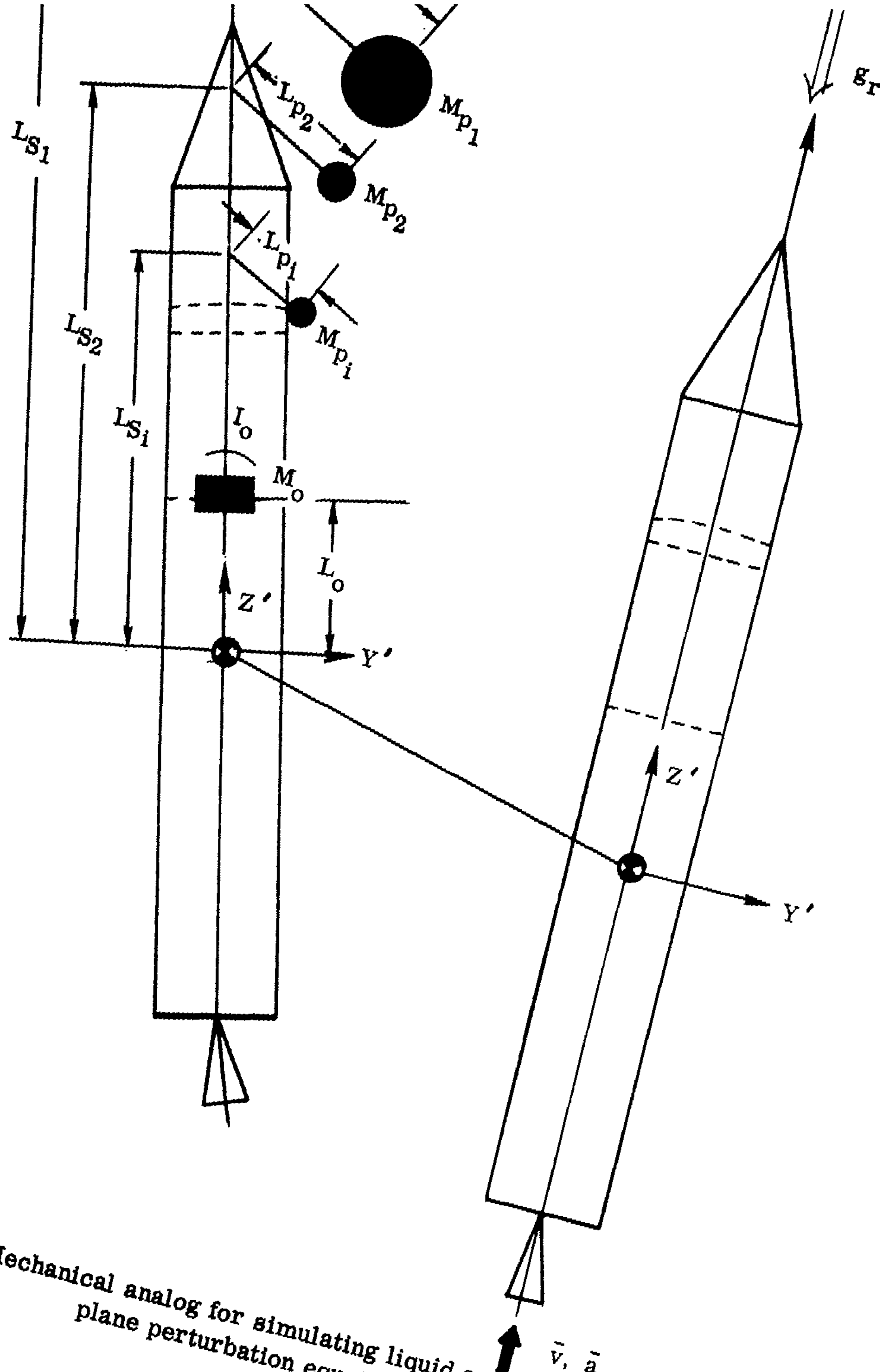


Figure 8. Mechanical analog for simulating liquid action in yaw plane perturbation equations.

$$+ \dot{\theta} \sum_{1}^{N} M_{pi} L_{pi} \{ u_z' \sin q_i + (u_y' - L_{si} \dot{\theta}) (\cos q_i - 1) \} \ .$$

The equations of motion may be obtained in manner similar to (3.2),

$$\frac{d}{dt} \left(\frac{\partial T}{\partial u_y'}\right) - \dot{\theta} \frac{\partial T}{\partial u_z'} = F_{ext_y}' - \iiint_{\tau^*} g_y' \, dm - \iiint_{\tau_e} g_y' \, dm - g_y' [M_O + \sum_{1}^{N} M_{pi}] \ , \tag{5.4}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial u_z'}\right) + \dot{\theta} \frac{\partial T}{\partial u_y'} = F_{ext_z}' - \iiint_{\tau^*} g_z' \, dm - \iiint_{\tau_e} g_z' \, dm - g_z' [M_O + \sum_{1}^{N} M_{pi}] \ ,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}}\right) + u_y' \frac{\partial T}{\partial u_z'} - u_z' \frac{\partial T}{\partial u_y'} = L_{ext_x}' - \iiint_{\tau^*} (y' g_z' - z' g_y') \, dm$$

$$- \iiint_{\tau_e} [(y_e \cos \beta - z_e \sin \beta) g_z' + (l_1 + z_e \cos \beta + y_e \sin \beta) g_y'] \, dm + M_O L_O \, g_y'$$

$$- \sum_{1}^{N} M_{pi} \{ g_z' L_{pi} \sin q_i - g_y' (L_{si} - L_{pi} \cos q_i) \} \ ,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = - \iiint_{\tau_e} (g_z' \sin \beta - g_y' \cos \beta) z_e \, dm + \iiint_{\tau_e} (g_z' \cos \beta + g_y' \sin \beta) y_e \, dm + Q_\beta \ ,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = - M_{pi} L_{pi} (g_y' \cos q_i + g_z' \sin q_i) \ ,$$

where

$$T = T_{mb} + T_e + T_{ms} \ .$$

T_{mb} and T_e are given in (3.1_1) and (3.1_2) .

Repeating the same simplifications and arguments used to obtain the perturbation equations for the motion of the liquid propellant vehicle, (5.4) gives, after considerable work,

$$(M_{mb} + M_e) \, \delta a_y' + 2 (S_{o_e y_e}^{e} + l_1 M_e) \ddot{\theta} - S_{o_e y_e}^{e} \ddot{\beta} = \delta F_{ext_y}' - \{[M_O + \sum_{1}^{N} M_{pi}] \, \delta a_y' \tag{5.5}$$

$$+ \sum_{1}^{N} M_{pi} L_{pi} \ddot{q}_i \ ,$$

$$(M_{mb} + M_e) \delta a_z' = \delta F_{ext_z}' - \{ [M_O + \sum_{1}^{N} M_{pi}] \, \delta a_z' \} \ ,$$

$$M_{pi} L_{pi} (L_{pi} - L_{si}) = \rho \left[\frac{2 (\cosh k_i h - 1)}{k_i \sinh k_i h} - 1 \right] \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} ,$$

$$I_O + M_O L_O^2 + \sum_1^N M_{pi} (L_{si} - L_{pi})^2 = I_O^{*\prime} .$$

Relations (5.7) contain six unknowns M_{pi}, L_{pi}, L_{si}, L_O, M_O, I_O. One more equation is needed to make the system determinate, namely, the first moment of mass about the center of rotation,

$$M_O L_O + \sum_1^N M_{pi} (L_{si} - L_{pi}) = M_1 (1 - h/2) . \tag{5.8}$$

solving (5.7, 8) simultaneously, we get

$$L_{pi} = \frac{1}{k_i \tanh k_i h} , \tag{5.9}$$

$$M_{pi} = \rho \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h ,$$

$$L_{si} = \frac{1}{k_i \tanh k_i h} + \left[1 - \frac{2 (\cosh k_i h - 1)}{k_i \sinh k_i h} \right] ,$$

$$M_O = M_1 - \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h ,$$

$$L_O = \frac{M_1 (1 - h/2) + \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left[\frac{2 (\cosh k_i h - 1)}{k_i \sinh k_i h} - 1 \right] k_i \tanh k_i h}{M_1 - \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h} ,$$

$$I_O = I_O^{*\prime} - \rho \sum \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left[1 - \frac{2 (\cosh k_i h - 1)}{k_i \sinh k_i h} \right]^2 k_i \tanh k_i h$$

$$- \frac{\left\{ M_1 (1 - h/2) + \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left[\frac{2 (\cosh k_i h - 1)}{k_i \sinh k_i h} - 1 \right] k_i \tanh k_i h \right\}^2}{M_1 - \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h} ,$$

in which

$$I_O^{*\prime} = \tilde{I}_O{}' - 4 \rho \iiint_\tau y^2 \, d\tau + 8 \rho \sum_1^N \frac{\cosh k_i h - 1}{k_i \sinh k_i h} \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} .$$

$$(I_{O}{}^{mb}_{x'x'} + I^{e}_{O}{}_{x'x'})\ddot{\theta} + 2\,(S_{O_e y_e} + l_1\ M_e)\,\delta a_y' - (I_{O_e y_e} + l_1\, S_{O_e y_e})\,\ddot{\beta}$$

$$- (a_r + g_r) S_{O_e y_e}\, \beta = \delta L_{ext_x}{}' - \{[\, I_O + M_O L_O^2$$

$$+ \sum_1^N M_{pi}\,(L_{si} - L_{pi})^2]\,\ddot{\theta} + \sum_1^N M_{pi} L_{pi}\,(L_{pi} - L_{si})\,\ddot{q}_i + (a_r + g_r) \sum_1^N M_{pi} L_{pi}\ q_i\} \ ,$$

$$M_{pi} L_{pi}{}^2\, \ddot{q}_i + (a_r + g_r)\, M_{pi} L_{pi}\, q_i + M_{pi} L_{pi}\,(L_{pi} - L_{si})\,\ddot{\theta}$$

$$+ M_{pi} L_{pi}\, \delta a_y' = 0 \ ,$$

$$I_{O_e y_e}\, \ddot{\beta} + (a_r + g_r)\, S_{O_e y_e}\, \beta - (I_{O_e y_e} + l_1\, S_{O_e y_e})\,\ddot{\theta} - \quad S_{O_e y_e}\, \delta a_y' = Q_\beta \ ,$$

in which the bracketed terms on the right hand members of ($5.5_{1,2,3}$) and (5.5_4) represent the action of the mechanical system. Explicitly,

$$- \delta F^{ms}_{y}{}' = [M_O + \sum_1^N M_{pi}\,]\,\delta a_y' + \sum_1^N M_{pi} L_{pi}\, \ddot{q}_i \ , \tag{5.6}$$

$$- \delta F^{ms}_{z}{}' = [\,M_O + \sum_1^N M_{pi}]\,\delta a_z' \ ,$$

$$- \delta L^{ms}_{x}{}' = [I_O + M_O\, L_O^2 + \sum_1^N M_{pi}\,(L_{si} - L_{pi})^2]\,\ddot{\theta} + \sum_1^N M_{pi} L_{pi}\,(L_{pi} - L_{si})\,\ddot{q}_i$$

$$+ (a_r + g_r) \sum_1^N M_{pi} L_{pi}\ q_i \ ,$$

$$M_{pi} L_{pi}{}^2\, \ddot{q}_i + (a_r + g_r)\, M_{pi} L_{pi}\, q_i + M_{pi} L_{pi}\,(L_{pi} - L_{si})\,\ddot{\theta}$$

$$+ M_{pi} L_{pi}\ \delta a_y' = 0 \ .$$

The forces, moment and surface wave height terms in formulae (4.18) will match formulae (5.6) for a finite (or infinite) number of pendula if the following associations are made:

$$M_O + \sum_1^N M_{pi} = M_1 \qquad , \tag{5.7}$$

$$M_{pi} L_{pi} = \rho\ \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2} \ ,$$

$$M_{pi} L_{pi}{}^2 = \frac{\rho(y,\ \varphi_i)^2}{k_i\, \|\varphi_i\|^2 \tanh k_i\, h} \ ,$$

The amplitudes of the wave height are related to the angular displacements of the pendula by

$$\xi_i = \frac{(y,\ \varphi_i)}{\|\varphi_i\|^2}\ q_i\ .$$

Thus, the action of the liquid represented by formulae (4.18) may be duplicated with a mechanical system--a system of pendula plus a concentrated mass and moment of inertia. A similar analogy may be effected with a system of springs and masses.

6/CORRELATION OF ANALYSIS OF VESSEL OF GENERAL SHAPE POSSESSING ROTATIONAL SYMMETRY WITH EXISTING MSFC ANALYSIS

CORRELATION OF ANALYSIS OF VESSEL OF GENERAL SHAPE POSSESSING ROTATIONAL SYMMETRY WITH EXISTING MSFC ANALYSIS

Formulae (4.8, 12_1) hold for containers of general shape possessing rotational symmetry about the z-axis, Figure 9, if we approximate the various volume integrals occurring in the analysis as

$$\iiint_{\tau(t)} (\ldots)\, d\tau = \iiint_{\tau} (\ldots)\, d\tau + \iint_{S} ds \int_0^{\zeta} (\ldots)\, dz \quad ,$$

and if we evaluate directional derivatives over the undisturbed surfaces Σ , S. Such simplifications can be justified. Thus, for the action of the liquid contained in the vessel of Figure 9, we have

$$-\delta F_y^{1\prime} = M_1\, \delta a_y{}' + \rho \iint_S y\, \zeta_{tt}\, ds \quad , \tag{6.1}$$

$$-\delta F_z^{1\prime} = M_1\, \delta a_z{}' \quad ,$$

$$-\delta L_x^{1\prime} = \rho\, \ddot{\theta} \iiint_{\tau} (\nabla \phi^*)^2 d\tau + \rho \iint_S \phi^*\, \zeta_{tt}\, ds + \rho(a_r + g_r) \iint_S y\, \zeta\, ds \quad ,$$

$$\rho \frac{\partial \psi}{\partial t} + \rho\, \ddot{\theta}\, \phi^* + \rho\, y\, \delta a_y{}' + \rho(a_r + g_r)\, \zeta = 0 \quad ,$$

where

$$\Delta \phi^* = 0 \quad , \quad P \epsilon \tau \qquad\qquad \Delta \psi = 0 \quad , \quad P \epsilon \tau \quad ,$$

$$\frac{\partial \phi^*}{\partial n} = y \cos(n,\ z) - (1 + z)\cos(n,\ y), \quad P \epsilon \Sigma\ ,\ S, \qquad \frac{\partial \psi}{\partial n} = \begin{cases} 0 & ,\ P \epsilon \Sigma\ , \\ \zeta_t & ,\ P \epsilon S\ . \end{cases}$$

This system may be brought into a form suitable for comparison with the analysis of [7] , if we substitute

$$z - L \text{ for } z \quad ,$$

$$L + L_1 \text{ for } l \quad ,$$

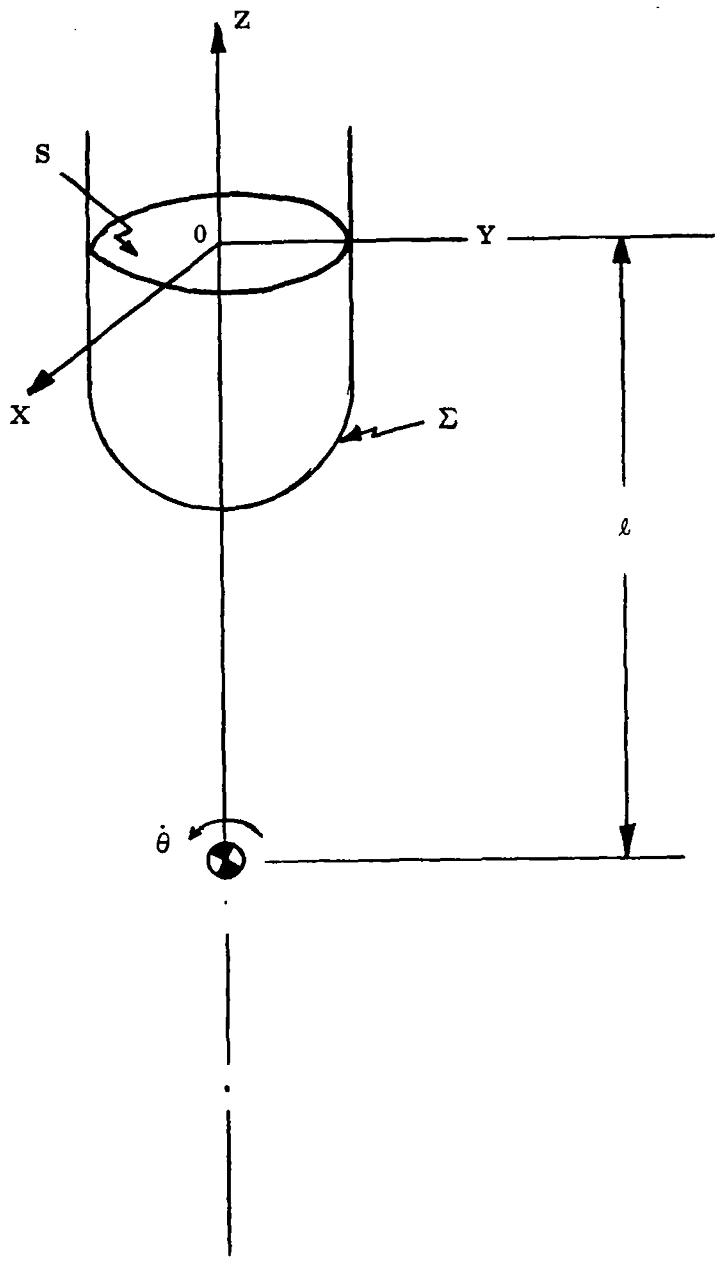


Figure 9. Arbitrary shaped container with rotational symmetry.

and let

$$\phi^* = y\,(z + L_1) - L^2\,\psi^* \ ,$$

where the coordinates are now reckoned from the center of gravity of the undisturbed liquid; L is the distance from the center of gravity of the undisturbed liquid to the quiescent free surface; L_1 is the distance from the center of rotation of the vehicle to the center of gravity of the undisturbed liquid (Figure 10). These substitutions give

$$-\,\delta F_{y'}^{1} = M_1\,\delta a_{y'} + \rho \iint_S y\,\zeta_{tt}\,ds \qquad (6.2)$$

$$-\,\delta F_{z'} = M_1\,\delta a_{z'} \ ,$$

$$-\,\delta L_{x'}^{1} = I_0^*\,\ddot{\theta} + \rho \iint_S [L\,(y - L\,\psi^*) + L_1\,y]\,\zeta_{tt}\,ds + \rho\,(a_r + g_r) \iint y\,\zeta\,ds \ ,$$

$$\rho\,\frac{\partial\psi}{\partial t} + \rho\,\ddot{\theta}\,[L(y - L\,\psi^*) + L_1\,y] + \rho y\,\delta a_{y'} + \rho\,(a_r + g_r)\,\zeta = 0 \ ,$$

in which

$$\Delta\psi^* = 0 \quad , \quad P \epsilon \tau \qquad (6.3)$$

$$\frac{\partial\psi^*}{\partial n} = \frac{2\,(L_1 + z)\cos(n,\,y)}{L^2} \ , \quad P \epsilon \Sigma_1\ S \ ,$$

$$I_{0'}^* = \rho \iiint_\tau (y^2 + z^2)\,d\tau - 4\,\rho \iiint_\tau z^2\,d\tau + 2\,\rho\,L^2 \iint_{S+\Sigma} \psi^* z\,\cos(n,\,y)\,ds + M_1 L_1^2$$

We can express the solution for ψ and ζ as a generalized fourier series of eigenfunctions ψ_i,

$$\zeta = \sum_1 \xi_i\,\psi_i \ , \qquad (6.4)$$

$$\psi = \sum_1 \frac{L}{K_i}\,\dot{\xi}_i\,\psi_i \ ,$$

where ψ_i satisfies

$$\Delta\psi_i = 0 \ , \quad P \epsilon \tau \ , \qquad (6.5)$$

$$\frac{\partial\psi_i}{\partial n} = \begin{cases} 0 \ , & P \epsilon \Sigma \ , \\ \equiv \dfrac{\partial\psi_i}{\partial z} = \dfrac{K_i}{L}\,\psi_i \ , & P \epsilon S \ , \end{cases}$$

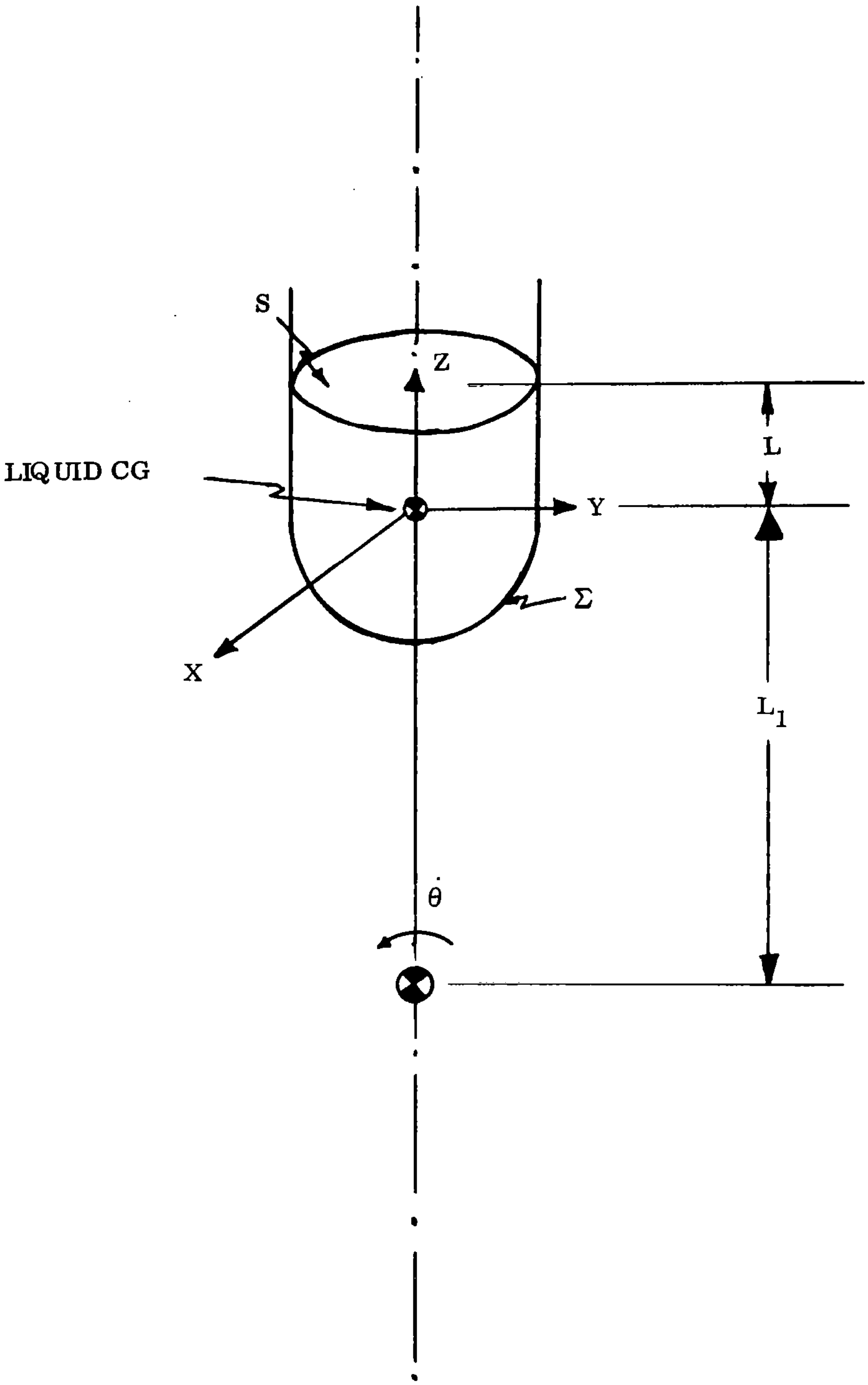


Figure 10. Arbitrary shaped container with coordinate system located at undisturbed liquid center-of-gravity.

$$\iint_S \psi_i \psi_j = 0 , \quad i \neq j .$$

Introducing (6.4) into (6.5), applying integral transformations, using boundary conditions (6.3_2, 5_2) and orthogonality condition (6.5_3), we get

$$- \delta F_y^{1\prime} = M_1 \ \delta a_y{}' + M_1 \sum_1^{\infty} \| \psi_i \|^2 b_i \ddot{\xi}_i \tag{6.6}$$

$$- \delta F_z^{1\prime} = M_1 \ \delta a_y{}'$$

$$- \delta L_x^{1\prime} = I_O^{*\prime} \ \ddot{\theta} + M_1 \sum_1^{\infty} \| \psi_i \|^2 \{ (a_r + g_r) b_i \xi_i + [L (b_i - h_i) - L_i b_i] \ddot{\xi}_i \} ,$$

$$\ddot{\xi}_i + \frac{(a_r + g_r)}{L} K_i \xi_i + K_i [L (b_i - h_i) - L_i b_i] \ddot{\theta} + K_i b_i \delta a_y{}' = 0 ,$$

where

$$\frac{\| \psi_i \|^2 V}{L} = \iint_S \psi_i^2 \, ds , \quad V = \iiint_\tau d\tau , \tag{6.7}$$

$$b_i \ V \| \psi_i \|^2 = \iint_S y \ \psi_i \, ds, \quad K_i h_i \ V \| \psi_i \|^2 = 2 \iint_{S+\Sigma} z \ \psi_i \cos(n, y) \, ds .$$

Formulae ($6.6_{1,2,3}$) are equivalent to formulae (3.47, 46, 48) of [7], except for terms involving the first moment of mass about the center of rotation of the vehicle. In our analysis, such terms have been included in the rigid vehicle dynamics. Formula (6.6_4) differs from (3.49) of [7] by the term $K_i (a_r + g_r) b_i \theta$. The reason for this is that the equations for the action of the liquid in [7] were not perturbed but arbitrarily linearized.

On the other hand if we compare (6.6) with (5.6), we get

$$M_O + \sum_1^N M_{pi} = M_1 , \tag{6.8}$$

$$M_{pi} L_{pi} = M_1 L \| \psi_i \|^2 b_i^2 ,$$

$$M_{pi} L_{pi}^2 = M_1 \frac{L^2 b_i^2}{K_i} \| \psi_i \|^2 ,$$

$$M_{pi} L_{pi} (L_{pi} - L_{si}) = M_1 L b_i \| \psi_i \|^2 [L(b_i - h_i) - L_1 b_i] ,$$

$$I_O + M_O L_O^2 + \sum_1^N M_{pi} (L_{si} - L_{pi})^2 = I_O^{*\prime} ,$$

and the first moment of mass about the center of rotation yields

$$M_o L_o + \sum_{1}^{N} M_{pi} (L_{si} - L_{pi}) = M_1 L_1 \ , \tag{6.9}$$

where

$$q_i = \frac{\xi_i}{Lb_i} \ . \tag{6.10}$$

Formulae (6.8, 9, 10) are equivalent in all respects to formulae (3.83) of [7] . In other words either analysis yields the same mechanical model even though they differ by a term in the free surface equation.

Methods for computing the eigenfunctions, eigenvalues and associated quantities are discussed in the next section.

7/FREE VIBRATIONS OF A HEAVY LIQUID ENCLOSED IN A FIXED VESSEL

FREE VIBRATIONS OF A HEAVY LIQUID ENCLOSED IN A FIXED VESSEL

When attempting an actual stability analysis of a liquid propellant space vehicle, the first problem the engineer encounters is that of determining the free vibrations of the liquid while the vessel is at rest.

VESSELS OF GENERAL SHAPE

Putting $\bar{\omega} = \bar{u} = 0$ in formulae (2.29, 31, 32), linearizing the resulting expressions, we find that the problem of linear oscillations of a heavy liquid enclosed in a rigid, immovable vessel is reduced to one of finding, in the class of functions satisfying conditions

$$\bar{v} = \nabla\varphi, \quad P\epsilon\tau, \tag{7.1}$$

$$\frac{\partial\varphi}{\partial n} = \begin{cases} 0, & P\epsilon\Sigma, \\ \zeta_t, & P\epsilon S, \end{cases}$$

$$\varphi_t + g\zeta = 0, \quad P\epsilon S,$$

all possible functions $\varphi(P, t)$, $P\epsilon\tau$ satisfying

$$\Delta\varphi = 0 \tag{7.2}$$

See Figure 11 for notation.

The solution of the Neumann problem--namely to determine a function φ harmonic in τ and possessing a normal derivative $\frac{\partial\varphi}{\partial n}$ that agrees with a specified function f defined on $\Sigma + S$ (subject to the restrictions $\iint_{\Sigma+S} \varphi ds = 0$, $\iint_{S+\Sigma} f\, ds = 0$[4])--is furnished by the formula

$$\varphi = Nf .$$

N is the integral operator

$$\varphi(Q) = \iint_{S+\Sigma} N(P, Q)\, f(P)\, dP$$

[4]See notes.

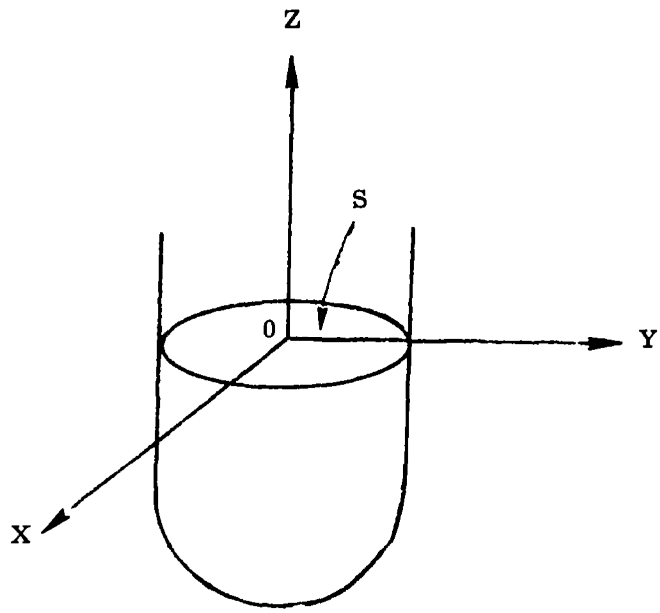


Figure 11. Partly filled vessel at rest.

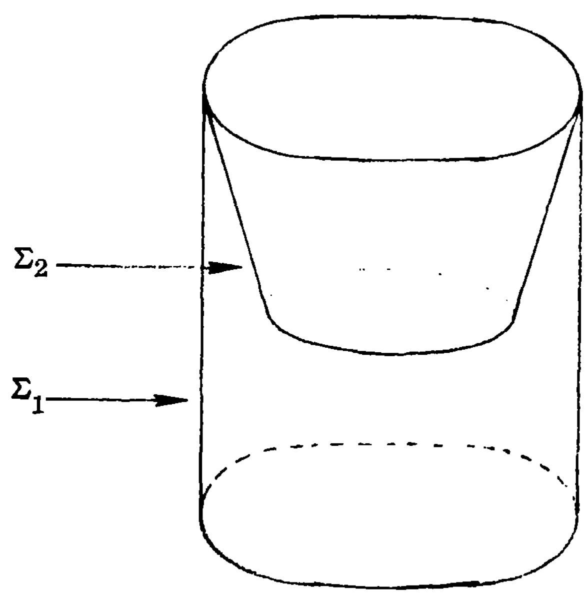


Figure 12. Two vessels partly filled with a heavy liquid, one enveloping the other.

whose kernel is Green's function for the Neumann problem. According to the general theory, the kernel is symmetrical and possesses for $P = Q$ a source-like singularity: $\log \rho^{-1}$ in the two-dimensional and ρ^{-1} in the three-dimensional problem, where

$$\rho = [(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2]^{\frac{1}{2}}$$

N is a completely continuous self-adjoint operator. Using (7.1_2) we get

$$\varphi(Q) = N\zeta_t \ , \tag{7.3}$$

$$\varphi(Q) = \iint N(P,Q)\, \zeta_t(P)\, dP \ .$$

φ can be eliminated from (7.1_3):

$$N\zeta_{tt} + g\zeta = 0 \tag{7.4}$$

Assuming that $Q \epsilon S$ in (7.4), we obtain an integro-differential equation for the determination of the linear oscillations. Formula (7.4) is especially useful for studying general properties of the system, but not too practical for actual numerical computations. To this end we resort to variational mechanics.

We begin by constructing the Lagrange function $L = T - \Pi$ for our problem. As is known, the kinetic energy is, to second order of smallness,

$$T = \tfrac{1}{2}\rho \iiint_{\tau} v^2\, d\tau \ ,$$

taken throughout the undisturbed volume τ occupied by the liquid. But $v^2 = (\nabla\varphi)^2$ and therefore from Green's theorem, if φ is single-valued, and since $\Delta\varphi = 0$,

$$T = \tfrac{1}{2}\rho \iiint_{\tau} (\nabla\varphi)^2\, d\tau = \tfrac{1}{2}\rho \iint_{S} \varphi\zeta_t\, ds \ , \tag{7.5}$$

because of (7.1_2). In addition, with (7.3) T may be represented as

$$T = \tfrac{1}{2}\rho \iint_{S} N\,\zeta_t \cdot \zeta_t\, ds \ . \tag{7.6}$$

The potential energy Π of the liquid is given by

$$\Pi = \rho g \iiint_{\tau(t)} z\, d\tau$$

This integral can be written as

$$\rho g \iiint_{\tau(t)} z \, d\tau = \rho g \iiint_{\tau} z \, d\tau + \rho g \iiint_{\tau_1(t)} z \, d\tau$$

in which τ is the volume occupied by the liquid in the equilibrium position and $\tau_1(t)$ is the volume enclosed between the free surface $z = \zeta\,(x, y, t)$ and the plane $S\,(z=0)$. The first term in the right-hand member of this expression is the potential energy possessed by the liquid if the free surface is replaced by a "lid". If we choose the zero potential to correspond to the liquid at rest, this term may be neglected. The last term may be approximated as

$$\rho g \iiint_{\tau(t)} z \, d\tau = \rho g \iint_S ds \int_0^{\zeta} z \, dz = \frac{\rho g}{2} \iint_S \zeta^2 \, ds \; .$$

Thus, the potential energy is

$$\Pi = \frac{\rho g}{2} \iint_S \zeta^2 \, ds \; . \tag{7.7}$$

The Lagrange function, using (7.5,7), is

$$L = \tfrac{1}{2}\rho \iiint_{\tau} (\nabla \varphi)^2 \, d\tau - \tfrac{1}{2}\rho g \iint_S \zeta^2 \, ds \quad , \tag{7.8}$$

or, using (7.6,7)

$$L = \tfrac{1}{2}\rho \iint_S (N\,\zeta_t \cdot \zeta_t - g\,\zeta^2) \, ds \; . \tag{7.9}$$

With (7.7) or (7.8), the time integral may be constructed,

$$I = \int_0^t L \, dt \; , \tag{7.10}$$

and, following Hamilton, the equations of motion may also be obtained from an isochronous variation of integral (7.10),

$$\delta I = 0 \; . \tag{7.11}$$

Consider the natural oscillations of the liquid. Accordingly, let

$$\varphi\,(P,t) = \psi\,(P) \cos \sigma t \quad , \tag{7.12}$$

$$\zeta\,(P,t) = \Phi\,(P) \sin \sigma t \quad ,$$

where the natural frequency σ is to be determined. With the appropriate substitution of (7.2) in (7.1,2) and (7.4), we get

$$\Delta\psi = 0 \quad , \quad P\epsilon\tau \ ,$$

$$\frac{\partial\psi}{\partial n} = \begin{cases} 0 \quad , & P\epsilon\Sigma \ , \\ \sigma\Phi \ , & P\epsilon S \ , \end{cases}$$

$$\sigma\psi = g\Phi \ , \quad P\epsilon S \ ,$$

or

$$\sigma^2 N\Phi = g\Phi,$$

for the determination of the natural oscillations of the liquid. Except for a few isolated cases and prismatic cylinders which we shall consider shortly, closed form solutions of the above differential systems are practically impossible to get.

However, we can make use of variational methods. To this end, substitute (7.2) into (7.10), recalling that L is given by (7.8) or (7.9), and integrate over t from 0 to $2\pi/\sigma$(full cycle). This gives, after omitting a non-essential multiplicative factor,

$$I = \iiint_\tau (\nabla\psi)^2 d\tau - \lambda \iint_S \varphi^2 ds \tag{7.13}$$

$$= \lambda \iint_S N\Phi \cdot \Phi ds - \iint_S \Phi^2 ds$$

where $\lambda = \sigma^2/g$. Thus, the determination of the natural oscillations of the liquid is reduced to a variational problem for functional (7.3). According to the differential system, the solution of the extremal problem should be sought in the class of harmonic functions. It can be shown that the extremum will nevertheless coincide with the value obtained if we consider any functions $\psi \epsilon$ l_2 as admissable functions, being the class of square-summable functions.

As is known from the theory, the lowest eigenvalue λ_1 is determined from

$$\lambda_1 = \min \frac{\iiint_\tau (\nabla\psi_1)^2 d\tau}{\iint_S \psi_1^2 ds} \tag{7.14}$$

The second eigenvalue λ_2 is determined as a solution of the variational problem

$$\lambda_2 = \min \frac{\iiint_\tau (\nabla\psi_2)^2 d\tau}{\iint_S \psi_2^2 ds}$$

in the class of functions orthogonal to ψ_1, if ψ_1 is a function solving the variational problem (7.4), and so on. Orthogonality as used here refers to the metric defined in l_2, the functions themselves being defined in S.

To solve the variational problem (7.11) for the functional (7.13) we apply the method of Ritz. Accordingly, let us introduce a system of coordinate functions $\{\chi_i(P)\}$ and seek solutions in the form

$$\psi = \sum_1^N \chi_i a_i .$$

Then, proceeding from (7.13) and (7.11), we arrive at the following system of algebraic equations

$$\sum_1^N a_i \, (\alpha_{ij} - \lambda \, \beta_{ij}) = 0 \ , \quad j = 1, 2, \ldots, N, \qquad (7.15)$$

where

$$\alpha_{ij} = \iiint_\tau \nabla \chi_i \nabla \chi_j \, d\tau \, , \quad \beta_{ij} = \iint_S \chi_i \ \chi_j \, ds$$

with

$$\alpha_{ij} = \alpha_{ji} \ , \ \beta_{ij} = \beta_{ji} \ .$$

For non trivial solutions of system (7.15) to exist it is necessary and sufficient that the eigenvalue λ satisfy the determinantal equation

$$|\alpha_{ij} - \lambda \, \beta_{ij}| = 0 \ . \qquad (7.16)$$

Denote the zeros of (7.16) by λ_i $(i = 1, 2, \ldots, N)$. In view of the symmetry of matrices α_{ij} and β_{ij}, the eigenvalues λ_i are real. The natural frequencies are determined from

$$\sigma_i^2 = \lambda_i \, g \qquad (7.17)$$

The procedure outlined above can be found in detail in books on the subject.

In most space vehicle applications, the determination of the natural frequencies of an oscillation is accomplished by use of digital computers. In such cases, the method of Ritz is advantageous because it meets the requirement of simplicity in standardizing a vast number of computations. The main difficulty encountered in the practical realization of this scheme is in the selection of the coordinate functions. Many ingenious selections have been made for specific problems. However, no general recommendations are available, but in the process of solving the problem several facts should be kept in mind.

(1) The value of λ_1 is relatively insensitive to the selection of the coordinate functions χ_i. Thus, if we replace ψ_1 which produces the minimum of

the functional (7.13) by another $\widetilde{\psi}_1$ (subject to the restriction $\iiint_\tau \nabla\psi_1 \nabla\widetilde{\psi}_1 \, d\tau \neq 0$), then λ_1 will change but slightly.

(2) The boundary conditions for ψ belong to the category of natural conditions and therefore it need not be required that the functions χ_i should strictly satisfy all boundary conditions.

(3) Thus, the system of coordinate functions χ_i may be chosen rather roughly. It suffices to provide only for the completeness of the system. Consequently, many schemes now in use select the χ_i as eigenfunctions of some volume containing the volume in question but having a simpler shape.

The method of Ritz is used at MSFC to solve for the eigenvalues K_i and the corresponding eigenfunctions ψ_i of differential system (6.5), as reported in [8]. The analysis is the same as that outlined above if we replace $\lambda_i = \frac{K_i}{L}$. The solution of the inhomogeneous boundary value problem (6.3) for ψ^* utilizing the method of Ritz is also contained in [8] .

Frequently, the engineer needs an estimate of the fundamental frequency of an oscillating liquid enclosed in a slightly irregular vessel without resorting to elaborate computations. We shall describe a method for doing this which is admirably suited to space vehicle tanks.

Accordingly, consider a heavy liquid enclosed in two vessels occupying volume τ_1 and τ_2 with equal free surface area S and Σ_1 enveloping Σ_2 (see Figure 12), i.e., $\tau_1 > \tau_2$.

Recall that in a free vibration, conservation of energy requires that

$$T + \Pi = \text{constant (independent of time)}$$

In a natural free vibration, by definition, the system varies with time in accordance with a common factor $\cos \sigma t$, so that for our system

$$T = \widetilde{T}(\psi) \cos^2 \sigma t, \quad \Pi = \widetilde{\Pi}(\psi) \lambda \sin^2 \sigma t,$$

where

$$\lambda = \sigma^2/g$$

and

$$\widetilde{T}(\psi) = \iiint_\tau (\nabla\psi)^2 \, d\tau, \quad \widetilde{\Pi}(\psi) = \iint_S \psi^2 \, ds \quad .$$

Then the energy equation requires that

$$\widetilde{T} = \lambda \widetilde{\Pi} .$$

Let $\widetilde{T}_1$ and $\widetilde{T}_2$ be associated with the liquid occupying the volume τ_1 and τ_2 respectively. Thus, for any function ψ, we have

$$\widetilde{T}_1(\psi) > \widetilde{T}_2(\psi) . \tag{7.18}$$

In addition, let λ^1 and ψ^1 solve the problem for volume τ_1, and λ^2, ψ^2 for τ_2. Then

$$\frac{1}{\lambda^1} = \frac{\widetilde{\Pi}(\psi^1)}{\widetilde{T}_1(\psi^1)} , \quad \frac{1}{\lambda^2} = \frac{\widetilde{\Pi}(\psi^2)}{\widetilde{T}_2(\psi^2)} . \tag{7.19}$$

But, according to Rayleigh's principle, we have, on replacing $\psi^{(2)}$ by $\psi^{(1)}$,

$$\frac{1}{\lambda^2} \geq \frac{\widetilde{\Pi}(\psi^1)}{\widetilde{T}_2(\psi^1)}$$

Using the first expression of (7.19), we get

$$\frac{1}{\lambda^2} \geq \frac{1}{\lambda^1} \quad \frac{\widetilde{T}_1(\psi^1)}{\widetilde{T}_2(\psi^1)} .$$

However, $\widetilde{T}_1(\psi^{(1)})/\widetilde{T}_2(\psi^{(1)}) > 1$ as can be seen from (7.18) so that

$$\lambda^1 > \lambda^2 \tag{7.20}$$

Thus, if we are given two vessels with the same free surface area but such that Σ_1 of the first container envelops Σ_2 of the second container, then the corresponding natural frequencies will be greater in the vessel whose volume is larger.

To illustrate the use of this theory, consider the fundamental frequency of the oscillating liquid which in the undisturbed position occupies the volume shown in Figure 13. Using known results (see pages 7-16), we get the following inequality:

$$\tanh(h_o + h_1) > \frac{\sigma_{01}^2}{g} > \tanh h_o$$

Similarly, for the volume shown in Figure 14, we get the following approximation to the fundamental frequency:

$$\tanh(h_o + h) > \frac{\sigma_{01}^2}{g} > \tanh h_o .$$

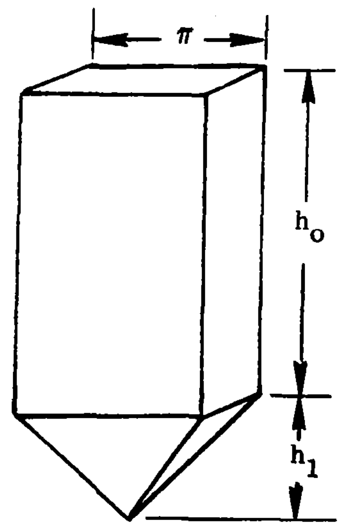


Figure 13. Slightly irregular rectangular cylinder.

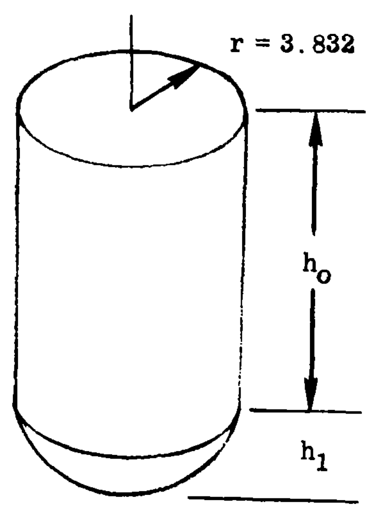


Figure 14. Slightly irregular circular cylinder.

For obvious reasons the natural oscillations of a heavy liquid enclosed in prismatic cylinders have received considerable attention in the literature. The problem is sufficiently important to warrant special attention.

With the notation of Figure 15 and the formulae from the general theory (7.1,2,), we see that the problem of linear oscillations of a heavy liquid enclosed in a fixed prismatic cylinder is reduced to one of finding, in the class of functions satisfying conditions

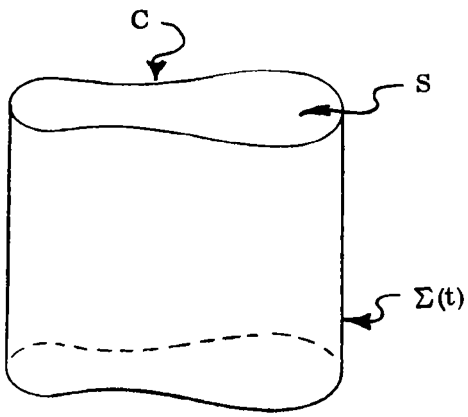


Figure 15. Prismatic cylinder.

$$v = \nabla\varphi, \ P\epsilon\tau \ , \tag{7.21}$$

$$\frac{\partial\varphi}{\partial n} = 0 \ , \ P\epsilon C \ ,$$

$$\frac{\partial\varphi}{\partial z} = 0 \ , \ P\epsilon\Sigma_2 \ , \ (z = -h) \ ,$$

$$\frac{\partial\varphi}{\partial z} = \zeta_t \ , \ P\epsilon S \ , \ (z = 0) \ ,$$

$$\frac{\partial\varphi}{\partial t} + g\,\zeta = 0 \ , \ P\epsilon S \ , \ (z = 0) \ ,$$

all possible functions $\varphi(P,t)$, $P\epsilon\tau$ satisfying

$$\Delta\varphi = 0, \ P\epsilon\tau \tag{7.22}$$

Using the method of separation of variables of Bernoulli, we find that for functions φ, ζ series

$$\varphi = \sum_1^\infty \dot{\xi}_i\,\varphi_i\,\frac{\cosh k_i\,(z+h)}{k_i \sinh k_i\,h} \ , \tag{7.23}$$

$$\zeta = \sum_1 \xi_i\,\varphi_i \ ,$$

may be constructed which will satisfy (7.21, 22) if

$$\Delta\varphi_i + k_i^2 \varphi_i = 0 \ , \ P\epsilon S \ , \qquad (7.24)$$

$$\frac{\partial\varphi_i}{\partial n} = 0 \ , \ P\epsilon C \ ,$$

and

$$\ddot{\xi}_i + \sigma_i^2 \xi_i = 0 \ , \qquad (7.25)$$

where

$$\sigma_i = (g\, k_i \tanh k_i h)^{\frac{1}{2}} \qquad (7.26)$$

is the i^{th} frequency of the natural oscillations of the liquid. Note that the φ_i , k_i are the same as those defined by the solution of differential system (3.11, 12). The properties (3.13, 15) hold.

Expansions (7.23) enable us to transform the kinetic energy and potential energy into sums of squares of the $\dot{\xi}_i$ and ξ_i respectively. Performing the transformation, i.e., substituting (7.23) into (7.5) and (7.7), we get

$$T = \frac{1}{2} \sum_1^\infty \frac{\|\varphi_i\|^2}{\sigma_i^2} \dot{\xi}_i^2 \ , \quad \Pi = \frac{1}{2} \sum_1^\infty g \|\varphi_i\|^2 \xi_i^2$$

Our variational principle (7.11) yields the familiar Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_i}\right) = - \frac{\partial \Pi}{\partial \xi_i} \ ; \ \text{hence} \ \ddot{\xi}_i = - \sigma_i^2 \xi_i \ .$$

T is a positive quadratic form and so is Π since we have a stable equilibrium. Thus, for every coordinate ξ_i, $i \neq 0$, we obtain a stable oscillation

$$\xi_i = C_i e^{\sqrt{-1}\ \sigma_i t} \ \text{with} \ \sigma_i^2 > 0 \ .$$

We see from the preceding that the determination of the natural frequencies and shapes of oscillation of a heavy liquid enclosed in a prismatic cylinder depend on the solution of the boundary value problem

$$\Delta\varphi + k^2\varphi = 0 \ , \ P\epsilon S$$

$$\frac{\partial\varphi}{\partial n} = 0, \ P\epsilon C \ .$$

We consider, summarily, solutions of this system for several boundary curves. Before proceeding, we should point out that the origin of the coordinate system for a given shape (cross-section) is not necessarily situated at the centroid of the figure. Thus, to conform to our previous assumptions the coordinate system should, in each case, be translated to the centroid of the figure.

(1) When the boundary curve C is a rectangle such as shown in Figure 16, the surface harmonics must satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) \varphi (x, y) = 0 \ ,$$

$$\frac{\partial \varphi}{\partial x} = 0 \ \text{ for } x = 0, \ x = a \ ,$$

$$\frac{\partial \varphi}{\partial y} = 0 \ \text{ for } y = 0, \ y = b \ ,$$

It is convenient to arrange the eigenfunctions and eigenvalues in a two-parameter set of solutions, and it is readily shown that

$$\varphi_{ij} = \cos \frac{i \pi x}{a} \cos \frac{j \pi y}{b} \ , \ (i, j = 0, 1, 2, \ldots) \ ,$$

$$k_{ij}^2 = \pi^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right) , \ \ (i, j = 0, 1, 2, \ldots) \ .$$

(2) When the curve is an isosceles right triangle, Figure 17, the surface harmonics must satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) \varphi (x, y) = 0 \ ,$$

$$\frac{\partial \varphi}{\partial x} = 0 \ \text{ for } x = 0 \ ,$$

$$\frac{\partial \varphi}{\partial y} = 0 \ \text{ for } y = 0 \ ,$$

$$\frac{\partial \varphi}{\partial x'} = 0 \ \text{ for } x' = a/\sqrt{2}$$

The eigenfunctions and eigenvalues obtained from the solution of this system are

$$\varphi_{ij} = \cos\left[\frac{\pi}{a} (i + j) x\right] \cos \left[\frac{\pi}{a} j y\right] + (-1)^i \cos \left[\frac{\pi}{a} (i + j)\right] \cos\left[\frac{\pi}{a} j x\right],$$

$$(i, j = 0, 1, 2, \ldots),$$

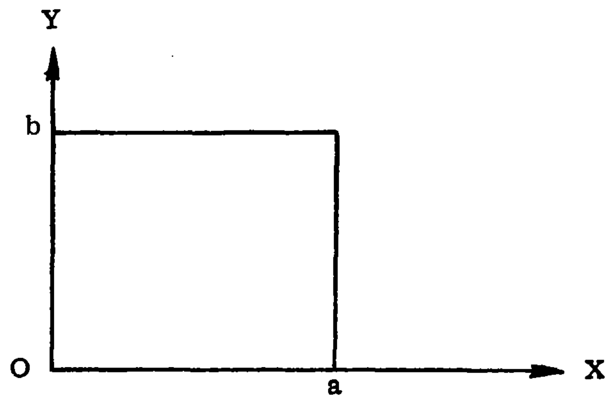


Figure 16. Rectangular boundary curve.

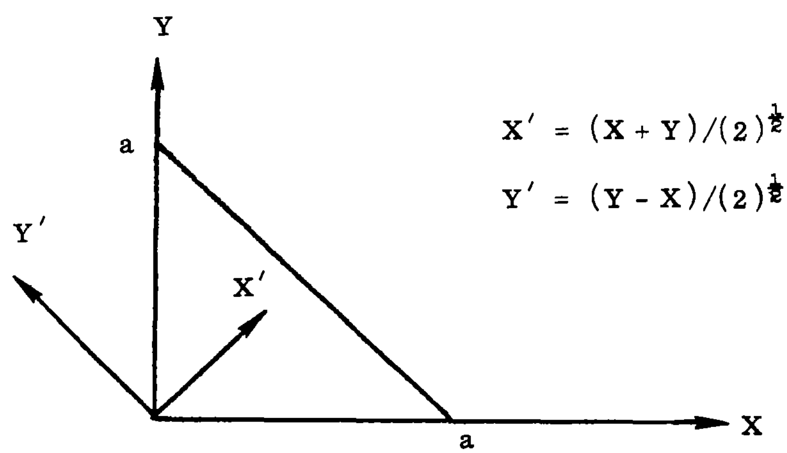


Figure 17. Isosceles right triangle boundary curve.

$$= \sin\left[\frac{\pi}{a\sqrt{2}}(i+2j)x'\right]\sin\left[\frac{\pi}{a\sqrt{2}}iy'\right] + \sin\left[\frac{\pi}{a\sqrt{2}}(i+2j)y'\right]\sin\left[\frac{\pi}{a\sqrt{2}}ix'\right],$$

$$(i = 1, 3, 5, \ldots;\ j = 0, 1, 2, \ldots),$$

$$= \cos\left[\frac{\pi}{a\sqrt{2}}(i+2j)x'\right]\cos\left[\frac{\pi}{a\sqrt{2}}iy'\right] + \cos\left[\frac{\pi}{a\sqrt{2}}(i+2j)y'\right]\cos\left[\frac{\pi}{a\sqrt{2}}ix'\right],$$

$$(i = 0, 2, 4, \ldots;\ j = 0, 1, 2, \ldots),$$

$$k_{ij}^2 = \left(\frac{\pi}{a}\right)^2\left[(i+j)^2 + j^2\right],\ (i, j = 0, 1, 2, \ldots)$$

(3) When the boundary curve is a circle, Figure 18, the surface harmonics must satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + k^2\right)\varphi(r,\theta) = 0,$$

$$\frac{\partial\varphi}{\partial r} = 0 \text{ for } r = a.$$

The solution of this system gives

$$\varphi_{ij} = J_i(k_{ij}r)\begin{matrix}\sin\\ \cos\end{matrix}\ j\theta,\quad (i = 0, 1, 2, \ldots;\ j = 1, 2, 3, \ldots)$$

$$\varphi_{oo} = \text{const.},$$

$$k_{oo} = 0,$$

$$J_i'(k_{ij}a) = 0,\quad (i = 0, 1, 2, \ldots;\ j = 1, 2, 3, \ldots).$$

k_{ij} is the j^{th} root of the derivative of Bessel's function of the first kind of order i.

(4) When the boundary curve is a circular annulus, Figure 19, the surface harmonics must satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + k^2\right)\varphi(r,\theta) = 0,$$

$$\frac{\partial\varphi}{\partial r} = 0 \text{ for } r = a,\ r = b.$$

The solution of this system gives

$$\varphi_{ij} = \left[Y_i'(k_{ij}a)J_i(k_{ij}r) - J_i'(k_{ij}a)Y_i(k_{ij}r)\right]\begin{matrix}\sin\\ \cos\end{matrix}\ j\theta,$$

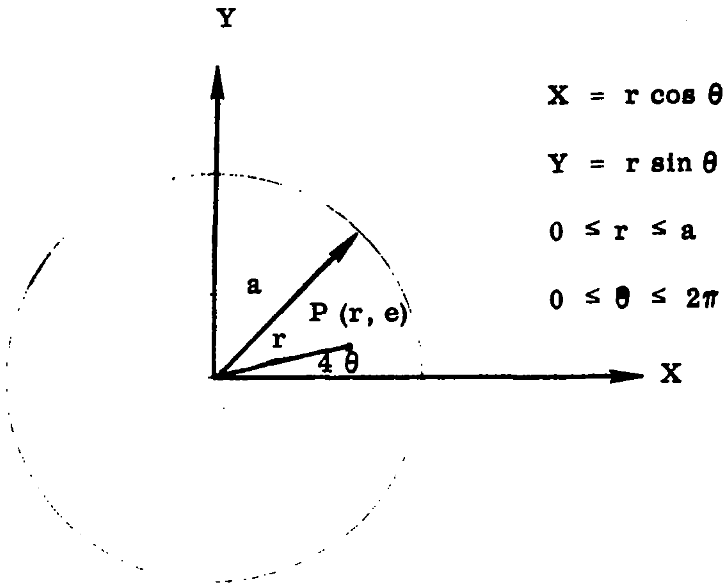


Figure 18. Circular boundary curve.

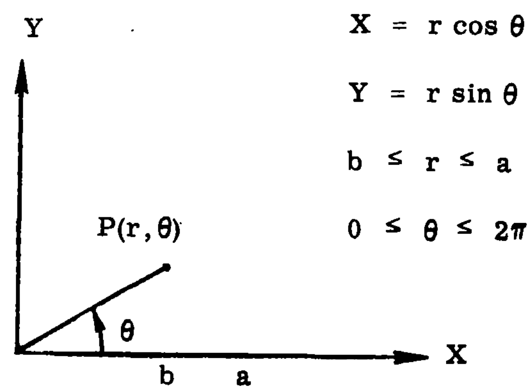


Figure 19. Circular annulus boundary curve.

$$(i,\ j = 0,\ 1,\ 2,\ \ldots;\ i \neq j = 0),$$

$$\varphi_{oo} = \text{const},$$

$$k_{oo} = 0\ ,$$

$$\begin{vmatrix} J_i'(k_{ij}\ a) & Y_i'(k_{ij}\ a) \\ J_i'(k_{ij}\ b) & Y_i'(k_{ij}\ b) \end{vmatrix} = 0\ ,\ (i,\ j = 0,\ 1,\ 2,\ \ldots;\ i \neq j = 0)\ .$$

Y_i is Bessel's function of the second kind of order i. Some zeros of the above determinant may be found in [9].

(5) When the boundary curve is a circular sector, Figure 20, the surface harmonics must satisfy

$$\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\ \frac{\partial}{\partial r} + \frac{1}{r^2}\ \frac{\partial^2}{\partial\theta^2} + k^2\right)\varphi(r,\theta) = 0$$

$$\frac{\partial\varphi}{\partial r} = 0 \text{ for } r = a\ ,$$

$$\frac{\partial\varphi}{\partial\theta} = 0 \text{ for } \theta = 0,\ \theta = \pi \text{ (note change of variable in Figure 20.)}$$

The solution is determined

$$\varphi_{ij} = J_{\frac{i}{2a}}\ (k_{ij}\ r)\cos j\,\theta,\ (i = 0,\ 1,\ 2,\ \ldots;\ j = 1,\ 2,\ 3,\ \ldots),$$

$$\varphi_{oo} = \text{const.}\ ,$$

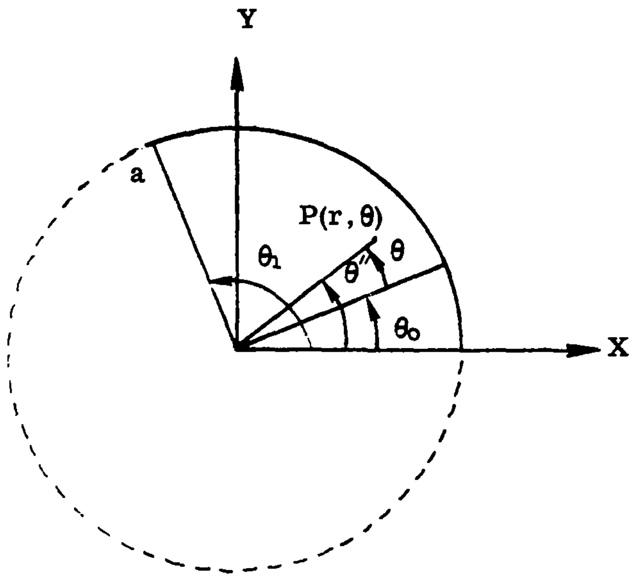
$$k_{oo} = 0\ ,$$

$$J'_{\frac{i}{2a}}\ (k_{ij}\,a) = 0,\ (i = 0,\ 1,\ 2,\ \ldots;\ j = 1,\ 2,\ 3,\ldots)\ .$$

(6) When the boundary curve is the sector of a circular annulus, Figure 21, the surface harmonics must satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial}{\partial r} + \frac{1}{(2\alpha\, r)^2}\ \frac{\partial^2}{\partial\theta^2} + k^2\right)\varphi\,(r,\ \theta) = 0\ ,$$

$$\frac{\partial\varphi}{\partial r} = 0 \text{ for } r = a,\ r = b\ ,$$



$$X = r \cos(2\alpha\theta + \theta_0)$$

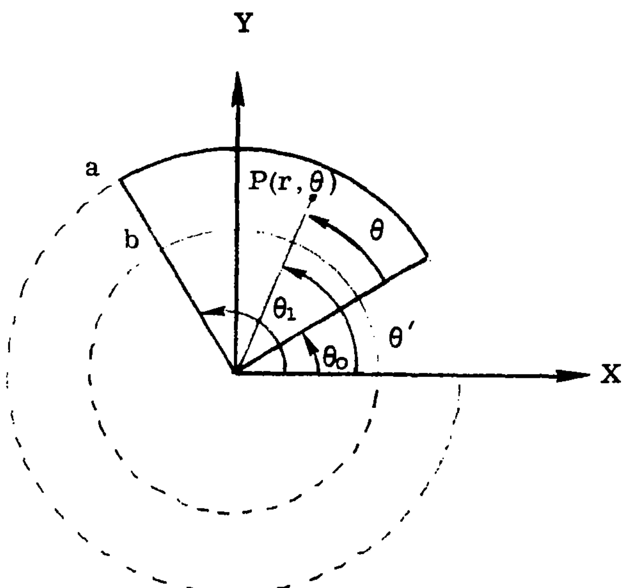
$$Y = r \sin(2\alpha\theta + \theta_0)$$

$$0 \leq r \leq a$$

$$0 \leq \theta \leq \pi$$

$$\alpha = \frac{\theta_1 - \theta_0}{2\pi}$$

Figure 20. Circular sector boundary curve.



$$X = r \cos(2\alpha\theta + \theta_0)$$

$$Y = r \sin(2\alpha\theta + \theta_0)$$

$$b \leq r \leq a$$

$$0 \leq \theta \leq \pi$$

$$\alpha = \frac{\theta_1 - \theta_0}{2\pi}$$

Figure 21. Sector of a circular annulus boundary curve.

$$\frac{\partial \varphi}{\partial \theta} = 0 \text{ for } \theta = 0,\ \theta = \pi,$$ (note change of variable in Figure 21).

The solution of this system gives

$$\varphi_{ij} = [Y'_{\frac{i}{2a}}(k \quad) J_{\frac{i}{2a}}(k_{ij} r) - J'_{\frac{i}{2a}}(k_{ij} a) Y_{\frac{i}{2a}}(k_{ij} r)] \cos j\theta$$

$$(i,\ j = 0,\ 1,\ 2,\ \ldots;\ i \neq j = 0),$$

$$\varphi_{oo} = \text{const.},$$

$$k_{oo} = 0,$$

$$\begin{vmatrix} J'_{\frac{i}{2a}}(k_{ij}\, a) & Y'_{\frac{i}{2a}}(k_{ij}\, a) \\ J'_{\frac{i}{2a}}(k_{ij}\, b) & Y'_{\frac{i}{2a}}(k_{ij}\, b) \end{vmatrix} = 0,\ (i,\ j = 0,\ 1,\ 2,\ \ldots;\ i \neq j - 0).$$

(7) When the boundary curve is comprised of two confocal parabolas such as shown in Figure 22, the surface harmonics must satisfy

$$\{\frac{1}{u^2 + v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + k^2\}\, \varphi(u,\ v) = 0$$

$$\frac{\partial \varphi}{\partial u} = 0 \text{ for } u = (2\,p)^{\frac{1}{2}}$$

$$\frac{\partial \varphi}{\partial v} = 0 \text{ for } v = \pm\ (2\,p)^{\frac{1}{2}}$$

The characteristic functions of this system are

$$\varphi_{ij} = H_e[\lambda_{ij}, \sqrt{k_{ij}}\ u] H_e[-\lambda_{ij}, \sqrt{k_{ij}}\ v],\ (i,\ j = 0,\ 2,\ 4,\ \ldots),$$

$$\varphi_{ij} = H_o[\lambda_{ij}, \sqrt{k_{ij}}\ u] H_o[-\lambda_{ij}, \sqrt{k_{ij}}\ v],\ (i,\ j = 1,\ 3,\ 5\ \ldots),$$

$$\varphi_{oo} = \text{const.},$$

While the eigenvalues are determined from

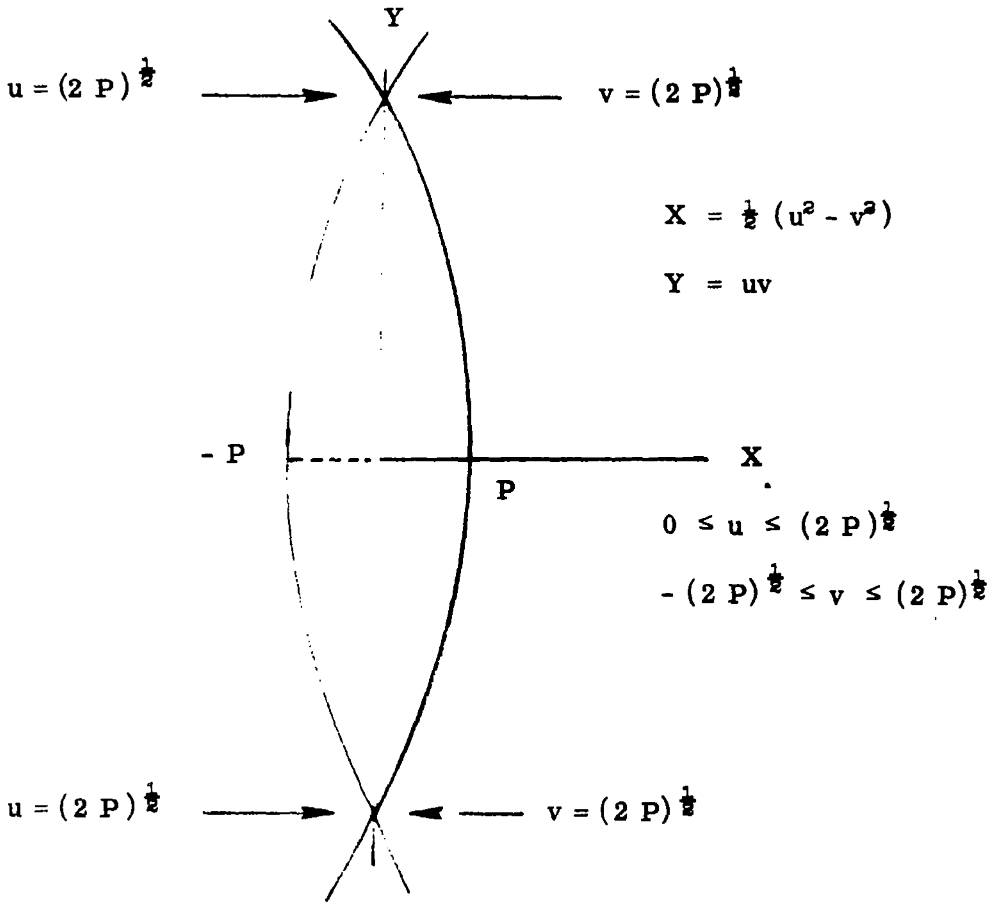


Figure 22. Two confocal parabolas boundary curve.

$$H_e'[\lambda_{ij}, \sqrt{2pk_{ij}}] = 0$$
$$H_e'[-\lambda_{ij}, \sqrt{2pk_{ij}}] = 0 \qquad (i, j = 0, 2, 4, \ldots),$$

$$H_o'[\lambda_{ij}, \sqrt{2pk_{ij}}] = 0$$
$$H_o'[-\lambda_{ij}, \sqrt{2pk_{ij}}] = 0 \qquad (i, j = 1, 3, 5, \ldots),$$

$$k_{oo} = 0,$$

H_e and H_o are defined as the sets of solutions given by the following differential systems:

$$H_e''(\lambda, x) + (\lambda + x^2) H_e(\lambda, x) = 0,$$
$$H_e(\lambda, 0) = 1, \; H_e'(\lambda, 0) = 0,$$

and

$$H_o''(\lambda, x) + (\lambda + x^2) H_o(\lambda, x) = 0,$$
$$H_o(\lambda, 0) = 0, \; H_o'(\lambda, 0) = 1,$$

respectively. Here $x^2 = ku^2$ or $x^2 = kv^2$, and λ is a separation constant (positive, zero or negative). Power series expansions for H_e and H_o are obtained by the method of Frobenius as

$$H_e(\lambda, x) = 1 - \frac{1}{2}\lambda x^2 + \frac{1}{24}(\lambda^2 - 2)x^4 - \frac{1}{720}(\lambda^3 - 14\lambda)x^6 + \ldots,$$

$$H_o(\lambda, x) = x - \frac{1}{6}\lambda x^3 + \frac{1}{120}(\lambda^2 - 6)x^5 - \frac{1}{5040}(\lambda^3 - 26\lambda)x^7 + \ldots,$$

in which non-zero coefficients of x^n are connected by

$$n(n-1)C_n + \lambda C_{n-2} + C_{n-4} = 0$$

H_e and H_o may also be expressed in terms of the confluent hypergeometric functions

$$H_e(\lambda, x) = \exp\left(-i\frac{x^2}{2}\right) F\left(\frac{1}{4} + \frac{1}{4} i\lambda \middle| \frac{1}{2} \middle| ix^2\right),$$

$$H_o(\lambda, x) = x \exp\left(-i\frac{x^2}{2}\right) F\left(\frac{3}{4} + \frac{1}{4} i\lambda \middle| \frac{3}{2} \middle| ix^2\right).$$

Some of the allowed values of k_{ij} (and λ_{ij}) are given in Figures 23 and 24 for both the even and odd quantum numbers (i, j). More information may be found in [10].

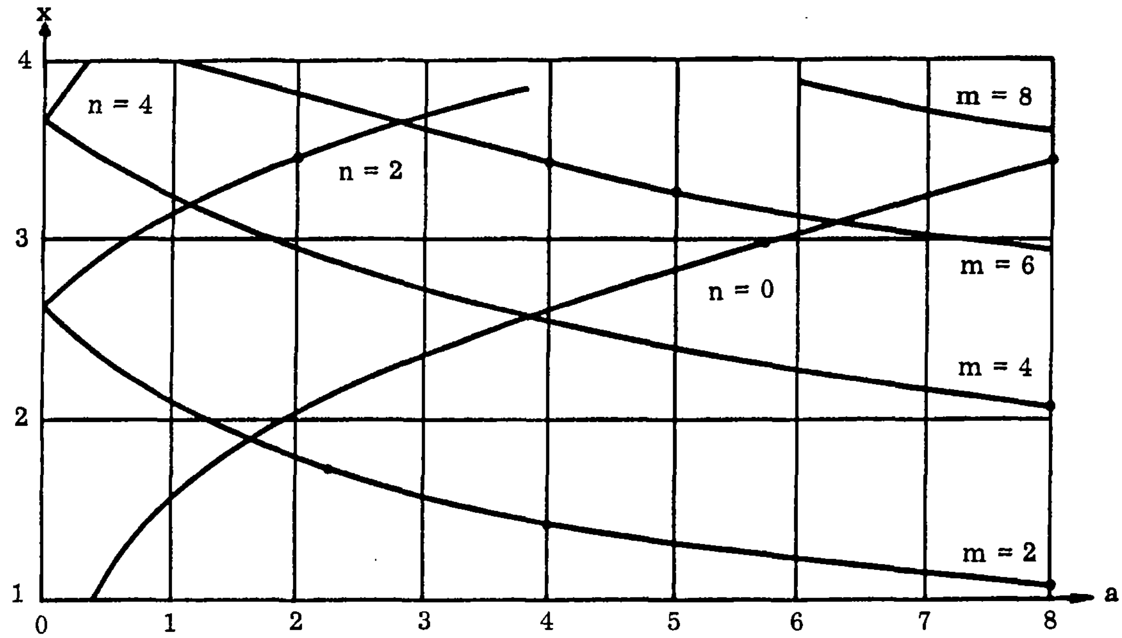


Figure 23. Solutions of $H_e'(z, x) = 0$ and $H_e'(-a, x) = 0$.

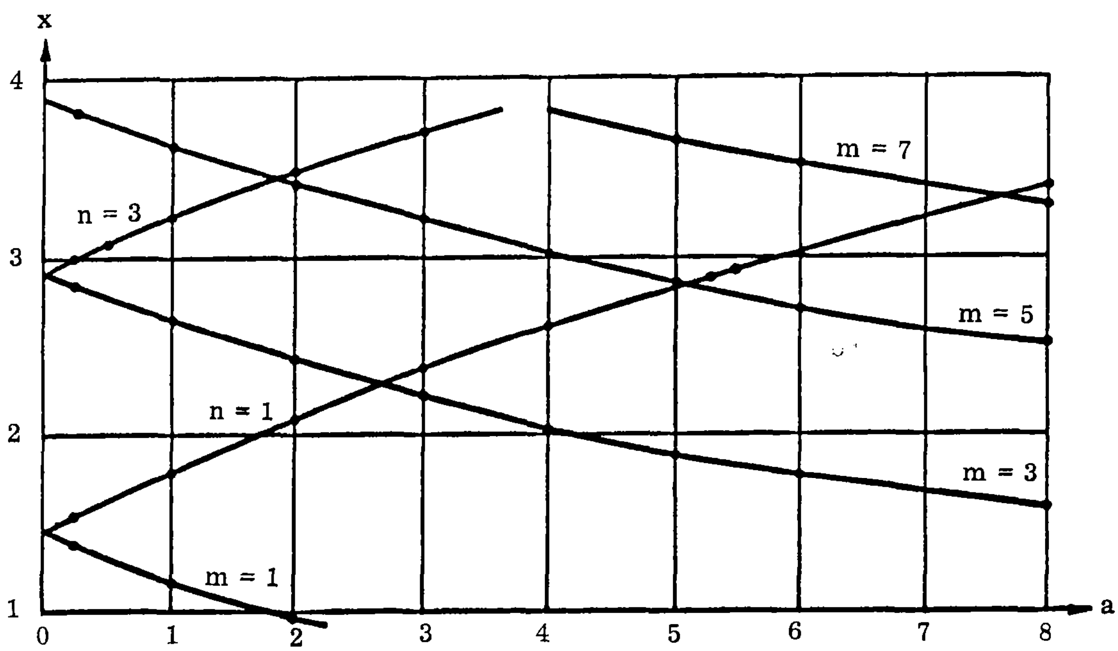


Figure 24. Solutions of $H_o'(z, x) = 0$ and $H_o'(-z, x) = 0$.

(8) **When the boundary curve is an ellipse, Figure 25, the surface harmonics must satisfy**

$$\left\{ \frac{1}{a^2(\cosh^2 u - \cos^2 v)} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + k^2 \right\} \varphi(u, v) = 0$$

$$\frac{\partial \varphi}{\partial u} = 0 \text{ for } u = u_0$$

The solution of this system gives

$$\varphi_{ij} = \varphi_{ij}^{(1)} + \varphi_{ij}^{(2)} ,$$

$$\varphi_{ij}^{(1)} = c\, e_i(v, k_{ij} \tfrac{a^2}{4})\, Ce_i(u, k_{ij} \tfrac{a^2}{4}), \; (i, j = 0, 1, 2, \ldots),$$

$$\varphi_{ij}^{(2)} = s\, e_i(v, k_{ij} \tfrac{a^2}{4})\, Se_i(u, k_{ij} \tfrac{a^2}{4}), \; (i, j = 1, 2, 3, \ldots),$$

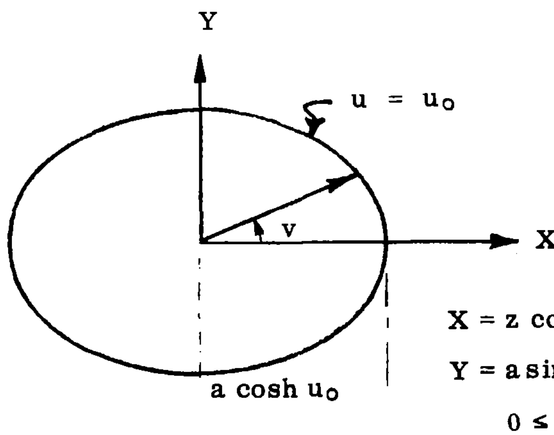
where the eigenvalues corresponding to $\varphi_{ij}^{(1)}$ and $\varphi_{ij}^{(2)}$ respectively, are determined from

$$C\dot{e}_i'(u_0, k_{ij} \tfrac{a^2}{4}) = 0, \; (i, j = 0, 1, 2, \ldots),$$

and

$$Se_i'(u_0, k_{ij} \tfrac{a^2}{4}) = 0, (i, j = 1, 2, 3, \ldots)$$

ce_i, se_i are the even and odd Mathieu functions of order i, respectively. Ce_i, Se_i are the even and odd modified Mathieu functions of order i respectively. Primes denote differentiation with respect to the argument. Some of the allowed values of k_{ij} may be obtained from [11]



$X = z \cosh u \cos v$

$Y = a \sinh u \sin v$

$0 \le u \le u_0$

$0 \le v \le 2\pi$

Figure 25. Ellipse boundary curve.

8/SMALL OSCILLATIONS OF CONSERVATIVE SYSTEMS POSSESSING LIQUID CAVITIES

SMALL OSCILLATIONS OF CONSERVATIVE SYSTEMS POSSESSING LIQUID CAVITIES

To facilitate an understanding of the general theory of small vibrations of conservative systems possessing liquid cavities, it is worthwhile first to consider several examples.

INVERTED PENDULUM PROBLEM

Let us analyze the plane vibrations of the liquid-containing body shown in Figure 26 about its position of equilibrium. The oscillating body is an open vessel, partly filled with liquid, which is attached to a fixed point o' by means of a weightless rod and a linear rotational spring.

To describe the motion of the system take two cartesian frames of reference $o'x'y'z'$ fixed at the point of suspension at a distance l below the "capped" free surface, and oxyz fixed relatively to the vessel. Reference oxyz is oriented in such a manner that oz is measured positively along the outward directed normal to the undisturbed free surface. Thus the free surface, denoted by S(t), coincides with plane xoy (the plane $z = 0$) when the vessel and liquid are at rest.

Let

$$z = \zeta\,(x, y, t)$$

be the equation of S(t) when it is displaced. Denote by $\Sigma(t)$ the wetted surface of the vessel, and by $\tau(t)$ the variable volume enclosed by $\Sigma(t)$ and S(t). Let Σ, τ and S represent the values of $\Sigma(t)$, $\tau(t)$ and S(t) in the undisturbed position. All surfaces are assumed to be piecewise smooth.

Coordinate systems oxyz and $o'x'y'z'$ are related as follows:

$$\begin{Bmatrix} x \\ y \\ z + l \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \tag{8. 1}$$

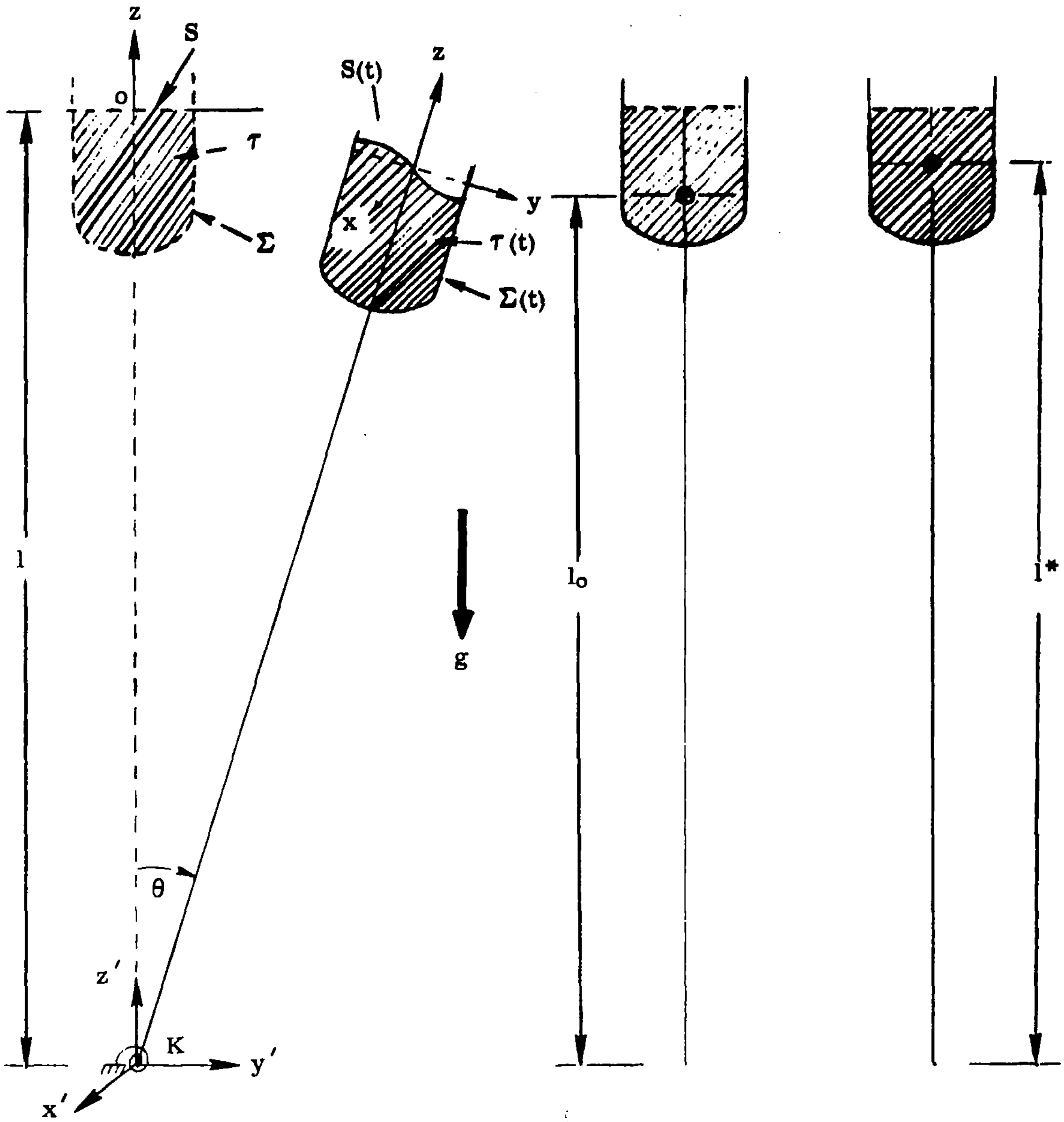


Figure 26. Liquid-containing inverted pendulum.

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z+1 \end{Bmatrix}$$

The vessel is moving relatively to inertial space with motion described by an observer in inertial space as a velocity

$$\bar{u} = (u_x', u_y', u_z') = (0, 1\dot{\theta}\cos\theta, -1\dot{\theta}\sin\theta), \quad (8.2)$$
$$= (u_x, u_y, u_z') = (0, 1\dot{\theta}, 0),$$

of $0'$ and an angular velocity

$$\bar{\omega} = (\omega_x', \omega_y', \omega_z') = (-\dot{\theta}, 0, 0) \quad (8.3)$$
$$= (\omega_x, \omega_y, \omega_z) = (-\dot{\theta}, 0, 0).$$

The velocity of an invariable point in the vessel, say P, is

$$\bar{v} = (v_x', v_y', v_z') = (0, \dot{\theta}z', -\dot{\theta}y') \quad (8.4)$$
$$= (v_x, v_y, v_z) = (0, \dot{\theta}(z+1), -\dot{\theta}y).$$

In particular, if cos (n, y), . . . denote the direction cosines of the outward directed normal to surface $\Sigma(t)$ at point P, we have

$$V_n = \dot{\theta}((1+z)\cos(n, y) - y\cos(n, z)), \quad (8.5)$$

when referred to moving axes oxyz.

Denote by $\bar{q}$ (P, t), $\bar{v}$ (P, t) the velocities of the liquid particle $P \epsilon \tau(t)$ at time t as estimated by observers in $o'x'y'z'$ and oxyz respectively. Then

$$\bar{q} = \{0, (z+1)\dot{\theta}, -\dot{\theta}y\} + \bar{v} \quad (8.6)$$

in which $\bar{q}$, $\bar{v}$ are referred to the moving reference oxyz.

Assume the liquid to be homogeneous and incompressible throughout the motion. Neglect surface and interfacial tension forces and capillary contact effects between liquid and boundary. Moreover, let the absolute motion of the liquid be irrotational. Then the motion of the system is completely described by the following formulae:

Equation for continuity of liquid

$$\bar{q} = \nabla\phi, \quad \nabla\cdot\bar{q} = 0, \quad P\epsilon\tau(t), \tag{8.7}$$

$$\Delta\phi = 0, \quad P\epsilon\,\tau(t),$$

Boundary conditions for liquid (kinematical)

$$\frac{\partial\phi}{\partial n} = \begin{cases} \dot{\theta}\,[(z+1)\cos(n, y) - y\cos(n, z)], & P\epsilon\,\Sigma(t), \\ \dot{\theta}\,[(z+1)\cos(n, y) - y\cos(n, z)] + \zeta_t\cos(n, z), & P\epsilon\,S(t), \end{cases} \tag{8.8}$$

Constancy of pressure at free surface

$$\rho\,\frac{\partial\phi}{\partial t} + \rho g\,(\zeta\cos\theta - y\sin\theta) + \tfrac{1}{2}\,\rho|\nabla\phi - \dot{\theta}(0, z+1, -y)|^2 \tag{8.9}$$

$$-\tfrac{1}{2}\,\dot{\theta}^2\,\rho\,[(\zeta+1)^2 + y^2] = 0, \quad P\epsilon S(t)$$

Equilibrium condition (sum of torques about o')

$$M_o\,(R^2 + l_o^2)\,\ddot{\theta} + K\,\theta - M_o\,g\,l_o\sin\theta + \rho\,\dot{\theta}\iiint_{\tau(t)}\{(z+1)\frac{\partial\varphi}{\partial z} + y\,\frac{\partial\varphi}{\partial y}\}\,d\tau \tag{8.10}$$

$$+\,\rho\iiint_{\tau(t)}\{(z+1)\frac{\partial}{\partial t}\,(\frac{\partial\varphi}{\partial y}) - y\,\frac{\partial}{\partial t}\,(\frac{\partial\varphi}{\partial z})\}\,d\tau + \rho\iint_{S(t)}\{(z+1)\,\frac{\partial\varphi}{\partial y} - y\,\frac{\partial\varphi}{\partial z}\}\,\zeta_t\cos(n, z)\,ds$$

$$-\,\rho\,g\sin\theta\iiint_{\tau(t)}(z+1)\,d\tau - \rho g\cos\theta\iiint_{\tau(t)} y\,d\tau = 0,$$

in which M_oR^2 is the moment of inertia of the vessel proper about an axis through the center of gravity, parallel to the axis of rotation. l_o is the distance from the center of gravity of the vessel proper to the point o'. Note that the liquid is referred to the moving frame of reference oxyz. Formulae (8.7-10) are sufficient to describe the dynamics of the complete system.

We now make certain simplifying assumptions. We suppose that the motion of the system consists of small oscillations about the equilibrium position. The deflection and slope of the free surface are presumed small. With these simplifications we arrive at the following linear description for our system:

$$q = \nabla\phi, \quad \nabla\cdot\bar{q} = 0, \quad P\epsilon\,\tau \tag{8.11}$$

$$\Delta\phi = 0, \quad P\epsilon\,\tau$$

$$\frac{\partial\phi}{\partial n} = \begin{cases} \dot\theta\,[(z+1)\cos(n,y) - y\cos(n,z)]\,, & P\epsilon\Sigma\,, \\ \dot\theta\,[(z+1)\cos(n,y) - y\cos(n,z)] + \zeta_t\,, & P\epsilon S\,, \end{cases} \tag{8.12}$$

$$\rho\,\frac{\partial\phi}{\partial t} + \rho g(\zeta - y\,\theta) = 0\,, \quad P\epsilon S\,, \tag{8.13}$$

$$M_o(R^2 + l_o^2)\,\ddot\theta + (K - M_o g l_o - \rho g \tau l^*)\,\theta + \rho \iint_{S+\Sigma} [(z+1)\cos(n,y) - y\cos(n,z)]\,\frac{\partial\phi}{\partial t}\,ds - \rho g \iint_S y\,\zeta\,ds = 0\,, \tag{8.14}$$

where l^* is the distance from the center of gravity of the undisturbed liquid to the point o' (center of rotation). In arriving at formula (8.14) the various volume integrals occurring in (8.10) were approximated as

$$\iiint_{\tau(t)} (\ldots)\,d\tau = \iiint_{\tau} (\ldots)\,d\tau + \iint_S ds \int_o^{\zeta} (\ldots)\,dz\,,$$

and directional derivatives evaluated over undisturbed surfaces.

Introduce functions φ^*, φ, harmonic in τ, such that

$$\phi = \dot\theta\,\varphi^* + \varphi \tag{8.15}$$

satisfying conditions

$$\frac{\partial\varphi^*}{\partial n} = (1+z)\cos(n,y) - y\cos(n,z)\,, \quad P\epsilon\,\Sigma,\ S \tag{8.16}$$

$$\frac{\partial\varphi}{\partial n} = \begin{cases} 0\,, & P\epsilon\Sigma\,, \\ \zeta_t\,, & P\epsilon S\,. \end{cases}$$

φ^* is the potential of Stokes for our problem, and as previously stated are determined solely from the geometry of τ (the plane S being replaced by a rigid lid).

With (8.15), the equilibrium condition (8.14) becomes

$$M_o(R^2 + l_o^2)\,\ddot\theta + (K - M_o g\,l_o - \rho g \tau l^*)\theta + \rho\ddot\theta \iint_{S+\Sigma} [(1+z)\cos(n,y) - y\cos(n,z)]\,\varphi^*\,ds + \rho \iint_{S+\Sigma} [(1+z)\cos(n,y) - y\cos(n,z)]\frac{\partial\varphi}{\partial t}\,ds - \rho g \iint_S y\,\zeta\,ds = 0\,.$$

But, from (8.16$_1$), $(1 + z)\cos(n, y) - y\cos(n, z) = \frac{\partial\varphi^*}{\partial n}$, and

$$\rho\,\ddot{\theta} \iint_{S+\Sigma} [(1 + z)\cos(n, y) - y\cos(n, z)]\varphi^*\,ds = \rho\,\ddot{\theta} \iint_{S+\Sigma} \varphi^* \frac{\partial\varphi^*}{\partial n}\,ds$$

$$\rho \iint_{S+\Sigma} [(1 + z)\cos(n, y) - y\cos(n, z)] \frac{\partial\varphi}{\partial t}\,ds = \rho \iint_{S+\Sigma} \frac{\partial\varphi^*}{\partial n}\frac{\partial\varphi}{\partial t}\,ds\,.$$

Application of Green's Theorem gives

$$\rho\,\ddot{\theta} \iint_{S+\Sigma} \varphi^* \frac{\partial\varphi^*}{\partial n}\,ds = \rho\,\ddot{\theta} \iiint_{\tau} (\nabla\varphi^*)^2\,d\tau$$

because $\Delta\varphi^* = 0$. Also, $\Delta\frac{\partial\varphi}{\partial t} = 0$, and

$$\frac{\partial}{\partial n}\left(\frac{\partial\varphi}{\partial t}\right) = \begin{cases} 0\,, & P\in\Sigma\,, \\ \zeta_{tt}\,, & P\in S\,, \end{cases}$$

we have

$$\rho \iint_{S+\Sigma} \frac{\partial\varphi}{\partial t}\frac{\partial\varphi^*}{\partial n}\,ds = \rho \iint_{S} \varphi^*\,\zeta_{tt}\,ds\,.$$

Thus we may write the equilibrium condition in the form

$$I\,\ddot{\theta} + \rho \iint_{S} \varphi^*\,\zeta_{tt}\,ds + K^2\theta - \rho g \iint_{S} y\,\zeta\,ds = 0 \tag{8.17}$$

where

$$I = M_0\,(R^2 + l_0^2) + \rho \iiint_{\tau} (\nabla\varphi^*)^2\,d\tau\,,$$

$$\varkappa^2 = K - M_0\,g\,l_0 - \rho g\,\tau\,l^*\,.$$

With (8.15), the condition of constancy of pressure at the free surface (8.13) becomes

$$\rho\,\ddot{\theta}\varphi^* + \rho\frac{\partial\varphi}{\partial t} + \rho g\,\zeta - \rho g\,y\,\theta = 0\,. \tag{8.18}$$

Functions φ and ζ appearing in system (8.17-18) are related by the kinematic condition (8.16$_2$). This relationship enables us to eliminate ζ. To this end, it is necessary to differentiate formula (8.18) with respect to t and make the substitution

(8.16_a). Hence

$$T = \tfrac{1}{2} I \dot{\theta}^2 + \rho \dot{\theta} \iiint_{\tau} (\nabla\varphi^*)(\nabla\varphi) d\tau + \tfrac{1}{2} \rho \iiint_{\tau} (\nabla\varphi)^2 \, d\tau \tag{8.21}$$

in which

$$J = M_0 (R^2 + l_0^2) + \rho \iiint_{\tau} (\nabla\varphi^*)^2 \, d\tau .$$

The integration indicated in (8.21) should be carried out over the volume which the liquid occupies in the position of equilibrium.

Next, we consider the potential energy of the system,

$$\Pi = \Pi_0 + \Pi_1 ,$$

$$\Pi_0 = \tfrac{1}{2} (K - m_0 \, g \, l_0) \, \theta^2 , \; \Pi_1 = \rho \, g \iiint_{\tau(t)} z' \, d\tau .$$

Π_0 is the potential energy of the vessel proper, and Π_1 the potential energy of the liquid. The integral appearing in the expression for Π_1 can be written as

$$\rho \, g \iiint_{\tau(t)} z' \, d\tau = \rho g \iiint_{\tau} z' d\tau + \rho g \iiint_{\tau_1(t)} z' \, d\tau$$

where τ is the volume occupied by the liquid in the equilibrium position, and $\tau_1(t)$ is the volume enclosed between the free surface $z = \zeta(x_1 , x_2 , t)$ and the plane $S \; (z = 0)$. The first integral in the right-hand member of this expression represents the potential energy of the liquid if the free surface were replaced by a rigid lid. Hence, we can write, without loss of generality,

$$\rho \, g \iiint_{\tau} z' \, d\tau = \rho \, g \, \tau \, z^{*\prime}$$

where $z^{*\prime}$ is the ordinate of the center of gravity of the liquid. But $z^{*\prime} = 1^* \cos\theta$, so that

$$\rho \, g \iiint_{\tau} z' \, d\tau = \rho \, g \, \tau \, 1^* \cos\theta \approx - \tfrac{1}{2} \rho \, g \, \tau 1^* \theta^2 + \text{const.}$$

Also,

$$\rho \, g \iiint_{\tau_1(t)} z' \, d\tau = \rho \, g \iiint_{\tau_1(t)} [(z + 1) \cos\theta - y \sin\theta] \, d\tau , \text{ (from 8.1)},$$

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial z} = \zeta_t$$

However, this leads to an artificial increase in the order of the system. This difficulty can be avoided by introducing a displacement potential determined by

$$\varphi = D_t ,$$

$$\frac{\partial D}{\partial n} = \begin{cases} 0 , & P \in \Sigma \\ \zeta , & P \in S \end{cases}$$

Using D, system (8.17-18) may be rewritten as follows:

$$I \ddot{\theta} + \rho \iint_S \varphi^* D_{z\,tt}\, ds + \varkappa^2 \theta - \rho g \iint_S D_z\, y\, ds = 0 , \tag{8.19}$$

$$\rho \ddot{\theta} \varphi^* + \rho D_{tt} - \rho g\, y\, \theta + \rho g\, D_z = 0 .$$

It is theoretically possible to eliminate the potential φ in (8.18), expressing it through the free boundary $\varphi = N\zeta_t$ as in (7.3). Accordingly, our system assumes the form

$$I \ddot{\theta} + \rho \iint_S \varphi^* \zeta_{tt}\, ds + \varkappa^2 \theta - \rho g \iint_S y\, \zeta\, ds = 0 , \tag{8.20}$$

$$\rho \ddot{\theta} \varphi^* + \rho N \zeta_{tt} + \rho g \zeta - \rho g\, y\, \theta = 0 .$$

Let us consider the variational formulation for the linear vibrations of the inverted pendulum. First we construct the Lagrange function $L = T - \Pi$ for the system. The kinetic energy is

$$T = T_0 + T_1 ,$$

$$T_0 = \tfrac{1}{2} M_0 (R^2 + l_0^2) \dot{\theta}^2 , \quad T_1 = \tfrac{1}{2} \rho \iiint_{\tau} (\nabla \phi)^2 d\tau .$$

T_0 is the kinetic energy of the vessel proper, and T_1 the kinetic energy of the vibrating liquid. With (8.15), we have

$$\nabla \phi = \dot{\theta} \nabla \varphi^* + \nabla \varphi ,$$

where φ^* is the potential of Stokes which is determined solely from the geometry of the cavity (8.16_1), and φ is the potential of wave motion in the vessel satisfying

$$\approx \rho\, g \iiint\limits_{\tau_1(t)} [(l + z) - y\, \theta\,]\, d\tau$$

$$= \rho g \iint\limits_S dS \int_0^{\zeta} [(l + z) - y\, \theta\,]\, dz$$

$$= \tfrac{1}{2} \rho g \iint\limits_S \zeta^2\, dS - \rho\, g\, \theta \iint\limits_S y\, \zeta\, dS\ .$$

The total potential energy of the system is therefore

$$\Pi = \tfrac{1}{2}\, \varkappa^2\, \theta^2 + \tfrac{1}{2}\, \rho\, g \iint\limits_S \zeta^2\, dS - \rho\, g\, \theta \iint\limits_S y\, \zeta\, dS \tag{8.22}$$

in which

$$\varkappa^2 = K - m_0\, g\, l_0 - \rho\, g\, \tau l^*\, .$$

The Lagrange function L is given as

$$L = T - \Pi = \tfrac{1}{2} I\, \dot{\theta}^2 + \rho\, \dot{\theta} \iiint\limits_{\tau} (\nabla\varphi^*)(\nabla\varphi) d\tau + \tfrac{1}{2}\, \rho \iiint\limits_{\tau} (\nabla\varphi)^2 d\tau \tag{8.23}$$

$$- \tfrac{1}{2}\, \varkappa^2\, \theta^2 - \tfrac{1}{2}\, \rho\, g \iint\limits_S \zeta^2\, dS + \rho\, g\, \theta \iint\limits_S y\, \zeta\, dS\ .$$

Following Hamilton, the equations of motion (8.17-18) are also obtained from an isochronous variation of

$$I = \int_0^t L\, dt\ .$$

Let us now consider the free oscillations of the system. To this end, we assume that

$$\theta = \theta_0 \sin \sigma t\ ,\ \varphi = \Phi \cos \sigma\, t\ ,\ \ \zeta = \Xi \sin \sigma\, t\ . \tag{8.24}$$

System (8.20) then becomes

$$(k^2 - I\, \sigma^2\,)\, \theta_0 - \rho \iint\limits_S (g\, y + \varphi^* \sigma^2)\ \Xi\, dS = 0\ , \tag{8.25}$$

$$- \theta_0\, \rho\, (g\, y + \varphi^*\, \sigma^2) + \rho\, g\, \Xi + \rho\, \sigma^2\, N\, \Xi = 0\ .$$

Assume that the natural frequencies ω_i and the principal modes of oscillations Ξ_i for the free vibration problem (stationary vessel) are known. They satisfy the obvious properties

$$N \Xi_i = \frac{g}{\omega_i^2} \Xi_i ,$$

$$(\Xi_i , \Xi_j) = \begin{cases} 0, & j \neq i , \\ |\Xi_i|^2 , & j = i , \end{cases}$$

$$(\Xi_i , \Xi_j) = \iint_S \Xi_i \Xi_j \, dS ,$$

$$|\Xi_i|^2 = \iint_S \Xi_i^2 \, dS .$$

Moreover, let us assume that

$$\Xi = \sum_i c_i \Xi_i ,$$

since Ξ_i is a system of functions complete with respect to integration over S. Substituting this expansion in (8.25$_2$), we get

$$\Xi = \frac{\theta_0}{g} \sum_i \frac{\omega_i^2}{\omega_i^2 - \sigma^2} \Xi_i \frac{\iint_S (g y + \varphi^* \sigma^2) \Xi_i \, dS}{|\Xi_i|^2} ,$$

after straightforward computation. Introducing this function in (8.25$_1$), we arrive at the following frequency equation:

$$(\varkappa^2 - I \sigma^2) = \rho / g \sum_i \frac{\omega_i^2 (a_i + b_i \sigma^2)^2}{\omega_i^2 - \sigma^2} \tag{8.26}$$

where

$$a_i = \frac{(g y , \Xi_i)}{|\Xi_i|} , \qquad b_i = \frac{(\varphi^* , \Xi_i)}{|\Xi_i|} ,$$

$$(g y , \Xi_i) = \iint_S g y \Xi_i \, dS , \text{ etc.}$$

Equation (8.26) can be solved graphically (see Figure 27). The unknown roots are the points of intersection of the straight line $Y_1 = \varkappa^2 - I \sigma^2$ with the curve

$$Y_2 = \rho / g \sum_i \frac{\omega_i^2 (a_i + b_i \sigma^2)^2}{\omega_i^2 - \sigma^2}$$

Designate by Γ_σ the resolvent operator for the operator $g\,E - \sigma^2 N$ where E is a unit operator. Then, from the second expression of (8.25),

$$\Xi = \theta_0 \Gamma_\sigma (g\,y + \varphi^* \sigma^2) \ .$$

Substituting this into the first expression of (8.25), we obtain the following equation for the determination of the natural frequencies:

$$(\varkappa^2 - I\,\sigma^2) - \rho \iint_S \Gamma_\sigma (g\,y + \varphi^*\sigma^2) \cdot (g\,y + \varphi^*\sigma^2)\, dS = 0 \ .$$

We know from the general theory of integral equations that the resolvent of an integral equation with a symmetric kernel is a meromorphic function over the whole complex plane of the parameter σ^2. All the poles of this function are simple and are the eigenvalues of the kernel. Hence our problem is reduced to finding the zeros of some meromorphic function. The inverted pendulum with the liquid possesses an enumerable set of natural frequencies. It can be seen that curve $Y_2(\sigma^2)$ has an enumerable set of poles.

If one can solve the free vibrations of a liquid in a stationary vessel, then the above process may be carried out effectively and the equation for the frequencies can be written down explicitly.

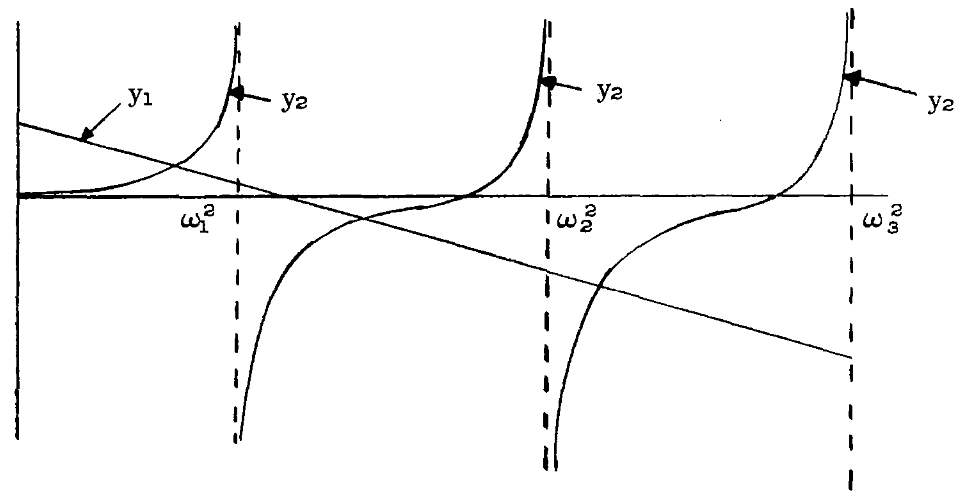


Figure 27. Typical plot of frequencies for equation (8.26).

For prismatic cylinders (Figure 28), formula (8.26) becomes, if we neglect the inertial properties of the vessel proper,

$$\left(\frac{\sigma_0}{\sigma_p}\right)^2 - \frac{\rho a b h(1-\frac{h}{2})}{I^*} - \left(\frac{\sigma}{\sigma_p}\right)^2 = \frac{\rho l}{I^*} \sum_{i=1}^{\infty} \frac{\left(\frac{\omega_i}{\sigma_p}\right)^2}{\left(\frac{\omega_i}{\sigma_p}\right)^2 - \left(\frac{\sigma}{\sigma_p}\right)^2} \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left\{1 + \left(\frac{\sigma}{\sigma_p}\right)^2 \cdot \right.$$

$$\left. \left[1 - \frac{2}{lk_i} \tanh k_i h\right]\right\}^2 \tag{8.27}$$

in which the surface harmonics φ_i and eigenvalues k_i are obtained from Section 7. Here

$$\sigma_0^2 = \frac{K}{I^*}, \quad \sigma_p = g/l, \tag{8.28}$$

$$I^* = \tilde{I} - 4\rho \iiint_\tau y^2\, d\tau + 8\rho \sum_{i=1}^{\infty} \frac{\tanh k_i h}{k_i} \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2},$$

$$\tilde{I} = \rho \iiint_\tau [y^2 + (l+z)^2]\, d\tau .$$

In particular, for a rectangular cylinder we have

$$Y_1\left[\left(\frac{\sigma}{\sigma_p}\right)^2\right] = \left(\frac{\sigma_0}{\sigma_p}\right)^2 - \frac{1 - \frac{1}{2}\left(\frac{h}{l}\right)}{1 - \left(\frac{h}{l}\right) + \frac{1}{3}\left(\frac{h}{l}\right)^3 - \frac{1}{4}\left(\frac{a}{l}\right)^2 \frac{64}{\pi^5}\left(\frac{a}{h}\right)\left(\frac{a}{l}\right)^2 \sum_{1,3,\ldots}^{\infty} \cdot \frac{1}{i^5} \tanh \frac{i\pi h}{a}}$$

$$- \left(\frac{\sigma}{\sigma_p}\right)^2,$$

$$Y_2\left[\left(\frac{\sigma}{\sigma_p}\right)^2\right] = \frac{8/\pi^4}{\left(\frac{l}{a}\right)\left(\frac{h}{a}\right) - \left(\frac{h}{a}\right)^2 - \frac{1}{4}\left(\frac{h}{l}\right) + \frac{1}{3}\left(\frac{h}{a}\right)^2\left(\frac{h}{l}\right) + \frac{64}{\pi^5}\left(\frac{a}{l}\right) \sum_{1,3,\ldots}^{\infty} \frac{1}{i^5} \tanh \frac{i\pi h}{a}} \cdot$$

$$\sum_{1,3,5,\ldots}^{\infty} \frac{1}{i^4} \frac{\left(\frac{\omega_i}{\sigma_p}\right)^2}{\left(\frac{\omega_i}{\sigma_p}\right)^2 - \left(\frac{\sigma}{\sigma_p}\right)^2} \left\{1 + \left(\frac{\sigma}{\sigma_p}\right)^2 \left[1 - \frac{2}{\pi}\left(\frac{a}{l}\right) \tanh \frac{i\pi h}{u}\right]\right\}^2 . \tag{8.29}$$

The solution of (8.27) for l/h = 1/2 (Figure 29) is shown in Figure 30.

A frequency equation similar to (8.26) may be obtained for the plane vibrations of the liquid-containing pendulum shown in Figure 31.

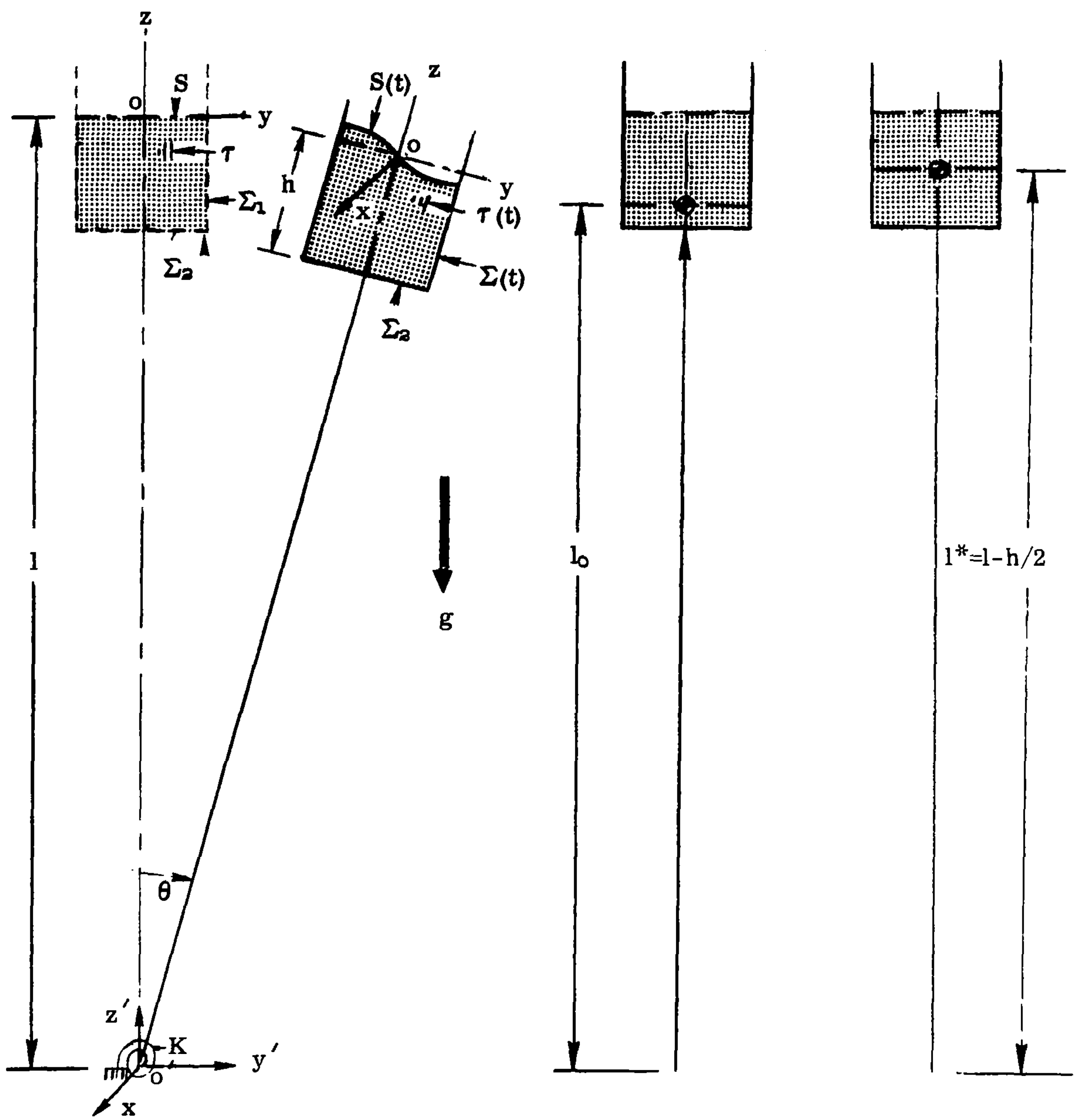


Figure 28. Inverted pendulum in the shape of a prismatic cylinder.

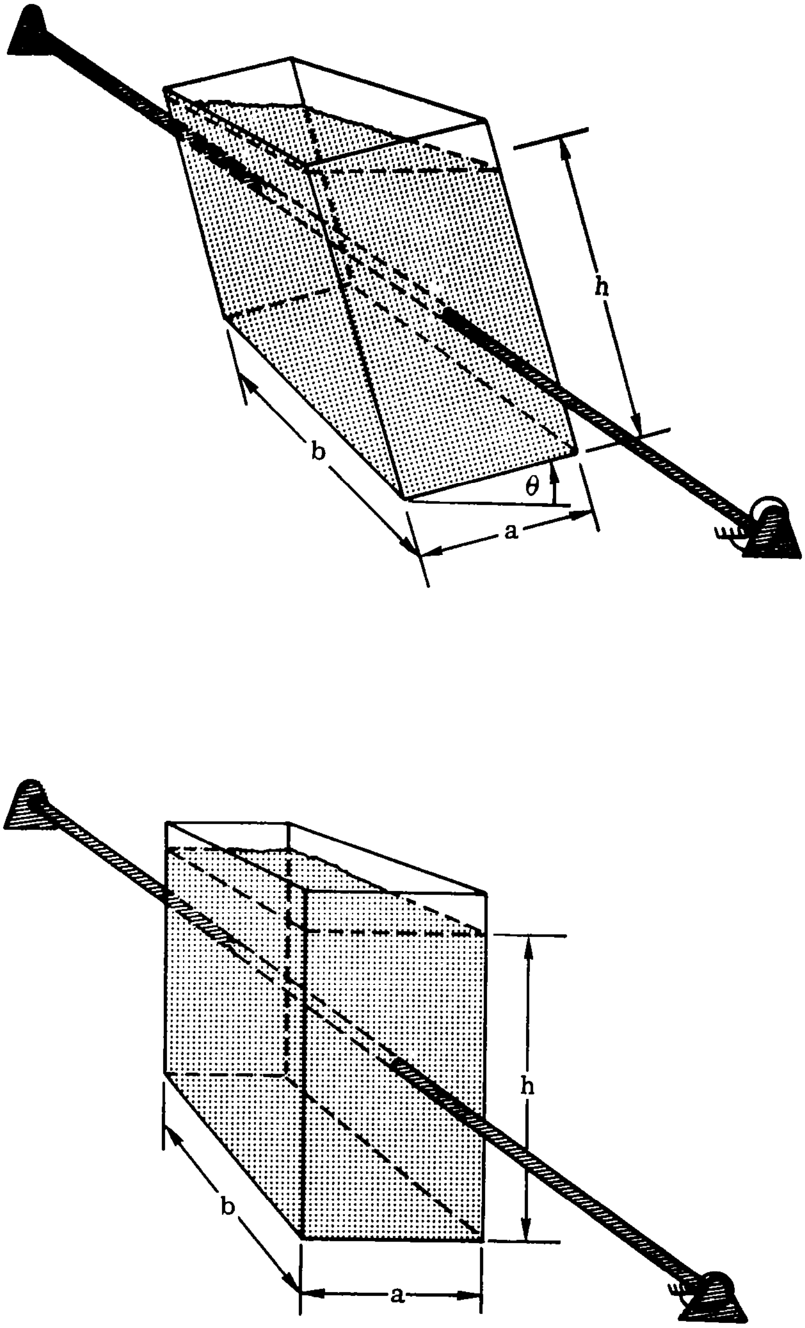


Figure 29. Rectangular cylinder with l/h = 1/2.

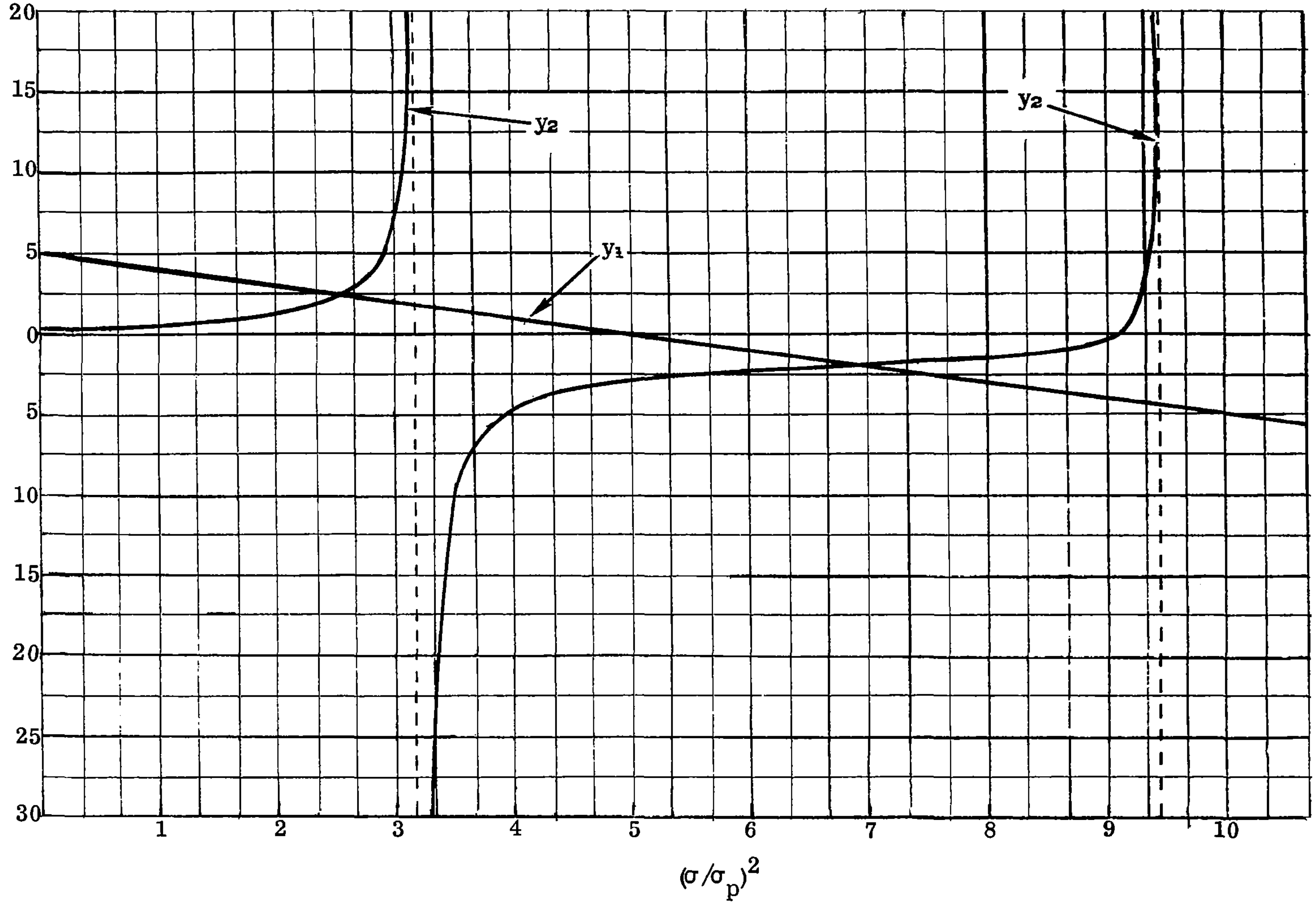


Figure 30. Solution of equation (8.27) when $l/h = 1/2$ (for $\sigma_o/\sigma_p = 5$).

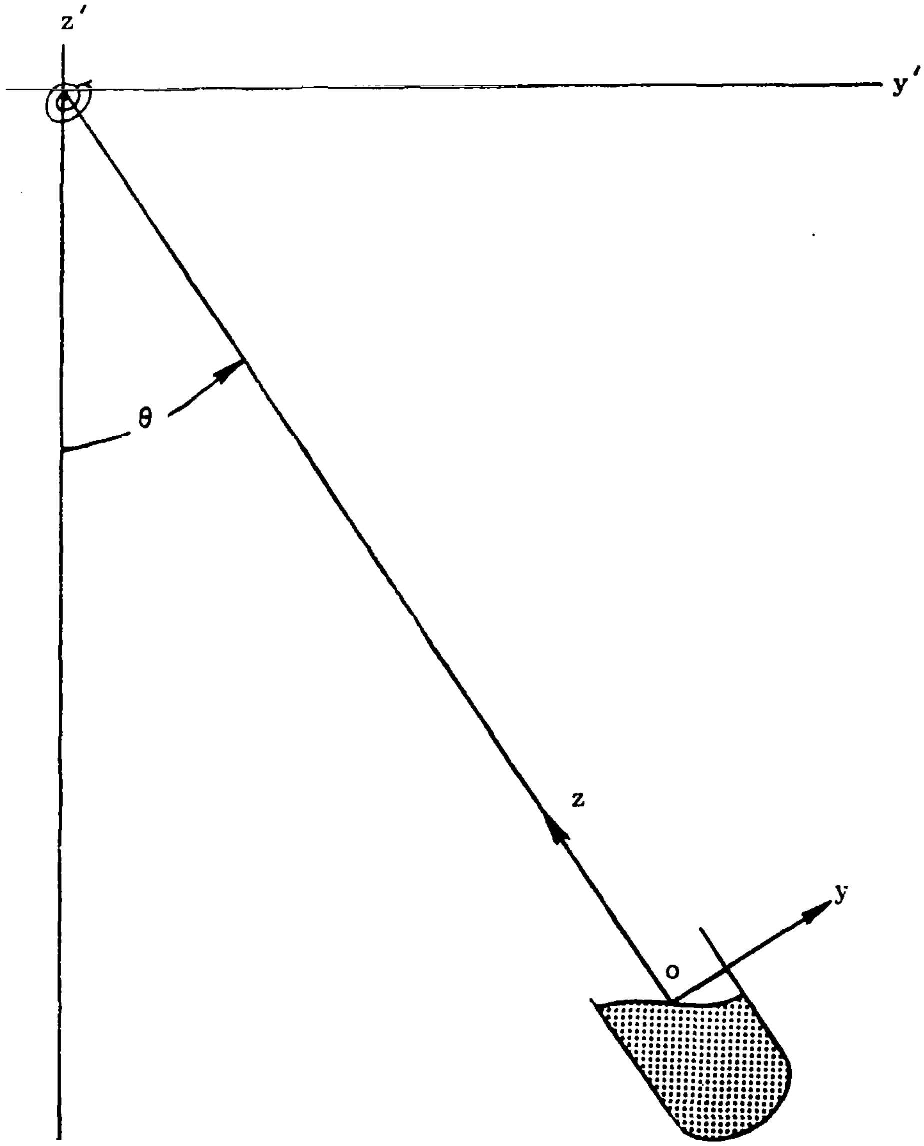


Figure 31. Liquid-containing plane pendulum.

To be sure, it can be shown that the required system of equations for this problem is

$$I \ddot{\theta} + \varkappa^2 \theta + \rho \iint_S \varphi^* \zeta_{tt} \, dS + \rho g \iint_S y \zeta \, dS = 0 , \tag{8.30}$$

$$\rho \varphi^* \ddot{\theta} + \rho N \zeta_{tt} + \rho g \zeta + \rho g y \theta = 0 ,$$

in which

$$I = M^\circ(R^2 + l_0^2) + \rho \iiint_\tau (\nabla \varphi^*)^2 \, d\tau ,$$

$$\varkappa^2 = M^\circ g l_0 + \rho g \tau l^* ,$$

with

$$\Delta \varphi^* = 0, \quad P \epsilon \tau$$

$$\frac{\partial \varphi^*}{\partial n} = (1 - z) \cos (n, y) + y \cos (n, z) , \quad P \epsilon S, \Sigma,$$

and

$$\varphi = N \zeta_t$$

$$\Delta \varphi = 0, \quad P \epsilon \tau ,$$

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 0, & P \epsilon \Sigma , \\ \zeta_t , & P \epsilon S . \end{cases}$$

Assuming simple harmonic motion as before and repeating similar arguments to those used for the inverted pendulum problem, we find

$$\varkappa^2 - I \sigma^2 = \rho / g \sum_i \frac{\omega_i^2 (a_i + b_i \sigma^2)^2}{\omega_i^2 - \sigma^2} , \tag{8.31}$$

$$a_i = \frac{(g y, \Xi_i)}{\| \Xi_i \|} , \quad b_i = - \frac{(\varphi^*, \Xi_i)}{\| \Xi_i \|} ,$$

exhibiting the same form as (8.26).

The plane vibrations of the spring-mass system sketched in Figure 32 gives rise to a somewhat similar frequency equation.

Indeed, the system of equations is given by

$$m\,\ddot{q} + K\,q + \rho \iint_S \varphi^* \zeta_{tt}\,dS = 0\,, \tag{8.32}$$

$$\rho\,\varphi^*\,\ddot{q} + \rho\,N\,\zeta_{tt} + \rho\,g\,\zeta = 0\,,$$

in which

$$m = M^{\circ} + \rho \iiint_{\tau} (\nabla\varphi^*)^2\,d\tau$$

with

$$\Delta\,\varphi^* = 0,\ P\epsilon\,\tau$$

$$\frac{\partial\varphi^*}{\partial n} = \cos\,(n,\ y)\ P\epsilon\,\Sigma,\ S,$$

and

$$\varphi = N\,\zeta_t\ ,$$

$$\Delta\varphi = 0,\ P\epsilon\,\tau$$

$$\frac{\partial\varphi}{\partial n} = \begin{cases} 0, & P\epsilon\,\Sigma \\ \zeta_t\ , & P\epsilon\,S \end{cases}$$

Assume that

$$q = Q\,\sin\sigma\,t,\ \zeta = \Xi\,\sin\sigma\,t\ ,$$

$$\Xi = \sum_i c_i\,\Xi_i\ .$$

Then, it follows that the frequency equation for the system is

$$(K - M\,\sigma^2) = \frac{\rho}{g} \sum_i \frac{\omega_i^2 (b_i\,\sigma^2)^2}{\omega_i^2 - \sigma^2}\ ,$$

$$b_i = \frac{(\varphi^*,\ \Xi_i)}{\|\Xi_i\|}$$

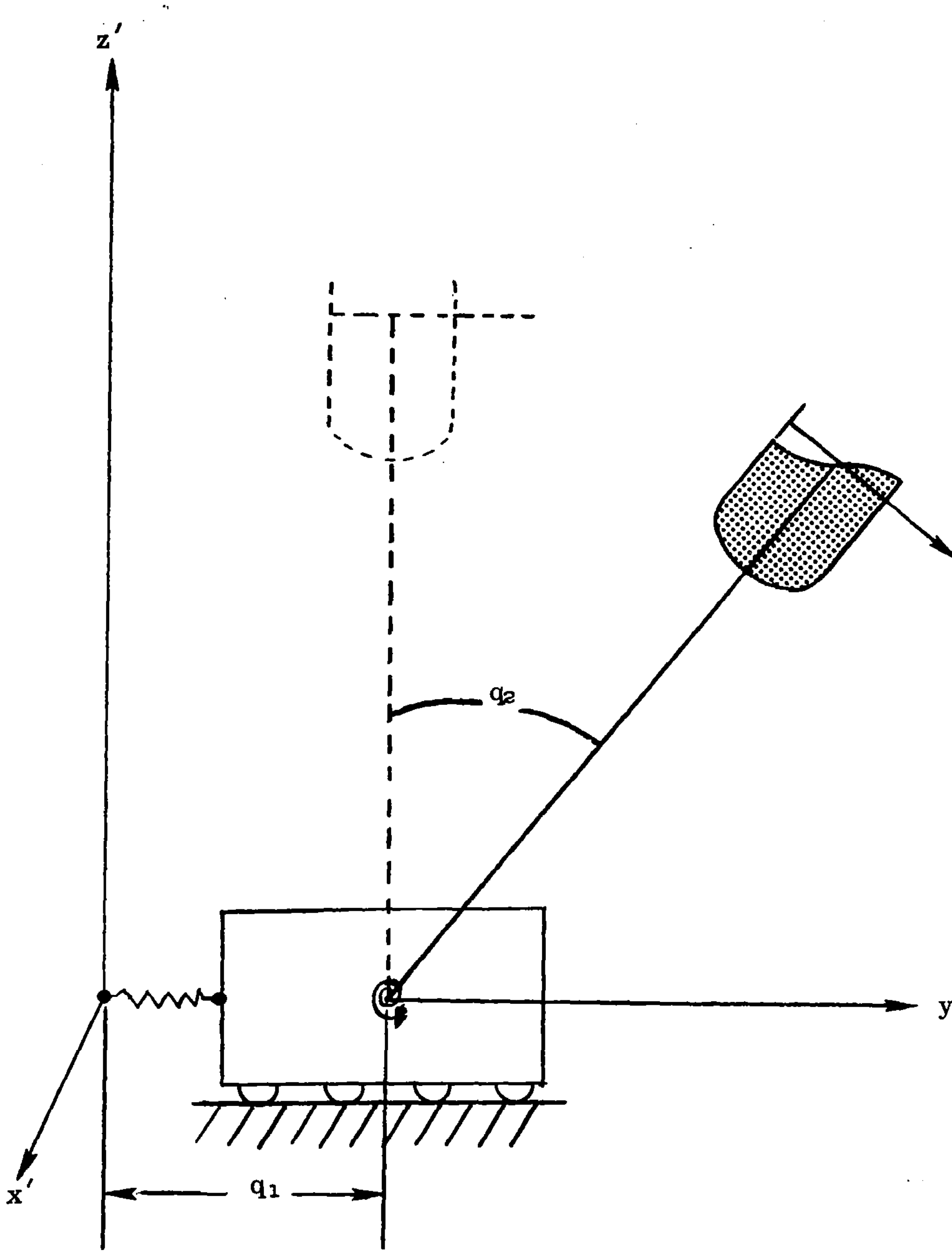


Figure 32. Inverted liquid-containing pendulum and spring-mass system.

PLANE VIBRATIONS OF A CONSTRAINED LAUNCH VEHICLE

Consider now the plane vibrations of the liquid-containing body illustrated in Figure 33 about its position of equilibrium. The oscillating body is an open vessel, partly filled with a heavy liquid, which is connected to a mount by means of a rigid weightless rod and linear rotational spring. The mount, in turn, is constrained to move horizontally by means of a linear translational spring. When viewed as a rigid body, the system has two degrees of freedom. Such an idealization has been used to approximate the vibrations of launch vehicles in a constrained condition (attached to the launcher). To simplify matters we consider only one tank in the following. The results are quite general and can be extended to two or more tanks.

To describe the motion of the system take two cartesian frames of reference $o'x'y'z'$ fixed at the point of suspension in the equilibrium position, and oxyz fixed relatively to the vessel. Reference oxyz is oriented in such a manner that oz is measured positively along the outward-directed normal to the undisturbed free surface. Thus the free surface, denoted by S(t), coincides with plane xoy (the plane $z = 0$) when the vessel and liquid are at rest.

As before let

$$z = \zeta\,(x,\ y,\ t)$$

be the equation of S(t) when it is displaced. Denote by $\Sigma(t)$ the wetted surface of the vessel, and by $\tau(t)$ the variable volume enclosed by $\Sigma(t)$ and S(t). Let Σ, τ and S represent the values of $\Sigma(t)$, $\tau(t)$ and S(t) in the undisturbed position. All surfaces are assumed to be piece-wise smooth.

Coordinate systems oxyz and $o'x'y'z'$ are related as follows:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_2 & -\sin q_2 \\ 0 & \sin q_2 & \cos q_2 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} - \begin{Bmatrix} 0 \\ q_1 \cos q_2 \\ q_1 \sin q_2 + l \end{Bmatrix}, \tag{8.33}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_2 & \sin q_2 \\ 0 & -\sin q_2 & \cos q_2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ (l+z) \end{Bmatrix} + \begin{Bmatrix} 0 \\ q_1 \\ 0 \end{Bmatrix}$$

with l the distance from the center of rotation to the undisturbed free surface.

The vessel is moving relatively to inertial space with motion described by an observer in inertial space as a velocity

$$\bar{u} = (u_x', u_y', u_z') = (0, \dot{q}_1 + l\,\dot{q}_2 \cos q_2, -l\,\dot{q}_2 \sin q_2) \qquad (8.34)$$

$$= (u_x, u_y, u_z) = (0, \dot{q}_1 \cos q_2 + l\,\dot{q}_2, \dot{q}_1 \sin q_2)$$

of $0'$ and angular velocity,

$$\bar{\omega} = (\omega_x', \omega_y', \omega_z') = (-\dot{q}_2, 0, 0) \qquad (8.35)$$

$$= (\omega_x, \omega_y, \omega_z) = (-\dot{q}_2, 0, 0)$$

The velocity of an invariable point in the vessel, say P, is

$$\bar{V} = (V_x', V_y', V_z') = (0, \dot{q}_1 + \dot{q}_2\, l \cos q_2 + \dot{q}_2\, z', -\dot{q}_2\, l \sin q_2 - \dot{q}_2\, y'), \qquad (8.36)$$

$$= (V_x, V_y, V_z) = (0, \dot{q}_1 \cos q_2 + l\,\dot{q}_2 + z\,\dot{q}_2, \dot{q}_1 \sin q_2 - \dot{q}_2\, y) .$$

In particular, if cos (n, y), . . . denote the direction cosines of the outward directed normal to surface $\Sigma(t)$ at point P, we have

$$V_n = \dot{q}_1 \left[\cos (n, y) \cos q_2 + \cos (n, z) \sin q_2\right] \qquad (8.37)$$

$$+ q_2'\left[(l + z) \cos (n, y) - y \cos (n, z)\right]$$

when referred to moving axes oxyz.

Assume the liquid to be homogeneous and incompressible throughout the motion. Neglect surface and interfacial tension forces and capillary contact effects between liquid and boundary. Moreover, let the absolute motion of the liquid be irrotational. Then the motion of the system is completely described by the following formulae:

Equation for continuity of liquid

$$\bar{q} = \nabla \phi, \quad \nabla \cdot \bar{q} = 0, \quad P \in \tau(t), \qquad (8.38)$$

$$\Delta \phi = 0, \quad P \in \tau(t),$$

Boundary conditions for liquid (kinematical)

$$\frac{\partial \phi}{\partial n} = \begin{cases} \dot{q}_1 [\cos(n, y) \cos q_2 + \cos(n, z) \sin q_2] \\ + \dot{q}_2 [(1 + z) \cos(n, y) - y \cos(n, z)], \quad P \in \Sigma(t) \\ \dot{q}_1 [\cos(n, y) \cos q_2 + \cos(n, z) \sin q_2] \\ + \dot{q}_2 [(1 + z) \cos(n, y) - y \cos(n, z)] \\ \quad + \zeta_t \cos(n, z), \quad P \in S(t) \end{cases} \tag{8.39}$$

Constancy of pressure at free surface

$$\rho \frac{\partial \phi}{\partial t} + \rho g (\zeta \cos q_2 - y \sin q_2) + \frac{\rho}{2} v^2 - \frac{\rho}{2} V^2 = 0, \quad P \in S(t) \tag{8.40}$$

$$v = |\nabla \phi - \bar{V}|$$

Equilibrium conditions

$$M^\circ \ddot{q}_1 + M^\circ l_0 \ddot{q}_2 + K_t q_1 - F_y' = 0, \tag{8.41}$$

$$M^\circ l_0 \ddot{q}_1 + M^\circ (R^2 + l_0^2) \ddot{q}_2 + K_r q_2 - M^\circ l_0 g \sin q_2 + M_x' = 0 .$$

F_y' and M_x' denote the force and moment produced by the liquid motion along the y'-axis and about the center of rotation respectively.

Utilizing the same arguments presented previously, we arrive at the following linear description of our system

$$\Delta \phi = 0, \quad P \in \tau \tag{8.42}$$

$$\frac{\partial \phi}{\partial n} = \begin{cases} \dot{q}_1 \cos(n, y) + \dot{q}_2 [(1 + z) \cos(n, y) - y \cos(n, z)], \quad P \in \Sigma, \\ \dot{q}_1 \cos(n, y) + \dot{q}_2 [(1 + z) \cos(n, y) - y \cos(n, z)] + \zeta_t, \quad P \in S, \end{cases}$$

$$\rho \frac{\partial \phi}{\partial t} + \rho g \zeta - \rho g y q_2 = 0, \quad P \in S$$

$$M^\circ \ddot{q}_1 + M^\circ l_0 \ddot{q}_2 + K_t q_1 - F_y' = 0 \tag{8.43}$$

$$M^\circ l_0 \ddot{q}_1 + M^\circ (R^2 + l_0^2) \ddot{q}_2 + (K_r - M^\circ l_0 g) q_2 + M_x' = 0$$

$$F_{y'} = -\rho \iiint_{\tau} \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) d\tau = -\rho \iint_{S+\Sigma} \frac{\partial \phi}{\partial t} \cos(n, y)\, dS \tag{8.44}$$

$$M_{x'} = \rho \iiint_{\tau} \left[(l + z)\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) - y\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right] d\tau$$

$$- \rho g \iint_{S} y \zeta\, dS - \rho g \tau l^* q_2$$

$$= \rho \iint_{S+\Sigma} \left[(l + z)\cos(n, y) - y \cos(n, z)\right] \frac{\partial \phi}{\partial t}\, dS$$

$$- \rho g \iint_{S} y \zeta\, dS - \rho g \tau l^* q_2 .$$

l^* is the distance from the undisturbed center of gravity of the liquid proper to the center of rotation.

Similarly, introduce functions φ_1^*, φ_2^*, φ, harmonic in τ, such that

$$\phi = \dot{q}_1 \varphi_1^* + \dot{q}_2 \varphi_2^* + \varphi \tag{8.45}$$

satisfying conditions

$$\frac{\partial \varphi_1^*}{\partial n} = \cos(n, y), \quad \frac{\partial \varphi_2^*}{\partial n} = \cos(n, y)(l + z) - \cos(n, z)\, y, \quad P\epsilon\Sigma, S,$$

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 0, & P\epsilon\Sigma , \\ \zeta_t , & P\epsilon S . \end{cases} \tag{8.46}$$

φ_i^* are the potentials of Stokes for our problem. They are determined solely from the geometry of τ , the plane S being replaced by a rigid lid.

With (8.45), the force expression (8.43_1) becomes

$$F_{y'} = \rho \ddot{q}_1 \iint_{\Sigma+S} \varphi_1^* \cos(n, y)\, dS - \rho \ddot{q}_2 \iint_{\Sigma+S} \varphi_2^* \cos(n, y)\, dS - \rho \iint_{\Sigma+S} \frac{\partial \varphi}{\partial t} \cos(n, y)\, dS$$

But, from (8.46), $\cos(n, y) = \dfrac{\partial \varphi_1^*}{\partial n}$, and

$$- \rho \ddot{q}_1 \iint_{\Sigma+S} \cos(n, y)\, \varphi_1^*\, dS = -\rho \ddot{q}_1 \iint_{\Sigma+S} \frac{\partial \varphi_1^*}{\partial n} \varphi_1^*\, dS ,$$

$$- \rho \ddot{q}_2 \iint_{\Sigma+S} \cos(n, y)\, \varphi_2^*\, dS = -\rho \ddot{q}_2 \iint_{\Sigma+S} \frac{\partial \varphi_1^*}{\partial n} \varphi_2^*\, dS ,$$

$$- \rho \iint_{\Sigma+S} \frac{\partial \varphi}{\partial t} \cos (n, y) \, dS = - \rho \iint_{\Sigma+S} \frac{\partial \varphi_1^*}{\partial n} \frac{\partial \varphi}{\partial t} \, dS .$$

Application of Green's theorem gives

$$- \rho \, \ddot{q}_1 \iint_{\Sigma+S} \frac{\partial \varphi_1^*}{\partial n} \varphi_1^* \, dS = - \rho \, \ddot{q}_1 \iiint_{\tau} \nabla \varphi_1^* \, \nabla \varphi_1^* \, d\tau ,$$

$$- \rho \, \ddot{q}_2 \iint_{\Sigma+S} \frac{\partial \varphi_1^*}{\partial n} \varphi_2^* \, dS = - \rho \, \ddot{q}_2 \iiint_{\tau} \nabla \varphi_1^* \, \nabla \varphi_2^* \, d\tau ,$$

because $\Delta \varphi_1^* = \Delta \varphi_2^* = 0$. Also since $\Delta \frac{\partial \varphi}{\partial t} = 0$ and

$$\frac{\partial}{\partial n} \left(\frac{\partial \varphi}{\partial t}\right) = \begin{cases} 0 & , \ P \epsilon \Sigma , \\ \zeta_{tt} & , \ P \epsilon S , \end{cases}$$

we have

$$- \rho \iint_{\Sigma+S} \frac{\partial \varphi_1^*}{\partial n} \frac{\partial \varphi}{\partial t} \, dS = - \rho \iint_{S} \varphi_1^* \zeta_{tt} \, dS .$$

Thus, the expression for the force (8.43_1) takes the form

$$F_2' = \ddot{q}_1 \, \rho \iiint_{\tau} \nabla \varphi_1^* \, \nabla \varphi_1^* \, d\tau - q_2 \, \rho \iiint_{\tau} \nabla \varphi_1^* \, \nabla \varphi_2^* \, d\tau - \rho \iint_{S} \varphi_1^* \zeta_{tt} \qquad (8.47)$$

Again, with (8.45), the moment expression (8.43_2) becomes

$$M_x' = \rho \, \ddot{q}_1 \iint_{\Sigma+S} [(l + z) \cos (n, y) - y \cos (n, z)] \varphi_1^* \, dS + \rho \, \ddot{q}_2 \iint_{\Sigma+S} [(l + z) \cos (n, y) - y \cos (n, z)] \varphi_2^* dS$$

$$+ \rho \iint_{\Sigma+S} [(l + z) \cos (n, y) - y \cos (n, z)] \frac{\partial \varphi}{\partial t} \, dS - \rho \, g \iint_{S} y \, \zeta \, dS - \rho \, \tau \, g \, l^* q_2 .$$

However, from (8.46), $(l + z) \cos (n, y) - y \cos (n, z) = \frac{\partial \varphi_2^*}{\partial n}$, and

$$\rho \, \ddot{q}_1 \iint_{\Sigma+S} [(l + z) \cos (n, y) - y \cos (n, z)] \varphi_1^* dS = \rho \, \ddot{q}_1 \iint_{\Sigma+S} \frac{\partial \varphi_2^*}{\partial n} \varphi_1^* \, dS = \rho \, \ddot{q}_1 \iiint_{\tau} \nabla \varphi_2^* \, \nabla \varphi_1^* \, d\tau ,$$

$$\rho \, \ddot{q}_2 \iint_{\Sigma+S} [(l + z) \cos (n, y) - y \cos (n, z)] \varphi_2^* \, dS = \rho \, \ddot{q}_2 \iint_{\Sigma+S} \frac{\partial \varphi_2^*}{\partial n} \varphi_2^* \, dS = \rho \, \ddot{q}_2 \iiint_{\tau} \nabla \varphi_2^* \, \nabla \varphi_2^* \, d\tau .$$

Also,

$$\rho \iint_{\Sigma+S} [(l+z)\cos(n, y) - y\cos(n, z)] \frac{\partial \varphi}{\partial t} dS = \rho \iint_{\Sigma+S} \frac{\partial \varphi_2^*}{\partial n} \frac{\partial \varphi}{\partial t} dS ,$$

and since $\Delta \frac{\partial \varphi}{\partial t} = 0$,

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 0 , & P \epsilon \Sigma , \\ \zeta_{tt} , & P \epsilon S , \end{cases}$$

we get

$$\rho \iint_{\Sigma+S} [(l+z)\cos(n, y) - y\cos(n, z)] \frac{\partial \varphi}{\partial t} dS = \rho \iint_{S} \varphi_2^* \zeta_{tt} dS .$$

Hence, the expression for the moment (8.44_2) takes the form

$$M_x{}' = \ddot{q}_1 \rho \iiint_{\tau} \nabla \varphi_2^* \nabla \varphi_1^* d\tau + \ddot{q}_2 \rho \iiint_{\tau} \nabla \varphi_2^* \nabla \varphi_2^* d\tau + \rho \iint_{S} \varphi_2^* \zeta_{tt} dS \tag{8.48}$$

$$- \rho g \iint_{S} y \zeta dS - \rho g \tau l^* q_2 .$$

With (8.47-48), we may write the equilibrium conditions (8.43) in the form

$$\sum_1^2 a_{mn} \ddot{q}_n + \sum_1^2 b_{mn} q_n + \rho \iint_{S} \varphi_m^* \zeta_{tt} dS + \iint_{S} e_m \zeta dS = 0 , \tag{8.49}$$

where

$$a_{mn} = M_{mn}^{\circ} + \rho \iiint_{\tau} \nabla \varphi_m^* \nabla \varphi_n^* d\tau ,$$

$$M_{mn}^{\circ} = \begin{pmatrix} M^{\circ} & M^{\circ} l_o \\ M^{\circ} l_o & M^{\circ} (l_o^2 + R^2) \end{pmatrix} ;$$

$$b_{mn} = K_{mn}^{\circ} - \rho g \tau l^* \delta_{m2} ,$$

$$K_{mn}^{\circ} = \begin{pmatrix} K_t & 0 \\ 0 & K_r - M^{\circ} g l_o \end{pmatrix}$$

$$e_n = - \rho g y \delta_{n2} .$$

The condition of constancy of pressure at the free surface (8.42_3) becomes

$$\rho \sum_1^2 \varphi_s^* \ddot{q}_s + \rho \frac{\partial \varphi}{\partial t} + \rho g \zeta + \sum_1^2 e_s q_s = 0 .$$

We can eliminate potential φ from this equality, expressing it through the free boundary $\varphi = N\zeta_t$ as in (7.4). Hence,

$$\rho \sum_1^2 \varphi_s^* q_s + \rho N\zeta_{tt} + \rho g \zeta + \sum_1^2 e_s q_s = 0 . \tag{8.50}$$

Variational Formulation: Let us construct the Lagrange function $L = T - \Pi$ for our system. The kinetic energy may be written as

$$T = T^* + T_1$$

$$T^* = 1/2 \, M^\circ (\dot{q}_1 + l \, q_2)^2 + 1/2 \, M^\circ R^2 q^2_2 , \quad T_1 = 1/2 \, \rho \iiint_\tau (\nabla \varphi)^2 \, d\tau .$$

T^* is the kinetic energy of the vessel proper, and T_1 the kinetic energy of the oscillating liquid. With (8.45), we have

$$\nabla \varphi = \dot{q}_1 \nabla \varphi_1^* + \dot{q}_2 \nabla \varphi_2^* + \nabla \varphi ,$$

where φ_1^*, φ_2^* are the potentials of Stokes which are determined solely from the geometry of the cavity (8.46_1) and φ is the potential of wave motion in the vessel satisfying (8.46_2). Thus

$$T = 1/2 \sum \sum a_{sn} \dot{q}_s \dot{q}_n + \sum \dot{q}_s \, \rho \iiint_\tau \nabla \varphi_s^* \nabla \varphi \, d\tau + 1/2 \, \rho \iiint_\tau (\nabla \varphi)^2 \, d\tau . \tag{8.51}$$

with a_{sn} previously defined. The integration indicated in (8.51) should be carried out over the volume which the liquid occupies in the position of equilibrium.

Consider the potential energy of the system,

$$\Pi = \Pi^* + \Pi_1$$

$$\Pi^* = 1/2 \, K_t^2 \, q_1^2 + 1/2 \, K_r \, q_2^2 + \frac{M^\circ g \, l_0}{2} q_2^2 , \quad \Pi_1 = \rho g \iiint_{\tau(t)} z' \, d\tau .$$

Π^* is the potential energy of the system in the absence of liquid, and Π_1 the potential energy of the liquid. The integral appearing in the expression for Π_1 can be written as

$$\rho\, g \iiint_{\tau(t)} z'\, d\tau = \rho\, g \iiint_{\tau} z'\, d\tau + \rho\, g \iiint_{\tau_1(t)} z'\, d\tau$$

$$= \rho\, g \iiint_{\tau} z'\, d\tau + \rho\, g \iiint_{\tau_1(t)} (l + z) \cos q_2 - y \sin q_2]\, d\tau ,$$

where τ is the volume occupied by the liquid in the equilibrium position, and $\tau_1(t)$ is the volume enclosed between the free surface $z = \zeta\,(y, z, t)$ and the plane $S(z = 0)$. The first integral in the right-hand member of this expression represents the potential energy of the liquid if the free surface were replaced by a rigid lid. Thus, we can write

$$\rho\, g \iiint_{\tau} z'\, d\tau = \rho\, g\, \tau\, z^{*\prime}$$

where $z^{*\prime}$ is the ordinate of the center of gravity of the liquid. However $z^{*\prime} = l^* \cos q_2$, so that

$$\rho\, g \iiint_{\tau} z'\, d\tau = \rho\, g\, \tau\, l^* \cos q_2 \approx -\, 1/2\, \rho\, g\, \tau l^*\, q_2^2 + \text{const.}$$

In addition,

$$\rho\, g \iiint_{\tau_1(t)} [\, (l + z) \cos q_2 - y \sin q_2\,]\, d\tau \approx \rho\, g \iiint_{\tau_1(t)} [\, (z + l) - q_2\, y\,]\, d\tau$$

$$= \rho\, g \iint_S dS \int_0^{\zeta} [(z + l) - q_2\, y]\, dz$$

$$\approx 1/2\, \rho\, g \iint_S \zeta^2\, dS - \rho\, g\, q_2 \iint_S y\, \zeta\, dS\, .$$

The total potential energy of the system is therefore

$$\Pi = 1/2 \sum_1^2 \sum_1^2 b_{mn}\, q_m\, q_n + \rho\, g/2 \iint_S \zeta^2\, dS + \sum_1^2 q_m \iint_S e_m\, \zeta\, dS\, , \tag{8.52}$$

in which b_{mn} and e_m are the same quantities defined earlier.

By definition, the Lagrange function L is simply

$$L = T - \Pi = 1/2 \sum_1^2 \sum_1^2 a_{mn}\, q_m{}'\, q_n{}' + \sum_1^2 q_m{}' \rho \iiint_{\tau} \nabla \varphi_m^* \,\nabla \varphi_m^* \,\nabla \varphi\, d\tau \tag{8.53}$$

$$+ 1/2\,\rho \iiint_{\tau} (\nabla\varphi)^2\, d\tau - 1/2 \,\Sigma\,\Sigma\, b_{sn}\, q_s\, q_n - 1/2\, \rho\, g \iint_S \zeta^2\, dS$$

$$- \sum_1^2 q_s \iint_S e_s\, \zeta\, dS \,.$$

According to Hamilton's principle

$$\delta I = 0 \tag{8.54}$$

where δI is an isochronous variation of the line integral

$$I = \int_0^t L\, dt \,.$$

Using (8.53) rewrite (8.54) as

$$\delta I = \int_0^t \{ \sum_1^2 \sum_1^2 a_{sn}\, q_n\, \delta q_s' - \sum_1^2 \sum_1^2 b_{sn}\, q_n\, \delta q_s + \rho \sum_1^2 \delta q_s' \iiint_{\tau} \nabla\varphi_s^{\#}\, \nabla\varphi\, d\tau \tag{8.55}$$

$$+ \rho \sum_1^2 q_s' \iiint_{\tau} \nabla\varphi_s^{\#}\, \nabla\delta\varphi\, d\tau + \rho \iiint_{\tau} \nabla\varphi\, \nabla\delta\varphi\, d\tau - \rho\, g \iint_S \zeta\, \delta\zeta\, dS$$

$$- \sum_1^2 \delta q_s \iint_S e_s\, \zeta\, dS - \sum_1^2 q_s \iint_S e_s\, \delta\zeta\, dS \}\, dt = 0 \,.$$

But, from (8.46a) and $\Delta\varphi = 0$, we have

$$\iiint_{\tau} \nabla\varphi\, \nabla\delta\varphi\, d\tau = \iint_S \varphi\, \frac{\partial\delta\varphi}{\partial n}\, dS \,.$$

Moreover, it follows from (8.46a) that

$$\frac{\partial\delta\varphi}{\partial n} = \delta\,\zeta_1 \,,\quad P \in S \,,$$

so that

$$\iiint_{\tau} \nabla\varphi\, \nabla\delta\varphi\, d\tau = \iint_S \varphi\, \delta\,\zeta_1\, dS \,.$$

Also, from (8.46a) and $\Delta\varphi_s^{\#} = 0$, we get

$$\iiint_{\tau} \nabla\varphi_s^{\#}\, \nabla\varphi\, d\tau = \iint_{\Sigma+S} \varphi_s^{\#}\, \frac{\partial\varphi}{\partial n}\, dS = \iint_S \varphi_s^{\#}\, \zeta_1\, dS$$

Similarly, we can show

$$\iiint_{\tau} \nabla \varphi_s^* \, \nabla \delta\varphi \, d\tau = \iint_{\Sigma+S} \varphi_s^* \frac{\partial \delta\varphi}{\partial n} \, dS = \iint_S \varphi_s^* \, \delta \zeta_t \, dS$$

Therefore (8.55) may be rewritten in the following way:

$$\int_0^t \{ \sum_1^2 \sum_1^2 (a_{sn} \, q_n' \, \delta q_s' - b_{sn} \, q_n \, \delta q_s) + \iint [\sum_1^2 (\rho \varphi_s^* (\zeta_t \, \delta q_s' + q_s' \, \delta \zeta_t)$$

$$- e_s (\zeta \, \delta q_s + q_s \, \delta \zeta)) + (\varphi \delta \zeta_t - \rho \, g \, \zeta \, \delta \zeta)] \, dS \} \, dt = 0 \, .$$

Integrating by parts and using the isochronism of the variations, we obtain

$$- \int_0^t \{ [\sum_1^2 \sum_1^2 (a_{sn} \, q_n'' + b_{sn} \, q_n) + \sum_1^2 \rho \iint_S \varphi_s^* \zeta_{tt} \, dS + \sum_1^2 \iint_S e_s \, \zeta \, dS] \, \delta q_s$$

$$+ \iint_S (\rho \sum_1^2 q_s'' \varphi_s^* + \rho \frac{\partial \varphi}{\partial t} + \rho \, g \, \zeta + \sum_1^2 e_s \, q_s) \, \delta \zeta \, dS \} \, dt = 0 \, .$$

By virtue of the arbitrariness of the variations, we get from this

$$\sum_1^2 a_{sn} \, q_n'' + \sum_1^2 b_{sn} \, q_n + \rho \iint_S \varphi_s^* \zeta_{tt} \, dS + \iint_S e_s \, \zeta \, dS = 0 \tag{8.56}$$

$$\rho \sum_1^2 q_s \, \varphi_s^* + \rho \frac{\partial \varphi}{\partial t} + \rho \, g \, \zeta + \sum_1^2 e_s \, q_s = 0$$

which is the same as before if we put $\varphi = N\zeta_t$ in the last expression.

Free Oscillations: Before we proceed to determine the free vibrations of the system, let us simplify the problem somewhat further. Let the free surface be replaced by a lid ($\zeta \equiv 0$ in (8.56)). Then, if the system is conservative and its position of equilibrium stable, there exist principal coordinates ξ_l such that the linear transformation

$$q_s = \sum_1^2 \phi_{sl} \, \xi_l(t) \tag{8.57}$$

reduces [12] to two independent second order differential equations in ξ_l . ϕ_{sl} satisfy the orthonormal conditions

$$\sum_1^2 \sum_1^2 a_{sn} \, \phi_{sk} \, \phi_{nl} = \delta_{kl}$$

Thus, according to vibration theory, we obtain

$$\ddot{\xi}_1 + \rho \iint_S \varphi_1^{**} \zeta_{tt} \, dS + \Omega_1^2 \xi_1 + \iint_S \nu_1 \zeta \, dS = 0 \,, \tag{8.58}$$

where Ω_1 are the natural frequencies of the system when the free surface is covered with a lid, and

$$\varphi_1^{**} = \sum_1^2 \phi_{s1} \varphi_s^* \,, \quad \nu_1 = \sum_1^2 \phi_{s1} e_s \,.$$

With (8.57), condition (8.50) becomes

$$\rho \sum_1^2 \varphi_1^{**} \ddot{\xi}_1 + \rho N \zeta_{tt} + \rho g \zeta + \sum_1^2 \nu_1 \xi_1 = 0 \,. \tag{8.59}$$

Equation (8.58-59) are sufficient to determine the free oscillations of the system, and are somewhat simpler than (8.56).

Suppose

$$\xi_1 = X_1 \sin \sigma t \,, \quad \zeta = \Xi \sin \sigma t \,. \tag{8.60}$$

With (8.60), system (8.58-59) becomes

$$(\sigma^2 - \Omega_1^2) X_1 - \iint_S (\nu_1 - \sigma^2 \rho \varphi_1^{**}) \cdot \Xi \, dS = 0 \tag{8.61}$$

$$\rho(\sigma^2 N - g) - \sum_1^2 (\nu_1 - \sigma^2 \rho \varphi_1^{**}) X_1 = 0$$

Let the eigenfunctions Ξ_k of the free oscillation problem (stationary vessel) be known. They satisfy the obvious properties

$$N \Xi_k = \frac{g}{\omega_k^2} \Xi_k$$

$$(\Xi_k, \Xi_j) = \begin{cases} 0 & , j \neq k \\ \|\Xi_k\|^2 & j = k \,. \end{cases}$$

Since Ξ_k is a system of functions complete with respect to integration over S, it is natural to assume that

$$\Xi = \sum_k c_k \Xi_k \,.$$

With the substitution of this expansion in (8.61) and appropriate use of the orthogonality relations, we arrive at the following system of algebraic equations:

$$(\sigma^2 - \Omega_l^2)\, X_l + \sum_k (\sigma^2 A_{lk} - B_{lk})\, c_k = 0\ ,\ (l = 1,\ 2)\ , \tag{8.62}$$

$$(\alpha_k\, \sigma^2 - \beta_k)\, c_k + \sum_1^2 (\sigma^2 A_{lk} - B_{lk})\, X_l = 0\ ,\ (k = 1,\ 2, \ldots)$$

where

$$A_{lk} = \rho \iint_S \varphi_l^{**} \Xi_k\, dS\ ,\ B_{lk} = \iint_S \nu_l\, \Xi_k\, dS\ ,$$

$$\alpha_k = \frac{\rho\sigma}{\omega_k^2} \|\Xi_k\|^2\ ,\ \beta_k = \rho\ g \|\Xi_k\|^2 .$$

Equations (8.62) hold for each l and k respectively, so that we have a set of linear simultaneous equations for the coefficients X_1, X_2, c_1, $c_2, \ldots$, which are homogeneous. Such a set of equations will have a non vanishing solution only if the determinant formed from the coefficients of the unknown X_l and c_k vanishes. Therefore

$$\begin{vmatrix} \sigma^2 - \Omega_1^2 & 0 & \sigma^2 A_{11} - B_{11} & \sigma^2 A_{12} - B_{12} & \ldots \sigma^2 A_{1n} - B_{1n} \\ 0 & \sigma^2 - \Omega_2^2 & \sigma^2 A_{21} - B_{21} & \sigma^2 A_{22} - B_{22} & \ldots \sigma^2 A_{2n} - B_{2n} \\ \sigma^2 A_{11} - B_{11} & \sigma^2 A_{21} - B_{21} & \alpha_1 \sigma^2 - B_1 & 0 & \ldots \\ \sigma^2 A_{12} - B_{12} & \sigma^2 A_{22} - B_{22} & 0 & \alpha_2 \sigma^2 - \beta_2 & \ldots \quad 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \quad \ldots \\ \sigma^2 A_{1n} - B_{1n} & \sigma^2 A_{2n} - B_{2n} & 0 & 0 & \ldots\ \alpha_n \sigma^2 - \beta_n \end{vmatrix} = 0\ . \tag{8.63}$$

The elements symmetrical to the main diagonal are equal; hence the roots of σ^2 are real.

We see that the computation of the natural frequencies and forms of oscillations for the system is a very laborious process. If the coefficients of the terms accounting for the reciprical oscillations are small, i. e., if the natural frequencies of the system differ but little from Ω_1, then we can simplify the calculations by using perturbation techniques.

Perturbation methods are particularly appropriate whenever the problem under consideration closely resembles one which is exactly solvable (such as this problem). It presumes that these differences are not singular in character, indeed, that one may change from the exactly solvable situation to the problem under consideration in a gradual fashion. This is expressed analytically by requiring that the perturbation be a continuous function of a parameter λ, measuring the strength of the perturbation.

With λ, we write (8.62) in the form

$$(\sigma^2 - \Omega_1^2) X_1 + \lambda \sum_k (\sigma^2 A_{1k}) c_k = 0$$

$$(\alpha_k \sigma^2 - \beta_k) c_k + \lambda \sum_k (\sigma^2 A_{1k} - B_{1k}) X_1 = 0$$

Solutions to this system of equations are sought as infinite series in powers of λ,

$$X_1 = \sum_0^\infty X_{1r} \lambda^r , \quad C_k = \sum_0^\infty c_{kr} \lambda^r , \quad \sigma^2 = \sum_0^\infty \sigma_r^2 \lambda^r .$$

Equating the coefficients of the successive powers of λ to zero gives, for zero order quantities,

$$X_{10} (\sigma_0^2 - \Omega_1^2) = 0 ,$$

$$C_{k0} (\sigma_0^2 \alpha_k - \beta_k) = 0 .$$

This system of equations has a solution if $\sigma_0^2 = \Omega_1^2$ or $\sigma_0^2 = \beta_k/\alpha_k$. Since we wish to determine the frequencies of the system close to those of the system when S is replaced by a rigid lid, the solution $\sigma_0^2 = \beta_k/\alpha_k$ must be rejected. To be specific, let $\sigma_0^2 = \Omega_1^2$; then $X_{s0} = C_{k0} = 0$, and we may take $X_{10} = 1$. For the first order quantities, we have the equations

$$X_{11} (\sigma_0^2 - \Omega_1^2) = - \sigma_1^2 ,$$

$$X_{s1} (\sigma_0^2 - \Omega_s^2) = 0 ,$$

$$c_{k1} (\sigma_0^2 \alpha_k - \beta_k) = - (A_{1k} \Omega_1^2 - B_{1k}) .$$

The condition for solvability of the first equation of this system is

$$\sigma_1^2 = 0 .$$

Hence we can always assume that $X_{11} = 0$. Also, it follows that

$$X_{s1} = 0, \quad c_{k1} = - \frac{A_{1k} \Omega_1^2 - B_{1k}}{\Omega_1^2 \alpha_k - \beta_k} .$$

For the second order quantities, we find that X_{12} and σ_2^2 satisfy the equation

$$X_{12}\,(\sigma_0^2 - \Omega_1^2) = -\sigma_2^2 - \sum_k C_{k1}\,(A_{1k}\,\Omega_1^2 - B_{1k})$$

For this equation to be solvable, it is necessary and sufficient that the right-hand member be equal to zero; i.e.,

$$\sigma_2^2 = \sum_k \frac{(A_{1k}\,\Omega_1^2 - B_{1k})^2}{\Omega_1^2\,\alpha_k - \beta_k}$$

wherein we have substituted the previously obtained value for C_{k1}. Therefore, we have to the second order

$$\sigma^2 = \Omega_1^2 + \lambda^2 \sum_k \frac{(A_{1k}\,\Omega_1^2 - B_{1k})^2}{\Omega_1^2\,\alpha_k - \beta_k} \tag{8.64}$$

we may use (8.64) for $\lambda = 1$. The foregoing scheme is a modification of the Feenberg perturbation formula [6].

Let us introduce the linear transformation (8.57) in the expressions for the kinetic energy (8.51) and the potential energy (8.52). Thus, in the new variables, we have

$$T = T^* + \sum_1^2 \xi_1\,\rho \iiint_\tau \nabla\varphi_1^{**}\,\nabla\varphi\,d\tau + 1/2\,\rho \iiint_\tau (\nabla\varphi)^2\,d\tau,$$

$$\Pi = \Pi^* + \sum_1^2 \xi_1 \iint_S \nu_1\,\zeta\,dS + 1/2\,\rho\,g \iint_S \zeta^2\,dS,$$

where

$$T^* = 1/2 \sum_1^2 \xi_1^{2\prime},\quad \Pi^* = 1/2 \sum_1^2 \Omega_1^2\,\xi_1^2$$

$$\varphi_1^{**} = \sum_1^2 \phi_{s1}\,\varphi_s^*,\quad \nu_1 = \sum \phi_{s1}\,e_s .$$

The Lagrange function becomes

$$L = T^* - \Pi^* + \sum_1^2 \xi_1\,\rho \iiint_\tau \nabla\varphi_1^{**}\,\nabla\varphi\,d\tau + 1/2\,\rho \iiint_\tau (\nabla\varphi)^2\,d\tau - \sum_1^2 \xi_1 \iint_S \nu_1\,\zeta\,dS \tag{8.65}$$

$$- 1/2\,\rho\,g \iint_S \zeta^2\,dS,$$

or, since

$$\iiint_{\tau} \nabla \varphi_1^{**} \nabla\varphi \, d\tau = \iint_S \varphi_1^{**} \zeta_t \, dS ,$$

$$\iiint_{\tau} (\nabla\varphi)^2 \, d\tau = \iint_{\Sigma+S} \varphi \frac{\partial\varphi}{\partial n} \, dS = \iint_S \varphi \, \zeta_t \, dS = \iint_S N\zeta_t \cdot \zeta_t \, dS ,$$

$$L = T^* - \Pi^* + \sum_1^2 \dot{\xi}_1 \rho \iint_S \varphi_1^{**} \zeta_t \, dS + 1/2 \, \rho \iint_S N \zeta_t \cdot \zeta_t \, dS - \sum_1^2 \xi_1 \iint_S \nu_1 \, \zeta \, dS \qquad (8.66)$$

$$- 1/2 \, \rho \, g \iint_S \zeta^2 \, dS$$

Substitute (8.60) into the line integral

$$I = \int_0^t L \, dt ,$$

using the form of L given by (8.66), and integrate over t from 0 to $\frac{2\pi}{\sigma}$.

This gives, after omitting a non-essential multiplicative factor,

$$I = \sigma^2 \left\{ 1/2 \sum_1^2 X_1^2 + \sum_1^2 X_1 \rho \iint_S \varphi_1^{**} \Xi \, dS + 1/2 \, \rho \iint_S N \, \Xi \cdot \Xi \, dS \right\} \qquad (8.67)$$

$$- \left\{ 1/2 \sum_1^2 \Omega_1^2 X_1^2 + \sum_1^2 X_1 \iint_S \nu_1 \, \Xi \, dS + 1/2 \, \rho \, g \iint_S \Xi^2 \, dS \right\} .$$

Thus the problem of free oscillations of our system is reduced to one of determining vector X_1, function Ξ , and parameter σ which make the variation of functional (8.67) vanish. To be sure, if the eigenfunctions of the free oscillation problem for a stationary vessel are known, the extremum of functional (8.67) gives the equations (8.62). If these are not available we can solve the variational problem by application of the method of Ritz.

Let there be a system of functions X_k complete with respect to integration over S, and assume that

$$\Xi = \sum_1^n c_k X_k .$$

After construction relations

$$\frac{\partial I}{\partial X_1} = 0 , \quad \frac{\partial I}{\partial c_k} = 0 ,$$

There results the system of algebraic equations

$$(\sigma^2 - \Omega_l^2)\, X_l + \sum_1^n (\sigma^2 A_{lk} - B_{lk})\, c_k = 0, \; (l = 1, 2), \tag{8.68}$$

$$\sum_1^n (\sigma^2 A_{lk} - B_{lk})\, X_l + \sum_1^n (\sigma^2 \alpha_{lk} - \beta_{lk})\, c_l = 0, \; (k = 1, 2, \ldots, n),$$

where

$$A_{lk} = \rho \iint_S \varphi_l^{*} X_k \, dS, \quad B_{lk} = \iint_S \nu_l X_k \, dS,$$

$$\alpha_{lk} = \rho \iint_S N X_l \cdot X_k \, dS, \quad \beta_{lk} = \rho g \iint_S X_l X_k \, dS,$$

and $\alpha_{lk} = \alpha_{kl}$ (from Green's formula).

Equations (8.68) hold for each l and k respectively, so that we have a set of linear simultaneous equations for the coefficients $X_1, X_2, c_1, c_2, \ldots, c_n$, which are homogeneous. Such a set of equations will have a non-vanishing solution if the determinant formed from the coefficients of the unknown X_l and c_k vanishes.

Therefore

$$\begin{vmatrix} \sigma^2 - \Omega_1^2 & 0 & \sigma^2 A_{11} - B_{11} & \sigma^2 A_{12} - B_{12} & \ldots & \sigma^2 A_{1n} - B_{1n} \\ 0 & \sigma^2 - \Omega_2^2 & \sigma^2 A_{21} - B_{21} & \sigma^2 A_{22} - B_{22} & \ldots & \sigma^2 A_{2n} - B_{2n} \\ \sigma^2 A_{11} - B_{11} & \sigma^2 A_{21} - B_{21} & \sigma^2 \alpha_{11} - \beta_{11} & \sigma^2 \alpha_{12} - \beta_{12} & \ldots & \sigma^2 \alpha_{1n} - \beta_{1n} \\ \sigma^2 A_{12} - B_{12} & \sigma^2 A_{22} - B_{22} & \sigma^2 \alpha_{21} - \beta_{21} & \sigma^2 \alpha_{22} - \beta_{22} & \ldots & \sigma^2 \alpha_{2n} - \beta_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma^2 A_{1n} - B_{1n} & \sigma^2 A_{2n} - B_{2n} & \sigma^2 \alpha_{n1} - \beta_{n1} & \sigma^2 \alpha_{n2} - \beta_{n2} & \ldots & \sigma^2 \alpha_{nn} - \beta_{nn} \end{vmatrix} = 0 . \tag{8.69}$$

The elements symmetrical to the main diagonal are equal; hence the roots σ^2 are real.

We now consider the general theory of small vibrations of a conservative system with a liquid cavity.

Now consider the motion of a solid body with a liquid-containing cavity about the position of equilibrium. As previously, but with slightly different notation, we assume the velocity of the liquid particles to be represented as

$$\bar{q} = \nabla \varphi + \bar{q}^*$$

It has been shown that

$$\bar{q}^* = \sum_1^3 u_i \nabla \varphi_m^* + \sum_1^3 \omega_m \nabla \varphi_{m+3}^* \qquad (8.70)$$

where $\varphi_m^* = (m = 1, 2, \ldots, 6)$ are Stokes potentials satisfying the boundary value problem (2.35). Hence, function φ must satisfy

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 0 & , \; P \epsilon \Sigma , \\ \zeta_t & , \; P \epsilon S \, . \end{cases}$$

$z = \zeta(P, t)$ is the equation of the free surface in the system of coordinates rigidly connected with the body.

Now suppose the motion of the solid body to be defined by the generalized coordinates $\alpha_1, \ldots, \alpha_6$. Then, by analogy with (8.70), we write

$$\bar{q} = \nabla \varphi + \sum_1^6 \dot{\alpha}_m \nabla \varphi_m^*$$

and assume the expression for the kinetic energy of the system as follows

$$T = 1/2 \sum_1^6 \sum_1^6 M_{ij} \, \dot{\alpha}_m \dot{\alpha}_n + \rho/2 \iiint_\tau (\nabla \varphi)^2 \, d\tau \qquad (8.71)$$
$$+ \sum_1^6 \dot{\alpha}_m \, \rho \iiint_\tau \nabla \varphi \, \nabla \varphi_i^* \, d\tau .$$

Here

$$M_{mn} = M_{mn}^{\circ} + \rho \iiint \nabla \varphi_m^* \, \nabla \varphi_n^* d\tau ,$$

and

$\{ M_{mn}^{\circ} \}$ is the matrix of the coefficients of the quadratic form representing the kinetic energy of the solid body.

Let us now consider the small oscillations of a conservative system K which has N degrees of freedom, and let α_m (m = 1, 2, ... N) be the generalized coordinates of the system. Suppose there is a solid body with a liquid cavity among the members of this oscillatory system. Then, without loss of generality, the kinetic energy of system K can be described by (8.71) in which the summation now extends from 1 to N.

We now compute the potential energy Π of system K. If the free surface is "capped" off (in this case we denote it by K^*), then

$$\Pi = \Pi^* = 1/2 \sum_1^N \sum_1^N b_{mn} \alpha_m \alpha_n \quad . \tag{8.72}$$

If we consider only the case when the equilibrium of system K^* is stable, Π^* is positive definite. This assumption is very important because if K^* is unstable the deviation rapidly ceases to be small and the theory makes no sense.

If the liquid does not fill the cavity completely, the potential energy of K is made up of (8.72) and of the potential energy of the oscillating liquid. The latter, in turn, may be represented as a sum of two terms, one of which is the potential energy of the liquid oscillating in a fixed vessel

$$\Pi_1 = 1/2 \rho g \iint_S \zeta^2 \, dS$$

As the liquid participates in the motion of the system through transport, its potential energy also depends on the coordinates α_m; consequently it should contain a term of the form

$$\Pi_2 = \sum_1^2 \alpha_m \iint_S e_m \zeta \, dS$$

where functions e_m are determined solely by the geometry of the cavity. Since $\iint_S \zeta \, dS = 0$ we may assume, again without loss of generality, that $\iint_S e_m \, dS = 0$.

Hence, we write the potential energy of system K as follows:

$$\Pi = 1/2 \sum_1^N \sum_1^N b_{mn} \alpha_m \alpha_n + 1/2 \rho g \iint_S \zeta^2 \, dS \tag{8.73}$$

$$+ \sum_1^N \alpha_m \iint_S e_m \zeta \, dS \; .$$

The problem may be simplified somewhat if we introduce new variables. Since system K* is conservative and its position of equilibrium stable, there exist principal coordinates $\xi_1(t)$ such that the linear transformation

$$\alpha_s = \sum_1^N \phi_{s1}\ \xi_1(t)$$

simultaneously diagonalizes Π^* and T^*:

$$T^* = 1/2 \sum_1^N \dot{\alpha}_s^2\ , \qquad \Pi^* = 1/2 \sum_1^N \Omega_s^2\ \alpha_s^2\ .$$

Ω_s are the natural frequencies of the system when the free surface is "capped-off".

In the new variables, we have

$$T = T^* + \sum_1^N \dot{\alpha}_s\ \rho \iiint_\tau \nabla\varphi\ \nabla\varphi_s^{**}\ d\tau + 1/2\ \rho \iiint_\tau (\nabla\varphi)^2\ d\tau\ , \tag{8.74}$$

$$\Pi = \Pi^* + \sum_1^N \alpha_s \iint_S \nu_s\ \zeta\ dS + 1/2\ \rho\ g \iint_S \zeta^2\ dS\ .$$

Here

$$\varphi_s^{**} = \sum_1^N \phi_{1s}\ \varphi_1^*\ , \qquad \nu_s = \sum_1^N \phi_{1s}\ e_1\ .$$

By eliminating φ and using Green's formula(8.74$_1$) becomes

$$T = T^* + \sum_1^N \rho \iint_S \varphi_s^{**}\ \zeta_t\ dS + 1/2\ \rho \iint_S N\,\zeta_t \cdot \zeta_t\ dS\ . \tag{8.75}$$

Applying Hamilton's principle and repeating the considerations used in the construction of the launch vehicle equations, we obtain the following system of equations for the oscillations of system K:

$$\ddot{\xi}_1 + \rho \iint_S \varphi_1^{**}\ \zeta_{tt}\ dS + \Omega_1^2\ \xi_1 + \iint_S \nu_1\ \zeta\ dS = 0 \tag{8.76}$$

$$\rho \sum_1^N \varphi_1^{**}\ \ddot{\xi}_1 + \rho\ N\,\zeta_{tt} + \rho\ g\,\zeta + \sum_1^N \nu_1\,\xi_1 = 0\ .$$

System (8.58, 59), which determines the oscillations of the launch vehicle, was a particular case of the system of integro-differential equations (8.76).

Using the methods of functional analysis, we can show that the following properties hold:

Theorem I. If system K consists of a finite number of conservative members and contains a finite number of cavities partly filled with liquid, and if the potential energy of the system has a minimum in the equilibrium position, then

(1) In the motion of this system about the equilibrium position there exist principal oscillations, and system (8.76) has a solution of the form

$$\alpha_n = X_n\, e^{i\sigma t}, \qquad \zeta = \Xi\, e^{i\sigma t};$$

(2) The frequencies of these oscillations are real quantities and $\sigma_n \to \infty$ with $n \to \infty$. This means that the position of equilibrium is stable;

(3) Any free motion of K may be represented as a superposition of oscillations, i.e., the system of principal oscillations is complete;

(4) Free oscillations and frequencies can be found by Ritz's method.

Theorem II. If the potential energy is not a minimum in the equilibrium position, then there is at least one negative quantity among the σ_n^2.

9/NOTES PERTAINING TO TEXT

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1 $$\left(\frac{d\bar{q}}{dt}\right)_{\text{inertial space}} = \left(\frac{d\bar{q}}{dt}\right)_{\text{moving space}} + \bar{\omega} \times \bar{q}$$

2 When S(t) is a free surface consisting of material particles moving with velocity $\bar{v}$. If $z - \zeta = 0$ is the equation of the surface, we must have $d(z - \zeta)/dt = 0$ so that

$$\zeta_t = - v_x \frac{\partial \zeta}{\partial x} - v_y \frac{\partial \zeta}{\partial y} + v_z$$

or

$$\zeta_t = (v_x, v_y, v_z) \cdot \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) .$$

But

$$\left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) = \nabla (z - \zeta) ,$$

and since

$$\nabla (z - \zeta) = (\cos(n, x), \cos(n, y), \cos(n, z))/\cos(n, z)$$

we get

$$v_n = \zeta_t \cos(n, z) ,$$

where

$$v_n = v_x \cos(n, x) + v_y \cos(n, y) + v_z \cos(n, z)$$

is the velocity along the normal to S(t) .

Similarly

$$q_n = V_n + \zeta_t \cos(n, z) .$$

[3] This could also be anticipated from the divergence theorem,

$$\iiint_{\tau(t)} \nabla \cdot \bar{v}\, d\tau = 0 = \iint_{S(t)+\Sigma(t)} v_n\, ds = \iint_{S(t)} \zeta_t \cos(n, z)\, ds = \iint_{S} \zeta_t\, dx\, dy\ .$$

[4] The first of these conditions is introduced to eliminate an arbitrary additive constant, and the second condition must be satisfied by the normal derivative of any function harmonic in the domain τ.

10/REFERENCES

REFERENCES

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