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# HODOGRAPHIC TREATMENT OF THE RESTRICTED THREE–BODY PROBLEM (FINAL REPORT – PHASE 2)

Prepared
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# FOREWORD

This report was prepared by the Spacecraft Department of the General Electric Missile and Space Division under Contract NAS 8-20360 on "Derivation of Analytical Methods Which Give Rapid Convergence to the Solution of Optimized Trajectories" for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under the technical direction of Resources Management Office, Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center with D. Chandler acting as project manager.

During the course of the study program, interim work reports were issued which comprise the sections of this Phase Final Report. The sections corresponding to the Phase Work Reports are as follows:

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# GENESIS OF THE PROBLEM
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SECTION 1

# GENESIS OF THE PROBLEM
# (PARAMETRIC FORM OF THE TWO-FIXED-CENTER PROBLEM)

Although the two-fixed-center problem has been solved in closed form by suitable selection of the problem coordinates, the restricted three-body problem has not yielded to comparable solution. Moreover, the classical solution form for the two-fixed-center problem is not amenable to direct and simple engineering application in feasibility and design studies.

The successful application of the hodograph theory of orbital mechanics to space trajectories generated in the presence of one force center suggests the possibility of its useful extension to problems with two force centers. In view of the simpler dynamics (compared with the restricted three-body problem) and the availability of the classical solution form with two-fixed-centers, the initial study effort will be directed to hodographic formulation and solution of this problem statement. The gravitational potentials will be assumed to be spherical harmonic functions. Also, the analysis will be limited to ballistic trajectories in two-dimensional space, at this time.

In essence, the hodographic analysis is a specific application of the vector space (or state space) theory of processes (Reference 1). The ballistic trajectory (or orbit) is defined by a unique vector locus in each vector space, with geometric transformations relating the respective trajectory loci (or maps). That is, a hodograph transformation enables us to map the trajectory representation in one vector space over into its corresponding representation in another vector space, by means of an algebraic function. Moreover, the orbit may be simply represented in parametric form; for example, with one force center, a conic section in position vector space, a circle in velocity vector space, or a functional variant of Pascal's limaçon in acceleration vector space.

## 1.1 VECTOR SPACE MAPS

Vector space maps for the two-fixed-center problem will be required in a form useful for synthesis of the desired transform space for solution, and the resulting transforms. Any new research on a major problem requires knowledge and understanding of the historical or classical precedents, in order to

a. Identify the uniqueness or novel form of the new formulation and solutions

b. Discover analytic clues to a new formulation or its consequent mathematical reduction

c. Derive analytic correlation with, and relation to, existing proofs and demonstrations.

Some of the most useful and immediately available references on classical developments in the two-fixed-center problem, which will be employed by the research team, are References 2, 3, 4, 5, and 6 which are identified at the end of this section.

At this time, three broad areas of useful information appear available as aids in obtaining the parametric solution form. For reference in discussion, let us briefly consider the two-fixed-center solution form presented in References 6 and 3 respectively.

Reference 6 (Whittaker):

$$\frac{d\xi}{du} = \left[h \cosh^2 \xi + \left(\frac{\mu_1 + \mu_2}{c}\right) \cosh \xi - \gamma\right]^{\frac{1}{2}} \quad (1\text{-}1)$$

$$\frac{d\chi}{du} = \left[-h \cos^2 \chi - \left(\frac{\mu_1 - \mu_2}{c}\right) \cos \chi + \gamma\right]^{\frac{1}{2}} \quad (1\text{-}2)$$

where

$\mu_1, \mu_2$ = gravitational constants of the respective force centers

2c = separation between the force centers, with x-y coordinate origin at its midpoint

h = orbital energy of the point-mass orbital body

$\gamma$ = constant of integration

u = auxiliary variable

$\xi, \chi$ = elliptic coordinates of the point-mass orbital body i.e.,

$$x = c \cosh \xi \cos \chi \qquad (1\text{-}3)$$

$$y = c \sinh \xi \sin \chi \qquad (1\text{-}4)$$

or

$$r_1 = c(\xi + \chi) \qquad (1\text{-}5)$$

$$r_2 = c(\xi - \chi) \qquad (1\text{-}6)$$

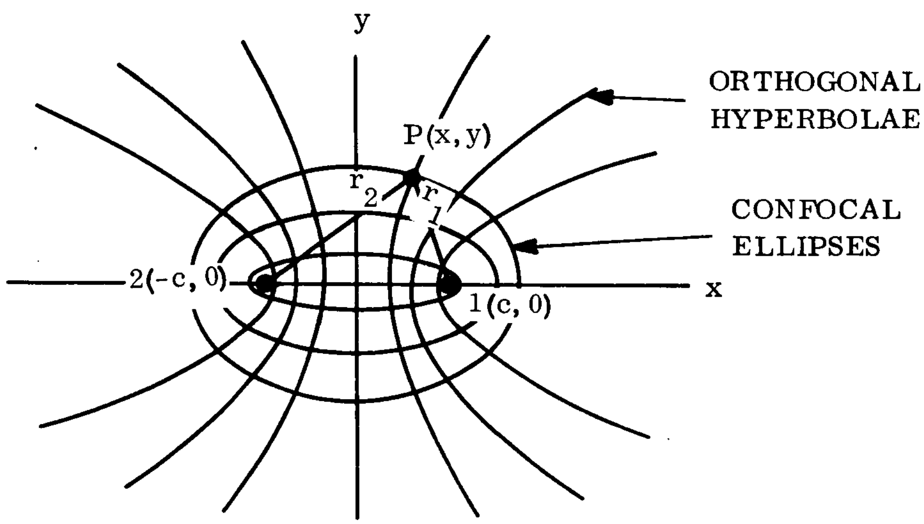



Figure 1-1. Elliptic Coordinate Contours in Position Vector Space

Reference 3 (Plummer):

$$\frac{d^2u}{dT^2} = (\mu_2 - \mu_1) c \sin u - c^2 h \sin 2u \qquad (1\text{-}7)$$

$$\frac{d^2v}{dT^2} = (\mu_2 + \mu_1) c \sinh v - c^2 h \sinh 2v \qquad (1\text{-}8)$$

where

$\mu_1, \mu_2$ = gravitational constants of the respective force centers

2c = separation between the force centers, with x-y coordinate origin at the midpoint

h = orbital energy of the point-mass orbital body

dT = dt/J

$$\left[\begin{array}{l} \text{i.e., } J = \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\cdot\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial y}{\partial u} \\ \text{and } t = \text{time} \end{array}\right] \qquad (1\text{-}9)$$

u, v = conjugate functions of x, y

$$\left[\begin{array}{l} \text{i.e., } x + i\,y = f\,(u + i\,v) \qquad (1\text{-}10) \\ \text{so that } \quad \dfrac{\partial x}{\partial u} = \dfrac{\partial y}{\partial v}\ ; \ \dfrac{\partial x}{\partial v} = -\dfrac{\partial y}{\partial u} \qquad (1\text{-}11,\ 1\text{-}12) \end{array}\right]$$

The coordinates (u, v) are identifiable as the elliptic coordinates ($\xi$, $\chi$), and the auxiliary variable u is identified with the normalized time variable T.

The result of the integration presented by Equations 1-1 and 1-2 contains two constants (h, $\gamma$). A parametric representation of a trajectory in a vector space requires the presence of two such parameters. Consequently, these constants are intimately related to the required parameters in the transform space for the desired parametric solution. Although the physical interpretation, dimensionality and functional definition of the orbital energy h is well-known, the comparable description of the constant $\gamma$ is not available. If such properties of this second parameter $\gamma$ were deduced, the results could be valuable in determining the vector space map of two-fixed-center trajectory in the required transform space.

The elliptic coordinates $(\xi, \chi)$ transform the position vector space contours of confocal ellipses and orthogonal hyperbolae (in (x, y) inertial coordinates) into the Cartesian representation; i.e., into a $(\xi, \chi)$ position space, as shown schematically in Figure 1-2. Then Equations 1-1 and 1-2 would represent the two-fixed-center trajectories in $(\dot{\xi}, \dot{\chi})$ velocity vector space, if we assume that the auxiliary variable u is the time variable t. This representation in $(\dot{\xi}, \dot{\chi})$ velocity vector space is dimensionally comparable to the hodographic map of a one-force-center trajectory in $(\dot{x}, \dot{y})$ velocity vector space. Study and analysis of this $(\dot{\xi}, \dot{\chi})$ map may provide useful information for the development of the required parametric transformation. Note that Equations 1-7 and 1-8 appear to define a trajectory map which is closely related to the map in $(\ddot{\xi}, \ddot{\chi})$ acceleration vector space.

The position and velocity vector space maps of realizable (or admissible) periodic orbits of simple geometric figure can be invaluable in complete study and development of the required transform space, and its mapping transformations to other vector spaces. In order to appreciate the utility of such data, it is noted that the development of the hodographic transformations (Reference 7) was accomplished only after definition and study of the Newtonian vector space maps (i.e., conics, circles and limaçon-like figures) and the point-to-point correspondence between these orbital figures. Bonnet's Theorem (see Reference 6) identifies confocal ellipses and the orthogonal hyperbolae, such as shown in Figure 1-1, as admissible orbits for the two-fixed-center problem. Consequently, these orbits and other simple geometric figures of orbit (when identifiable) with their respective velocity hodographs will be studied, as special forms of vector space maps which must be obtainable by means of the desired transformations. Of course, such transformations would also be required then to meet additional tests which assure that all admissible orbits are described by their use.

It is observed that the orbits of confocal ellipses and orthogonal hyperbolae are represented in $(\xi, \chi)$ vector space by lines parallel to the $\xi$ and $\chi$ axes respectively. Consequently, the velocity paths (of such orbits) in $(\dot{\xi}, \dot{\chi})$ space will lie only on the $\dot{\xi}$ and $\dot{\chi}$ axes respectively.

## 1.2 SYNTHESIS OF THE REQUIRED VECTOR SPACE AND TRANSFORM FUNCTIONS

The hodographic study of the two-fixed-center problem requires the development of transformations which will map the trajectory of any admissible orbit (either periodic or nonperiodic)

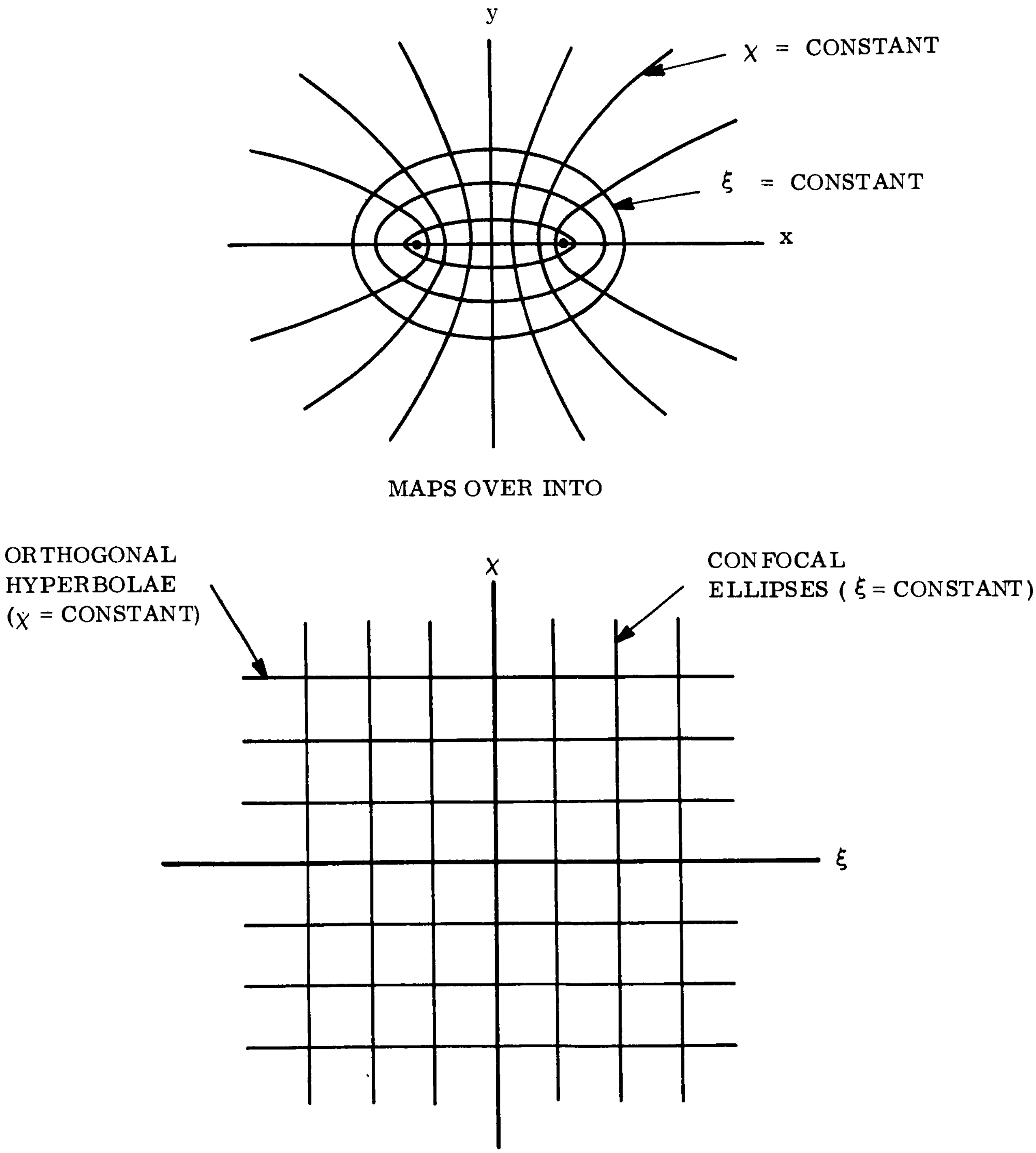



Figure 1-2. Elliptic Coordinate Transformation

between any given Newtonian vector spaces (i.e., position, velocity and acceleration). Aside from the vector space maps which will be obtained for various classes of admissible trajectories, the nature and functional form of the hodograph transformations for one force center are additional essential clues to the required vector spaces and transform functions for fixed force centers. Consider the admissible orbits in the Newtonian vector spaces (References 1, 7) as defined by

$$z = \left[\mu/C\,(C + R\cos\Phi)\right]\exp\left(i\Phi_z\right) \tag{1-13}$$

$$w = C^2\exp\left(i\left\{\Phi_z + \frac{\pi}{2}\right\}\right) + iCR \tag{1-14}$$

$$q = \left[(C^2 + CR\cos\Phi)^2/\mu\right]\exp\left(i\left\{\Phi_z + \pi\right\}\right), \tag{1-15}$$

and the orbital hodograph transformations

$$w = \left[\mu\left(\frac{\sqrt{1+y'^2}}{y - xy'}\right)\right]\exp\left(i\Phi_w\right) \tag{1-16}$$

$$q = \left[\frac{1}{\mu}\,\frac{(v - uv')^2}{(1 + v'^2)}\right]\exp\left(i\Phi_q\right) \tag{1-17}$$

$$q = \left[\frac{\mu}{r^2}\right]\exp\left(i\Phi_q\right) \tag{1-18}$$

for these orbits, where

| | | |
|---|---|---|
| $\mu$ | = | the gravitational constant of the one force center |
| C, R | = | invariant scalars of the velocity hodograph vectors $\bar{C}$, $\bar{R}$ |
| i | = | unit vector of the imaginary coordinate of a complex vector |
| x, y, y' | = | real, imaginary and slope coordinates of the orbital trajectory in position vector space. |
| u, v, v' | = | real, imaginary and slope coordinates of the orbital trajectory in velocity vector (or potential) space. |
| r | = | complex (or radius) vector in position vector space ($\equiv z = x + i\,y$) |
| w | = | complex vector in velocity (or potential) vector space ($\equiv u + iv \equiv C\dot{x} + i\,C\dot{y}$) |
| q | = | complex vector in acceleration vector space ($= ¢ + id \equiv \ddot{x} + i\,\ddot{y}$) |
| $\phi_z$, $\phi_w$, $\phi_q$ | = | arguments of the complex vectors z, w, q respectively in their vector spaces. |

The functional form of the orbital hodograph transformations of Equations 1-16 and 1-17 is symbolically identified in the suggested tensor form* of the transform moduli:

order of the vector space

geometric inversion

$$T^{1}_{mn} = \left[ \underbrace{\mu}_{\text{magnification}} \left( \underbrace{\frac{\sqrt{1+y_m'^{2}}}{y_m - x_m y_m'}}_{\text{pedal}} \right)^{n} \right]^{-1^{m}} \qquad (1\text{-}19)$$

*Note that this tensor form of the transform moduli has not been proven as generally valid for transformation between all Newtonian vector spaces. However, it is valid for the defined transformations of Equations 1-16 and 1-17 between successive vector spaces of position/velocity and velocity/acceleration.

where $m = n + 1$. It is clear that the hodograph transformation is comprised of the basic transformation properties of the pedal, geometric inversion and magnification.

The following hypothesis is stated for use in this transformation synthesis:

> The orbital hodograph transformations are degenerate (or reduced) forms of a general form of transformation for n-body vector spaces of orbital dynamics.

As a corollary, the general transformation function (with $n > 1$) will reduce to the corresponding hodograph transform when $n = 1$. Consequently, the required hodograph transformation for two fixed force centers will be functionally formed at least with the transform properties of the pedal, geometric inversion and magnification. However, additional transform properties will be present, which must undergo reduction whenever either of the two force centers

a. Reduces to zero

b. Recedes to infinity in position vector space or

c. Approaches the other so that the position displacement reduces to zero (thereby coalescing into one force center).

The additional transform properties, which arise due to the presence of two force centers rather than only one, must be valid not only for the two-fixed-center problem but also, upon suitable analytic extension, for the restricted three-body problem. That is, such transform properties and their specific functional forms must not be limited, by their inherent nature, to two-dimensional space and position-fixed potential fields. For example, the pedal, geometric inversion, and magnification (such as present in the hodograph transforms) are valid in three-dimension, as well as two-dimensional space. Consequently, the additional transform properties must enable or admit the compatibility of hodographic solutions with the classical theorems concerning solutions of the restricted three-body problem; in particular, Bruns' and Poincaré's theorems (Reference 6). Although these points will be discussed and analyzed extensively in Work Phase IIB, it is timely to note here, that the conclusions of Poincaré's theorem are directed to assumed solution forms (of the trajectory) which are single-valued functions. The vector space transformation for trajectories subject to more than one force center will necessarily be a multiple-valued function which maps the single-valued solution in the transform space into multiple branches in position vector space; only one of these multiple branches is the valid solution, as identified by the physical (or boundary) conditions.

The parametric solutions in higher orders of vector space than position vector space require the use either of translation ($w = z + \alpha$) or reflection ($w = \bar{z}$)*. Inversion ($w = 1/z$) provides geometric inversion (operating upon the modulus) and reflection (operating upon the argument), as illustrated by the following:

In complex form,

$$w = \frac{1}{z} \; ; \qquad (1\text{-}20A)$$

in polar form,

$$z = \rho e^{i\lambda} \qquad (1\text{-}21)$$

so that

$$w = \left[\frac{1}{\rho}\right] e^{-i\lambda} \; ; \qquad (1\text{-}20B)$$

that is,

$$\left[\frac{1}{\rho}\right] = \text{geometric inversion} \qquad (1\text{-}22)$$

$$e^{-i\lambda} = \text{reflection.} \qquad (1\text{-}23)$$

*Reflection may be taken "in a circle" or "in a straight line"; here, the reflection is defined as "a reflection in the real axis" only for descriptive purposes.

Since the hodograph transformation requires geometric inversion, it appears probable that the additional transform property is reflection, so that the required complete transform will provide inversion. It is noted that inversion ($w = 1/z$) is a conformal transformation, without reversion of angles.

Having decided to initiate transform synthesis with inversion, the coordinates for inversion must now be considered. That is, inversion can be taken with respect to lines or circles. Also, the conditions upon the correspondence of the vector space maps, as described previously, must be fulfilled by the transformation. Upon study of Reference 2, it appears that the required solution must be a biharmonic function, due to the presence of two force centers. But every biharmonic function can be expressed by two functions of a complex variable, as defined by Hurse's formula (Reference 8). This biharmonic characteristic suggests the use of a transformation which maps circles about the respective force centers, over into circles about the origin of the required transform space, by inversion.

Referring to Figure 1-3, it is obvious that each circle in the transform space represents two circles in the original vector space, one about force center A and the other about force center B. Consequently the inverse transformation (i.e., from transform to original vector space) is a multiple-valued function. Also, force centers A and B map over into the "points at infinity" in the transform space, whereas the "points at infinity" in the original vector space map over into the transform space origin. The selected coincidence of the concentric circles transformed from the original space will be determined by the magnification (or isomorphism) of the transform. For example, the magnification may be selected so that those circles (about centers A and B) which represent a given potential V may map over into coincidence; that is, $\mu_A/r_A = \mu_B/r_B$. As an alternative, those circles of equal radius about centers A and B may be mapped over into coincidence; that is, $r_A = r_B$. Note that the ordering of the circles in the transform space is inverse to the ordering of the sets of circles in the original space, relative to the origin of the given space; this property is shown by the directions of the arrows in Figure 1-3.

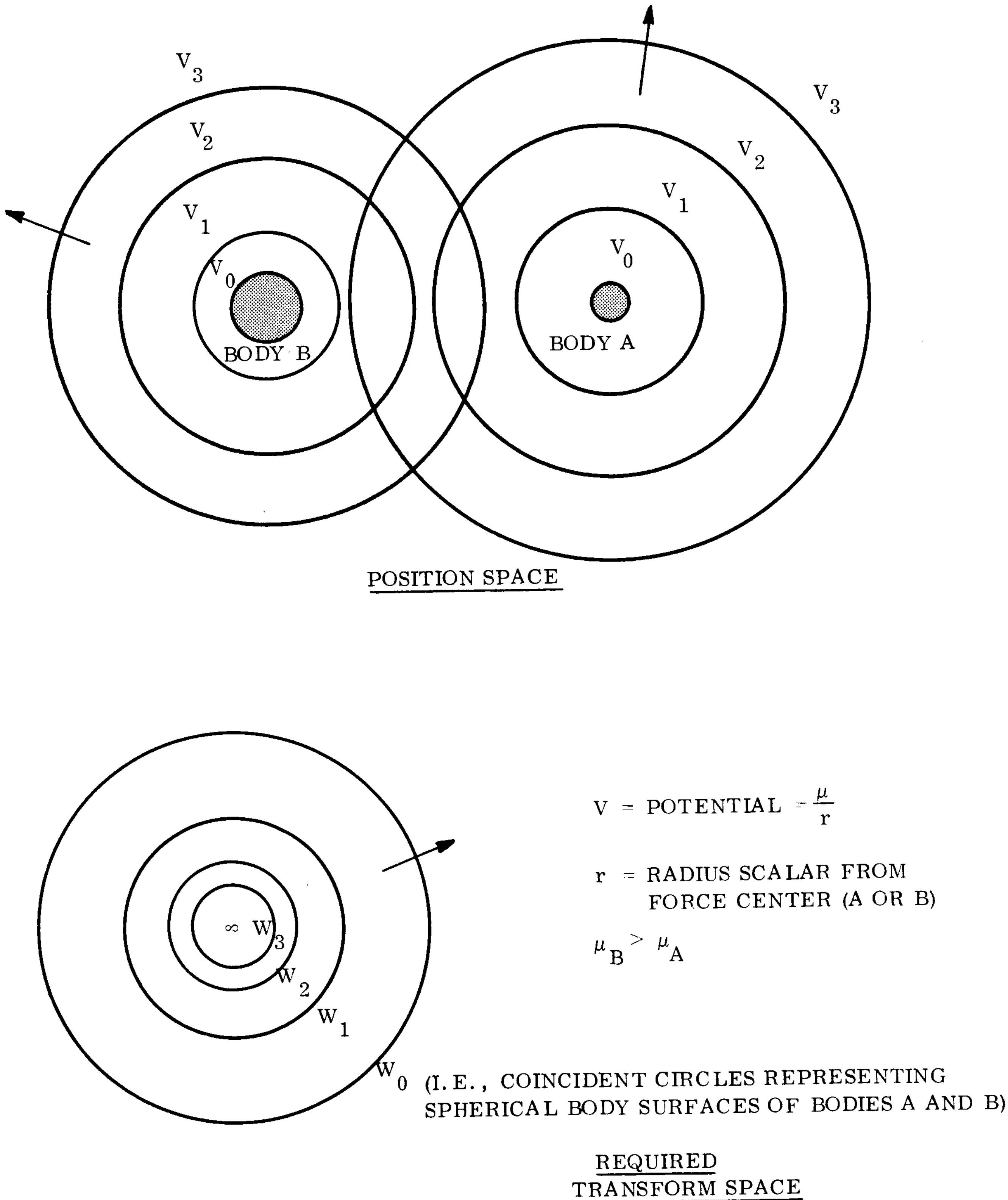



Figure 1-3. Proposed Biharmonic Transformation

The region of admissible trajectories in the original (i.e., position) space is the entire volume outside the two attracting celestial bodies. The required transform space maps this region within a bounded spherical volume about its origin; the spherical bound must represent the surfaces of the attracting bodies. This unique representation of two closed surfaces, which are spheres of different radii, requires the unique (rather than arbitrary) selection of magnification in the transformation. However, this element of the synthesis will be deferred until later in the work task, since it can be easily determined after successful synthesis of all preceding conditions.

Finally, it is noted that other transformations with potentially useful properties are not being used in the initial synthesis. For example, the specific form (Reference 9).

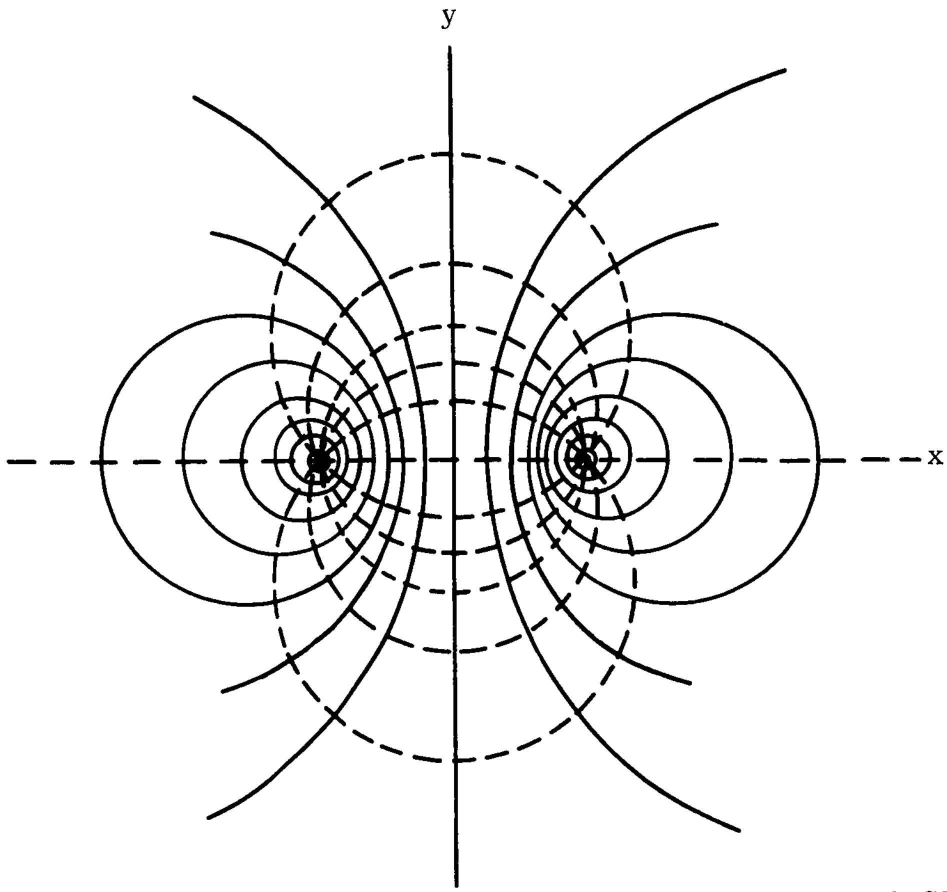
$$w = \frac{z - z_2}{z - z_1} \tag{1-24}$$

of the linear fractional transformation* provides the mapping shown in Figure 1-4. Note that the point $z_1$ maps over into the "points at infinity", whereas the point $z_2$ maps over into the origin. As another closely related example, bipolar coordinates have proven useful in past analyses of the celestial mechanics; for example, in approximating the "surfaces of zero relative velocity" in the restricted three-body problem (Reference 10). The transformation function for bipolar coordinates is

$$w = i \ln \left( \frac{z - z_2}{z - z_1} \right) . \tag{1-25}$$

This transform maps the point $z_2$ into $v = -\infty$, the point $z_1$ into $v = +\infty$, and the x-coordinate axis beyond the $z_2$-$z_1$ strip (i.e., $z < z_2$ and $z > z_1$) into the v-coordinate axis. However, these transforms do not meet the conditions a-c for the synthesis of the required transform function.

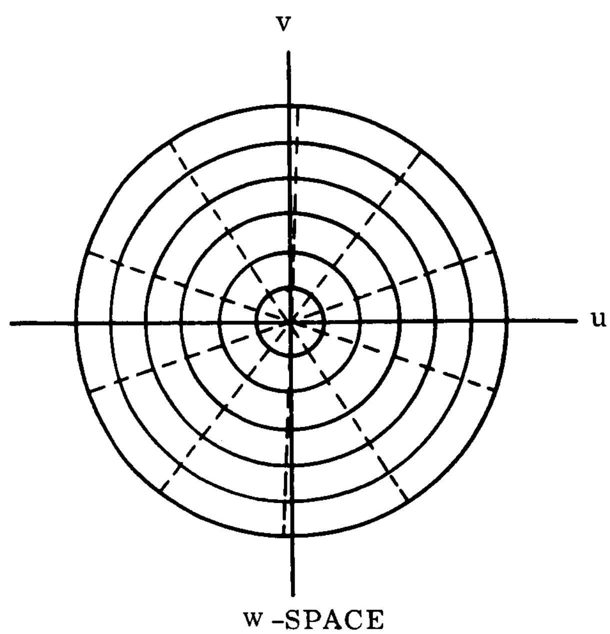
*Also termed a bilinear, or a Möbius transformation.




z -SPACE

NOTE: DOTTED CIRCLES ARE ORTHOGONAL TO SOLID CIRCLES.

DOTTED LINES ARE ORTHOGONAL TO SOLID CIRCLES.




w -SPACE

Figure 1-4. A Linear Fractional Transformation

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SECTION 2

# THE TWO INVARIANTS OF TWO-FIXED-CENTER ORBITS (PERIODIC AND APERIODIC)

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## SECTION 2

## THE TWO INVARIANTS OF TWO-FIXED-CENTER ORBITS

Classical techniques of advanced dynamics have provided analytic solution to the two-fixed-center problem, in terms of elliptic functions (i.e., by use of elliptic coordinates) (References 1 to 3). However, this form of solution is not only quite complex, but is neither tractable in application nor provides substantial insight into the more general (and realistic) model of the restricted three-body problem.

The vector space theory for trajectories in the presence of one force center has shown that a parametric form of trajectory solution is available not only in position vector space (i.e., by use of conic parameters), but also in velocity and acceleration vector spaces, as well as all relevant state spaces (References 4 to 6). That is, the locus of the trajectory in any two-dimensional state space can be specified in terms of parameters defined in the given state space (e.g., conic parameters a, e, $\Psi$, $\phi$ in position vector space, velocity parameters C, R, $\Psi$, $\phi$ in velocity vector space, etc.). The basic objective of Phase IIA is to develop the parametric form of two-fixed-center trajectory solution, comparable or analogous to the parametric form of one-force-center trajectory solution; specifically, by use of velocity parameters. A parametric form of trajectory solution must necessarily exist. It is speculated that, when available, this form of solution will not only be more tractable to mission application by machine computation or manual analysis, but will also be amenable to final transformation to a state space and form of locus suitable for extension to the restricted three-body problem.

Any given trajectory (as well as a given class of trajectories) is characterized by certain invariant properties (Reference 7). In particular, a ballistic trajectory will be defined by specific invariants and at least one time-dependent (or space-dependent) variable: cyclic for periodic motion, noncyclic for aperiodic motion. The invariants of the two-fixed-center problem, as obtained in classical theory, could be valuable in developing the equivalent invariants required for the parametric form of trajectory solution.

## 2.1 NATURE OF THE REQUIRED PARAMETRIC INVARIANTS

For the moment, let us consider the orbital trajectory in the presence of one force center. At a given instant of time, the complete orbital state (and consequently the attendant trajectory) is definable by, for example, the vector set ($\bar{r}$, $\bar{v}$). Each vector provides two scalars, modulus and argument, so that the vector definition of trajectory at one instant of time provides four variables. The equivalent parametric definition of trajectory must, necessarily, require four parameters; for example, C, R, $\Psi$, $\phi$. Note that three (C, R, $\Psi$) are invariants for a ballistic orbit with a simple spherical harmonic function of gravitational potential field; the true anomaly $\phi$ is the cyclic variable.

Now let us consider the orbital trajectory in the presence of two fixed centers, as shown schematically in Figure 2-1. At a given instant of time, the complete orbital state is definable by the vector set ($\bar{R}$, $\bar{V}$) referred to the barycenter of the system. That is, eight terms define the motion:

a. Two scalars (modulus and argument) due to each vector $\rightarrow$ 4 variables and

b. The distance of each force center from the barycenter, the direction of the axis of fixed centers, and the gravitational constant for each force center $\rightarrow$ 4 variables.

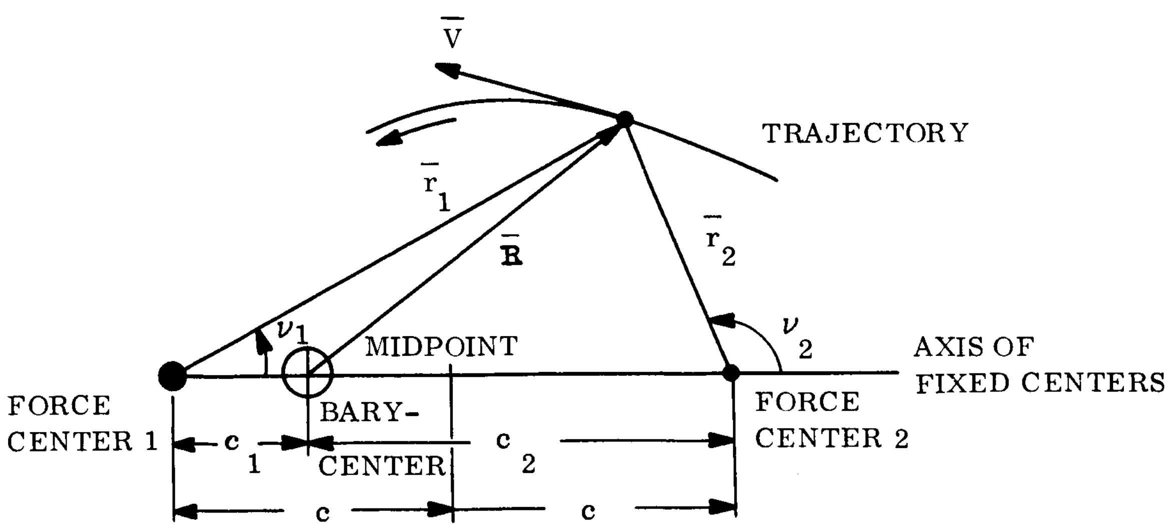


Figure 2-1. Space Geometry of the Two-Fixed-Center Problem

An alternative set of eight terms which do not refer to the barycenter directly can define the motion:

a. Two scalars (modulus and argument) due to each vector $\rightarrow$ 4 variables,

b. The bipolar angles $\nu_1$, $\nu_2 \rightarrow$ 2 variables, and

c. The gravitational constant for each force center $\rightarrow$ 2 variables.

With a hypothetical valid decomposition of the total velocity vector $\bar{V}$ into two components $\bar{v}_1$, $\bar{v}_2$ which, together with the given position vector $\underline{\bar{R}}$ (or $\bar{r}_1$, $\bar{r}_2$), will define the complete trajectory, the vector sets ($\bar{r}_1$, $\bar{v}_1$) and $\bar{r}_2$, $\bar{v}_2$) due to each force center will provide 4 + 4 = 8 terms. In the proposed parametric form of solution, eight corresponding parameters will be required: four due to each force center, yielding a total of eight parameters, e.g., ($C_1$, $R_1$, $\Psi_1$, $\phi_1$) and ($C_2$, $R_2$, $\Psi_2$, $\phi_2$). While $C_{( )}$, $R_{( )}$, $\Psi_{( )}$ are invariants, $\phi_1$ and $\phi_2$ are time-dependent (or space-dependent) cyclic variables.

Although $\phi_1$ and $\phi_2$ are cyclic variables, the orbital trajectory generated by use of the parametric generating functions is not necessarily cyclic (i.e., not necessarily a periodic orbit). A periodic orbit will be generated only if the period of the generating function due to one force center is a multiple of the period of the generating function due to the other force center. In other words, quoting Charlier (Reference 1): "The motion is periodic in time as often as $\omega_{21}$ and $\omega_{22}$ (the biperiodic function) are commensurable with one another." For example, if the periods are identical, the simplest figure of periodic orbit will be generated, i.e., a conic figure. In general, all other periodic orbits will be of complex geometric figure, not simply described in analytic form by differential geometry.

The parametric invariants ($C_1$, $R_1$, $\Psi_1$) and ($C_2$, $R_2$, $\Psi_2$) are required for the parametric form of solution. Moreover, the corresponding anomalies $\phi_1$, $\phi_2$ at any given instant of time must be identified by a valid governing algorithm to be determined. Both requirements might be deduced from

a. The algorithm or law for decomposition of the velocity vector $\overline{V}$ into the desired $\overline{v}_1$, $\overline{v}_2$, or

b. A useful form of the invariants for the classical solution.

Since the algorithm for velocity vector decomposition is not yet available, the useful form of the classical invariants would be desirable.

## 2.2 THE INVARIANT h'

The system energy per unit mass (h') is invariant, so that

$$h' = \frac{T+V}{m} = \frac{V^2}{2} - \left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) = \text{constant} . \tag{2-1}$$

In Cartesian coordinates (x, y),

$$T = m\left[(\dot{x})^2 + (\dot{y})^2\right]/2 \tag{2-2}$$

$$V = -\mu_1 m\left[(x-c)^2 + y^2\right]^{-\frac{1}{2}} - \mu_2 m\left[(x+c)^2 + y^2\right]^{-\frac{1}{2}} ; \tag{2-3}$$

in elliptic coordinates ($\xi$, $\chi$),

$$T = mc^2\left(\cosh^2\xi - \cos^2\chi\right)\left[(\dot{\xi})^2 + (\dot{\chi})^2\right]/2 \tag{2-4}$$

$$V = -\mu_1 m\left[c\left(\cosh\xi - \cos\chi\right)\right]^{-1} - \mu_2 m\left[c\left(\cosh\xi + \cos\chi\right)\right]^{-1} . \tag{2-5}$$

The invariant h' may be decomposed into any two scalars $h_1'$, $h_2'$ such that

$$h' = h_1' + h_2' \quad . \tag{2-6}$$

However, the required decomposition of the invariant h' for the parametric form of solution is not arbitrary, but depends upon the algorithm of velocity vector decomposition. For example, if the valid decomposition components of the velocity vector $\overline{V}$ were $\overline{v}_1$, $\overline{v}_2$, then

$$2h' = (v_1^2 + v_2^2) - 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) \tag{2-7A}$$

or

$$2h' = \left(v_1^2 - \frac{2\mu_1}{r_1}\right) + \left(v_2^2 - \frac{2\mu_2}{r_2}\right) \quad . \tag{2-7B}$$

Consequently,

$$\frac{h_1'}{h_2'} = \frac{v_1^2 - \frac{2\mu_1}{r_1}}{v_2^2 - \frac{2\mu_2}{r_2}} \tag{2-8}$$

so that

$$2h_1' + 2h_2' = \left(v_1^2 - \frac{2\mu_1}{r_1}\right) + \left(v_2^2 - \frac{2\mu_2}{r_2}\right) \tag{2-9A}$$

or

$$2h_1' + 2h_2' = \left(v_{1r}^2 + v_{1\nu}^2 - \frac{2\mu_1}{r_1}\right) + \left(v_{2r}^2 + v_{2\nu}^2 - \frac{2\mu_2}{r_2}\right) \quad . \tag{2-9B}$$

Now, for the purpose of demonstration of the decomposition, (or superposition) principle to be sought, let us assume that the contribution (to the total angular momentum $\underline{\overline{R}}\ V_\nu$) due to each force center is a constant. Then

$$(2h_1' + C_1^2) + (2h_2' + C_2^2) = \left[v_{1r}^2 + (v_{1\nu} - C_1)^2\right] + \left[v_{2r}^2 + (v_{2\nu} - C_2)^2\right] \tag{2-10A}$$

or

$$R_1^2 + R_2^2 = \left[v_{1r}^2 + (v_{1\nu} - C_1)^2\right] + \left[v_{2r}^2 + (v_{2\nu} - C_2)^2\right] \tag{2-10B}$$

where

$$C_1 = \mu_1 / r_1 v_{1\upsilon} \tag{2-11}$$

$$C_2 = \mu_2 / r_2 v_{2\upsilon} \tag{2-12}$$

$$R_1 = \sqrt{2h_1' + C_1^2} \tag{2-13}$$

$$R_2 = \sqrt{2h_2' + C_2^2} \quad . \tag{2-14}$$

Note that the demonstrative assumption implies the hypothesis of a valid algorithm for the decomposition of the total velocity vector $(\overline{V})$ into the component vectors $\overline{v}_1$, $\overline{v}_2$ which are each valid for one-force-center treatment, independent of one another. The subsequent investigation for the parametric solution for the two-fixed-center problem requires the determination of this required decomposition (or the superposition principle). The demonstrative assumption is not itself considered valid, at this time.

## 2.3 THE INVARIANT δ

A second invariant δ has been derived as a function of the concurrent angular momenta relative to each force center (References 3* and 8). That is,

$$\frac{\ell_1' \ell_2'}{2c} + \mu_1 \left( \frac{x+c}{r_1} \right) - \mu_2 \left( \frac{x-c}{r_2} \right) = \delta \tag{2-15}$$

or

$$\frac{\ell_1' \ell_2'}{2c} + \left( \mu_1 \cos \upsilon_1 - \mu_2 \cos \upsilon_2 \right) = \delta \tag{2-16}$$

where

$\ell_1' \equiv (x+c)\dot{y} - y\dot{x}$ = angular momentum per unit mass, relative to force center 1 (2-17)

$\ell_2' \equiv (x-c)\dot{y} - y\dot{x}$ = angular momentum per unit mass, relative to force center 2. (2-18)

*p. 283

Note that the variables $\ell_1'$, $\ell_2'$ are not mutually exclusive; that is,

$$\ell_1' \neq r_1 v_{1\nu}$$

$$\ell_2' \neq r_2 v_{2\nu} \quad .$$

From a physical viewpoint of the dynamics, both variables ($\ell_1'$, $\ell_2'$) encompass a common part of the total kinetic energy T.

The analytic equivalent of Equations 2-15 or 2-16, in terms of mutually exclusive (or component) angular momenta due to each force center, is required in order to assist development of the parametric formulation. That is, the invariant condition should be expressed in terms of $\ell_{1\nu}$, $\ell_{2\nu}$ such that

$$\ell_{1\nu} = m v_{1\nu} \tag{2-19}$$

$$\ell_{1r} = m v_{1r} \tag{2-20}$$

$$\ell_{2\nu} = m v_{2\nu} \tag{2-21}$$

$$\ell_{2r} = m v_{2r} \tag{2-22}$$

where

$$\overline{V} = f(\overline{v}_1, \overline{v}_2) \tag{2-23}$$

or

$$\overline{V} = f(v_{1\nu}, v_{1r}; v_{2\nu}, v_{2r}) \; . \tag{2-24}$$

The functional relation f is required for parametric representation of the two-fixed-center trajectory. One approach to determination of this functional relation (or algorithm) is study of the form of the two orbital invariants for particular solutions of known analytical form.

Note that "analytical form" may refer to the trajectory hodograph in velocity or acceleration vector space, rather than the trajectory figure in position vector space.

The orbital invariants of the parametric solutions will be expressed as functions of $\ell_{1\nu}$, $\ell_{2\nu}$ or $\ell_{1\nu}$, $\ell_{1r}$, $\ell_{2\nu}$, $\ell_{2r}$ rather than $\ell_1'$ $\ell_2'$.

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SECTION 3

# ELLIPTIC ORBITS OF THE TWO-FIXED-CENTER PROBLEM

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## SECTION 3

## ELLIPTIC ORBITS OF THE TWO-FIXED-CENTER PROBLEM

In the initial report (Section 1) which identified the problem genesis and major aspects of the new analytic approach to ballistic trajectories of the two-fixed-center problem, three major avenues of study effort were described. The preceding report (Section 2) discussed one such study direction: the nature and properties of the invariants (often referred to as "integral invariants"). Another study direction which holds great promise considers the detailed vector space analysis of admissible orbits of simple geometric form- periodic or aperiodic. This section presents the study results on such analysis of the class of elliptic orbits which have been proven, in the classical literature (References 1 and 2), to be admissible geometric figures of orbit in position vector space. Although such orbits are special cases, their properties may provide some knowledge about the required general trajectory. In any case, the properties of the general trajectory must necessarily define the properties of this special trajectory as a subclass.

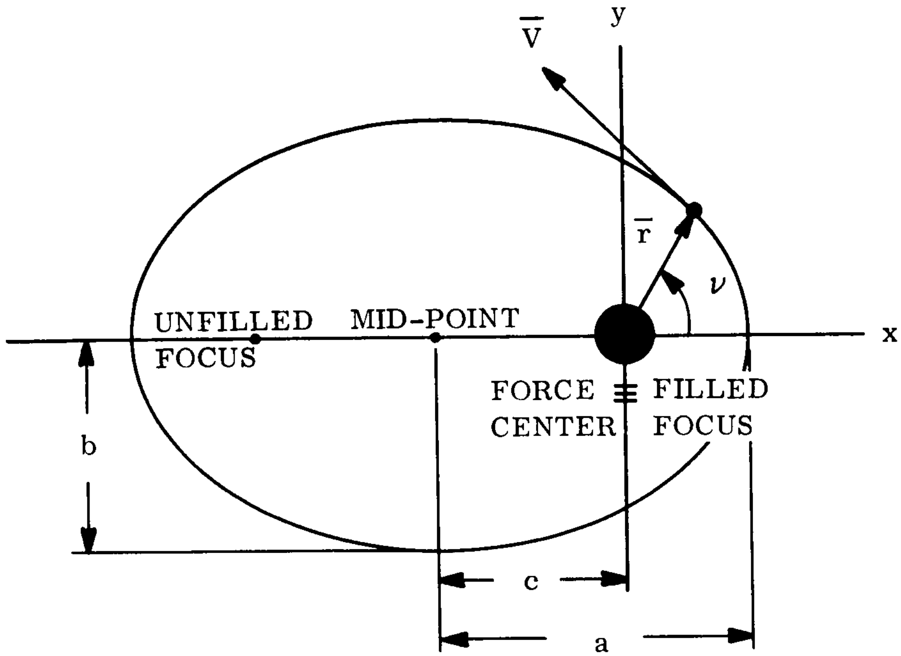
The elliptic orbit of the two-fixed-center problem is the simplest geometric figure of periodic (or cyclic) occurrence. However, classical literature contains little information on its properties or its relation to other special orbits or the general class of ballistic trajectory. Consequently, the research effort has required considerable time on exploratory analysis of such trajectories, prior to the vector space analysis itself. Apparently, the new concepts of the vector space theory are keystones to recognition of the trajectory characteristics which may be subsequently useful.

### 3.1 THE ELLIPSE AS AN ADMISSIBLE TWO-FIXED-CENTER ORBIT

It is well-known that the ellipse is an admissible geometric figure of orbit in the presence of one force center.* The properties of the ellipse (both as a geometric figure and as a single-force-center orbit) are used here for this analytical study. Consequently, for orbits about one force center, some of the definitive equations expressed in terms of the parameters of the ellipse are listed in Table 3-1. It is clearly advantageous to define the coordinate origin as the force center, with the apsidal line of the orbit (or major axis of

* It is understood that the gravitational potential field of the attracting force center is a simple spherical harmonic function.

Table 3-1. Definitive Equations of the Orbital Ellipse (One-Force-Center)




$$r = \frac{p}{1 + e \cos \nu} \tag{3-1}$$

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{3-2}$$

$$\ell^2 = \mu m^2 p \tag{3-3}$$

$$E = -\frac{\mu m}{2a} \tag{3-4}$$

$$V = -\frac{\mu m}{r} \tag{3-5}$$

where

$$\mu = GM \tag{3-6}$$

$$p = a\,(1 - e^2) \tag{3-7}$$

$$c = ae \tag{3-8}$$

the ellipse) as one coordinate axis. While Equations 3-1 through 3-5 are unique to the orbit about one force center, Equations 3-7 and 3-8 are geometric equations for the ellipse as a general conic.

It is easily shown by means of Bonnet's theorem (Reference 2) which is described in Subsection 3.6, that confocal ellipses (in which each force center is situated at a focus of the ellipse) are admissible orbits. The definitive equations of the confocal ellipse as a two-fixed-center orbit are listed in Table 3-2. In order to obtain the most tractable and compact forms of the equations for general analysis, the coordinate origin is defined as the geometric center, or the point (on the major axis of the ellipse) which is midway between the foci (i.e., the fixed force centers)*. As shown in the figure of Table 3-2, each of the two force centers (where, in general, $\mu_1 \neq \mu_2$) is located at opposing foci. Theoretically, both force centers could be located at either one of the foci; the two-fixed-center problem would then have degenerated to the one-force-center problem as covered by Table 3-1. In such a case, $\mu = \mu_1 + \mu_2$, $\bar{r} = \bar{r}_1 = \bar{r}_2$, $\nu = \nu_1 = \nu_2$.

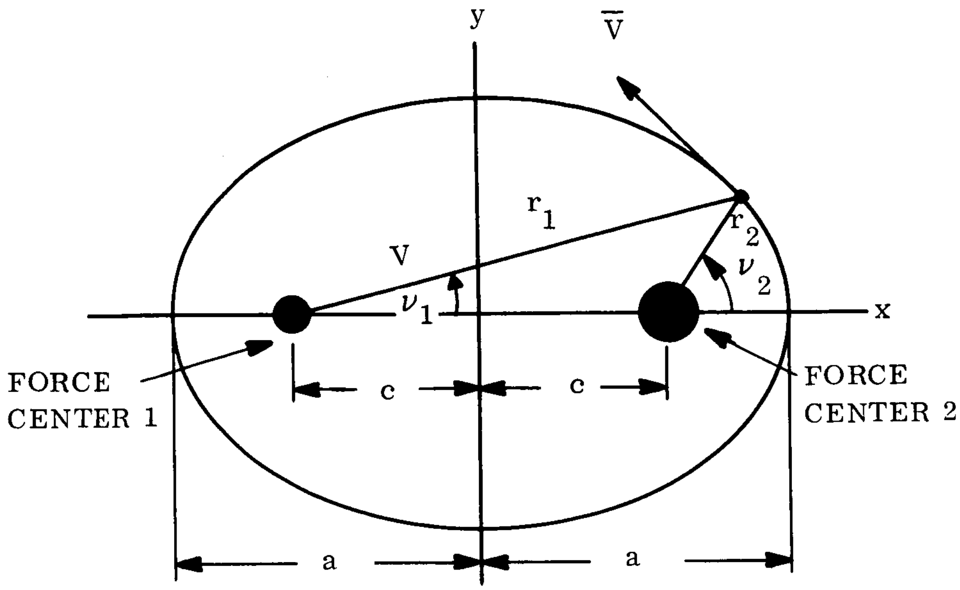
The angular momentum of the satellite in a two-fixed-center system is not simply described, as for the one-force-center system. The total angular momentum is not the sum of the angular momenta due to each force center alone, as for the scalars of energy E, potential V, and kinetic energy T. In the one-force-center system, angular momentum referred to the force center as the center of rotation is obviously conserved. Consequently, the analytic function, as presented in Equation 3-3, is simple. In the two-fixed-center system, the reference point is no longer simply located; in essence, angular momentum in a bipolar system is complex in functional form. This very point is the key obstacle to significant analytical advances in trajectory problems for two or more force centers. As discussed in Section 2, the invariant $\delta$ is presently defined in terms of concurrent angular momenta of the satellite, referred to each of the two force centers alone. However, the physical significance and utility of these concurrent momenta $\ell_1'$, $\ell_2'$ has not yet been determined.

## 3.2 TRAJECTORY HODOGRAPHS IN VELOCITY AND ACCELERATION VECTOR SPACES

The trajectory "hodograph" (in position vector space) of the two-fixed-center orbit is an ellipse. With the selected coordinate convention, the elliptic orbit in position vector

*However, the selection of the system barycenter as the coordinate origin might be advantageous in some analytic treatments. The use of a focus as the coordinate origin may be employed in specialized problem statements or for use of series approximation techniques in analysis.

Table 3-2. Definitive Equations of the Orbital Ellipse (Two-Fixed-Center)




$$r_1 + r_2 = 2a \tag{3-9}$$

$$\cos \nu_1 = \left[\frac{K + \cos \nu_2}{1 + K \cos \nu_2}\right] \tag{3-10}$$

where

$$K = \frac{2e}{1+e^2} \quad ; \tag{3-11}$$

$$V = V_1 + V_2 = -m\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) \tag{3-12}$$

$$E = E_1 + E_2 = -\frac{m}{2a}(\mu_1 + \mu_2) \tag{3-13}$$

$$T = T_1 + T_2 = \frac{m}{2}\left(v_1^2 + v_2^2\right) \tag{3-14}$$

$$V^2 = -\frac{(\mu_1 + \mu_2)}{a} + \frac{2}{p}\left[(1 - e\cos \nu_1)\mu_1 + (1 + e\cos \nu_2)\mu_2\right] \tag{3-15}$$

space is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3-16}$$

where

a = semi-major axis

b = semi-minor axis .

As noted in Table 3-1,

$$c = ae$$

where

$$e^2 = 1 - \frac{b^2}{a^2} \quad . \tag{3-17}$$

Knowing that this geometric figure is an admissible orbit, the velocity and acceleration hodographs must be developed so that subsequent study may reveal unique properties of the trajectory in the given vector space, as well as the transformations between vector spaces. The equations of the hodographs will be functions only of the parameters of the ellipse and satellite position in orbit.

Analytic development of the definitive equations of the velocity and acceleration hodographs as functions of the trajectory state in position vector space has been accomplished by means of the following algorithm, or set of sequential operations:

STEP 1: The derivative of the definitive equation in position vector space, with respect to time, is developed so that an equation as a function of $x$, $y$; $\dot{x}$, $\dot{y}$ is obtained. In general, this equation will not define velocity as a function of position explicitly.

STEP 2: Consequently, the derivative of the final equation of STEP 1, with respect to time, is developed so that an equation as a function of x, y; $\dot{x}$, $\dot{y}$; $\ddot{x}$, $\ddot{y}$ is obtained.

STEP 3: It is known that

$$\ddot{x} = -\frac{1}{m}\frac{\partial V}{\partial x} \tag{3-18}$$

$$\ddot{y} = -\frac{1}{m}\frac{\partial V}{\partial y} \quad , \tag{3-19}$$

where

$$V = f(\mu_1, \mu_2, a; x, y) \quad , \tag{3-20}$$

define valid relations between acceleration and position in separate, explicit form. These relations are the identical equations of the acceleration hodograph. Proceeding further to obtain the required velocity hodograph equations, the final equation of STEP 2 is reduced to a function of x, y; $\dot{x}$, $\dot{y}$ by means of Equations 3-18 and 3-19.

STEP 4: With the given definitive equation in position vector space (i.e., Equation 3-16) and the final equation of STEP 1, the final equation of STEP 3 is reduced to the required equations of the velocity hodograph.

In accordance with the above logic, the hodograph equations are developed as follows:

IMPL 1: Differentiating Equation 3-16 with respect to time,

$$\frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} = 0 \tag{3-21A}$$

or

$$\left(\frac{x}{a^2}\right)\dot{x} = -\left(\frac{y}{b^2}\right)\dot{y} \quad . \tag{3-21B}$$

IMPL 2: Differentiating Equation 3-21B with respect to time,

$$\frac{1}{a^2}\left[\dot{x}^2 + x\ddot{x}\right] = -\frac{1}{b^2}\left[\dot{y}^2 + y\ddot{y}\right] \tag{3-22}$$

IMPL 3: Since

$$V = -m\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right]$$

where

$$r_1 = \left[(x+c)^2 + y^2\right]^{\frac{1}{2}} \tag{3-23}$$

$$r_2 = \left[(x-c)^2 + y^2\right]^{\frac{1}{2}} , \tag{3-24}$$

then

$$V = -m\left\{\frac{\mu_1}{\left[(x+c)^2 + y^2\right]^{\frac{1}{2}}} + \frac{\mu_2}{\left[(x-c)^2 + y^2\right]^{\frac{1}{2}}}\right\} \tag{3-25}$$

so that, according to Equations 3-18 and 3-19,

$$\ddot{x} = -\left\{\frac{\mu_1(x+ae)}{\left[(x+ae)^2 + y^2\right]^{3/2}} + \frac{\mu_2(x-ae)}{\left[(x-ae)^2 + y^2\right]^{3/2}}\right\} \tag{3-26}$$

$$\ddot{y} = -y\left\{\frac{\mu_1}{\left[(x+ae)^2 + y^2\right]^{3/2}} + \frac{\mu_2}{\left[(x-ae)^2 + y^2\right]^{3/2}}\right\} . \tag{3-27}$$

Noting that

$$A^2 = \ddot{x}^2 + \ddot{y}^2 \quad , \tag{3-28}$$

it is seen that Equations 3-26 and 3-27 are the acceleration hodograph equations. Now, substituting Equations 3-26 and 3-27 into Equation 3-22 and rearranging,

$$\frac{\dot{x}^2}{a^2} + \frac{\dot{y}^2}{b^2} = \frac{y^2}{b^2}\left\{\frac{\mu_1}{\left[(x+ae)^2+y^2\right]^{3/2}} + \frac{\mu_2}{\left[(x-ae)^2+y^2\right]^{3/2}}\right\} +$$

$$+ \frac{x}{a^2}\left\{\frac{\mu_1(x+ae)}{\left[(x+ae)^2+y^2\right]^{3/2}} + \frac{\mu_2(x-ae)}{\left[(x-ae)^2+y^2\right]^{3/2}}\right\} . \tag{3-29A}$$

IMPL 4: Rearranging Equations 3-16 and 3-21B to obtain

$$y^2 = b^2\left[1 - \frac{x^2}{a^2}\right] \tag{3-30}$$

$$\dot{y} = -\left(\frac{b^2}{a^2}\right)\left(\frac{x}{y}\right)\dot{x} \tag{3-31}$$

respectively, Equations 3-30 and 3-31 are inserted into Equation 3-29A to obtain

$$\boxed{\dot{x}^2 = \left[\frac{\mu_1}{(a+ex)^2} + \frac{\mu_2}{(a-ex)^2}\right]\frac{(a^2-x^2)}{a}} \quad . \tag{3-32}$$

In similar fashion, we obtain

$$\boxed{\dot{y}^2 = \left[\frac{\mu_1}{(a+ex)^2} + \frac{\mu_2}{(a-ex)^2}\right]\frac{(1-e^2)x^2}{a}} \quad . \tag{3-33}$$

Noting that

$$V^2 = \dot{x}^2 + \dot{y}^2 \quad , \tag{3-34A}$$

it is seen that Equations 3-32 and 3-33 are the velocity hodograph equations, which, upon insertion into Equation 3-34A, provide

$$V^2 = \frac{a^2-(ex)^2}{a}\left[\frac{\mu_1}{(a+ex)^2} + \frac{\mu_2}{(a-ex)^2}\right] \quad . \tag{3-34B}$$

Note that the velocity hodograph Equations 3-32 and 3-33 are functions of x, in which

$$-a \leqslant x \leqslant a \tag{3-35}$$

is the region of solution. By means of Equation 3-30, the acceleration hodograph Equations 3-26 and 3-27 may be obtained in comparable form as

$$\boxed{\ddot{x} = -\left[\frac{\mu_1(x+ae)}{(a+ex)^3} + \frac{\mu_2(x-ae)}{(a-ex)^3}\right]} \tag{3-36}$$

$$\boxed{\ddot{y} = -\left[(1-e^2)(a^2-x^2)\right]^{\frac{1}{2}}\left[\frac{\mu_1}{(a+ex)^3} + \frac{\mu_2}{(a-ex)^3}\right]} \quad . \tag{3-37}$$

Upon examination of Equations 3-32 and 3-33, it is seen that a more concise form of Equation 29A is possible, as follows:

$$\frac{\dot{x}^2}{a^2} + \frac{\dot{y}^2}{b^2} = \frac{1}{a}\left[\frac{\mu_1}{(a+ex)^2} + \frac{\mu_2}{(a-ex)^2}\right] \quad . \tag{3-29B}$$

A few typical velocity hodographs and one acceleration hodograph were generated for demonstration and study of the hodograph characteristics. The hodographs of the following classes of elliptic orbits are presented in Figures 3-1 through 3-5:

VELOCITY HODOGRAPHS

$$\mu_1 = \mu_2 \begin{cases} \text{constant a , variable e} & \text{(Figure 3-1)} \\ \text{constant e , variable a} & \text{(Figure 3-2)} \\ \text{constant c(=ae), variable a and e} & \text{(Figure 3-3)} \end{cases}$$

$\mu_1 \neq \mu_2$: constant a , variable e (Figure 3-4)

ACCELERATION HODOGRAPH

$\mu_1 = \mu_2$ : $a = 10$ , $e = 0.50$ . (Figure 3-5)

Although these hodographs were generated by means of the explicit analytic equations for these state space vectors, typical sections of the various runs were verified by means of the complete dynamical equations of motion. Many observations may be made about the hodographs. As shown in Figure 3-6, the position space intercepts of the ellipse with the x-axis (or major axis) correspond with the velocity space intercepts of the velocity hodograph with the $\dot{y}$-axis; similarly, the y-axis intercepts of the ellipse correspond with the $\dot{x}$-axis intercepts of the velocity hodograph.

In all cases, orbital hodographs closer to the origin in velocity vector space represent orbits in position vector space farther from the origin (and the force centers). That is, geometric inversion occurs in the transformation from position to higher order vector spaces, just as in one-force-center dynamics. Also, the orbital energy of those trajectory hodographs closer to the velocity space origin is smaller.

Referring to the constant-energy class of orbital hodographs shown in Figure 3-1, it is seen that all hodographs coincide at common intersections with the $\dot{x}$-axis. The coincidence of the hodograph intersections with the $\dot{x}$-axis represents the constant-energy constraint for this

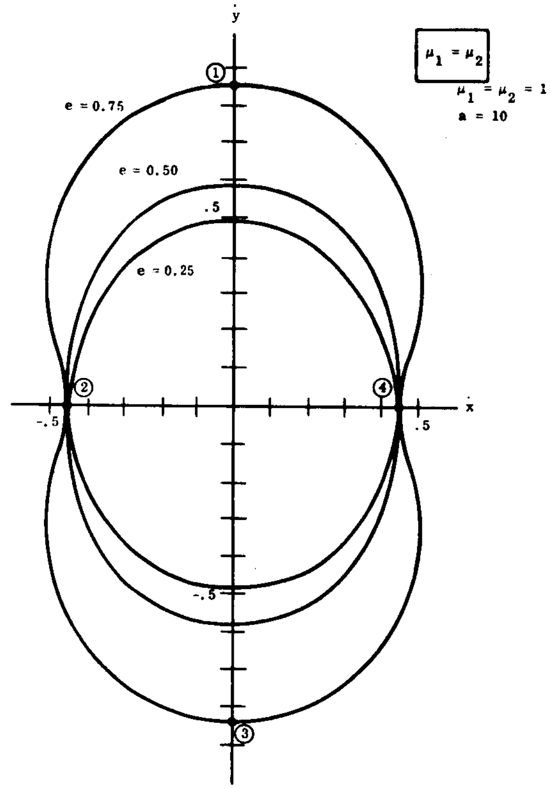



Figure 3-1. Velocity Hodographs for Two-Fixed-Center Elliptic Orbit (Constant Energy, $\mu_1 = \mu_2$)

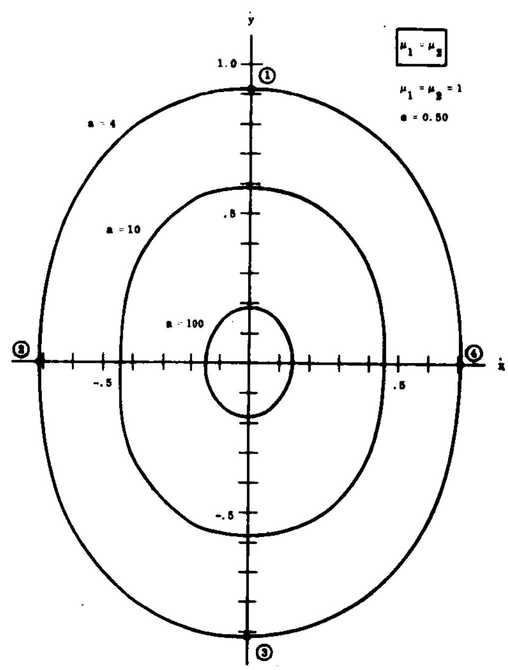



Figure 3-2. Velocity Hodograph for Two-Fixed-Center Elliptic Orbit (Constant Eccentricity, $\mu_1 = \mu_2$)

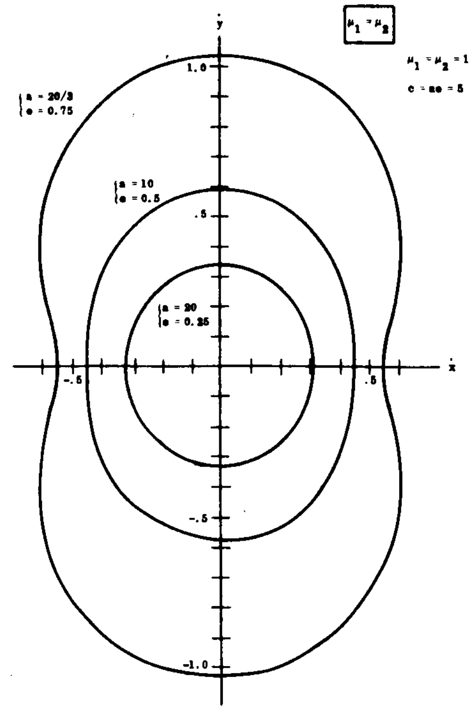



Figure 3-3. Velocity Hodograph for Two-Fixed-Center Elliptic Orbit (Constant Separation Distance, $\mu_1 = \mu_2$)

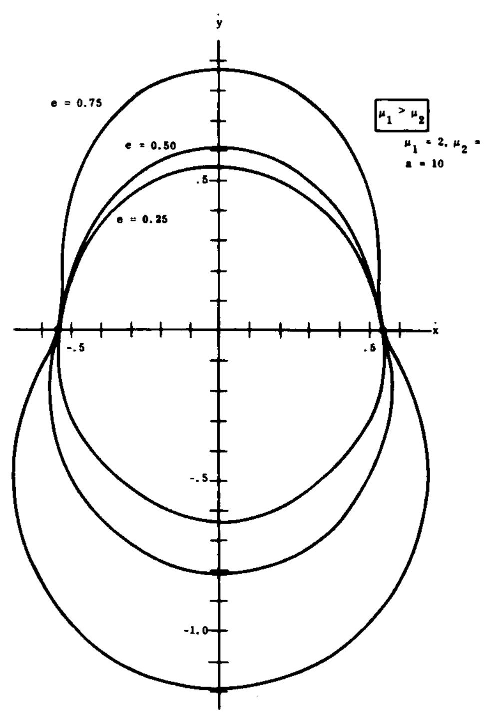



Figure 3-4. Velocity Hodographs for Two-Fixed-Center Elliptic Orbit (Constant Energy, $\mu_1 \neq \mu_2$)

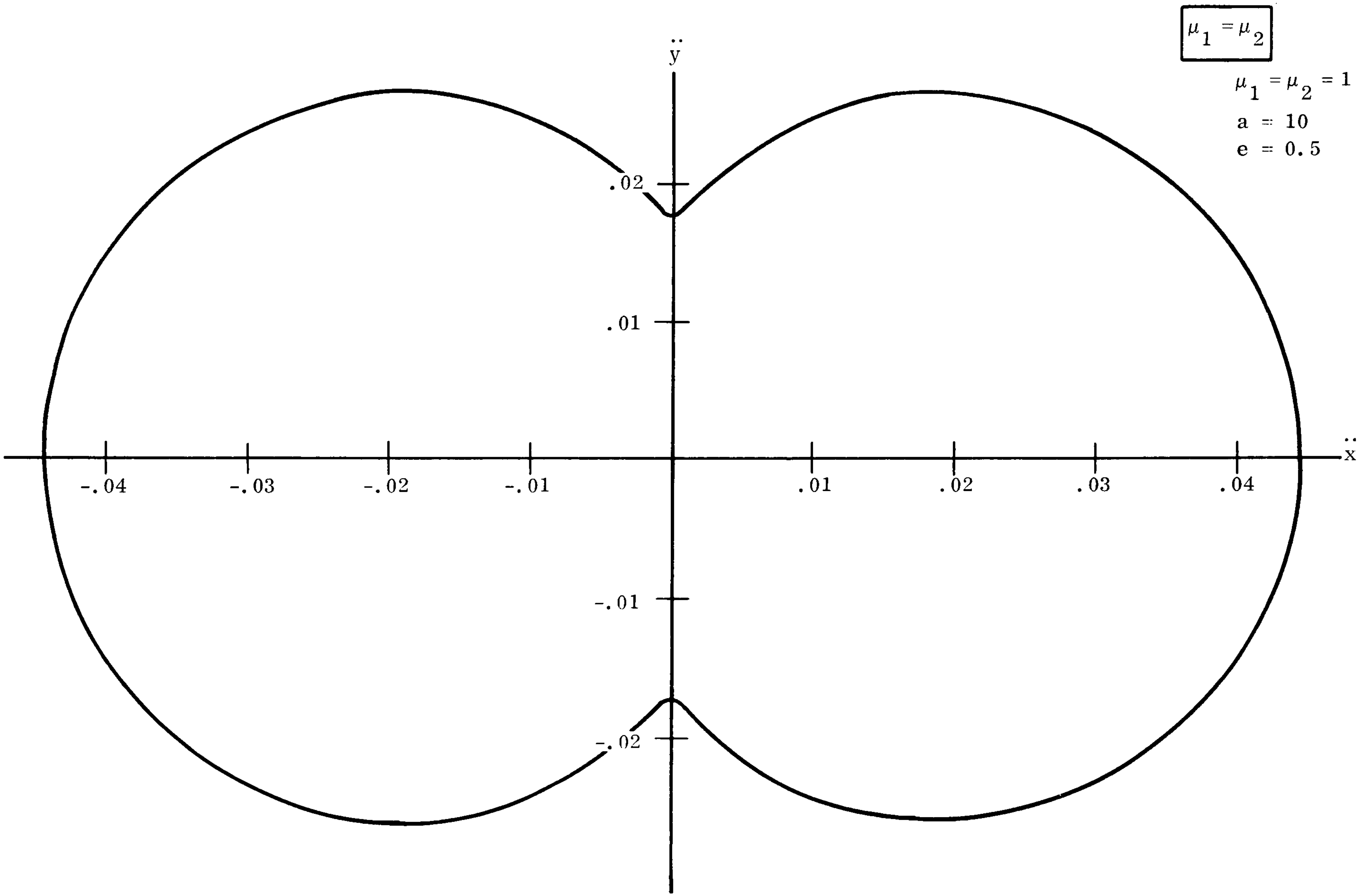



Figure 3-5. Acceleration Hodograph for Two-Fixed-Center Elliptic Orbit ($\mu_1 = \mu_2$)

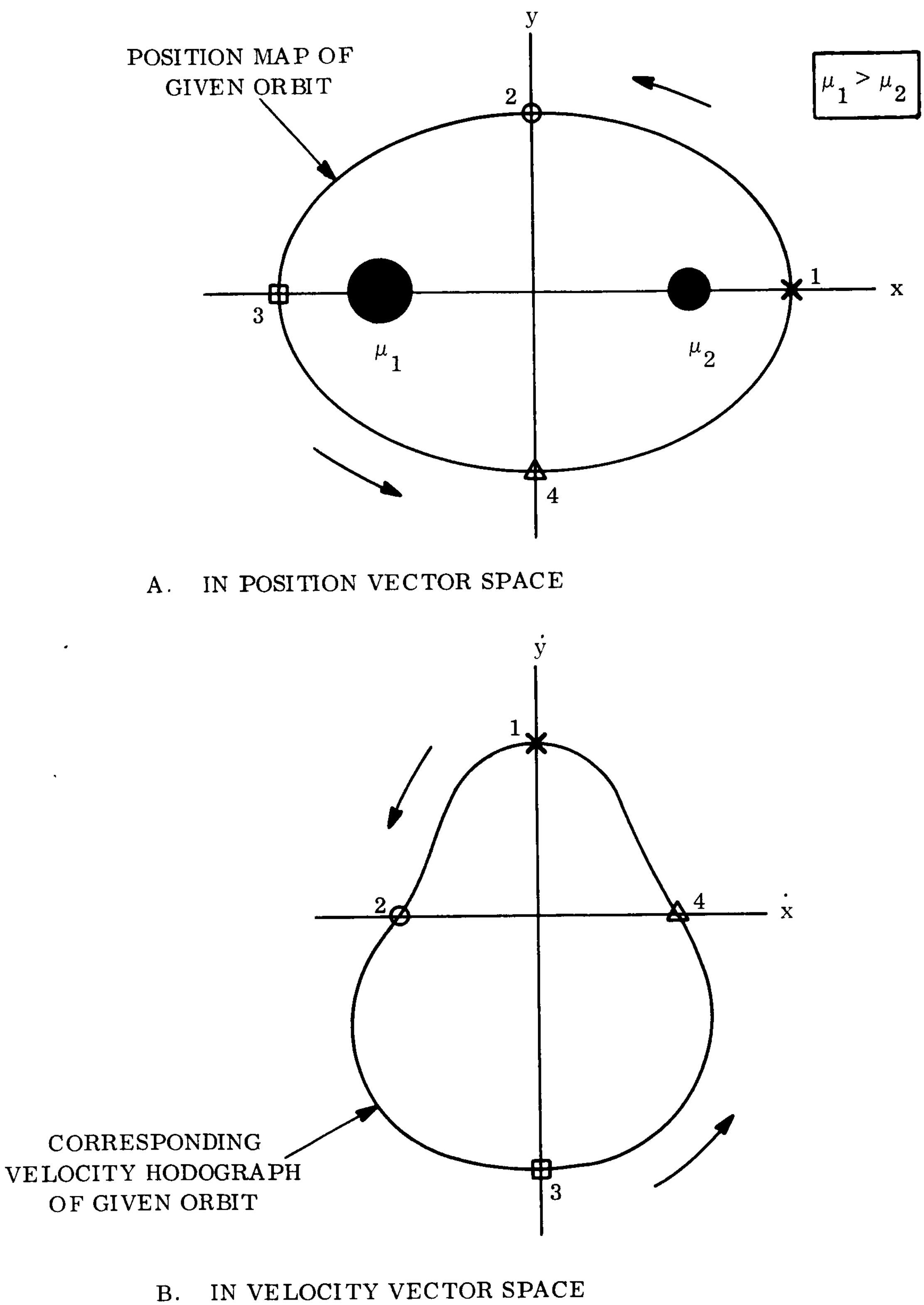



Figure 3-6. Correspondence of Vector Space Maps for Two-Fixed-Center Elliptic Orbits

family of orbits. This functional relation is exactly the same as for the one-force-center problem (Reference 3) as shown in Figure 3-7. Referring to the constant-energy hodographs for $\mu_1 \neq \mu_2$ as shown in Figure 3-4, this geometric condition is seen valid for general cases, even though the hodographs are no longer symmetrical about the $\dot{x}$-axis.

The velocity hodographs for $\mu_1 = \mu_2$, shown in Figures 3-1 through 3-3, are symmetrical about both the coordinate axes, whereas the velocity hodographs for $\mu_1 \neq \mu_2$, shown in Figure 3-4, are symmetrical only about the $\dot{y}$-axis. The $\dot{y}$-axis is the velocity vector space counterpart of the x-axis, which is the axis of the two fixed centers. That is, the transformation from position to velocity vector space produces a phase advance of $\pi/2$, just as in the one-force-center problem. To substantiate this deduction, the acceleration hodograph shown in Figure 3-5 produces a further phase advance of $\pi/2$; of course, graphical demonstration of this condition (rather than a phase retardation or lag of $\pi/2$) would be provided by an acceleration hodograph for $\mu_1 \neq \mu_2$. However, analytical examination of Equations 3-36 and 3-37 for the acceleration hodograph confirms this conclusion.

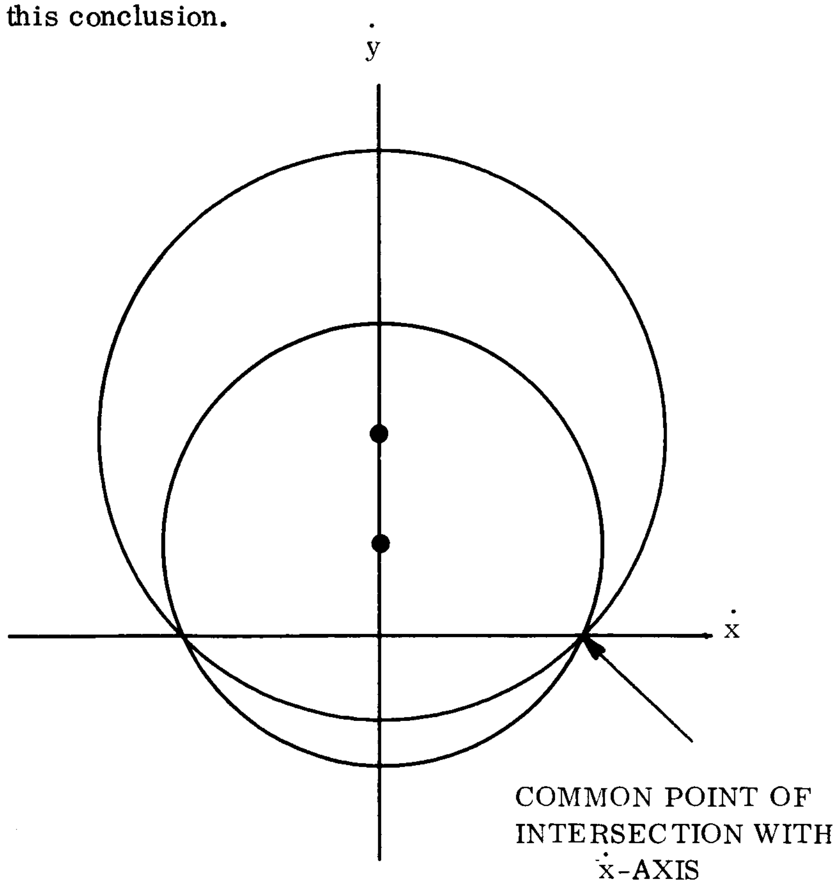



Figure 3-7. Constant-Energy Family of One-Force-Center Elliptic Orbits

Concurrent study of Equation 3-15 and the velocity hodographs (Figures 3-1 through 3-4) show that a constant component of the velocity is always present, due to the term

$$-\frac{\mu_1 + \mu_2}{a} + \frac{2(\mu_1 + \mu_2)}{p},$$

with periodic variation with $\nu_1$, $\nu_2$ which are of equal period. The first part of this constant term (which could be shown by a circle about the origin in velocity vector space) represents the energy invariant h (Reference 4). It is speculated that, upon comparable study of Equations 3-36 and 3-37 for the acceleration hodograph, a constant or invariant term of the total acceleration scalar (i.e., of the geometric figure of the acceleration hodograph) may also be identified.

Since two invariants of the two-fixed-center trajectory exist, the other invariant must also be definable in the velocity vector space. That is, the geometric figure of the velocity hodograph must be a function of two invariants which are dimensionally velocities (e.g., ft/sec ). This functional relation is identified in a later section of this report.

## 3.3 HODOGRAPH SUPERPOSITION TECHNIQUE FOR GENERATING THE ELLIPTIC TWO-FIXED-CENTER ORBIT

The development of the general parametric formulation of the two-fixed-center trajectory solution requires the definition and use of the second invariant as discussed briefly above. Although this analysis objective has not yet been attained, the utility of the consequent parametric formulation should be apparent with this special case of confocal elliptic orbits. For example, the parametric formulation should enable generation of the trajectory hodograph (and consequently its corresponding map in any other vector or state space) by means of the hodographs due to each force center alone. That is, the hodographic solution for the one-force-center problem will provide the two-fixed-center solution by an algorithm of hodographic superposition. The existence of such a superposition principle is assured by the fact that the required solution is a biharmonic function (Reference 1); every biharmonic function can be expressed by two analytic functions of a complex variable (Reference 5). Although we do not yet have the general parametric formulation, the well-defined knowledge of the special class of confocal ellipses must enable the analytic development of the special form of the superposition principle for the elliptic orbit. Not only would the existence of the hodograph superposition technique for this special class of orbit be essential in order for the general technique to exist, but the unique properties of the algorithm must be embodied within the algorithm for superposition generation of the general solution. That is, the algorithm of the special case will provide essential clues about the general case.

According to Bonnet's Theorem, two conditional relations between the velocity vectors ($\bar{v}_1$, $\bar{v}_2$, $\bar{V}$) must be fulfilled at each and every point of the orbit:

a. The arguments or directions of all velocity vectors must be identical; that is,

$$\arg \bar{v}_1 \equiv \arg \bar{v}_2 \tag{3-38}$$

$$\equiv \arg \bar{\bar{V}} \tag{3-39}$$

b. The magnitudes of the velocity vectors must be related by their squares, as the sum of the composite vectors; that is,

$$V^2 = v_1^2 + v_2^2 \quad . \tag{3-40}$$

Aside from the essential value in synthesizing the algorithm of the superposition technique for the confocal elliptic orbit itself, these conditions are significant in two other respects. First, these conditions are the direct result of superposition of the fields of force. Consequently, the existence of the required superposition for this special class of orbit is obviously established. Second, Equation 3-40 shows that the velocity vector decomposition suggested in Reference 4, i.e., that

$$\bar{\bar{V}} = \bar{v}_1 + \bar{v}_2 \quad , \tag{3-41}$$

cannot be valid* since it would require that

$$V^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \cos \alpha \tag{3-42}$$

where $\alpha$ = direction angle between the vectors $\bar{v}_1$, $\bar{v}_2$. Obviously, Equation 3-42 can reduce to Equation 3-40 only when $\alpha = n\pi/2$ (n = odd integer); but this reduction condition is impossible by virtue of Equations 3-38 and 3-39.

The above noted conditions define the following superposition algorithm to generate the velocity hodograph of the elliptic orbit, as shown schematically in Figure 3-8:

a. Given an initial position of the spacecraft in the two-fixed-center system and the definitive parameters of the confocal ellipse through that point in position vector space, derive the hodograph parameters for each force center alone**.

b. The velocity hodographs due to each force center will then be defined in velocity vector space (Figure 3-8.)

c. For any one value of the angle variable $\nu_1$, the corresponding angle variable $\nu_2$ is determined by the collinearity of $\bar{v}_1$ and $\bar{v}_2$ (Figure 3-8).

d. For any one value of the angle variable $\nu_1$, the magnitudes of $\bar{v}_1$ and $\bar{v}_2$ are determined by the intersection of the line with each of the single-force-center hodographs.

e. The magnitude of the total velocity vector (i.e., the two-fixed-center orbit velocity) is then

$$V = \sqrt{v_1^2 + v_2^2} \quad .$$

*Note that this statement does not invalidate the hypothesis of mutually exclusive angular momenta due to each force center.

**Naturally, the conic parameters of the confocal ellipse can also be directly derived, if desired.

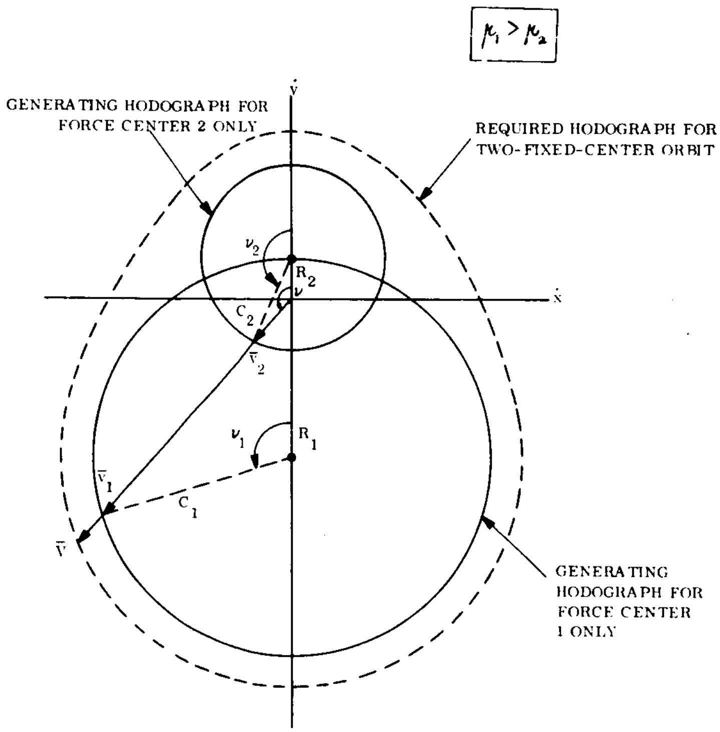



Figure 3-8. Algorithm Geometry for Superposition Technique of Generating Two-Fixed-Center Elliptic Orbits

This superposition technique is valid for unequal as well as equal force centers (i.e., $\mu_1 \neq \mu_2$). Note that the angle variable $\nu$ which refers the velocity vector $\overline{V}$ to the $\dot{y}$ -axis maps over into the angle (in position vector space) which refers the radial line between the position space origin and the spacecraft point on the elliptic orbit, to the x-axis.

The interdependence of $\nu_1$ and $\nu_2$ occurs as the result of the bipolar nature of the problem. However, it is noted that these two angle variables are always related by a function of the hodograph parameters ($C_1$, $R_1$, $C_2$, $R_2$), which are invariant. Consequently, the interdependence of the angle variables is a space-dependent, not a time-dependent, relation. This functional property is essential for extension and development of the superposition principle.

## 3.4 THE INVARIANTS h, δ FOR THE ELLIPTIC ORBIT

The two invariants, h and δ, of the two-fixed-center problem have been briefly discussed in Reference 4. It is extremely valuable to study these invariants for the elliptic orbit, since superposition implies, necessarily, the separation of component terms of these two invariants, each component due to the individual force centers alone. That is, determination of h (hence E) and $\delta$ (hence $\ell_\nu'$)* for each of the two generating hodographs, due to the individual force centers, will define the hodograph parameters C and R, since

$$C = \frac{\mu}{\ell_\nu'} \tag{3-43}$$

$$R = \sqrt{-h + \left(\frac{\mu}{\ell_\nu'}\right)^2} \quad . \tag{3-44}$$

The invariant h is directly defined by the total orbital energy E, as

$$h = \frac{2E}{m} \quad . \tag{3-45}$$

Consequently, the invariant h for the elliptic orbit has, in essence, been treated by the preceding work, since

$$E = E_1 + E_2 = -\frac{m}{2a}(\mu_1 + \mu_2) \ .$$

That is, the orbital energy is decomposed into

$$E_1 = -\frac{\mu_1 m}{2a} \tag{3-46}$$

$$E_2 = -\frac{\mu_2 m}{2a} \tag{3-47}$$

or, for the invariant h,

$$h_1 = -\mu_1/a \tag{3-48}$$

$$h_2 = -\mu_2/a \quad . \tag{3-49}$$

*Note that, in accordance with the notation convention established in Reference 4, $\ell_{\nu 1}'$, $\ell_{\nu 2}'$ define mutually exclusive angular momenta about each force center due to the component velocity vectors $\bar{v}_1$, $\bar{v}_2$, whereas $\ell_1'$, $\ell_2'$ define the non-exclusive angular momenta about each force center, each due to the total velocity vector $\bar{V}$.

Actually, this decomposition for E or h is the immediate aspect of the energy conservation principle. Upon comparison of Equations 3-46 and 3-47, it is seen that

$$\frac{E_1}{E_2} = \frac{\mu_1}{\mu_2} \qquad (3\text{-}50)$$

Similarly, the invariant $\delta$ should decompose into the components $\delta_1$ and $\delta_2$ due to each force center. First, the complete invariant $\delta$ may be expressed in terms of the conic parameters of the orbit. The general forms (Reference 4) of the invariant $\delta$ are

$$\frac{\ell_1'\ell_2'}{2c} + \mu_1\left(\frac{x+c}{r_1}\right) - \mu_2\left(\frac{x-c}{r_2}\right) = \delta \qquad (3\text{-}51)$$

or

$$\frac{\ell_1'\ell_2'}{2c} + \left(\mu_1 \cos v_1 - \mu_2 \cos v_2\right) = \delta \qquad (3\text{-}52)$$

where

$\ell_1' = r_1 V_{v1} = (x+c)\dot{y} - y\dot{x} =$ angular momentum per unit mass, relative to force center 1 (3-53)

$\ell_2' = r_2 V_{v2} = (x-c)\dot{y} - y\dot{x} =$ angular momentum per unit mass, relative to force center 2. (3-54)

Note that

$$V^2 = V_{r1}^2 + V_{v1}^2 \qquad (3\text{-}55)$$

or

$$V^2 = V_{r2}^{\ 2} + V_{\nu 2}^{\ 2} \quad . \tag{3-56}$$

As shown in Subsection 3.7,

$$\frac{\ell_1'\ell_2'}{2c} = \frac{pV^2}{2e} \tag{3-57}$$

and

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \frac{E}{em}(p-2a) - \frac{pV^2}{2e} \tag{3-58A}$$

or

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \left(\frac{1+e^2}{2e}\right)(\mu_1+\mu_2) - \frac{pV^2}{2e} \quad . \tag{3-58B}$$

Consequently, upon use of Equations 3-57 and 3-58 in Equation 3-52, the invariant $\delta$ is expressed as

$$\delta = \frac{E}{em}(p-2a) \tag{3-59}$$

$$= -\frac{Ea}{m}\left[\frac{1}{e^2}+1\right] \quad . \tag{3-60}$$

Now, let us decompose the invariant $\delta$ into the components due to each force center. Upon substituting Equation 3-40 into Equation 3-57,

$$\frac{\ell_1'\ell_2'}{2c} = \frac{pV_1^{\ 2}}{2e} + \frac{pV_2^{\ 2}}{2e} \quad . \tag{3-61}$$

Consequently, Equation 3-52 for the total invariant $\delta$ can now be expressed as

$$\delta = \delta_1 + \delta_2 \tag{3-62}$$

where

$$\delta_1 = \frac{pv_1^2}{2\epsilon} + \mu_1 \cos \nu_1 \tag{3-63A}$$

$$\delta_2 = \frac{pv_2^2}{2\epsilon} - \mu_1 \cos \nu_2 \quad . \tag{3-64A}$$

But, from Equation 3-130,

$$\mu_1 \cos \nu_1 = - \frac{\mu_1 p}{\epsilon r_1} + \frac{\mu_1}{\epsilon} \quad . \tag{3-65}$$

The energy equation

$$E_1 = \frac{mv_1^2}{2} - \frac{\mu_1 m}{r_1} \tag{3-66}$$

provides

$$\frac{\mu_1}{r_1} = \frac{v_1^2}{2} - \frac{E_1}{m} \tag{3-67}$$

so that, upon substituting Equation 3-67 into Equation 3-65,

$$\mu_1 \cos \nu_1 = - \frac{pv_1^2}{2\epsilon} + \frac{pE_1}{\epsilon m} + \frac{\mu_1}{\epsilon} \quad . \tag{3-68}$$

Consequently, substituting Equation 3-68 into Equation 3-63A,

$$\delta_1 = \frac{1}{e}\left[\mu_1 + \frac{pE_1}{m}\right] \quad . \tag{3-63B}$$

Then, by use of Equations 3-7 and 3-46,

$$\delta_1 = \frac{\mu_1}{2e}\left(1 + e^2\right) \quad . \tag{3-63C}$$

Similarly,

$$\delta_2 = \frac{1}{e}\left[e_2 + \frac{pE_2}{m}\right] \tag{3-64B}$$

or

$$\delta_2 = \frac{\mu_2}{2e}\left(1 + e^2\right) \quad . \tag{3-64C}$$

Upon comparison of Equations 3-63C and 3-64C, it is seen that

$$\frac{\delta_1}{\delta_2} = \frac{\mu_1}{\mu_2} \quad . \tag{3-69}$$

Finally, the relation between the invariants $\delta$, $\delta_1$, $\delta_2$ and the angular momenta $\ell_\nu'$, $\ell_{\nu 1}'$, $\ell_{\nu 2}'$ must be established, in order that the hodograph parameters C and R be defined by means of $\delta_{(\ )}$ (Equations 3-43 and 3-44) . It is known (Reference 6) that

$$p = \frac{\ell_\nu'^2}{\mu} \tag{3-70}$$

so that Equation 3-63C can be expressed in terms of $\ell_{\nu 1}'$ rather than $\mu_1$, by

$$\delta_1 = \left[\frac{1 + e^2}{2pe}\right] \ell_{\nu 1}'^2 \quad ; \tag{3-63D}$$

similarly,

$$\delta_2 = \left[\frac{1+e^2}{2pe}\right] \ell_{v2}'^{\,2} \quad . \tag{3-64D}$$

Consequently,

$$\delta = \left[\frac{1+e^2}{2pe}\right] \left(\ell_{v1}'^{\,2} + \ell_{v2}'^{\,2}\right) \tag{3-71}$$

and

$$\frac{\delta_1}{\delta_2} = \left(\frac{\ell_{v1}'}{\ell_{v2}'}\right)^2 \quad . \tag{3-72}$$

In summary, it has been found that the hodograph superposition for the two-fixed-center elliptic orbit is due to the analytical decomposition of the basic invariants, as follows:

$$\boxed{\begin{gathered} E = E_1 + E_2 \\ \delta = \delta_1 + \delta_2 \\ \frac{E_1}{E_2} = \frac{\delta_1}{\delta_2} = \frac{\mu_1}{\mu_2} = \left(\frac{\ell_{v1}'}{\ell_{v2}'}\right)^2 \end{gathered}} \quad . \qquad \begin{matrix}(3\text{-}50)\\(3\text{-}69)\\(3\text{-}72)\end{matrix}$$

The hodograph parameters (which are also invariants) can now be determined. Since

$$\ell_{v1}' = \frac{\mu_1}{C_1} \tag{3-73}$$

and

$$\ell_{v2}' = \frac{\mu_2}{C_2} \quad , \tag{3-74}$$

then

$$\frac{\ell_{v1}'}{\ell_{v2}'} = \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{C_2}{C_1}\right) \tag{3-75}$$

so that, upon substitution of Equation 3-75 into

$$\left(\frac{\ell_{v1}'}{\ell_{v2}'}\right)^2 = \frac{\mu_1}{\mu_2} \tag{3-76}$$

and subsequent reduction, we obtain

$$\left(\frac{C_1}{C_2}\right)^2 = \frac{\mu_1}{\mu_2} \quad . \tag{3-77}$$

Also, since

$$R_1^2 - C_1^2 = -\frac{2E_1}{m} \tag{3-78}$$

and

$$R_2^2 - C_2^2 = -\frac{2E_2}{m} \quad , \tag{3-79}$$

then

$$\frac{R_1^2 - C_1^2}{R_2^2 - C_2^2} = \frac{E_1}{E_2} = \frac{\mu_1}{\mu_2} \qquad . \tag{3-80}$$

so that, upon substitution of Equation 3-77 into Equation 3-80 and subsequent reduction,

$$\left(\frac{R_1}{R_2}\right)^2 = \frac{\mu_1}{\mu_2} \qquad . \tag{3-81}$$

The hodograph parameters (or invariants) are summarized as follows:

$$\boxed{\left(\frac{C_1}{C_2}\right)^2 = \frac{\mu_1}{\mu_2} = \left(\frac{R_1}{R_2}\right)^2} \qquad . \qquad \begin{matrix}(3\text{-}77)\\(3\text{-}81)\end{matrix}$$

It is clear that the ratio $(\mu_1 : \mu_2)$ will determine the geometric constraints which the generating hodographs due to each force center must fulfill, as discussed in the preceding section and shown in Figure 3-8.

## 3.5 SUMMARY CONCLUSIONS

The equations of the velocity and acceleration hodographs for the confocal elliptic orbit of the two-fixed-center problem have been developed. The functional dependence of the hodograph geometry upon the various parameters of the elliptic orbit has been briefly explored by mapping a few typical cases (Figures 3-1 through 3-5).

It has been shown that a superposition algorithm for generating the orbit in velocity vector space, by means of the hodographs due to each force center alone, exists; also, this superposition technique has been completely defined. Moreover, simple and basic relations between the invariants - the classical "integral invariants" $(h\ ,\ \delta)$, the hodograph parameters (C, R), and the mutually exclusive angular momenta $(\ell'_{\nu 1},\ \ell'_{\nu 2})$ - have been established. Obviously, the desired algorithm of superposition for generating any admissible orbit must encompass or include this special algorithm as a subclass. Since the basic elements of the algorithm are nonsimple due to the vector conditions expressed by Equations 3-38 to 3-40, the difficulties of the analytical search become quite apparent. That is, the algorithm is fundamentally nonlinear, so that many more candidate logical elements are available in development of the general algorithm of superposition for all admissible orbits. However, all analytical properties are consistent with the dynamical and geometric principles which must be fulfilled, if a general superposition principle does exist. Moreover, the laws of analytical decomposition for the basic invariants, expressed by Equations 3-13, 3-50, 3-62, 3-69 and 3-72, provide promising clues to further research development of the parametric form of the general trajectory solution.

Further study of the properties of the velocity hodograph of the elliptic orbit is possible and desirable. However, the equations and hodographs of the orbit in acceleration vector space should first (or concurrently) be studied. Reference to Equations 3-18 and 3-19 indicates that functional analysis in acceleration vector space may prove quite fruitful.

## 3.6 BONNET'S THEOREM

Bonnet's Theorem identified a particular class of trajectory occurring in the presence of a force field due to more than one force center or source. The trajectory must fulfill specific conditions of motion in the presence of each force center or source alone. The theorem may be stated as follows (Reference 2):

> "If a given orbit can be described in each of m given fields of force, taken separately, the velocities at any point P of the orbit being $v_1, v_2, \ldots, v_j$ respectively, then the same orbit can be described in the field of force which is obtained by superposing all these fields, the velocity at the point P being $(v_1^2 + v_2^2 + \ldots + v_j^2)^{1/2}$."

In order to understand this theorem and its potential application, let us develop its logical proof.

Consider the typical physical problem shown in Figure 3-9. The particular trajectory path in position vector space occurs in the presence of "k" bodies producing a composite field due to "j" force centers, in which k ≥ j. The intrinsic equations of the orbital path are, in general,

$$A_t = V\frac{dV}{ds} \quad = \text{tangential component of total acceleration} \tag{3-82}$$

$$A_n = \frac{V^2}{\rho} \quad = \text{principal normal component of total acceleration} \tag{3-83}$$

where, at any given point on the trajectory, the instantaneous radius of curvature is defined as "$\rho$". Then

$$\overline{F}_{T_n} = -\frac{mV^2}{\rho}\frac{\overline{\rho}}{|\overline{\rho}|} \tag{3-84}$$

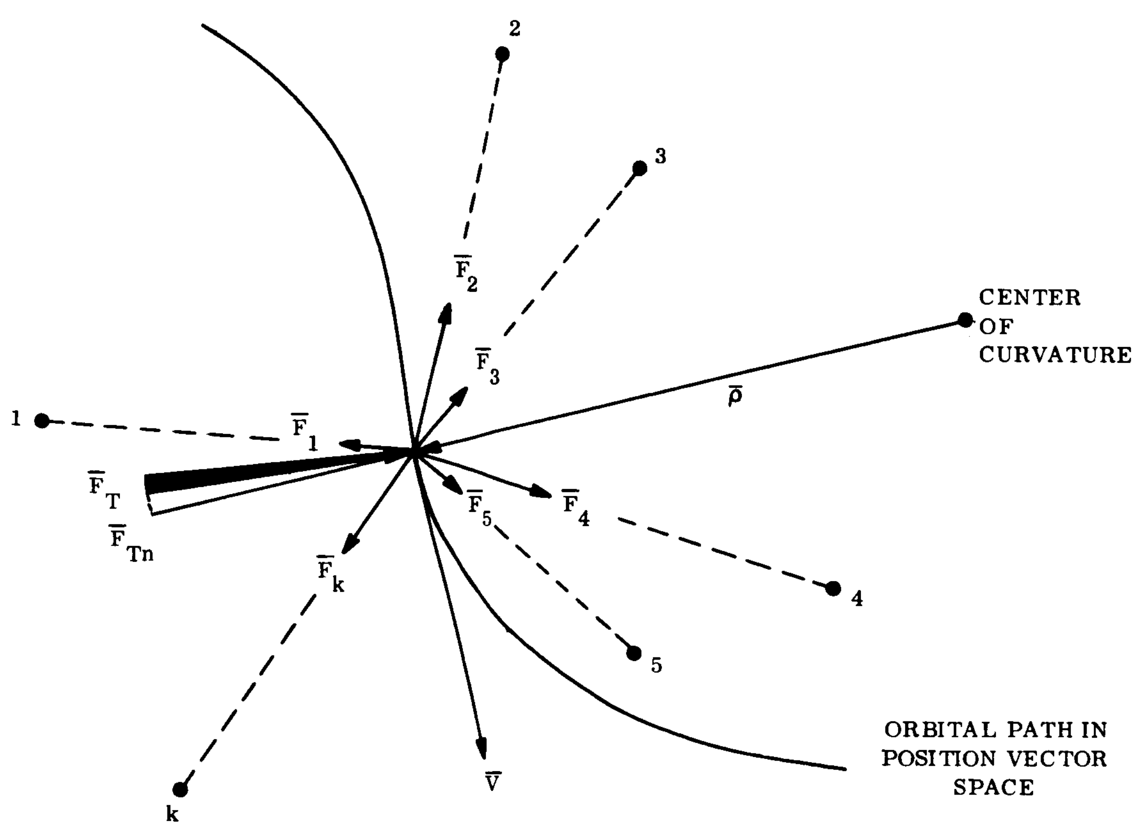



Figure 3-9. An Orbital Path Resulting from the Composite Field of Several Force Centers

where

$\bar{F}_{Tn}$ = normal component of the total apparent force vector, for the observed path curvature $\rho$

V = scalar of the velocity vector $\bar{V}$ (which is necessarily tangential to the trajectory path).

As stated in Bonnet's Theorem, it is known that this trajectory is identically realizable in each of the "j" force fields, taken independently. Then the radius of curvature "$\rho$" at any one point on the trajectory must obviously be the same in each case, even though the sets of vector scalars ($F_i$, $V_i$) may differ from field to field. Consequently, it must be true that, at any one point on the trajectory, the normal force components are

$$\left.\begin{aligned} \bar{F}_{1n} &= -\frac{mV_1^2}{\rho}\frac{\bar{\rho}}{|\bar{\rho}|} \\ \bar{F}_{2n} &= -\frac{mV_2^2}{\rho}\frac{\bar{\rho}}{|\rho|} \\ &\vdots \\ \bar{F}_{jn} &= -\frac{mV_j^2}{\rho}\frac{\bar{\rho}}{|\bar{\rho}|} \end{aligned}\right\} \tag{3-85}$$

Since field forces at a given point may be summed vectorially to define the total effective force $\bar{F}$ active upon a test mass at the point,

$$\bar{F}_n = \bar{F}_{1n} + \bar{F}_{2n} + \cdots + \bar{F}_{jn} \quad . \tag{3-86}$$

Also, the trajectories in the presence of each of the force fields are known to be identical. Then, must the trajectory in the presence of all such force fields be also the identical same path? Let us prove that such is the case, by indirect or "reductio ad absurdum" proof.

That is, assume that an additional instantaneous force $\bar{R}$ (directed along the normal to the path) were required at a given point, in order that the particle would stay on the specified trajectory. This additional normal force is defined as

$$\bar{R} = -R \frac{\bar{\rho}}{|\bar{\rho}|} \tag{3-87}$$

so that, now,

$$\bar{F}_{Tn} = \bar{F}_n + \bar{R} \quad . \tag{3-88}$$

Since all the normal force vectors ($\bar{F}_{1n}$, $\bar{F}_{2n}$, ....., $\bar{F}_{jn}$, R) have the same vector direction, then the scalar equation must be

$$\bar{F}_{Tn} = F_{1n} + F_{2n} + \cdots + F_{jn} + R \tag{3-89}$$

so that, obtaining the force scalars from Equation 3-83,

$$F_{Tn} = \frac{m}{\rho}\left(v_1^2 + v_2^2 + \cdots + v_j^2\right) + R \quad . \tag{3-90}$$

But the total velocity scalar V is related to the total normal force vector $\bar{F}_{Tn}$ by Equation 3-84; consequently,

$$\frac{mV^2}{\rho} = \frac{m}{\rho}\left(v_1^2 + v_2^2 + \cdots + v_j^2\right) + R \tag{3-91}$$

or

$$V^2 = (v_1^2 + v_2^2 + \cdots + v_j^2) + \left(\frac{\mathcal{E}}{m}\right) R . \tag{3-92}$$

Now, the principle of energy conservation is valid in each of the force fields as well as in the total (or composite) force field, in which the total energy consists of kinetic energy T and potential energy V. Consequently,

$$\left.\begin{aligned} E_1 &= T_1 + V_1 \\ E_2 &= T_2 + V_2 \\ &\vdots \\ E_j &= T_j + V_j \end{aligned}\right\} \tag{3-93}$$

and

$$E = T + V . \tag{3-94}$$

Then, in accordance with the energy conservation principle,

$$E = E_1 + E_2 + \cdots + E_j \tag{3-95}$$

and

$$V = V_1 + V_2 + \cdots + V_j \tag{3-96}$$

in accordance with the superposition principle for fields, so that

$$T = T_1 + T_2 + \cdots + T_j \tag{3-97}$$

which means that

$$V^2 = v_1^2 + v_2^2 + \cdots + v_j^2 \quad . \tag{3-98}$$

Upon comparing Equations 3-90 and 3-96, it is seen that

$$R \equiv 0 \tag{3-99}$$

since, in general, $\rho \neq 0$ and the particle mass is finite. Consequently, Bonnet's theorem must be true, as presented analytically by Equations 3-84, 3-85, and 3-98.

Note that, although the velocity vectors are collinear, the total velocity vector is not the linear sum (as for vectorial addition), but is the square root of the sums of the squares of the vector scalars. This superposition principle for velocities (contrasted with the superposition principle for fields, employed previously) is strikingly different from the conventional principles which we find so powerful with other methods or theory.

### 3.7 REDUCTION OF THE MAJOR FUNCTIONAL TERMS OF THE INVARIANT δ FOR ELLIPTIC ORBITS

One general form of the invariant δ is

$$\frac{\ell_1' \ell_2'}{2c} + (\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2) = \delta \tag{3-100}$$

where

$$\ell_1' = r_1 V_{\nu 1} = (x+c)\dot{y} - y\dot{x}$$ = angular momentum per unit mass, relative to force center 1 (3-101)

$$\ell_2' = r_2 V_{\nu 2} = (x-c)\dot{y} - y\dot{x}$$ = angular momentum per unit mass, relative to force center 2 (3-102)

and all other terms are defined graphically in Figure 3-10. Note that

$$V^2 = V_{r1}^2 + V_{\nu 1}^2 \tag{3-103}$$

or

$$V^2 = V_{r2}^2 + V_{\nu 2}^2 \quad . \tag{3-104}$$

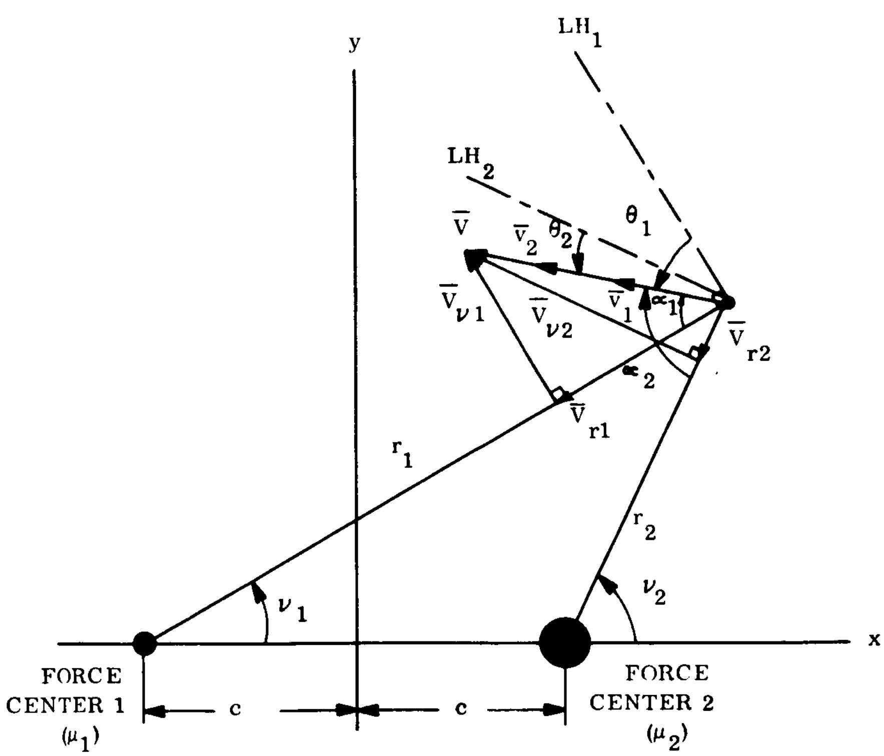



Figure 3-10. Vector Geometry of the Two-Fixed-Center Problem

Each of the two major terms on the left-hand side of Equation 3-100 will be reduced to functions of the given constants (c, $\mu_1$, $\mu_2$), the conic parameters (p , e) of the elliptic figure of orbit and the velocity scalars ($v_1$, $v_2$, V), where

$$V^2 = v_1^2 + v_2^2 \quad . \tag{3-105}$$

The velocity vectors $\bar{v}_1$, $\bar{v}_2$, $\bar{V}$ are collinear, as shown in Figure 3-10.

### 3.7.1 THE TERM ($\ell_1' \ell_2'/2c$)

Referring to Figure 3-10, it is seen that

$$V_{v1} = V \sin \alpha_1 = \sin \alpha_1 \sqrt{v_1^2 + v_2^2} \tag{3-106}$$

and

$$V_{v2} = V \sin \alpha_1 = \sin \alpha_2 \sqrt{v_1^2 + v_2^2} \quad . \tag{3-107}$$

Also,

$$r_1 (v_1 \cos \theta_1) = \sqrt{\mu_1 p} \tag{3-108}$$

and

$$r_2 (v_2 \cos \theta_2) = \sqrt{\mu_2 p} \quad , \tag{3-109}$$

so that

$$\cos \theta_1 = \frac{\sqrt{\mu_1 p}}{r_1 V_1} \tag{3-110}$$

and

$$\cos \theta_2 = \frac{\sqrt{\mu_2 p}}{r_2 V_2} \quad . \tag{3-111}$$

But

$$\alpha_1 - \theta_1 = \pi/2 \tag{3-112}$$

and

$$\alpha_2 - \theta_2 = \pi/2 \tag{3-113}$$

so that

$$\sin \alpha_1 = \cos \theta_1 = \frac{\sqrt{\mu_1 p}}{r_1 V_1} \tag{3-114}$$

and

$$\sin \alpha_2 = \cos \theta_2 = \frac{\sqrt{\mu_2 p}}{r_2 V_2} \quad . \tag{3-115}$$

Consequently, upon substitution of Equation 3-114 and 3-115 into Equation 3-106 and 3-107, we obtain

$$V_{v1} = \sqrt{v_1^2 + v_2^2} \quad \frac{\sqrt{\mu_1 P}}{r_1 v_1} \tag{3-116}$$

and

$$V_{v2} = \sqrt{v_1^2 + v_2^2} \quad \frac{\sqrt{\mu_2 P}}{r_2 v_2} \tag{3-117}$$

respectively, so that

$$\ell_1' = r_1 V_{v1} = \sqrt{v_1^2 + v_2^2} \quad \frac{\sqrt{\mu_1 P}}{v_1} \tag{3-118}$$

$$\ell_2' = r_2 V_{v2} = \sqrt{v_1^2 + v_2^2} \quad \frac{\sqrt{\mu_2 P}}{v_2} \quad . \tag{3-119}$$

Consequently,

$$\frac{\ell_1' \ell_2'}{2c} = \frac{P\sqrt{\mu_1 \mu_2}}{2c} \left[ \frac{v_1^2 + v_2^2}{v_1 v_2} \right] \quad . \tag{3-120A}$$

Since

$$v_1^2 = \mu_1 \left( \frac{2}{r_1} - \frac{1}{a} \right) = \frac{\mu_1}{r_1} \left( 2 - \frac{r_1}{a} \right) \tag{3-121}$$

and

$$v_2^2 = \mu_2 \left( \frac{2}{r_2} - \frac{1}{a} \right) = \frac{\mu_2}{r_2} \left( 2 - \frac{r_2}{a} \right) \quad , \tag{3-122}$$

then

$$v_1^2 v_2^2 = \frac{\mu_1 \mu_2}{r_1 r_2} \left[ 4 - \frac{2}{a}(r_1 + r_2) + \frac{r_1 r_2}{a^2} \right] \quad . \tag{3-123A}$$

But

$$r_1 + r_2 = 2a \tag{3-124}$$

so that

$$v_1^2 \, v_2^2 = \frac{\mu_1 \mu_2}{a^2} \tag{3-123B}$$

or

$$v_1 v_2 = \frac{\sqrt{\mu_1 \mu_2}}{a} \quad . \tag{3-125}$$

Substituting Equation 3-125 into Equation 3-120A,

$$\frac{\ell_1' \ell_2'}{2c} = \frac{pV^2}{2e} \quad . \tag{3-120B}$$

### 3.7.2 THE TERM ($\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2$)

For a conic orbit referred to one focus,

$$r = \frac{p}{1 + e \cos \phi} \tag{3-126}$$

so that

$$\cos \phi = \frac{1}{e}\left[\frac{p}{r} - 1\right] \quad . \tag{3-127}$$

Referring to the figure in Table 3-1, it is seen that

$$\nu_1 = \pi - \phi_1 \tag{3-128}$$

for the orbit referred to the focus at which force center 1 is located, so that

$$\cos \phi_1 = \cos(\pi - \nu_1) = -\cos \nu_1 \tag{3-129}$$

and consequently

$$\cos \nu_1 = -\frac{1}{e}\left[\frac{p}{r_1} - 1\right] \tag{3-130}$$

Also,

$$\cos \nu_2 \equiv \cos \phi_2 = \frac{1}{e}\left[\frac{p}{r_2} - 1\right] . \tag{3-131}$$

By means of Equations 3-130 and 3-131, the required term is

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \frac{\mu_1 + \mu_2}{e} - \frac{p}{e}\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right] . \tag{3-132}$$

But

$$\frac{V}{m} = -\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right] = \frac{E}{m} - \frac{V^2}{2} \tag{3-133}$$

so that

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \frac{1}{e}\left[(\mu_1 + \mu_2) + p\left(\frac{E}{m} - \frac{V^2}{2}\right)\right] \tag{3-134A}$$

or

$$\mu_1 \cos v_1 - \mu_2 \cos v_2 = \frac{E}{em}(p - 2a) - \frac{pV^2}{2e} \quad . \tag{3-134B}$$

Equation 3-134A could also be obtained directly from Equation 3-15 in Table 3-1, by simple rearrangement of the functional terms of the equation.

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2. E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, 1964.

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5. V. I. Smirnov, A Course of Higher Mathematics, Vol. III, Part 2, Pergamon Press, New York, 1964.

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SECTION 4

# HYPERBOLIC ORBITS OF THE TWO-FIXED-CENTER PROBLEM (PERIODIC AND APERIODIC)

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## SECTION 4

## HYPERBOLIC ORBITS OF THE TWO-FIXED-CENTER PROBLEM (PERIODIC AND APERIODIC)

Vector space analysis of admissible orbits of the two-fixed-center problem is a new and promising approach to development of the general parametric formulation of two-fixed-center trajectory solutions (Reference 1). By means of Bonnet's Theorem, confocal conics have been identified as admissible orbits; that is, confocal ellipses, and confocal hyperbolas which must necessarily then be orthogonal to the ellipses. Initial study results on the confocal ellipse have been reported in Reference 2. Since the ellipse is a closed figure, the confocal elliptic orbit must obviously be periodic (or cyclic). However, the dynamics of the hyperbolic orbits are quite different from the dynamics of the elliptic orbit for the two-fixed-center problem, even though the geometric figures, as conic sections, are intimately related.

In fact, the two-fixed-center hyperbolas do not meet all conditions of Bonnet's Theorem for force centers of identical sense (i.e., solely attracting or solely repulsive), although Charlier (Reference 3) has positively identified two classes of hyperbolic motion, as follows:

> "The planet moves along a hyperbola
>
> either so that it departs to infinity along the hyperbola. Then the focus of this hyperbola lies in the larger mass;
>
> or so that it oscillates back and forth pendulum-like along the hyperbola around the x-axis. Then the focus of this hyperbola lies in the smaller mass."

That is, the admissible two-fixed-center orbits of hyperbolic figure in position vector space may be aperiodic or periodic. In contrast, the admissible one-force-center orbits of hyperbolic figure are aperiodic only.

Surprisingly, very scant information about the hyperbolic orbits is available, upon search of the classical and contemporary literature. Apparently, the detailed study of specific classes of solution for the two-fixed-center problem has been ignored, as an intermediate means of study leading to more comprehensive and physically realistic problem models such as the restricted-three-body problem. Although Charlier (Reference 3) has employed an analytic criteria to classify the different regions and/or geometric shapes of such admissible orbits, specific and detailed properties of these orbits in position vector space (as well as other state spaces, obviously) were not analyzed or presented. This report presents not only the results of vector space analysis of the confocal hyperbolic orbits (periodic and aperiodic), but also new, definitive data on the regions of occurrence admissible to such orbits.

## 4.1 THE HYPERBOLA AS AN ADMISSIBLE TWO-FIXED-CENTER ORBIT

It might be tacitly assumed that hyperbolic orbits for the two-fixed-center problem are obviously admissible by virtue of Bonnet's Theorem (see Reference 2, Appendix A). Such an assumption is made in Whittaker* (Reference 4) concerning "the problem of two centres of attraction" in stating: "Now any ellipse or hyperbola with the two centres of force as foci is a possible orbit when either centre of force acts alone, and therefore by Bonnet's theorem it is a possible orbit when both centres of force are acting." While the hyperbola as a position state space locus meets the statements of geometric condition, the dynamical conditions are not fulfilled. The theorem is stated as follows:

> "If a given orbit can be described in each of m given fields of force, taken separately, the velocities at any point P of the orbit being $v_1$, $v_2$, ......, $v_j$ respectively, then the same orbit can be described in the field of force which is obtained by superposing all these fields, the velocity at the point P being $(v_1{}^2 + v_2{}^2 + \ldots + v_j{}^2)^{1/2}$."

*Contemporary and easily accessible references are employed wherever possible, even though the original work has often preceded such publications. Original sources are usually noted in such references.

The theorem is invalid for hyperbolic motion in the field of two attracting centers, since only the branch about the filled focus (of one force field alone) is physically realizable. The other branch cannot "be described," although that branch is identical with the realizable orbit for the other force field alone. Moreover, the study results have conclusively shown that Bonnet's Theorem is not applicable here, since the velocity relation

$$V = (v_1^2 + v_2^2 + \cdots + v_j^2)^{1/2} \tag{4-1}$$

is not fulfilled.

Various techniques of analysis for two-fixed-center orbits in general have been devised. All such known approaches synthesize the actual field due to the given force centers, by an equivalent, composite field of other force centers and/or field laws which enable a specialized mathematical reduction technique to be employed. As far as is known, this study has employed a different approach. This approach is comprised of two logical techniques carried out concurrently:

a. Development of the state space equations (which define the hodographs) from the dynamical equations of motion;

b. Synthesis of a force center model (restricted to the given state space sites) which fulfills the state space equations and hodographs in all respects.

This analytical approach has resulted in a remarkably compact and effective model which has enabled extension of the superposition technique for elliptic orbits (Reference 2) without ambiguity. Most striking of all, this superposition technique and the attendant decomposition of the two invariants are valid for all problem parameter variations, continuously and without singularity. That is, the problem parameters $\mu_1$, $\mu_2$, c (i.e., the separation distance between centers 1 and 2) may be selected freely without change in the algorithm of the superposition technique, or the equations of the invariant components.

Since the properties of the hyperbola (both as a geometric figure and a single-force-center orbit) are referred to in this analytical study, some of the definitive equations expressed in terms of the parameters of the hyperbola are listed in Figure 4-1, for orbits about one force center*. It is clearly advantageous (for the single-force-center orbit) to define the coordinate origin as the force center, with the transverse axis of the hyperbola as one coordinate axis. While Equations 4-2 through 4-6 are unique to the orbit about one force center, Equations 4-8 and 4-9 are geometric equations for the hyperbola as a general conic. The basic geometry and terms of the orbital hyperbola for two force centers are presented in Figure 4-2. In this case, it is quite convenient to define the coordinate origin as the geometric center, or the point (on the transverse axis of the hyperbola) which is midway between the foci (i.e., the fixed force centers). Aside from the analytic symmetry of the equations, the consequent y-axis (which is then the conjugate axis of the hyperbola) is the boundary between the regions of periodic and aperiodic orbits of hyperbolic figure for $\mu_1 \neq \mu_2$, as will be shown later.

## 4.2 EQUATIONS OF TRAJECTORY HODOGRAPHS IN VELOCITY AND ACCELERATION VECTOR SPACES

The trajectory "hodograph" (in position vector space) of the two-fixed-center orbit is a hyperbola. Three classes of orbital motion along a hyperbolic path (or segment thereof) are admissible (Reference 3):

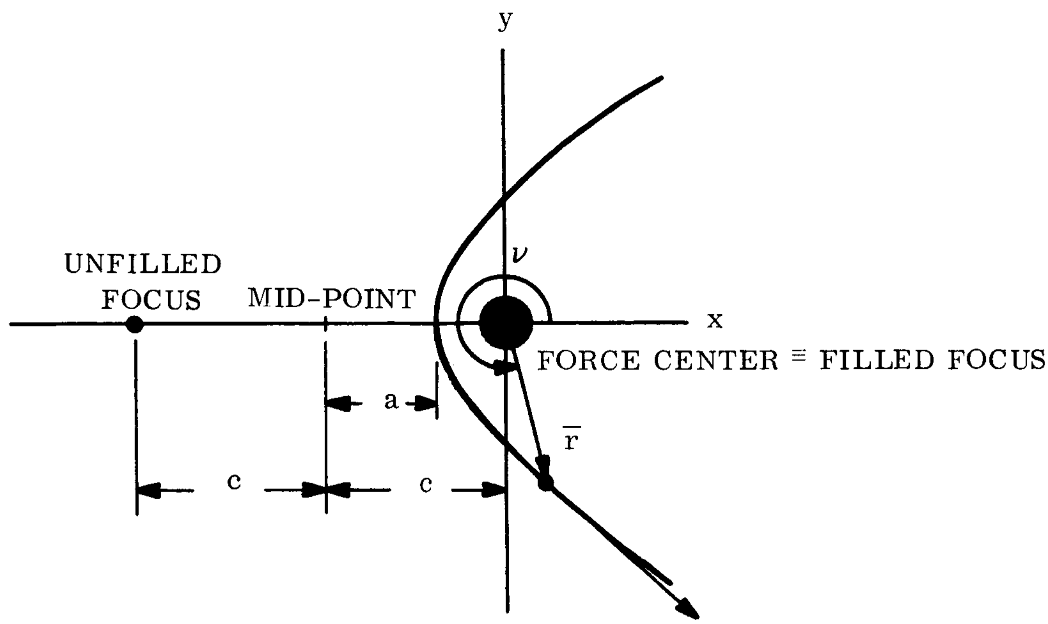
a. The satellite mass will proceed aperiodically to infinity and finite velocity, along the hyperbolic branch about the focus in which the stronger force center is located;

b. The satellite mass will oscillate periodically in pendulum-like motion along a finite segment of the hyperbolic branch about the focus in which the weaker force center is located; or,

c. For equal force centers, the satellite mass will proceed to infinity and zero velocity along either hyperbolic branch.

---

*It is understood that the gravitational potential field of the attracting force center is a simple spherical harmonic function.




$$r = \frac{p}{1 + e\cos\nu} \tag{4-2}$$

$$V^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right) \tag{4-3}$$

$$\ell^2 = \mu m^2 p \tag{4-4}$$

$$E = \frac{\mu m}{2a} \tag{4-5}$$

$$V = -\frac{\mu m}{r} \tag{4-6}$$

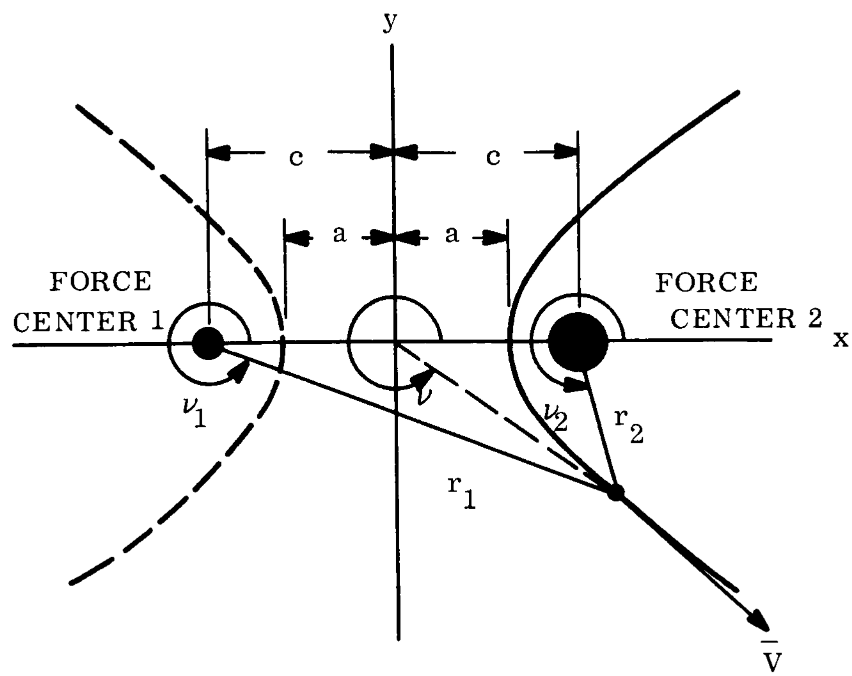
WHERE

$$\mu = GM \tag{4-7}$$

$$p = a(e^2 - 1) \tag{4-8}$$

$$c = ae \tag{4-9}$$

Figure 4-1. Definitive Equations of the Orbital Hyperbola (One Force Center)




$$r_1 - r_2 = 2a \qquad (4\text{-}10)$$

$$r_1^2 = (x+c)^2 + y^2 \qquad (4\text{-}11)$$

$$r_2^2 = (x-c)^2 + y^2 \qquad (4\text{-}12)$$

Figure 4-2. The Orbital Hyperbola (Two-Fixed-Center)

Class c is the bounding (or transition) case between the aperiodic and periodic motion of classes a and b , respectively. With the selected coordinate convention, the hyperbolic orbit in position vector space is defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (4\text{-}13)$$

where

a = semitransverse axis

b = semiconjugate axis.

As noted in Figure 4-1,

$$c = ae$$

where

$$e^2 = 1 + \frac{b^2}{a^2} > 1 \quad . \tag{4-14}$$

Knowing that this geometric figure is an admissible orbit, the velocity and acceleration hodographs must be developed so that subsequent study may reveal unique properties of the trajectory in the given vector space, as well as the transformations between vector spaces. The equations of the hodographs will be functions only of the parameters of the hyperbola and satellite position in orbit.

Analytic development of the definitive equations of the velocity and acceleration hodographs as functions of the trajectory state in position vector space has been accomplished by means of the following algorithm or set of sequential operations:

STEP 1: The derivative of the definitive equation in position vector space, with respect to time, is developed so that an equation as a function of $x$, $y$; $\dot{x}$, $\dot{y}$ is obtained. In general, this equation will not define velocity as a function of position explicitly.

STEP 2: Consequently, the derivative of the final equation of Step 1, with respect to time, is developed so that an equation as a function of $x$, $y$; $\dot{x}$, $\dot{y}$; $\ddot{x}$, $\ddot{y}$ is obtained.

STEP 3: It is known that

$$\ddot{x} = -\frac{1}{m} \frac{\partial V}{\partial x} \tag{4-15}$$

$$\ddot{y} = -\frac{1}{m} \frac{\partial V}{\partial y} \quad , \tag{4-16}$$

where

$$V = V_1 + V_2 = f(\mu_1, \mu_2, c; x, y) \quad , \tag{4-17A}$$

define valid relations between acceleration and position in separate, explicit form. These relations are the identical equations of the acceleration hodograph. Proceeding further to obtain the required velocity hodograph equations, the final equation of Step 2 is reduced to a function of x, y; $\dot{x}$, $\dot{y}$ by means of Equations 4-15 and 4-16.

STEP 4: With the given definitive equation in position vector space (i.e., Equation 4-13) and the final equation of Step 1, the final equation of Step 3 is reduced to the required equations of the velocity hodograph.

This logical procedure has been described and implemented also in Reference 2.

In accordance with the above logic, the hodograph equations are developed as follows:

IMPL 1: Differentiating Equation 4-13 with respect to time,

$$\frac{x\dot{x}}{a^2} - \frac{y\dot{y}}{b^2} = 0 \tag{4-18A}$$

or

$$\left(\frac{x}{a^2}\right)\dot{x} = \left(\frac{y}{b^2}\right)\dot{y} \quad . \tag{4-18B}$$

IMPL 2: Differentiating Equation 4-18B with respect to time,

$$\frac{1}{a^2}\left[\dot{x}^2 + x\ddot{x}\right] = \frac{1}{b^2}\left[\dot{y}^2 + y\ddot{y}\right] \quad . \tag{4-19}$$

IMPL 3: Since

$$V = V_1 + V_2 = -m\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) , \tag{4-17B}$$

then

$$V = -m\left\{\frac{\mu_1}{\left[(x+c)^2 + y^2\right]^{\frac{1}{2}}} + \frac{\mu_2}{\left[(x-c)^2 + y^2\right]^{\frac{1}{2}}}\right\} \tag{4-20}$$

so that, according to Equations 4-15 and 4-16

$$\ddot{x} = -\left\{ \frac{\mu_1 (x+ae)}{\left[(x+ae)^2 + y^2\right]^{3/2}} + \frac{\mu_2 (x-ae)}{\left[(x-ae)^2 + y^2\right]^{3/2}} \right\} \tag{4-21}$$

$$\ddot{y} = -y \left\{ \frac{\mu_1}{\left[(x+ae)^2 + y^2\right]^{3/2}} + \frac{\mu_2}{\left[(x-ae)^2 + y^2\right]^{3/2}} \right\} . \tag{4-22}$$

Noting that

$$A^2 = \ddot{x}^2 + \ddot{y}^2 \quad , \tag{4-23}$$

it is seen that Equations 4-21 and 4-22 are the acceleration hodograph equations. Now, substituting Equations 4-21 and 4-22 into Equation 4-19 and rearranging,

$$\frac{\dot{x}^2}{a^2} - \frac{\dot{y}^2}{b^2} = -\frac{y^2}{b^2} \left\{ \frac{\mu_1}{\left[(x+ae)^2 + y^2\right]^{3/2}} + \frac{\mu_2}{\left[(x-ae)^2 + y^2\right]^{3/2}} \right\} +$$

$$+ \frac{x}{a^2} \left\{ \frac{\mu_1 (x+ae)}{\left[(x+ae)^2 + y^2\right]^{3/2}} + \frac{\mu_2 (x-ae)}{\left[(x-ae)^2 + y^2\right]^{3/2}} \right\} . \tag{4-24A}$$

But

$$\left[(x+ae)^2 + y^2\right]^{3/2} = (a+ex)^3 \tag{4-25}$$

$$\left[(x-ae)^2 + y^2\right]^{3/2} = (a-ex)^3 \tag{4-26}$$

so that

$$\frac{\dot{x}^2}{a^2} - \frac{\dot{y}^2}{b^2} = -\frac{y^2}{b^2}\left[\frac{\mu_1}{(a+ex)^3} + \frac{\mu_2}{(a-ex)^3}\right] +$$

$$+\frac{x}{a^2}\left[\frac{\mu_1(x+ae)}{(a+ex)^3} + \frac{\mu_2(x-ae)}{(a-ex)^3}\right] . \tag{4-24B}$$

For $x \geq a > 0$,

$$\frac{\dot{x}^2}{a^2} - \frac{\dot{y}^2}{b^2} = -\frac{y^2}{b^2}\left[\frac{\mu_1}{|a+ex|^3} - \frac{\mu_2}{|a-ex|^3}\right] +$$

$$+\frac{|x|}{a^2}\left[\frac{\mu_1|x+ae|}{|a+ex|^3} - \frac{\mu_2|x-ae|}{|a-ex|^3}\right] ; \tag{4-27}$$

for $x \leq -a < 0$,

$$\frac{\dot{x}^2}{a^2} - \frac{\dot{y}^2}{b^2} = -\frac{y^2}{b^2}\left[-\frac{\mu_1}{|a+ex|^3} + \frac{\mu_2}{|a-ex|^3}\right] +$$

$$+\frac{|x|}{a^2}\left[-\frac{\mu_1|x+ae|}{|a+ex|^3} + \frac{\mu_2|x-ae|}{|a-ex|^3}\right] . \tag{4-28}$$

Upon comparison of Equations 4-27 and 4-28, it is seen that

$$\frac{\dot{x}^2}{a^2} - \frac{\dot{y}^2}{b^2} = \left\{-\frac{y^2}{b^2}\left[\frac{\mu_1}{(a+ex)^3} - \frac{\mu_2}{(a-ex)^3}\right] +\right.$$

$$\left.+\frac{x}{a^2}\left[\frac{\mu_1(x+ae)}{(a+ex)^3} - \frac{\mu_2(x-ae)}{(a-ex)^3}\right]\right\} \operatorname{sgn} x \tag{4-24C}$$

where "sgn x" denotes the signum function; that is,

$\operatorname{sgn} x = 1$ for $x > 0$

$\operatorname{sgn} x = -1$ for $x < 0$

Upon the use of Equation 4-13 and subsequent reduction of Equation 4-24C

$$\frac{\dot{x}^2}{a^2} - \frac{\dot{y}^2}{b^2} = \left[\frac{\mu_1}{a(a+ex)^2} - \frac{\mu_2}{a(a-ex)^2}\right] \operatorname{sgn} x \tag{4-24D}$$

IMPL 4: Rearrangement of Equation 4-13 provides

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) \tag{4-29}$$

so that, upon substitution in Equation 4-18B,

$$\dot{y}^2 = \frac{b^2}{a^2}\left(\frac{x^2}{x^2-a^2}\right)\dot{x} \quad . \tag{4-30}$$

Substitution of Equation 4-30 in Equation 4-24D with subsequent reduction, yields

$$\dot{x}^2 = \left[\frac{\mu_1}{(a+ex)^2} - \frac{\mu_2}{(a-ex)^2}\right]\left[\frac{a^2-x^2}{a}\right] \operatorname{sgn} x \, . \tag{4-31}$$

In similar fashion, we obtain

$$\dot{y}^2 = \left[\frac{\mu_1}{(a+ex)^2} - \frac{\mu_2}{(a-ex)^2}\right]\left[\frac{(1-e^2)x^2}{a}\right] \operatorname{sgn} x \, . \tag{4-32}$$

Since

$$V^2 = \dot{x}^2 + \dot{y}^2 \quad , \tag{4-33A}$$

it is seen that Equations 4-31 and 4-32 are the velocity hodograph equations, which, upon insertion into Equation 4-33A provide

$$\boxed{V^2 = \left[\frac{\mu_1}{(a+ex)^2} - \frac{\mu_2}{(a-ex)^2}\right]\left[\frac{a^2-(ex)^2}{a}\right] \operatorname{sgn} x \, .} \tag{4-33B}$$

Note that Equations 4-31 through 4-33 are functions of x, which lie in two distinct regions of solution defined by

$$x > 0$$

and

$$x < 0$$

that is, in the right half-plane and left half-plane respectively. By means of Equations 4-25, 4-26, and 4-29, the acceleration hodograph Equations 4-21 and 4-22 may be obtained in comparable form as

$$\ddot{x} = -\left[\frac{\mu_1(x+ae)}{(a+ex)^3} - \frac{\mu_2(x-ae)}{(a-ex)^3}\right] \operatorname{sgn} x \tag{4-34}$$

$$\ddot{y} = -\left[(e^2-1)(x^2-a^2)\right]^{\frac{1}{2}}\left[\frac{\mu_1}{(a+ex)^3} - \frac{\mu_2}{(a-ex)^3}\right] \operatorname{sgn} x \ . \tag{4-35}$$

The potential energy V, expressed previously by Equation 4-20, can be reduced to

$$\boxed{V = -m\left[\frac{\mu_1}{(a+ex)} - \frac{\mu_2}{(a-ex)}\right] \operatorname{sgn} x} \ . \tag{4-36}$$

Substituting Equations 4-33B and 4-36 into the basic energy relation

$$E = T + V \ , \tag{4-37}$$

the orbital energy is determined to be

$$\boxed{E = \frac{m}{2a}(\mu_2 - \mu_1) \operatorname{sgn} x} \ . \tag{4-38}$$

It is important to note that the total energy E is a constant independent of the coordinate system (x, y); that is, sgn x is a notation convention used to identify that the total energy of hyperbolic orbits about the larger or smaller force center is positive or negative respectively.

## 4.3 REGIONS AND CLASSES OF HYPERBOLIC ORBITS

The results of the preceding section show that the orbits of confocal hyperbolas in the two-fixed-center problem are defined discretely within two regions of solution: the right half-plane ($x > 0$) and the left half-plane ($x < 0$). Also, the terms due to each force center ($\mu_1$, $\mu_2$) always appear as differences. Consequently, the orbital maps fall into three different classes: $\mu_1 \neq \mu_2$ (periodic), $\mu_1 \neq \mu_2$ (aperiodic), $\mu_1 = \mu_2$. Study of Equation 4-38 shows that, for $\mu_1 \neq \mu_2$ (i.e., $\mu_1 < \mu_2$ or $\mu_1 > \mu_2$), the total energy E will be negative for orbits about the smaller force center and positive for orbits about the larger force center, due to the signum function (sgn x). The total energy is zero for $\mu_1 = \mu_2$.

The total energy for hyperbolic orbits of the one-force-center problem (see Figure 4-1, Equation 4-5) is always positive. Then the orbital mass will always escape from the system (i.e., aperiodic departure to infinity) so that the kinetic energy never becomes zero. In the one-force-center problem, the orbital energy is negative only for the periodic orbits of elliptic (or circular) figure, whereas zero orbital energy defines the aperiodic orbit of the parabola in which the kinetic energy of the orbital mass approaches zero as the mass departs to infinity. If these characteristic relations between orbital energy and the orbital figures for the one-force-center problem were assumed to be valid for the two-fixed-center orbits, then the confocal hyperbolic orbits would be classified as follows:

a. aperiodic hyperbolas with positive kinetic energy at "infinity", for $E > 0$;

b. aperiodic hyperbolas with zero kinetic energy only at "infinity", for $E = 0$; and

c. periodic hyperbolic sections with zero kinetic energy at "turning points" located in finite space, for $E < 0$.

Obviously, Classes b and c would then be uniquely different from the one-force-center hyperbolas, with characteristic differences in their dynamics (e.g., "position," velocity and acceleration hodographs).

The three classes of hyperbolic orbit postulated above do, in fact, occur as described, as revealed upon study of the hodographic equations. In the next section, some typical hodograph solutions are presented, which demonstrate these properties. At this point, the periodic hyperbolic orbit of Class c merits special study, particularly of the "turning points" which define the endpoints of the hyperbolic section. That is, the velocity of the orbital mass decreases to zero as it comes to momentary rest at a turning point, before returning along its prior path with regained velocity. As a degenerate case, the turning point may be a libration point, if the orbital mass is continually at rest at one such point, without possible forces causing a displacement.

Since the velocity is zero at the turning points $x_t$, Equation 4-33B provides the conditions* that, when $V^2 = 0$,

$$\frac{\mu_1}{(ex_t + a)^2} = \frac{\mu_2}{(ex_t - a)^2} \qquad (4\text{-}39)$$

where

$$e > 1$$

$$x_t \geq a \quad .$$

*Note that, if the original form of Equation 4-33B were employed,

$$\frac{\mu_1}{(a + ex_t)^2} = \frac{\mu_2}{(a - ex_t)^2} \quad .$$

With $e > 1$, $x_t \geq a$,

$$(a + ex_t) = + |a + ex_t|$$

$$(a - ex_t) = - |a - ex_t|$$

so that $\sqrt{\mu_1/\mu_2}$ must be negative in sign, with this convention. The convention for Equation 4-39 is for convenience only; the analytical results with the original form of Equation 4-33B are identical.

Consequently

$$x_t = \frac{a}{e}\left[\frac{\tau+1}{\tau-1}\right] \tag{4-40A}$$

where

$$\tau = \sqrt{\frac{\mu_1}{\mu_2}} \tag{4-41}$$

or

$$x_t = \left(\frac{c}{e^2}\right)f = \left(\frac{a^2}{c}\right)f \tag{4-40B}$$

where

$$f = \frac{\tau+1}{\tau-1} \quad . \tag{4-42}$$

Equations 4-40A or 4-40B express the relation between the x-coordinate of the turning points, and the physical parameters $\mu_1$, $\mu_2$, c (which are given) and the trajectory parameters a or e (which are dependent upon the energy state or initial conditions). Upon replacement of the parameters a, b in the definitive equation of the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad ,$$

by means of Equations 4-14 and 4-40B, the following relation between the x, y coordinates of the turning points is obtained:

$$\frac{x_t^2}{x_t\left(\frac{c}{f}\right)} - \frac{y_t^2}{c^2 - x\left(\frac{c}{f}\right)} = 1 \quad . \tag{4-43}$$

Reduction of Equation 4-43 provides

$$\left[x_t - \frac{c(f^2+1)}{2f}\right] + y_t^2 = \left[\frac{c(f^2-1)}{2f}\right]^2 \quad , \tag{4-44}$$

which is recognizable as the equation of a circle with radius $\rho_o$ and circle center coordinates $(x_o, y_o = 0)$ where

$$\boxed{\rho_o = \left| \frac{c(f^2-1)}{2f} \right|} \tag{4-45}$$

$$\boxed{x_o = \frac{c(f^2+1)}{2f}} \tag{4-46}$$

or

$$\frac{x_o}{c} = \frac{f^2+1}{2f} \quad . \tag{4-47}$$

This means that the locus of the turning points of periodic hyperbolic solutions for a given set of physical parameters $\mu_1$, $\mu_2$, c must be a circle, as shown in Figure 4-3. All periodic solutions about the smaller force center, possible for various values of a or e, must lie within this bounding circle.

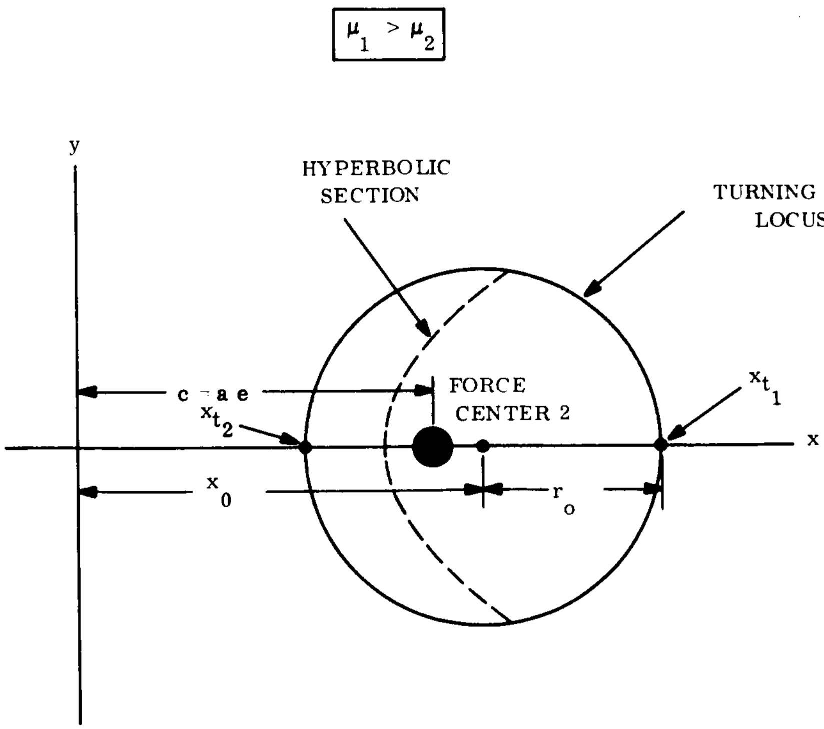



Figure 4-3. Characteristics of the Turning Point Locus

Since $f \geq 1$ in all cases of periodic solution (i.e., $\mu_1 \neq \mu_2$), then

$$|x_0| \geqslant c \quad . \tag{4-48}$$

That is, the center of the circular locus of the turning points is displaced from the force center along the x-coordinate axis in a direction away from the coordinate origin (see Figure 4-3) by the distance

$$|x_0 - c| = c\left[\frac{(|f|-1)^2}{2|f|}\right] \quad . \tag{4-49}$$

Also, since

$$\left|\frac{x_0}{\rho_0}\right| = \left|\frac{f^2+1}{f^2-1}\right| \quad , \tag{4-50}$$

then

$$|x_0| > |\rho_0| \quad ; \tag{4-51}$$

that is, the locus of turning points is always located in the right (or left) half-plane for $\mu_1 \neq \mu_2$, never crossing or intersecting the y-axis. For $\mu_1 = \mu_2$, the locus of turning points becomes the y-axis identically.

The intersection points of the locus and the x-axis are determined by Equation 4-43 for $y_t = 0$, thereby resulting in

$$x_t - \frac{c(f^2+1)}{2f} = \pm\left[\frac{c(f^2-1)}{2f}\right] \tag{4-52}$$

so that

$$x_t = \frac{c}{2f}\left[(f^2+1) \pm (f^2-1)\right] \quad . \tag{4-53}$$

Consequently, the two intersection points ($x_{t_1}$, $x_{t_2}$) are

$$\boxed{x_{t_1} = c f} \tag{4-54}$$

and

$$\boxed{x_{t_2} = \frac{c}{f}} \tag{4-55}$$

Note that the turning point $x_{t_2}$ is itself a libration point; that is, the hyperbolic orbit containing the point $x_{t_2}$ is a degenerate case in which the turning point is the only admissible point of that degenerate hyperbola. In this case, the force vectors due to the force centers are exactly equal in magnitude, and opposite in direction.

Let us now survey the characteristic fields of conic orbit for the one-force-center and two-force-center problems in position vector space. The elliptic orbit of the one-force-center problem occurs regardless of which focus is "filled" by the force center. On the other hand, the one-force-center hyperbolic orbit occurs only in that half-plane in which the force center is located. An equivalent yet more significant interpretation, when comparing the conic orbits for one- and two-force-center problems, is that one force center alone (fixed in one focus) acts as an attractive center for hyperbolas in one half-plane as shown in Figure 4-4A, and as a repulsive center for hyperbolas in the other half-plane as shown in Figure 4-4B. Although a repulsive force center is not encountered in gravitational systems, this viewpoint is useful for effective understanding of the two-fixed-center hyperbolic orbits.

The admissible conic figures of two-fixed-center orbit are confocal ellipses and hyperbolas. As shown in Figure 4-5, the field of confocal hyperbolas is orthogonal to the field of confocal ellipses. This unique property enables the use of these fields as elliptic coordinates ($\xi$, $\chi$) for the general solution in the classical literature (Reference 4). The three classes of two-fixed-center hyperbolic orbit occur as shown schematically in Figure 4-6. While the aperiodic hyperbolas may occur for any values of $\mu_1$ and $\mu_2$, the periodic hyperbolic section of orbit can occur only for $\mu_1 \neq \mu_2$, about the smaller force center. The admissible

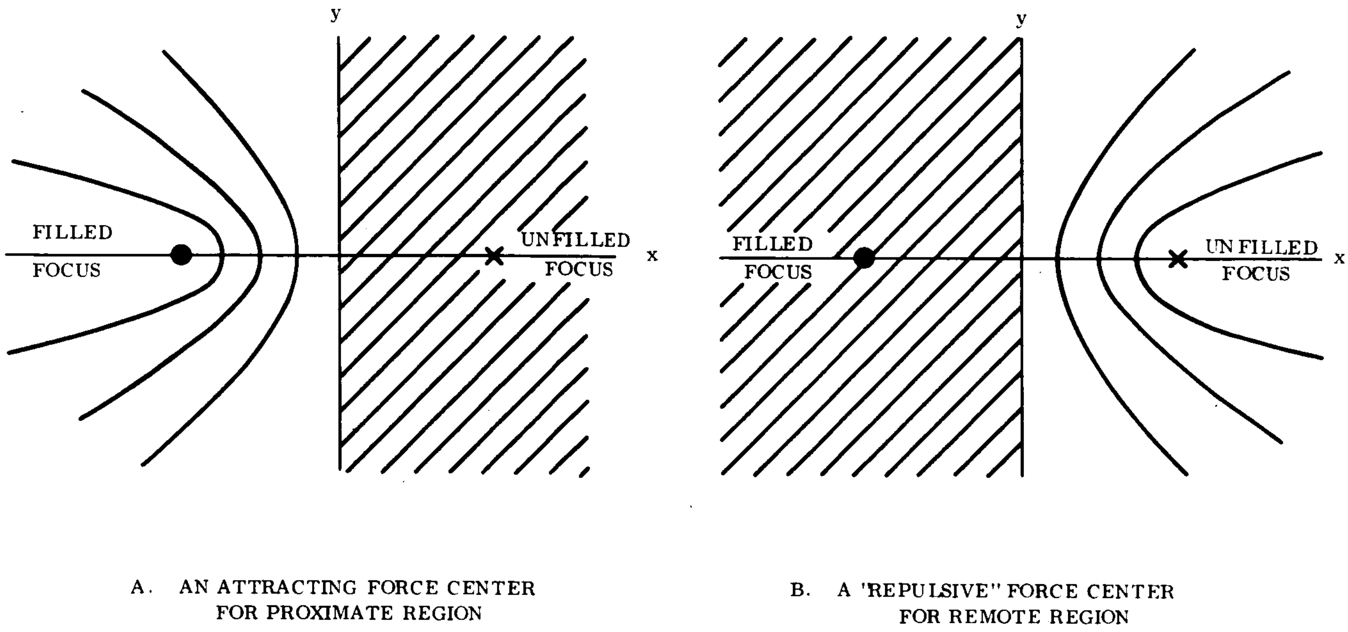



Figure 4-4. Hyperbolic Branches for One Force Center

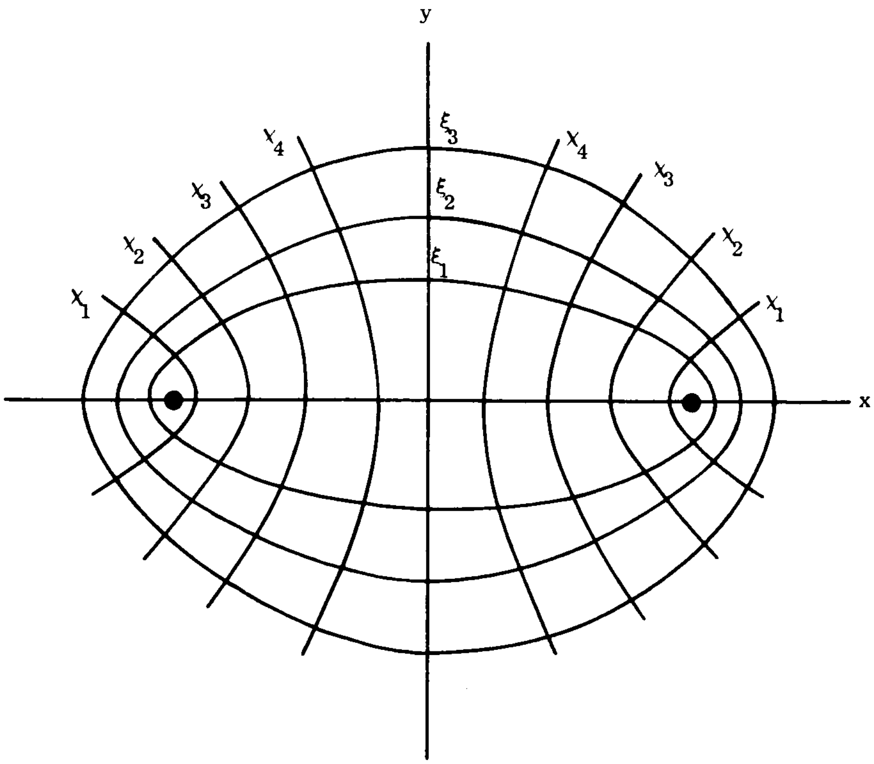
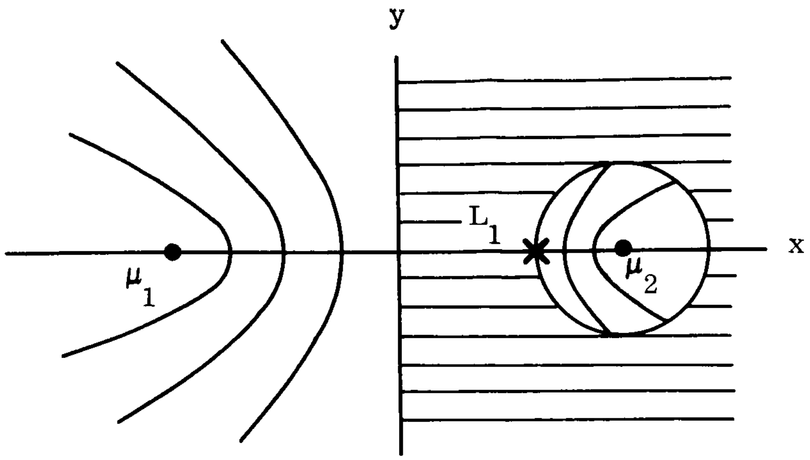
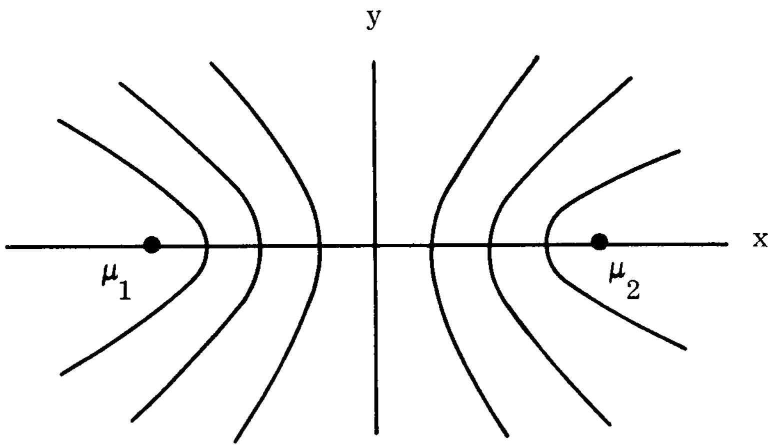


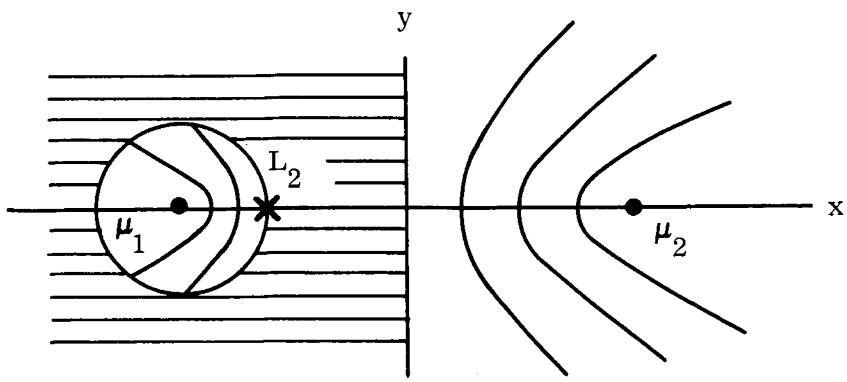

Figure 4-5. Confocal Ellipses and Orthogonal Hyperbolas




A. APERIODIC (E > 0) AND PERIODIC (E < 0) ORBITS ($\mu_1 > \mu_2$)




B. APERIODIC ORBITS ONLY (E = 0, $\mu_1 = \mu_2$)




C. PERIODIC (E < 0) AND APERIODIC (E > 0) ORBITS ($\mu_1 < \mu_2$)

Figure 4-6. Hyperbolic Orbits for Two-Fixed Centers region of periodic solution is a circle; the remaining region in that half-plane is an inadmissible region for orbital solution of hyperbolic figure. For each circular region of periodic solution, one libration point $L_{(\ )}$ exists, located as shown in Figure 4-6A and 4-6C. All periodic hyperbolic sections of orbit are unstable, whereas all the aperiodic hyperbolas are stable orbits, as noted in Reference 3.

Consider the circular region of periodic solution. The weaker the smaller force center (compared with the stronger force center), the smaller will be the radius of the circular region. In the limit, if the smaller force becomes smaller until it is nonexistent, the problem has degenerated to the one-force-center problem. On the other hand, if the weaker force center increases until $\mu_1 = \mu_2$, then the circular region becomes an entire half-plane so that aperiodic hyperbolas may occur in either half-plane of position vector space.

## 4.4 HODOGRAPH MAPS IN VELOCITY AND ACCELERATION VECTOR SPACES

A few typical velocity hodographs and one acceleration hodograph for aperiodic and periodic hyperbolic orbits were generated for demonstration and study of the hodograph characteristics.

### 4.4.1 APERIODIC HYPERBOLIC ORBITS

The hodographs of the following families of aperiodic hyperbolic orbits are presented in Figures 4-7 through 4-11:

VELOCITY HODOGRAPHS

| | | |
|---|---|---|
| $\mu_1 < \mu_2$ | constant a, variable e | (Figure 4-7) |
| | constant e, variable a | (Figure 4-8) |
| | constant c (=ae), variable a and e | (Figure 4-9) |
| $\mu_1 \leq \mu_2$: | constant a and e, variable $\mu_1/\mu_2$ | (Figure 4-10) |

ACCELERATION HODOGRAPH

| | | |
|---|---|---|
| $\mu_1 < \mu_2$: | a = 25, e = 0.50 | (Figure 4-11). |

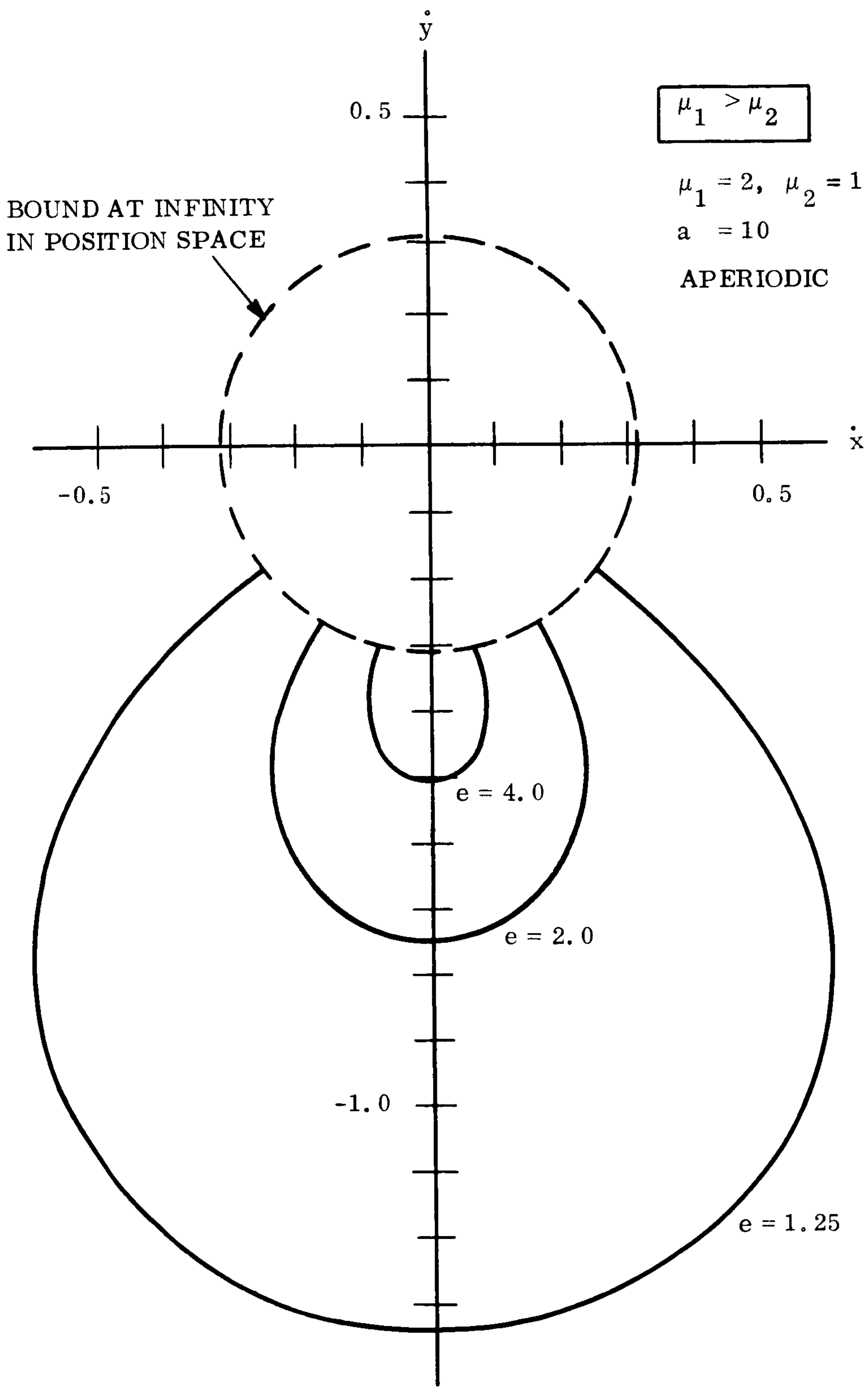



Figure 4-7. Velocity Hodographs of Two-Fixed-Center Hyperbolic Orbits About $\mu_1$ (Constant Energy $\mu_1 > \mu_2$)

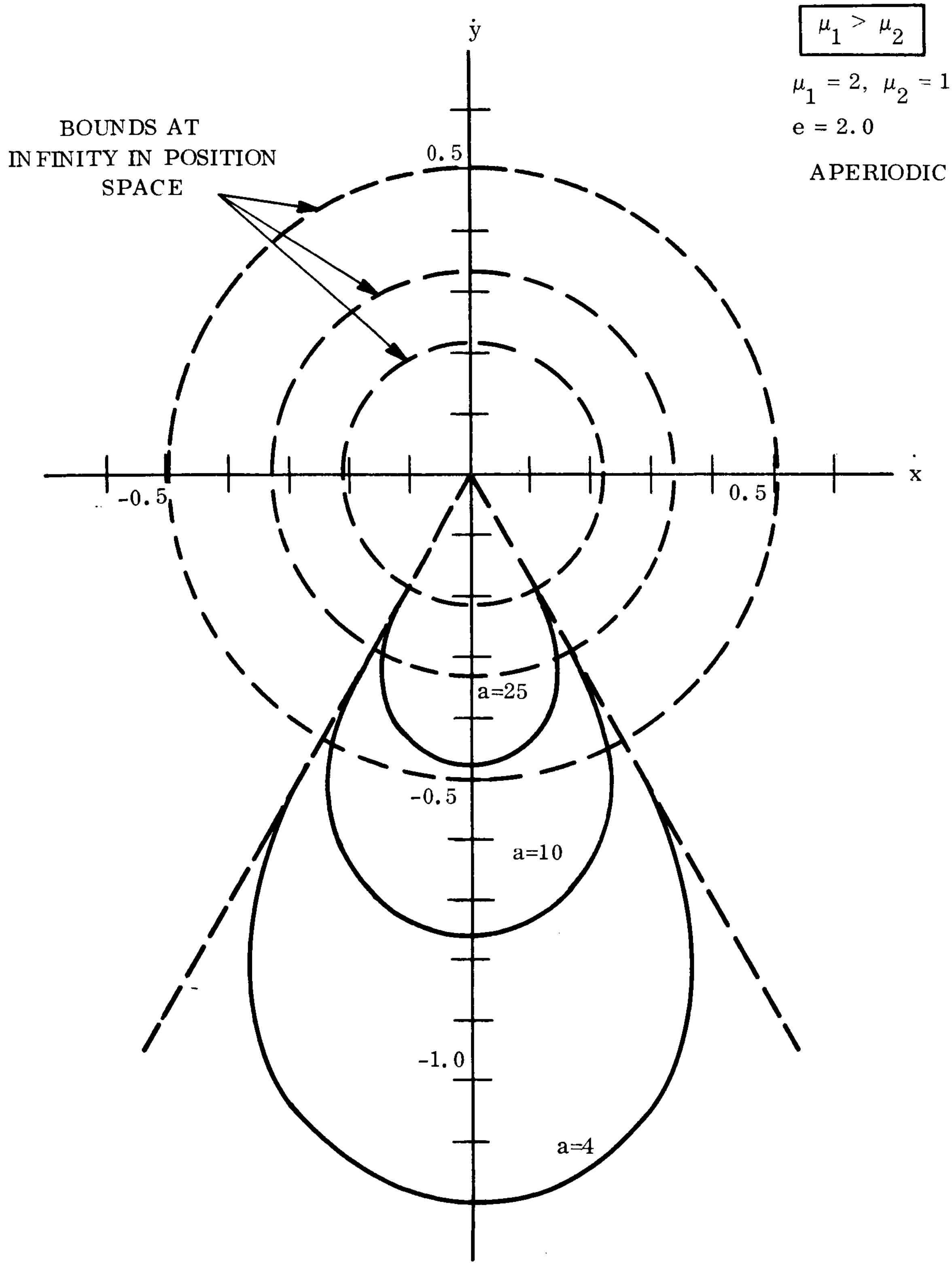



Figure 4-8. Velocity Hodographs of Two-Fixed-Center Hyperbolic Orbits (Constant Eccentricity, $\mu_1 > \mu_2$)

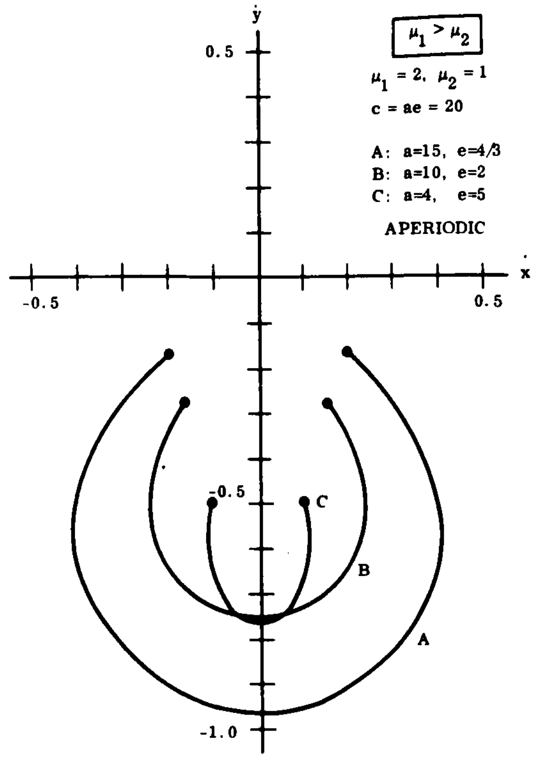



Figure 4-9. Velocity Hodographs of Two-Fixed-Center Hyperbolic Orbits (Constant Separation, $\mu_1 > \mu_2$)

Many observations may be made about the functional characteristics of the hodographs. As shown in Figure 4-12, the position space intercept of the hyperbola with the x-axis (or transverse axis) corresponds with the velocity space intercept of the velocity hodograph with the $\dot{y}$-axis. Although it is not directly apparent, the transformation from position to velocity vector space produces a phase advance of $\pi/2$, just as in the one-force-center problem and for two-fixed-center elliptic orbits. However, referring to the acceleration hodograph in Figure 4-11, it is apparent that further transformation from velocity to acceleration vector space must then produce a further phase advance of $\pi/2$, since each point of the acceleration hodograph is $\pi$ radians leading the corresponding point in position vector space. Note that, just as in the vector space theory for one force center, the phase in velocity vector space is not defined relative to the origin, but to the center of a generating hodograph (Reference 5). (The generating hodographs of these hyperbolic orbits will be discussed in the next section).

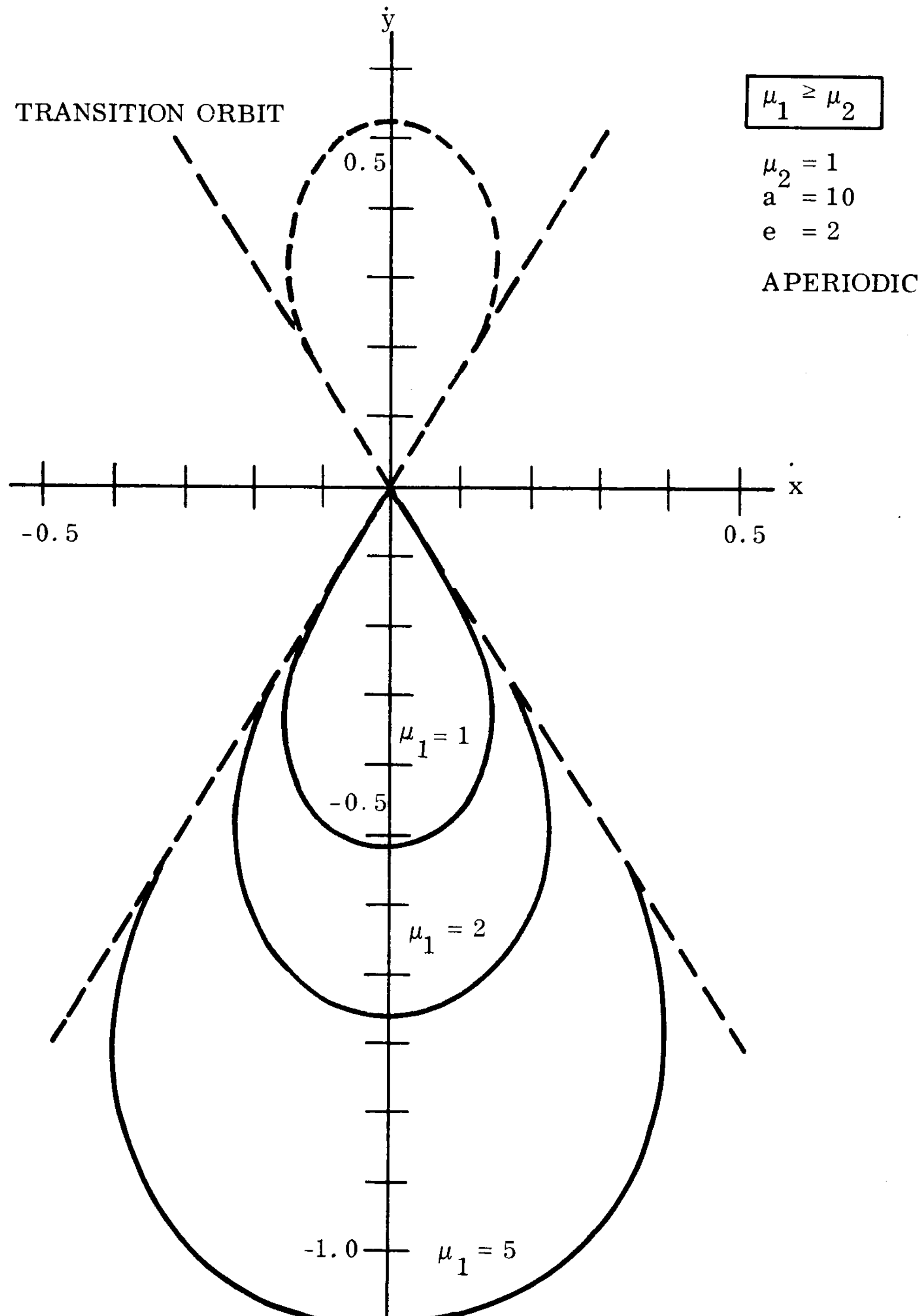



Figure 4-10. Velocity Hodographs of Two-Fixed-Center Hyperbolic Orbits (For Various Source Strengths $\mu_1$)

In all cases, orbital hodographs closer to the origin in velocity vector space represent orbits in position vector space farther from the origin (and the force centers). That is, geometric inversion occurs in the transformation from position to higher order vector spaces, just as in the one-force-center dynamics. Also, the orbital energy of those trajectory hodographs closer to the velocity space origin is smaller.

Referring to the constant-energy family of orbital hodographs shown in Figure 4-7, it is seen that all hodographs terminate on a circle of constant radius about the origin. The hodographs approach the endpoints located on this circle, along a radius. This geometric condition represents the constant-energy constraint which defines the given family of orbits. This functional relation is exactly the same as for the one-force-center problem (Reference 6), as shown in Figure 4-13.

Referring to the constant-eccentricity family of orbital hodographs shown in Figure 4-8, it is seen that all hodographs terminate on two lines of constant slope (slope of one = +m, slope of the other = -m) from the origin. The hodographs approach the endpoints located on these lines, tangentially. This geometric condition represents the constant-eccentricity constraint which defines the given family of orbits. This functional relation is exactly the same as for the one-force-center problem (Reference 6), as shown in Figure 4-14.

Figure 4-10 presents the velocity hodographs for hyperbolic orbits with identical parameters, but with different field strengths for the force center about which the aperiodic orbits occur. That is, all such orbits are entirely coincident in position vector space, even though the potential fields are different. Of course, the velocity hodographs terminate on lines of constant slope from the origin just as in Figure 4-8, since the eccentricity is constant. However, the endpoints occur at different radial distances from the origin even though the parameter a is constant, because the energy (for the various orbits) is not the same (see Equation 4-38). As the field strength of the proximate center increases, the velocity hodograph expands; that is, the velocity scalar (point-for-point) is larger.

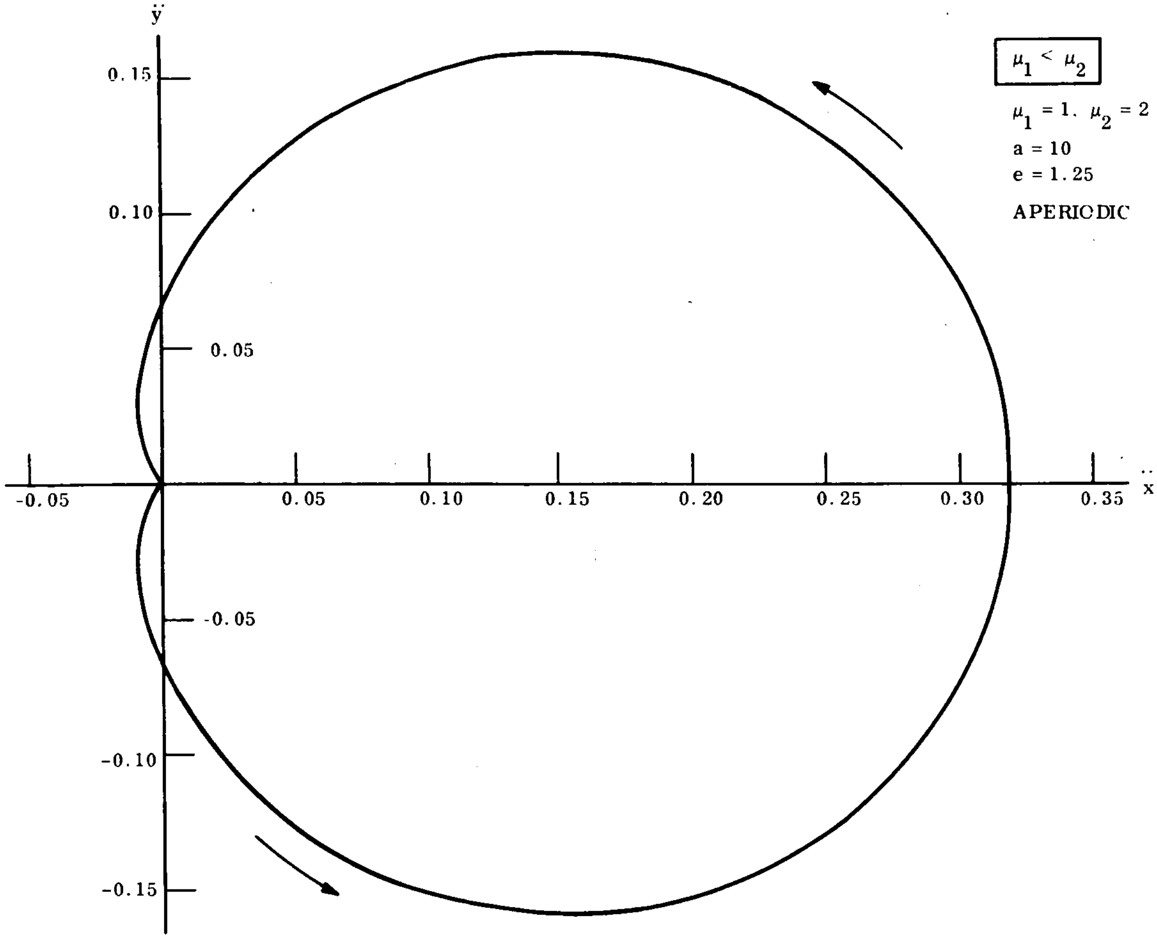



Figure 4-11. Acceleration Hodograph for Two-Fixed-Center Hyperbolic Orbit

The velocity hodograph of a "transition" hyperbola (i.e., $\mu_1 = \mu_2$) has also been presented in Figure 4-10, since it is the limiting hodograph of this family. This "transition" hodograph originates and terminates at the origin. The hodograph of the corresponding hyperbola, which lies in the half-plane of position space containing the other force center, will be a mirror image (about the $\dot{x}$-axis) of the "transition" hyperbola, as shown in the lower half-plane.

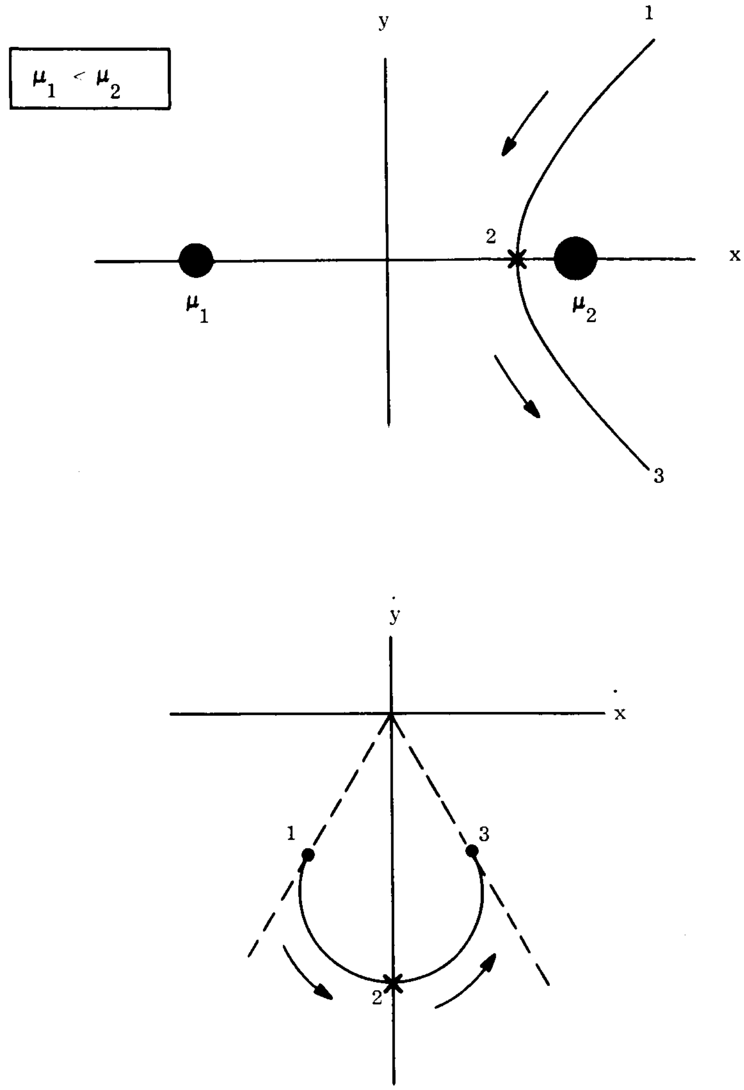



Figure 4-12. Correspondence of Vector Space Maps for Two-Fixed-Center Hyperbolic Hyperbolas ($\mu_1 \neq \mu_2$)

### 4.4.2 PERIODIC HYPERBOLIC ORBITS

The hodographs of the following families of periodic hyperbolic orbits are presented in Figures 4-15 through 4-19:

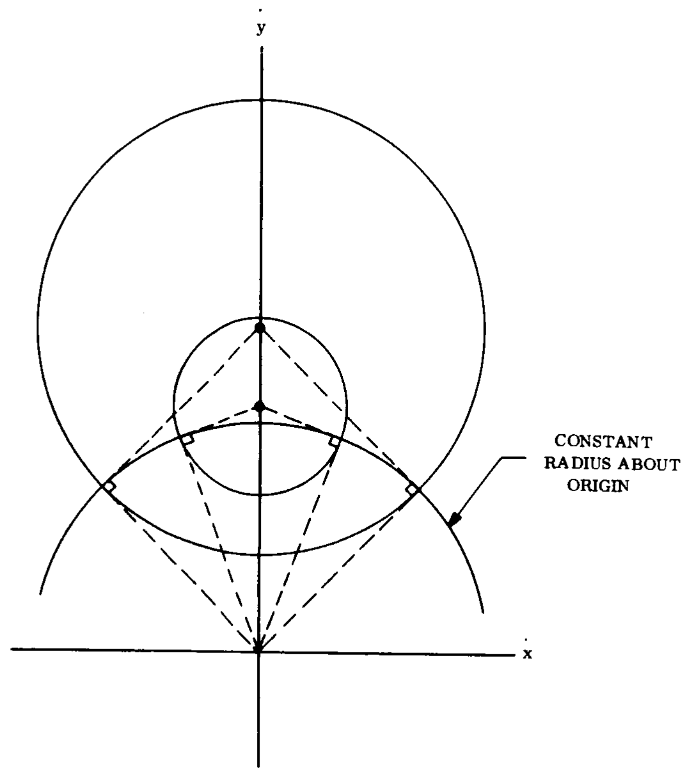



Figure 4-13. Velocity Hodographs for Constant Energy Family of One-Force-Center Hyperbolas

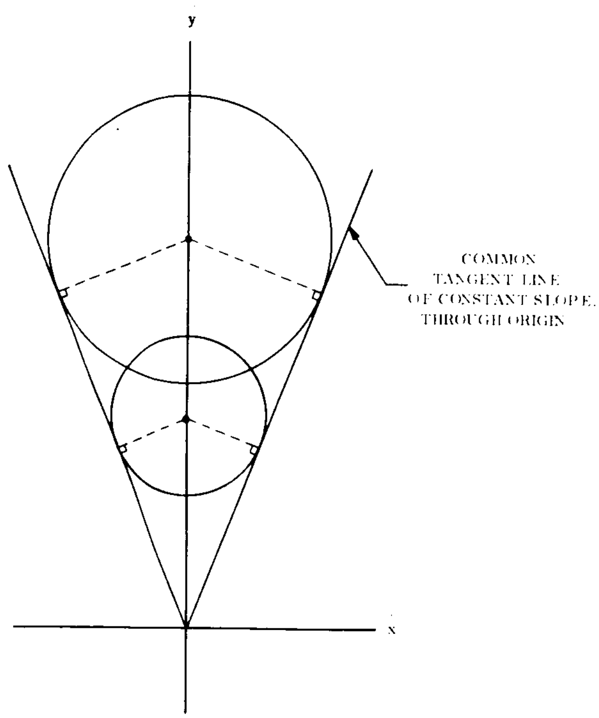
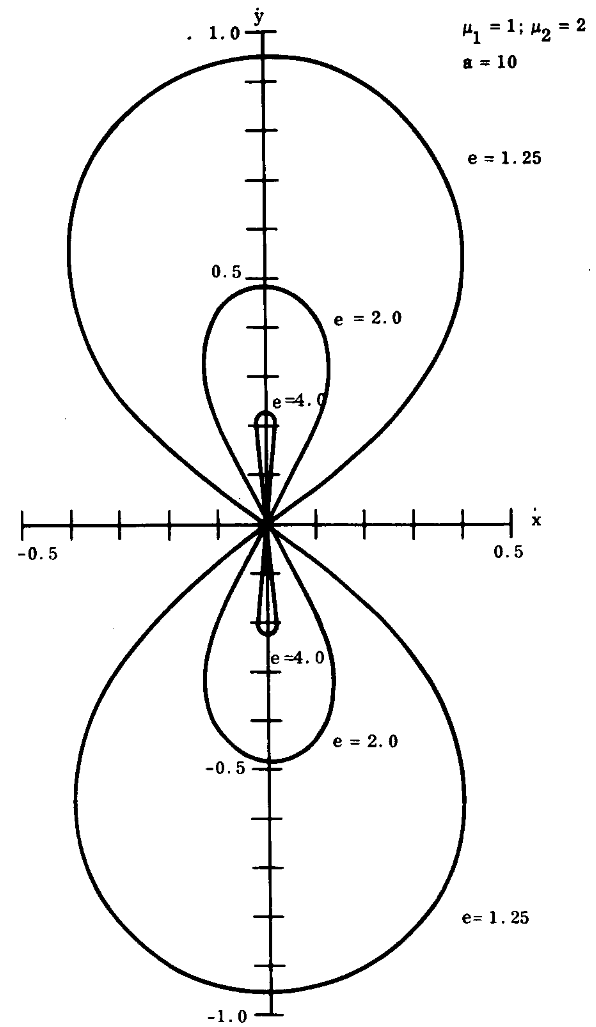



Figure 4-14. Velocity Hodographs for Constant Eccentricity Family of One-Force-Center Hyperbolas




PERIODIC HYPERBOLAS ABOUT $\mu_1$ ($\mu_1 < \mu_2$)

Figure 4-15. Velocity Hodographs of Two-Fixed-Center Hyperbolas (Constant Energy Curves)

VELOCITY HODOGRAPHS

| | | |
|---|---|---|
| $\mu_1 < \mu_2$ | constant a, variable e | (Figure 4-15) |
| | constant e, variable a | (Figure 4-16) |
| | constant c (= ae), variable a and e | (Figure 4-17) |
| $\mu_1 \leq \mu_2$ | constant a and e, variable $\mu_1/\mu_2$ | (Figure 4-18) |

ACCELERATION HODOGRAPH

| | | |
|---|---|---|
| $\mu_1 < \mu_2$: | a = 20, e = 0.50 | (Figure 4-19) |

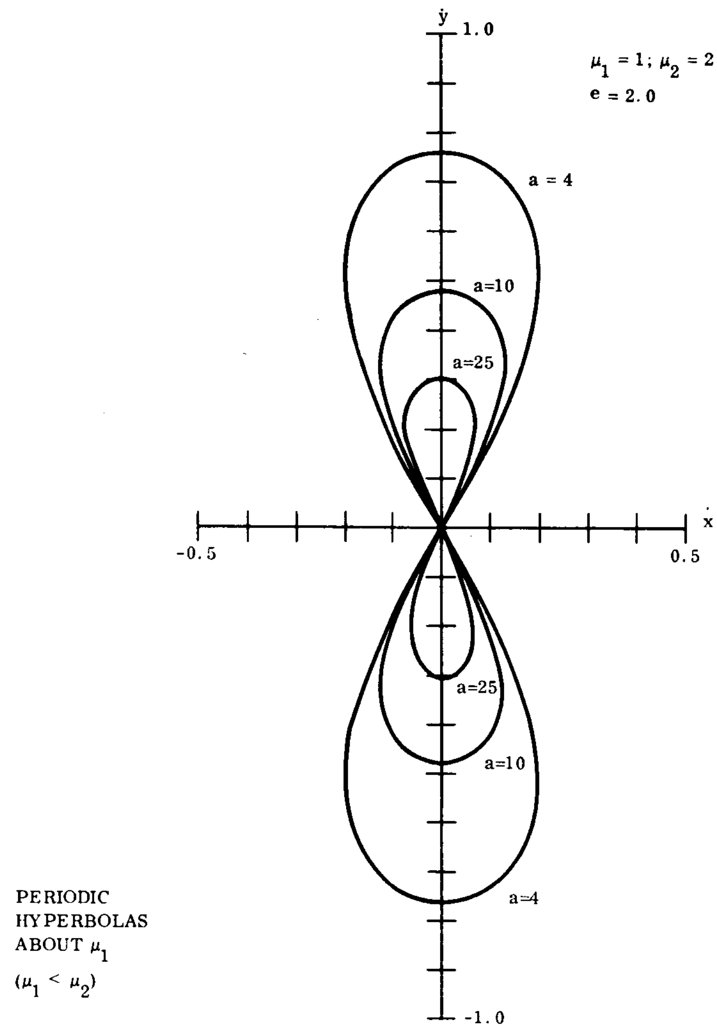



Figure 4-16. Velocity Hodographs of Two-Fixed-Center Hyperbolas (Constant Eccentric Curves)

Useful observations about the functional characteristics of the families of hodographs for periodic hyperbolas are not as apparent or easily provided as for the previous classes (i.e., ellipses and aperiodic hyperbolas). Of course, all velocity hodographs are periodic figures shaped like the figure eight (8). As shown in Figure 4-20, the transformation between vector spaces provides progressive phase advance of $\pi/2$ for each successive higher-order vector space, just as for all previously explored classes of orbit. Moreover, the origin of the velocity vector space maps over into the turning points in position vector space, or into the corresponding endpoints of the acceleration hodograph.

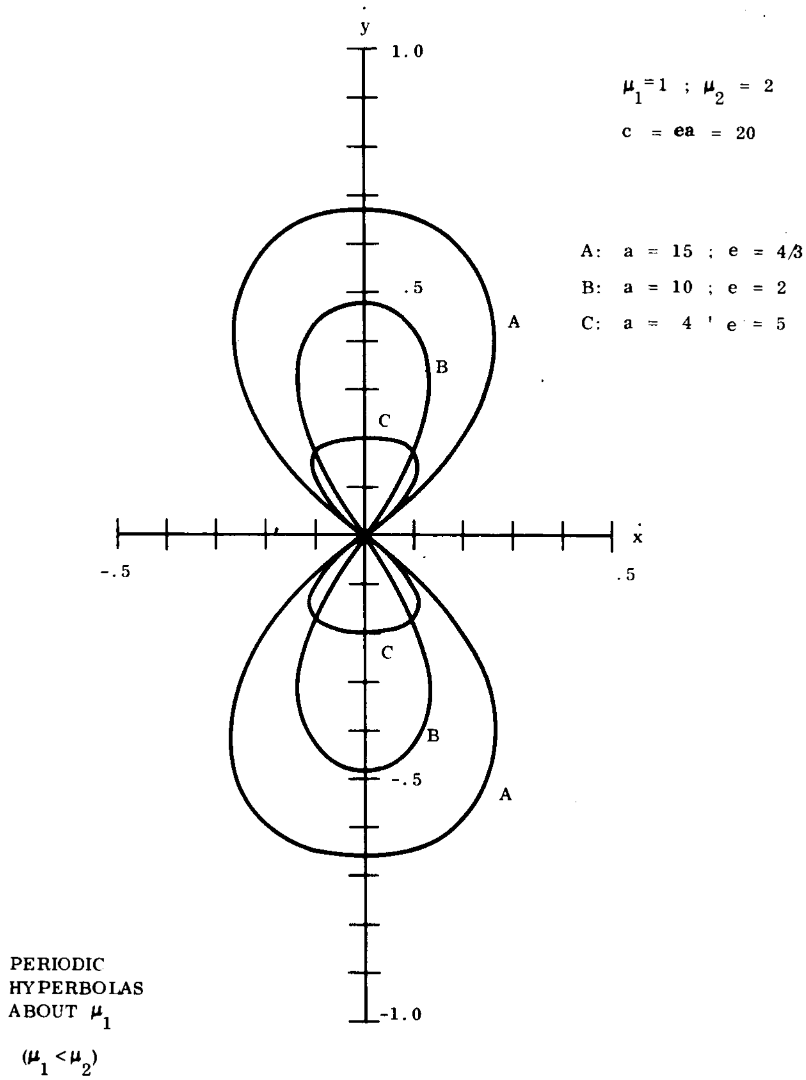



Figure 4-17. Velocity Hodographs of Two-Fixed-Center Hyperbolas (Constant Separation Curves)

The sense of progression along the orbit in the various vector space is well-defined, as shown. If one were to reflect the velocity hodograph upon the upper half-plane, about the $\dot{x}$-axis, the ambiguity of sense would no longer be present. Then all vector space loci of the orbit would have one sense only: counterclockwise.

Comparison of the corresponding hodographs for aperiodic, constant-energy hyperbolas (Figure 4-7) and periodic, constant-energy hyperbolas (Figure 4-15) indicates that the circle of constant radius (which represents "points at infinity" in position vector space)

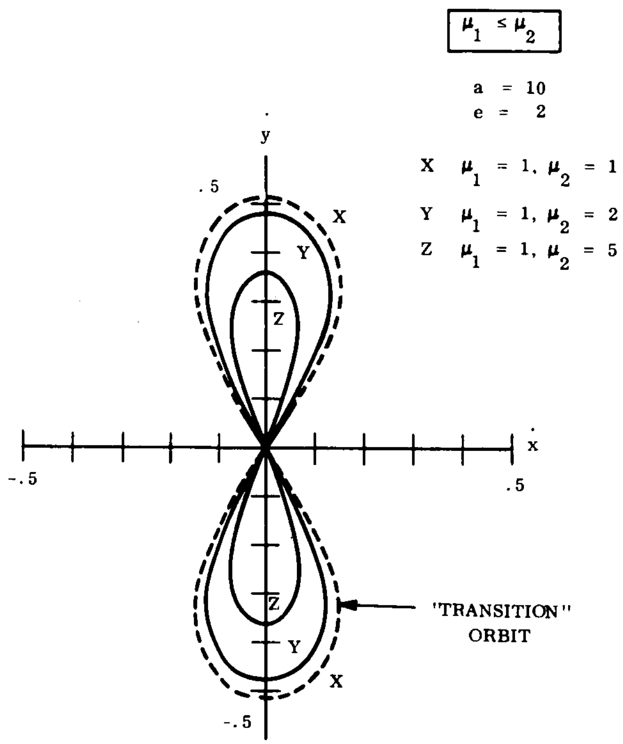



Figure 4-18. Velocity Hodographs of Two-Fixed-Center Periodic Hyperbolas (For Various Source Strengths $\mu_2$)

vanishes to the origin, thereby constricting or "shrinking" the hodographs in both coordinates. In fact, preliminary investigation indicates a direct constriction may occur, as a non-linear function. If this were true, then any other hyperbolic orbit could be directly generated from one given and defined hyperbolic orbit, periodic or aperiodic. However, further analytical and/or computer study would be required before this hypothesis could be accepted conclusively.

Referring to the constant-eccentricity family of orbital hodographs shown in Figure 4-16, it is seen that all hodographs terminate on constant-slope lines, just as for aperiodic orbits. In this family, however, all orbital hodographs contact the constant slope lines tangentially at one common point, the origin.

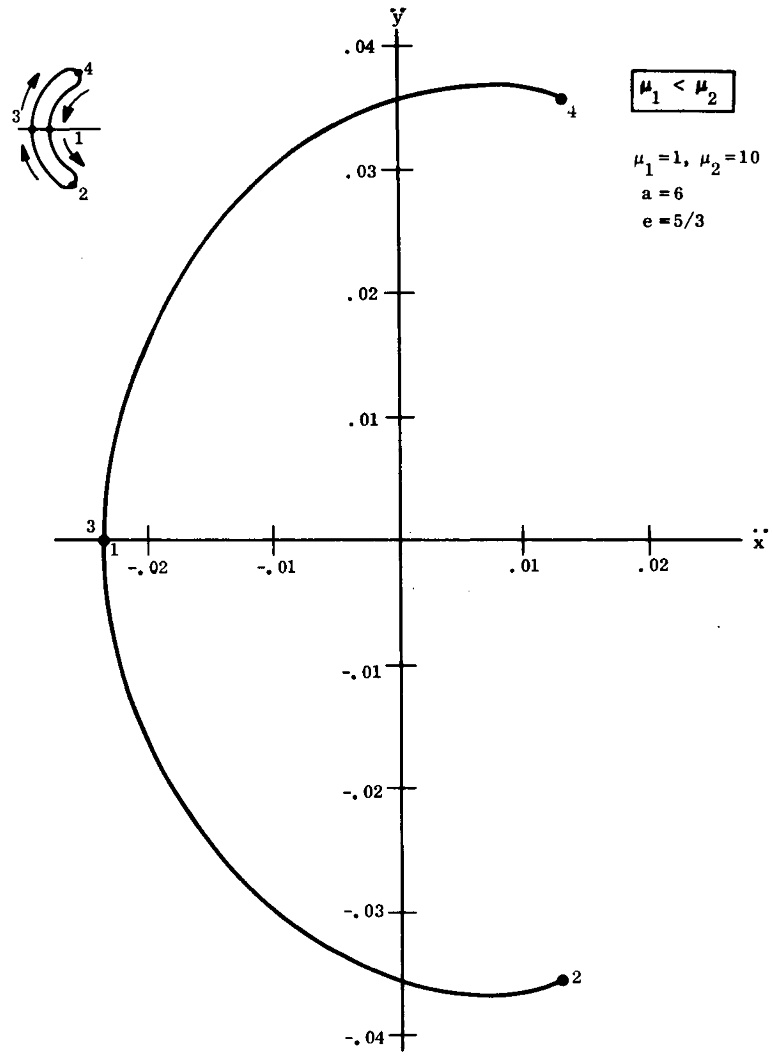



Figure 4-19. Acceleration Hodograph for Two-Fixed-Center Periodic Hyperbolic Orbit ($\mu_1 < \mu_2$)

Figure 4-18 presents the velocity hodographs for periodic hyperbolic orbits with identical parameters, but with different field strengths for the force center most remote from the periodic orbits. Consequently, all such orbits are entirely coincident in position vector space. Moreover, all functional statements about the constant-eccentricity family shown in Figure 4-14 apply here also. In contrast with the hodograph of the aperiodic hyperbola occurring in the other half-plane (see Figure 4-10), the velocity hodograph of the periodic orbit shrinks as the field strength ($\mu_2$) of the remote force center increases. Naturally, all comments on the "transition" hyperbola are identical with those noted previously (see Figure 4-10).

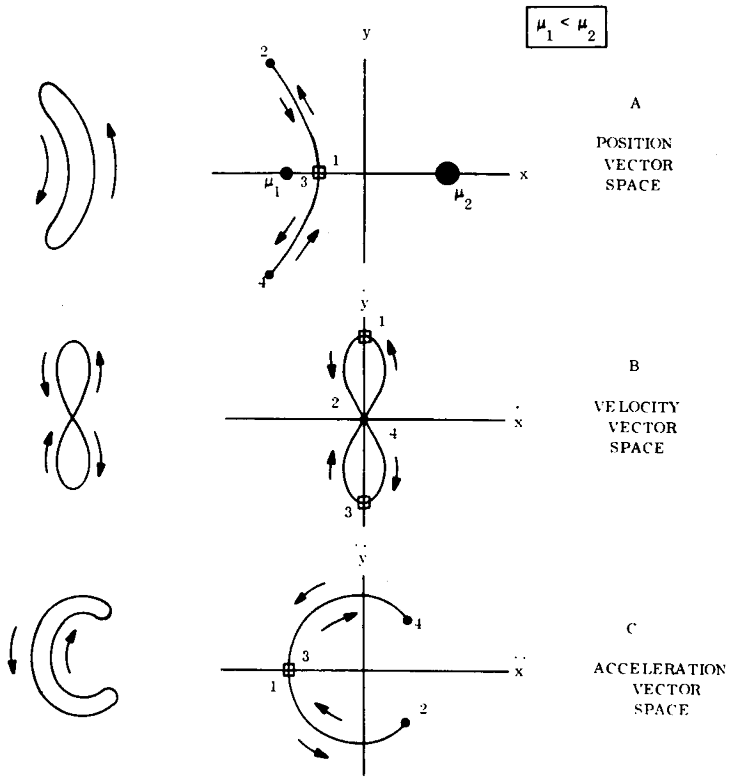



Figure 4-20. Correspondence of Vector Space Maps for Two-Fixed-Center Periodic Hyperbolas

An alternative function for the total velocity, other than Equation 4-33B, may be simply obtained by means of the basic energy relation of Equation 4-37. The differences between orbital energy and potential energy, by use of Equations 4-36 and 4-38 in Equation 4-37, provide

$$V^2 = \left[\left(\frac{\mu_2 - \mu_1}{a}\right) - 2\left(\frac{\mu_2}{a - ex} - \frac{\mu_1}{a + ex}\right)\right] \operatorname{sgn} x \ . \qquad \text{(4-33C)}$$

Equation 4-33C shows that a constant component of velocity is always present, due to the term

$$\left(\frac{\mu_2 - \mu_1}{a}\right)$$

even though it is not directly apparent in Figures 4-15 to 4-17. This constant velocity component, which could be shown by a circle about the origin in velocity vector space, represents the energy invariant h (Reference 7).

Since two invariants of the two-fixed-center trajectory exist, the other invariant must also be definable in the velocity vector space. That is, the geometric figure of the velocity hodograph must be a function of two invariants which are dimensionally velocities (e.g., ft/sec.). This functional relation has been identified in a later section of this report.

## 4.5 HODOGRAPH SUPERPOSITION TECHNIQUE FOR GENERATING THE HYPERBOLIC TWO-FIXED-CENTER ORBIT

The development of the general parametric formulation of the two-fixed-center trajectory solution requires the definition and use of the second invariant as discussed briefly above. Although this analysis objective has not yet been attained, the utility of the consequent parametric formulation should be apparent with this special case of confocal hyperbolic orbits. For example, the parametric formulation should enable generation of the trajectory hodograph (and consequently its corresponding map in any other vector or state space) by means of the hodographs due to each force center alone. That is, the hodographic solution for the one-force-center problem will provide the two-fixed-center solution by an algorithm of hodographic superposition. The existence of such a superposition principle is assured by the fact that the required solution is a biharmonic function (Reference 1); every biharmonic function can be expressed by functions of a complex variable (Reference 8). Although we do not yet have the general parametric formulation, the well-defined knowledge of the special class of confocal hyperbolas must enable the analytic development of the special form of the superposition principle for the hyperbolic orbits. Not only would the existence of the hodograph superposition technique for this special class of orbit be essential in order that the general technique exist, but the unique properties of the algorithm must be embodied within the algorithm for superposition generation of the general solution. That is, the algorithm of the special case will provide essential clues about the general case.

As shown in Reference 2, superposition for the elliptic orbit of the two-fixed-center problem is provided (in conformance with Bonnet's Theorem) by the following conditions for the velocity vectors ($\overline{V}_e$, $\overline{v}_{1e}$, $\overline{v}_{2e}$):

a. The arguments or directions of all velocity vectors must be identical; that is,

$$\arg \overline{V}_e \equiv \arg \overline{v}_{1e} \equiv \arg \overline{v}_{2e} \tag{4-56}$$

b. The magnitudes of the velocity vectors must be related by the sum of their squares; that is,

$$V_e^2 = v_{1e}^2 + v_{2e}^2 \quad . \tag{4-57}$$

In accordance with the results presented in the preceding sections and Paragraph 4. 8, it has been determined that a comparable set of conditions between the velocity vectors ($\overline{V}$, $\overline{v}_1$, $\overline{v}_2$) of the confocal hyperbolic orbits must also be fulfilled. These conditions are as follows:

a. The magnitudes of the velocity vectors must be related by their squares, as the <u>difference</u> between the composite vectors; that is,

$$V^2 = (v_2^2 - v_1^2) \operatorname{sgn} x \tag{4-58}$$

b. The arguments or directions of all velocity vectors are identical or opposed by $\pi$ radians; that is,

$$\left.\begin{array}{ll} \arg \overline{V} \equiv \arg \overline{v}_1 = \pi + \arg \overline{v}_2 & \text{for } |\overline{v}_1| > |v_2| \\ \text{or} & \\ \arg \overline{V} \equiv \arg \overline{v}_2 = \arg \overline{v}_1 - \pi & \text{for } |\overline{v}_1| < |v_2| \end{array}\right\} . \tag{4-59}$$

The above noted conditions define the following superposition algorithm to generate the velocity hodograph of the hyperbolic orbit, as shown schematically in Figures 4-21 and 4-22:

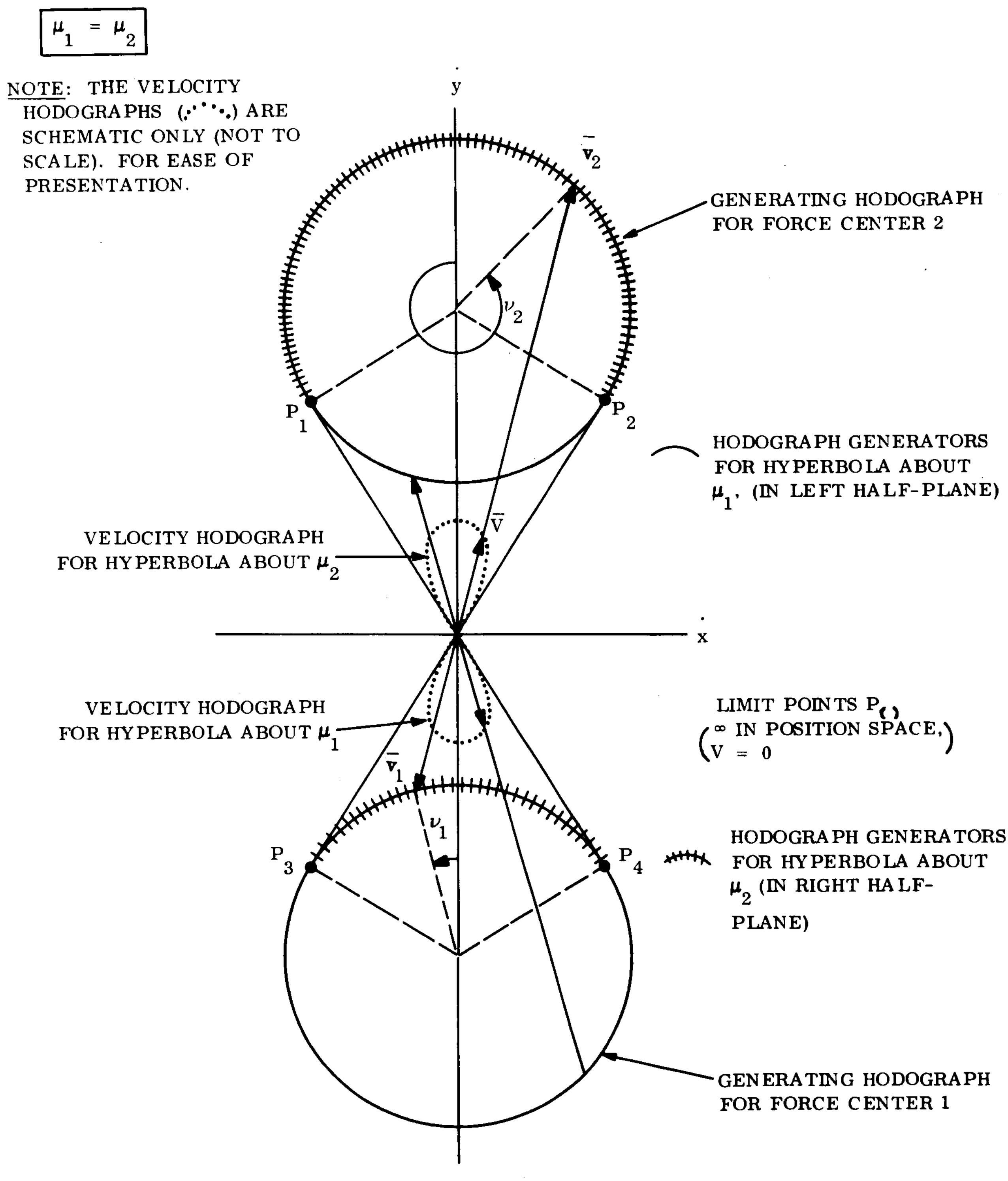



Figure 4-21. Superposition Geometry for Generating Two-Fixed- Center Hyperbolic Orbits ($\mu_1 = \mu_2$)

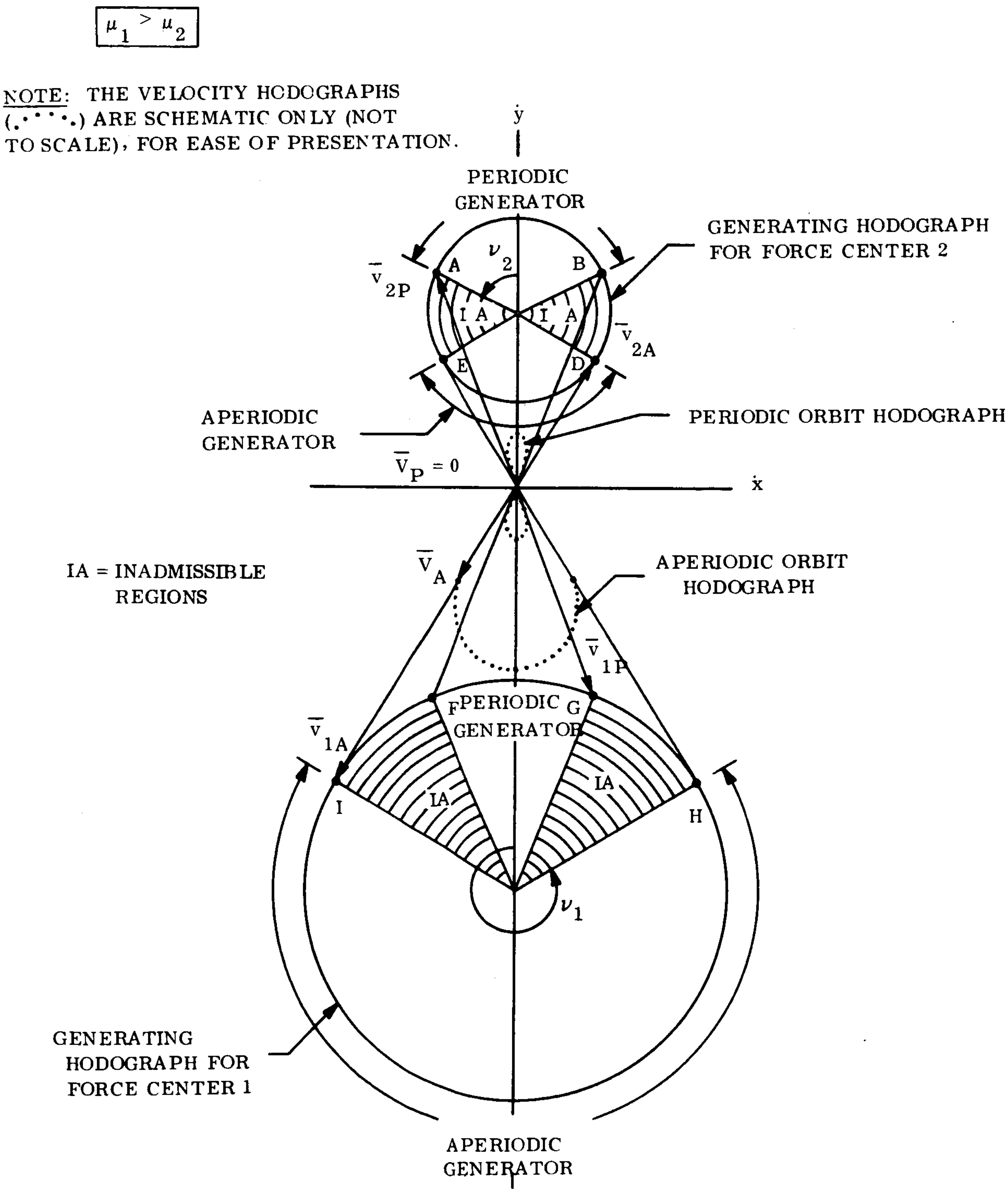



Figure 4-22. Superposition Geometry for Generating Two-Fixed-Center Hyperbolic Orbits ($\mu_1 > \mu_2$)

a. Given an initial position of the spacecraft in the two-fixed-center system and the definitive parameters of the confocal hyperbola through that point in position vector space, derive the hodograph parameters for each force center alone.

b. The velocity hodographs due to each force center will then be defined in velocity vector space (see Figures 4-21 and 4-22).

c. For any one value of the angle variable $\nu_1$, the corresponding angle variable $\nu_2$ is determined by the collinearity of $\bar{v}_1$ and $\bar{v}_2$ (see Figures 4-21 and 4-22).

d. For any one value of the angle variable $\nu_1$, the magnitudes of $v_1$ and $v_2$ are determined by the intersection of the line with each of the single-force-center hodographs.

e. The magnitude of the total velocity vector (i.e., the two-fixed-center orbit velocity) is then

$$V = \sqrt{(v_2^2 - v_1^2) \operatorname{sgn} x} \quad .$$

This superposition technique is valid for unequal as well as equal force centers (i.e., $\mu_1 \neq \mu_2$). However, the algorithm geometry is no longer as simple as the comparable algorithm geometry for the elliptic orbit, as would be anticipated because the corresponding equations of dynamic constraint are also more complex.

In order to understand and interpret the algorithm geometry more easily, let us consider the simplest case: equal force centers. As shown in Figure 4-21, the velocity hodographs due to each force center are equal in radius, and displaced from the state space origin along the $\dot{y}$-axis by equal and opposite center displacements. The limit points ($P_1$, $P_2$) and ($P_3$, $P_4$) of the two sectors of each generating hodograph are defined by the two lines which pass through the state space origin and, necessarily, are tangent to each hodograph, as shown. The hodograph sectors $\overset{\frown}{P_1 P_2}$, due to force center 2, and $\overset{\frown}{P_3 P_4}$, due to force center 1, generate the velocity hodograph for the hyperbolic orbit for center 2, whereas sectors $\underset{\smile}{P_1 P_2}$ and $\underset{\smile}{P_3 P_4}$ generate the other orbit about center 1. As shown, the total velocity vector $\bar{V}$ is directed along the larger vector ($\bar{v}_2$ as shown), with magnitude $V = \sqrt{v_2^2 - v_1^2}$. At the corresponding sets of limit points, ($P_1$, $P_4$) and ($P_2$, $P_3$), which represent "points of infinity" in position space, $v_1 = v_2$ so that $V = 0$. The corresponding position and velocity space loci are shown schematically in Figure 4-23, with the related sense of progression for each orbit.

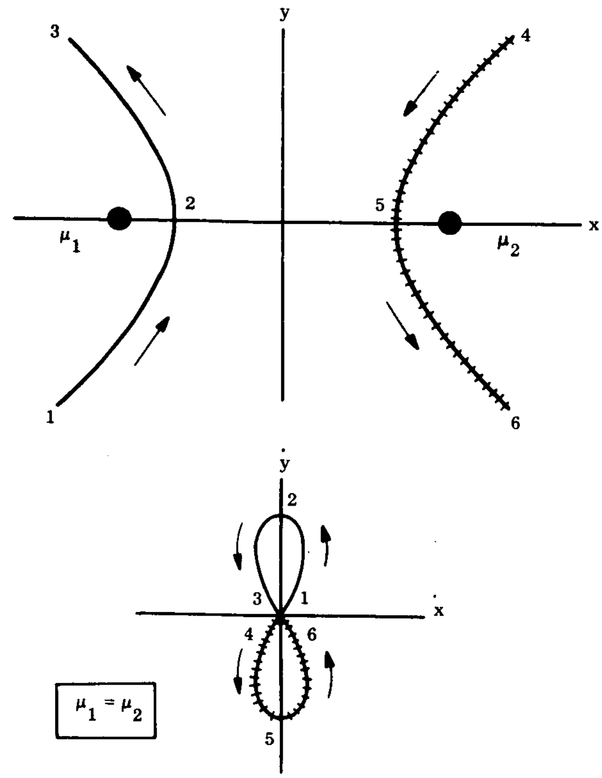



Figure 4-23. Correspondence of Vector Space Maps for Two-Fixed-Center Hyperbolas ($\mu_1 = \mu_2$)

Let us now continue to the general case, for example, as shown in Figure 4-22 for $\mu_1 > \mu_2$. Many characteristic properties of this superposition geometry may be observed. First, the generating hodograph for the weaker force center (2) has smaller radius ($C_2$) and $\dot{y}$-axis center displacement ($R_2$) than for the stronger force center (1). Second, the aperiodic hyperbolic orbit (which is proximate to center 1) is generated by the hodograph sectors $\underset{\smile}{\mathrm{ED}}$ (due to force center 2) and $\underset{\smile}{\mathrm{IH}}$ (due to force center 1). Since, for each possible set of $\bar{v}_1$, $\bar{v}_2$, it is apparent that $|\bar{v}_1| > |\bar{v}_2|$, the resulting aperiodic orbit hodograph lies necessarily in the lower half-plane of velocity vector space, without approaching the space origin. Third, the periodic hyperbolic orbit (which is proximate to center 2) is generated by the hodograph sectors $\widehat{\mathrm{AB}}$, due to force center 2, and $\widehat{\mathrm{FG}}$, due to force center 1. At each set of limit points (A, B) and (F, G), the corresponding velocity vectors $\bar{v}_1$, $\bar{v}_2$ have equal magnitudes (e.g., $|\bar{v}_{1P}| = |\bar{v}_{2P}|$) so that the endpoints of the periodic orbit hodograph must lie at the velocity space origin. Note that the directions of motion along the hodograph segments $\widehat{AB}$ and $\widehat{FG}$ determine the symmetrical lobes (of the figure eight 8) for the periodic orbit hodograph. As the two corresponding points representing the orbital mass proceed counterclockwise (on each of the generating hodographs), the lobe of the periodic orbit hodograph is generated in the upper half-plane. When these points then retrogress in clockwise motion after reaching limit points A, G, the other hodograph lobe is generated in the lower half-plane by point symmetry $\left[\text{i.e., } f(w') = -f(-w') \text{ where } w' = \dot{x} + i\dot{y}\right]$, as indicated schematically in Figure 4-22. Fourth, the generating hodograph sectors EA, BD, IF and GH represent inadmissible or inaccessible regions of generation. As the field strength $\mu_1 \rightarrow \mu_2$ until $\mu_1 = \mu_2$, the inadmissible sectors vanish (i.e., EA, BD, IF and GH $\rightarrow$ 0) until $E \equiv A$, $B \equiv D$, $I \equiv F$ and $G \equiv H$ for $\mu_1 = \mu_2$. Then the geometry of Figure 4-22 would have reduced to the geometry of Figure 4-21. Finally, the corresponding position and velocity space loci for $\mu_1 \neq \mu_2$ are shown schematically in Figure 4-24, with the related sense of progression for each orbit.

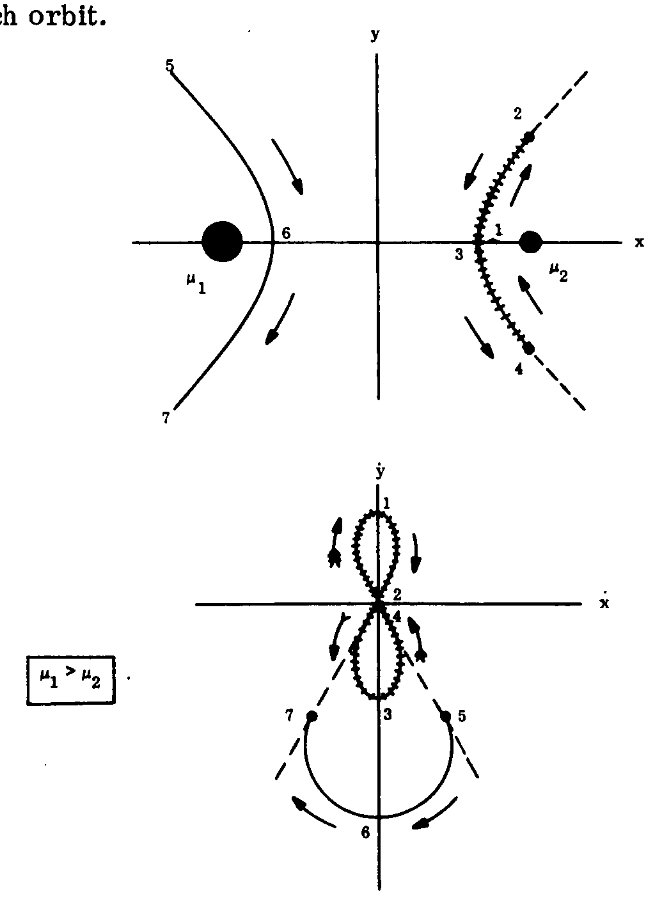



Figure 4-24. Correspondence of Vector Space Maps for Two-Fixed-Center Hyperbolas ($\mu_1 \neq \mu_2$)

## 4.6 INVARIANTS FOR THE HYPERBOLIC ORBITS

The two invariants, h and δ, of the two-fixed-center problem have been briefly discussed in Reference 7. It is extremely valuable to study these invariants for the hyperbolic orbits, since superposition implies, necessarily, the separation of component terms of these two invariants, each component due to the individual force centers alone. That is, determination of h (hence E) and δ (hence $\ell_\nu'$)* for each of the two generating hodographs, due to the individual force centers, will define the hodograph parameters C and R since

$$C = \frac{\mu}{\ell_\nu'} \tag{4-60}$$

$$R = \sqrt{-h + \left(\frac{\mu}{\ell_\nu'}\right)^2} \tag{4-61}$$

The invariant h is directly defined by the total orbital energy E, as

$$h = \frac{2E}{m} \tag{4-62}$$

Consequently, the invariant h for the hyperbolic orbits has, in essence, been treated by the preceding work, since**

$$E = E_1 + E_2 = \frac{m}{2a}\left(\mu_2 - \mu_1\right) \quad . \tag{4-63}$$

That is, the orbital energy is decomposed into

$$E_1 = -\frac{\mu_1 m}{2a} \tag{4-64}$$

$$E_2 = \frac{\mu_2 m}{2a} \tag{4-65}$$

---

* Note that, in accordance with the notation convention established in Reference 7, $\ell_{\nu 1}'$, $\ell_{\nu 2}'$ define mutually-exclusive angular momenta about each force center due to the component velocity vectors $\underline{v}_1$, $\underline{v}_2$, whereas $\ell_1'$, $\ell_2'$ define the non-exclusive angular momenta about each force center - each due to the total velocity vector $\underline{V}$.

** In this section, the signum function (sgn x) will not be used, but is understood to govern the selection of the hyperbolic branches, periodic or aperiodic. The analytical results are completely valid for both cases. As presented in the following work, the equations describe hyperbolic orbits proximate to force center 2.

or, for the invariant h,

$$h_1 = -\frac{\mu_1}{a} \tag{4-66}$$

$$h_2 = \frac{\mu_2}{a} \tag{4-67}$$

Actually, this decomposition for E or h is the immediate aspect of the energy conservation principle. Upon comparison of Equations 4-64 and 4-65, it is seen that

$$\frac{E_1}{E_2} = -\frac{\mu_1}{\mu_2} \tag{4-68}$$

Similarly, the invariant $\delta$ should decompose into the components $\delta_1$ and $\delta_2$ due to each force center. First, the complete invariant $\delta$ may be expressed in terms of the conic parameters of the orbit. The general forms (Reference 7) of the invariant $\delta$ are

$$\frac{\ell_1'\ell_2'}{2c} + \mu_1\left(\frac{x+c}{r_1}\right) - \mu_2\left(\frac{x-c}{r_2}\right) = \delta \tag{4-69}$$

or

$$\frac{\ell_1'\ell_2'}{2c} + \left(\mu_1 \cos v_1 - \mu_2 \cos v_2\right) = \delta \tag{4-70}$$

where

$\ell_1' = r_1 V_{v1} = (x+c)\dot{y} - y\dot{x}$ = angular momentum per unit mass, relative to force center 1 (4-71)

$\ell_2' = r_2 V_{v2} = (x-c)\dot{y} - y\dot{x}$ = angular momentum per unit mass relative to force center 2. (4-72)

As shown in Subsection 4.8,

$$\frac{\ell_1'\ell_2'}{2c} = -\frac{pV^2}{2e} \tag{4-73}$$

and

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = -\frac{E}{em}(p + 2a) + \frac{pV^2}{2e} \tag{4-74A}$$

or

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \left[\frac{1+e^2}{2e}\right](\mu_1 - \mu_2) + \frac{pV^2}{2e} \quad . \tag{4-74B}$$

Consequently, upon use of Equations 4-73 and 4-74 in Equation 4-70, the invariant $\delta$ is expressed as

$$\delta = -\frac{E}{em}(p+2a) = -\frac{Ec}{m}\left(\frac{1}{e^2}+1\right) \quad . \tag{4-75A:4-75B}$$

Now, let us decompose the invariant $\delta$ into the components due to each force center. Upon using Equation 4-58 with Equation 4-73,

$$\frac{\ell_1' \ell_2'}{2c} = \frac{pV_1^2}{2e} - \frac{pV_2^2}{2e} \tag{4-76}$$

Consequently, Equation 4-70 for the total invariant $\delta$ can now be expressed as

$$\delta = \delta_1 + \delta_2 \tag{4-77A}$$

where

$$\delta_1 = \frac{pV_1^2}{2e} + \mu_1 \cos \nu_1 \tag{4-78A}$$

$$\delta_2 = -\frac{pV_2^2}{2e} - \mu_2 \cos \nu_2 \tag{4-79A}$$

But, by means of Equation 4-133,

$$\mu_1 \cos \nu_1 = \frac{\mu_1 p}{er_1} + \frac{\mu_1}{e} \tag{4-80}$$

The energy equation

$$E_1 = -\frac{mv_1^2}{2} - \frac{\mu_1 m}{r_1} \tag{4-81}$$

provides

$$\frac{\mu_1}{r_1} = -\frac{v_1^2}{2} - \frac{E_1}{m} \tag{4-82}$$

so that, upon substituting Equation 4-82 into Equation 4-80,

$$\mu_1 \cos \nu_1 = -\frac{pv_1^2}{2e} - \frac{pE_1}{em} + \frac{\mu_1}{e} \tag{4-83}$$

Consequently, substituting Equation 4-83 into Equation 4-78A,

$$\delta_1 = \frac{1}{e}\left[\mu_1 - \frac{pE_1}{m}\right] \quad . \tag{4-78B}$$

Then, by use of Equation 4-8 and 4-64,

$$\delta_1 = \frac{\mu_1}{2e}\left(1 + e^2\right) \quad . \tag{4-78C}$$

Similarly,

$$\delta_2 = -\frac{1}{e}\left[\mu_2 + \frac{pE_2}{m}\right] \tag{4-79B}$$

or

$$\delta_2 = -\frac{\mu_2}{2e}\left(1 + e^2\right) \quad . \tag{4-79C}$$

Consequently, *

$$\delta = -\frac{(1+e^2)}{2e}(\mu_2 - \mu_1) \quad . \tag{4-77B}$$

Upon comparison of Equations 4-78C and 4-79C, it is seen that

$$\frac{\delta_1}{\delta_2} = -\frac{\mu_1}{\mu_2} \quad . \tag{4-84}$$

Finally, the relation between the invariants $\delta$, $\delta_1$, $\delta_2$ and the angular momenta $\ell_\nu'$, $\ell_{\nu 1}'$, $\ell_{\nu 2}'$ must be established, in order that the hodograph parameters C and R be defined by the terms $\delta_{(\,)}$ (see Equations 4-60 and 4-61).

It is known (Reference 9) that

$$p = \frac{\ell_\nu'^2}{\mu} = \frac{\mu}{C^2} \tag{4-85}$$

so that Equation 4-78C can be expressed in terms of $\ell_\nu'$ rather than $\mu_1$, by

$$\delta_1 = \frac{(1+e^2)}{2pe}\,\ell_{\nu 1}'^2 \quad ; \tag{4-78D}$$

similarly,

$$\delta_2 = -\frac{(1+e^2)}{2pe}\,\ell_{\nu 2}'^2 \tag{4-79D}$$

Consequently,

$$\delta = -\frac{(1+e^2)}{2pe}\left(\ell_{\nu 2}'^2 - \ell_{\nu 1}'^2\right) \tag{4-77C}$$

---

* Again, attention is called to the footnote on page 4-43 re "sgn x" which has been omitted here, for ease and clarity in reporting.

and

$$\frac{\delta_1}{\delta_2} = -\left(\frac{\ell_{v1}'}{\ell_{v2}'}\right)^2 \quad . \tag{4-86}$$

In summary, it has been found that the hodograph superposition for the two-fixed-center elliptic orbit is due to the analytical decomposition of the basic invariants (E, δ) as follows:

$$\boxed{\begin{gathered} E = E_1 + E_2 \\ \delta = \delta_1 + \delta_2 \\ \frac{E_1}{E_2} = \frac{\delta_1}{\delta_2} = -\frac{\mu_1}{\mu_2} = -\left(\frac{\ell_{v1}'}{\ell_{v2}'}\right)^2 \end{gathered}}$$

The hodograph parameters (which are also invariants) can now be determined. Since

$$\ell_{v1}' = \frac{\mu_1}{C_1} = r_1 v_{v1} \tag{4-87}$$

and

$$\ell_{v2}' = \frac{\mu_2}{C_2} = r_2 v_{v2} \quad , \tag{4-88}$$

then

$$\frac{\ell_{v1}'}{\ell_{v2}'} = \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{C_2}{C_1}\right) \tag{4-89}$$

so that, upon substitution of Equation 4-89 into

$$\left(\frac{\ell_{v1}'}{\ell_{v2}'}\right)^2 = \frac{\mu_1}{\mu_2} \tag{4-90}$$

and subsequent reduction, we obtain

$$\left(\frac{C_1}{C_2}\right)^2 = \frac{\mu_1}{\mu_2} \tag{4-91}$$

Also, since

$$R_1^2 - C_1^2 = -\frac{2E}{m} \tag{4-92}$$

and

$$R_2^2 - C_2^2 = \frac{2E_2}{m} \quad , \tag{4-93}$$

then

$$\frac{R_1^2 - C_1^2}{R_2^2 - C_2^2} = -\frac{E_1}{E_2} = \frac{\mu_1}{\mu_2} \tag{4-94}$$

so that, upon substitution of Equation 4-91 into Equation 4-94 and subsequent reduction,

$$\left(\frac{R_1}{R_2}\right)^2 = \frac{\mu_1}{\mu_2} \tag{4-95}$$

The hodograph parameters (or invariants) are summarized as follows:

$$\boxed{\left(\frac{C_1}{C_2}\right)^2 = \frac{\mu_1}{\mu_2} = \left(\frac{R_1}{R_2}\right)^2}$$

It is clear that the ratio ($\mu_1 : \mu_2$) will determine the geometric constraints which the generating hodographs due to each force center must fulfill, as discussed in the preceding section and shown in Figures 4-21 and 4-22.

## 4.7 SUMMARY CONCLUSIONS

The equations of the velocity and acceleration hodographs for the confocal hyperbolic orbits of the two-fixed-center problem have been developed. The functional dependence of the hodograph geometry upon the various parameters of the hyperbola has been briefly explored by mapping a few typical cases (Figure 4-7 to 4-11, and 4-15 to 4-19). Aside from the new and previously unexplored state space characteristics which have been revealed, the regions and conditions of constraint upon two-fixed-center orbits of hyperbolic section have been defined in detail. The regions and classes of hyperbolic orbits for $\mu_1 \neq \mu_2$ are summarized graphically in Figure 4-25. All trajectories are hyperbolas aperiodic (or nonrecurring) about the larger force center, or hyperbolic sections periodic about the smaller force centers. Note that this means the half-plane in which the smaller force center is imbedded is a nonadmissible region of solution, except for the circular region about the force center. As the field strengths of the force centers approach equality, the circular region expands until it completely fills the half-plane for $\mu_1 = \mu_2$. Consequently, the orbits for $\mu_1 = \mu_2$ are hyperbolas (aperiodic solutions only) mapped over the entire position vector space.

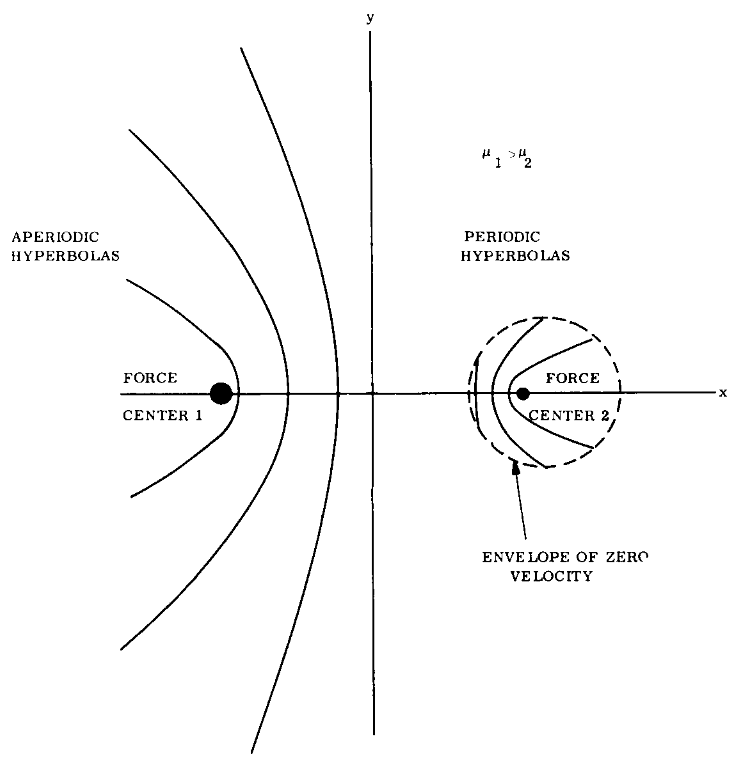



Figure 4-25. Hyperbolic Orbits of the Two-Fixed-Center Problem

It has been shown that a superposition algorithm for generating the orbit in velocity vector space, by means of the hodographs due to each force center alone, exists; also, this superposition technique has been completely defined. Consequently, two special classes of orbits, elliptic and hyperbolic, have been shown to be determinate solely by the generating hodographs for each force center alone. Moreover, simple and basic relations between the invariants: the classical "integral invariants" (h, $\delta$), the hodograph parameters (C, R) and the mutually exclusive angular momenta ($\ell_{\nu 1}'$, $\ell_{\nu 2}'$), have been established.

The study results have demonstrated that libration points, or zero velocity states, are not singular points in state space analysis, but points of continuous variation (in the mathematical sense) in velocity state space. Moreover, the hodograph parameters are always finite and well-defined.

Further study of the velocity hodographs, acceleration hodographs and invariants (both classical and hodographic) is strongly indicated as promising and desirable in achieving a parametric representation for the general orbit of the two-fixed-center problem.

## 4.8 REDUCTION OF THE MAJOR FUNCTIONAL TERMS OF THE INVARIANT $\delta$, FOR HYPERBOLIC ORBITS

One general form of the invariant $\delta$ is

$$\frac{\ell_1' \ell_2'}{2c} + (\mu_1 \cos v_1 - \mu_2 \cos v_2) = \delta \tag{4-96}$$

where

$$\ell_1' = r_1 V_{\nu 1} = (x+c)\dot{y} - y\dot{x} \tag{4-97}$$

= angular momentum per unit mass, relative to force center 1

$$\ell_2' = r_2 V_{\nu 2} = (x-c)\dot{y} - y\dot{x} \tag{4-98}$$

= angular momentum per unit mass, relative to force center 2 and all other terms are defined graphically in Figure 4-26. Note that

$$V^2 = V_{r1}^2 + V_{\nu 1}^2 \tag{4-99}$$

or

$$V^2 = V_{r2}^2 + V_{\nu 2}^2 \quad . \tag{4-100}$$

Each of the two major terms on the left-hand side of Equation 4-96 will be reduced to functions of the given constants (c, $\mu_1$, $\mu_2$), the conic parameters (p, e) of the hyperbolic figure of orbit and the velocity scalars (V, $v_1$, $v_2$). According to Equation 4-33B in the main body of the report,

$$V^2 = \left[\frac{\mu_1}{(a+ex)^2} - \frac{\mu_2}{(a-ex)^2}\right]\left[\frac{a^2-(ex)^2}{a}\right] \operatorname{sgn} x \tag{4-101A}$$

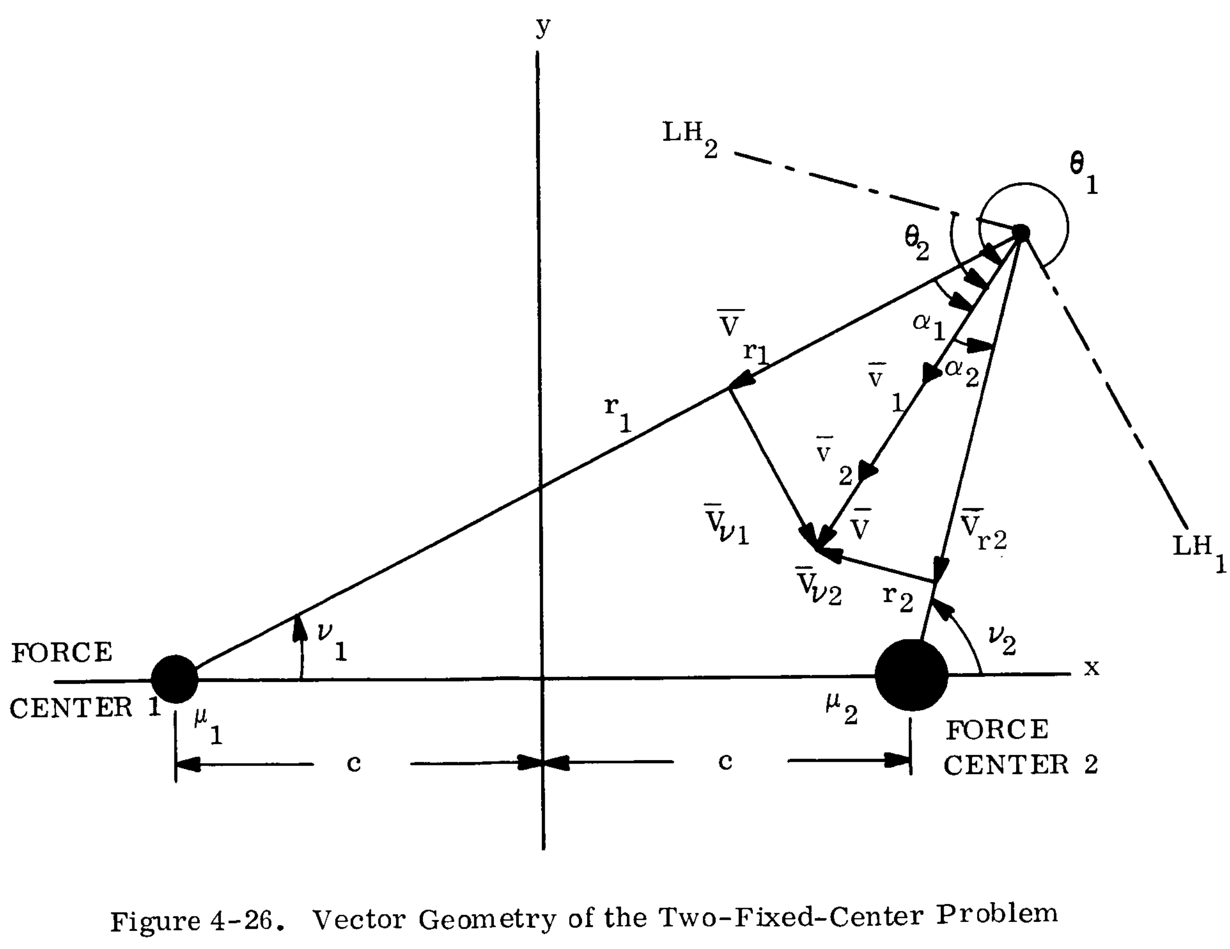



Figure 4-26. Vector Geometry of the Two-Fixed-Center Problem

An alternative form is

$$V^2 = \left[\frac{\mu_1}{a}\left(\frac{a-ex}{a+ex}\right) - \frac{\mu_2}{a}\left(\frac{a+ex}{a-ex}\right)\right] \operatorname{sgn} x \tag{4-101B}$$

which may be shown to be

$$V^2 = v_2^2 - v_1^2 \tag{4-102}$$

for the hyperbolic branch proximate to force center 2. (See footnote on page 4-43). The velocity vectors $\bar{V}$, $\bar{v}_1$, $\bar{v}_2$ are collinear, as shown in Figure 4-26.

### 4.8.1 THE TERM ($\ell_1' \ell_2'/2c$)

Referring to Figure 4-26, it is seen that

$$V_{v1} = V \sin \alpha_1 = \sin \alpha_1 \sqrt{v_2^2 - v_1^2} \tag{4-103}$$

and

$$V_{v2} = V \sin \alpha_2 = \sin \alpha_2 \sqrt{v_2^2 - v_1^2} \quad . \tag{4-104}$$

Also,

$$r_1 (v_1 \cos \theta_1) = - \sqrt{\mu_1 p} \tag{4-105}$$

and

$$r_2 (v_2 \cos \theta_2) = \sqrt{\mu_2 p} \tag{4-106}$$

so that

$$\cos \theta_1 = - \frac{\sqrt{\mu_1 p}}{r_1 v_1} \tag{4-107}$$

and

$$\cos \theta_2 = \frac{\sqrt{\mu_2 P}}{r_2 V_2} \tag{4-108}$$

But

$$\theta_1 - \alpha_1 = \frac{3\pi}{2} \tag{4-109}$$

and

$$\theta_2 + \alpha_2 = \frac{\pi}{2} \tag{4-110}$$

so that

$$\sin \alpha_1 = \cos \theta_1 = - \frac{\sqrt{\mu_1 P}}{r_1 V_1} \tag{4-111}$$

and

$$\sin \alpha_2 = \cos \theta_2 = \frac{\sqrt{\mu_2 P}}{r_2 V_2} \tag{4-112}$$

Consequently, upon substitution of Equations 4-111 and 4-112 into Equations 4-103 and 4-104, we obtain

$$V_{v1} = - \sqrt{V_2^2 - V_1^2} \quad \frac{\sqrt{\mu_1 P}}{r_1 V_1} \tag{4-113}$$

and

$$V_{v2} = \sqrt{V_2^2 - V_1^2} \quad \frac{\sqrt{\mu_2 P}}{r_2 V_2} \tag{4-114}$$

respectively, so that

$$\ell_1' = r_1 V_{v1} = -\sqrt{v_2^2 - v_1^2}\ \frac{\sqrt{\mu_1 p}}{v_1} \tag{4-115}$$

$$\ell_2' = r_2 V_{v2} = \sqrt{v_2^2 - v_1^2}\ \frac{\sqrt{\mu_2 p}}{v_2} \tag{4-116}$$

Consequently,

$$\frac{\ell_1' \ell_2'}{2c} = -\frac{p\sqrt{\mu_1 \mu_2}}{2c}\left[\frac{v_2^2 - v_1^2}{v_1 v_2}\right] \tag{4-117A}$$

Since

$$T = E - V \quad , \tag{4-118}$$

then Equations 4-17B, 4-38 and 4-102 lead to

$$\frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \left[-\frac{\mu_1 m}{2a} + \frac{\mu_2 m}{2a}\right] - \left[-\frac{\mu_1 m}{r_1} - \frac{\mu_2 m}{r_2}\right] \tag{4-119}$$

so that

$$-\frac{mv_1^2}{2} = -\frac{\mu_1 m}{2a} + \frac{\mu_1 m}{r_1} \tag{4-120}$$

or

$$v_1^2 = -\mu_1\left[\frac{2}{r_1} - \frac{1}{a}\right] = -\frac{\mu_1}{r_1}\left[2 - \frac{r_1}{a}\right] \tag{4-121}$$

and

$$\frac{mv_2^2}{2} = \frac{\mu_2 m}{2a} + \frac{\mu_2 m}{r_2} \tag{4-122}$$

or

$$V_2^2 = \mu_2\left[\frac{2}{r_2} + \frac{1}{a}\right] = \frac{\mu_2}{r_2}\left[2 + \frac{r_2}{a}\right] \tag{4-123}$$

Then

$$V_1^2\, V_2^2 = -\frac{\mu_1\mu_2}{r_1 r_2}\left[4 - \frac{2}{a}(r_1 - r_2) - \frac{r_1 r_2}{a^2}\right] \tag{4-124A}$$

But

$$r_1 - r_2 = 2a \tag{4-125}$$

so that

$$V_1^2\, V_2^2 = \frac{\mu_1\mu_2}{a^2} \tag{4-124B}$$

or

$$V_1 V_2 = \frac{\sqrt{\mu_1\mu_2}}{a} \tag{4-126}$$

Substituting Equation 4-126 into Equation 4-117A,

$$\frac{\ell_1'\ell_2'}{2c} = -\frac{pV^2}{2\epsilon} \tag{4-117B}$$

4.8.2 THE TERM ($\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2$)

Referring to Figure 4-2 of the main body of the report, let us consider the hyperbolic branch proximate to force center 2, as shown. This hyperbolic branch referenced to force center 1 is defined by

$$r_1 = -\frac{P}{1 - e\cos\phi_1} \tag{4-127}$$

and

$$r_2 = \frac{p}{1 + e \cos \phi_2} \tag{4-128}$$

referenced to force center 2. Then

$$\cos \phi_1 = \frac{1}{e}\left[\frac{p}{r_1} + 1\right] \tag{4-129}$$

and

$$\cos \phi_2 = \frac{1}{e}\left[\frac{p}{r_2} - 1\right] \tag{4-130}$$

But

$$\upsilon_1 = 2\pi - \phi_1 \tag{4-131}$$

so that

$$\cos \upsilon_1 = \cos \phi_2 \tag{4-132}$$

and consequently

$$\cos \upsilon_1 = \frac{1}{e}\left[\frac{p}{r_1} + 1\right] . \tag{4-133}$$

Also,

$$\upsilon_2 = \pi + \phi_2 \tag{4-134}$$

so that

$$\cos \nu_2 = -\cos \phi_2 \tag{4-135}$$

and consequently

$$\cos \nu_2 = -\frac{1}{e}\left[\frac{p}{r_2} - 1\right] . \tag{4-136}$$

By means of Equations 4-133 and 4-136, the required term is

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \frac{\mu_1 - \mu_2}{e} + \frac{p}{e}\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right] . \tag{4-137A}$$

But

$$\frac{V}{m} = -\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right] = \frac{E}{m} - \frac{\mathcal{V}^2}{2} \tag{4-138}$$

so that

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = \frac{1}{e}\left[(\mu_1 - \mu_2) - p\left(\frac{E}{m} - \frac{\mathcal{V}^2}{2}\right)\right] \tag{4-137B}$$

or

$$\mu_1 \cos \nu_1 - \mu_2 \cos \nu_2 = -\frac{E}{em}(p + 2a) + \frac{p\mathcal{V}^2}{2e} . \tag{4-137C}$$

<u>References</u>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## SECTION 5

## CANDIDACY OF EQUIPOTENTIAL CONTOURS AS TWO-FIXED-CENTER ORBITS

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## SECTION 5

## CANDIDACY OF EQUIPOTENTIAL CONTOURS AS TWO-FIXED-CENTER ORBITS

The candidacy of isotachs (i.e., constant velocity orbits) as admissible orbits in a conservative force field has been considered previously. Szebehely has discussed isotach orbits and the relation of Hill (or zero-velocity) curves and orbits. A detailed analytical study of this subject has been presented in References 1 and 2. Continuing these previous investigations, our present objective is to explore the complete class of isotach curves (zero and non-zero velocity) in order to determine their admissibility as orbits. In particular, the analysis of this section is restricted to the problem of two-fixed-centers of attraction.

### 5.1 EQUATIONS OF MOTION AND NECESSARY CONDITIONS

Let us consider a system of two attracting masses $m_1$ and $m_2$ of fixed location on the x-axis of an inertial system of Cartesian coordinates, as shown in Figure 5-1. The coordinates of these masses are $m_1(+c, 0)$; $m_2(-c, 0)$. We will consider the motion of a third body of infinitesimal mass m which moves under the influence of the combined gravitational field of $m_1$ and $m_2$. In particular, we want to determine whether m may move in the space about $m_1$ and $m_2$ along an equipotential curve or space contour. Thus, our question simply phrased is, "Are the equipotential curves orbits?"

In the following discussion, the gravitational constants of the attracting bodies will be indicated by $\mu_1 = km_1$, and $\mu_2 = km_2$, where k is the universal gravitational constant. As known, the equations of motion of m are

$$\ddot{x} = - U_x = Q_x \tag{5-1}$$

$$\ddot{y} = - U_y = Q_y \tag{5-2}$$

where U(x, y) and Q(x, y) are the potential function and the force function respectively, per unit mass of the moving body. In our specific case, these functions are defined as

$$U = - Q = - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{5-3}$$

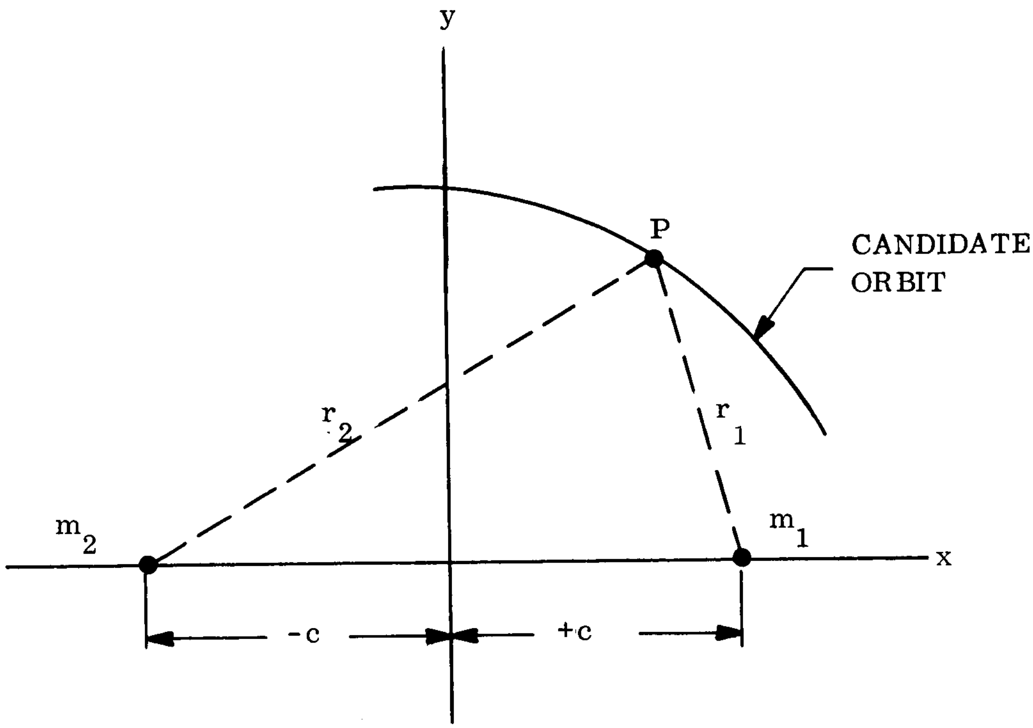


Figure 5-1. Geometry of the Orbital Path in the Two-Fixed Center Problem

where

$$r_1 = \left[(x-c)^2 + y^2\right]^{\frac{1}{2}} \quad , \quad r_2 = \left[(x+c)^2 + y^2\right]^{\frac{1}{2}} \quad . \tag{5-4}$$

Thus, the equation of the equipotential curves may be expressed as

$$\Phi(x,y) \equiv \mu_1 r_2 + \mu_2 r_1 - K r_1 r_2 = 0 \quad , \tag{5-5}$$

where K is a positive constant ($K > 0$).

Due to the fact that the Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x,y) \tag{5-6}$$

is explicitly independent of the time, the variational equation

$$\frac{dH}{dt} = H_t \tag{5-7}$$

leads to the integral of energy, H = constant. Consequently, along equipotential orbits, it must be true that

$$\dot{x}^2 + \dot{y}^2 = V^2 = 2(H - U) = k^2 = \text{const.} \tag{5-8}$$

This implies that the equipotential orbits must be isotachs in order to be admissible. Furthermore, as known, any admissible equipotential orbit in a conservative field must satisfy the second order, parabolic, partial differential equation

$$U_y^2 U_{xx} - 2U_x U_y U_{xy} + U_x^2 U_{yy} - \left[\frac{U_x^2 + U_y^2}{k}\right]^2 = 0 \tag{5-9}$$

where the constant $k = \sqrt{2(H - U)}$. Equation 5-9 is a necessary and sufficient condition for the existence of equipotential orbits. This property may be clearly demonstrated upon solving Equation 5-9 by means of the method of characteristics. In such case, the characteristic solution requires that

$$\Delta = \begin{vmatrix} U_y^2 & -2U_x U_y & U_x^2 \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \tag{5-10}$$

which, along an orbit x = x(t), y = y(t), leads directly to the well-known tangency condition of the equipotential orbits; that is,

$$U_y \dot{y} + U_x \dot{x} = 0; \text{ i.e., } U(x,y) = \text{const.} \tag{5-11}$$

Consequently, the characteristic solution of the partial differential equation (Equation 5-9) is an equipotential orbit. This fact seems to be responsible for the difficulties encountered in the determination and analysis of the properties of such curves. That is, the solution of the partial differential equation (Equation 5-9) considered (as known from the available theory) is not determined solely by knowledge of U and its first partial derivatives $U_x$ and $U_y$ along the characteristics. In terms of our problem, this fact implies that knowledge of the equations of the equipotential curves and the equations of motion ($-U_x = \ddot{x}$, $-U_y = \ddot{y}$) is not sufficient to provide a solution satisfying the partial differential equation proposed, and consequently, to define the desired orbits. Moreover, along the characteristics, the second partial derivatives of U are indeterminate, as shown by the condition that the determinant $\Delta$ must vanish.

In view of these preceding considerations, we have chosen to analyze the problem of the existence of equipotential orbits by the use of a test of the necessary isotach condition which such orbits must satisfy.

## 5.2 NECESSARY CONDITIONS ALONG ISOTACH ORBITS

From the tangency condition along equipotential (or isotach) orbits, it follows that

$$\dot{x} = -\frac{U_y}{U_x}\dot{y} \quad , \quad \dot{y} = -\frac{U_x}{U_y}\dot{x} \quad . \qquad \begin{matrix}(5\text{-}12)\\(5\text{-}13)\end{matrix}$$

Substituting Equations 5-12 and 5-13 in the integral of energy Equation 5-8, we obtain

$$\dot{x} = \pm\frac{k\,U_y}{\sqrt{U_x^2 + U_y^2}} \quad , \quad \dot{y} = \mp\frac{k\,U_x}{\sqrt{U_x^2 + U_y^2}} \quad . \qquad \begin{matrix}(5\text{-}14)\\(5\text{-}15)\end{matrix}$$

Differentiating the left-hand member of Equation 5-5 and using the preceding expressions, it can be shown that

$$\pm\Phi_x U_y \mp \Phi_y U_x = 0 \quad , \qquad (5\text{-}16)$$

which is the tangency condition for the curves defined by Equation 5-5, since $\sqrt{U_x^2 + U_y^2} \neq 0$. Consequently, upon comparing Equation 5-16 with the previous tangency condition (Equation 5-11), we obtain the following velocity components

$$\dot{x} = \mp \alpha \frac{\partial \Phi}{\partial y} \tag{5-17}$$

$$\dot{y} = \pm \alpha \frac{\partial \Phi}{\partial x} \quad , \tag{5-18}$$

in which $\alpha$ is a constant of proportionality which, for the purpose of the following analysis, may be set equal to $\alpha = 1$. Consequently, Equations 5-4, 5-5, 5-17 and 5-18 lead to the following expressions for the velocity components:

$$\dot{x} = \mp \left\{ y \left[ \frac{\mu_1 - K r_1}{r_2} + \frac{\mu_2 - K r_2}{r_1} \right] \right\} \tag{5-19}$$

$$\dot{y} = \pm \left[ (\mu_1 - K r_1) \frac{x+c}{r_2} + (\mu_2 - K r_2) \frac{x-c}{r_1} \right] . \tag{5-20}$$

By use of these equations, we can now determine the coordinates and velocity at the points of intersection of the equipotential curves with the coordinate axes. Consequently, after simple reductions, we obtain

a. <u>At intersection with the y-axis</u>:

$$x = 0 \tag{5-21}$$

$$r_1 = r_2 = \frac{\mu_1 + \mu_2}{K} \tag{5-22}$$

$$y = \pm \left[ \left( \frac{\mu_1 + \mu_2}{K} \right)^2 - c^2 \right]^{\frac{1}{2}} \tag{5-23}$$

$$\dot{y} = \pm \left( \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \right) K c \tag{5-24}$$

$$\dot{x} = \pm K \left[ \left( \frac{\mu_1 + \mu_2}{K} \right)^2 - c^2 \right]^{\frac{1}{2}} . \tag{5-25}$$

As shown by Equation 5-24, the velocity vector will cross the y-axis normally (i. e., be parallel to the x-axis) only if the gravitational constants, $\mu_1$ and $\mu_2$, are equal.

b. <u>At intersection with the x-axis:</u>

$$y = 0 \tag{5-26}$$

$$r_1 = x - c, \quad r_2 = x + c \tag{5-27}$$

$$x = \frac{\mu_1 + \mu_2}{2K} \pm \sqrt{\left(\frac{\mu_1 + \mu_2}{2K}\right)^2 + c\left(\frac{\mu_1 - \mu_2}{K} + c\right)} \tag{5-28}$$

$$\dot{x} = 0 \tag{2-29}$$

$$\dot{y} = \pm \left[\mu_1 + \mu_2 - 2K\alpha\right] . \tag{5-30}$$

Equation 5-29 indicates that the velocity vector crosses the x-axis normally in all cases. The corresponding sign in Equations 5-21 to 5-30 is determined by the sense of rotation with which the particle moves on its orbit. For the purpose of the present analysis, however, the sense of rotation is not important, as will be seen in the following.

## 5.3 NONEXISTENCE OF EQUIPOTENTIAL ORBITS

The equations derived in Sections 5.1 and 5.2 may readily be used in a direct verification of the <u>isotach</u> condition. Since, along a candidate equipotential orbit,

$$V^2 = \dot{x}^2 + \dot{y}^2 = 2(H - U) = k^2 = \text{const.},$$

then the magnitude of the velocity vector must be the same at any point of the equipotential orbit. In particular, the following condition must be satisfied:

$$\left|\overline{V}\right|_{x=0} = \left|\overline{V}\right|_{y=0} . \tag{5-31}$$

Consequently, using the equations of the preceding section, it follows that

$$|\overline{V}|_{x=0} = \sqrt{\dot{x}^2 + \dot{y}^2}\Big|_{x=0} =$$

$$= K \sqrt{\left(\frac{\mu_1+\mu_2}{K}\right)^2 + c^2\left[\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^2 - 1\right]} \tag{5-32}$$

and

$$|\overline{V}|_{y=0} = \sqrt{\dot{x}^2 + \dot{y}^2}\Big|_{y=0} =$$

$$= \sqrt{(\mu_1+\mu_2)^2 + 4K^2 c\left(\frac{\mu_1-\mu_2}{K} + c\right)} \quad . \tag{5-33}$$

Therefore, upon equating Equations 5-32 and 5-33 in accordance with Equation 5-31, we obtain a necessary condition for the existence of equipotential orbits, as follows:

$$c\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^2 = 4\left(\frac{\mu_1-\mu_2}{K}\right) + 5c \quad . \tag{5-34}$$

Without loss of generality, let us now assume that $\mu_1 - \mu_2 > 0$, since if an orbit exists for $\mu_1 - \mu_2 = k_1 > 0$, then the reflected orbit exists for $\mu_2 - \mu_1 = k_1 > 0$. Consequently, if an orbit is admissible, it must be admissible for $\mu_1 - \mu_2 > 0$.

However, upon rearranging Equation 5-34, we obtain

$$K = - \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2)^2}{c\left[(\mu_1 + \mu_2)^2 + \mu_1 \mu_2\right]} \quad . \tag{5-35}$$

Consequently, for $\mu_1 - \mu_2 > 0$, then it must be that $K < 0$, which is not admissible since $K > 0$. This contradiction implies that no trajectory satisfies the isotach condition. That is, no equipotential curve can be an orbit in the problem of two-fixed-centers.

References


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# APPENDIX A

## THE MECHANICS OF THE HEAVENS

### CHAPTER II - VOLUME 1

TRANSLATED FROM "DIE MECHANIK DES HIMMELS" BY CARL L. CHARLIER, VOLUMES 1 AND 2, VEIT AND CO., LEIPZIG, 1902.

TRANSLATED BY: SAMUEL P. ALTMAN

CHAPTER II, SECTION 1

A.1 "INTEGRATION OF THE HAMILTON-JACOBI DIFFERENTIAL EQUATION THROUGH SEPARATION OF THE VARIABLES, THEOREM BY STÄCKEL."

If the time does not occur explicitly in the characteristic function H, then the Hamilton-Jacobi partial differential equations read

$$H\left(q_1, \ldots, q_n; \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = h \tag{1}$$

according to formula (11), §9 in the first chapter.

Here, H is a function of second degree in the partial differential quotients of $W_1$.

One can now present the problem: When can this equation be integrated through separation of the variables, i.e., when is it possible to integrate the Equation (1) in such a way that one assumes the following form for W:

$$W = \sum_{i=1}^{n} W^{(i)}(q_i) \tag{2}$$

where each term of the right side, $W^{(i)}(q_i)$, is therefore dependent only on one of the variables $-q_i$.

The solution of this problem in its general form appears to be combined with not insignificant difficulties, at least when it is a question of searching for the necessary as well as the sufficient conditions for solution. However, Mr. P. Stäckel succeeded in solving the problem in a fairly broad case.

This case is the one in which, in (1), besides the variables $q_1, \ldots, q_n$, only the squares of $\frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}$ occur. One can then find not only the relevant conditions for the solution, but also discuss the motion completely.

Accordingly, assume that (1) is of the following form:

$$H = \frac{1}{2}\sum_{K=1}^{n} A_K \left(\frac{\partial W}{\partial q_K}\right)^2 - (U + \alpha_1) = 0 \tag{3}$$

where U denotes the force function, $\alpha_1$ a constant. The problem now is to investigate: how must $A_1, \ldots, A_n$ and U depend upon $q_1, \ldots, q_n$, in order that complete solution of (3) be of the form (2).

If one sets

$$W_\kappa = \frac{\partial W}{\partial q_\kappa} , \tag{3*}$$

then

$$\sum A_\kappa W_\kappa^2 = 2(U + \alpha_1) . \tag{3**}$$

Now if $W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n)$ is the complete solution of this equation, then upon substitution of this function in (3**), this equation transforms into an identity, which then is satisfied for arbitrary values of the integration constants $\alpha_1, \ldots, \alpha_n$.

Consequently, one can differentiate this equation with respect to any one of these quantities $\alpha_1, \ldots, \alpha_n$, and one thereby gets the n equations

$$\left.\begin{array}{l} A_1 \frac{\partial W_1^2}{\partial \alpha_1} + A_2 \frac{\partial W_2^2}{\partial \alpha_1} + \cdots + A_n \frac{\partial W_n^2}{\partial \alpha_1} = 2 \\ A_1 \frac{\partial W_1^2}{\partial \alpha_2} + A_2 \frac{\partial W_2^2}{\partial \alpha_2} + \cdots + A_n \frac{\partial W_n^2}{\partial \alpha_2} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_1 \frac{\partial W_1^2}{\partial \alpha_n} + A_2 \frac{\partial W_2^2}{\partial \alpha_n} + \cdots + A_n \frac{\partial W_n^2}{\partial \alpha_n} = 0 . \end{array}\right\} \tag{4}$$

Since, according to the assumption, the function $W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n)$ is a <u>complete</u> integral of (1), then according to §9 in the preceding chapter, the Hessian determinant of W is different from zero. That is,

$$E = \begin{vmatrix} \frac{\partial^2 W}{\partial q_1 \partial \alpha_1}, & \cdots, & \frac{\partial^2 W}{\partial q_n \partial \alpha_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 W}{\partial q_1 \partial \alpha_n}, & \cdots, & \frac{\partial^2 W}{\partial q_n \partial \alpha_n} \end{vmatrix} \neq 0 . \tag{5}$$

If one now considers the determinant

$$\Delta = \begin{vmatrix} \frac{\partial W_1^2}{\partial x_1}, & \cdots, & \frac{\partial W_n^2}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial W_1^2}{\partial x_n}, & \cdots, & \frac{\partial W_n^2}{\partial x_n} \end{vmatrix}, \tag{6}$$

then one finds that

$$\Delta = 2^n W_1, W_2, \ldots, W_n E \tag{7}$$

and since E does not vanish identically, then this must also be the case with $\Delta$.

But if this is so, then one can solve (4) for the quantities $A_1, \ldots, A_n$, and one then obtains, according to I §1 (9),

$$\left.\begin{aligned} A_1 &= \frac{2}{\Delta} \frac{\partial \Delta}{\partial \frac{\partial W_1^2}{\partial x_1}} \\ A_2 &= \frac{2}{\Delta} \frac{\partial \Delta}{\partial \frac{\partial W_2^2}{\partial x_1}} \\ &\cdots\cdots \\ A_n &= \frac{2}{\Delta} \frac{\partial \Delta}{\partial \frac{\partial W_n^2}{\partial x_1}} \end{aligned}\right\} . \tag{8}$$

Now, according to (3**),

$$U = -x_1 + \sum_{K=1}^{n} W_K^2 \frac{1}{\Delta} \frac{\partial \Delta}{\partial \frac{\partial W_K^2}{\partial x_1}} . \tag{9}$$

Since, according to I §1 (2),

$$\Delta = \sum_{K=1}^{n} \frac{\partial W_K^2}{\partial x_1} \frac{\partial \Delta}{\partial \frac{\partial W_K^2}{\partial x_1}} ,$$

then one can also write

$$U = \sum_{K=1}^{n} \left( W_K^2 - x_1 \frac{\partial W_K^2}{\partial x_1} \right) \frac{1}{\Delta} \frac{\partial \Delta}{\partial \frac{\partial W_K^2}{\partial x_1}} \tag{9*}$$

instead of (9).

The formulae (8) and (9*) are always valid. We have here made no other assumption about the function W, than that it constitutes the complete integral of (1).

We now subject W to the condition that the variables in W are separable, so that W is then of the form (2). We will furthermore see what results from the formulae (8) and (9*).

First, it follows from (3*), that $W_\kappa$ is a function of $q_\kappa$ alone. Furthermore, since we have found that $\Delta$ does not vanish identically, there are always values for the integration constants $\alpha_1$, ..., $\alpha_n$ for which $\Delta$ is different from zero. Let $\alpha_1^o, \ldots, \alpha_n^o$ be such a system of values.

For these values of $\alpha_1, \ldots, \alpha_n$, the functions

$$\frac{\partial W_\kappa^2}{\partial \alpha}$$

transform into certain functions of $q_\kappa$.

Then set

$$\Phi_{\kappa\nu}(q_\kappa) = \frac{\partial W_\kappa^2}{\partial \alpha_\nu} \qquad (\kappa, \nu = 1, 2, \ldots, n)$$

so that

$$\Delta = \begin{vmatrix} \Phi_{11}(q_1), & \ldots, & \Phi_{n1}(q_n) \\ \cdots & \cdots & \cdots \\ \Phi_{1n}(q_1), & \ldots, & \Phi_{nn}(q_n) \end{vmatrix} . \qquad (10)$$

For the considered values of $\alpha_1, \ldots, \alpha_n$, the expressions

$$W_\kappa^2 - \alpha_1 \frac{\partial W_\kappa^2}{\partial \alpha_1} \qquad (\kappa = 1, 2, \ldots, n)$$

transform into certain functions of $q_\kappa$ ( $= 1, 2 \ldots, n$). If one then sets

$$\Psi_\kappa(q_\kappa) = W_\kappa^2 - \alpha_1 \frac{\partial W_\kappa^2}{\partial \alpha_1} ,$$

then one is led to the theorem of Stäckel:

When the Hamilton-Jacobi differential equation

$$H = \sum_{K=1}^{n} A_K \left(\frac{\partial W}{\partial q_K}\right)^2 - 2\left(U + \varkappa_1\right) = 0 \tag{11}$$

permits separation of the variables, then there is necessarily a system of $n^2$ functions

$$\phi_{K\nu}(q_\nu) \qquad (K, \nu = 1, 2, \ldots, n)$$

and a system of n functions

$$\Psi_1(q_1), \ldots, \Psi_n(q_n)$$

with the property, that the coefficients $A_1, A_2, \ldots, A_n$ can be described by the equations

$$A_K = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \phi_{K1}} \qquad (K = 1, 2, \ldots, n) \tag{12}$$

and the force function U by the equation

$$U = \sum_{K=1}^{n} \Psi_K \frac{1}{\Delta} \frac{\partial \Delta}{\partial \phi_{K1}} \quad . \qquad \text{(Reference 1)} \tag{13}$$

Here, $\Delta$ is defined by (10).

(The factor 2 occurs in the expressions (8) for $A_K$. These can obviously be omitted, whereby one uses only the arbitrary functions $\Psi_K$ suitable for transforming.)

These conditions are necessary. Whether the choice of the functions $\phi_{K\nu}$ and $\Psi_K$ can be done arbitrarily, is hence not directly obvious. But this is, in fact, the case. In order to show this, the most direct method would be evident-if one could show that the equation (11), where the coefficients are given by (12) and (13), can always be solved through separation of the variables-which values give the functions $\phi$ and $\Psi$. That is just what Stäckel (cited elsewhere) has shown.

1. P. Stäckel: On the Integration of the Hamilton-Jacobi Differential Equation by means of Separation of the Variables. Inaugural Dissertation. Halle 1891.

The solution of the equation in question is, in fact, the following:

$$W = \sum_{K=1}^{n} \int \sqrt{2\Psi_K(q_K) + \sum_{\lambda=1}^{n} 2\alpha_\lambda \Phi_{K\lambda}(q_K)}\; dq_K \,. \tag{14}$$

Hence follows

$$\left(\frac{\partial W}{\partial q_K}\right)^2 = 2\,\Psi_K(q_K) + \sum_{\lambda=1}^{n} 2\alpha_\lambda \Phi_{K\lambda}(q_K) \,. \tag{15}$$

But if one inserts the expressions (12) and (13) for A and U into (11), and considers the relation

$$1 = \sum_{K=1}^{n} \Phi_{K1} \frac{1}{\Delta} \frac{\partial \Delta}{\partial \Phi_{K1}} \quad ,$$

then this reads:

$$\frac{1}{\Delta} \sum_{K=1}^{n} \frac{\partial \Delta}{\partial \Phi_{K1}} \left[ \left(\frac{\partial W}{\partial q_K}\right)^2 - 2\Psi_K - 2\alpha_1 \Phi_{K1} \right] = 0 \,. \tag{16}$$

But this equation is satisfied by (15). That is, if one used this expression for $\frac{\partial W}{\partial q}$, then one obtains

$$\frac{1}{\Delta} \sum_{K=1}^{n} \left\{ \frac{\partial \Delta}{\partial \Phi_{K1}} \sum_{\lambda=2}^{n} 2\alpha_\lambda \cdot \Phi_{K\lambda}(q_K) \right\} = \frac{1}{\Delta} \sum_{\lambda=2}^{n} \left\{ 2\alpha_\lambda \sum_{K=1}^{n} \Phi_{K\lambda} \frac{\partial \Delta}{\partial \Phi_{K1}} \right\}$$

instead of (16).

Now, according to I § 1 (2),

$$\sum_{K=1}^{n} \Phi_{K\lambda} \frac{\partial \Delta}{\partial \Phi_{K1}} = \begin{cases} 1 & \text{for } \lambda = 1 \\ 0 & \text{for } \lambda \neq 1 \end{cases}$$

so that the coefficients of $\alpha_1$ vanish identically.

Therefore, the partial differential equation (11) will be satisfied through (14). This equation contains the necessary number of constants $\alpha_1, \ldots, \alpha_n$ and these are not dependent upon each other. In fact, it follows from (15), that

$$\frac{\partial W_\kappa^2}{\partial \alpha_\nu} = 2\phi_{\kappa\nu}$$

and consequently

$$\left|\frac{\partial W_\kappa^2}{\partial \alpha_\nu}\right| = 2^n \left|\phi_{\kappa\nu}\right| \qquad (\kappa, \nu = 1, 2, \ldots, n) .$$

Otherwise

$$\begin{aligned}\left|\frac{\partial W_\kappa^2}{\partial \alpha_\nu}\right| &= 2^n W_1 W_2 \ldots W_n \left|\frac{\partial W_\kappa}{\partial \alpha_\nu}\right| \\ &= 2^n W_1 W_2 \ldots W_n \left|\frac{\partial^2 W}{\partial q_\kappa \partial \alpha_\nu}\right| .\end{aligned}$$

Now, since it is assumed that $\left|\phi_{\kappa\nu}\right|$ does not vanish identically, then this must also be the case with $\left|\frac{\partial^2 W}{\partial q_\kappa \partial \alpha_\nu}\right|$, and the integration constants $\alpha_1, \ldots, \alpha_n$ are then independent of each other.

The partial differential equation

$$H = 0$$

corresponds with the system of canonical differential equations

$$\begin{aligned}\frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}\end{aligned} \qquad (i = 1, 2, \ldots, n) \tag{17}$$

where now, if separation of the variables occurs,

$$H = \frac{1}{2}\sum_{\kappa=1}^{n} A_\kappa p_\kappa^2 - U \tag{18}$$

and $A_\kappa$ and $U$ are given through the relations (12) and (13). According to I § 9 (12), the solution of (17) reads as follows:

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \sum_{\kappa=1}^{n} \int \frac{\phi_{\kappa 1}(q_\kappa)\, dq_\kappa}{\sqrt{2\Psi_\kappa(q_\kappa) + 2\sum_{\lambda=1}^{n} \alpha_\lambda \phi_{\kappa\lambda}(q_\kappa)}} \tag{19}$$

$$\frac{\partial W}{\partial \alpha_\mu} = \quad \beta_\mu = \sum_{\kappa=1}^{n} \int \frac{\phi_{\kappa\mu}(q_\kappa)\, dq_\kappa}{\sqrt{2\Psi_\kappa(q_\kappa) + 2\sum_{\lambda=1}^{n} \alpha_\lambda \phi_{\kappa\lambda}(q_\kappa)}} \tag{19*}$$

$$(\mu = 2, 3, \ldots, n).$$

These equations (19) and (19*) contain the complete solution of the canonical differential equations (17) under the assumptions made here.

Therefore, if we compile the results, then they read as follows:

If n (n + 1) functions of every one of the variables

$$\phi_{\kappa\nu}(q_\kappa) \qquad (\kappa, \nu = 1, 2, \ldots, n)$$

and

$$\Psi_\kappa(q_\kappa) \qquad (\kappa = 1, 2, \ldots, n)$$

are given, with the condition that the determinant

$$\Delta = |\phi_{\kappa\nu}| \qquad (\kappa, \nu = 1, 2, \ldots, n)$$

does not vanish identically, but also are chosen arbitrarily; if furthermore we assume that the n + 1 functions $A_1, \ldots, A_n$ and U are determined so that

$$A_n = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \phi_{\kappa 1}}$$

$$U = \sum_{\kappa=1}^{n} \Psi_\kappa \frac{1}{\Delta} \frac{\partial \Delta}{\partial \phi_{\kappa 1}} ,$$

then the solution of the canonical differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, \ldots, n)$$

where

$$H = \frac{1}{2} \sum A_\kappa p_\kappa^2 - U$$

is expressed by equations (19) and (19*).

Note: If one of the minor determinants $\frac{\partial \Delta}{\partial \phi_{\kappa 1}}$ is identically equal to zero, then the corresponding coefficient A would disappear. But then it follows from the differential equations, that the corresponding value for $q_{\kappa}$ is a constant, so that the order of the differential equations can be reduced by one unit. I have assumed that this reduction of the order will surely be accomplished so that none of the coefficients $A_{\kappa}$ vanishes.

One can discuss the trajectory determined by equations (19) and (19*) completely with the help of these equations. But before I pass on to this discussion for an arbitrary number of variables, I will look at the simplest example, namely, the case that only one single variable q is present.

CHAPTER II, SECTION 2

A.2 "TRAJECTORIES WHICH ARE DEFINED BY ONE DEGREE OF FREEDOM. LIBRATION AND LIMITATION."

Concerning a trajectory which is determined by the canonical differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial q_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, n)$$

one says that this possesses n degrees of freedom. If only one degree of freedom is present, then

$$\left.\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned}\right\} \tag{1}$$

where

$$H = \frac{1}{2} A(q) p^2 - U(q) \quad . \tag{1*}$$

The corresponding Hamilton-Jacobi differential equation here reads simply

$$A(q)\left(\frac{\partial W}{\partial q}\right)^2 = 2(U + \alpha)$$

and consequently

$$W = \int \sqrt{\frac{2(U+\alpha)}{A(q)}}\, dq \quad . \tag{2}$$

Hence one obtains, furthermore,

$$t + \beta = \frac{\partial W}{\partial \alpha} = \int \frac{dq}{\sqrt{2(U+\alpha)A(q)}} \quad . \tag{3}$$

We now want to investigate the trajectory determined by this equation.

In order to shorten the form of the expression somewhat, I will introduce the designation "mechanical quantity." By a mechanical quantity, I mean a variable which is real, continuous and finite for each finite value of time, in addition to its first differential quotient.

Of this kind are, for example, the right-angle coordinates in the three-body problem in the case that a collision does not occur in finite time. One can also regard the osculating elliptical elements of a planet, etc., as such mechanical quantities.

Instead of time, one can hereby introduce any other independent variable with the property that it: (1) increases continuously with time, and (2) becomes infinite with time.

Now let a mechanical quantity q be defined by a differential equation of the form (3). If one sets

$$2(U+\alpha)A(q) = F(q) \quad ,$$

then this equation reads

$$\left(\frac{dq}{dt}\right)^2 = F(q) \quad . \tag{4}$$

Now it can be shown, that q can never exceed such a value q = a for which the function F(q) disappears.

Let F(a) = 0 and the degree of this root be equal to m where m denotes a whole, positive number. One can then write

$$F(q) = (q-a)^m \, \phi(q) \tag{4*}$$

where $\phi(a)$ is different from zero. We now assume that $\phi(q)$ can be developed in a convergent exponential series in the neighborhood of x = a, where therefore, according to the assumptions, the constant term does not vanish. Then, from (4), we obtain

$$\frac{dq}{(q-a)^{\frac{m}{2}}}\left(c_0+c_1(q-a)+c_2(q-a)^2+\cdots\right)=\pm\, dt\,. \tag{5}$$

Upon integration, we must here differentiate between two cases, according to whether m is an even or an odd number.

For odd m, the integral of (5) reads

$$\frac{2}{(q-a)^{\frac{m}{2}-1}}\left\{\frac{c_0}{m-2}+\frac{c_1}{m-4}(q-a)+\frac{c_2}{m-6}(q-a)^2+\cdots\right\}=\mp(t+\tau) \tag{6}$$

and for even m

$$\left.\begin{aligned}-c_{\frac{m}{2}-1}\log(q-a)+\frac{2}{(q-a)^{\frac{m}{2}-1}}\left\{\frac{c_0}{m-2}+\frac{c_1}{m-4}(q-a)+\right.\\ \left.+\frac{c_2}{m-6}(q-a)^2+\cdots\right\}=\mp(t+\tau)\,,\end{aligned}\right\} \tag{7}$$

where $\tau$ denotes an integration constant and is omitted inside the bracket of the term with the coefficient $C_{\frac{m}{2}-1}$.

Now if, in this equation, we let q approach the value q = a, and indeed, so that the left sides of (6) and (7) remain real, then the absolute value of the left sides and therefore also of t increase simultaneously over all limits, whenever $m \geq 2$. Conversely, we can also assert that, whenever $m \geq 2$, there is no finite value t' of t, for which q assumes the value a. Furthermore, since q can only be transformed from real values greater than a over to real values smaller than a - because q is real and continuous - through the presence of the value q = a, then q never exceeds the root q = a, for $m \geq 2$.

Finally, if m = 1, then one has - according to (6) - the following relation between q and t in the neighborhood of q = a:

$$\pm(t+\tau)=2(q-a)^{\frac{1}{2}}\left\{c_0+\frac{c_1}{3}(q-a)+\frac{c_2}{5}(q-a)^2+\cdots\right\}\,. \tag{8}$$

Hence one now obtains

$$q - a = A_1 (t+\tau)^2 + A_2 (t+\tau)^4 + \dots \quad . \tag{8*}$$

For $t = -\tau$ follows $q = a$, which value one therefore reaches for a finite value of t. But the point $q = a$ can not be passed through. In fact, it follows from (8*), that $q > a$ for $A_1$ positive $q < a$ always for $A_1$ negative. In both cases, the point $q = a$ can not be passed.

Consequently, we arrive at the following result:

If q is a mechanical quantity which is determined by the differential equation

$$\left(\frac{dq}{dt}\right)^2 = F(q) \tag{a}$$

then for no finite value of t can q pass through a point $q = a$, for which

$$F(q) = 0 . \tag{b}$$

If $q = a$ is a simple root of (b), then this value is obtained for a finite value of t and the sign of the differential quotient of q changes at $q = a$. If the order of the root is greater than one, then the value $q = a$ will be attained for no finite value of t.

Now let a and b ($b > a$) be two neighboring roots of (b) with the orders m and n, so that

$$\left(\frac{dq}{dt}\right)^2 = (q-a)^m (b-q)^n \psi(q) \tag{9}$$

where $\psi(q)$ vanishes for no value of q between a and b, including the boundaries.

If q lies between a and b at the beginning of the motion, then-according to the above theorem - q must always lie between the limits a and b.

In order to investigate the trajectory defined by (9), we introduce an auxiliary quantity w, which is so defined, that

$$\left(\frac{dq}{dw}\right)^2 = \beta^2 (q-a)^m (b-q)^n \tag{10}$$

and then

$$\left(\frac{dw}{dt}\right)^2 = \frac{1}{\beta^2} \Psi(q) \quad . \tag{11}$$

Since Equation (9) is to be replaced by two others, each with their special integration constants, it is permitted to determine one of these constants with ease. We arrange that in (11) the integration constants are defined so that w obtains the value zero whenever t itself is equal to zero. Then it follows - from the same equation - that w must always have a real value. But we cannot specify only the value of w at the inception of motion, but, since it is only necessary that the equation

$$\left(\frac{dq}{dw}\right)^2 \left(\frac{dw}{dt}\right)^2 = \left(\frac{dq}{dt}\right)^2$$

occur between the squares of the differential quotients, then it is also permitted to give $\frac{dw}{dt}$ an arbitrary sign at the inception of motion.

Then if we give dw the same sign as t at the inception of motion, then the signs of both these quantities will always coincide, since $\frac{dw}{dt}$ can never be zero (or infinite). Therefore, we have - for all values of t -

$$\frac{dw}{dt} = + \frac{1}{\beta} \sqrt{\Psi(q)}$$

where $\beta$ denotes a still undetermined positive constant.

From this equation between w and t, we can deduce an important property of w. That is, since q in its variations is determined so that always $b \geq q \leq a$, and in accordance with the assumptions made, $\psi$ (q) is continuous in this region and is never zero or infinite, then $\psi$ (q) must necessarily have a finite upper bound and a positive lower bound different from zero, for the named q-value. Then this is also true for $\frac{1}{\beta}\sqrt{\psi(q)}$ so that it is always possible to find two positive numbers $L_1$ and $L_2$ of the kind so that, for all q which are under consideration here,

$$\beta L_1 < \sqrt{\phi(q)} < \beta L_2$$

and consequently it follows that

$$L_1 t < \omega < L_2 t \quad . \tag{12}$$

Since we have found two definite limits within which the value of w must be included, we can pass on to the investigation of the equation (10), which contains the relation between q and w.

Here we notice, at once, the great advantage which the introduction of the auxiliary quantity w provides us. In fact, the entire discussion of the motion now lies in the equation (10), which can be treated by elementary methods. Assuming that the solution of this equation were

$$q = f(w) \ ,$$

then here we need only to permit w to pass through all values between $-\infty$ and $+\infty$, in order to describe the entire motion of q. And moreover, if we identify w with a constant times t, we can obtain an approximate description of the motion. This is an approximation of the same kind as that which one obtains in the general pendulum problem, if one substitutes the sine amplitude occurring there for a general sine.

Let us first assume that

$$m = n = 1,$$

which case was treated first by Weierstrass in an important paper (Monthly Report of the Berlin Academy 1866).

Now

$$\frac{dq}{dw} = \beta \sqrt{(q-a)(b-q)} \quad . \tag{13}$$

The solution of this equation reads

$$q = a \cos^2 \tfrac{1}{2}\beta w + b \sin^2 \tfrac{1}{2}\beta w \quad . \tag{14}$$

Therefore, in this case, q is a periodic function of w with the period $\frac{2\pi}{\beta}$. But it will be shown that q is then also periodic in t. Concerning the constant $\beta$, we can, for example, arrange it so that the duration of the period in reference to w coincides with the length of period in t.

Now, according to (11),

$$\int_0^w \frac{\beta\, dw}{\sqrt{\psi(q)}} = t \quad .$$

If w increases by $\frac{2\pi}{\beta}$, then q remains unchanged. If we name the corresponding increase in t as 2T, then

$$2T = \int_w^{w+\frac{2\pi}{\beta}} \frac{\beta\, dw}{\sqrt{\psi\left(a \cos^2 \frac{1}{2}\beta w + b \sin^2 \frac{1}{2}\beta w\right)}} \quad .$$

But this expression is independent of w, since $\psi$ is periodic in w with the period $\frac{2\pi}{\beta}$; consequently, 2T becomes a constant, and q is therefore periodic in t with the period

$$2T = \int_0^{\frac{2\pi}{\beta}} \frac{\beta\, dw}{\sqrt{\psi\left(a \cos^2 \frac{1}{2}\beta w + b \sin^2 \frac{1}{2}\beta w\right)}} \tag{15}$$

or

$$T = \int_0^{\frac{\pi}{\beta}} \frac{\beta\, dw}{\sqrt{\Psi(a \cos^2 \frac{1}{2}\beta w + b \sin^2 \frac{1}{2}\beta w)}} \tag{16}$$

an equation which, according to (14), can also be written as follows:

$$T = \int_a^b \frac{dq}{\sqrt{(q-a)(b-q)\Psi(q)}} \quad . \tag{16*}$$

If the length of the period in u and in t should be the same, then one has

$$2T = \frac{2\pi}{\beta}$$

or

$$\beta = \frac{\pi}{T} \quad . \tag{17}$$

If the roots a and b of the equation

$$F(q) = 0$$

are simple roots, then q becomes a periodic function of the time. This function is also a direct function of t. The analytic description of this function can easily be found.

According to the theorem by Fourier, one has

$$q = \frac{1}{2}B_0 + B_1 \cos \frac{\pi t}{T} + B_2 \cos \frac{2\pi t}{T} + \cdots \tag{18}$$

where

$$TB_n = 2 \int_0^T q \cos \frac{n\pi t}{T}\, dt \quad . \tag{18*}$$

Hence one obtains - upon integration by parts -

$$n\pi B_n = -2\int_a^b \sin\frac{n\pi t}{T}\, dq \tag{18**}$$

where one has expressed t by means of q.

The computation of (18**) occurs best, when one expresses t and q by means of the auxiliary quantity w.

According to (11) and (17) , one has

$$\left.\begin{aligned} dt &= \frac{\beta\, dw}{\sqrt{\psi\left(a\cos^2\frac{\pi w}{2T} + b\sin^2\frac{\pi w}{2T}\right)}} \\ &= \left[\frac{1}{2}c_0 + c_1\cos\frac{\pi w}{T} + c_2\cos\frac{2\pi w}{T} + \cdots\right] dw \end{aligned}\right\} \tag{19}$$

and here

$$Tc_n = 2\int_0^T \frac{\beta\cos\frac{n\pi w}{T}\, dw}{\sqrt{\psi\left(a\cos^2\frac{\pi w}{2T} + b\sin^2\frac{\pi w}{2T}\right)}} . \tag{19*}$$

In particular, according to (16),

$$\frac{1}{2}c_0 = 1 . \tag{19**}$$

Through integration of (19), one now obtains

$$t = w + \frac{T}{\pi}c_1\sin\frac{\pi w}{T} + \frac{T}{2\pi}c_2\sin\frac{2\pi w}{T} + \cdots . \tag{20}$$

By means of this equation, t is expressed by w, and the coefficients in this series can always be calculated from (19*), for example, by means of the so-called mechanical quadrature.

On the other hand, according to (13) and (14),

$$dq = \frac{\pi}{2T}(b-a)\sin\frac{\pi w}{T}\,dw \quad . \tag{21}$$

If we now introduce the expression (20) and (21) into (18**), then one obtains

$$nTB_n = -(b-a)\int_0^T \sin\frac{n\pi t}{T}\sin\frac{nw}{T}\,dw \tag{22}$$

or

$$\frac{2nTB_n}{b-a} = \int_0^\pi \cos\frac{\pi}{T}(nt+w)\,dw - \int_0^\pi \cos\frac{n}{T}(nt-w)\,dw \tag{22*}$$

where the expression (20) for t is now introduced.

The numerical computation of $B_n$ according to this formula can be carried out in different ways: by mechanical quadrature, through formulation with powers of $C_1$, $C_2$, ...., by means of Bessel functions, etc.

If the order of the roots of the equation

$F(q) = 0$

are greater than one, then the discussion of the trajectory may be easily carried out with the help of the equation

$$\frac{dq}{dw} = \beta(q-a)^{\frac{m}{2}}(b-q)^{\frac{n}{2}} \quad .$$

First, let one of the orders, for example n, be 1, but the other - m - greater than one. Now when dq is positive for t = 0, then q increases until the upper limit, q = b, is reached. Now here, dq:dw changes sign, q begins to decrease and approaches the value q = a without limit, with increasing w (therefore also t), without reaching it in finite time.

Second, if m as well as n is ≥ 2, then-during the motion - dq:dw never changes sign and q gradually approaches one of the limits a or b (the latter limit if, for t = 0, dq:dw is positive, the former if dq:dw is negative), without reaching it in finite time.

I have elsewhere (Reference 1) used the following notations in order to characterize the trajectories occurring upon solution of differential equation (4):

a. One says that a quantity possesses a libration motion if it oscillates periodically back and forth between two fixed limits. These limits were named libration limits.

b. One says that a quantity possesses a limitation motion if it gradually approaches a definite limit-value, without ever reaching it in finite time. The limit-value in question was named limitation limit.

From the preceding analysis, it now results immediately, that the motion in the present case can only be of both these kinds. And indeed libration occurs whenever the roots a and b are both simple - otherwise only limitation.

The general analytical expression for q is given [(18) and (22*)] for the libration case above (mainly according to Weierstrass, cited previously). The corresponding expressions in the limitation case are, as I know well, still not yet given.

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1"On the Solution of Mechanics Problems Which Lead to Hyperelliptical Differential Equations," Bulletin of the Academy of St. Petersburg, 1888.

The trajectories denoted here by "limitation" are of the same kind as the asymptotic trajectories investigated by Poincaré. However, the analytical means of development of the latter does not correspond with that considered here, which is valid for the limitation trajectory, and I have therefore continued to use the name limitation motion.

The investigations on the equation

$$\left(\frac{dq}{dt}\right)^2 = F(q) \tag{a}$$

have received a unique treatment, in the history of mathematics. If we let F(q) denote a polynomial in q of degree s, then it was proven in the beginning of the preceding century, that, whenever $s = 4$, q is a so-called elliptic function of t, which has two periods, of which at least one must be imaginary. But if $s > 4$, then q, regarded as a function of the complex variable t, must have more than two periods. However, it was shown by Jacobi in a celebrated treatise ("De Functionibus duarum Variabilium, quadrupliciter periodicis," Part II)* that, when a function has 3 (or more) periods, either these periods can be composed from two periods, or the function must be constituted so that it is invariant with an infinitesimal increase of the argument. From this, Jacobi deduced (as cited previously) the conclusion that, whenever F(q) is of higher degree than 4, q cannot be considered an analytic function of t.

The fallacy (which I am permitted to so call it) lies in that Jacobi has assumed in his argument, that q is unconditionally variable in the entire imaginary plane. However, if one limits q to only real values, as Weierstrass has done in his treatise cited above, then - as we have already seen - q is considered as a well-defined function of the equally real variable t. One can go even further in this way (Reference 2), when one separates out four - let them be $a_1$, $a_2$, $a_3$, $a_4$ - instead of only two roots from the equation $F(q) = 0$ and introduce an auxiliary quantity w defined by means of the following equation:

*Translator's footnote: "On Functions of two Variables, of quadratic periodicity."
2. See a paper by Dillner in the "Mem. de la Societe des Sc. phys. et math. of Bordeaux."

$$\left(\frac{dq}{dw}\right)^2 = \beta^2 (q-a_1)^{s_1} (q-a_2)^{s_2} (q-a_3)^{s_3} (q-a_4)^{s_4}$$

which can be of use when, in a mechanics problem, q can - under certain circumstances - lie between different pairs of roots $a_i$ and $a_j$ at the inception of motion.

Example. The simple pendulum.

If the vertical coordinate of the pendulum, measured from the suspension point to the nadir, is denoted by z, and the length of the pendulum by $\ell$, the acceleration of gravity by g, then

$$\frac{\ell\, dz}{\sqrt{2g(\ell^2 - z^2)(z - z_0)}} = dt$$

where $z_0$ denotes an integration constant.

It now follows directly from the preceding arguments that when

(1) $-\ell < z_0 < \ell$, libration occurs between the limits $z_0$ and $+\ell$; when

(2) $z_0 < -\ell$, libration occurs between the limits $-\ell$ and $+\ell$, that is, the pendulum always moves in the same direction, when

(3) $z_0 = -\ell$, limitation occurs, and the pendulum approaches arbitrarily near to the highest point with increasing time, without reaching it in finite time; when

(4) $z_0 = +\ell$, the pendulum remains stationary at the lowest point.

For the period of 2T, the motion in cases (1) and (2) yields - according to (16*) - the values

$$2T = 2\int_{z_0}^{\ell} \frac{\ell\, dz}{\sqrt{2g(\ell^2 - z^2)(z - z_0)}}$$

and

$$2T = 2\int_{-\ell}^{+\ell} \frac{\ell\, dz}{\sqrt{2g(\ell^2 - z^2)(z - z_0)}}$$

## CHAPTER II, SECTION 3

## A.3 "CONCERNING THE DIFFERENTIAL EQUATIONS IN MECHANICS. CONDITIONAL PERIODIC MOTIONS"

Limited Periodic Motions

I now proceed to the general case, that the motion possesses n degrees of freedom. Consequently, let a canonical system of differential equations

$$\left.\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} \end{aligned} \quad (i = 1,\ 2,\ \ldots,\ n) \right\} \tag{1}$$

be proposed, and we assume that the corresponding Hamilton-Jacobi partial differential equation can be integrated upon separation of the variables, so that $q_1, \ldots, q_n$ are given-according to § 1 (19) and (19*) - by the following differential equations.

$$t + \beta_1 = \int \sum_{i=1}^{n} \frac{\Phi_{i1}(q_i)\, dq_i}{\sqrt{2\psi_i(q_i) + 2\alpha_1 \Phi_{i1}(q_i) + \cdots + 2\alpha_n \Phi_{in}(q_i)}} \tag{2}$$

$$\beta_j = \int \sum_{i=1}^{n} \frac{\Phi_{ij}(q_i)\, dq_i}{2\psi_i(q_i) + 2\alpha_1 \Phi_{i1}(q_i) + \cdots + 2\alpha_n \Phi_{in}(q_i)} \cdot \tag{2*}$$

$(\mu = 2,\ 3,\ \ldots,\ n)$*.

The variations of $q_1, \ldots, q_n$ should be investigated under the assumption that they describe <u>mechanical quantities,</u> according to the definition given in the preceeding paragraph.

*Translator's note: "$\mu$" should apparently be "j"

For n = 2, these equations were first investigated by Staude (Math. Annals 1887), who has also introduced the remarkably productive concept "conditional periodic motions", about which it is spoken below. For n > 2, Staude's investigations were conducted elsewhere further by Stäckel, who has also revealed an especially important property of these motions (Math. Annals, Volume 54).

If one sets

$$\Phi_i(q_i) = 2\psi_i(q_i) + 2\alpha_1\varphi_{i1}(q_i) + \cdots + 2\alpha_n\varphi_{in}(q_i) \tag{3*}$$

then the equations (2) and (2*) after their differentiation read:

$$\left.\begin{aligned}
dt &= \frac{\varphi_{11}(q_1)\,dq_1}{\sqrt{\Phi_1(q_1)}} + \cdots + \frac{\varphi_{n1}(q_n)\,dq_n}{\sqrt{\Phi_n(q_n)}} \\
0 &= \frac{\varphi_{12}(q_1)\,dq_1}{\sqrt{\Phi_1(q_1)}} + \cdots + \frac{\varphi_{n2}(q_n)\,dq_n}{\sqrt{\Phi_n(q_n)}} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \frac{\varphi_{1n}(q_1)\,dq_1}{\sqrt{\Phi_1(q_1)}} + \cdots + \frac{\varphi_{nn}(q_n)\,dq_n}{\sqrt{\Phi_n(q_n)}} \quad .
\end{aligned}\right\} \tag{3}$$

At the beginning of the motion, $q_1, \ldots, q_n$ may have the values $q_1^0, \ldots, q_n^0$. Let us now consider the equations:

$$\Phi_i(q_i) = 0 \qquad (i = 1, 2, \ldots, n) \quad . \tag{4}$$

Let $a_i$ and $b_i$ be two values of $q_i$, for which (4) is fulfilled, and constituted so that $a_i < q_i^0 < b_i$. These roots may be further constituted so that no other root of the equation $\Phi_i = 0$ lies between them. Let the ordinal numbers of the roots be $m_i$ and $n_i$. We now set

$$\frac{dq_i}{(q_i - a_i)^{\frac{m_i}{2}} (b_i - q_i)^{\frac{n_i}{2}}} = \beta_i \, dw_i \qquad (i = 1, 2, \ldots, n) \; . \tag{5}$$

When we introduce the notations

$$\Phi_i(q_i) = (q_i - a_i)^{m_i} (b_i - q_i)^{n_i} \Psi_i(q_i) \qquad (i = 1, 2, \ldots, n) \tag{6}$$

and

$$F_{ij}(q_n) = \frac{\phi_{ij}}{\sqrt{\Psi_i(q_i)}} \quad , \tag{7}$$

then we obtain the equations

$$\left.\begin{aligned} dt &= F_{11}(q_1)\beta_1 \, dw_1 + \cdots + F_{n1}(q_n)\beta_n \, dw_n \; , \\ 0 &= F_{12}(q_1)\beta_1 \, dw_1 + \cdots + F_{n2}(q_n)\beta_n \, dw_n \; , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= F_{1n}(q_1)\beta_1 \, dw_1 + \cdots + F_{nn}(q_n)\beta_n \, dw_n \; . \end{aligned}\right\} . \tag{8}$$

According to § 1, we know that the determinant

$$\Delta = \left| \phi_{ij} \right| \qquad (i, j = 1, 2, \ldots n)$$

is nonzero. Then it must be the same case for the determinant

$$E = \left| F_{ij} \right| \qquad (i, j = 1, 2, \ldots n).$$

In fact, according to (7), the relation

$$E = \frac{\Delta}{\sqrt{\Psi_1 \Psi_2 \ldots \Psi_n}} \tag{10}$$

exists between both these determinants.

If one differentiates this equation with respect to $\phi_{ij}$ and thereby consider $\psi_1, \ldots, \psi_n$ as constants, then one obtains

$$\frac{1}{E} \frac{\partial E}{\partial \Phi_{ij}} = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \Phi_{ij}} .$$

But now, according to (7),

$$\frac{\partial E}{\partial \Phi_{ij}} = \frac{\partial E}{\partial F_{ij}} \cdot \frac{1}{\sqrt{\Psi_i(q_i)}}$$

and consequently

$$\frac{1}{E} \frac{\partial E}{\partial F_{ij}} = \sqrt{\Psi_i(q_i)} \; \frac{1}{\Delta} \frac{\partial \Delta}{\partial \Phi_{ij}} . \tag{11*}$$

In the special case of j = 1,

$$\frac{1}{E} \frac{\partial E}{\partial F_{i1}} = \sqrt{\Psi_i(q_i)} \; \frac{1}{\Delta} \frac{\partial \Delta}{\partial \Phi_{i1}} . \tag{11}$$

In the first paragraph, however, we have proven that

$$\frac{\partial \Delta}{\partial \Phi_{i1}} \neq 0,$$

and consequently the determinants

$$\frac{\partial E}{\partial F_{i1}} \qquad (i = 1, 2, \ldots, n)$$

cannot be identically equal to zero.

Let us assume that these determinants will be zero or infinite for no value of $q_i$ $(i = 1, 2, \ldots n)$ between $a_i$ and $b_i$ $(i = 1, 2, \ldots, n)$.

After we have reviewed these observations about the determinant E and its minor determinant of first order, now we have to analyze the system (8) with respect to $dw_1, \ldots, dw_n$, which can happen since at this point we know that the determinant E is other than zero.

According to I §1 (9),

$$\frac{dt}{E} = \frac{\beta_1 dw_1}{\frac{\partial E}{\partial F_{11}}} = \ldots = \frac{\beta_n dw_n}{\frac{\partial E}{\partial F_{n1}}} \tag{12}$$

If we now properly select the values of the coefficients $\beta_1, \ldots, \beta_n$, then it follows that, since t increases (through real values) from $-\infty$ to $+\infty$, then $w_1, \ldots, w_n$ must always increase.

Hence it follows secondly now from (5), in accordance with the analyses in the preceding paragraphs, that $q_i$ must always remain between the bounds $a_i$ and $b_i$.

The functions

$$\frac{1}{E}\ \frac{\partial E}{\partial F_{i1}}$$

must always have an upper and a lower bound and then at last, it follows that $w_i$ must increase with t beyond all bounds.

The motions defined by the equations (2) and (2*) are therefore analogous to the motions investigated in the preceding paragraphs in so far as $q_1, \ldots, q_n$ must have either a libratory or a bounded motion. Nothing prevents that some of the quantities $q_i$ have a bounded motion, the rest a libratory motion.

If

$$m_i = n_i = 1 \qquad (i = 1, 2, \ldots, n),$$

then libration by all variables occurs, and since this case is of special interest, we will especially pursue it further.

Now the equation (5) reads

$$\frac{dq_i}{\sqrt{(q_i - a_i)(b_i - q_i)}} = \beta_i \, dw_i \qquad (i = 1, 2, \ldots, n), \tag{13}$$

and yields

$$q_i = a_i \cos^2 \tfrac{1}{2}\beta_i w_i + b_i \sin^2 \tfrac{1}{2}\beta_i w_i \quad . \tag{14}$$

We insert this value of $q_i$ into (8). If we make use of the notation

$$G_{ij}(q_i) = \int F_{ij}(q_i)\,\beta_i\,dw_i \quad , \tag{15}$$

then we obtain the system

$$\left.\begin{aligned} t + A_1 &= G_{11}(w_1) + \cdots + G_{n1}(w_n) \\ A_2 &= G_{12}(w_1) + \cdots + G_{n2}(w_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n &= G_{1n}(w_1) + \cdots + G_{nn}(w_n) \end{aligned}\right\} \tag{16}$$

by integration of (8), where $A_1, \ldots, A_n$ are the constants of integration. If we let $w_1$ increase by $2\pi$ in this equation, then it follows–when we assume the simplest form with

$$\beta_i = \pm 1 \qquad (i = 1, 2, \ldots, n) \quad - \tag{17}$$

that all together

$$\left.\begin{aligned} t + A_1 + 2\omega_{11} &= G_{11}(w_1 + 2\pi) + G_{21}(w_2) + \cdots + G_{n1}(w_n) \\ A_2 + 2\omega_{12} &= G_{12}(w_1 + 2\pi) + G_{22}(w_2) + \cdots + G_{n2}(w_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n + 2\omega_{1n} &= G_{1n}(w_1 + 2\pi) + G_{2n}(w_2) + \cdots + G_{nn}(w_n) \end{aligned}\right\} \tag{18}$$

and here

$$2\omega_{ij} = G_{1j}(w_1 + 2\pi) - G_{ij}(w_1) = \int_{w_1}^{w_1+2\pi} F_{1j}(q_1)\,dw_1 \quad .$$

Since $q_1$ is a periodic function of $w_1$, <u>then $\omega_{ij}$ is consequently a constant, and we have</u>

$$2\omega_{1j} = \int_0^{2\pi} F_{1j}(q_1)\, dw_1 = 2\int_0^{\pi} F_{1j}(q_1)\, dw_1$$

or

$$\omega_{1j} = \int_{a_1}^{b_1} \frac{\phi_{1j}(q_1)\, dq_1}{\sqrt{(q_1-a_1)(b_1-q_1)\,\Psi_1(q_1)}} \;.$$

One obtains similar formulae, when $w_2$, $w_3$ etc. are increased by $2\pi$.

Consequently, we are lead to the result that, if $w_i$ $(i = 1, 2, \ldots, n)$ is increased by $2\pi$, the integration constants $A_1, A_2, \ldots, A_n$ increase by constant quantities $\omega_{i1}, \omega_{i2}, \cdots, \omega_{in}$. On the other hand, we can assert that, if one regards $w_1, w_2, \ldots, w_n$ as functions of the integration constants $A_1, A_2, \ldots, A_n$, etc. These functions are of the character that, if $A_1$, $A_2, \ldots, A_n$ etc, are increased respectively by $\omega_{i1}, \omega_{i2}, \ldots, \omega_{in}$, $w_1, w_2, \ldots, w_{i-1}$, $w_{i+1}, \ldots, w_n$ remain unchanged, whereas $w_i$ will increase by $2\pi$.

The quantities $w_1, \ldots, w_n$ are, in that case, clearly functions of the integration constants $A_1, \ldots, A_n$. In order to show this, we differentiate the equations (16), since $A_1, \ldots, A_n$ will be treated as variables. If one considers (15), then one obtains

$$\left.\begin{aligned} dA_1 &= F_{11}\, dw_1 + \cdots + F_{n1}\, dw_n \\ dA_2 &= F_{12}\, dw_1 + \cdots + F_{n2}\, dw_n \\ &\ldots\ldots\ldots\ldots\ldots \\ dA_n &= F_{1n}\, dw_1 + \cdots + F_{nn}\, dw_n \end{aligned}\right\} \qquad (18^*)$$

which equations yield

$$\left.\begin{aligned} E\, dw_1 &= \frac{\partial E}{\partial F_{11}}\, dA_1 + \cdots + \frac{\partial E}{\partial F_{1n}}\, dA_n \\ &\ldots\ldots\ldots\ldots\ldots \\ E\, dw_n &= \frac{\partial E}{\partial F_{n1}}\, dA_1 + \cdots + \frac{\partial E}{\partial F_{nn}}\, dA_n \end{aligned}\right\} \qquad (18^{**})$$

when solved for $dw_1, \ldots, dw_n$.

Now since the functional determinant E vanishes for no value of $q_i$ which lies within the admissible region for this quantity, then a unique, determinate system of values for the differentials $dw_1, \ldots, dw_n$ belongs obviously to an arbitrary system of values $dA_1, \ldots, dA_n$. Therefore, if the integration constants $A_1, \ldots, A_n$ pass through an arbitrary continuous series of finite values, then $w_1, \ldots, w_n$ assume also a completely determinate continuous series of values.

The quantities $q_1, \ldots, q_n$ are specific functions of $w_1, \ldots, w_n$, therefore also of $A_1, \ldots, A_n$. If we now let

$$\left.\begin{array}{l} q_1 = f_1(t+A_1, A_2, \ldots, A_n) \\ \ldots\ldots\ldots\ldots\ldots \\ q_n = f_n(t+A_1, A_2, \ldots, A_n) \end{array}\right\}, \qquad (19^*)$$

then, according to the foregoing, the functions f have the following properties. This is,

$$\left.\begin{array}{l} f_i(t+A_1+2\omega_{11}, A_2+2\omega_{12}, \ldots, A_n+2\omega_{1n}) = \\ \quad = f_i(t+A_1, A_2, \ldots, A_n), \\ f_i(t+A_1+2\omega_{21}, A_2+2\omega_{22}, \ldots, A_n+2\omega_{2n}) = \\ \quad = f_i(t+A_1, A_2, \ldots, A_n), \\ \ldots\ldots\ldots\ldots\ldots \\ f_i(t+A_1+2\omega_{n1}, A_2+2\omega_{n2}, \ldots, A_n+2\omega_{nn}) = \\ \quad = f_i(t+A_1, A_2, \ldots, A_n), \qquad (i=1,2,\ldots,n). \end{array}\right\} \qquad (19)$$

These functions are thus, according to a notation employed by Weierstrass, <u>n-periodic functions of the n variables $t+A_1, A_2, \ldots, A_n$.</u>

The periods $\omega_{ij}$ are given by the following formula

$$\omega_{ij} = \int_0^\pi F_{ij}(q_i)\,dw_i = \int_{a_i}^{b_i} \frac{\Phi_{ij}(q_i)\,dq_i}{\sqrt{(q_i-a_i)(b_i-q_i)\Psi_i(q_i)}} \quad . \tag{20}$$

In place of $t+A_1$, $A_2, \ldots, A_n$, we introduce n new, definite (Reference 1) variables $u_1, \ldots, u_n$ in the following manner.

$$\left.\begin{aligned} \pi(t+A_1) &= \omega_{11}u_1 + \omega_{21}u_2 + \cdots + \omega_{n1}u_n \;, \\ \pi A_2 &= \omega_{12}u_1 + \omega_{22}u_2 + \cdots + \omega_{n2}u_n \;, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi A_n &= \omega_{1n}u_1 + \omega_{2n}u_2 + \cdots + \omega_{nn}u_n \;, \end{aligned}\right\} \tag{21}$$

where we assume that the determinant

$$\Omega = |\omega_{ij}| \qquad (i,j = 1,2,\ldots,n) \tag{22}$$

is different from zero.

Now from the relations (21), it follows directly that, if $u_i$ $(i = 1, 2, \ldots, n)$ is increased by $2\pi$,

$$\begin{aligned} t+A_1 &\text{ increases by } 2\omega_{i1} \;, \\ A_2 &\quad\text{"}\quad\text{"}\quad 2\omega_{i2} \;, \\ &\cdots\cdots\cdots\cdots \\ A_n &\quad\text{"}\quad\text{"}\quad 2\omega_{in} \;. \end{aligned}$$

1. See Weierstrass: "Some theorems relating to the theory of analytic functions of several variables."

On the other hand, we can state-since the relations (21) are linear and the determinant $\Omega$ is different from zero-that, since $t+A_1$, $A_2, \ldots, A_n$ increase by $2\omega_{i1}$, $2\omega_{i2}, \ldots, 2\omega_{in}$ respectively, at the same time $u_i$ will increase by $2\pi$, while $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$ remain unchanged.

It now follows according to (19), that $f_i$ ($i = 1, 2, \ldots, n$) periodic functions of $u_i$ ($i = 1, 2, \ldots, n$) have period of $2\pi$.

If we now denote those functions into which the $f_i$ transform, by $g_i$, if the variable $u_1, \ldots, u_n$ are introduced instead of $t+A_1$, $A_2, \ldots, A_n$, then one has

$$
\begin{gathered}
g_i(u_1+2\pi, u_2, \ldots, u_n) = g_i(u_1, u_2, \ldots, u_n) \\
g_i(u_1, u_2+2\pi, \ldots, u_n) = g_i(u_1, u_2, \ldots, u_n) \\
\cdots\cdots\cdots\cdots\cdots \\
g_i(u_1, u_2, \ldots, u_n+2\pi) = g_i(u_1, u_2, \ldots, u_n) \\
(i = 1, 2, \ldots, n)
\end{gathered}
$$

and in general

$$
\begin{gathered}
g_i(u_1+2m_1\pi, u_2+2m_2\pi, \ldots, u_n+2m_n\pi) = \\
= g_i(u_1, u_2, \ldots, u_n), \\
(i = 1, 2, \ldots, n),
\end{gathered} \tag{23}
$$

where $m_1, m_2, \ldots, m_n$ denote arbitrary positive or negative whole integers.

However, functions of this kind let themselves be expressed (Weierstrass in the place cited), by means of a generalized Fourier series, with n variables:

$$g_i = \sum C^{(i)}_{\nu_1, \nu_2, \ldots, \nu_n} e^{(\nu_1 u_1 + \nu_2 u_2 + \cdots + \nu_n u_n)\sqrt{-1}} \tag{24}$$

$$(\nu_1, \nu_2, \ldots, \nu_n = -\infty, \ldots, +\infty)$$

where the coefficients are given by the formula

$$\pi^n C^{(i)}_{\nu_1, \nu_2, \ldots, \nu_n}$$

$$= \int_0^{\pi} \cdots \int_0^{\pi} g_i(u_1, u_2, \ldots, u_n) e^{-(\nu_1 u_1 + \nu_2 u_2 + \cdots + \nu_n u_n)\sqrt{-1}} du_1 \, du_2 \ldots du_n . \tag{24*}$$

The numerical values of this integral let themselves always be calculated, and in reference hereon, the same comments which were made above in § 2 for one variable, are valid - mutatis mutandis.

The coordinates $q_i$ ($i = 1, 2, \ldots, n$) are therefore, in this case, ($m_i = n_i = 1$) n-periodic functions of the n variables $u_1, u_2, \ldots, u_n$. It can also happen that they may be periodic functions of time. But, in general, this is not so.

According to (21), $u_1, u_2, \ldots, u_n$ are linear functions of time. If one solves this equation, one obtains

$$\left.\begin{aligned}
\Omega u_1 &= \frac{\partial\Omega}{\partial\omega_{11}}\pi(t+A_1) + \frac{\partial\Omega}{\partial\omega_{12}}\pi A_2 + \cdots + \frac{\partial\Omega}{\partial\omega_{1n}}\pi A_n \\
\Omega u_2 &= \frac{\partial\Omega}{\partial\omega_{21}}\pi(t+A_1) + \frac{\partial\Omega}{\partial\omega_{22}}\pi A_2 + \cdots + \frac{\partial\Omega}{\partial\omega_{2n}}\pi A_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\Omega u_n &= \frac{\partial\Omega}{\partial\omega_{n1}}\pi(t+A_1) + \frac{\partial\Omega}{\partial\omega_{n2}}\pi A_2 + \cdots + \frac{\partial\Omega}{\partial\omega_{nn}}\pi A_n \; .
\end{aligned}\right\} \quad (25)$$

Now if t is increased by 2T, then at the same time

$u_1$ is increased by $\frac{1}{\Omega}\frac{\partial\Omega}{\partial\omega_{11}}\cdot 2\pi T$

$u_2$ is increased by $\frac{1}{\Omega}\frac{\partial\Omega}{\partial\omega_{21}}\cdot 2\pi T$

. . . . . . . . . .

$u_n$ is increased by $\frac{1}{\Omega}\frac{\partial\Omega}{\partial\omega_{n1}}\cdot 2\pi T$ .

When the motion is said to be periodic in time with period of 2T, then the preceding additions to $u_1, u_2, \ldots, u_n$ must be whole multiples of $2\pi$; consequently, the following relations must occur, in which $m_1, m_2, \ldots, m_n$ denote whole numbers,

$$\begin{aligned}
\frac{1}{\Omega}\frac{\partial\Omega}{\partial\omega_{11}} &= \frac{m_1}{T} \\
\frac{1}{\Omega}\frac{\partial\Omega}{\partial\omega_{21}} &= \frac{m_2}{T} \\
&\cdots\cdots \\
\frac{1}{\Omega}\frac{\partial\Omega}{\partial\omega_{n1}} &= \frac{m_n}{T} \; .
\end{aligned}$$

If these equations are now multiplied respectively by $\omega_{11}, \omega_{22}, \ldots, \omega_n$, and likewise by $\omega_{12}, \omega_{22}, \ldots, \omega_{n2}$, etc., and the equations (so obtained) added, then the following system of equations results:

$$\left.\begin{aligned} T &= m_1\omega_{11} + m_2\omega_{21} + \cdots + m_n\omega_{n1} \\ 0 &= m_1\omega_{12} + m_2\omega_{22} + \cdots + m_n\omega_{n2} \\ &\cdots\cdots\cdots\cdots \\ 0 &= m_1\omega_{1n} + m_2\omega_{2n} + \cdots + m_n\omega_{nn} \quad . \end{aligned}\right\} \tag{26}$$

In order that the motion be periodic in t, the above relations must therefore exist between the periods $\omega_{ij}$. Then the period (in t) is given by means of the first of these formulae.

Because the coordinates $q_i$ can be periodic functions of t under certain circumstances, Staude introduced the name conditional periodic motions, for these motions.

Since the determinant $\Omega$ does not vanish, then according to (26), one unique system of values for the quantities $m_1, m_2, \ldots, m_n$ belongs to each value of 2T. If the quantities so determined are whole numbers, then the motion is periodic with period of 2T.

In reference to the conditional periodic motion, Stäckel has proven an interesting and important theorem which I will now explain.

First of all, I note that an infinitely small variation of the numbers $A_1, \ldots, A_n$ corresponds, according to (18**) to an infinitely small variation of $w_1, \ldots, w_n$ and therefore also to an infinitely small variation of $q_1, \ldots, q_n$.

We now have

$$q_i = f_i(t + A_1, A_2, \ldots, A_n) \qquad (i = 1, 2, \ldots, n),$$

and these functions are, according to (19), of the character that

$$\left.\begin{aligned} & f_1\left(t+A_1+\sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha 1},\ A_2+\sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha 2},\ \ldots\right. \\ & \qquad \left. A_n+\sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha n}\right) = f_i\left(t+A_1, A_2, \ldots, A_n\right) . \end{aligned}\right\} \qquad (27)$$

Now however, one can show (Reference 2) that one can always find infinitely many systems of whole numbers $m_1, \ldots, m_n$ with the property that any one of the n-1 expressions

$$\sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha j}, \qquad (j=2,3,\ldots,n)$$

is smaller than any arbitrary quantity $\epsilon$, however small $\epsilon$ may be chosen.

Consequently, let the number $m_\alpha$ be chosen so that

$$\sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha j} = \epsilon_j \qquad (j=1,2,\ldots,n)$$

where $|\epsilon_j| < \epsilon$. Then one has, according to (27),

$$\begin{aligned} & f\left(t+A_1+\sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha 1},\ A_2+\epsilon_2, \ldots, A_n+\epsilon_n\right) \\ & \quad = f\left(t+A_1, A_2, \ldots, A_n\right) . \end{aligned}$$

If one here set

$$t = t_1 - \sum_{\alpha=1}^{n} 2m_\alpha\omega_{\alpha 1} = t_1 - P$$

2. See Jacobi elsewhere cited; Kronecker, Proceedings of the Berlin Academy, 1884.

where t denotes an arbitrary value of time, then

$$f(t_1+A_1, A_2+\varepsilon_2, \ldots, A_n+\varepsilon_n) = f(t_1-P+A_1, A_2, \ldots, A_n) \; .$$

Now however, I have really shown that, on account of the continuity of the functions f, they undergo-upon an infinitely small variation of the integration constants $A_1$, $A_2, \ldots,$ $A_n$-a corresponding infinitely small variation. Therefore,

$$f(t_1+A_1, A_2+\varepsilon_2, \ldots, A_n+\varepsilon_n)$$

differs from f ($t_o$ + $A_1$, $A_2, \ldots,$ $A_n$) by an infinitely small amount, and therefore it follows also- from the last equation- that the functions

$$f(t_1+A_1, A_2, \ldots, A_n)$$

and

$$f(t_1-P+A_1, A_2, \ldots, A_n)$$

differ from one another by arbitrarily small amounts.

Expressed in another way, this means that the coordinates $q_1$, $q_2, \ldots,$ $q_n$ at time $t_1$ and at time $t_1$-P differ from one another by arbitrarily small amounts.

Therefore, if the coordinates at time $t_1$ have the values $q_1^{(1)}, \ldots, q_n^{(1)}$, then there is always another value for t - and even, because the numbers $m_\alpha$ can be chosen in infinitely many different ways, infinitely many other values for t - for which the trajectory approaches arbitrarily near to the point $q_1^{(1)}, \ldots, q_n^{(1)}$.

One can go further with Stäckel and show that one always can find those values for the time, for which the trajectory approaches arbitrarily near to some one point, which generally lies within the admissible region for the coordinates $q_1, \ldots, q_n$. This region B is determined, according to the foregoing, so that

$$a_i \lessgtr q_i \leqq b_i \qquad (i = 1, 2, \ldots, n) \; .$$

Now if $q_1^o, q_q^o, \ldots, q_n^o$ is any one point of the region B, then according to (16) a specific system of values for $t + A_1, A_2, \ldots, A_n$ belong to it - which values we wish to denote with $B_1, B_2, \ldots, B_n$.

Now the function

$$Q_i = f_i(B_1, B_2, \ldots, B_n)$$

does not necessarily determine one value of $q_i$. However, we want to show that one can always find - in the expression for $q_i$

$$q_i = f_i(t + A_1, A_2, \ldots, A_n) \quad -$$

one value for t, for which $Q_i$ ($i = 1, 2, \ldots, n$) differs from $q_i$ by an arbitrarily small amount.

In that case, there is the relation (27). But now it follows, from the above noted theorem by Jacobi and Kronecker, that-as long as the periods $\omega_{ij}$ are independent of each other - one can always determine the numbers $m_1, m_2, \ldots, m_n$ so that

$$A_i + \sum_{\alpha=1}^{n} 2 m_\alpha \omega_{\alpha i} \qquad (i = 2, 3, \ldots, n)$$

differs from $B_i$ by arbitrarily small amounts. If we then set

$$t = B_1 - A_1 - \sum_{\alpha=1}^{n} 2 m_\alpha \omega_{\alpha 1} \; ,$$ then the theorem is verified.

This theorem expresses in qualitative theoretical terms, that the trajectory fills the entire region B everywhere dense.

The above explanations undergo only one exception then, whenever relations of the form (26) exist between the periods $\omega_{ij}$. Then the motion will be periodic, as has already been described.

The division between the periodic and nonperiodic paths is a question of the greatest significance for these motions. The stability question is certainly intimately connected with the nature of this division. In a following paragraph, I will have occasion to return to this question.

In the above discussion, it was assumed that the functions $\Phi_i(q_i)$ in formula (4) actually have two roots $a_i$ and $b_i$, between which the variable $q_i$ lies at inception of motion, and then must always remain. It can also occur, that this is not so, but that one or more of these functions vanish for no real value of $q_i$. The treatment of this case is, however, analogous to the preceding treatment. That is, it first follows that the corresponding q increases continuously (or decreases continuously) with time. The variable q then has the same property as the corresponding auxiliary quantity w previously. Of special interest is the case, that the coefficient of dq is periodic. That is, whenever the considered system of the following is

$$\left.\begin{aligned} t + A_1 &= \int \frac{\phi_{11}(q_1)\,dq_1}{\sqrt{\Phi_1(q_1)}} + \cdots + \int \frac{\phi_{n1}(q_n)\,dq_n}{\sqrt{\Phi_n(q_n)}} \\ &\;\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n &= \int \frac{\phi_{1n}(q_1)\,dq_1}{\sqrt{\Phi_1(q_1)}} + \cdots + \int \frac{\phi_{nn}(q_n)\,dq_n}{\sqrt{\Phi_n(q_n)}} \end{aligned}\right\} \qquad (28)$$

and the coefficients of-for example $-dq_1$ have the property of disappearing for no value of $q_1$ or of being infinite and moreover are periodic with period of $2\pi$, then it follows that, since $q_1$ increases by $2\pi$, $A_1, \ldots, A_n$ will increase by constant quantities $\omega_{11}, \ldots, \omega_{1n}$. The quantity $q_i$ then has here the same property as $w_1$ in equation (16).

If

$$\frac{\phi_{1i}(q_1)}{\sqrt{\Phi_1(q_1)}}$$

is a constant, then the period is arbitrary.

It often occurs in mechanics, that the mutual position of the moving bodies remains invariant, whenever one or more quantities $q_i$ (so-called angle magnitude) are increased by multiples of $2\pi$. Then one introduces the auxiliary quantities

$$\sin q_i = x_i$$

in place of these $q_i$, and the discussion will then be completely analogous to the argument given above.

Example. The conic pendulum.

If one takes the Z-axis as vertical, denotes by $\theta$ the angle which the projection of the pendulum on the XY-plane makes with a fixed line, the Z-coordinate by z, the length of the pendulum by $\ell$, the acceleration of the gravity force by g, and finally two integration constants by c and c', then the variations of z and $\theta$ are given (see, for example, Despeyrous: Cours de Mécanique, Chapter II, page 70) by the following formulae

$$\left.\begin{aligned} dt &= \frac{\ell\, dz}{\sqrt{(2gz+c')(\ell^2-z^2)-c^2}} \\ 0 &= d\theta + \frac{c\ell\, dz}{(\ell^2-z^2)\sqrt{(2gz+c')(\ell^2-z^2)-c^2}} \end{aligned}\right\} \qquad (29)$$

and from this form of the differential equations, we can immediately conclude that we are dealing with a conditional periodic motion.

For c = 0, one obtains the usual pendulum problem in the plane. If we exclude this case, then one easily finds that the equation

$$(2gz+c^2)(\ell^2-z^2)-c^2 = 0$$

has three real roots, one root negative and numerically greater then 1, both the others numerically smaller than 1.

Therefore, the coordinate z must always remain between both the latter two roots. Hence it now follows that $\ell^2-z^2$ can never be zero. The degree of the root is one, and we can directly solve the formulae (13) and the following.

As often as z retains the same value and $\theta$ increases simultaneously by a multiple of $2\pi$, the pendulum returns to the same position.

Therefore if we set

$$y = \cos\theta \quad , \tag{30}$$

then the motion is periodic whenever y and z regain their original values. In these coordinates, the differential equations after integration read

$$\left.\begin{aligned} t + A_1 &= \int \frac{\ell\, dz}{(2gz+c')(\ell^2-z^2)-c^2} \\ A_2 &= \int \frac{dy}{\sqrt{1-y^2}} + \int \frac{c\ell\, dz}{(\ell^2-z^2)\sqrt{(2gz+c')(\ell^2-z^2)-c^2}} \end{aligned}\right\} \tag{31}$$

If we set

$$h(z) = (2gz+c')(\ell^2-z^2)-c^2$$

and denote both roots of h(z) = 0 by $\alpha$ and $\beta$ ($\alpha > \beta$), which are numerically smaller than $\ell$, and denote the third root by $-\gamma$, then one has to insert

$$F_{11} = 0 \quad ; \quad F_{21} = \frac{\ell}{\sqrt{2g(z+\gamma)}}$$

$$F_{12} = 1 \quad ; \quad F_{22} = \frac{c\ell}{(\ell^2-z^2)\sqrt{2g(z+\gamma)}}$$

into formula (6) and the following.

For the periods $\omega_{ij}$, one obtains the expression

$$\omega_{11} = 0 \quad ; \quad \omega_{21} = \int_\alpha^\beta \frac{\ell\, dz}{\sqrt{h(z)}}$$

$$\omega_{12} = -\pi \quad ; \quad \omega_{22} = \int_\alpha^\beta \frac{c\ell\, dz}{(\ell^2-z^2)\sqrt{h(z)}}$$

and for the determinant $\Omega$, one obtains the value

$$\Omega = \pi\omega_{21}$$

which consequently is different from zero.

The variables $u_1$ and $u_2$ and (21) are determined by the following formulae

$$\pi(t+A_1) = \qquad \omega_{21} u_2$$
$$\pi A_2 = -\pi u_1 + \omega_{22} u_2$$

so that

$$u_1 = \frac{\omega_{22}}{\omega_{21}}(t+A_1) - A_2$$
$$u_2 = \frac{\pi}{\omega_{21}}(t+A_1) \quad .$$

The coordinates y and z can be described as biperiodic functions of both variables $u_1$ and $u_2$ with the help of equations (24) and (24*), and it is worth noting that these descriptions have considerable merit over the conventional solution of this problem in application of elliptic functions.

According to (26), the motion will be purely periodic whenever

$$0 = m_1 \pi - m_2 \omega_{22}$$

and the period 2T is then

$$2T = 2 m_2 \omega_{21} \quad .$$

The complete theory of the spherical pendulum is contained in these formulae.

# APPENDIX B

## "THE MECHANICS OF THE HEAVENS"

## CHAPTER III - VOLUME 1

TRANSLATED FROM "DIE MECHANIK DES HIMMELS" BY
CARL L. CHARLIER, VOLUMES 1 AND 2, VEIT AND CO., LEIPZIG, 1902.

TRANSLATED BY: SAMUEL P. ALTMAN

## B.1 GENERAL CONSIDERATIONS

In the future I will confine myself to the case, that motion occurs in one plane. However, it is worth noting that the general case - in three dimensions - can also be treated in completely analogous fashion.

If one makes use of the elliptical coordinates $\lambda$ and $\mu$ in I §7, then we obtain

$$2T = (\lambda^2-\mu^2)\left[\frac{\lambda'^2}{\lambda^2-c^2} + \frac{\mu'^2}{c^2-\mu^2}\right] , \tag{1}$$

$$U = \frac{1}{\lambda^2-\mu^2}\left[(K+K')\lambda - (K-K')\mu\right] . \tag{2}$$

If we want to write the differential equations in canonical form, then we can set

$$q_1 = \lambda , \quad q_2 = \mu$$

and then obtain, according to I §8

$$p_1 = \frac{\partial T}{\partial q_1'} = \frac{\lambda^2-\mu^2}{\lambda^2-c^2}\lambda' ,$$

$$p_2 = \frac{\partial T}{\partial q_2'} = \frac{\lambda^2-\mu^2}{c^2-\mu^2}\mu' ,$$

so that

$$2T = \frac{1}{q_1^2-q_2^2}\left[(q_1^2-c^2)p_1^2 + (c^2-q_2^2)p_2^2\right] ,$$

$$U = \frac{1}{q_1^2-q_2^2}\left[(K+K')q_1 - (K-K')q_2\right] .$$

The canonical differential equations now become

$$\left.\begin{aligned} \frac{dq_1}{dt} &= \frac{\partial H}{\partial p_1}, & \frac{dp_1}{dt} &= -\frac{\partial H}{\partial q_1}, \\ \frac{dq_2}{dt} &= \frac{\partial H}{\partial p_2}, & \frac{dp_2}{dt} &= -\frac{\partial H}{\partial q_2}, \end{aligned}\right\} \tag{3}$$

where

$$H = T - U. \tag{4}$$

The differential equations are obviously of the form that is necessary for the use of the theorem of Stäckel. In fact, if one sets |II §1 (10)|

$$\Phi_{11} = \frac{q_1^2}{q_1^2 - c^2}, \quad \Phi_{21} = \frac{q_2^2}{q_2^2 - c^2},$$

$$\Phi_{12} = \frac{1}{q_1^2 - c^2}, \quad \Phi_{22} = \frac{1}{q_2^2 - c^2},$$

$$\Psi_1 = \frac{K + K'}{(q_1^2 - c^2)} q_1, \quad \Psi_2 = \frac{K - K'}{(q_2^2 - c^2)} q_2,$$

so that

$$\Delta = \frac{q_1^2 - q_2^2}{(q_1^2 - c^2)(q_2^2 - c^2)},$$

then we obtain the above form.

If we introduce the symbols $\lambda$ and $\mu$ (instead of $q_1$ and $q_2$) again, then one obtains - according to II § 1 (19) and (19*) - the following equations:

$$\left.\begin{aligned}\int\frac{\lambda^2\,d\lambda}{\sqrt{2(\lambda^2-c^2)[(K+K')\lambda+h\lambda^2+\alpha]}}+\int\frac{\mu^2\,d\mu}{\sqrt{2(\mu^2-c^2)[(K-K')\mu+h\mu^2+\alpha]}}&=t+\beta_1\,,\\ \int\frac{d\lambda}{\sqrt{2(\lambda^2-c^2)[(K+K')\lambda+h\lambda^2+\alpha]}}+\int\frac{d\mu}{\sqrt{2(\mu^2-c^2)[(K-K')\mu+h\mu^2+\alpha]}}&=\beta_2\,,\end{aligned}\right\}\quad(5)$$

corresponding with I §7 (43), and read the intermediate integrals:

$$\left.\begin{aligned}(\lambda^2-\mu^2)\frac{d\lambda}{dt}&=\sqrt{2(\lambda^2-c^2)[(K+K')\lambda+h\lambda^2+\alpha]}\,,\\ (\lambda^2-\mu^2)\frac{d\mu}{dt}&=-\sqrt{2(\mu^2-c^2)[(K-K')\mu+h\mu^2+\alpha]}\,.\end{aligned}\right\}\quad(5^*)$$

The characteristic function H is independent of t and therefore here one has the integral

$$H = h$$

or

$$\left.\begin{aligned}&\frac{1}{2}(\lambda^2-\mu^2)\left[\frac{\lambda'^2}{\lambda^2-c^2}+\frac{\mu'^2}{c^2-\mu^2}\right]=\\ &=\frac{1}{\lambda^2-\mu^2}\left[(K+K')\lambda-(K-K')\mu\right]+h\,.\end{aligned}\right\}\quad(6)$$

From these equations one finds that the quantities $\lambda$ and $\mu$ are–in general – biperiodic functions of $t+\beta_1$ and $\beta_2$. For the periods $\omega_{ij}$, one obtains – according to II §3 (20) – the following values

$$\left.\begin{aligned} \omega_{11} &= \int_{a_1}^{b_1} \frac{\lambda^2\, d\lambda}{\sqrt{R(\lambda)}} , \quad \omega_{21} = \int_{a_2}^{b_2} \frac{\mu^2\, d\mu}{\sqrt{S(\mu)}} , \\ \omega_{12} &= \int_{a_1}^{b_1} \frac{d\lambda}{\sqrt{R(\lambda)}} , \quad \omega_{22} = \int_{a_2}^{b_2} \frac{d\mu}{\sqrt{S(\mu)}} , \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} R(\lambda) &= 2(\lambda^2 - c^2)\left[(K+K')\lambda + h\lambda^2 + \alpha\right] , \\ S(\mu) &= 2(\mu^2 - c^2)\left[(K-K')\mu + h\mu^2 + \alpha\right] , \end{aligned}\right\} \tag{8}$$

and $a_1$, $b_1$ and $a_2$, $b_2$ denote (respectively) the two roots of the equations $R(\lambda) = 0$ and $S(\mu) = 0$.

One finds that the determinants $\Delta$ and $\Omega$ in II § 3 are different from zero, as often as not $\lambda = \mu$, in which case a conflict occurs.

If one introduces the auxiliary quantities $u_1$ and $u_2$ according to II § 3 (21), then one has

$$\pi(t+\beta_1) = \omega_{11} u_1 + \omega_{21} u_2 ,$$
$$\pi\beta_2 = \omega_{12} u_1 + \omega_{22} u_2$$

or

$$\left.\begin{aligned} \Omega u_1 &= \pi\omega_{22}(t+\beta_1) - \pi\omega_{21}\beta_2 , \\ \Omega u_2 &= -\pi\omega_{12}(t+\beta_1) + \pi\omega_{11}\beta_2 , \end{aligned}\right\} \tag{9}$$

and $\lambda$ and $\mu$ now become biperiodic functions of $u_1$ and $u_2$ with the period $2\pi$, which can be developed in Fourier series in multiples of $u_1$ and $u_2$.

The motion is periodic <u>in the time</u>, as often as $\omega_{21}$ and $\omega_{22}$ are <u>commensurable</u> with one another, so that

$$0 = m_1 \omega_{12} + m_2 \omega_{22} \quad , \tag{10}$$

where $m_1$ and $m_2$ denote whole numbers, and one obtains - for the period 2T - the value

$$2T = 2 m_1 \omega_{11} + 2 m_2 \omega_{21} \quad . \tag{10*}$$

If the roots of the equations R $(\lambda)$ = 0 and S $(\mu)$ = 0 are not simple roots, then bounded motions occur.

Aside from these cases, only two cases occur with different values of h and $\alpha$, namely

1) that $\lambda$ or $\mu$ retains a constant value, or
2) that each motion is impossible.

In reference to the quantities $\lambda$ and $\mu$, I call attention to our definitions

$$2\lambda = r + r' \quad ,$$
$$2\mu = r - r' \quad ,$$

from which it follows, that the following inequalities must always be fulfilled

$$\lambda \geqslant c \geqslant \mu \geqslant -c \quad . \tag{11}$$

I call further attention to the fact that $\lambda$ signifies the semimajor axis of an ellipse, whose foci lie in the two fixed centres and which goes through the movable point. The quanitity $\mu$ denotes the corresponding defining-values of a hyperbola. 2c is the distance between both foci.

If $\lambda$ = c, then this means that the foci coincide with the end-points of the major axis of the ellipse; that is, the ellipse is transformed into the straight line K'K for $\lambda$ = c. Consequently, the movable point must always be found on the line K'K for $\lambda$ = c; I will name it simply the planet.

If on the other hand $\mu = c$, then the hyperbola consists of that part of the negative X-axis which lies on the other side of the mass K'. For $\mu = -c$ one obtains the corresponding part of the positive X-axis on the other side of the mass K. For $\mu = \pm c$ the planet is then found on one of these parts of the X-axis.

If $\mu = 0$, then the hyperbola coincides with the Y-axis.

If the coordinates of the planet are $\lambda_o$ and $\mu_o$ at the inception of motion, then we must obtain-according to (5*)

$$\left.\begin{aligned} L(\lambda_0) &= (K+K')\lambda_0 + h\lambda_0^2 + \alpha \geq 0 \quad , \\ M(\mu_0) &= (K-K')\mu_0 + h\mu_0^2 + \alpha \leq 0 \end{aligned}\right\} \tag{12}$$

from which it follows that $\lambda_o$ and $\mu_o$ must comply also with the inequalities (11). Furthermore the relations (11) and (12) must be fulfilled not only with inception of the motion, but always with all value-pairs generally compatible with the problem.

I proceed now to more detailed consideration of the different conditions of motion. It appears to be appropriate to differentiate between the following cases:

1) h negative,
2) h positive,
3) h = 0,
4) bounded motion,
5) pure periodic motion.

In 1) and 2) I assume that the roots of the equations $R(\lambda) = 0$ and $S(\lambda) = 0$ are simple.

## B.2 THE CONSTANT h (OF THE KINETIC ENERGY) NEGATIVE. LIBRATION CASE

The equation $R(\lambda) = 0$ always has both roots $\lambda = \pm c$. Since we want to assume here that the constant h is negative, we set

$$h = -h_1, \tag{1}$$

where $h_1$ denotes a positive value. If we now set

$$L(\lambda) = (K+K')\lambda - h_1\lambda^2 + \alpha \quad , \tag{2}$$

then we must always obtain - according to § 1 (12) -

$$L(\lambda) \geq 0 \ . \tag{3}$$

If $\lambda$ could be increased beyond all limits, then $L(\lambda)$ would obviously become negative for sufficiently large values of $\lambda$, which is not permitted according to (3). Therefore, it follows that - for negative h - the quantity $\lambda$ must always have a finite upper limit. Consequently, the planet cannot deviate arbitrarily far from K' and K.

We now call $r_1$ and $r_2$ the roots of the equation $L(\lambda) = 0$ and therefore have

$$L(\lambda) = h_1(r_1 - \lambda)(\lambda - r_2) \quad , \tag{4}$$

where it shall be assumed that

$$r_1 > r_2 \tag{5}$$

for real values of $r_1$, $r_2$.

One can then recognize four cases:

$r_1$ and $r_2$ imaginary,
$r_1$ and $r_2$ real, but smaller than c,
$r_1$ real and greater than c,
$r_1$ and $r_2$ real and greater than c.

However, the first two of these cases give rise to similar states of motion and may therefore be treated simultaneously. In both cases the sign of the same $L(\lambda)$ cannot change, because no real root, which is greater than c, exists. In both cases - then - $L(\lambda)$ must necessarily remain negative, because $L(+\infty)$ is negative.

Consequently, we recognize the following three cases

(a) $r_1$ and $r_2$ either imaginary, or real and smaller than c,

(b) $r_1 > c > r_2$,

(c) $r_1 > r_2 > c$.

If we make

$$M(\mu) = (K-K')\mu - h_1\mu^2 + \alpha \quad , \tag{6}$$

and designate the roots of the equation

$$M(\mu) = 0 \tag{7}$$

as $\rho_1$ and $\rho_2$, where $\rho_1 > \rho_2$ for real roots, then

$$M(\mu) = h_1(\rho_1 - \mu)(\mu - \rho_2) \quad ,$$

and one must always have

$$M(\mu) \leq 0 \ . \tag{8}$$

Here we have to investigate four cases:

$\alpha$) $\rho_1$ and $\rho_2$ imaginary,

$\beta$) $\rho_1$ and $\rho_2$ real and greater than c in absolute magnitude,

$\gamma$) $\rho_1 > c > \rho_2 > -c$,

$\delta$) $c > \rho_1 > \rho_2 > -c$.

In ($\gamma$) is also included the case

$$c > \rho_1 > -c > \rho_2$$

as will be discussed shortly, below.

Since now each of the cases a), b) and c) will combine with the last four, one obtains here 12 different cases, which we will consider successively.

Case Ia. The roots $r_1$ and $r_2$ either imaginary, or real and smaller than c.

According to § 1 (5*)

$$(\lambda^2 - \mu^2) \frac{d\lambda}{dt} = \sqrt{2(\lambda^2 - c^2) L(\lambda)} \quad . \tag{9}$$

Since now - in the case - $L(\lambda)$ always remains negative and moreover $\lambda$ cannot become smaller than c, then one can satisfy this equation only if one sets

$$\lambda \equiv c \quad . \tag{10}$$

Therefore, the planet must always remain on the line K'K. The motion will be straight. Dependent upon the value of $\rho$, we obtain now:

Case Ia$\alpha$. $\rho_1$ and $\rho_2$ imaginary.

The function $M(\mu)$ does not vary in sign for real values of $\mu$ and is then negative, because it will be negative for sufficiently large values of $\mu$.

We have

$$(\lambda^2 - \mu^2) \frac{d\mu}{dt} = \sqrt{2(\mu^2 - c^2) M(\mu)} \quad , \tag{11}$$

and since $M(\mu)$ is now negative, then $\mu$ would oscillate periodically between +c and -c. However, since the planet coincides with one of the masses K or K' for $\mu = \pm c$ and $\lambda = c$, then the validity of the differential equations ceases.

The case Ia$\alpha$ is therefore characterized by the fact, that the planet is located on the line K'K with the inception of the motion and its initial velocity lies along the X-axis. The planet moves in this direction until coincidence with K' or K occurs.

Case Ia$\beta$. $\rho_1$ and $\rho_2$ real and greater than c in absolute magnitude.

The function $M(\mu)$ does not change in sign during the motion.

IF $M(\mu)$ is negative, then one has the same motion as in Ia$\alpha$. If - on the other hand - $M(\mu)$ is positive, then it follows from (11), that $\mu$ must necessarily be constant +c or -c.

Consequently, the planet has now the same coordinates as one of the masses and motion is not possible.

Case Ia $\gamma$. $\rho_1 > c > \rho_2 > -c$.

We have

$$(\mu^2 - c^2)M = (\mu^2 - c^2)h_1(\rho_1 - \mu)(\mu - \rho_2) \quad . \tag{12}$$

In order that this expression remains positive, it is obvious that

$$-c \leq \mu \leq \rho_2 \quad .$$

If $\frac{d\mu}{dt}$ is positive at the inception of motion, then $\mu$ increases until the value $\mu = \rho_2$, then reverses and coincides at last with the mass K.

Case I a $\delta$. $c > \rho_1 > \rho_2 > -c$.

It follows from (12), that one has either

$$-c \leq \mu \leq \rho_2 \quad ,$$

or

$$\rho_1 \leq \mu \leq c \quad .$$

The planet coincides with K or K'.

As a fifth possibility, one can properly consider the case, that

$$c > \rho_1 > -c > \rho_2 \quad ,$$

which is obviously similar to Ia $\gamma$, only here the planet must coincide with the mass K'.

We now turn our attention to

Case Ib. $r_1 > c > r_2$.

From

$$(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = \sqrt{2(\lambda^2 - c^2)h_1(r_1 - \lambda)(\lambda - r_2)} \quad , \tag{13}$$

it follows that

$$r_1 \geqslant \lambda \geqslant c\ .$$

If $d\lambda$ is positive at the start of the motion, then $\lambda$ increases until the value $\lambda = r_1$ is obtained. then begins to decrease, and decreases continually until $\lambda = c$, in order then to increase again. Therefore, <u>libration in $\lambda$</u> occurs. Geometrically speaking, the planet must always be found inside that ellipse which has its foci in K' and K, and whose semi-major axis is constant $r_1$. The complete description of the motion depends upon the values of the roots $\rho_1$ and $\rho_2$.

<u>Case Ib$\alpha$. $\rho_1$ and $\rho_2$ imaginary.</u>

The function $M(\mu)$ does not vary in sign and remains negative. It now follows from (11), that $\mu$ swings between the limits +c and -c. Therefore, libration occurs in $\gamma$ as well as in $\mu$, and we can here apply the results of limited periodic motion. <u>Therefore, the trajectory curve consists either of a continuous line</u> - nearly in the form of a lemniscate - <u>or it fills the entire interior space of the ellipse $r_1$ with uniform density.</u>

For the fundamental periods $\omega_{ij}$, one has here the values

$$\omega_{11} = \int_c^{r_1} \frac{\lambda^2\, d\lambda}{\sqrt{R(\lambda)}} \quad ; \quad \omega_{21} = \int_{-c}^{+c} \frac{\mu^2\, d\mu}{\sqrt{S(\mu)}} \quad ;$$

$$\omega_{12} = \int_c^{r_1} \frac{d\lambda}{\sqrt{R(\lambda)}} \quad ; \quad \omega_{22} = \int_{-c}^{+c} \frac{d\mu}{\sqrt{S(\mu)}} \quad .$$

The quantities $\lambda$ and $\mu$ can be described as Fourier series constantly converging to the multiples of both arguments $u_1$ and $u_2$, determined by means of §1 (9).

<u>Case Ib$\beta$. $\rho_1$ and $\rho_2$ real and greater than c in absolute magnitude.</u>

The function $M(\mu)$ does not vary in sign during the motion. If $M(\mu)$ is negative, then we revert to Ib$\alpha$. On the other hand, if $M(\mu)$ is positive, then it follows - according to (11) - that

$$\mu \equiv +c \quad \text{or} \quad -c\ .$$

The motion occurs on that part of the X-axis which, computed from the coordinate origin, lies on the other side of the masses K' and K. The planet coincides at last with one of the masses.

Case Ib$\gamma$. $\rho_1 > c > \rho_2 > -c.$

As in the case Ia$\gamma$, one now finds that

$$-c \leq \mu \leq \rho_2 \quad .$$

Then libration occurs in $\mu$, as well as in $\lambda$. The path lies inside the space situated around K, which is bounded by the ellipse $\lambda = r_1$ and the hyperbola $\mu = \rho_2$. The motion is then conditionally periodic. For the fundamental periods $\omega_{11}$ and $\omega_{12}$ one obtains the same expressions as in Ib$\alpha$; the values of the rest are

$$\omega_{21} = \int_{-c}^{\rho_2} \frac{\mu^2 \, d\mu}{\sqrt{S(\mu)}} \quad ; \quad \omega_{22} = \int_{-c}^{\rho_2} \frac{d\mu}{\sqrt{S(\mu)}} \quad .$$

Then the body changes to a satellite of mass K. Here it is of the greatest interest to note, that the trajectory curve of the satellite fills the admissible region with uniform density, and that there is - according to that - no lower limit for the distance of the satellite from the mass K. However, an upper limit is present.

To this case also belongs the one that

$$c > \rho_1 > -c > c_2 \quad .$$

Then the body moves as a satellite around the mass K'.

Case Ib$\delta$. $c > \rho_1 > \rho_2 > -c.$

The body changes to a satellite which moves either about K' or about K. The treatment is the same as in the preceding case.

Case Ic. $r_1$ as well as $r_2$ greater than c.

According to (13) it now follows, that either $\lambda \equiv c$, which case we can pass over, or

$$r_2 \leq \lambda \leq r_1 \quad .$$

The motion occurs inside two ellipses, whose semi-major axes are constant $r_2$ and $r_1$. The magnitude $\lambda$ possesses a libratory motion between the limits $r_1$ and $r_2$. More detailed determination of the motion depends upon the values of the roots $\rho_1$ and $\rho_2$.

Case Ic$\alpha$. $\rho_1$ and $\rho_2$ imaginary.

As in Case Ib$\alpha$, so it follows here, that $\mu$ oscillates between the limits +c and -c. The planet moves in a path which encloses both masses K' and K, and which is either continuous, or fills the space enclosed between both ellipses $r_1$ and $r_2$ with uniform density.

Case Ic$\beta$. $\rho_1$ and $\rho_2$ real and greater than c in absolute magnitude.

According to the presupposition, $r_1 > r_2 > c$ and

$$L(r_1) = L(r_2) = 0 \quad .$$

Further since L $(+\infty)$ is negative, then

$$L(c) < 0 \quad ,$$

which means

$$(K+K')c - h_1 c^2 + \alpha < 0 \quad .$$

However, there now is

$$M(c) = (K-K')c - h_1 c^2 + \alpha < L(c) \quad .$$

Then M (c) must also be negative. But according to the presupposition, the roots of the equation $M(\mu) = 0$ are both real and numerically greater than c. Then $M(\mu)$ must be negative during the motion - for which the condition $|\mu| \leq c$ always occurs - and one finds accordingly, that we will be led back to the Case Ic$\alpha$.

Case Ic$\gamma$. $\rho_1 > c > \rho_2 > -c$.

This case cannot occur. In fact, one finds, as in the preceding case, that

$$M(c) < L(c) < 0 \quad .$$

But we have now, furthermore,

$$M(-c) < M(c) \quad .$$

Therefore, the function M ($\mu$) is subject to the conditon that it is negative for $\mu = +c$ and $\mu = -c$. But it follows therefrom, that the equation

$$M(\lambda) = 0$$

has either two roots or no roots between the values $\mu = +c$ and $\mu = -c$, which is contrary to the presupposition.

Also, two roots cannot exist between these limits. If we let a given x denote some real and positive quantity which is smaller than c, then we know that - as a result of the presupposition - L (x) is negative. But it is now true that

$$M(x) - L(x) = -2K'x \quad ,$$

and then M (x) must also be negative for all positive x, which are smaller than c. Otherwise

$$M(-x) = -(K-K')x - h_1 x^2 + \alpha = M(x) - 2(K-K')x \, ,$$

and since we have here assumed $K > K'$, then M (-x) must also be negative.

Consequently, no roots can occur here (if $r_1 > r_2 > c$) between +c and -c.

Case Ic$\delta$, that $c_1 > \rho_1 > \rho_2 > c$, cannot occur then.

## B.3 THE CONSTANT h POSITIVE

Now we go on to the second main division, namely to the case, that the constant h is positive, and here make the same sub-divisions in reference to the roots.

Since now

$$L(\lambda) = (K+K')\lambda + h\lambda^2 + \alpha \quad , \tag{1}$$

and since further, $L(\lambda)$ must always be positive (or zero) for $\lambda > c$ in accordance with the nature of the motion, then it follows that now $\lambda$ can assume any large value. It is even necessary that $\lambda$ increase limitlessly with time. Since

$$(\lambda^2-\mu^2)\left(\frac{d\lambda}{dt}\right)^2 = 2(\lambda^2-c^2)L(\lambda) = 2(\lambda^2-c^2)h(\lambda-r_1)(\lambda-r_2), \qquad (2)$$

or

$$(\lambda^2-\mu^2)\left(\frac{d\lambda}{dt}\right)^2 = 2h(\lambda+c)(\lambda-c)(\lambda-r_1)(\lambda-r_2),$$

then one finds, if we exclude the coinciding roots for the present, that $\lambda$ must be either larger than the three roots

$$c, \; r_1, \; r_2$$

or smaller than all three. But now $\lambda$ can never be smaller than c, so that $\lambda$ must always be greater than the largest of these three roots or at least equal to this root. Therefore, if $d\lambda$ is negative at the beginning of the motion, then $\lambda$ decreases until this largest root is reached. Then $d\lambda$ changes in sign and $\lambda$ grows continuously without limit. All motions which occur with positive h are of this class. One is then concerned here only with the determination of the minimum value of $\lambda$.

The motion will be influenced, moreover, by the values of the roots $\rho_1$ and $\rho_2$.

Case IIa. $r_1$ and $r_2$ either imaginary, or real and smaller than c.

The lower limit of $\lambda$ is here equal to c. Depending upon the values of $\rho_1$ and $\rho_2$, we obtain

Case II$a\alpha$. $\rho_1$ and $\rho_2$ imaginary.

$M(\mu)$ is always positive, because $M(\infty)$ is positive. Then we must necessarily have

$$\mu \equiv +c \text{ or } -c.$$

The planet moves itself along the X-axis and coincides with one of the masses K or K' or withdraws to infinity, depending upon the sign of $d\lambda$ at the beginning of the motion.

Case IIa$\beta$. $\rho_1$ and $\rho_2$ real and greater than c in absolute magnitude.

If M ($\mu$) is here also positive, then the motion will be as in the previous case. On the other hand, if M ($\mu$) is negative, then the planet can intersect the line K'K once and then departs from K'K in ever greater spirals, since $\mu$ oscillates periodically between +c and -c.

Case IIa$\gamma$. $\rho_1 > c > \rho_2 > -c$ (or $c > \rho_1 > -c > \rho_2$).

The planet makes one revolution around K' (or K) and then departs gradually to infinity, since $\mu$ oscillates periodically between the negative (or positive) X-axis and the hyperbola $\mu = \rho_2$ (or $\rho_1$).

Case IIa$\delta$. $c > \rho_1 > \rho_2 > -c$.

Here $\mu$ oscillates periodically between the values $\rho_1$ and $\rho_2$. The planet intersects the line K'K and then departs to infinity, oscillating periodically between both hyperbolas $\rho_1$ and $\rho_2$.

Case IIb. $r_1 > c > r_2$.

The lower limit for $\lambda$ is here equal to $r_1$. The planet approaches the ellipse $\lambda = r_1$, touches it tangentially and then departs to infinity, since $\lambda$ grows continuously. Here the constant $\alpha$ must be negative.

Both roots $r_1$ amd $r_2$ cannot be greater than c for positive h. One has, namely,

$$r_1 = -\frac{K+K'}{2h} + \sqrt{\frac{(K+K')^2}{4h^2} - \frac{\alpha}{h}}$$

$$r_2 = -\frac{K+K'}{2h} - \sqrt{\frac{(K+K')^2}{4h^2} - \frac{\alpha}{h}} \quad .$$

If the roots are real, then $r_2$ must necessarily be negative.

Case IIb$\alpha$. The roots $\rho_1$ and $\rho_2$ imaginary or real and greater than c in absolute magnitude.

Here $M(\mu)$ will always be negative because $M(0) = \alpha$ is negative. The quantity $\mu$ oscillates between +c and -c. The planet revolves, as it departs, about the masses K and K' in ever greater spirals. The path is a type of externally running spiral.

Case IIb$\beta$. One of the roots $\rho_1$ and $\rho_2$, or both, smaller than +c or greater than -c.

This case cannot occur. Since $r_1 > c$ according to the presupposition, and it is immediately evident that the absolute magnitude of $r_2$ is greater than the absolute magnitude of $r_1$, then $r_2$ must be negative and numerically greater than c. Therefore, the function L(x) does not vanish for those x-values which lie between +c and -c, and remains negative for these values. Now

$$M(x) = L(x) - 2K'x \quad ,$$

from which equation it follows that M(x) must also be negative (and different from zero) for positive x. Otherwise

$$M(-x) = M(x) - 2(K-K')x \quad ,$$

and consequently M(-x) is also negative in the considered region. Therefore, none of the roots $\rho_1$ and $\rho_2$ can lie between +c and -c.

## B.4 h EQUAL TO ZERO

We now have

$$L(\lambda) = (K+K')\lambda + \alpha = (K+K')(\lambda - r) \quad ,$$

$$M(\mu) = (K-K')\mu + \alpha = (K-K')(\mu - \rho) \quad .$$

Here we have two cases to investigate in reference to the value of r, namely

1) $r < c$ ; 2) $r > c$.

Case IIIa. $r < c$.

The function $L(\lambda)$ remains positive during the motion. The quantity $\lambda$ can then decrease continuously, until it has reached the value $\lambda = c$. Then $\lambda$ begins to increase and grows to infinity.

We have

$$r = -\frac{\alpha}{K+K'} < c \quad ,$$

$$\rho = -\frac{\alpha}{K-K'} \quad .$$

Therefore, the absolute magnitude of $\rho$ can, depending upon the conditions, become smaller or larger than c.

Case IIIa$\alpha$. $|\rho| > c$.

Here the function $M(\mu)$ remains negative always, and since

$$(\lambda^2 - \mu^2)\frac{d\mu}{dt} = \sqrt{2(\mu^2 - c^2)M(\mu)} \quad ,$$

then $\mu$ must oscillate periodically between +c and -c.

The planet intersects the line K'K once and then departs to infinity in an externally running spiral which winds about the line K'K. The case is similar to IIb$\alpha$.

Case IIIa$\beta$. $|\rho| < c$.

In order that $M(\mu)$ be negative, $\mu$ must here be smaller than $\rho$. Then the quantity $\mu$ oscillates periodically between $\rho$ and $-c$. The planet intersects the line KK' once and departs to infinity, since it oscillates back and forth periodically between the hyperbola and the positive X-axis. The motion is very peculiar and unexpected.

Case IIIb. $r > c$.

Here the lower limit for $\lambda$ will be equal to r, and the motion takes place outside the ellipse $\lambda = r$. The constant $\alpha$ must be negative $(= -\alpha_1)$, and, according to the presupposition, one has

$$\frac{\alpha_1}{K+K'} > c \quad .$$

Consequently, one must also have

$$\rho = \frac{\alpha_1}{K-K'} > c \quad ,$$

so that only one case remains to consider.

Case IIIb $\alpha$. $r > c$. $\rho > c$.

$M(\mu)$ always remains negative during the motion, and therefore oscillates periodically between the limits +c and -c. The planet touches the ellipse $r = c$ tangentially and then departs to infinity in an externally running spiral which winds about the subject ellipse.

## B.5 TWO OR MORE ROOTS OF THE EQUATION $R(\lambda) = 0$ OR THE EQUATION $S(\mu) = 0$ COINCIDENT. BOUNDED MOTIONS.

If two roots coincide, then bounded motion occurs. We recognize the following cases:

A) $r_1 = r_2 > c$,

B) $r_1 > r_2 = c$,

C) $r_1 = c > r_2$,

D) $r_1 = c = r_2$.

Later we will investigate the different cases in which two roots of the equation $S(\mu) = 0$ coincide.

Case IVA. $r = r_1 = r_2 > c$.

We have

$$(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = \sqrt{2(\lambda^2 - c^2)h(\lambda - r)^2} \quad .$$

For the possibility of some motion, it is necessary that h is positive or $\lambda = r$. I will later analyze the latter case.

Consequently, we assume h positive.

Since the equation

$$(K+K')\lambda + h\lambda^2 + \alpha = 0$$

possesses the double root $\lambda = r$, then

$$(K+K')r + hr^2 + \alpha = 0$$

and $$(K+K') + 2hr = 0 ,$$

so that

$$r = -\frac{K+K'}{2h} .$$

Of course the root must be negative, and consequently cannot be greater than c. This case can thus only occur if h is negative, and then $\lambda$ must be equal to r. It will be investigated in the next paragraph.

<u>Case IVB.</u> <u>$r_1 > r_2 = c$.</u>

Here there is

$$L(c) = (K+K')c + hc^2 + \alpha = 0 ,$$

and one has

$$(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = (\lambda - c)\sqrt{2(\lambda+c)(\lambda-r_1)} .$$

If h is positive, then we must have $\lambda > r_1$, and we return to Case IIb.

Therefore, we assume that h is <u>negative</u> ($= -h_1$). Then the quantity $\lambda$ must be smaller than $r_1$ and approaches the value $\lambda = c$ asymptotically with increasing time.

On reference to the value of the root $\rho$, which can enter into the question here, we note that

$$M(c) < L(c) = 0 \quad ,$$

and - furthermore - that

$$M(-c) = M(c) - 2(K-K')c < M(c) \quad .$$

Consequently, $M(\mu)$ is negative for $\mu = \pm c$, and then either both roots $\rho$ must be situated between the limits +c and -c, or both must lie outside these limits (or be imaginary).

<u>Case IVBα.</u> <u>$\rho_1$ and $\rho_2$ imaginary, or real and greater than c in absolute magnitude.</u>

$M(\mu)$ always remains negative during the motion. Then the quantity $\mu$ oscillates periodically between the limits +c and -c. The planet describes a spiral, which approaches the line K'K asymptotically. This spiral is limited externally by the ellipse $\lambda = r_1$.

<u>Case IVBβ.</u> <u>The roots $\rho_1$ and $\rho_2$ real and smaller than c in absolute magnitude.</u>

Since now

$$M(\mu) = h_1 (\rho_1 - \mu)(\mu - \rho_2) \quad ,$$

then, in order that $M(\mu)$ be negative, $\mu$ must be either greater than $\rho_1$, or smaller than $\rho_2$. In the latter case $\mu$ oscillates periodically between $\rho_2$ and -c, in the former case between $\rho_1$ and +c. The planet describes a pendulum-like motion around K' or K and thereby approaches the line K'K asymptotically.

<u>Case IVC.</u> <u>$r_1 = c_1 > r_2$.</u>

As in the former case, one now obtains

$$(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = (\lambda - c)\sqrt{2(\lambda + c)h(\lambda - r_2)} \quad .$$

Since here $\lambda > r_2$, then h must be <u>positive</u>. The semi-major axis of the ellipse can be optionally large and approaches the value $\lambda = c$ asymptotically with increasing time.

In reference to the value of the root $\rho$, which can enter into the question here, one obtains - as in the former case - the inequalities

$$M(-c) < M(c) < L(c) = 0 \quad .$$

Since now $M(\pm\infty)$ is positive, then one finds that

$$\rho_1 > c \quad ; \quad -c > \rho_2 . \quad .$$

Consequently, the function $M(\mu)$ cannot change in sign during the motion, but remains negative, and then $\mu$ oscillates between the limits $+c$ and $-c$. The motion is similar to that investigated in Case IVB$\alpha$, only here there is no upper limit for $\lambda$.

The case

IVD. $r_1 = r_2 = c$

cannot occur for the same reasons as noted in reference to Case IVA.

Now we come to those cases in which various roots of the equation

$$\left(\frac{d\lambda}{dt}\right)^2 = 0$$

occur. The function $S(\mu) = 0$ has four (4) roots, and all four can appear as limiting points of the allowable region. Therefore we have to investigate the following cases:

K) $\rho_1 = \rho_2$,

L) $\rho_1 = c$ ,

M) $\rho_1 = -c$,

N) $\rho_2 = c$ ,

O) $\rho_2 = -c$.

Since it can happen that three (3) roots coincide:

P) $\rho_1 = \rho_2 = c$ ,

Q) $\rho_1 = \rho_2 = -c$ .

Case VK. $\rho_1 = \rho_2$.

Here one must assume that the root $\rho$ lies between +c and -c, because it can have no effect upon the motion, in other respects. Now we have

$$S(\mu) = 2(\mu^2 - c^2)h(\mu - \rho)^2 ,$$

from which it follows, that either h is positive and then $\mu^2 \equiv c^2$ must be true[1], in which case the motion takes place along the X-axis, or h is negative ($= -h_1$). Let us consider this case. It is

$$M(\mu) = (K - K')\mu - h_1\mu^2 + \alpha .$$

Since $\rho$ is a double root, then one has

$$\begin{aligned} (K-K')\rho - h_1\rho^2 + \alpha &= 0 , \\ K - K' - 2h_1\rho &= 0 , \end{aligned}$$

so that

$$\rho = \frac{K-K'}{2h_1} . \tag{1}$$

If one inserts this value in the first equation, then

$$(K-K')^2 + 4h_1\alpha = 0 , \tag{2}$$

which relation must consequently exist with coefficients so that $\alpha$ is negative ($= -\alpha_1$).

We have further

$$L(\lambda) = (K+K')\lambda - h_1\lambda^2 - \alpha_1 ,$$

and the roots of the equation $L(\lambda) = 0$ are

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{K+K' \pm \sqrt{(K+K')^2 - 4\alpha_1 h_1}}{2h_1} .$$

[1] The case $\mu = \rho$ can then also occur and will be investigated later.

If the quantity $(K - K')^2 - 4\alpha_1 h_1$, which is equal to zero, is now subtracted under the square root, then one obtains:

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{K+K' \pm 2\sqrt{KK'}}{2h_1} = \frac{(\sqrt{K} \pm \sqrt{K'})^2}{2h_1} . \tag{3}$$

Now

$$\begin{aligned}(\sqrt{K} - \sqrt{K'})^2 &= K + K' - 2\sqrt{KK'} = \\ &= K - K' + 2K' - 2\sqrt{KK'} = \\ &= K - K' + 2\sqrt{K'}(\sqrt{K'} - \sqrt{K}) < K - K' ,\end{aligned}$$

and consequently one has

$$r_2 < \frac{K-K'}{2h_1} < c . \tag{4}$$

Only one root r can then be greater than c. Consequently, we have to consider two cases here:

Case VKa. $r_1 < c$.

Here it is necessarily true that

$$\lambda \equiv c ,$$

the motion occurs along the line K'K, and with increasing time the planet approaches the point $\mu = \rho$ without limit, where $\rho$ is determined on this line from (1), and this can be done from one or the other side.

Case VKb. $r_1 > c$.

The motion is bounded by the ellipse $\lambda = r_1$. The planet oscillates between this ellipse and the line K'K, since it gradually approaches the hyperbola $\mu = \rho$ asymptotically with constantly increasing or constantly decreasing $\mu$.

Case VI. $\rho_1 = c > \rho_2$.

The planet approaches the line K' ∞ asymptotically.

There is

$$(K-K')c + hc^2 + \alpha = 0 \quad , \tag{5}$$

and one has

$$S(\lambda) = 2(\lambda - c)^2 h (\lambda - \rho_2)(\lambda + c) \quad .$$

If h is negative, the $\mu$ must oscillate between $\rho_2$ and -c, and we return to the case treated earlier in § 2.[1]

If h is positive, then it follows that one must always have $\mu > \rho_2$. The planet touches the hyperbola $\mu = \rho_2$ once and then approaches the negative X-axis (or a line parallel to it) asymptotically.

Now as for the roots $r_1$ and $r_2$, it can be proven that they cannot be greater than c, for positive h. In fact, according to (5)

$$L(c) = (K+K')c + hc^2 + \alpha > 0 \quad ,$$

and one has

$$L(\lambda) > L(c)$$

for all $\lambda$ greater than c.

Consequently, the function $L(\lambda)$ does not change in sign during the motion, but always remains positive. The quantity $\lambda$ takes its minimum value $\lambda = c$ once. The planet intersects the line K'K once and then asymptotically approaches a line parallel to the X-axis.

[1] If $\rho_2 < -c$, then the motion occurs on the X-axis.

Case VM. $\rho_1 = -c > \rho_2$.

The planet approaches the positive X-axis asymptotically.

One has

$$S(\mu) = 2(\mu + c)^2 h(\mu - \rho_2)(\mu - c) .$$

Since $\mu < c$, then h must be negative ($= -h_1$). Consequently,

$$M(-c) = -(K-K')c - hc^2 + \alpha = 0 ,$$

so that $\alpha$ is positive.

Furthermore

$$L(\lambda) = (K+K')\lambda - h_1\lambda^2 + \alpha ,$$

and L(c) is then positive. However, since $L(\pm\infty)$ is negative, it is obvious that

$$r_1 > c > r_2 .$$

The motion is bounded by the ellipse $\lambda = r_1$. The quantity $\lambda$ oscillates between $\lambda = r_1$ and $\lambda = c$. The negative X-axis - beyond c - can be crossed once and the trajectory curve then approaches the positive X-axis asymptotically in pendulum-like swings with constantly increasing $\mu$- values.

Case VN. $\rho_1 > \rho_2 = c$.

Now one has

$$S(\mu) = 2(\mu - c)^2 h(\mu - \rho_1)(\mu + c) .$$

Here h must be negative (= -h). Consequently,

$$M(c) = (K-K')c - h_1c^2 + \alpha = 0 .$$

Hence it follows that L(c) is positive, and since $L(\pm\infty)$ will be negative, then

$$r_1 > c > r_2.$$

The case is in accordance with the previous VM, only now the planet approaches the negative X-axis asymptotically in this case.

Case VO. $\rho_1 > \rho_2 = -c$.

The planet approaches the positive X-axis asymptotically. In other respects, this case corresponds to VL.

Case VP. $\rho_1 = \rho_2 = c$.

We have

$$M(c) = (K-K')c + hc^2 + \alpha = 0 \quad ,$$
$$M'(c) = K-K' + 2hc = 0 \quad ,$$

so that

$$c = -\frac{K-K'}{2h} \quad ,$$

and hence it follows that h must be negative ($h = -h_1$).

As in Case VK, one finds now

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{(\sqrt{K} \pm \sqrt{K'})^2}{2h_1} \quad ,$$

so that - as in the named case -

$$r_2 < c.$$

Otherwise one finds that here always

$$r_1 > c.$$

The motion is bounded by the ellipse $r_1 = c$, and the trajectory curve will be the same as in Case VN.

Finally, in Case VQ

$$\rho_1 = \rho_2 = -c \; .$$

Here

$$-c = -\frac{K-K'}{2h} \quad ,$$

so that h is positive. The same expressions for $r_1$ and $r_2$ apply as in the former case, so that

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{(\sqrt{K} \pm \sqrt{K'})^2}{2h} \quad ,$$

and both roots are negative.

The function $L(\lambda)$ does not change its sign during the motion and remains positive, and since

$$(\lambda^2-\mu^2)\frac{d\lambda}{dt} = 2(\lambda^2-c^2)\,L(\lambda) \quad ,$$

the quantity $\lambda$ diminishes once to its minimum value $\lambda = c$. Then the trajectory curve crosses the line K'K once, to approach, asymptotically, a line parallel to the X-axis.

This case is similar to VM.

Concerning the remarkable trajectory shape which can occur in this problem, those appearing in IIa $\gamma$ and IIa $\delta$ are perhaps the most peculiar, and they even appear very improbable. Therefore, I will investigate them quite completely.

Here we have in (IIa)

h positive,

$r_1$ and $r_2$ either imaginary or, since they are real, smaller than c.

I will investigate those values of the root $\rho$, which can then occur.

One has

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{-(K+K') \pm \sqrt{(K+K')^2 - 4\alpha h}}{2h} ,$$

and

$$\left.\begin{matrix} \rho_1 \\ \rho_2 \end{matrix}\right\} = \frac{-(K-K') \pm \sqrt{(K-K')^2 - 4\alpha h}}{2h} .$$

First, we intend to assume that $r_1$ and $r_2$ are imaginary. Then one has

$$(K+K')^2 < 4\alpha h .$$

$\alpha$ must therefore be positive, and - further - one has

$$(K-K')^2 < 4\alpha h .$$

Then the roots $\rho_1$ and $\rho_2$ are imaginary, and we find ourselves under the same conditions as in Case II$a\alpha$.

Second, we assume that <u>$r_1$ and $r_2$ are real and smaller than c.</u>

Consequently,

$$(K+K')^2 > 4\alpha h$$

and

$$2hr_2 < 2hr_1 = \sqrt{(K+K')^2 - 4\alpha h} - (K+K') < 2ch ,$$

or

$$(K+K'+2ch)^2 > (K+K')^2 - 4\alpha h > 0 ,$$

or

$$4ch(K+K') + 4c^2h^2 > -4\alpha h ,$$

where $\alpha$ can be negative. One can divide out by 4h and consequently obtain

$$(K+K')c + hc^2 > -\alpha ,$$

which one can also write in the form $L(c) > 0$.

The Forms IIa$\gamma$ and IIa$\delta$ presuppose that $\rho_1$ and $\rho_2$ can assume real values, and that one of these values or both are smaller than c in absolute magnitude.

For the reality of the roots $\rho_1$ and $\rho_2$, the above inequalities present no obstacles in its way. If both roots should be smaller than c in absolute magnitude, then one has

$$K-K'+\sqrt{(K-K')^2-4\alpha h} < 2ch ,$$

or

$$(K-K')^2-4\alpha h < \left[2ch-(K-K')\right]^2 ,$$

and hence one obtains - after some reduction

$$-(K-K')c + hc^2 + \alpha > 0 ,$$

which one can also write in the form M (-c) > 0. It is now evident that

$$M(-c) = L(c) - 2Kc ,$$

and nothing prevents us from having M (-c) > 0 for L (c) > 0. Then both roots $\rho_1$ and $\rho_2$ can be real and numerically smaller than c.

We now consider the corresponding differential equations

$$(\lambda^2-\mu^2)\frac{d\lambda}{dt} = \sqrt{2(\lambda^2-c^2)h(\lambda-r_1)(\lambda-r_2)} ,$$

$$(\lambda^2-\mu^2)\frac{d\mu}{dt} = \sqrt{2(\mu^2-c^2)h(\mu-\rho_1)(\mu-\rho_2)} ,$$

where

$$c > r_1 > r_2$$

$$c > \rho_1 > \rho_2 > -c .$$

In order that the quantities under the square root become positive now, the quantities $\lambda$ and $\mu$ must obviously fulfill the following inequalities

$$\lambda \geq c$$

$$\rho_1 \geq \mu \geq \rho_2 .$$

We establish

$$\frac{dw_1}{dt} = \frac{\sqrt{2(\lambda+c)h(\lambda-r_1)(\lambda-r_2)}}{\lambda^2-\mu^2},$$

$$\frac{dw_2}{dt} = \frac{\sqrt{2(c^2-\mu^2)h}}{\lambda^2-\mu^2}.$$

The quantities under the radical sign can never become zero, and the auxiliary quantities $W_1$ and $W_2$ consequently increase constantly with time.

Now

$$\frac{d\lambda}{dw_1} = \sqrt{\lambda-c}$$

$$\frac{d\mu}{dw_2} = \sqrt{(\rho_1-\mu)(\mu-\rho_2)},$$

and hence one obtains-through integration

$$\lambda = c + \frac{1}{4} w_1^2$$

$$\mu = \rho_1 \cos^2 \frac{1}{2} w_2 + \rho_2 \sin^2 \frac{1}{2} w_2 .$$

The planet oscillates between both hyperbolas $\rho_1$ and $\rho_2$ and departs simultaneously and constantly to infinity.

Consequently, the case IIa$\delta$ exists. As for Case IIa$\beta$, it appears again in IIIa$\beta$.

## B. 6 PERIODIC MOTIONS

Motions which are periodic in time, can occur in the cases treated in §2, as often as the condition §1 (10) is fulfilled. Moreover, the motion will be periodic, whenever the body moves along a curve

$$\lambda = \text{constant},$$

and also under the conditions when the trajectory curve has the equation

$$\mu = \text{constant}.$$

We want to first consider these cases.

Case VIa. $\lambda$ = constant.

Since

$$(\lambda^2-\mu^2)\frac{d\lambda}{dt} = \sqrt{2(\lambda^2-c^2)L(\lambda)} \quad , \tag{1}$$

it is evident that $\lambda$ is either equal to c, or coincides with a root of the equation

$$L(\lambda) = 0 \quad . \tag{2}$$

In the first case, the body moves along the line K'K and we already know that–in so doing–only three (3) cases can occur, either that a collision with one of the attractive masses occurs, or that the planet approaches a point on the line K'K asymptotically (VKa) or that the planet departs to infinity. Consequently, we can pass over this case.

There still remains the case, that $\lambda$ coincides with a root $r_1$ and $r_2$.

If $r_1 > c > r_2$, then it is not possible that $\lambda$ coincide identically with $r_1$. Namely, we know, that libration between $r_1$ and c must then occur for negative h in $\lambda$ (Ib) and $\lambda$ increases to infinity for positive h (IIb).

If $r_1 > r_2 > c$, then h must necessarily be negative, because at least one root must be negative for positive h. Thus, we have

$$(\lambda^2-\mu^2)\frac{d\lambda}{dt} = \sqrt{2(\lambda^2-c^2)h_1(r_1-\lambda)(\lambda-r_2)} \tag{3}$$

If we exclude the case $\lambda \equiv c$, then here a libratory motion between $r_1$ and $r_2$ always occurs, and $\lambda$ can only remain constant under the single condition, that

$$r_1 = r_2 \quad .$$

Since a double root occurs here, then

$$(K+K')^2 - 4\alpha h = 0$$

and it follows that

$$(K-K')^2 - 4\alpha h < 0 \quad ,$$

so that $\rho_1$ and $\rho_2$ become imaginary. The quantity $\mu$ oscillates accordingly between $+c$ and $-c$, and the motion proceeds along the ellipse $\lambda = r$, where now (according to IVA)

$$r = \frac{K+K'}{2h_1} \quad .$$

A singular solution of the differential equation (3) is always

$$\lambda = r_1$$

or $\lambda = r_2$. However one finds that the <u>second</u> differential quotient of $\lambda$ with respect to time then has a finite value. However, if $\lambda = r$ denotes a double root, then not only the second differential quotient but also all differentials of higher order for $\lambda = r$ will vanish.

Case VIb. $\mu$ = constant.

If we here exclude the cases $\mu = \pm c$, then the differential equation

$$(\lambda^2-\mu^2)\frac{d\mu}{dt} = \sqrt{2(\mu^2-c^2)h\,(\mu-\rho_1)(\mu-\rho_2)} \quad , \tag{6}$$

is satisfied only through the values $\mu = \rho_1 = \rho_2 = \rho$, if $\mu$ should be constant.

VIb$\alpha$. Then one has

$$(\lambda^2-\mu^2)\frac{d\mu}{dt} = \sqrt{2(c^2-\mu^2)h_1(\mu-\rho)^2}$$

for negative h $(= -h_1)$, and here $\mu = \rho$ is an unstable solution.

Because $\rho$ is a double root, one has

$$\rho = \frac{K-K'}{2h_1} < c \tag{7}$$

and therefrom

$$(K-K')^2 - 4\alpha h = 0 \ . \tag{8}$$

Furthermore

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{-(K+K') \pm \sqrt{(K+K')^2 - 4\alpha h}}{2h}$$

or - according to (8)

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{-(K+K') \pm 2\sqrt{KK'}}{2h} = \frac{(\sqrt{K} \pm \sqrt{K'})^2}{2h_1} \quad .$$

Then we have

$$\rho = \sqrt{r_1 r_2} \quad . \tag{9}$$

Hence it follows that $r_1$ and $r_2$ cannot <u>both</u> be greater than c. However, it is possible that

$$r_1 > c > r_2.$$

Now one has

$$L(\lambda) = h_1 (r_1 - \lambda)(\lambda - r_2)$$

and $\lambda$ oscillates between $r_1$ and c.

In this case the planet will perform a pendulum-like motion along the hyperbola

$$\mu = \frac{K-K'}{2h_1} \quad .$$

Whenever $K = K'$, then $\mu = 0$, and the planet moves back and forth along the Y-axis to both sides from the coordinate origin.

VIb$\beta$. <u>If h is positive</u>, then one obtains - for <u>coinciding</u> $\rho$ - values

$$\rho = -\frac{K-K'}{2h_1}$$

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{(\sqrt{K} \pm \sqrt{K'})^2}{2h} \quad .$$

Here both roots r are negative. The function $L(\lambda)$ remains positive during the motion. The magnitude $\lambda$ increases to infinity.

In this case the planet moves along the hyperbola

$$\rho = -\frac{K-K'}{2h}$$

to infinity.

Periodic trajectory shapes can also occur whenever the fundamental periods $\omega_{12}$ and $\omega_{22}$ have such values that

$$m_1\omega_{12} + m_2\omega_{22} = 0 \quad ,$$

where $m_1$ and $m_2$ denote whole numbers. Since one can select $\omega_{12}$ and $\omega_{22}$ as positive, the numbers $m_1$ and $m_2$ are of different signs. Consequently, we write - as a better choice -

$$m_1\omega_{12} - m_2\omega_{22} = 0 \quad , \tag{10}$$

where $m_1$ and $m_2$ now denote positive numbers. Then one obtains the value

$$2T = 2m_1\omega_{11} - 2m_2\omega_{21} \tag{11}$$

for the corresponding period.

If one assumes a specific value for the integration constant h - among the values possible in each case - then the other integration constant $\alpha$ will satisfy the equation (10) for an infinite series of (discrete) values, and vice versa. The periodic cases then comprise a two-fold infinite set.

The smaller the numbers $m_1$ and $m_2$ become, the "simpler" will the corresponding periodic motion be. Consequently, the trajectory curve will take the simplest form whenever $m_1 = m_2 = 1$. In general, one can also select the integration constants so that this case occurs. As - for example - in Ib$\gamma$ or Ic$\alpha$. Then we obtain a periodic curve which does not cross itself. But in Case Ib$\alpha$, such a choice will hardly be possible. It appears that the lowest values for $m_1$ and $m_2$ are here $m_1 = 2$, $m_2 = 1$.

If we consider the quotient

$$\nu = \frac{\omega_{12}}{\omega_{22}} ,$$

which we can assume smaller than unity - since otherwise similar conclusions apply for $\omega_{22}/\omega_{12}$ - then a periodic motion will occur as often as $\nu$ denotes a rational number. Since now many rational numerical values are present always without limit inside an arbitrarily small section, as small as this section is selected, so will the periodic trajectory curves, among all trajectory forms which can occur for $0 < \nu < 1$, be distributed with uniform density. Consequently, one needs only to establish an infinitely small variation of $\nu$, in order to change from an arbitrary periodic trajectory curve to an adjacent one which is not periodic, and vice versa.

According to Cantor, one terms an infinite set of things as enumerable, if one can number the elements with 1, 2, . . . . , n. . . so that no element will be omitted. The rational numbers between 0 and 1 constitute a numerical aggregate; in fact one can write them in the following order

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6} \cdots\cdots ,$$

so that one writes down-after each other-those numbers whose denominator is 2, 3, 4, 5, 6, etc., and-for each denominator-the values 1, 2, 3, etc. are assigned to the numerator, with the omission of those values whose numerators and denominators are relative prime numbers.

The periodic trajectory curves comprise a numerical aggregate; on the other hand, this is not the case with the non-periodic trajectory curves. Hence the aggregate of the latter is - according to the terminology of Cantor - of higher order than the aggregate of the periodic curves.

If we consider the non-periodic trajectories, then we know, according to § 2 in the second section, that one can consequently develop the distances r and r' (of the planet) from the attracting masses - in a Fourier series of the form

$$\sum_{\substack{i_1\\i_2}\}=-\infty}^{+\infty} C_{i1,i2} \cos(i_1 u_1 + i_2 u_2) \quad . \tag{12}$$

These series are uniformly convergent[1] and $u_1$ and $u_2$ designate linear functions of time and of the form

$$\begin{cases} u_1 = n_1 t + \tau_1 \quad , \\ u_2 = n_2 t + \tau_2 \quad , \end{cases} \tag{13}$$

where

$$n_1 = \frac{\pi\omega_{12}}{\Omega} \quad , \qquad n_2 = -\frac{\pi\omega_{22}}{\Omega} \quad . \tag{14}$$

For these series one can make some interesting observations, which appear to be of great interest for the solution of the problem of three bodies. In order to obtain the series (12), one can,namely, make use of an approximation method similar to that customary in the "perturbation theory". If we keep in mind, for example, the "satellite" of Case Ib$\delta$, in which the moving body must always remain in the vicinity of the mass K, and we assume the mass K' as relatively small; then in order to obtain the expression for the coordinates we can make use of a development for the potential of the small mass K'. With the determination of the successive approximation values of the coefficient $C_{i1', i2'}$ one would then (probably) have to contend with the difficulties developed through the so-called small divisors - of which more will be said in the following - which are of the form $i_1 n_1 + i_2 n_2$, and the series in the different approximations would not be uniformly convergent, although this is the case with the real series (12). Once can even say in advance, why such difficulties must be encountered here. The explanation appears to lie in the fact that - as was proven in § 3 in the second section - the distance r from the body K in this case possesses no lower limit different from zero. For this reason, there exists no average value for this distance in the sense that one uses this concept in the perturbation theory.

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[1] One finds the proof for this by Weierstrass elsewhere.

In fact, it appears to me possible that one has to look into the perturbation theory for the explanation of the remarkable convergence condition of the series, under similar circumstances.

## B.7 CLASSIFICATION OF THE DIFFERENT TRAJECTORY FORMS, WHICH CAN OCCUR WITH THE ATTRACTION OF A BODY TO TWO FIXED CENTRES

1) Straight line motion: The planet moves along the line K'K or its extension.

The planet moves in the initial direction, until it coincides with one of the masses. Ia$\alpha$, Ia$\alpha$, Ia$\beta$, IIa$\alpha$ among others;

or the planet moves in the initial direction, until it reaches a certain point; then turns about and coincides with a mass after some time. Ia$\gamma$, Ia$\delta$ among others;

or departs to infinity along the X-axis. IIa$\alpha$;

or approaches - without limit - a point situated between K' and K, without reaching it in a finite time. VKa.

2) Lemniscate motion: If the motion is not nearly periodic, the trajectory curve fills the entire space enclosed within an ellipse with uniform density, Ib$\alpha$, Ib$\beta$. Figure 2.

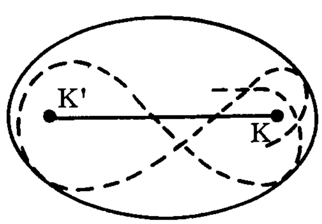


Figure 2

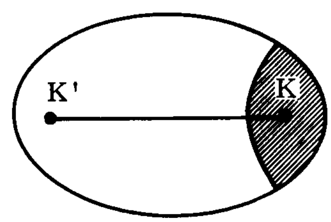


Figure 3

3) Satellite motion: The planet moves within a distinct space, in which one mass K or K' is located. The boundary of this space constitutes part of an ellipse and a hyperbola. The trajectory curve is either periodic or fills the space in question with uniform density. Ib$\gamma$, Ib$\delta$. Figure 3.

4) Planetary motion: The planet moves (as an outer planet) in a space enclosed by two confocal ellipses. The trajectory curve is either periodic, in which case it touches both ellipses, or it fills the space in question with uniform density. Ic$\alpha$, Ic$\beta$. Figure 4.

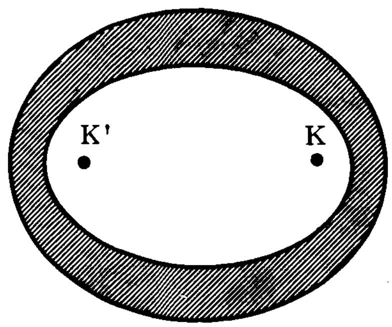


Figure 4

5) Diverging pendulum-motion: The planet departs to infinity from either centre, while it oscillates back and forth in ever larger pendulum-like swings between both branches of a hyperbola to both sides of the X-axis. IIa$\gamma$, IIIa$\beta$. Figure 5.

6) Hyperbolic sinusoid-motion: The planet departs to infinity, while it oscillates back and forth periodically between two confocal hyperbolas. IIa$\delta$. Figure 6.

7) Diverging spiral-motion: The planet departs to infinity, while it performs ever larger spirals around the line K'K. IIa$\beta$, IIb$\alpha$, IIIa$\alpha$, IIIb$\alpha$ Figure 7.

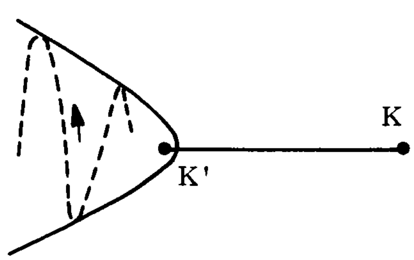


Figure 5

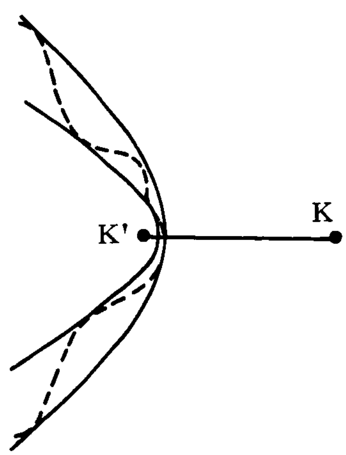


Figure 6

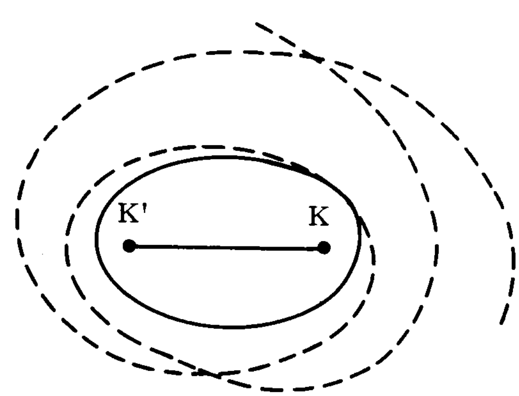


Figure 7

8) <u>Convergent spiral-motion</u>: The planet approaches the line K'K asymptotically in ever smaller spirals, without reaching it in finite time. IVB$\alpha$, IVC. Figure 8.

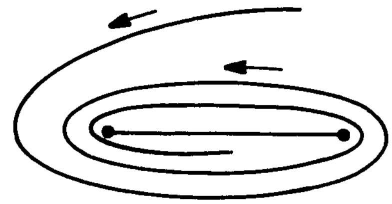

Figure 8

9) <u>Converging pendulum-motion</u>: The planet approaches either a hyperbola (VKb) or the X-axis asymptotically in pendulum-like swings on both sides of the X-axis, without reaching these limits in finite time. VKb, IVB$\beta$, VM, VN, VP. Figures 9 and 10.

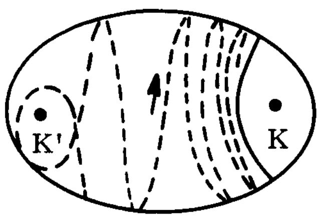


Figure 9

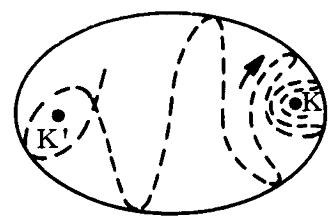


Figure 10

10) <u>Asymptotic-straight line motion:</u> The planet approaches a line parallel to the X-axis asymptotically. VL, VO. Figure 11.

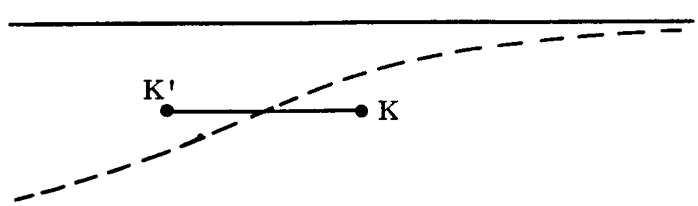


Figure 11

11) <u>Elliptical motion</u>: The planet moves in a certain direction along an ellipse, whose equation is

$$\lambda = \frac{K+K'}{2h_1} . \qquad \text{VIa.}$$

12) <u>Hyperbolic motion</u>: The planet moves along a hyperbola

<u>either</u> so that it departs to infinity along the hyperbola. Then the focus of this hyperbola lies in the <u>larger</u> mass (K). VIb$\beta$;

<u>or</u> so that it oscillates back and forth pendulum-like along the hyperbola around the X-axis. Then the focus of this hyperbola lies in the <u>smaller</u> mass (K'). VIb$\alpha$. Figure 12.

<u>All forms of motion considered here (with the exception of VIb$\alpha$) are stable, and one cannot change the integration constants (h and $\alpha$) from one form to another, with an infinitesimally small variation.</u>

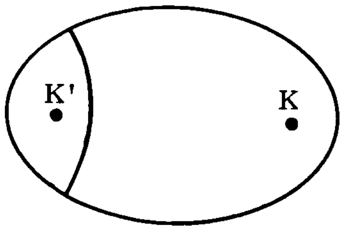


Figure 12

The names (introduced above by me) for the different motion forms do not always give - with certainty - an adequate expression for the corresponding trajectory curves. However, they appear to me to be not too confusing. For detailed description, I refer to the preceding paragraphs.

In our classical work on the elliptic integral, Legendre has devoted a detailed investigation of the problem of the attraction to two fixed centres.[1]

He has limited himself therein to the case that h is <u>negative</u>, in which case as we have seen, the trajectory curves are finite. He has treated Numbers 1, 2, 3, 4, 8, 9, 11, 12 of the

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[1] "Traité des Fonctions elliptiques" T.I.

motions possible here. In reference to the converging pendulum motion Number 9, however, he has only treated the case that the planet approaches the X-axis asymptotically (Fig. 10). He appears to have overlooked the general case (Fig. 9), that the planet approaches a hyperbola asymptotically. It is noteworthy that the important characteristic of the non-periodic trajectories, to fill the admissible region with uniform density, were not unknown to Legendre. At least he mentions this explicitly in reference to the planetary motion Number 4.

The straight-line motion was treated by Legendre under the supposition that the planet can pass through the masses K and K'. I have considered it advisable to let the motion terminate with the coincidence, because the physical sense of the motion and the validity of the differential equations cease to apply here.

In the treatment of this problem, Legendre relied upon his profound investigations of the elliptic integrals. However, the treatment was thereby unnecessarily detailed and difficult.

On the other hand, discussion of the motion progresses - as we have seen - almost without calculation work and without detailed formulae. It would provide no advantage, if one were to introduce the elliptic functions instead of the integrals of Legendre. This is superfluous for the discussion of the forms of motion and it is only a greater detour in the computation of the value of the coordinates at an arbitrary time, since one does not hereby obtain the coordinates through the time, but the time is expressed through the coordinates. But through the formula (12), the coordinates will be expressed directly as functions of the time and the coefficients in the series can be calculated always and with relative ease.

## B.8 EXAMPLES

Although no examples are known in nature, in which the motion of a body is determined by the attraction of two fixed centres, whenever we are concerned with three bodies which are mutually attracted according to the Newtonian law, in certain cases it may be justifiable to assume that the problem of the attraction of two fixed centres can approximately lead to an understanding of the trajectory curve.

* Such a case would be present, for example, if one were to investigate the motion of a small body which passes through a double-star system with high velocity.

Also, in our planetary system, examples are not lacking in which it would be possible to obtain an approximation of the true trajectory. For example, if one considers the system which consists of the sun, a planet and a satellite belonging to it, then the angular velocity of the planet around the sun can be considered as very small whenever the satellite lies sufficiently near the planet, and one could then consider the sun as stationary at least for a short time, and the satellite as attracted by two fixed centres. If we are dealing with the motion of a small planet under the attraction of the sun and a large planet - Jupiter or Saturn -, then the coordinates of the planet can be generated according to the power of the angular velocity of the large planet (as will be shown in one of the following sections) and one will thereby be led to an approximation method in which the problem of the attraction of two fixed centres would give the first approximation. Of course, the convergence of this approximation will not be investigated; nevertheless it may be of interest to undertake an examination of the trajectories which one would get in the first approximation.

Suppose that a body is found on the connecting line K'K between the sun (K) and a planet (K'), and that at the beginning of the motion this body is thrown out with a velocity perpendicular to this line, whereby K' and K will be considered as stationary; its motion should be investigated under the assumption that the body will be attracted by K' and K according to the Newtonian law.

Consequently, we have to determine the integration constants h and $\alpha$, and then to calculate therefrom the roots $r_1$, $r_2$, $\rho_1$, $\rho_2$.

According to formula (6) and (5*) § 1, one has the following formulae for the computation h and $\alpha$:

$$h = \frac{1}{2}(\lambda^2-\mu^2)\left[\frac{\lambda'^2}{\lambda^2-c^2}+\frac{\mu'^2}{c^2-\mu^2}\right]-\frac{(K+K')\lambda}{\lambda^2-\mu^2}+\frac{(K-K')\mu}{\lambda^2-\mu^2}, \tag{1}$$

$$\alpha = \frac{(\lambda^2-\mu^2)\lambda'^2}{2(\lambda^2-c^2)} - (K+K')\lambda - h\lambda^2 \quad , \tag{2}$$

or

$$-\alpha = \frac{(\lambda^2-\mu^2)\mu'^2}{2(c^2-\mu^2)} + (K-K')\mu + h\mu^2 \quad . \tag{2*}$$

In the above, we have to insert the values $\lambda_o$, $\mu_o$, $\lambda_o'$ and $\mu_o'$ for the inception of motion, in order to obtain the values of h and $\alpha$.

We select the unit of length so that c = 1. The distance K'K is then equal to 2. If the body lies in the interval a from K', then

$$\lambda_0 = 1 \quad ; \quad \mu_0 = 1 - a \quad . \tag{3}$$

In order to obtain the values of the differential quotients $\lambda_o'$ and $\mu_o'$, we make use of the equations I §7 and consequently obtain - since according to the given assumptions

$$x_0' = 0 \quad , \quad -$$

the following equations:

$$0 = \mu_0 d\lambda_0 + \lambda_0 d\mu_0$$

$$dy_0 = \sqrt{c^2-\mu_0^2}\,\lambda_0 \frac{d\lambda_0}{\sqrt{\lambda_0^2-c^2}} - \sqrt{\lambda_0^2-c^2}\,\mu_0 \frac{d\mu_0}{\sqrt{c^2-\mu_0^2}} \quad ,$$

or according to the values obtained for $\lambda_o$ and $\mu_o$

$$0 = (1-a)\lambda_0' + \mu_0'$$

$$y_0' = \sqrt{1-(1-a)^2}\,\frac{\lambda_0'}{\sqrt{\lambda_0'-1}} \quad .$$

Then

$$\lambda_0' = \mu_0' = 0 \quad , \tag{4}$$

and

$$\frac{\lambda_0'}{\sqrt{\lambda_0^2-1}} = \frac{y_0'}{\sqrt{1-(1-a)^2}} \quad . \tag{4*}$$

If one inserts the values (3), (4) and (4*) in 1 and 2, then

$$h = \frac{1}{2} y_0'^2 - \frac{K}{2-a} - \frac{K'}{a} \quad , \tag{5}$$

$$-\alpha = (K-K')(1-a) + h(1-a)^2 . \tag{5*}$$

Whenever the values of the masses K and K', the distance a and the initial velocity $y_o'$ are given, then the values of h and $\alpha$ are hereby determined.

In reference to a, I will now make two different assumptions, the one corresponding to the case that we are dealing with a planet which moves between the sun and the disturbed planet, the other corresponding to a satellite case.

However, I note first that the equation (5*) can be written in the following form:

$$M(1-a) = 0 \quad ,$$

where $M(\mu)$ has the same meaning as in the previous paragraph. Consequently, one of the roots $(\rho_1, \rho_2)$ is equal to 1–a, namely equal to that value which $\mu$ has at the beginning of the motion. Then the one boundary of the admissible region for the trajectory curve goes through that point in which the trajectory curve (at the beginning of the motion) crosses the line K'K at right angles.

In order to obtain a simple planetoid-like example, we set

a = 1. [1]

Now according to (5*)

$$\left.\begin{aligned} \alpha &= 0 \\ h &= \tfrac{1}{2} y_o'^2 - K - K' \quad . \end{aligned}\right\} \tag{6}$$

We determine the initial velocity $y_o'$ in such a way that the planetoid would move in a circle about the sun (K) whenever the attraction of the planet K' ceases. Then – as one finds in the following paragraph –

$$y_o'^2 = \frac{K}{r} = K \quad , \tag{7}$$

[1] This would approximately describe a small planet which is located in the mean distance of the asteroids, and which will be disturbed by Jupiter.

so that

$$h = -\frac{K}{2} - K' \quad . \tag{8}$$

Now in order to find the boundary curve of the admissible region, one has to calculate the roots of the equations

$$L(\lambda) = 0$$

and

$$M(\mu) = 0 \quad ,$$

according to the previous paragraph. Since $\alpha = 0$, we simply obtain

$$L(\lambda) = (K+K')\lambda + h\lambda^2 \quad ,$$
$$M(\mu) = (K-K')\mu + h\mu^2$$

or

$$L(\lambda) = (K+K')\lambda - (\tfrac{1}{2}K+K')\lambda^2 \quad ,$$
$$M(\mu) = (K-K')\mu - (\tfrac{1}{2}K+K')\mu^2 \quad .$$

Then we have

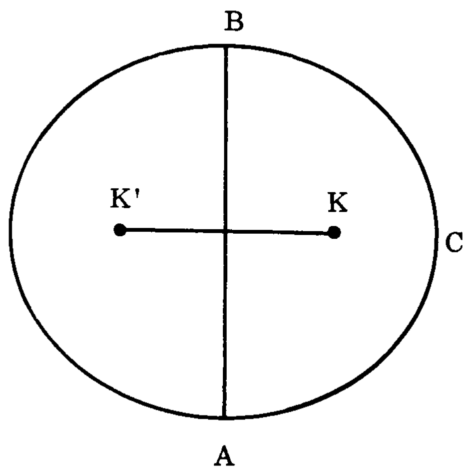
$$r_2 = \frac{2(K+K')}{K+2K'}$$
$$r_1 = 0$$
$$\rho_1 = \frac{2(K-K')}{K+2K'}$$
$$\rho_2 = 0 \quad .$$

According to the notation of the preceding paragraph, one has

$$r_1 > c > r_2$$
$$\rho_1 > c > \rho_2 > -c \quad ,$$

and we find ourselves in Case Ib$\gamma$ in § 2.

Now we find the following limits for $\lambda$ and $\mu$ -




$$\frac{2(K+K')}{K+2K'} \geq \lambda \geq 1$$

$$0 \geq \mu \geq -1 .$$

Figure 13

The motion is a libratory motion. For $\mu = 0$, one obtains the Y-axis A O B. The ellipse

$$\lambda = \frac{2(K+K')}{K+2K'}$$

has approached the semi-major axis equal to 2, because the mass K' is small. The motion occurs inside the region ABC, which is filled by the trajectory curve with uniform density. It is noteworthy that for each value of K' the one boundary of the admissible region will always be formed from the Y-axis.

Let us now pass on to the second case, that the small body very near to the planet K' is thrown out perpendicular to the line K'K. If we assume that the body possesses such an initial velocity that it would move in a circle about K' if the attraction of the sun could be neglected, then one has

$$y_0'^2 = \frac{K'}{a} ,$$

and consequently

$$h = -\frac{K}{2-a} - \frac{K'}{2a} , \tag{9}$$

where $\alpha$ now denotes a small quantity.

Will the body move about K or about K'? In order to investigate this, we must calculate the roots $r_1$, $r_2$, $\rho_1$, $\rho_2$. Now,

$$M(\mu) = (K-K')\mu + h\mu^2 + \alpha$$
$$M(1-a) = (K-K')(1-a) + h(1-a^2) + \alpha = 0 \;,$$

so that - if M(1-a) is subtracted from $M(\mu)$-

$$M(\mu) = \left[K - K' + h(\mu + 1 - a)\right]\left[\mu - (1-a)\right] .$$

For the one hyperbola which bounds the region inside which the body can move, the semi-major axis is equal to 1 - a. Now the other hyperbola is located near K, so the body must move around K' or else around K. The condition for the fact that the body will be a satellite around K' is

$$-\frac{K-K'+h(1-a)}{h} < 1-a \quad ,$$

or if one inserts the value for h,

$$\frac{K'}{a^2} > \frac{K}{2-a} \quad . \tag{10}$$

For the earth's moon -

$$K = 320\;000\,K'$$
$$200\,a = 1 \quad ,$$

and one finds that here

$$\frac{K'}{a^2} < \frac{K}{2-a} \quad ,$$

and consequently a body which would be found (under the mentioned conditions) in the interval from the moon, will move around the sun and not around the earth, if both were considered stationary. This is true also, if one makes use of the synodic instead of the sidereal angular velocity of the moon.

This seems somewhat surprising at first sight. Perhaps one would expect that the body would surely move around the earth. However, with more careful reflection one finds that the result cannot be otherwise, since one has here neglected the attraction of the sun on the earth. In fact the direct attraction of the sun on the moon is almost twice as large as the direct attraction of the earth upon the same body. The inequality (10) may be simply stated: that the body moves around the sun whenever the twofold value of the attraction of the sun is greater than the attraction of the planet.

Consequently, one would not be able to use the attraction of two fixed centres as a first approximation for our earth's moon.

If one were to select - instead of this - for example the inner Mars' moon Phobos, then one would find that the attraction of Mars on Phobos is 200 times greater than the attraction of the sun on this satellite. For the Neptune satellite, the attraction of the major planet is more than 8000-fold of the direct attraction of the sun. In this case, one can then consider the sun as stationary in the first approximation.