Growth and Decay of Magnetic Fields
in Turbulent Fluids

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ABSTRACT

We investigate the magnetohydrodynamic properties of random, stationary turbulence. If the fluid is highly conducting and the magnetic field weak, the logarithm of the field strength B is shown to be a random-walk variable to which the central limit theorem applies. The expectation value of B is then shown to increase exponentially with time. The behavior of infinitesimal material displacements which move frozen in the fluid, and which are, on the average, stretched by the turbulence is investigated. It is well-known that for high conductivity the quantity $\mathbf{B}$/density in a given fluid element remains proportional to the length of such a displacement chosen initially parallel to $\mathbf{B}$. Further, we show, with the use of a special gauge condition, that the magnitude of the vector potential A/density remains proportional to the area of an infinitesimal material parallelogram chosen perpendicular to $\mathbf{A}$. The central limit theorem for two random variables then gives the joint probability density for A and B. If $\mathbf{B}$ has no large-scale Fourier component (imposed by boundary conditions), the scale of the field is $\approx$ A/B. The resulting ohmic dissipation grows much faster than the random-walk energy input from the fluid. Plausible extension to high ohmic dissipation is made, and we conclude that in the absence of a large-scale Fourier component, the ohmic loss destroys the field. An experiment is proposed to test these conclusions.

I. INTRODUCTION

The problem of the amplification and maintenance of magnetic fields in turbulent conducting fluids has long interested hydrodynamicists and astrophysicists. One opinion about the behavior of such systems that is generally agreed upon is that under suitable conditions an initial magnetic "seed" field will, on the average, be amplified by the turbulent motion (Biermann and Schlüter[1], Batchelor[2,3], Syrovatski[4], Saffman[5,6], Kraichnan and Nagarajan[7], and Pao[8]). However, the amplification process is also accompanied by a steady decrease in the scale of the field, so that the ohmic losses may in the long run overtake the amplification and wash the magnetic field out of existence.

Many authors have conjectured that turbulence will indeed finally destroy a magnetic field (Zeldovich[9], Saffman[5], and Parker[10]), but no convincing proof of this conjecture has appeared. Zeldovitch[9] proved that two-dimensional turbulence will, under certain rather restrictive boundary conditions, destroy a magnetic field, but it seems impossible to generalize his method to three dimensions.

The present paper is concerned with setting up a statistical formalism for stationary random turbulence which is exact in the case of high conductivity and weak magnetic fields.

By "weak" we mean that the Lorentz forces may be neglected in the fluid equations of motion. We use the central limit theorem to derive an exponential law of magnetic field increase with time by showing that the logarithm of the field strength in a given fluid element is a good random-walk variable. Biermann and Schlüter[1] were the first to surmise the exponential increase of the field strength. Our analysis is related in some ways to that of Parker[10], who also derived an exponential increase, and we are able to provide a more nearly rigorous basis for some of his results, with which we are in substantial agreement.

We derive a similar exponential law for the magnitude of the vector potential and are then able to show that the ohmic dissipation increases at a much faster rate than the magnetic field energy. Whether or not a final steady state can be maintained is shown to depend in an important way on boundary conditions, and we argue plausibly, although not rigorously, that if the magnetic field is not imposed from the outside by boundary conditions, the turbulence must eventually destroy the field, no matter how large the conductivity.

If the field is initially strong enough to suppress the turbulence to some extent, this conclusion may not necessarily be valid.

A feasible laboratory experiment using liquid sodium is proposed which might show whether or not the general conclusions of this paper are valid. It would also provide a test of Batchelor's criterion[2] for the maintenance of turbulent fields, with which the present paper disagrees.

2. LOGARITHMIC FIELD STRENGTH AND VECTOR POTENTIAL AS RANDOM WALK VARIABLES

The equations of motion of the magnetic field $\underset{\sim}{B}$ are, in Gaussian electromagnetic units,

$$\frac{d\underset{\sim}{B}}{dt} \equiv \frac{\partial \underset{\sim}{B}}{\partial t} + (\underset{\sim}{u}.\underset{\sim}{\nabla})\underset{\sim}{B} = (\underset{\sim}{B}.\underset{\sim}{\nabla})\underset{\sim}{u} - \underset{\sim}{B}(\underset{\sim}{\nabla}.\underset{\sim}{u}) - \underset{\sim}{\nabla}\times(\lambda\underset{\sim}{\nabla}\times\underset{\sim}{B}), \tag{1}$$

where $\underset{\sim}{u}$ is the fluid velocity field and $\lambda = (4\pi\sigma)^{-1}$ is the magnetic diffusivity, σ being the conductivity in sec/cm^2. If we assume that the conductivity is infinite, then the equations become, in three-dimensional Cartesian tensor form, with a comma indicating partial differentiation,

$$\frac{dB_a}{dt} = B_n u_{a,n} - B_a u_{n,n} = T_{an} B_n, \tag{2}$$

where $T_{ab} \equiv u_{a,b} - \delta_{ab} u_{n,n}$.

The material derivative d/dt follows the motion of a given fluid element as it experiences the various velocity shears and compressions. In what follows we investigate the behavior of the field and vector potential in the given fluid element as it moves about in the turbulent medium. Thus we suppress the spatial dependence of the dependent variables and consider explicitly only their time dependence.

2.1 The Field Strength

We consider the fluid motions to be turbulent and random, and thus we assume that the tensor $T_{ab}(t)$ is a random function and that it is, for weak fields, independent of the field strength B(t) in the fluid element. Let us substitute $B_a = Bn_a$ in Eqn. (2), where $n_a(t)$ is a unit vector, and then form the scalar product with n_a, using the fact that $n_a dn_a/dt = 0$, to obtain

$$\frac{d \ln B}{dt} = n_a n_b T_{ab}. \tag{3}$$

It is thus apparent that lnB(t) can be used as a random-walk variable, since the right-hand side of this equation depends only on $T_{ab}(t)$ and on the direction of the field. We now break up the time axis into steps of constant length δt, such that the change in lnB over δt is essentially uncorrelated with the changes over the previous steps. If τ is the time scale of the larger components of the turbulence, we might take $\delta t \gtrsim 2\tau$. It is clear that since $T_{ab}(t)$ should be continuous in t, we can never completely lose the correlation with previous values of T_{ab}. However, for $\delta t \gtrsim 2\tau$, these correlations should become very small[11].

We can now use the central limit theorem to obtain the probability density for $\ln B \equiv b$ after a large number N of time steps. Let $\delta_j b \equiv \delta_j \ln B$ be the change in $\ln B$ over the jth time interval. For stationary turbulence, the probability density function for $\delta_j b$ will not depend on j, and we thus write $\mu \equiv \langle \delta_j b \rangle$ and $\theta_1{}^2 \equiv \langle (\delta_j b)^2 \rangle - \mu^2 > 0$, where μ and θ_1 do not depend on j. Under these circumstances, the central limit theorem[12] states that the probability density for the sum $\Delta b \equiv \sum_{j=1}^{N} \delta_j b = b(N) - b(0)$ for large N converges to the normal density

$$f(\Delta b;N) = (2\pi\theta_1{}^2 N)^{-\frac{1}{2}} \exp\left[-(\Delta b - N\mu)^2/2\theta_1{}^2 N\right].$$

We now wish to argue that $\mu = 0$. Since the turbulence is random, the components of the tensor T_{ab} should have zero expectation value, and since in any case all "memory" of the initial field direction $n_a{}^0$ is very quickly lost, it seems impossible even in the case of anisotropic turbulence that the right-hand side of Eqn. (3) should show any statistical tendency to be either positive or negative. Thus we set $\mu = 0$, and the normal density for Δb simplifies to

$$f(\Delta b;N) = (2\pi\theta_1{}^2N)^{-\frac{1}{2}} \exp\left[-(\Delta b)^2/2\theta_1{}^2N\right]. \tag{4}$$

This is at variance with the assumption of Batchelor[3] that $\mu > 0$.

We may now derive the exponential law of increase for the expectation value of the field strength. If B_0 is the initial field strength in a given fluid element, then after N time steps, the expectation value of B in the same fluid element is $\langle B\rangle = B_0\langle e^{\Delta b}\rangle$, and we easily obtain

$$\langle B\rangle = B_0(2\pi\theta_1{}^2N)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} dx\, \exp\left(-\frac{x^2}{2\theta_1{}^2N} + x\right) = B_0\exp(\tfrac{1}{2}\theta_1{}^2N). \tag{5}$$

Since $N = t/\delta t$, we see that $\langle B\rangle$ increases exponentially with the time.

What is the order of magnitude of θ_1? If we take $\delta t = 2\tau$, then probably $\theta_1 \gtrsim 1$, the exact value depending on the type of turbulence.

It is easy to find the law of increase for the magnetic energy density. We write

$$\langle\varepsilon\rangle = \frac{1}{8\pi}\langle B^2\rangle = \varepsilon_0\langle e^{2\Delta b}\rangle = \varepsilon_0\exp(2\theta_1{}^2N). \tag{6}$$

Thus $\langle B^2\rangle > (\langle B\rangle)^2$.

Note, by contrast, that the conclusion $\mu = 0$ implies that the expectation value of the change of lnB vanishes: $\langle \Delta \ln B \rangle = \langle \Delta b \rangle = 0$. Thus, in a given fluid element, the field strength is just as apt to decrease as to increase. However, given that an increase in B occurs, the expected change in $|B|$ is much larger than when a decrease occurs.

2.2 The Vector Potential

Having found the normal distribution for lnB, we now do the same for the logarithm of the vector potential magnitude A, again under the assumption of infinite conductivity. We first show that with the choice of a special gauge condition the vector potential $\underset{\sim}{A}$ in a given fluid element remains perpendicular to an infinitesimal parallelogram formed by two material line elements which move frozen in the fluid element. I.e., if we define the parallelogram by two infinitesimal material displacements $\delta\underset{\sim}{y}$ and $\delta\underset{\sim}{z}$ such that initially $\underset{\sim}{A} \propto \delta\underset{\sim}{y} \times \delta\underset{\sim}{z}$, then this proportionality holds for all time. Furthermore, the quantity A/ρ remains proportional to the area of the parallelogram.

The special gauge condition chosen is written $\underset{\sim}{u}.\underset{\sim}{A} = c\psi$, where c is the velocity of light and ψ is the scalar potential. Writing Ohm's law as $\sigma^{-1}\underset{\sim}{J} = \underset{\sim}{E} + c^{-1}\underset{\sim}{u} \times \underset{\sim}{B}$, taking $\sigma \to \infty$, and using $\underset{\sim}{E} = -\,\partial\underset{\sim}{A}/\partial ct - \underset{\sim}{\nabla}\psi$ and the above gauge condition, one finds

$$\frac{d\underset{\sim}{A}}{dt} = -\,(\underset{\sim}{A}.\underset{\sim}{\nabla})\underset{\sim}{u} - \underset{\sim}{A}\times(\underset{\sim}{\nabla}\times\underset{\sim}{u}) = -\,(\underset{\sim}{\nabla}\underset{\sim}{u}).\underset{\sim}{A}.$$

However it is easily shown[13] that the equation of motion of the quantity $\rho\delta\underset{\sim}{S} = \rho\delta\underset{\sim}{y} \times \delta\underset{\sim}{z}$ is identical to the above equation for $\underset{\sim}{A}$. Therefore $\underset{\sim}{A}/\rho$ remains proportional to the infinitesimal surface element $\delta\underset{\sim}{S}$, provided that the above gauge condition is employed.

We may immediately recover the conclusion of Zeldovitch[9] that for two-dimensional incompressible motion, $\underset{\sim}{A}$ is not changed by the motion itself, but only by the ohmic dissipation. In his analysis the only important component of $\underset{\sim}{A}$ is the one perpendicular to the plane of the motion, and since the motion is supposed incompressible, the areas of the parallelograms remain constant, and by our analysis A then remains constant.

We now derive the probability density for the area of the parallelogram in the three-dimensional case. Since the parallelogram is infinitesimal, the shears and strains become, in the limit $\delta\underset{\sim}{S}\to 0$, spatially uniform over its surface, so that it remains a parallelogram. In Figure 1 we have indicated how the parallelogram may change over the time interval δt, and we define three parameters which specify this change: (1) A homogeneous horizontal scale change, which transforms $\overline{AB}\to\overline{A'B'}$; (2) a homogeneous vertical scale change, which transforms $\overline{CE}\to\overline{C'E'}$; and (3) a pure transverse shear which changes $\overline{AD}\to\overline{A'D'}$, but does not change $\overline{AB}$ or $\overline{CE}$. Of course, the orientation of the parallelogram changes as well, but this is of no interest here.

The area of the parallelogram is $S = \overline{AB}.\overline{CE}$, $S' = \overline{A'B'}.\overline{C'E'}$, and thus the transverse shear does not change the area, and we may ignore it. We are left with the two perpendicular scale changes $\delta\alpha$ and $\delta\beta$, which change $\ln S$ additively over the time interval δt; i.e., a given scale change $(\delta\alpha,\delta\beta)$ always induces the same $\delta\ln S$, independently of the initial value of S. Then defining $\delta\alpha$ and $\delta\beta$ such that each represents the change of the ln of a unit length, we set $\delta\ln S = \delta\alpha + \delta\beta$, and therefore $\delta\ln\,(A/\rho) = \delta\alpha + \delta\beta$. Now, for stationary turbulence, the density may fluctuate, but in the long run we expect $\rho \approx$ constant, since $\delta\ln\rho$ is negatively correlated with previous $\delta\ln\rho$. Thus we ignore the density fluctuations and treat the fluid as incompressible. Therefore, we conclude finally, for our purposes,

$$\delta\ln A \approx \delta\alpha + \delta\beta. \tag{7}$$

However, it is well-known[14] that the quantity B/ρ is proportional to the length of an infinitesimal displacement parallel to $\underset{\sim}{B}$ which moves frozen in the fluid. Therefore, since we have set $\rho \approx$ constant, the δb introduced in Sec. 2.1 is the same type of parameter as the $\delta\alpha$ and $\delta\beta$ defined directly above, since it is likewise the change in the ln of the length of a material displacement.

Therefore, except for the fact that $\delta\alpha$ and $\delta\beta$ are mutually perpendicular, the parameters $\delta\alpha$, $\delta\beta$, and δb are statistically identical. Hence $\langle(\delta\alpha)^2\rangle = \langle(\delta\beta)^2\rangle = \langle(\delta b)^2\rangle = \theta_1^{\,2}$. Also, $\langle\delta\alpha\rangle = \langle\delta\beta\rangle = \langle\delta b\rangle = 0$, as in Sec. 2.1. We define the correlation $\rho_1 \equiv \langle\delta\alpha\delta\beta\rangle/\theta_1^{\,2}$, where $-1 < \rho_1 < +1$.

Following the procedure in Sec. 2.1, we can now use a two-dimensional form of the central limit theorem[15] to obtain the joint probability density of the sum of $N \gg 1$ two-dimensional variables $(\Delta\alpha,\Delta\beta) = \sum_{j=1}^{N} (\delta_j\alpha,\delta_j\beta)$. Since the joint single-step density for $(\delta_j\alpha,\delta_j\beta)$ does not depend on previous values of these variables, the joint density for $(\Delta\alpha,\Delta\beta)$ converges for large N to the bivariate normal density, which for this case simplifies to[16]

$$f(\Delta\alpha,\Delta\beta;N) = \left[2\pi\theta_1^{\,2}N(1-\rho_1^{\,2})^{\frac{1}{2}}\right]^{-1} \times \exp\left\{-\left[2\theta_1^{\,2}N(1-\rho_1^{\,2})\right]^{-1}\left[(\Delta\alpha)^2-2\rho_1\Delta\alpha\Delta\beta+(\Delta\beta)^2\right]\right\} \qquad (8)$$

What is the sign of ρ_1? Since the volume of the fluid element is approximately conserved, we expect $\delta_j\alpha$ and $\delta_j\beta$ to have different signs statistically, and hence ρ_1 is negative.

If we assume that the correlation is caused only by volume conservation, we can derive an exact value for ρ_1. Let $\delta\S$ be the third perpendicular logarithmic scale change. Since pure shears do not change the volume of the fluid element, the joint probability density $f(\Delta\alpha,\Delta\beta,\Delta\S;N)$ contains a delta-function factor $\delta(\Delta\alpha+\Delta\beta+\Delta\S)$; and since the arguments must appear symmetrically, we can write

$$f(\Delta\alpha,\Delta\beta,\Delta\S;N)=C\ \exp\left\{-D\left[(\Delta\alpha)^2+(\Delta\beta)^2+(\Delta\S)^2\right]\right\}\delta(\Delta\alpha+\Delta\beta+\Delta\S).$$

We then integrate over $\Delta\S$ to get the marginal density

$$f(\Delta\alpha,\Delta\beta;N) = \int d\Delta\S f(\Delta\alpha,\Delta\beta,\Delta\S;N)$$

$$= C\ \exp\left\{-2D\left[(\Delta\alpha)^2+\Delta\alpha\Delta\beta+(\Delta\beta)^2\right]\right\}.$$

Comparison with Eqn. (8) gives the result

$$\rho_1 = -\tfrac{1}{2}\ . \tag{9}$$

At this point we can make contact with some of the results of Parker[10], who also derived the exponential increase of the magnetic field by considering the stretching of material line elements. He set up a Fokker-Planck equation by supposing that the relative length change of a line element over a time step (large enough for the correlation with adjacent time steps to be small) was very small, but he recognized this assumption to be erroneous. Our use of the central limit theorem does not involve such an assumption, and also has the advantage of treating the local perpendicular scale changes by means of a joint probability density, instead of having to discuss the approximate evolution of a macroscopic flux rope.

It is useful to derive an expression for the probability density of $\Delta a \equiv \Delta\alpha + \Delta\beta$. One may change variables in the $(\Delta\alpha,\Delta\beta)$ plane to $\Delta a = \Delta\alpha + \Delta\beta$, $\Delta\alpha' = \Delta\alpha$ and then integrate over $\Delta\alpha'$, the Jacobian of the transformation being unity, to get

$$f(\Delta\alpha;N) = \left[4\pi\theta_1{}^2 N(1+\rho_1)\right]^{-\frac{1}{2}}\exp\left\{-(\Delta a)^2\left[4\theta_1{}^2 N(1+\rho_1)\right]^{-1}\right\},$$

which is the normal density with variance $2\theta_1{}^2 N(1+\rho_1) \equiv \theta_2{}^2 N$. If Eqn. (9) holds, then $\theta_2 = \theta_1$.

Since the magnitude A of the vector potential in a given fluid element is proportional to the area S of the perpendicular parallelogram, we have $A/A_o = S/S_o$, and therefore, from Eq. (7), $\Delta \ln A = \Delta\alpha + \Delta\beta = \Delta a$. Hence $\langle A\rangle = A_o\langle e^{\Delta a}\rangle$, or

$\langle A\rangle = A_o \exp(\frac{1}{2}\theta_2{}^2 N)$, which is similar to Eqn. (5) for the field strength.

We have used the same assumptions here as those employed in Sec. 2.1; namely, that both the dynamic reaction of the field on the fluid and the ohmic dissipation may be neglected.

3. THE JOINT DISTRIBUTION FOR A AND B, AND THE OHMIC DISSIPATION

In this section we bring together the results of the previous section in order to gain information about the rate of ohmic dissipation of the field. The previous results depend on the assumption that the ohmic losses are negligible, but we will be able to extrapolate, with some reservations, to the regime where the ohmic losses are considerable. The conclusion will be that, except when the boundary conditions force a permanent large-scale Fourier component of the field, the field must ultimately be destroyed.

3.1 The Local Scale and the Correlations Between lnA and lnB

We begin by finding an expression for the local scale ℓ of the magnetic field. Dimensionally, the relation $A \approx \ell B$ suggests itself. Now, this ℓ is really the scale of $\underset{\sim}{A}$, but if $\underset{\sim}{B}$ possesses no large-scale Fourier component, so that the polarity of $\underset{\sim}{B}$ alternates randomly in space, then $\ell \approx A/B$ is, on the average, the scale of $\underset{\sim}{B}$ as well. Thus we may state that for each $(\underset{\sim}{r},t)$, there exists a point $\underset{\sim}{r} + \Delta\underset{\sim}{r}(\underset{\sim}{r},t)$, such that

$$\ell(\underset{\sim}{r},t)B(\underset{\sim}{r},t) = c_0 A(\underset{\sim}{r} + \Delta\underset{\sim}{r},t), \tag{10}$$

where c_0 is a constant geometric factor of order unity.

It may happen that at a particular $(\underset{\sim}{r},t)$, the field is flat, so that $\ell \to \infty$ there. Then $\Delta\underset{\sim}{r}$ will not exist. However, we interpret ℓ as an average scale over a finite region, and since we are interested only in average quantities in what follows, our conclusions will still be valid. It is important to emphasize that if $\underset{\sim}{B}$ has a large-scale Fourier component, Eqn.(10) is not valid. In Sec. 3.3 we present a counterexample to results derived from Eqn. (10) and show how other aspects of our reasoning may break down as well.

Let us now discuss the correlation between A and B. Since $\underset{\sim}{A}\cdot\underset{\sim}{B}/\rho \propto \rho\,\delta\underset{\sim}{x}\cdot\delta\underset{\sim}{y} \times \delta\underset{\sim}{z}$, which is the total conserved mass in the material volume element defined by the three infinitesimals, the quantity $\underset{\sim}{A}\cdot\underset{\sim}{B}$ is time-independent except for fluctuations of the density, which as we have said we may ignore for stationary turbulence. Thus $\underset{\sim}{A}\cdot\underset{\sim}{B} = AB\cos\theta \approx$ constant in a given fluid element, where θ is the angle between $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Therefore, except for the anomalous case $\underset{\sim}{A}\cdot\underset{\sim}{B} = 0$, it follows from $\ln A + \ln B + \ln|\cos\theta| = \ln|\text{constant}|$ that, for a given change of $\cos\theta$, $\ln A$ and $\ln B$ will be negatively correlated, and we may use the generalized central limit theorem as in Sec. 2.2 to write the joint probability density, after $N \gg 1$ time steps, of $\Delta\ln A \equiv \Delta a$ and $\Delta\ln B \equiv \Delta b$ as the bivariate normal density

$$f(\Delta a,\Delta b;N) = \left[2\pi\theta_1\theta_2 N(1-\rho_2{}^2)^{\frac{1}{2}}\right]^{-1} \times$$

$$\exp\left\{-\left[2N(1-\rho_2{}^2)\right]^{-1}\left[\frac{(\Delta a)^2}{\theta_2{}^2} - \frac{2\rho_2\Delta a\Delta b}{\theta_2\theta_1} + \frac{(\Delta b)^2}{\theta_1{}^2}\right]\right\} \tag{11}$$

where $\theta_2{}^2 = 2\theta_1{}^2(1+\rho_1)$, and $-1 < \rho_2 < 0$.

3.2 The Ohmic Dissipation Rate

Using the expression for ℓ provided by Eqn. (10), we now find the expectation value of the ohmic loss rate. At a point $\underset{\sim}{r}$ the loss of energy in erg/cm^{3}sec is then

$$\frac{\lambda}{4\pi} J^2(\underset{\sim}{r}) \approx \frac{\lambda B(\underset{\sim}{r})^2}{4\pi \ell(\underset{\sim}{r})^2} = \frac{\lambda B(\underset{\sim}{r})^4}{4\pi c_o^2 A(\underset{\sim}{r}+\Delta\underset{\sim}{r})^2} \quad .$$

Now in order to find the expectation value of this quantity, we must weaken the correlation between Δa and Δb somewhat, since they are taken a distance $\Delta\underset{\sim}{r}$ apart. Thus in Eqn. (11), we substitute $\rho_2 \to \rho_3$, where $\rho_2 < \rho_3 < 0$, and obtain, for a given fluid element,

$$\left\langle \frac{\lambda}{4\pi} J^2 \right\rangle \approx \frac{\lambda}{4\pi} J_o^2 \left\langle e^{4\Delta b - 2\Delta a} \right\rangle$$
$$= \frac{\lambda}{4\pi} J_o^2 \exp\left[(2\theta_1^2 - 8\rho_3\theta_1\theta_2 + 8\theta_2^2)N\right]. \qquad (12)$$

But Eqn. (6) implies that the rate of energy density increase from the random walk is, on the average, $d\langle\varepsilon\rangle/dt = 2\,\varepsilon_o\theta_1^2(\delta t)^{-1}\exp(2\theta_1^2 N)$. Since $\rho_3 < 0$, Eqn. (13) tells us that no matter how small J_o^2 is, the ohmic losses eventually become comparable to the random walk gain, and the field ceases to grow. In particular, if $\rho_1 = -\frac{1}{2}$ and $\theta_1 = \theta_2$ (Eqn. (9)), then $\langle J^2 \rangle = J_o^2 \exp\left[\theta_1^2 N(10-8\rho_3)\right]$, which grows much faster than $\exp(2\theta_1^2 N)$.

May we now conclude from this that the field eventually destroys itself? Not yet, for the probability densities derived for A and B depended on the assumption that the ohmic losses were negligible. Indeed, one possibility is that the ohmic dissipation can serve to increase the scale of the field, and thus an equilibrium state might be possible. This possibility is discussed more thoroughly in the next subsection in connection with a counter-example, which also entails the possibility that the reasoning leading to Eqn. (10) breaks down.

It is reasonable to conclude, however, that if the ohmic losses do not counter the tendency for the scale to decrease, and if Eqn. (10) holds, then the field must be destroyed. The probabilistic analysis given above casts considerable light on the action of the turbulent fluid motion on the field, and it is fair to say that the random walk process which tends to increase the field strength must at the same time inexorably be accompanied by a more drastic increase in dissipation: If $\langle B^2 \rangle$ increases, then $\langle B^2/\ell^2 \rangle$ must increase even faster, since the dispersion associated with the ℓ^{-2} part of the probability density is increasing at the same time.

3.3 A Counterexample

We now discuss a special case, in which a large-scale Fourier component is forced upon the field by boundary conditions, and we show how the preceding analysis breaks down.

Consider the situation depicted in Fig. 2, where the fluid is broken up into three regions: regions I and III, where $\lambda = 0$, $\underset{\sim}{u} = 0$, and region II, where $\lambda > 0$, $\underset{\sim}{u} \neq 0$. Let Σ_1 be a part of the boundary surface between I and II such that the magnetic flux $\phi_1 = \iint_{\Sigma_1} \underset{\sim}{B} \cdot d\underset{\sim}{\Sigma} > 0$.

Then ϕ_1 is constant in time, and the flux conservation law $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$ implies that the flux ϕ_2 through the "cap" surface Σ_2 in region II is the same as ϕ_1. Therefore, no matter what the nature of the turbulence in region II, and no matter how low the conductivity is there, the magnetic field in region II must persist. Thus an equilibrium state is possible.[17]

Machine integrations by Weiss[18] suggest that in such a system the magnetic field may be expelled out of the main body of the fluid into a boundary layer. However, Weiss dealt with time-dependent velocity fields, and one can imagine that in the random turbulent case the expulsion tendency might be countered by turbulent transport of flux back out of the boundary layer into the main body of region II.

Where does the analysis in Sec. 3.2 break down in this case? As mentioned before, there is the possibility that the ohmic dissipation tends to keep the scale of the field from decreasing. This can be shown to be related to the fact that in this counterexample the field has a permanent large-scale component.

Consider the very simple case of a diffusion equation $\partial B/\partial t = \partial^2 B/\partial x^2$. If the field is purely sinusoidal, then $B = B_1 e^{-\kappa t} \sin(x\sqrt{\kappa})$ is a solution, and the scale of the field $\ell = B/(\partial B/\partial x) = (\kappa)^{-\frac{1}{2}} \tan(x\sqrt{\kappa})$ is time-independent. Now suppose the field has a large-scale constant Fourier component, so that $B = B_0 + B_1 e^{-\kappa t} \sin(x\sqrt{\kappa})$ is a solution. Then the scale becomes $\ell = B(t)/\left[B_1 e^{-\kappa t} \cos(x\sqrt{\kappa})\sqrt{\kappa}\right]$ which diverges as $t \to \infty$. We see that, at least in this crude example, the diffusion term leads to an increase in scale if there is a large-scale component.

There is also another possibility that we must reckon with, which may also operate to invalidate the conclusions of Sec. 3.2, and which deals with the assumption of Eqn. (10) and its application to the ohmic loss estimate. In writing Eqn. (10), we assumed, in essence, that the field has a random character such that the contribution to $\underset{\sim}{A}$ coming from the field in distant volume elements was very small. However, if $\underset{\sim}{B}$ has a large-scale component superimposed on a random, alternating-polarity component, then the analysis might break down because $\Delta\underset{\sim}{r}$ could conceivably introduce unexpected positive correlations between $\ln A$ and $\ln B$.

It is not clear which of these two effects plays the greater role in the above counterexample.

3.4 Extension to High Dissipation and Strong Fields

Let us now go back to the situation in which there is no permanent large-scale Fourier component of the field, and for which the ohmic losses presumably do not operate to increase the scale of the field.

As the ohmic losses grow, the field is no longer frozen in the fluid, and begins to "slip" relative to the fluid. Thus the random-walk build-up and the scale decrease both become less efficient. However, the general probabilistic analysis for the random-walk transfer of energy from the fluid to the field and for the ohmic dissipation should still hold. Presumably θ_1 then decreases, and J_o^2 must be continually renormalized to larger values. But with each random-walk extraction of energy from the fluid, the scale decreases. Thus the energy is renewable up to a point, but in the absence of a large-scale Fourier component the scale of the field seems doomed to a continual decrease, until the field is extinguished.

We may also try to extend the reasoning to the case where the field strength is high enough to control the motion of the fluid. Where there is no large-scale component of the field, the same conclusion may result: It may be impossible to put random-walk energy into the field without a continual decrease in scale. However, since the magnetic force term in the fluid equations of motion is $\approx B^2(4\pi \ell)^{-1}$, the field is more amenable to distortion where ℓ is large, and hence large fields and large scales may become positively correlated. It is not evident that this tendency can counter the general reduction of scale implicit in the random-walk process.

4. A NECESSARY CONDITION FOR TURBULENT AMPLIFICATION, AND A POSSIBLE EXPERIMENT

The conclusions of the preceding sections are, as we have said, by no means rigorously derived. Therefore, it would be very useful to devise an experiment to test whether or not they are valid. In order to do this, we first show a necessary condition, elucidated also by Syrovatski[4], for initial amplification of a magnetic field by turbulence. It is then shown that a technologically feasible experiment using liquid sodium as the conducting medium may be of considerable interest in testing our results.

Suppose that we are given a medium of conductivity σ, in which there exists turbulence of root-mean-square velocity u and characteristic scale ℓ_t, defined perhaps by the expression[19] $\ell_t \equiv \int_0^\infty E(k)dk / \int_0^\infty E(k)kdk$, where $E(k)$

is the turbulent energy spectrum, and k is the wave number. ℓ_t is thus the scale associated with the energy-bearing part of the spectrum and is presumably the scale most significant in distorting the magnetic field.

Now, any magnetic field carried with the fluid must, if it is to be amplified or maintained by the turbulence, have a scale which is less than or equal to the turbulent scale. Further, Eqn. (1) shows that for amplification we must have $|(\underset{\sim}{B}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}| > |\underset{\sim}{\nabla}\times(\lambda\underset{\sim}{\nabla}\times\underset{\sim}{B})|$, or in approximate form $u/\ell_t > \lambda/\ell^2$. But since $\ell \lesssim \ell_t$, we must have $u/\ell_t > \lambda/\ell_t^{\,2}$, or

$$4\pi\sigma u\ell_t > 1. \tag{13}$$

Batchelor[2] used the analogy between vorticity and the magnetic field to derive a necessary and sufficient condition for turbulent amplification and maintenance of the field. His condition is that the kinematic viscosity ν be greater than λ, or $4\pi\sigma\nu > 1$. The physical reasoning behind this inequality is as follows: The smaller is ν, the smaller is the scale of the turbulence. The scale of the field is inherited from the scale of the turbulence, and thus if ν is small enough the ohmic losses will overwhelm the growth rate. However, our analysis shows that if the field has no large-scale Fourier component, the scale of the field in any case becomes ultimately so small that the field is extinguished.

Let us now see under what circumstances we may satisfy Eqn. (13), which is at least a necessary condition for turbulent amplification. One of the most highly conducting fluids available in the laboratory is liquid sodium, for which[20] $\sigma \gtrsim 10^{-4}$ sec/cm^2. If we can generate turbulence for which $\ell_t = 1$ cm, then Eqn. (13) will be satisfied for $u > 10^3$cm/sec.

Suppose that the liquid sodium be contained in a rotating cylindrical vessel of radius 16 cm and a counter-rotating set of paddles be situated inside to stir up the liquid. The scale of the energy-containing eddies is presumably inherited (but somewhat reduced) from the boundary scales. Thus in this case $\ell_t \gtrsim 1$ cm. If the container and the paddles are counter-rotated each at a speed of 100 rev/sec, then the boundary speed of each is $u \approx 10^4$cm/sec. This should insure the satisfaction of Eqn. (13), but higher speeds might of course be used.

The magnetic amplification properties of the arrangement might be tested with various externally impressed magnetic fields, but it would be difficult to set up the situation described in the counterexample, where the external conductivity must be much higher than in the turbulent fluid.

The viscous dissipation in the fluid would amount to roughly $\nu u^2/\ell_t^{\,2}$ erg/gm sec. For liquid sodium[21], $\nu \approx 10^{-2}$ cm^2/sec, and if we assume 10^3 gm of material, the total viscous power is then 10^9 erg/sec = 100 watts.

The turbulent kinetic energy density is $\frac{1}{2}\rho u^2 \approx 5 \times 10^7$ erg/cm^3, and if equipartion with the fluid is ever reached, the field strength would amount to $B \approx 3 \times 10^4$ gauss. The associated ohmic dissipation is then $\lambda B^2 (4\pi \ell_t^{\,2})^{-1}$ erg/cm^{3}sec, and the total ohmic power for 10^3 cm^3 is 10^{13} erg/sec = 10^6 watts. Clearly, it would be impossible to reach this high power rate required for equipartion. If, however, turbulent maintenance of the field is possible, contrary to the results of this paper, then an average mechanical power input of 200 watts should maintain a field of 300 gauss.

One might also test the possibility that a field which is initially strong enough to control the turbulence could be maintained. This possibility is discussed at the end of the previous section. Since large-scale fields of 3×10^4 gauss are not easy to produce in the laboratory, a more tractable combination of u and ℓ_t might be tried.

A negative result for this series of experiments would substantiate the conclusions of this paper, but would not show that Batchelor's criterion is wrong, since $4\pi\sigma\nu \approx 10^{-5}$. However, a positive result would show that both Batchelor's criterion and our conclusions are incorrect, and that turbulent maintenance is possible.

ACKNOWLEDGEMENTS

It is a pleasure to thank Dr. Douglas O. Gough for many critical comments and suggestions which substantially improved the paper. Financial support was provided by the National Aeronautics and Space Administration in the form of a National Academy of Sciences-National Research Council Research Associateship at the Goddard Institute for Space Studies.

FOOTNOTES

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11. We wish to use a simple form of the central limit theorem, for which it is necessary that these correlations vanish. However, G. K. Batchelor, *Theory of Homogeneous Turbulence* (Cambridge University Press, London, 1959), p. 98, states that there are more general forms of the central limit theorem for which this requirement is relaxed.
12. W. Feller, *Introduction to Probability Theory*, Vol. I (J. Wiley & Sons, Inc., New York 1957), 2nd ed., p.229.
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17. With the use of Eqns. (6) and (11), we could derive a field amplification factor from an initial state (B_o, J_o) to a final equilibrium B in terms of an initial magnetic Reynolds number. The result is (with $\theta_1 = \theta_2$) $\langle B^2\rangle^{\frac{1}{2}} = (R_m{}^o)^{1/[8(1-\rho_3)]}B_o$, where $R_m{}^o = 4\pi\theta_1{}^2\sigma B_o{}^2/(J_o{}^2\Delta t)$. It is not easy to reconcile this with Parker's result (Ref. 10) $\langle B^2\rangle^{\frac{1}{2}} = R_m{}^{1/4}B_o$, but both results are very approximate.

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21. The dynamic viscosities $(\rho\nu)$ of liquid metals are slightly greater than that of water, for which $\nu \approx 10^{-2}$ cgs. See Ref. 20, pp. 319, 327.

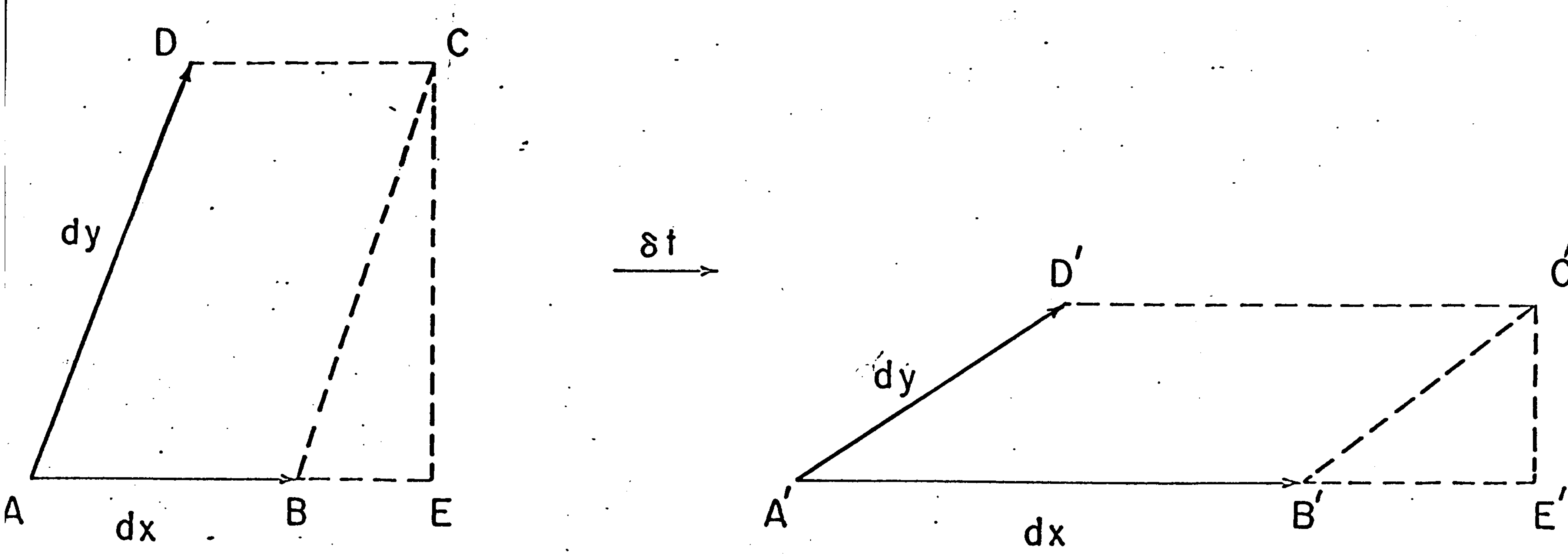

FIG. 1. The change of an infinitesimal parallelogram over the time interval δt. The change in orientation is neglected.

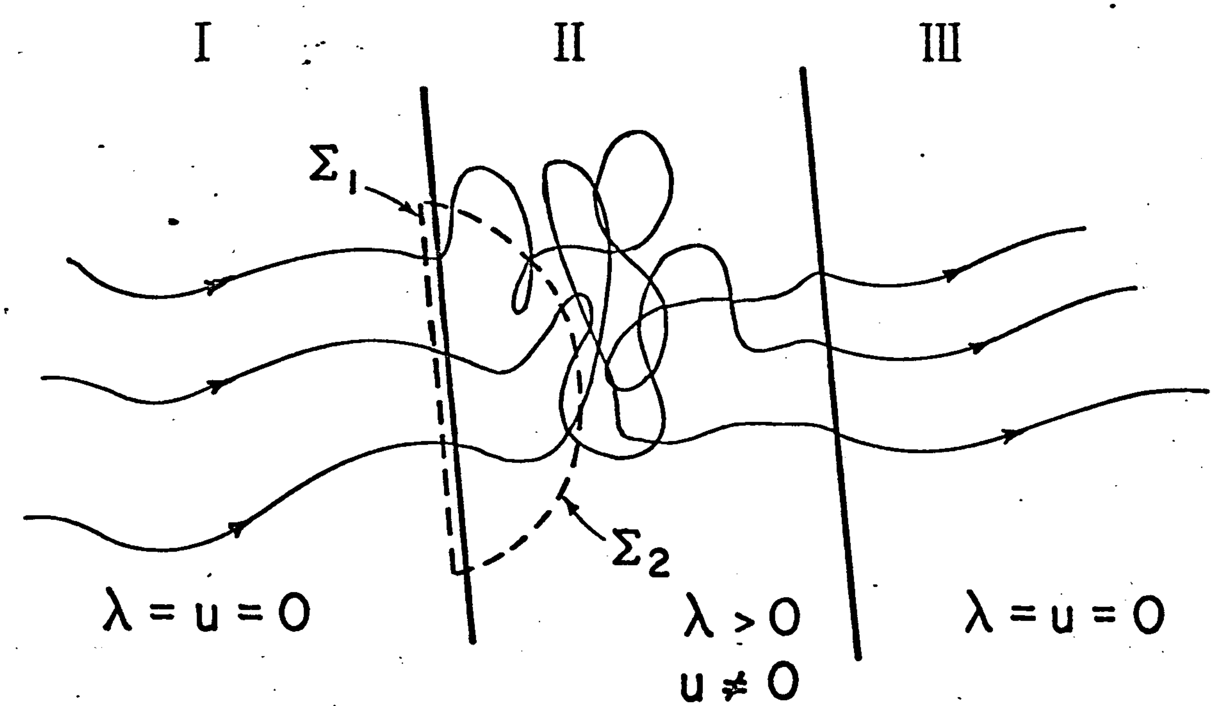

FIG. 2. The counterexample to the results of Sec. 3.2. A large-scale permanent Fourier component is forced on the field in region II by boundary conditions.