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KINETIC THEORY OF INHOMOGENEOUS SYSTEMS

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A theory is developed which treats the coupled equations of the various hierarchies as simultaneous equations in time. This scheme proceeds by successive approximations rather than a power series expansion in the small parameter $\left([n\lambda_d^3]^{-1}\right.$ in a plasma, $\left. n r_0^3 \text{ in a Boltzmann gas}\right)$. The theory is suitable for the derivation of equations for non-uniform and force driven systems. Examples are given for a plasma and a Boltzmann gas.

I. Introduction

Although the goal of kinetic theory is the description of general non-equilibrium systems, most present work is directed toward obtaining small corrections to the behavior of infinite homogeneous systems. However for an inhomogeneous system the domain of validity of such corrections becomes smaller and smaller as their accuracy increases. Thus while the first approximation (to the collision integral) may be presumed accurate when $L \gg r_0$ or λ_d, where L is a characteristic macroscopic length and r_0 or λ_d a typical interaction length, the next correction will be quantitatively significant only for $L > \lambda_{\text{mean free path}}$, etc. Since spatial gradients and external forces are generally used to create non-equilibrium systems, the experimental verification of the theories will prove difficult.

In addition the assumptions needed to obtain kinetic equations (Section III) are weak enough so that equations for non-uniform and force driven systems should be easily derivable. The fact that they cannot be derived indicates that some aspect of the problem has been overlooked.

Finally it appears that most expansions in kinetic theory diverge, so that the mathematical simplifications must be underlaid by a physical error.

In such a case one has reason to reexamine the basic procedures of kinetic theory, and to seek a first order theory (the first approximation to the collision term) which is as simple and general as possible.

Most modern work in this subject is based on the formulation of Bogoliubov,[1] who set up the hierarchy of equations and a procedure based on expansion in a small parameter for solving them. In the case of a Boltzmann gas (Section VI) Bogoliubov's procedure breaks down for an inhomogeneous system because the boundary conditions (at $r = \infty$) which he imposed are not satisfied, and are not relevant to the physical problem.

His theory with some modification can be applied to yield a first order plasma kinetic equation for an inhomogeneous system, because three particle effects (shielding) are included in the plasma theory. However in this case the weakening of correlations described by Bogoliubov has been omitted,[2] so that the theory is unsuitable for non-uniform systems.

At higher orders Bogoliubov's procedure corresponds to the calculation of correlation functions in successive intervals of time, which disagrees with his statement of the time scales involved (Section III). Nevertheless we believe that Bogoliubov obtained the first order theory almost correctly, and in the present work we attempt to correct the difficulties which appear in the higher orders.

Sections II and III, which are non mathematical, discuss the motivation for the work, the significance of the small parameter, and the role of time. The statement of the procedure to be used appears in Section III-D.

In Section IV the first order plasma theory is developed, and in Section V the second order theory is compared to the quasilinear theory[3,4] and the recent work of Dupree.[5]

Section VI contains a discussion of a Boltzmann gas, while Section VII contains a brief application to an equilibrium plasma.

In the conclusion the general outlook for the theory is discussed.

II. The Purpose of Kinetic Theory

In a mathematical sense the description of a complex physical system ($N \sim 10^{23}$ particles) by one or several continuous functions obeying relatively simple equations is not justified. The equations of motion are known (ignoring quantum mechanical effects, and in the present treatment, radiation) so that in principle there is no need for further approximations.

In fact, of course, one cannot solve these equations for the motion of the N particles comprising the system. In addition the necessary boundary condition (typically the initial state of the system) cannot be given by experiment, so that even a formal solution to the equations of motion is not useful.

Thus our theory is necessarily statistical. We believe that the justification for our procedures will ultimately come from statistical mechanics, but this justification is lacking at present, except in the case of thermal equilibrium. It follows that the theory is completely ad hoc until experimental evidence is available. In this respect verification of the Boltzmann and Vlasov equations represents the major evidence that the theory is headed in the right direction. For the same reason we attempt to drive theory toward experiment by considering non-uniform and force driven systems, as they typically represent non-equilibrium experimental situations better than homogeneous field free systems.

One may conceive of experimental verification of the theory at many levels. A small detector $\left(\Delta V \sim \lambda^3_{\text{mean free path}},\ \Delta V \sim \lambda^3_{\text{Debye}}\right)$ may measure the directed flow of particles in order to measure the one particle distribution as a function of time. A pair of such detectors spaced closely together may be used to obtain the evolution of the pair correlation function, etc.

In fact such detailed measurements are virtually impossible, so that only crude comparisons with the predictions of kinetic theory may be achieved. For this reason we treat basic physical principles as internal constraints on the theory, even though direct experimental verification may prove difficult. In the present work the conservation laws of particle number, momentum and energy will be used as a test of the theory.

III. Fundamental Assumptions

A. Mathematical Framework

Since the procedures to be used in setting up a kinetic theory are a matter of choice we must be careful to justify the equality sign in any equations we use. For this reason we work with a hierarchy of equations which can be derived rigorously. The Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy is natural for a Boltzmann gas (Section VI), while the Klimontovich-Dupree,[6,7] equations are more convenient for describing a plasma (Section IV). We also work with another (quasilinear) hierarchy (Section V) first mentioned by Dupree,[8] because it is mathematically simple. We believe that these latter equations are unsuitable for describing real systems because of the absence of source terms in f, but they permit a direct comparison with quasilinear theory and its extension by Dupree.[5] It should be emphasized that the functions satisfying the above equations are not the quantities of physical interest. Each closed set of equations obtained by setting the $(n + 1)$th correlation function equal to zero is fully time reversible, while we seek equations describing an irreversible approach to a limiting state, e.g. thermal equilibrium.

In what follows we justify the continued use of the equality sign by keeping our equations formally identical to those of the respective hierarchy. We obtain successive approximations to a set of correlation functions, where the difference between the exact functions and the approximate functions remains in the equation and may, in principle, be determined.

B. The Use of the Small Parameter

We consider each of the three hierarchies as a time reversible (and hence improper) approximation to a different set of equations (a kinetic hierarchy) which describes the quantities of physical interest. The arguments that follow apply to the physical quantities rather than the solution by direct integration of the various time reversible equations.

We may begin by specifying the relative importance of terms in each equation by scaling velocities, lengths, times, etc. to those which are believed to dominate the physical behavior. We do not carry out the procedure here, but simply state that it leads to the appearance of a small parameter. This parameter (to be called β) is essentially nr_0^3 in a Boltzmann gas, and $(n\lambda_{Debye^3})^{-1}$ in a plasma. In most physical situations the numerical value of β is 10^{-2} or less, but it may be somewhat larger without destroying the significance of the theory. We choose to work with unscaled quantities, and insert a coefficient $\beta = 1$ where the small parameter appears in the scaled equations. The significance of β in the theory we wish to set up is the following:

We demand that the sequence of functions obtained by including more and more physical effects (i.e., additional terms in β in the equations) be convergent. Thus

$$\frac{\Psi(\beta^{n+1}) - \Psi(\beta^n)}{\Psi(\beta^n)} = o(\beta^n)$$

This severe requirement is weakened by several conditions which cannot be avoided.

1. The requirement cannot be satisfied over all of phase space. For example the ordering in β breaks down for short distances $|r_i - r_j| \sim q^2/mv_{thermal}^2$ in a plasma, and for large distances $|r_i - r_j| \sim \lambda_{mean\ free\ path}$ in both a plasma and a Boltzmann gas. Thus convergence is strictly required only in the region specified in the original ordering.
2. For short times $(t < t_{collision})$ the ordering procedure is meaningless. Here we use procedures discussed in Part C.
3. It is possible that solutions may diverge after long times. This is unimportant provided the system effectively reaches an end state before

the divergence occurs. We find asymptotic behavior by special procedures such as transport theory (e.g., Chapman-Enskog[9] theory) rather than by direct integration in time.

Although the demand for convergence does not specify a mathematical procedure, we believe it eliminates the one most commonly used. In general a perturbation expansion in β maps the flow and scattering of particles onto higher and higher order terms, while in fact they are moved about within the same function. The expansion breaks down after the displacement of a substantial number of particles in some region of phase space. Because expansions may be expected to diverge, we do not expand. Instead, we truncate at a given level consistent with the "small" statistical effect in the next higher equation, and attempt to solve the resulting equations exactly. At a given level a perturbation expansion may be convergent for numerical estimates, but the expansion must be reappraised at each higher level of approximation.

Obviously convergence does not determine a mathematical procedure, it only puts a "boundary condition" on the techniques that may be used. We believe that the physical behavior is represented by an intricate mathematical dependence on the small parameter, so that a statement of a general mathematical procedure is not possible.

C. The Significance of Time

Our conclusions regarding time are similar to those of Bogoliubov.[1] Our statements apply to quantities which we might conceivably measure in the laboratory, for which we wish to obtain equations. If we consider the behavior of a physical system which is set in motion at some instant of time, then the evolution may generally be broken into three phases.

1. In a time of the order of a collision time t_c an arbitrary initial state will relax so that the correlation functions become substantially functions of the one particle distribution f, and hence <u>of one other</u>. Some effects of the detailed initial conditions may persist (e.g., in the case of plasma instabilities); these must be included in the theory. During the initial relaxation no general procedure (including ordering) is valid, and kinetic theory is entirely an initial value problem.

   In the mathematical theory integrals over the product of a potential and a correlation function appear frequently. If the initial conditions are reasonable then these integrals may reach their asymptotic form much more rapidly ($t \sim r_0/v_{thermal}$ or $t \sim \lambda_{Debye}/v_{thermal}$). This is the relaxation time described by Bogoliubov.

2. Following the initial relaxation f decays roughly as $\partial f/\partial t = 1/t_c(f - f_0)$, i.e., rapidly at first and then more and more slowly while approaching a local end state f_0. In the early stages the separation of times 1 and 2 breaks down and the theory we develop is less valid. This is not important, as we are generally interested in the asymptotic behavior of f, rather than the precise way it gets there. The aim of the present paper is to find equations valid during this second regime. These equations

should describe the final relaxation of the correlation functions as they become functionals of f, and the subsequent evolution of f.

3. The third time is often given by $L/v_{thermal}$, but it is better described as the decay time for a macroscopic state, e.g., a thermal gradient or a gaseous shock. In this regime a description by transport theory should be adequate.

The second and third time intervals are described by the solution of the kinetic equation, while only the first time interval is relevant for the derivation of kinetic equations. In particular we consider the end of the first time interval, as the system approaches asymptotically the kinetic regime. Difficulties (e.g., the breakdown of ordering) early in the first phase are avoided simply by considering t always in this asymptotic limit.

In a plasma some effects of initial conditions may persist in this limit. If a general equation might be written

$$\left(\frac{\partial}{\partial t} + H\right) \Psi + \Phi(t = 0) \frac{\partial \Psi}{\partial x} = \mathcal{L}(t) \Phi(t = 0)$$

then the term on the right is permissible, providing the operator $\mathcal{L}(t)$ eventually carries $\mathcal{L}(t)\Phi(t = 0)$ to 0. The term $\Phi(t = 0)\,\partial\Psi/\partial x$ is improper as it continues to affect the evolution indefinitely. This is contrary to our demand (relaxed perforce in Section VI) that the evolution of the system depend only on f after sufficient time. In addition this term must vanish if the evolution is to be independent of the origin of time. This reasonable requirement also implies the condition $\mathcal{L}(t)\,\mathcal{L}(\tau) = \mathcal{L}(t+\tau)$.

In what follows all equations are treated as simultaneous equations in time.

D. Formal Procedure

The procedure we advocate is one of successive approximations. For simplicity we discuss the first order statistical correction to the $\beta = 0$ equations.

1. The equations of the hierarchy are integrated for the case $\beta = 0$. It is important to keep the streaming (homogeneous) solutions for the correlation functions, although these terms frequently may be neglected in the solution of kinetic equations. The resultant expressions for the correlation functions $\Psi(\beta = 0)$ are then substituted into the small (β) terms in the hierarchy. The difference between these expressions and the (unknown) exact solutions remains in the equation, but is now higher order $\Psi - \Psi(\beta = 0) = O(\beta\Psi)$, and would be considered in the second order theory.

2. The small term $\beta\Psi_{n+1}$ will, in general, contain an integral over the earlier behavior of lower order correlation functions

$$\beta\Psi_{n+1} \sim \int_0^t d\tau \left[\Psi_n(\tau),\ \Psi_{n-1}(\tau) \cdots f(\tau)\right]$$

The behavior of the lower order functions is now approximated by the $\beta = 0$ solutions such that all correlation functions are evaluated at the same time t. Thus

$$\Psi_n(\tau) = \mathcal{L}_n(\tau, t)\Psi_n(t) + \int_t^\tau \{\ \} d\tau$$

where the operator $\mathcal{L}_n$ is the streaming operator that carries Ψ_n from time t to time τ.

3. The $\Psi_n(t)$, $\Psi_{n-1}(t) \cdots$ are now considered to be the exact correlation functions (rather than the $\beta = 0$ correlation functions) so that the approximation is thrown over from the functions Ψ to the operators $\mathcal{L}$. As we are deriving equations rather than solving them, we are approximating (scattering) operators rather than functions.

4. The scattering operators involving time integrals are now evaluated. There is now a difficulty which appears in the first order theory, and presumably in the higher orders. The operators just defined are not mathematically proper, e.g., they may diverge. This must be corrected by making each operator consistent with the small (order β) operators in the next higher equation. The reason for this difficulty is found from examination of the resulting equations: we are using the approximation $\beta \doteq 0$ in order to get a grasp on the problem, but the desired result is not analytic at $\beta = 0$. In general this consistency is the most difficult part of the theory to obtain, but since it represents a small (order β) effect on an operator which is already small, the theory is not sensitive to the exact method chosen.

The resultant equations represent the kinetic hierarchy including the first (order β) statistical effects.

Although we can easily generate a formal statement and notation for going to arbitrary order in β, the added weight and complexity add nothing to the theory. Instead we describe the general procedure only so far as we are able to carry it out explicitly, and to the level of the first equation we cannot solve.

We state two features that are not immediately apparent. Firstly, we are obtaining successive approximations for a correlation function Ψ_n as a function of lower order functions; $\Psi_n = \Psi_n(\Psi_{n-1}, \Psi_{n-2}, \cdots, f)$. It follows that the kinetic hierarchy closes naturally at each level n. Secondly, the analytic solution at the level β^{ℓ} is required for the construction of the $\beta^{\ell+1}$ equations. This illustrates forcibly that it is not possible to obtain corrections to equations which are already insoluble.

Despite the fact that the theory is convergent by definition (if this cannot be arranged a new ordering is necessary), so that one may in principle go to any order, we have no proof that this is a correct procedure. The question of validity lies outside the theory.

IV. First Order Plasma Theory

Our techniques follow the methods developed by Dupree[7] in his brilliant paper of 1963. The equations are those developed by Klimontovich and Dupree; they are equivalent to those of the BBGKY hierarchy. The one particle distribution for species μ, $f_\mu(r_1, v_1, t)$ is normalized so that

$$\int dv_1\, f_\mu(r_1, v_1, t) = \frac{N_\mu(\mathbf{r}_1, t)}{\bar{n}_\mu}$$

where N_μ is the local particle density and $\bar{n}_\mu$ is the system average density.

f satisfies the equation

$$\frac{\partial f_\mu}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1} f_\mu + \frac{q_\mu}{m_\mu}\left[\mathbf{E}(\mathbf{r}_1, t) + \frac{\mathbf{v}_1}{c} \times \mathbf{B}(\mathbf{r}_1, t)\right] \cdot \frac{\partial f_\mu}{\partial \mathbf{v}_1}$$

$$= -\beta \frac{q_\mu}{m_\mu} \frac{\partial}{\partial \mathbf{v}_1} \cdot \sum_\nu \bar{n}_\nu q_\nu \int d\mathbf{X}_2\, \nabla_1 \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} h_{2\mu\nu}(\mathbf{X}_1, \mathbf{X}_2, t)$$

$$= -\beta \frac{q_\mu}{m_\mu} \frac{\partial}{\partial \mathbf{v}_1} \cdot \langle \delta f_\mu \, \delta \mathbf{E} \rangle \quad (1)$$

where $\mathbf{X}_i = (\mathbf{r}_i, \mathbf{v}_i)$, β is the ordering parameter, and the second notation for the collision term is often convenient.

The electric and magnetic fields (**E** and **B**) are determined from Maxwell's equations, and will be regarded as known. We shall treat the forces as constant in time (an adiabatic hypothesis), which is valid if the correlation functions reach their asymptotic values rapidly compared to the variation of the fields.[10] In our examples we consider forces which are constant in space, although they may in general vary over distances much greater than a Debye length.

The zero order ($\beta = 0$) solution to (1) is $f_\mu \left(\mathbf{X}_1, t\right) = e^{-V(1)t} f_\mu \left(\mathbf{X}_1, t = 0\right)$ where the operator e^{-Vt} runs the particles backward on their orbits

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t), \qquad \dot{\mathbf{v}}(t) = \frac{q_\mu}{m_\mu}\left[\mathbf{E}(\mathbf{r}) + \frac{\mathbf{v}(t)}{c} \times \mathbf{B}(\mathbf{r})\right]$$

from the initial point $\mathbf{r}(0) = \mathbf{r}$; $\mathbf{v}(0) = \mathbf{v}$. If the solution to the zero order (Vlasov-Maxwell) equations is not known it is not possible to go to first order.

As implied by the right side of Eq. (1), we use the Coulomb approximation in treating fluctuations in the system. Electromagnetic effects may easily be included in the theory,[11] but they complicate calculation considerably. For present purposes any fluctuating field $\overline{\mathbf{E}}$ is related to the fluctuation density $\overline{f}$ by Poisson's equation

$$\overline{\mathbf{E}}(\mathbf{r}, t) = -\sum_\nu \overline{n}_\nu q_\nu \int d\mathbf{X}' \, \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \overline{f}_\nu (\mathbf{X}', t)$$

In order to permit comparison with Section V, we write the equation for the general correlation function (ignoring particle species[12]) $h_n(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n; t)$. We define the Vlasov operator

$$V(i) = \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{r}_i} + \frac{q}{m}\left[\mathbf{E} + \frac{\mathbf{v}_i}{c} \times \mathbf{B}\right] \cdot \frac{\partial}{\partial \mathbf{v}_i}$$

the fluctuation operator

$$T(i) = V(i) - \frac{q}{m}\frac{\partial f}{\partial \mathbf{v}_i} \cdot \int d\mathbf{X}_i' \, \nabla_i \frac{\bar{n}q}{|\mathbf{r}_i - \mathbf{r}_i'|}$$

and the interaction operator

$$O(i, j) = \frac{-q}{m}\frac{\partial}{\partial \mathbf{v}_i} \int d\mathbf{X}_j \, \nabla_i \frac{\bar{n}q}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Then h_n satisfies the equation

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{n} T(i)\right] h_n + \sum_{i=1}^{n} O(i, n+1) \left\{ \sum_{m=2}^{n-1} h_m(\cdots, \mathbf{X}_i, \cdots)\, h_{n-m+1}(\cdots, \mathbf{X}_{n+1}) + \beta h_{n+1} \right\} = 0 \qquad (2)$$

In the sum on m the remaining coordinates are distributed in all ways such that each appears once, and rearrangements within a function are not distinct. The correlation functions h_n are related to those of the BBGKY (g_n, with g_2 = "g", g_3 = "h", etc.) by

$$h_n = g_n + \sum_{i \neq j} g_{n-1}(\cdots, \mathbf{X}_{i-1}, \mathbf{X}_{i+1}, \mathbf{X}_j, \cdots)\, \Delta(\mathbf{X}_i, \mathbf{X}_j) + \sum g_{n-2}(\cdots)\Delta\Delta$$

$$+ \cdots + f(\mathbf{X}_1)\, \Delta(\mathbf{X}_1, \mathbf{X}_2)\, \Delta(\mathbf{X}_1, \mathbf{X}_3) \cdots \Delta(\mathbf{X}_1, \mathbf{X}_n) \qquad (3)$$

where[13]

$$\Delta(\mathbf{X}_i, \mathbf{X}_j) = \frac{1}{\bar{n}} \delta(\mathbf{r}_i - \mathbf{r}_j)\, \delta(\mathbf{v}_i - \mathbf{v}_j)$$

The first correction in β to the equation for f may be found by calculating h_2 to zero order. We set $\beta = 0$ and $n = 2$ in Eq. (2), and use the more familiar notation $h_2 = \langle \delta f \delta f \rangle$.

$$\left[\frac{\partial}{\partial t} + T(1) + T(2)\right] \langle \delta f \delta f \rangle = 0 \tag{4}$$

As pointed out by Dupree a product solution is possible. We define a propagation operator $P(\mathbf{X}, t)$ by

$$\left[\frac{\partial}{\partial t} + T(1)\right] P = 0 \tag{5}$$

with the initial condition $P(\mathbf{X}, 0) = I$. It is convenient to define an auxiliary (electric field) operator

$$P(\mathbf{r}, \mathbf{E}, t) = -\bar{n}q \int d\mathbf{X}' \, \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} P(\mathbf{r}', \mathbf{v}', t) \tag{6}$$

so that 5 may be written

$$\left(\frac{\partial}{\partial t} + V\right) P + \frac{q}{m} P(\mathbf{E}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

We first integrate along the orbits as described previously.

$$P = e^{-\nu t} - \frac{q}{m}\int_0^t d\tau\, e^{-\nu(t-\tau)}\, P(\mathbf{r}, \mathbf{E}, \tau)\cdot \frac{\partial}{\partial \mathbf{v}}\, f(\mathbf{r}, \mathbf{v}, \tau)$$

This expresses P in terms of f at other positions and earlier times. However from the $\beta = 0$ approximation for f we have $f(\tau) = e^{\nu(t-\tau)} f(t)$. Thus

$$P = e^{\nu t} - \frac{q}{m}\int_0^t d\tau\, e^{-\nu(t-\tau)} P(\mathbf{r}, \mathbf{E}, \tau)\cdot \frac{\partial}{\partial \mathbf{v}}\, e^{\nu(t-\tau)} f(\mathbf{X}, t) \tag{7}$$

Operator Eqs. (6) and (7) may be solved by a Fourier transform in space ($\mathbf{k}$) and a Laplace transform in time (ω), provided we use an adiabatic hypothesis to ignore the space and time variation of f in the transforms. Then the transforms of the operators are given by

$$P(\mathbf{k}, \mathbf{v}, \omega) = \int_0^\infty d\tau\, e^{i\mathbf{k}\cdot[\mathbf{r}(-\tau)-\mathbf{r}]}\, e^{-\nu\tau}\, e^{i\omega\tau} + \frac{q}{m}\frac{i\mathbf{k}}{k^2}\cdot \left[\int_0^\infty d\tau\, e^{i\omega\tau}\, e^{i\mathbf{k}\cdot[\mathbf{r}(-\tau)-\mathbf{r}]}\right.$$

$$\left.\times\left\{\frac{\partial \mathbf{r}(\tau)}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{\partial \mathbf{v}(\tau)}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}\right\} f(\mathbf{r}, \mathbf{v}, t)\right] \frac{4\pi \bar{n} q}{\epsilon(\mathbf{k}, \omega)}\int d\mathbf{v}' \int_0^\infty d\tau\, e^{i\mathbf{k}\cdot[\mathbf{r}'(-\tau)-\mathbf{r}']} e^{i\omega\tau}\, e^{-\nu\tau} \tag{8}$$

$$P(\mathbf{k}, \mathbf{E}, \omega) = -\frac{i\mathbf{k}}{k^2}\frac{4\pi \bar{n} q}{\epsilon(\mathbf{k}, \omega)}\int d\mathbf{v}\int_0^\infty d\tau\, e^{i\omega\tau}\, e^{i\mathbf{k}\cdot[\mathbf{r}(-\tau)-\mathbf{r}]}\, e^{-\nu\tau} \tag{9}$$

where the generalized dielectric function is given by

$$\epsilon(\mathbf{k}, \omega, \mathbf{r}, t) = 1 - \frac{\omega_p^2}{k^2} i\mathbf{k} \int d\mathbf{v} \int_0^\infty d\tau \, e^{i\omega\tau} e^{i\mathbf{k}\cdot[\mathbf{r}(-\tau)-\mathbf{r}]}$$

$$\times \left[\frac{\partial \mathbf{r}(\tau)}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{\partial \mathbf{v}(\tau)}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{r}, \mathbf{v}, t) \quad (10)$$

To use the **P** operators we Fourier transform the operand, apply the operator, and invert the Fourier and Laplace transforms. For example the solution to Eq. (4) is given by

$$\langle \delta f \delta f \mid \mathbf{X}_1, \mathbf{X}_2, t \rangle = P(\mathbf{X}_1, t) \, P(\mathbf{X}_2, t) \, \langle \delta f \delta f \mid \mathbf{X}_1, \mathbf{X}_2, t = 0 \rangle \quad (11)$$

This solution is improper in several respects. For example one part of the solution is given by $\langle \delta f \delta f \mid \mathbf{X}_1(-t), \mathbf{X}_2(-t), t = 0 \rangle$, which does not vanish as $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$. This defect is caused by the omission of three particle effects, which tend to smear out the orbits and prevent the particles from being correlated to $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$. This represents non-analytic behavior in β, and should be corrected in the second order theory.

A second defect arises in the boundary conditions which should be applied to the correlation functions. The difficulties of using the Fourier transform illustrate this, for in general $\langle \delta f \delta f \rangle$ is not simply a function of $\mathbf{r}_i - \mathbf{r}_j$. One cannot seek the asymptotic behavior (in time) of wave effects[14] for they correspond to the propagation over large distance of the initial fluctuations, which may be determined only by specifying a boundary condition. Once again we believe that the proper resolution should be found in the treatment of three particle correlations.

Despite these difficulties we may find a first approximation to the kinetic equation, for the potential $q/|\mathbf{r}_1 - \mathbf{r}_2|$ in the collision integral cuts off most of the dependence on $|\mathbf{r}_1 - \mathbf{r}_2| > \lambda_{\text{Debye}}$. We ignore the "slow" dependence on space and approximate

$$\langle \delta f \delta f \mid \mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_1, \mathbf{v}_2, t = 0 \rangle = f\left(X_1, t = 0\right) \frac{(2\pi)^3}{\bar{n}} \delta\left(\mathbf{v}_1 - \mathbf{v}_2\right) \delta\left(\mathbf{k}_1 + \mathbf{k}_2\right)$$

$$+ g_2\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_1, \mathbf{v}_2, t = 0\right) (2\pi)^3 \delta\left(\mathbf{k}_1 + \mathbf{k}_2\right)$$

The function $f\left(X_1, t = 0\right)$ is written $e^{V(1)t} f\left(X_1, t\right)$ in the collision integral, where the operator $e^{V(1)t}$ must be inserted in the time integrals defining the transformed (ω) P operators. If we shorten notation by defining

$$\begin{Bmatrix} P_r \ (\mathbf{k}, \mathbf{v}, \omega) \\ \\ P_v \ (\mathbf{k}, \mathbf{v}, \omega) \\ \\ P_0 \ (\mathbf{k}, \mathbf{v}, \omega) \\ \\ P_I \ (\mathbf{k}, \mathbf{v}, \omega) \end{Bmatrix} = \int_0^\infty d\tau \, e^{i\omega\tau} \, e^{i\mathbf{k}\cdot\left[\mathbf{r}(-\tau)-\mathbf{r}\right]} \begin{Bmatrix} \mathbf{k} \cdot \dfrac{\partial \mathbf{r}(\tau)}{\partial \mathbf{v}} \\ \\ \mathbf{k} \cdot \dfrac{\partial \mathbf{v}(\tau)}{\partial \mathbf{v}} \\ \\ 1 \\ \\ e^{-V\tau} \end{Bmatrix} \tag{12}$$

then the kinetic equation is given by

$$\frac{\partial f}{\partial t} + V(1)\, f = -\frac{q}{m}\frac{\partial}{\partial \mathbf{v}_1} \cdot \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 t}$$

$$\times \left[\left\{ P_0\left(\mathbf{k}, \mathbf{v}_1, \omega_1\right) + \frac{q}{m}\left[\mathbf{P}_\mathbf{r}\left(\mathbf{k}, \mathbf{v}_1, \omega_1\right) \cdot \frac{\partial}{\partial \mathbf{r}_1} + \mathbf{P}_\mathbf{v}\left(\mathbf{k}, \mathbf{v}_1, \omega_1\right) \cdot \frac{\partial}{\partial \mathbf{v}_1}\right] f\left(\mathbf{r}_1, \mathbf{v}_1, t\right)\right.\right.$$

$$\left.\times \frac{4\pi \bar{n} q i}{k^2 \epsilon\left(\mathbf{k}, \omega_1\right)} \int d\mathbf{v}_1' \, P_0\left(\mathbf{k}, \mathbf{v}_1', \omega_1\right)\right\} \left\{\frac{i\mathbf{k}}{k^2} \frac{4\pi \bar{n} q}{\epsilon\left(-\mathbf{k}, \omega_2\right)} \int d\mathbf{v}_2 \, P_0\left(-\mathbf{k}, \mathbf{v}_2, \omega_2\right)\right\}$$

$$\times f\left(\mathbf{r}_1, \mathbf{v}_2, t\right) \frac{\delta\left(\mathbf{v}_1 - \mathbf{v}_2\right)}{\bar{n}} + \left\{ P_I\left(\mathbf{k}, \mathbf{v}_1, \omega_1\right) + \frac{q}{m}\left[\mathbf{P}_\mathbf{r}\left(\mathbf{k}, \mathbf{v}_1, \omega_1\right) \cdot \frac{\partial}{\partial \mathbf{r}_1}\right.\right.$$

$$\left.\left. + \mathbf{P}_\mathbf{v}\left(\mathbf{k}, \mathbf{v}_1, \omega_1\right) \cdot \frac{\partial}{\partial \mathbf{v}_1}\right] f\left(\mathbf{r}_1, \mathbf{v}_1, t\right) \int d\mathbf{v}_1' \, P_I\left(\mathbf{k}, \mathbf{v}_1', \omega_1\right)\right\} \left\{\frac{i\mathbf{k}}{k^2} \frac{4\pi \bar{n} q}{\epsilon\left(-\mathbf{k}, \omega_2\right)}\right.$$

$$\left.\left. \times \int d\mathbf{v}_2 \, P_I\left(-\mathbf{k}, \mathbf{v}_2, \omega_2\right)\right\} g_2\left(\mathbf{k}, \mathbf{v}_1, \mathbf{v}_2, t = 0\right)\right] \quad (13)$$

The Laplace transforms are evaluated by considering t in the asymptotic limit. We illustrate the procedure with a simple example.

A. Inhomogeneous System

For a system with no fields the velocity $\mathbf{v}$ is a constant and $\mathbf{r}(t) = \mathbf{r} + \mathbf{v}t$. The operators P_0, $\mathbf{P}_r$ and $\mathbf{P}_v$ are given by

$$P_0 = (-i\omega + i\mathbf{k}\cdot\mathbf{v})^{-1} \qquad \mathbf{P}_r = -(\omega - \mathbf{k}\cdot\mathbf{v})^{-2}\,\mathbf{k} \qquad \mathbf{P}_v = (-i\omega + i\mathbf{k}\cdot\mathbf{v})^{-1}\,\mathbf{k} \tag{14}$$

and the dielectric function is

$$\epsilon(\mathbf{k}, \omega, \mathbf{r}, t) = 1 + \frac{\omega_p^2}{k^2}\int \frac{d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} + \frac{\omega_p^2}{k^2}\int \frac{d\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{r}}}{(\omega - \mathbf{k}\cdot\mathbf{v})^2} \tag{15}$$

As the methods of evaluation have been discussed elsewhere[15] we simply give the result

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1} f = \frac{2q^2}{\pi m}\frac{\partial}{\partial \mathbf{v}_1}\cdot\int d\mathbf{k}\,\frac{\mathbf{kk}}{k^4}\Bigg[\bar{n}q^2\int d\mathbf{v}_2\Bigg\{\frac{\pi\delta(\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2)}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_2)|^2}\,\Theta(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_2)$$

$$\cdot\frac{1}{m}\left[\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2}\right] f(\mathbf{r}_1, \mathbf{v}_1)\, f(\mathbf{r}_1, \mathbf{v}_2) + \frac{1}{(\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2 - i\rho)^2}\left[\frac{\Theta(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)|^2}\frac{1}{m} f(\mathbf{r}_1, \mathbf{v}_1)\right.$$

$$\left.\cdot\frac{\partial}{\partial \mathbf{r}_1} f(\mathbf{r}_1, \mathbf{v}_2) - \frac{\Theta(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_2)}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_2)|^2}\frac{1}{m} f(\mathbf{r}_1, \mathbf{v}_2)\frac{\partial}{\partial \mathbf{r}_1} f(\mathbf{r}_1, \mathbf{v}_1)\right] + \mathbf{j}(\mathbf{k}, \mathbf{r}_1, \mathbf{v}_1)\left[\frac{\delta(\mathbf{k}\cdot\mathbf{v}_1 - \Omega_{\mathbf{k}})}{\gamma_k\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\right.$$

$$\left.\left.\times f(\mathbf{r}_1, \mathbf{v}_1) + 2\int_0^t d\tau'\,\frac{E(\mathbf{k}, t', t)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} f(\mathbf{r}_1, \mathbf{v}_1, t')\,\delta[\mathbf{k}\cdot\mathbf{v}_1 - \Omega_k(t')]\right]\right\} + \frac{\bar{n}q}{\pi}\int d\mathbf{v}_2\,\bar{n}q\int d\mathbf{v}_3$$

$$\left.\times\frac{E(\mathbf{k}, 0, t)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\mathbf{j}(\mathbf{k}, \mathbf{r}_1, \mathbf{v}_1)\,\frac{g_2(\mathbf{k}, \mathbf{v}_2, \mathbf{v}_3, t = 0)}{(\Omega_k - \mathbf{k}\cdot\mathbf{v}_2 - i\,|\gamma_k|)(\Omega_k - \mathbf{k}\cdot\mathbf{v}_3 + i\,|\gamma_k|)}\right] \tag{16}$$

where ρ is a real infinitesmal, $\omega_k = \Omega_k + \gamma_k$, and we have defined

$$j(\mathbf{k}, \mathbf{r}_1, \mathbf{v}_1) = \left[\frac{\gamma_k}{|\omega_k - \mathbf{k}\cdot\mathbf{v}|^2} + 2\pi\delta\left(\mathbf{k}\cdot\mathbf{v} - \Omega_k^-\right)\right]\frac{1}{m}\frac{\partial f(\mathbf{r}_1, \mathbf{v}_1)}{\partial \mathbf{v}_1} + \frac{1}{(\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2 - i\rho)^2}\frac{1}{m}\frac{\partial f(\mathbf{r}_1, \mathbf{v}_1)}{\partial \mathbf{r}_1}$$

$$\Theta(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}) = \pm 1 \qquad \mathrm{Im}\,\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}) \gtrless 0$$

$$E(k, t', t) = \exp\left[2\int_{t'}^{t} \gamma_k(\tau)\, d\tau\right] \tag{17}$$

The collision integral conserves number density and momentum, but it does not yield a correct energy law, for the energy transport cannot be written as the divergence of a flux. Although the difficulty is easily traced to the spatial adiabatic hypothesis in the Fourier transform, we believe that a more correct mathematical treatment is not justified, as this will describe the flow over large distances of the microscopic fluctuations originally present in the system. Instead a treatment of three particle correlations should cut off the dependence of the collision term on distant processes, as well as conserve energy.

B. Plasma in a Magnetic Field

For simplicity we assume a plasma in a uniform magnetic field $B\hat{e}_z$ although slow variation of **B** in space, and weak electric fields may be included easily.[16] We express velocities and wavevectors in cylindrical coordinates, taking the x direction as origin for angular variables; $\mathbf{v} = (v_\perp, \theta, v_z)$, $\mathbf{k} = (k_\perp, \alpha, k_z)$. For physical applications it is appropriate to keep spatial dependence in cartesian coordinates, so our notation is sometimes mixed. The particle orbit is given by

$$\mathbf{r}(t) = \vec{r} + \hat{e}_z v_z t + \hat{e}_x \frac{v_\perp}{\omega_c}\left[\sin\left(\theta + \omega_c t\right) - \sin\theta\right] + \hat{e}_y \frac{v_\perp}{\omega_c}\left[\cos\theta - \cos\left(\theta + \omega_c t\right)\right]$$

where

$$\omega_c = \frac{qB}{mc}$$

The P_0, $\mathbf{P_r}$ and $\mathbf{P_v}$ operators are given by

$$\begin{Bmatrix} P_0 \\ \\ \mathbf{P_v} \\ \\ \mathbf{P_r} \end{Bmatrix} = \sum_n \frac{e^{-ik_\perp v_\perp/\omega_c^{-1}\sin(\theta-\alpha)}\, e^{in(\theta-\alpha)}}{x_n} \begin{Bmatrix} i J_n(a) \\ \\ \mathbf{k}_z J_n(a) + \frac{n\omega_c}{v_\perp} J_n(a)\,\hat{e}_\perp + ik_\perp \frac{\partial J_n}{\partial a}\,\hat{e}_\theta \\ \\ \frac{i\mathbf{k}_z J_n(a)}{x_n} + \frac{L_2\,\mathbf{k}_\perp}{\omega_c} + \frac{L_1 - J_n(a)}{\omega_c}\mathbf{k}\times\hat{B} \end{Bmatrix}$$

where

$$L_1(n, a, \theta, \alpha) = \frac{n}{a} J_n(a)\cos(\theta-\alpha) - i\,\frac{\partial J_n}{\partial a}\sin(\theta-\alpha)$$

$$L_2(n, a, \theta, \alpha) = \frac{n}{a} J_n(a)\sin(\theta-\alpha) + i\,\frac{\partial J_n}{\partial a}\cos(\theta-\alpha) \tag{18}$$

J_n is the Bessel function of order n, and we have defined

$$a = \frac{k_\perp v_\perp}{\omega_c}, \qquad x_n(\mathbf{k}, \mathbf{v}, \omega) = \omega - k_z v_z - n\omega_c$$

and the dielectric function is given by

$$\epsilon(\mathbf{k}, \omega, \mathbf{r}, t) = 1 + \frac{\omega_p^2}{k^2} \int d\mathbf{v}\, e^{-ik_\perp v_\perp \omega_c^{-1} \sin(\theta-\alpha)} \sum_n \frac{J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{in(\theta-\alpha)}}{x_n}$$

$$\times \left[\left(k_z J_n \frac{\partial}{\partial v_z} + \frac{n\omega_c}{v_\perp} J_n \frac{\partial}{\partial v_\perp} + \frac{ik_\perp}{v_\perp} \frac{\partial J_n}{\partial a} \frac{\partial}{\partial \theta}\right) f + \left(\frac{ik_z J_n}{x_n} + \frac{L_2 (k_x + k_y)}{\omega_c}\right.\right.$$

$$\left.\left. + \frac{L_1 - J_n}{\omega_c} \mathbf{k} \times \hat{\mathbf{B}}\right) \cdot \frac{\partial}{\partial \mathbf{r}} f \right] \qquad (19)$$

The Laplace inversion and analytic continuation are now straightforward but the result is so intricate the evaluation should be performed after approximations suitable to the problem at hand. A single difficulty arises, which is the appearance of terms oscillating at multiples of the cylotron frequency. These should be eliminated by time averaging over one period, for if the adiabatic hypothesis is not satisfied (in effect making $\partial f/\partial\theta$ small) then our analysis is not relevant to the rapid time variation of the system.

For purposes of approximating the **k** integral one must know the dependence of the dielectric function α. However this lies within the linear theory and will not be discussed here. Even in the case of general fields leading to particle drifts the collision integral conserves particle number and canonical momentum. The transport of electrostatic energy is not described properly so that a more thorough study of wave effects is needed.

V. Toward Second Order Plasma Theory

Our treatment in this section is very incomplete because of the great complexity of the mathematics. We simplify the problem by working with the quasi-linear hierarchy described by Dupree.[8] The equations are formally identical to those of the preceding section, but the correlation functions (w_n) have no delta function between spaces; they are non-singular. One obtains the same equations by dropping from the BBGKY equations all terms in $\nabla\phi$, while keeping integrals over such terms. Because the w_n equations have no sources in f the correlation functions do not relax to become functionals of f. In this respect the system never becomes "kinetic" for knowledge of f alone is not sufficient to determine the subsequent evolution.

Using the operators defined in Section IV, we have

$$\left[\frac{\partial}{\partial f} + \sum_{i=1}^{n} T(i)\right] w_n = -\sum_{i=1}^{n} O(i, n+1)\left\{\sum_{m=2}^{n-1} w_m\left(\cdots, \mathbf{X}_i, \cdots\right) w_{n-m+1}\left(\cdots, \mathbf{X}_{n+1}\right) + \beta w_{n+1}\left(\mathbf{X}_1, \cdots \mathbf{X}_{n+1}\right)\right\} \quad (20)$$

The streaming solution (including shielding) is given by

$$w_{n,0} = P_n \left(\mathbf{X}_1, \cdots \mathbf{X}_n;\ t \right) w_n (t = 0) \tag{21}$$

where we have abbreviated $P_n \left(\mathbf{X}_1, \mathbf{X}_2, \cdots \mathbf{X}_n;\ t \right) = P\left(\mathbf{X}_1, t \right) P\left(\mathbf{X}_2, t \right) \cdots P\left(\mathbf{X}_n, t \right)$. Equation 20 has been integrated by Dupree for the special case $\nabla f = 0$, i.e. for a homogeneous system with no forces, or else $\partial f / \partial \theta = 0$ in the case of a uniform magnetic field. We shall use his solution, despite the fact that it is not strictly correct in general.[17]

For $\beta = 0$ we have

$$w_n = w_{n,0} - \int_0^t d\tau\, P_n (t - \tau) \sum_{i=1}^{n} O(i, n+1) \sum_{m=2}^{n-1} w_m \left(\cdots \mathbf{X}_i, \cdots; \tau \right) w_{n-m+1} \left(\cdots, \mathbf{X}_{n+1}; \tau \right) \tag{22}$$

Since the source terms in the integral are of lower index an iterative solution for w_n is now possible. Instead we use successive approximations in the source terms, starting with the streaming solution. We approximate the behavior of the w_m by the streaming solution, after which the w_m are treated as the exact functions. We then use the commutative properties of the P operators acting on different spaces to write

$$w_{n,1} = P_n(t)\,w_n(t=0) - \int_0^t d\tau\, P_n(t-\tau) \sum_{i=1}^{n} O(i, n+1) \sum_{m=2}^{n-1} P_m^{-1}(t$$

$$-\tau)\, P_{n-m+1}^{-1}(t-\tau)\, w_m(t)\, w_{n-m+1}(t) = P_n(t)\,w_n(t=0) - \sum_{i=1}^{n} \int_0^t d\tau\, P(\mathbf{X}_i, t$$

$$-\tau)\, O(i, n+1)\, P^{-1}(\mathbf{X}_i, t-\tau)\, P^{-1}(\mathbf{X}_{n+1}, t-\tau) \sum_{m=2}^{n-1} w_m(\cdots X_i, \cdots; t)\, w_{n-m+1}(\cdots X_{n+1}, t) \tag{23}$$

Substitution of $w_{n+1,1}$ into the equation for w_n yields

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{n} T(i)\right] w_n = -\sum_{i=1}^{n} O(i, n+1) \left\{ \sum_{m=2}^{n-1} w_m(\cdots \mathbf{X}_i, \cdots)\, w_{n-m+1}(\cdots, \mathbf{X}_{n+1}) \right.$$

$$\left. + w_{n+1,1} \right\} - \beta \sum_{i=1}^{n} O(i, n+1) \left\{ w_{n+1} - w_{n+1,1} \right\} \tag{24}$$

The first $w_{n+1,1}$ should be considered order 1, for while it is quantitatively small its effect on w_n cannot be found by expansion. The terms with factor β should then be evaluated from the new $\beta = 0$ equations.

Because $w_{2,0}$ is the $\beta = 0$ solution for the source terms of w_3, it follows that $w_{3,1}$ is the correct $\beta = 0$ solution for w_3. Thus $\beta(w_3 - w_{3,1})$ is formally order β^2. For $n > 3$ the source terms must be iterated $n - 2$ times before

$\beta\left(w_n - w_{n,n-2}\right)$ is order β^2. We pursue the general case no farther, but consider $n = 2$, and drop the β^2 terms.

$$\left[\frac{\partial}{\partial t} + T(1) + T(2)\right] w_2 = \sum_{j=1}^{2} O(j, 3) \sum_{i=1}^{3} \int_0^t d\tau \, P\left(\mathbf{X}_i, t-\tau\right) O(i, 4) P^{-1}\left(\mathbf{X}_i, t-\tau\right)$$

$$\times P^{-1}\left(\mathbf{X}_4, t-\tau\right) w_2\left(-, \mathbf{X}_i; t\right) w_2\left(-, \mathbf{X}_4; t\right) - \sum_{j=1}^{2} O(j, 3) w_{3,0} \quad (25)$$

Although $w_{3,0}$ depends explicitly on the initial value of w_3, the evolution of w_2 should not depend on a particular choice of the origin of time. To ensure this we must give a consistent treatment of the initial value problem.

Because of the asymptotic analysis (Section III-C) and equation (23) we have

$$w_{3,1}(t) = \underset{t\uparrow}{P_3(t)} w_3(t=0) - \int_0^{t\to\infty} d\tau \sum_{i=1}^{3} P\left(\mathbf{X}_i, t-\tau\right) O(i, 4) P^{-1}\left(\mathbf{X}_i, t-\tau\right) P^{-1}\left(\mathbf{X}_4, t-\tau\right)$$

$$\times \; w_2(t) w_2(t)$$

so that $w_3(t)$ is a functional of $w_2(t) w_2(t)$. The operators should be evaluated in the asymptotic limit, which does not exist at this level because of particle streaming. This defect should be corrected by the inclusion of order β effects in the solution for w_3, but we may indicate the formal procedure. At the initial instant we have

$$w_3(t=0) = \underset{t\uparrow;\, t=0}{P_3(t)} w_3(t=0) - \sum_{i=1}^{3} \mathcal{L}(i) w_2(t = 0) w_2(t = 0) \qquad (26)$$

The left side denotes the measurable initial value for w_3, from which the dependence on $w_2\ (t = 0)$ (also measurable) must be subtracted to yield the proper initial value for the term $w_{3,0}$ of equation 25.

Due to the integrations of the operators $O(1, 3)$, $O(2, 3)$ on the initial value of w_3 and the subsequent integration $O(1, 2)$ in the evaluation of the collision term for f, we may obtain an estimate of the effect of $w_3\ (t = 0)$ on the evolution of f despite the unboundedness of the unintegrated quantities.

The term $w_3\ (t = 0)$ contributes an order β^2 term to the kinetic equation so that a rough estimate is satisfactory. By ignoring the non linear terms in Eq. (25), we may integrate to find this correction.

$$\left(\frac{\partial f}{\partial t}\right)\left(w_3\ (t = 0)\right) = O(1, 2)\int_0^t d\tau\, P_2\,(t-\tau)\sum_{j=1}^{2} O(j, 3)\, P_3\,(\tau)\, w_3\ (t = 0)$$

We do not evaluate this term here, for it has the general form of the collision integral produced by the initial value of g_2 of Section IV, and is formally order β smaller.

Henceforth we omit $w_3\ (t = 0)$ and use the convenient notation $w_2 = \langle \delta f\, \delta f \rangle_c$ where the subscript denotes a continuous rather than a singular function.

$$\left[\frac{\partial}{\partial t} + T(1) + T(2)\right]\langle \delta f(\mathbf{X}_1, t)\ \delta f(\mathbf{X}_2, t)\rangle_c = \sum_{j=1}^{2} O(j, 3)\sum_{i=1}^{3}\int_0^t d\tau\, P(\mathbf{X}_i, t-\tau)$$

$$\times\, O(i, 4)P^{-1}(\mathbf{X}_i, t-\tau)\, P^{-1}(\mathbf{X}_4, t-\tau)\langle \delta f(-, t)\, \delta f(\mathbf{X}_i, t)\rangle_c \langle \delta f(-, t)\, \delta f(\mathbf{X}_4, t)\rangle_c \qquad (27)$$

Due to the sums

$$\sum_{j=1}^{2}\sum_{i=1}^{3}$$

and the two ways of distributing the remaining coordinates the right side contains 12 terms in $\langle \delta f \delta f \rangle_c \ \langle \delta f \delta f \rangle_c$.

We consider first the four terms containing both O(1, 3) and O(2, 4). Upon evaluation we find that these terms have the form

$$\int dk \int dk' \, \mathcal{L}(\mathbf{k}, \mathbf{k}') \, e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \, e^{i\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)} \langle \delta f \delta f \mid \mathbf{k} \rangle_c \langle \delta f \delta f \mid \mathbf{k}' \rangle_c$$

while all remaining terms have the general form

$$\int dk \, e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \int dk' \, \mathcal{L}\left(\mathbf{k}, \mathbf{k}'\right) \langle \delta f \, \delta f \mid \mathrm{k} \rangle_c \ \langle \delta f \delta f \mid \mathbf{k}' \rangle_c$$

The former terms correspond to an interaction between $\delta f(\mathbf{X}_1)$ and $\delta f(\mathbf{X}_2)$ over a range $|\mathbf{r}_1 - \mathbf{r}_2| \sim \lambda_d$, and will be neglected compared to the latter, which represent the interaction of $\delta f(\mathbf{X}_2)$ and $\delta f(\mathbf{X}_2)$ separately with a fluctuation background. This approximation, which is similar to the neglect of close encounters in the equations for h_n, causes a great simplification, for it follows that the behavior of $\{\mathbf{X}_1\}$ and $\{\mathbf{X}_2\}$ separate as in Section V. It is convenient to abbreviate notation by treating only the $\mathbf{X}_1$ dependence, written $|\,\delta f(\mathbf{X}_1, t)\rangle$. We have

$$\left[\frac{\partial}{\partial t} + T(1)\right] |\,\delta f(\mathbf{X}_1, t)\rangle = O(1, 3) \int_0^t d\tau \left[P(\mathbf{X}_1, \tau)\, O(1, 4)\, P^{-1}(\mathbf{X}_1, \tau)\, P^{-1}(\mathbf{X}_4, \tau)\right.$$

$$\times \left\{|\,\delta f(\mathbf{X}_1, t)\rangle \langle \delta f(\mathbf{X}_3, t)\, \delta f(\mathbf{X}_4, t)\rangle_c + |\,\delta f(\mathbf{X}_4, t)\rangle \langle \delta f(\mathbf{X}_1, t)\, \delta f(\mathbf{X}_3, t)\rangle_c\right\}$$

$$+ P(\mathbf{X}_3, \tau)\, O(3, 4)\, P^{-1}(\mathbf{X}_3, \tau)\, P^{-1}(\mathbf{X}_4, \tau) \left\{\langle \delta f(\mathbf{X}_1, t)\, \delta f(\mathbf{X}_4, t)\rangle_c\, |\,\delta f(\mathbf{X}_3, t)\rangle\right.$$

$$\left.\left. + \langle \delta f(\mathbf{X}_1, t)\, \delta f(\mathbf{X}_3, t)\rangle_c\, |\,\delta f(\mathbf{X}_4, t)\rangle\right\}\right] \qquad (28)$$

The operators $O(1, 4)$ and $O(3, 4)$ create an electric field from the fluctuation density $\delta f(\mathbf{X}_4)$. In order to proceed we assume that this field may be expressed as

$$\int dk\, \delta\mathbf{E}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega_k t}$$

Although this is consistent with the asymptotic behavior from linear theory,[18] it does not necessarily give the proper connection to the initial state of the system, contrary to the procedures of Section III. Finally because the correlation function $\langle \delta f(\mathbf{X}_i)\ \delta f(\mathbf{X}_j)\rangle$ is not a function of $\mathbf{r}_i - \mathbf{r}_j$ except in the case of spatial homogeneity we use an adiabatic hypothesis to omit the dependence on slow spatial variation.

$$\left[\frac{\partial}{\partial t} + T(1)\right] \mid \delta f(\mathbf{X}_1, t)\rangle = O(1, 3) \int_0^t d\tau \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\omega_k \tau} \Big[P(\mathbf{X}_1, \tau)\, e^{i\mathbf{k}\cdot\mathbf{r}_1} \frac{q}{m} \frac{\partial}{\partial \mathbf{v}_1}$$

$$\times\ P^{-1}(\mathbf{X}_1, \tau) \Big\{ \langle \delta f(-\mathbf{k}, \mathbf{v}_3, t)\ \delta \mathbf{E}(\mathbf{k}, t)\rangle_c\, e^{-i\mathbf{k}\cdot\mathbf{r}_3} \mid \delta f(\mathbf{X}_1, t)\rangle + \langle \delta f(\mathbf{X}_1, t)\ \delta f(\mathbf{X}_3, t)\rangle_c$$

$$\times \mid \delta \mathbf{E}(\mathbf{k}, t)\rangle \Big\} + P(\mathbf{X}_3, \tau)\, e^{i\mathbf{k}\cdot\mathbf{r}_3} \frac{q}{m} \frac{\partial}{\partial \mathbf{v}_3} P^{-1}(\mathbf{X}_3, \tau) \Big\{ \langle \delta f(-\mathbf{k}, \mathbf{v}_1, t)\ \delta \mathbf{E}(\mathbf{k}, t)\rangle_c\, e^{-i\mathbf{k}\cdot\mathbf{r}_1}$$

$$\times \mid \delta f(\mathbf{X}_3, t)\rangle + \langle \delta f(\mathbf{X}_1, t)\ \delta f(\mathbf{X}_3, t)\rangle \mid \delta \mathbf{E}(\mathbf{k}, t)\rangle \Big\} \Big] \qquad (29)$$

The P operator contains two terms, a flow operator and a shielding operator. Because evaluation of the latter requires once again an assumption about the behavior of the electric field we leave these terms for further study and approximate

$$P(\mathbf{X}, \tau)\, e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{\partial \mathbf{v}} P^{-1}(\mathbf{X}, \tau) \qquad \text{by} \qquad e^{-V\tau}\, e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{\partial \mathbf{v}} e^{V\tau}$$

in Eq. (29).

$$\left[\frac{\partial}{\partial t} + T(1)\right] | \delta f(X_1, t)\rangle = \frac{q}{m} O(1, 3) \int_0^t d\tau \int \frac{dk}{(2\pi)^3} e^{i\omega_k \tau} \left[e^{ik\cdot r_1(-\tau)} \left\{ \frac{\partial r_1(\tau)}{\partial v_1} \right.\right.$$

$$\left. \cdot \frac{\partial}{\partial r_1} + \frac{\partial v_1(\tau)}{\partial v_1} \cdot \frac{\partial}{\partial v_1} \right\} \left\{ e^{-ik\cdot r_3} \langle \delta f(-k, v_3, t)\ \delta E(k, t)\rangle_c \,|\, \delta f(X_1, t)\rangle + \langle \delta f(X_1, t) \right.$$

$$\left. \times\ \delta f(X_3, t)\rangle_c \,|\, \delta E(k, t)\rangle \right\} + e^{ik\cdot r_3(-\tau)} \left\{ \frac{\partial r_3(\tau)}{\partial v_3} \cdot \frac{\partial}{\partial r_3} + \frac{\partial v_3(\tau)}{\partial v_3} \cdot \frac{\partial}{\partial v_3} \right\} \left\{ \langle \delta f(-k, v_1, t) \right.$$

$$\left.\left. \times\ \delta E(k, t)\rangle_c\, e^{-ik\cdot r_3} \,|\, \delta f(X_3, t)\rangle + \langle \delta f(X_1, t)\ \delta f(X_3, t)\rangle \,|\, \delta E(k, t)\rangle \right\} \right] \quad (30)$$

The first term may be evaluated by carrying the τ integration to infinity

$$\frac{\partial}{\partial v_1} \cdot \int \frac{dk}{(2\pi)^3} \frac{q^2}{m^2} \langle \delta E(k, t)\ \delta E(-k, t)\rangle \int_0^\infty d\tau\, e^{i\omega_k \tau} e^{ik\cdot[r_1(-\tau) - r_1]} \left[\frac{\partial r_1(\tau)}{\partial v_1} \right.$$

$$\left. \cdot \frac{\partial}{\partial r_1} + \frac{\partial v_1(\tau)}{\partial v_1} \cdot \frac{\partial}{\partial v_1} \right] | \delta f(X_1, t)\rangle$$

This term describes the diffusion of particle orbits due to background fluctuations in the electric field. This effect was first described by Dupree,[19] who based his theory on the Vlasov equation.

Before proceeding we discuss the consistency relations which modify the right side of Eq. (30). It should be noted that the desired consistency should be sought between the new equations for w_n (Eq. (24)) where the behavior of each w_n now differs from the $\beta = 0$ approximation because of interactions with w_2. This modified behavior alters the P_n which propagate the w_n in time, which in turn leads to a correction of the "collision operator" relating $\partial w_n / \partial t$ to $w_m(t)\, w_{n-m+2}(t)$.

In the approximation that interactions between spaces $\mathbf{X}_i$, $\mathbf{X}_j$ are neglected the consistency for the w_n equations simplifies greatly because of the possibility of treating each "δf" separately. The desired consistency among the equations for w_n now reduces to <u>self consistency</u> of the $|\delta f\rangle$ equation, as in the case of the Vlasov theory (Section B).

We wish to modify P, which propagates $|\delta f\rangle$ for $\beta = 0$, to absorb the corrections produced by the right side of Eq. (30). These corrections should alter in a self consistent way the collision operator relating $\partial |\delta f\rangle / \partial t$ to $\langle \delta f(t)\, \delta f(t)\rangle\, |\delta f(t)\rangle$. Since this appears very difficult to carry out we simply indicate how some of the consistency relations might be included.

Thus if we modify the orbits and ignore the other order β corrections, then in Eq. (30) we replace (symbolically) the operator

$$e^{-V\tau}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{\partial}{\partial \mathbf{v}}\, e^{V\tau}$$

by

$$e^{-V'\tau}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{\partial}{\partial \mathbf{v}}\, e^{V'\tau}$$

where

$$V' = V - \frac{\partial}{\partial \mathbf{v}} \cdot D_{\mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{v}} \cdot D_{\mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}}$$

and

$$D_{\mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} + D_{\mathbf{r}} \frac{\partial}{\partial \mathbf{r}} = \frac{q^2}{m^2} \int \frac{d\mathbf{k}}{(2\pi)^3} \langle \delta \mathbf{E}(\mathbf{k}, t)\, \delta \mathbf{E}(-\mathbf{k}, t) \rangle \int_0^{\infty} d\tau\, e^{i\omega_k \tau}\, e^{-V'\tau}\, e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{\partial \mathbf{v}}\, e^{V'\tau}\, e^{-i\mathbf{k}\cdot\mathbf{r}}$$

This correction has been considered in the limit of short time[19] and long time[20] in other work. This orbit correction may be applied to any other resonance term to eliminate mathematical difficulties, but its effect may be small compared to the other order β terms in a particular physical problem.

The fourth term may also be evaluated by carrying the τ integration to infinity.

$$- \frac{q^2}{m^2} \frac{\partial}{\partial \mathbf{v}_1} \cdot \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int d\mathbf{v}_3\, 4\pi \bar{n} q \frac{i(\mathbf{k} + \mathbf{k}')}{|\mathbf{k} + \mathbf{k}'|^2} \mid \delta \mathbf{E}(\mathbf{k}, t) \rangle\, e^{i\mathbf{k}\cdot\mathbf{r}_1} \int_0^{\infty} d\tau\, e^{i\omega_k \tau}$$

$$\times\, e^{i\mathbf{k}\cdot[\mathbf{r}_3(-\tau) - \mathbf{r}_3]} \left[\frac{\partial \mathbf{r}_3(\tau)}{\partial \mathbf{v}_3} \cdot i\mathbf{k}' + \frac{\partial \mathbf{v}_3(\tau)}{\partial \mathbf{v}_3} \cdot \frac{\partial}{\partial \mathbf{v}_3} \right] \langle \delta f(-\mathbf{k}', \mathbf{v}_1, t)\, \delta f(\mathbf{k}', \mathbf{v}_3, t) \rangle_c$$

This term represents a correction to the shielding of $\mid \delta \mathbf{E} \rangle$ due to non-uniformity in the background plasma. The principal effect is a correction to the dielectric function, leading to a change in the frequency ω_k.

The second and third terms describe effects similar to large angle scattering in a Boltzmann gas. The third

$$-\frac{q^2}{m^2}\frac{\partial}{\partial \mathbf{v}_1}\cdot\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\mathbf{k}'}{(2\pi)^3}\int d\mathbf{v}_3\, 4\pi\bar{n}q\,\frac{i(\mathbf{k}+\mathbf{k}')}{(\mathbf{k}+\mathbf{k}')^2}\int_0^\infty d\tau\, e^{i\omega_k\tau}\, e^{i\mathbf{k}\cdot[\mathbf{r}_3(-\tau)-\mathbf{r}_3]}$$

$$\times\,\langle\delta f(-\mathbf{k},\mathbf{v}_1,t)\,\delta\mathbf{E}(\mathbf{k},t)\rangle_c\cdot\left[\frac{\partial\mathbf{r}_3(\tau)}{\partial\mathbf{v}_3}\cdot i\mathbf{k}'+\frac{\partial\mathbf{v}_3(\tau)}{\partial\mathbf{v}_3}\cdot\frac{\partial}{\partial\mathbf{v}_3}\right]|\,\delta f(\mathbf{k}',\mathbf{v}_3,t)\rangle\, e^{i\mathbf{k}'\cdot\mathbf{r}_1}$$

represents the correction to the electric field caused by the modified orbits of the fluctuation particles. The second cannot be evaluated as written in Eq. (30) because the time integral is not well defined at its upper limit. This may be corrected by including the diffusion of orbits, but the self consistent correction due to scattering is probably more important physically. In both terms the non-linearity must be taken into account in correcting the P operator, because $\partial\,|\,\delta f\rangle/\delta t$ depends on the velocity distribution of the correlation functions through

$$\int d\mathbf{v}'\int d\mathbf{k}'\,\langle\delta f(\mathbf{k}',\mathbf{v})\,\delta f(-\mathbf{k}',\mathbf{v}')\rangle_c$$

We expect these scattering terms to be important for the description of acoustic phenomena where phase velocities are essentially constant over a range of wave numbers.

It is obvious that second order plasma theory is very complicated even when particle discreteness effects are omitted. It appears desirable to seek criteria for the relative importance of terms in a given physical situation, and means of approximating them, rather than a significantly better analytic treatment.

A. Comparison with Quasilinear Theory

In the analysis of quasilinear theory and mode coupling one derives an equation for a spatially averaged distribution $f_0(\mathbf{v}, t)$ by solving the equations for the fluctuations (Fourier transformed) $f_{\mathbf{k}}(\mathbf{v}, t)$, $\mathbf{E}_{\mathbf{k}}(t)$.

Thus the right side of the equation

$$\frac{\partial f_0}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{d\mathbf{k}}{(2\pi)^3} \langle f_{\mathbf{k}}(\mathbf{v}, t)\,\mathbf{E}_{-\mathbf{k}}(t)\rangle_{\mathbf{r}} \tag{31}$$

is to be found from the equations

$$\frac{\partial f_{\mathbf{k}}}{\partial t} + i\mathbf{k}\cdot\mathbf{v}\, f_{\mathbf{k}} + \frac{q}{m}\mathbf{E}_{\mathbf{k}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + \frac{q}{m}\frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{d\mathbf{k}'}{(2\pi)^3} \mathbf{E}_{\mathbf{k}'}\, f_{\mathbf{k}-\mathbf{k}'} = 0 \tag{32}$$

$$i\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}} = 4\pi\bar{n}q \int d\mathbf{v}\, f_{\mathbf{k}} \tag{33}$$

The usual procedure is to solve Eq. (32) by perturbation theory, treating the convolution integral as small. Thus the first approximation arises from the solution of the linearized Vlasov equation, while the corrections may be written as the sum of two series: 1) Terms proportional to $\partial f_0/\partial \mathbf{v}$ which we call shielding terms and shall discuss later. 2) Terms containing $e^{-i\mathbf{k}\cdot\mathbf{v}t}\, f_{\mathbf{k}}(\mathbf{v},\, t = 0)$ which are usually called the "initial value terms." We shall discuss these terms first. Dropping the shielding term from Eq. (32), and performing a power series expansion in $\mathbf{E}$, we find

$$f^{0}_{\mathbf{k},\text{i.v.}} = e^{-i\mathbf{k}\cdot\mathbf{v}t}\, f_{\mathbf{k}}(\mathbf{v},\, t = 0)$$

$$f^{n\geq 1}_{\mathbf{k},\text{i.v.}} = -\frac{q}{m}\int_0^t d\tau\; e^{-i\mathbf{k}\cdot\mathbf{v}(t-\tau)}\,\frac{\partial}{\partial \mathbf{v}}\cdot\int \frac{d\mathbf{k}'}{(2\pi)^3}\,\mathbf{E}_{\mathbf{k}'}(t = 0)\, e^{-i\omega_{k'}\tau}\, f^{n-1}_{\mathbf{k}-\mathbf{k}',\text{i.v.}}(\mathbf{v},\, \tau)$$

where the dependence on $f(t = 0)$ arises from the convolution over $f_{\mathbf{k}-\mathbf{k}'}$.

Because

$$\left(\frac{\partial}{\partial t} + i\mathbf{k}\cdot\mathbf{v}\right) f_{\mathbf{k}} + \frac{q}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\int \frac{d\mathbf{k}'}{(2\pi)^3}\,\mathbf{E}_{\mathbf{k}'}\, f_{\mathbf{k}-\mathbf{k}'}$$

is the transform of

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{q}{m}\,\delta\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\delta f$$

which may be identified with particle orbits, we draw the following conclusion: Neglecting initial value terms is equivalent to neglecting the effect of electric fields on the orbits (or statistical orbits) of particles in the fluctuations, and the modification of the electric fields due to these particle deflections.

As an example we consider $f^2_{\mathbf{k},\,\mathrm{i.v.}}$ from the perturbation analysis and average over the electric fields in the random phase approximation.

$$f^2_{\mathbf{k},\mathrm{i.v.}} = -\frac{q}{m}\int_0^t d\tau\, e^{-i\mathbf{k}\cdot\mathbf{v}(t-\tau)}\,\frac{\partial}{\partial \mathbf{v}}\int \frac{d\mathbf{k}'}{(2\pi)^3}\,\mathbf{E}_{\mathbf{k}'}(t=0)\,e^{-i\omega_{k'}\tau}\left(-\frac{q}{m}\right)\int_0^\tau d\tau'$$

$$\times\; e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}(\tau-\tau')}\,\frac{\partial}{\partial \mathbf{v}}\,\mathbf{E}_{-\mathbf{k}'}(t=0)\,e^{-i\omega_{-k'}\tau'}\,e^{-i\mathbf{k}\cdot\mathbf{v}\tau'}\,f_{\mathbf{k}}(\mathbf{v},\,t=0)$$

The quantity is equal to the first correction (in $\mathsf{D}_{\mathbf{r}}, \mathsf{D}_{\mathbf{v}}$) to the solution of

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla - \frac{\partial}{\partial \mathbf{v}}\cdot \mathsf{D}_{\mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}}\cdot \mathsf{D}_{\mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\right]\delta f = 0$$

where $\mathsf{D}_{\mathbf{r}}$ and $\mathsf{D}_{\mathbf{v}}$ are given without the consistency requirement, and with upper limit t in the time integration. By dropping initial value terms we omit this correction.

We next consider the effect of the perturbation treatment on the shielding terms. A simple way to estimate some of these effects is to expand (in $\mathsf{D}_{\mathbf{r}}, \mathsf{D}_{\mathbf{v}}$) the equation

$$\left(\frac{\partial}{\partial t} + i\mathbf{k}\cdot\mathbf{v}\right) f_{\mathbf{k}} - \frac{\partial}{\partial \mathbf{v}}\cdot \mathsf{D}_{\mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}} f_{\mathbf{k}} - \frac{\partial}{\partial \mathbf{v}}\cdot \mathsf{D}_{\mathbf{r}}\cdot i\mathbf{k}\, f_{\mathbf{k}} + \frac{q}{m}\mathbf{E}_{\mathbf{k}}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0$$

The lowest order term

$$f_{\mathbf{k},s}^{0} = \frac{q}{m}\left(i\omega_k - i\mathbf{k} - \mathbf{v}\right)^{-1}\left(e^{-i\omega_k t} - e^{-i\mathbf{k}\cdot\mathbf{v}t}\right)\mathbf{E}_{\mathbf{k}}(t = 0)\frac{\partial f_0}{\partial \mathbf{v}}$$

varies as

$$-\frac{q}{m}\, t\, e^{-i\omega_k t}\,\mathbf{E}_{\mathbf{k}} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

in the resonance region $\omega_k = \mathbf{k}\mathbf{v}$. We may estimate the fastest growing correction from the term in $D_{\mathbf{v}}$, and overestimate the time of validity of the expansion by treating $D_{\mathbf{v}}$ as constant.

$$f_{\mathbf{k},s}^{1} = \int_0^t d\tau\, e^{-i\mathbf{k}\cdot\mathbf{v}(t-\tau)}\left[\frac{\partial}{\partial \mathbf{v}} \cdot D_{\mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \cdot D_{\mathbf{r}} \cdot i\mathbf{k}\right] f_{\mathbf{k},s}^{0}(\tau)$$

$$f_{\mathbf{k},s}^{1}(\text{fastest growing}) = \frac{q}{m}\mathbf{E}_{\mathbf{k}}(t = 0) \cdot \frac{\partial f_0}{\partial \mathbf{v}} D_{\mathbf{v}} : \int_0^t d\tau\, e^{-i\mathbf{k}\cdot\mathbf{v}(t-\tau)}\frac{\partial^2}{\partial \mathbf{v}^2}\int_0^{\tau} d\tau' e^{-i\mathbf{k}\cdot\mathbf{v}(\tau-\tau')} \times e^{-i\omega_k \tau'}$$

$$f_{\mathbf{k},s}^{1}(\text{resonance}) \sim \frac{q}{m}\mathbf{E}_{\mathbf{k}}(t = 0) \cdot \frac{\partial f_0}{\partial \mathbf{v}}\left(i\mathbf{k} \cdot D_{\mathbf{v}} \cdot i\mathbf{k}\right)\frac{t^4}{12} e^{-i\omega_k t}$$

By comparing $f_{\mathbf{k},s}'$ and $f_{\mathbf{k},s}^{0}$ we see that the perturbation series breaks down[21] in the resonance region for $t \sim \left(k^2 D_{\mathbf{v}}\right)^{-1/3}$. From the equation

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot D_{\mathbf{v}} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

we may estimate the time for f_0 to change as $t \sim v^2/D_v$. Thus the perturbation expansion may break down before f_0 changes significantly.

Although this breakdown need not affect the electric fields strongly (the secularities vanish on velocity integration) it should modify the behavior of f_0 significantly. We believe that corrections to the f_0 equation (through $f_{\mathbf{k}}$) should be included if the mode coupling analysis is to be considered valid.

B. Comparison with Dupree's Theory

In Dupree's perturbation theory for plasma turbulence the starting point is the Vlasov equation together with Poisson's Equation (33).

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{q}{m} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{r}, \mathbf{v}, t) = 0 \tag{34}$$

The electric field and distribution function are then expanded in Fourier series in space ($\mathbf{k}$) followed by an additional Fourier series in the phases $(\beta_{\mathbf{k}}, \beta_{\mathbf{k}'} \cdots)$ relative to the arbitrary initial phases of the electric field.

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t) \exp\left(i\mathbf{k} \cdot \mathbf{r} + i\beta_{\mathbf{k}}\right)$$

$$f(\mathbf{r}, \mathbf{v}, t) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \sum_{n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots = -\infty}^{\infty} F\left(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots\right) e^{i n_{\mathbf{k}} \beta_{\mathbf{k}}} e^{i n_{\mathbf{k}'} \beta_{\mathbf{k}'}} e \cdots$$

Before proceeding we point out that Dupree's statement that at the initial instant

$$F\left(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots\right) = 0 \quad \text{unless} \quad n_{\mathbf{k}}, n_{\mathbf{k}'}, n_{\mathbf{k}''} \cdots = 0$$

is not consistent with Poisson's equation, so that the electric field is not uniquely determined. The statement regarding $f(t = 0)$ is essentially that of dropping the initial value terms of quasilinear theory,[22] except that the diffusion correction $D_{\mathbf{v}}$ to the orbits is produced by the statistical fields. The consequences of this approach have been discussed by Orszog and Kraichnan,[19] who have noted the similarity to the stochastic acceleration problem.

Instead of discussing the difficulties of Dupree's approach, we give a simple procedure based on the Vlasov equation which substantially duplicates the theory based on the equations for w_n, while permitting a comparison with Dupree's result.

We may ensemble average Eq. (34) in any convenient way to find

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \langle f \rangle + \frac{q}{m} \langle \mathbf{E} \rangle \cdot \frac{\partial}{\partial \mathbf{v}} \langle f \rangle = -\frac{q}{m} \frac{\partial}{\partial \mathbf{v}} \cdot \langle \delta f \delta \mathbf{E} \rangle \tag{35}$$

The difference quantity $f - \langle f \rangle \equiv \delta f$ then satisfies the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{q}{m} \langle \mathbf{E} \rangle \cdot \frac{\partial}{\partial \mathbf{v}}\right) \delta f = -\beta \frac{q}{m} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\delta f \delta \mathbf{E} - \langle \delta f \delta \mathbf{E} \rangle\right] \tag{36}$$

where we have inserted β in order to follow the procedure described in Sections II and III. We wish to obtain a statistical estimate of the order β terms by using a self consistent perturbation theory. From a power series expansion in β we find

$$\delta f^0 = P(t)\,\delta f(t=0)\,; \qquad \delta \mathbf{E}^0 = P(\mathbf{E},t)\,\delta f(t=0) \Rightarrow \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{-i\omega_{\mathbf{k}} t}\, \delta \mathbf{E}(\mathbf{k}, t=0)$$

$$\delta f^1 = -\frac{q}{m}\int_0^t d\tau\, P(t-\tau)\,\frac{\partial}{\partial \mathbf{v}}\cdot\left[\delta f(\tau)\,\delta\mathbf{E}(\tau) - \left\langle \delta f(\tau)\,\delta\mathbf{E}(\tau)\right\rangle\right]$$

$$\delta \mathbf{E}^1 = -\bar{n}q \int d\mathbf{r}' \int d\mathbf{v}'\, \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\, \delta f^1(\mathbf{r}', \mathbf{v}', t)$$

By using the $\beta = 0$ solutions we may write $\delta f^1(t)$ and $\delta\mathbf{E}^1(t)$ in terms of $\delta f^0(t)$ and $\delta\mathbf{E}^0(t)$. Then multiplying δf^1 by $\delta\mathbf{E}$ and $\delta\mathbf{E}^1$ by δf and dropping subscripts yields

$$\delta f(\mathbf{X},t)\,\delta\mathbf{E}(\mathbf{r},t) = -\frac{q}{m}\int_0^t d\tau\, P(\mathbf{X}, t-\tau)\,\frac{\partial}{\partial\mathbf{v}}\cdot\int\frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, e^{i\omega_{\mathbf{k}}(t-\tau)}\, P^{-1}(\mathbf{X}, t-\tau)$$
$$\times\left[\delta\mathbf{E}(\mathbf{k},t)\,\delta f(\mathbf{X},t)\,\delta\mathbf{E}(\mathbf{r},t) - \left\langle\delta\mathbf{E}(\mathbf{k},t)\,\delta f(-\mathbf{k},\mathbf{v},t)\right\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\,\delta\mathbf{E}(\mathbf{r},t)\right]$$

$$\delta\mathbf{E}(\mathbf{r},t)\,\delta f(\mathbf{X},t) = \frac{\bar{n}q^2}{m}\int d\mathbf{r}'\int d\mathbf{v}'\,\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\int_0^t d\tau\, P(\mathbf{X}', t-\tau)\,\frac{\partial}{\partial\mathbf{v}}\cdot\int\frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}'}$$
$$\times\, e^{i\omega_{\mathbf{k}}(t-\tau)}\, P^{-1}(\mathbf{X}', t-\tau)\left[\delta\mathbf{E}(\mathbf{k},t)\,\delta f(\mathbf{X}',t)\,\delta f(\mathbf{X},t) - \left\langle\delta\mathbf{E}(\mathbf{k},t)\,\delta f(-\mathbf{k},\mathbf{v}',t)\right\rangle\right.$$
$$\left.\times\, e^{-i\mathbf{k}\cdot\mathbf{r}'}\,\delta f(\mathbf{X},t)\right]$$

We now average in all ways so as to obtain pair correlations, and add to obtain all statistical contributions to the right side of Eq. (36).

$$\left(\frac{\partial \delta f}{\partial t}\right)\left(\begin{matrix}\text{statistical}\\ \text{collisions}\end{matrix}\right) = \frac{q^2}{m^2}\frac{\partial}{\partial \mathbf{v}} \cdot \int_0^t d\tau \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\omega_k \tau} \Big[P(\mathbf{X}, \tau)\, e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{\partial \mathbf{v}} P^{-1}(\mathbf{X}, \tau)$$

$$\times \Big\{ \langle \delta \mathbf{E}(\mathbf{k}, t)\, \delta \mathbf{E}(-\mathbf{k}, t) \rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\, \delta f(\mathbf{X}, t) + \langle \delta f(\mathbf{X}, t)\, \delta \mathbf{E}(\mathbf{r}, t) \rangle\, \delta \mathbf{E}(\mathbf{k}, t) \Big\}$$

$$- \frac{\bar{n} q}{m} \int d\mathbf{r}' \int d\mathbf{v}'\, \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} P(\mathbf{X}', \tau)\, e^{i\mathbf{k}\cdot\mathbf{r}'} \frac{\partial}{\partial \mathbf{v}'} P^{-1}(\mathbf{X}', \tau)$$

$$\times \Big\{ \langle \delta \mathbf{E}(\mathbf{k}, t)\, \delta f(-\mathbf{k}, \mathbf{v}, t) \rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\, \delta f(\mathbf{X}', t) + \langle \delta f(\mathbf{X}', t)\, \delta f(\mathbf{X}, t) \rangle\, \delta \mathbf{E}(\mathbf{k}, t) \Big\} \Big]$$

Although we are unable to motivate the above procedures, the result is identical to Eq. (29). As discussed previously one of these terms (modified by the consistency relation) is the diffusion correction to the orbits first obtained by Dupree.

VI. Boltzmann Gas

In this section we use the BBGKY equations for $g_n(\mathbf{X}, \cdots \mathbf{X}_n)$, the n particle correlation function $(g_1 = f)$. The force field F is produced externally and will be considered independent of time. Although some results may be obtained more directly by considering the n particle distribution f_n, we follow the procedure described previously. g_n satisfies the equation

$$\left[\frac{\partial}{\partial t} + H_n\right] g_n - \sum_{m=1}^{n-1} \sum_{i \neq j} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} g_m(\cdots, \mathbf{X}_i, \cdots) \, g_{n-m}(\cdots, \mathbf{X}_j, \cdots)$$

$$- \beta \bar{n} \sum_{i=1}^{n} \int d\mathbf{X}_{n+1} \nabla_i \phi_{i,n+1} \cdot \frac{\partial}{\partial \mathbf{v}_i} \left[\sum_{m=1}^{n} g_m(\cdots, \mathbf{X}_i, \cdots) \, g_{n-m+1}(\cdots, \mathbf{X}_{n+1}) \right.$$

$$\left. + \, g_{n+1}(\cdot, \mathbf{X}_i, \cdots, \mathbf{X}_{n+1}) \right] = 0 \quad (37)$$

where

$$H_n = \sum_{i=1}^{n} \left[\mathbf{v}_i \cdot \nabla_i + \frac{\mathbf{F}(\mathbf{r}_i)}{m} \cdot \frac{\partial}{\partial \mathbf{v}_i} \right] - \sum_{i<j=1}^{n} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i}$$

ϕ is the potential, and β is the ordering parameter. The sum

$$\sum_{m=1}^{n-1} \sum_{i \neq j} g_m \, g_{n-m}$$

includes all possible ways of distributing the n coordinates such that each appears once, and the i, j sum runs over all values which take one coordinate index from each correlation function.

Instead of attempting to write the full solution for $\beta = 0$, we consider the effect of the source terms for g_n as successive corrections, and use a second subscript to indicate the order of approximation of the source terms. Thus $g_{n,0}$ (no source term) is equal to $e^{-H_n t}\, g_n(\mathbf{X}_1, \cdots \mathbf{X}_n, t = 0)$, where $e^{-H_n t}$ runs the particles backward on their orbits for time t. By using the $g_{m,0}$ solutions in the source terms we may find $g_{n,1}$. Since

$$-\sum_{i,j} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i}\, g_m(\cdots, \mathbf{X}_i, \cdots)\, g_{n-m}(\cdots, \mathbf{X}_j, \cdots) = \Big[H_n(\mathbf{X}_1, \cdots, \mathbf{X}_n)$$

$$- H_m(\mathbf{X}_a, \mathbf{X}_b, \mathbf{X}_c, \cdots) - H_{n-m}(\mathbf{X}_p, \mathbf{X}_q, \mathbf{X}_r \cdots)\Big]\, g_m(\mathbf{X}_a, \mathbf{X}_b, \mathbf{X}_c, \cdots)\, g_{n-m}(\mathbf{X}_p, \mathbf{X}_q, \mathbf{X}_r, \cdots)$$

we may write (with $n \geq 2$, $\beta = 0$).

$$\left[\frac{\partial}{\partial t} + H_n\right] g_{n,1} + \sum_{m=1}^{n-1} \left[H_n - H_m - H_{n-m}\right] g_{m,0}\, g_{n-m,0} = 0$$

$$\left[\frac{\partial}{\partial t} + H_n\right] g_{n,1} + \sum_{m=1}^{n-1} \left[H_n + \frac{\partial}{\partial t}\right] g_{m,0}\, g_{n-m,0} = 0$$

$$g_{n,1} = e^{-H_n t} \sum_{m=1}^{n-1} g_{m,0}(t = 0)\, g_{n-m,0}(t = 0) - \sum_{m=1}^{n-1} g_{m,0}(t)\, g_{n-m,0}(t) + e^{-H_n t}\, g_n(t = 0) \tag{38}$$

Note that the sum on m contains all distinct ways of distributing the n coordinates between g_m and g_{n-m}. We use the relation $g_{m,0}(t = 0) = e^{H_m t}\, g_{m,0}(t)$ and then treat the $g_{m,0}$ as the exact functions, thereby throwing the approximation over to the operator.

$$g_{n,1}(t) = \sum_{m=1}^{n-1} \left\{ e^{-H_n t} e^{H_m t} e^{H_{n-m} t} - 1 \right\} g_m(t)\, g_{n-m}(t) + e^{-H_n t} g_n(t=0)$$

The limit $t \to \infty$ on the operator does not exist unless boundary conditions at $r = \infty$ are specified on the functions g_m (including $g_1 = f$!), for this operator brings in particles from greater distances as t increases. This should not occur because the orbits are altered by statistical encounters with particles in the space $(\mathbf{r}_{n+1}, \mathbf{v}_{n+1})$. However these effects are excluded by the ordering of terms in the hierarchy, so that order β effects should be included to give a proper result. Nevertheless $g_{2,1}(\beta = 0)$ may be used to obtain the first approximation to the collision integral.

A. Boltzmann Equation

For $n = 2$ we have

$$g_2 = e^{-H_2 t} g_2(t = 0) + \left\{ e^{-H_2 t} e^{H_1 t} e^{H_1 t} - 1 \right\} f(\mathbf{X}_1, t)\, f(\mathbf{X}_2, t) \qquad (39)$$

We substitute this result into the equation for f, and discard the initial value term $g_2(t = 0)$, which will vanish for long times if order β effects are included in the g_2 equation.

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_1 + \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}_1}\right] f = \bar{n} \int dr_2\, dv_2\, \nabla_1 \phi_{12} \cdot \frac{\partial}{\partial \mathbf{v}_1} \left[e^{-H_2 t}\, e^{H_1 t}\, e^{H_1 t} - 1\right] f(\mathbf{X}_1, t)$$

$$\times\ f(\mathbf{X}_2, t) + \bar{n} \int d\mathbf{r}_2\, d\mathbf{v}_2\, \nabla_1 \phi_{12} \cdot \frac{\partial}{\partial \mathbf{v}_1} f(\mathbf{X}_1, t)\, f(\mathbf{X}_2, t) + \bar{n}\beta \int d\mathbf{r}_2\, d\mathbf{v}_2\, \nabla_1 \phi_{12}$$

$$\cdot \frac{\partial}{\partial \mathbf{v}_1} \left[g_2 - g_2\,(\beta = 0)\right] \qquad (40)$$

The term $g_2 - g_2\,(\beta = 0)$ is formally order β, so the corresponding term in the equation is order β^2 and may be neglected. We cut off t in the operator at some large time T, use the consequent relation

$$H_2\, g_2 = -\nabla_1 \phi_{12} \cdot \left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2}\right) f f$$

and the fact that

$$- \int dX_2\, \nabla_1 \phi_{12} \cdot \frac{\partial}{\partial \mathbf{v}_1} = \int dX_2 \left[H_2 - H_1\,(1) - H_1\,(2)\right]$$

to write

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_1 + \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}_1}\right] f = \bar{n} \int d\mathbf{r}_2\, d\mathbf{v}_2 \left[\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2\right] \left[e^{-H_2 T}\, e^{H_1 T}\, e^{H_1 T} - 1\right] f(\mathbf{X}_1, t)\, f(\mathbf{X}_2, t)$$

Although Bogoluibov[1] obtained essentially this result he evaluated it incorrectly. The result is improper as written because the correlation extend to large distances $|\mathbf{v}_1 - \mathbf{v}_2|$ T, or ∞ in Bogoluibov's derivation. Since interactions with a statistically distributed third particle (an order β effect) will destroy the correlation we cut off the integral at a distance λ somewhat less than a mean free path. At distances of this order the collisional correction to g_2, and to f (since f does not evolve as $e^{-H_1 t} f(t = 0)$ for a mean free time) must be taken into account by the second order theory.

We first change the variable of integration to $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (b, \phi, z)$ where z is parallel to $\mathbf{v}_2 - \mathbf{v}_1$, and use the fact that

$$\int d\mathbf{X}\, g = 0$$

$$\left[\frac{\partial}{\partial t} + H_1\right] f = \bar{n} \int d\mathbf{v}_2 \int b\,db\,d\phi\,dz\, |\mathbf{v}_1 - \mathbf{v}_2| \frac{\partial}{\partial z} \left[e^{-H_2 T} e^{H_1 T} e^{H_1 T} - 1\right] \times f(\mathbf{r}_1, \mathbf{v}_1, t)\, f(\mathbf{r}_1 + \mathbf{r}, \mathbf{v}_2, t)$$

The external force affects only the motion of the center of mass and drops out of the operators in the collision integral. The z integral is cut off as specified above.

$$\left[\frac{\partial}{\partial t} + H_1\right] f = \bar{n} \int d\mathbf{v}_2 \int b\,db\,d\phi\, |\mathbf{v}_1 - \mathbf{v}_2| \left\{f(\mathbf{r}_1', \mathbf{v}_1')\, f(\mathbf{r}_2', \mathbf{v}_2') - f(\mathbf{r}_1, \mathbf{v}_1)\, f(\mathbf{r}_1 + \lambda, \mathbf{v}_2)\right\}$$

Here $\mathbf{r}_1'$, $\mathbf{v}_1'$, $\mathbf{r}_2'$, $\mathbf{v}_2'$ scatter in the absence of an external field to produce $\mathbf{r}_1$, $\mathbf{v}_1$, $\mathbf{r}_1 + \boldsymbol{\lambda}$, $\mathbf{v}_2$. We expand $\mathbf{r}_1'$ and $\mathbf{r}_2'$ above $\mathbf{r}_1$, keeping terms in ∇f, and find by evaluation

$$\Delta \mathbf{r}_1 = \frac{\lambda(\mathbf{v}_1' - \mathbf{v}_1)}{|\mathbf{v}_1 - \mathbf{v}_2|}, \qquad \Delta \mathbf{r}_2 = \frac{\lambda|\mathbf{v}_2' - \mathbf{v}_2|}{|\mathbf{v}_1 - \mathbf{v}_2|}$$

so that the resultant equation is

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \nabla_1 f + \frac{F}{m} \cdot \frac{\partial f}{\partial \mathbf{v}_1} = \bar{n} \int d\mathbf{v}_2 \int b db d\phi \, |\mathbf{v}_1 - \mathbf{v}_2| \left[f(\mathbf{r}_1, \mathbf{v}_1') \, f(\mathbf{r}_1, \mathbf{v}_2') \right.$$

$$\left. - f(\mathbf{r}_1, \mathbf{v}_1) \, f(\mathbf{r}_1, \mathbf{v}_2) \right] + \bar{n}\lambda \int d\mathbf{v}_2 \int b db d\phi \left[(\mathbf{v}_1' - \mathbf{v}_1) \cdot \frac{\partial f(\mathbf{r}_1, \mathbf{v}_1')}{\partial \mathbf{v}_1} \, f(\mathbf{r}_1, \mathbf{v}_2') \right.$$

$$\left. + (\mathbf{v}_2' - \mathbf{v}_1) \cdot f(\mathbf{r}_1, \mathbf{v}_1') \frac{\partial f(\mathbf{r}_1, \mathbf{v}_2')}{\partial \mathbf{r}_1} \right] - \bar{n}\lambda \int d\mathbf{v}_2 \int b db d\phi \, (\mathbf{v}_1 - \mathbf{v}_2)$$

$$\cdot f(\mathbf{r}_1, \mathbf{v}_1) \frac{\partial f(\mathbf{r}_1, \mathbf{v}_2)}{\partial \mathbf{r}_1} \qquad (41)$$

The first term on the right is the usual Boltzmann integral, and the corrections due to a spatial gradient may be identified with the orbits of the scattering particles. The first represents a correction to the number scattered into $\mathbf{r}_1$, $\mathbf{v}_1$ because the number of collisions at a distance λ is not generally the same as the number at $\mathbf{r}_1$. Since particles scattering into $\mathbf{r}_1, \mathbf{v}_1$ suffered collisions at an earlier time, for consistency the number scattered out of $\mathbf{r}_1, \mathbf{v}_1$ should be evaluated at this earlier time. This gives rise to the second term in ∇f, a correction to the number scattered out of $\mathbf{r}_1, \mathbf{v}_1$, because the target particle density must be evaluated at $\mathbf{r}_1 + \boldsymbol{\lambda}$.

The collision integral conserves local number density, but it contributes to a momentum and energy flux.

$$\int d\mathbf{v}_1\, \chi(\mathbf{v}_1)\ [\text{collision integral}] = n\lambda\nabla_1 \int d\mathbf{v}_1\, \mathbf{v}_1\, \chi(\mathbf{v}_1) \int d\mathbf{v}_2 \Big\{ f(\mathbf{r}_1, \mathbf{v}_1) \times f(\mathbf{r}_1, \mathbf{v}_2) - f(\mathbf{r}_1, \mathbf{v}_1') \, f(\mathbf{r}_1, \mathbf{v}_2') \Big\}$$

where

$$\chi(\mathbf{v}_1) = \left\{ \bar{n} m \mathbf{v}_1, \ \frac{1}{2}\bar{n} m v_1^2 \right\}$$

We have not pointed out several minor difficulties in the derivation of the kinetic equation, as their resolution should be based on the inclusion of three particle effects.

B. Higher Order Effects

As in Part A we substitute each $g_{n+1,1}$ into the order β term of the equation for g_n and calculate the collisional correction to behavior of g_n.

$$\left[\frac{\partial}{\partial t} + H_n\right] g_n - \sum_{m=1}^{n-1} \sum_{i,j} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} g_m(\cdots, \mathbf{X}_i, \cdots)\, g_{n-m}(\cdots, \mathbf{X}_j, \cdots)$$

$$- \bar{n} \sum_{m=1}^{n} \sum_{i \neq j=1}^{n} \int d\mathbf{X}_{n+1} \nabla_i \phi_{i,n+1} \cdot \frac{\partial}{\partial \mathbf{v}_i} g_m(\cdots, \mathbf{X}_i, \cdots)\, g_{n-m+1}(\cdots, \mathbf{X}_{n+1}) + \bar{n} \int d\mathbf{X}_{n+1}$$

$$\times \left[H_{n+1} - H_n(\mathbf{X}_1, \cdots \mathbf{X}_n) - H_1(\mathbf{X}_{n+1})\right] \sum_{m=1}^{n} \left\{ e^{-H_{n+1}t} e^{H_m t} e^{H_{n-m+1}t} - 1 \right\}_{t\uparrow} g_m\, g_{n-m+1}$$

$$= -\beta \bar{n} \sum_{i=1}^{n} \int d\mathbf{r}_{n+1} \nabla_i \phi_{i,n+1} \cdot \frac{\partial}{\partial \mathbf{v}_i} \left[g_{n+1,1} - g_{n+1}\right] \quad (42)$$

The collision term is evaluated as before.

$$\left[\frac{\partial}{\partial t} + H_n\right] g_n - \sum_{m=1}^{n-1} \sum_{i,j} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} g_m(\cdots X_i \cdots)\, g_{n-m}(\cdots X_j \cdots) - \bar{n} \int d\mathbf{X}_{n+1} \left[\sum_{i=1}^{n+1} \mathbf{v}_i \cdot \nabla_i\right]$$

$$\times \sum_{m=1}^{n} \left\{ e^{-H_{n+1}T} e^{H_m T} e^{H_{n-m+1}T} - 1 \right\} g_m\, g_{n-m+1} = -\bar{n}\beta \int d\mathbf{X}_{n+1} \nabla_i \phi_{i,n+1} \cdot \frac{\partial}{\partial \mathbf{v}_i} \left[g_{n+1,1} - g_{n+1}\right]$$

Each integral over $\partial/\partial \mathbf{r}_{n+1}$ should be cut off consistent with the collision operator in the next higher equation. We are unable to perform this operation at present, but progress is still possible for $n = 2$.

For $n = 2$ we may write

$$\left[\frac{\partial}{\partial t} + H_2\right] g_2 - \sum_{i \neq j = 1}^{2} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} f f - \bar{n} \int d\mathbf{X}_3 \sum_{i=1}^{3} \left[\mathbf{v}_i \cdot \nabla_i\right] \sum_{\text{permutations}}$$

$$\times \left\{e^{-H_3 T} e^{H_2 T} e^{H_1 T} - 1\right\} g_2(\mathbf{X}_i, \mathbf{X}_j, t) \; f(\mathbf{X}_k, t) = -\bar{n}\beta \int d\mathbf{X}_3 \left[H_3 - H_2(\mathbf{X}_1, \mathbf{X}_2) - H_1(\mathbf{X}_3)\right]$$

$$\times \left\{g_{3,2} - g_{3,1}\right\} + \bar{n}\beta^2 \int d\mathbf{X}_3 \left[H_3 - H_2(\mathbf{X}_1, \mathbf{X}_2) - H_1(\mathbf{X}_3)\right] \left\{g_{3,2} - g_3\right\} \quad (43)$$

where $g_{3,2}$ includes the second approximation to the source terms of the g_3 equation. The difference between g_3 and $g_{3,2}$ is regarded as order β and will be neglected here. (In general $g_{n,n-1}$ is a functional of f, and the difference $g_n - g_{n-1}$ should be treated as one order smaller, i.e. $g_n - g_{n,n-1} = O(\beta g_n)$). We may calculate $g_{3,2}$ to order 1 by using the first approximation $g_{2,1}$ in the source term.

$$\left[\frac{\partial}{\partial t} + H_3\right] g_{3,2} = + \sum_{i,j} \sum_{\text{permutations}} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} g_{2,1} f$$

$$= - \sum_{p} \left[H_3 - H_2 - H_1\right] g_{2,1} f$$

Since

$$H_1 f = -\frac{\partial f}{\partial t}$$

and

$$H_2 g_{2,1} = -\frac{\partial g_{2,1}}{\partial t} + \sum \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} ff$$

we find

$$\left[\frac{\partial}{\partial t} + H_3\right] g_{3,2} = -\sum_p \left[H_3 + \frac{\partial}{\partial t}\right] g_{2,1} f - \left[H_3 + \frac{\partial}{\partial t}\right] fff$$

$$g_{3,2}(t) = e^{-H_3 t} \left[\sum_p g_{2,1}(t = 0)\, f(t = 0) + f(t = 0)\, f(t = 0)\, f(t = 0)\right]$$

$$- \sum_p g_{2,1}(t)\, f(t) - f(t)\, f(t)\, f(t)$$

$$g_{3,2} = \left\{e^{-H_3 t}\, e^{H_1 t}\, e^{H_1 t}\, e^{H_1 t} - \sum_p e^{-H_2 t}\, e^{H_1 t}\, e^{H_1 t} + 2\right\}$$

$$\times\ f(X_1, t)\, f(X_2, t)\, f(X_3, t)$$

In calculating the difference $\beta\left(g_{3,2} - g_{3,1}\right)$ to order β we may approximate $g_{3,1}\left(g_2\right)$ to order one by $\dot{g}_{3,1}\left(g_{2,1}\right)$

$$g_{3,1}\left(g_{2,1}\right) = \sum_{p} \left\{ e^{-H_3 t} e^{H_2(\mathbf{X}_i,\mathbf{X}_j)t} e^{H_1(\mathbf{X}_k)t} - 1 \right\} g_{2,1}\left(\mathbf{X}_i, \mathbf{X}_j, t\right) f\left(\mathbf{X}_k, t\right)$$

$$= \sum_{p} \left\{ e^{-H_3 t} e^{H_2(\mathbf{X}_i,\mathbf{X}_j)t} e^{H_1(\mathbf{X}_k)t} - 1 \right\} \left\{ e^{-H_2(\mathbf{X}_i,\mathbf{X}_j)t} e^{H_1(\mathbf{X}_i)t} \right.$$

$$\left. \times\ e^{H_1(\mathbf{X}_j)t} - 1 \right\} f\left(\mathbf{X}_i, t\right) f\left(\mathbf{X}_j, f\right) f\left(\mathbf{X}_k, t\right)$$

Taking the difference we find

$$g_{3,2}\left(g_{2,1}\right) - g_{3,1} = \left\{ \sum_{p} e^{-H_3 t} e^{H_2(\mathbf{X}_i,\mathbf{X}_j)t} e^{H_1(\mathbf{X}_k)t} - 2e^{-H_3 t} e^{H_1(\mathbf{X}_i)t} \right.$$

$$\left. \times\ e^{H_1(\mathbf{X}_j)t} e^{H_1(\mathbf{X}_k)t} + 1 \right\} f\left(\mathbf{X}_i, t\right) f\left(\mathbf{X}_j, t\right) f\left(\mathbf{X}_k, t\right)$$

We substitute this result into Eq. (43) to find the equation for g_2, correct to order β.

$$\left[\frac{\partial}{\partial t} + H_2\right] g_2 - \sum_{i \neq j} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} ff - \bar{n} \int d\mathbf{X}_3 \sum_{i=1}^{3} \left[v_i \cdot \nabla_i\right] \sum_{p} \left\{ e^{-H_3 T} e^{H_2 T} e^{H_1 T} - 1 \right\}$$

$$\times\ g_2 f = -\bar{n} \int d\mathbf{X}_3 \sum_{i=1}^{3} \left[v_i \cdot \nabla_i\right] \left\{ \sum_{p} e^{-H_3 T} e^{H_2 T} e^{H_1 T} - 2e^{-H_3 T} e^{H_1 T} e^{H_1 T} e^{H_1 T} + 1 \right\} fff \qquad (44)$$

The integrals should be cut off by considering the effect of a statistically distributed fourth particle. By initial hypothesis g_2 should depend on this cutoff in a weak way (order β^2). As before g_2 should be obtained in terms of an arbitrary f, after which the dependence on f at earlier times should be eliminated using the formal solution for the Boltzmann equation (Eq. (41)). The latter requirement may be avoided if we consider a system sufficiently near equilibrium so that the change of f due to flow is canceled to order β by the effects of collisions. Even in this (transport) regime the solution of the g_2 equation appears difficult, for the collisional shielding terms cannot be found by expansion.

VII. Plasma Equilibrium

For the study of thermal equilibrium the cluster expansion discussed recently by Ramanathan[23] is most suitable. In equilibrium the one particle distribution is simply $f(\mathbf{X}_1) = f_m(v_1)$, the Maxwellian velocity distribution. The pair and triplet correlations $\alpha_{ij}(\mathbf{r}_i - \mathbf{r}_j)$, $\alpha_{ijk}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)$ are defined by

$$f_2 = f_m(v_1)\, f_m(v_2)\, (1 + \alpha_{12})$$

$$f_3 = f_n(v_1)\, f_m(v_2)\, f_m(v_3)\, (1 + \alpha_{12})\, (1 + \alpha_{13})\, (1 + \alpha_{23})\, (1 + \alpha_{123})$$

The pair correlation is related to that of the BBGKY by $g_2 = f_m(v_1)\, f_m(v_2)\, \alpha_{12}$ while the triplet correlations are quite different. For either hierarchy of equations the usual expansion procedures are not uniformly valid; they break down for large $\mathbf{r}_{ij}$. In this section we find a convergent order β correction to α_{12}.

The function α_{12} satisfies the equation

$$C(1)\,\alpha_{12} = -\frac{1}{KT}\nabla_1 \phi_{12} - \frac{\beta}{KT}\left[\nabla_1 \phi_{12}\alpha_{12} + \bar{n}\int d\mathbf{r}_3 \nabla_1 \phi_{13}\left\{\alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{23} + \alpha_{123}\right\}\right] + O\left(\beta^2\right) + \cdots \quad (45)$$

where the operator C is defined by

$$C(i)\,\psi\left(\mathbf{X}_i, \mathbf{X}_j, \cdots\right) = \nabla_i \psi + \frac{\bar{n}}{KT}\int d\mathbf{r}_{i'} \nabla_i \phi_{i,i'}\,\psi\left(\mathbf{r}_{i'}, \mathbf{r}_j, \cdots\right)$$

and ϕ is the potential $\phi_{ij} = -q^2/|\mathbf{r}_i - \mathbf{r}_j|$. The triplet correlation α_{ijk} satisfies the equation

$$C(i)\,\alpha_{ijk} = -\frac{\bar{n}}{KT}\int d\mathbf{r}_4 \nabla_i \phi_{i4}\,\alpha_{j4}\,\alpha_{k4} + O(\beta) + \cdots \quad (46)$$

and all $\beta = 0$ equations for $n > 2$ have the form

$$C(i)\,\alpha_{ij\cdots n} = -\frac{\bar{n}}{KT}\int dr_{n+1} \nabla_i \phi_{i,n+1}\begin{Bmatrix}\text{no dependence}\\ \text{on } \mathbf{r}_i\end{Bmatrix}$$

We now define χ_{ij}, which is the zero order approximation to the pair correlation, by

$$C(i)\,\chi_{ij} = -\frac{1}{KT}\nabla_i \phi_{ij} \qquad \chi_{ij}(\infty) = 0$$

In a plasma

$$\chi_{ij} = -\frac{q^2}{KT|\mathbf{r}_i - \mathbf{r}_j|} e^{-|\mathbf{r}_i - \mathbf{r}_j|/\lambda_d}$$

where $\lambda_d^2 = KT/4\pi\bar{n}q^2$. We see at once that Eq. (46) has the solution

$$\alpha_{ijk} = \bar{n}\int dr_4\, \chi_{i4}\, \alpha_{j4}\, \alpha_{k4}$$

We substitute this result into Eq. (45), with i = 3, and use the approximation $\alpha_{23} \Rightarrow \chi_{23}$ in the term $\alpha_{12}\,\alpha_{23}$, to find the equation for α_{12} correct to order β.

$$C(1)\,\alpha_{12} = -\frac{1}{KT}\nabla_1 \phi_{12} + \beta\left[\nabla_1 \chi_{12}\alpha_{12} + \bar{n}\int dr_3\, \nabla_1 \chi_{13}\alpha_{13}\alpha_{23}\right] \qquad (47)$$

If we use perturbation theory we substitute $\alpha_{ij} = \chi_{ij}$ on the right and use the integral

$$C(i)\,\alpha_{i\ldots} = \nabla_i \psi(\mathbf{r}_i, \cdots) \Rightarrow \alpha_{i\ldots} = \left[\psi(\mathbf{r}_i, \cdots) + \bar{n}\int d\mathbf{r}_{i'}\, \chi_{i,i'}\, \psi(\mathbf{r}_{i'}, \cdots)\right] = I(i)\,\psi(\mathbf{r}_i, \cdots)$$

then we find at once

$$\alpha_{12} = \chi_{12} + \frac{1}{2} I(1) \left[\chi_{12}^2 + \bar{n} \int dr_3 \, \chi_{13}^2 \, \chi_{23} \right]$$

which is the result obtained previously by Rostoker and O'Neil[24] for $r_{12} \sim \lambda_d$. This result breaks down for $r_{12} \sim q^2/KT$, where $\alpha_{12} = e^{\chi_{12}} - 1$, and for $r_{12} >> \lambda_d$. In the latter case we approximate α_{12} and α_{13} by χ in the order β terms of Eq. (47), to find

$$C(1)\, \alpha_{12} - \frac{n}{2} \int d\mathbf{r}_3 \, \nabla_1 \chi_{13}^2 \, \alpha_{23} = -\frac{1}{KT} \nabla_1 \phi_{12} + \frac{1}{2} \nabla_1 \chi_{12}^2 \tag{48}$$

We use a Fourier transform to solve Eq. (48).

$$\alpha_{12}(k) = -\frac{1}{n} \frac{\left[\frac{1}{k^2 \lambda_d^2} - \frac{i}{4} \left(\frac{q^2}{KT} \right) \frac{1}{k \lambda_d^2} \ln \left(\frac{1 - ik\lambda_d/2}{1 + ik\lambda_d/2} \right) \right]}{\left[1 + \frac{1}{k^2 \lambda_d^2} - \frac{i}{4} \left(\frac{q^2}{KT} \right) \frac{1}{k \lambda_d^2} \ln \left(\frac{1 - ik\lambda_d/2}{1 + ik\lambda_d/2} \right) \right]}$$

The inverse transformation is evaluated by analytic continuation. The result comes from the pole at

$$k_0 = i \left[\frac{1}{\lambda_d} + \frac{q^2}{8\, KT \lambda_d^2} \ln 3 \right]$$

where the contribution from a branch cut starting at $k = 2ik_d$ may be neglected for large r_{12}.

$$\alpha_{12}(r) = -\frac{q^2}{KTr} e^{-k_0 r} \left\{ \frac{1 + \frac{3}{8}\frac{q^2}{KT\lambda_d} \ln 3}{1 - \frac{q^2}{6\,KT\lambda_d}} \right\} \tag{49}$$

It is apparent that the asymptotic behavior of α_{12} remains undetermined.

VIII. Conclusion

Although the procedures described in this paper appear adequate for the calculation of a reasonably accurate (first order) collision term, it may be desirable to recast the theory in less mathematical form for the calculation of higher order corrections. Since the solution of the first order theory is necessary for the calculation of higher order terms, we must consider the development of procedures for solving kinetic equations as a primary goal. For the important case of asymptotic solutions (in the domain of transport theory) the knowledge of first order solutions may not be necessary, but it is not clear that second order corrections will be important in this regime.

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10. The inclusion of time variation causes great difficulties, for even the operator P defined later must be generalized. These difficulties appear inevitably in the second order theory.

11. T. H. Dupree, ref. 7. The equation for the particle fluctuation "δf" is augmented by the Maxwell curl equations for $\delta\mathbf{E}$ and $\delta\mathbf{B}$. The propagation operator for δf must then be generalized to a 3×3 matrix which propagates $(\delta f, \delta\mathbf{E}, \delta\mathbf{B})$.

12. In general one may think of each coordinate as having an associated particle index. Thus $h_{3\mu\nu\alpha}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = h_3(\mathbf{X}_1, \mu; \mathbf{X}_2, \nu; \mathbf{X}_3, \alpha)$, etc. The integration creating an electric field becomes integration plus

summation over particle species.

$$\bar{n}q \int d\mathbf{X}' \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} h_n (\cdots, \mathbf{X}', \cdots) \Rightarrow \sum_{\nu} \bar{n}_{\nu} q_{\nu} \int d\mathbf{X}' \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} h_n (\cdots; \mathbf{X}', \nu; \cdots)$$

Henceforth any velocity integration shall imply summation over the associated particle species.

13. In the multispecies case

$$\Delta(\mathbf{X}_i, \mathbf{X}_j) \Rightarrow \Delta(\mathbf{X}_i, \mu; \mathbf{X}_j, \nu) = \frac{\delta_{\mu\nu}}{\bar{n}_{\mu}} \delta(\mathbf{r}_i - \mathbf{r}_j) \delta(\mathbf{v}_i - \mathbf{v}_j)$$

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17. Because of the time dependence of f, the propagator depends on t as well as $t - \tau$, so that a more general notation is needed. Lacking an explicit form for the operator we continue as before, for the notation remains consistent after the approximations we are forced to make.

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21. The series converges, but one must include more and more terms as t increases.

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