The Identification of Moving Magnetic Field Lines

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ABSTRACT

Magnetic field line motion and its relationship with particle $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts are considered. It is shown that a unique and significant field line identification can be made whenever all field lines in a region pierce a conducting, but arbitrarily shaped surface. If the region in question is free space, particle electric drifts, in general, bear no relation with field line motion identified in this fashion. However, with a conducting plasma filling the region and allowing no parallel component of electric field ($\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$), particles drift so as to always remain on the same magnetic line of force. These points are specifically illustrated for a time-dependent model of the earth's magnetosphere.

INTRODUCTION

A widely used concept in the electrodynamics of a guiding center plasma is that of frozen-in magnetic field line motion. This concept asserts (Newcomb, 1958) that the velocity field constructed instantaneously of particle $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocities is, in the appropriate circumstances, a legitimate (flux preserving) field line motion.

Theoretical analyses of particle diffusion in the earth's magnetosphere have used this concept (Kellogg, 1959; Parker, 1960; Davis and Chang, 1962; Nakada and Mead, 1965; Fälthammar, 1965; Conrath, 1967). During model, time-dependent, magnetospheric disturbances, a magnetic field line is identified (in these studies) by the time invariant longitude and latitude with which it enters the earth's dipole. Each field line is consequently traced through space (using the field line equations) as a function of time. The frozen-in field line concept is then applied in the identification of particle $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts with the motion of field lines as so defined.

Our interest in this problem was stimulated by Kavanagh's (1967) recent criticism of the work of Nakada and Mead. Kavanagh's criticism is equally applicable to all of the previously cited work and is based upon a direct calculation of particle electric drifts, using a vacuum electric field characteristic of the time dependent magnetospheric models. Kavanagh finds that the particle drifts which he calculates differ grossly from the field line motions of the earlier work. Seemingly, either Kavanagh or the earlier authors are in error.

In this paper we show how a magnetic field line velocity of significance can be uniquely defined whenever all magnetic lines cut an arbitrarily shaped but perfectly conducting surface. It is found that particle $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts coincide with the field line motion whenever no parallel component of electric field exists ($\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$). Our results indicate that the field line identification and the frozen-in field assumption previously used are indeed valid, but in a model which differs from that assumed by Kavanagh by the presence of a conducting plasma. We further show explicitly how the electric field found by Kavanagh is modified by the presence of this conducting plasma, the ensuing $\underset{\sim}{E} \times \underset{\sim}{B}$ motions being then identical with those derived by the earlier workers.

Field Line Identification In the Presence of a Conducting Surface

Consider a volume of space V in which there exists a magnetic field $\underset{\sim}{B}(\underset{\sim}{r}, t)$. Let the field be sufficiently regular that it may be described by Euler potentials $\alpha(\underset{\sim}{r}, t)$ and $\beta(\underset{\sim}{r}, t)$, where $\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$. Define the magnetic vector potential as $\underset{\sim}{A} = \alpha\underset{\sim}{\nabla}\beta$.

The choice of α and β is not unique, for any transformation $\alpha' = \alpha'(\alpha, \beta, t)$, $\beta' = \beta'(\alpha, \beta, t)$ leads to equally acceptable Euler potentials provided that the Jacobian, $J\left(\frac{\alpha', \beta'}{\alpha, \beta}\right)$, of the transformation is unity. Such transformations of Euler potentials are completely equivalent to gauge transformation of the vector potential $A \rightarrow A + \nabla\Lambda$.

Now consider a stationary surface S sufficiently large that every line of force cuts S at least once. A choice of α and β on S (subject, of course, to $\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$) fixes the gauge throughout V.

The electric field $\underset{\sim}{E}(\underset{\sim}{r}, t)$ is given by

$$\underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t} - \underset{\sim}{\nabla}\varphi$$
$$= \frac{1}{c}\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right) - \nabla(\varphi + \psi) \qquad (1)$$
$$= -\frac{\underset{\sim}{W}\times\underset{\sim}{B}}{c} - \underset{\sim}{\nabla}(\varphi + \psi),$$

where

$$\psi = \frac{1}{c}\alpha\frac{\partial\beta}{\partial t} . \qquad (2)$$

We have introduced into the last form of Eq. (1) the velocity

$$\underset{\sim}{W} = \left(\frac{\partial\beta}{\partial t}\underset{\sim}{\nabla}\alpha - \frac{\partial\alpha}{\partial t}\underset{\sim}{\nabla}\beta\right)\times\frac{\underset{\sim}{B}}{B^2} . \qquad (3)$$

It may be verified by direct substitution that $\underset{\sim}{w}$ satisfies the equation

$$\underset{\sim}{\nabla}\times\left(\underset{\sim}{E} + \frac{\underset{\sim}{W}\times\underset{\sim}{B}}{c}\right) = 0 \qquad (4)$$

and is therefore a perfectly valid flux preserving field line velocity. Further, with this field line identification, lines move in such a way as to preserve their α - β labels. This follows from the easily proven relations

$$\frac{\partial \alpha}{\partial t} + \underset{\sim}{W} \cdot \underset{\sim}{\nabla} \alpha = 0 , \tag{5a}$$

$$\frac{\partial \beta}{\partial t} + \underset{\sim}{W} \cdot \underset{\sim}{\nabla} \beta = 0 , \tag{5b}$$

Now let us demand that the surface S be a perfect conductor so that we have the requirement that the normal component of $\underset{\sim}{B}$ at the surface be constant in time,

$$\underset{\sim}{n} \cdot \underset{\sim}{B} = f(X_1, X_2)\, J\left(\frac{\alpha, \beta}{X_1, X_2}\right) = \text{const.} \tag{6}$$

Here $\underset{\sim}{n}$ is the unit normal to the surface, X_1 and X_2 are general coordinates defined on the surface, $f(X_1, X_2)$ is a function which depends on the curvilinear nature of the X_1, X_2 coordinates, and J is the Jacobian. Equation 6 implies $J\left(\frac{\alpha', \beta'}{X_1, X_2}\right) = J\left(\frac{\alpha, \beta}{X_1, X_2}\right)$ and hence $J\left(\frac{\alpha', \beta'}{\alpha, \beta}\right) =$ $J\left(\frac{\alpha', \beta'}{X_1, X_2}\right) \Big/ J\left(\frac{\alpha, \beta}{X_1, X_2}\right) = 1.$

Any changes with time of α and β on S thus correspond to relabeling transformations, which we eliminate by choosing $\frac{\partial \alpha}{\partial t} = \frac{\partial \beta}{\partial t} = 0$ on the surface S.

A unique identification of magnetic field lines is thus established: a field line is permanently labeled by the time independent values of α and β at the fixed point (or points) where it crosses S. This is, in fact, the identification of field lines made by the earlier authors, the role of the conducting surface being here assumed by the requirement that the same field line always enter the earth origin at the same longitude and latitude.

The question now arises as to what relationship this motion has to the particles which might be trapped in the magnetic field. The drift of low energy trapped particles perpendicular to $\underset{\sim}{B}$ is predominantly the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocity

$$\underset{\sim}{V} = c \, \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} . \tag{7}$$

(Magnetic gradient and curvature drifts are proportional to the energy of a particle.) This drift velocity is itself an acceptable field line velocity only if

$$\nabla \times \left[\underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} \right] = \underset{\sim}{\nabla} \times \left[\frac{\underset{\sim}{E} \cdot \underset{\sim}{B} \, \underset{\sim}{B}}{B^2} \right] = 0 . \tag{8}$$

Our (now uniquely specified) field line velocity $\underset{\sim}{w}$ differs from this drift velocity by

$$\underset{\sim}{V} - \underset{\sim}{W} = \frac{c\, \underset{\sim}{B} \times \underset{\sim}{\nabla}(\phi + \psi)}{B^2} . \qquad (9)$$

Since the tangential electric field vanishes at a conducting surface and $\frac{\partial \beta}{\partial t} = \frac{\partial \alpha}{\partial t} = 0$ on S by construction, we see from Eq. (1) that $(\varphi + \psi)$ is constant everywhere on S. Without further restriction on the electric field in the volume V, the term $c \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla}(\varphi + \psi)}{B^2}$ does not, in general, vanish; $\underset{\sim}{E}$ is yet undetermined; and $\underset{\sim}{v}$ and $\underset{\sim}{w}$ differ. It is just this difference between $\underset{\sim}{v}$ and $\underset{\sim}{w}$ which is the object of Kavanagh's criticism.

Impose now the condition that $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$ by filling the volume V with a conducting plasma. We find from Eq. (1) that $(\varphi + \psi)$ is constant along each field line. Furthermore, since all field lines intersect S and $(\varphi + \psi)$ is constant on S, we conclude that when $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$, $(\varphi + \psi)$ is constant throughout space. From Eq. (9) it is then immediately evident that the electric drift velocity $\underset{\sim}{v}$ and this uniquely defined field line velocity $\underset{\sim}{w}$ are identical. The electric field is directly given as $\underset{\sim}{E} = \frac{1}{c}\left[\frac{\partial \beta}{\partial t} \underset{\sim}{\nabla}\alpha - \frac{\partial \alpha}{\partial t} \underset{\sim}{\nabla}\beta \right]$. This assumption that $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$ distinguishes the models developed in previous work from Kavanagh's vacuum configuration.

<u>Application to a Model Magnetosphere</u>

In illustration of the field line identification proposed in the previous section, we here consider a model for time dependent magnetospheric distortions. The model is illustrated in Figure 1 and consists of a spherical perfectly conducting earth of radius a and a dipole of moment μ offset at the distance $\ell(t)$. Internal to the sphere it is assumed that there exists the static, uniform magnetic field $\underset{\sim}{B}_i = -\frac{2\mu}{a^3}\,\hat{e}_z$. The conducting sphere is the surface at which field line identification is made.

The magnetic field external to the sphere has two sources: the moving dipole, instantaneously at ℓ, and surface currents, both permanent and induced, on the sphere. At points $r \ll \ell$, this magnetic field has the form

$$
\begin{aligned}
\underset{\sim}{B} = \frac{\mu}{\ell^3}\Big\{ & \hat{e}_r \cos\theta\left[-\frac{2\ell^3}{r^3} + \left(1-\frac{a^3}{r^3}\right) - \frac{6r}{\ell}\sin\theta\cos\phi\left(1-\frac{a^5}{r^5}\right)\right] \\
& + \hat{e}_\theta\left[-\sin\theta\left(\frac{\ell^3}{r^3} + 1 + \frac{a^3}{2r^3}\right) - \frac{3r}{\ell}\left(\sin^2\theta - \cos^2\theta\right)\cos\phi\left(1+\frac{2a^5}{3r^5}\right)\right] \\
& + \hat{e}_\phi\left[\frac{3r}{\ell}\cos\theta\sin\phi\left(1+\frac{2a^5}{3r^5}\right)\right] \\
& + O\left(\frac{r}{\ell}\right)^2 \Big\}
\end{aligned}
\tag{10}
$$

Note that the undistorted magnetic field ($\ell \to \infty$) is also dipole in nature. Longitudinally asymmetric compressions of the magnetosphere are caused by dipole motion toward the sphere $\left(\frac{d\ell}{dt} < 0\right)$, while expansions are the results of the reverse dipole motion $\left(\frac{d\ell}{dt} > 0\right)$.

It can be verified by direct differentiation ($\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$) that this magnetic field is represented to the indicated asymptotic order by the Euler potentials

$$\alpha = -\frac{\mu}{r}\sin\vartheta\Bigg(\sin\vartheta - \frac{3r^3}{\ell^3}\Bigg\{\frac{\sin\vartheta}{6}\left(1-\frac{a^3}{r^3}\right)$$

$$+\frac{2r}{\ell}\cos\phi\left[\frac{1}{7}-\frac{2a^5}{9r^5}+\frac{5}{63}\left(\frac{a}{r}\right)^{7/2}-\frac{\sin^2\vartheta}{3}\left(1-\frac{a^5}{r^5}\right)\right]\Bigg\} \tag{11a}$$

$$+\,O\left(\frac{r}{\ell}\right)^5\Bigg),$$

$$\beta = \phi + \frac{3r^4}{\ell^4}\frac{\sin\phi}{\sin\vartheta}\left[\frac{1}{7}-\frac{2a^5}{9r^5}+\frac{5}{63}\left(\frac{a}{r}\right)^{7/2}\right]+O\left(\frac{r}{\ell}\right)^5 \tag{11b}$$

Note that at the surface of the sphere, α and β reduce to the time invariant values $\alpha = -\frac{\mu}{a}\sin^2\vartheta$, $\beta = \phi$. The unique identification of field lines introduced in the previous section becomes for this example a question of mapping in the region $a < r \ll \ell$ the curves $\alpha =$ const., $\beta =$ const, as a function of time (with, of course, α and β defined by Eq. 11).

Associated with the time varying magnetic field Eq. 10, is an electric field. Assuming the region surrounding the sphere to be a vacuum, we obtain an analytic representation for this field by solving the equation $\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$, subject to the boundary conditions (1) that the tangential electric field be zero on the surface of the sphere and (2) that in the limit of vanishing sphere radius ($a \to 0$) the field be simply represented as

$\underset{\sim}{E} = \frac{1}{c}\frac{d\ell}{dt}\,\hat{e}_x \times \underset{\sim}{B}$. Such an electric field is

$$\underset{\sim}{E} = -\frac{\mu}{c\ell^3}\frac{d\ell}{dt}\Bigg(\hat{e}_r \sin\theta \sin\phi\left\{\left(1 + \frac{2a^3}{r^3}\right) - \frac{3r}{\ell}\sin\theta\cos\phi\left(1 + \frac{a^5}{2r^5}\right)\right\}$$

$$+\,\hat{e}_\theta \cos\theta\sin\phi\left\{\left(1 - \frac{a^3}{r^3}\right) - \frac{3r}{\ell}\sin\theta\cos\phi\left(1 - \frac{a^5}{r^5}\right)\right\}$$

$$+\,\hat{e}_\phi\left\{\cos\phi\left(1 - \frac{a^3}{r^3}\right) - \frac{r}{\ell}\sin\theta\left[\ldots\left(1 - \frac{a^5}{r^5}\right) + \frac{1}{2}\left(\frac{a}{r}\right)\left(1 - \frac{r}{a}\right)\right]\right\}$$

$$+\,O\left(\frac{r}{\ell}\right)^2\Bigg) \tag{12}$$

Note from Eqs. (10) and (12) that at all longitudes ϕ, low latitude mirroring particles excute $\underset{\sim}{E} \times \underset{\sim}{B}$ motions away from the sun during magnetospheric compression $\frac{d\ell}{dt} < 0$) and toward the sun during magnetospheric expansion. One can directly verify from Eqs. (11), however, that all magnetic field lines at low latitudes move toward the earth during the compressional phase. Herein is exemplified the difference between $\underset{\sim}{w}$ and $\underset{\sim}{v}$ discussed in the previous section.

That the electric drift is not a legitimate field line velocity in this case may be verified directly:

$$\nabla \times \left(\frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \mathbf{B} \right) = \frac{3\mu}{r} \frac{1}{c\ell^3} \frac{d\ell}{dt} \frac{\cos\theta}{(1+3\cos^2\theta)}$$

$$\left\{ \left(\hat{e}_r \sin\theta - 2\hat{e}_\theta \cos\theta \right) \cos\phi \left(1 + \frac{a^3}{r^3} \right) \right. \tag{13}$$

$$\left. - \hat{e}_\phi \sin\phi \left[3\sin^2\theta + 2\left(\frac{1-3\cos^2\theta}{1+3\cos^2\theta} \right)\left(1 + \frac{a^3}{r^3} \right) \right] + O\left(\frac{r}{\ell}\right) \right\} .$$

The right hand side of Eq. (13) is, in general, non-vanishing in the region $a < r \ll \ell$.

Witness now the result of adding a plasma to the region external to the sphere. Charge separation in the plasma is assumed the source of an electrostatic field which cancels the component of $\mathbf{E}$ (as given by Eq. 12) parallel to $\mathbf{B}$. So long as $\frac{d\ell}{dt} \ll c$, currents induced in a low β plasma are negligible, and the magnetic field and Euler potentials as given by Eqs. (10) and (11) are virtually unaltered.

Thus, in the model as modified by this presence of plasma, the Euler potentials may be used to directly calculate the electric field

$$\underset{\sim}{E}^{*} = \frac{1}{c}\left[\frac{\partial\beta}{\partial t}\,\underset{\sim}{\nabla}\alpha - \frac{\partial\alpha}{\partial t}\,\underset{\sim}{\nabla}\beta\right]$$

$$= -\frac{\mu r}{c\ell^4}\frac{d\ell}{dt}\Bigg(\left[\hat{e}_r \sin\vartheta - \hat{e}_\vartheta 2\cos\vartheta\right]\frac{12r}{\ell}\sin\phi\left[\frac{1}{7} - \frac{2a^5}{7r^5} + \frac{5}{63}\left(\frac{a}{r}\right)^{7/2}\right]$$

$$- \hat{e}_\phi\Big\{\frac{3}{2}\sin 2\vartheta\left(1 - \frac{a^3}{r^3}\right) + \frac{24r}{\ell}\cos\phi\left[\frac{1}{7} - \frac{2}{7}\frac{a^5}{r^5} + \frac{5}{63}\left(\frac{a}{r}\right)^{7/2}\right. \tag{14}$$

$$\left. - \frac{\sin^2\vartheta}{3}\left(1 - \frac{a^5}{r^5}\right)\right]\Big\} + \mathcal{O}\left(\frac{r^2}{\ell^2}\right)\Bigg).$$

Note that to leading order in $(\frac{r}{\ell})$ an azimuthally symmetric longitudinally directed electric field from Eq. (14) combines with the undistorted dipole magnetic field component of Eq. (10) to drive particles - together with field lines - toward the earth during magnetospheric compression.

The electric field as given by Eq. (14) is, of course, not divergence-free. Poisson's Equation may be used to deduce the plasma charge density needed to effect the drastic change from the vacuum electric configuration

$$\rho = \frac{\underset{\sim}{\nabla}\cdot\underset{\sim}{E}^{*}}{4\pi} = -\frac{5r}{14\pi}\frac{\mu}{c\ell^5}\frac{d\ell}{dt}\sin\phi\,\sin\vartheta\left[4 + 2\left(\frac{a}{r}\right)^{7/2}\right] \tag{15}$$

For values representative of the earth's magnetosphere $\left[\mu = 10^{26}\text{ gauss cm}^3,\right.$ $\left.\ell = 10^{10}\text{ cm } (\sim 20\text{ earth radii}),\ \frac{d\ell}{dt} = \frac{2 \times 10^6\text{ cm}}{\text{sec}}\right]$ the maximum charge density needed at six earth radii in the equatorial plane is approximately $3 \times 10^{-20}\ \frac{\text{esu}}{(\text{cm})^3}$. This corresponds to an order of magnitude difference in the number densities of protons and electrons of one part in 10^{10}.

Summary

Our main point in this paper has been to indicate that in any region permeated by a magnetic field $\underset{\sim}{B}$, the presence of a conducting surface through which all field lines pass permits a unique, legitimate, and significant identification of the motion of magnetic field lines. The field line motion so defined is the $\underset{\sim}{E} \times \underset{\sim}{B}$ particle drift motion if $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$; otherwise the two are different. For the case $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$ low energy particles may be regarded as moving frozen to the field lines as identified by this prescription.

In illustration of these points a time dependent model magnetosphere has been considered. Distinction between the situations where the magnetosphere is free space and filled with plasma is made. The latter case is found to correspond to models adopted in diffusion analyses where the frozen in field assumption is - justifiably - used.

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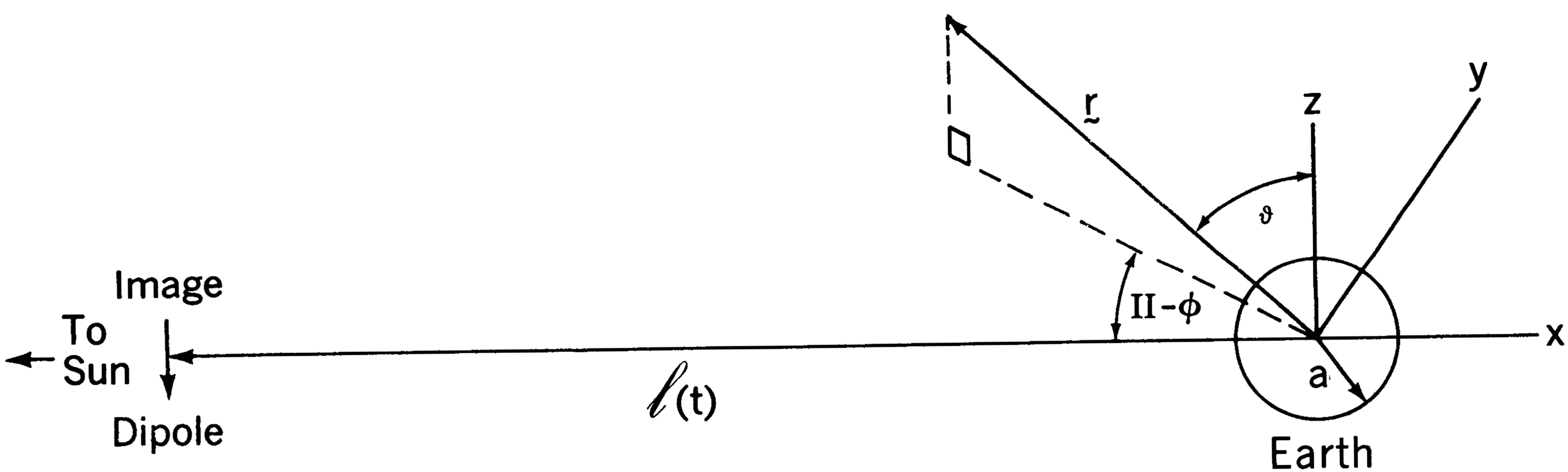
Figure Captions

Figure 1. Geometry of the Compressed Magnetosphere.



y
z
x
r
ϑ
II-ϕ
a
Earth
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To
Sun
Dipole