Brown

Geometric Theory of Functional Differential Equations

Jack K. Hale

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1. Introduction.

Functional differential equations provide a mathematical model for a physical system in which the rate of change of the system may depend upon its past history; that is, the future state of the system depends not only upon the present but also a part of its past history. A special case of such an equation is a differential difference equation

$$\dot{x}(t) = f(t, x(t), x(t-r))$$

where r is a nonnegative constant. For $r = 0$, this is an ordinary differential equation. A more general equation, which we choose to call a functional differential equation, is one of the form

$$\dot{x}(t) = f(t, x_t) \tag{1}$$

where x is an n-vector and the symbol x_t is defined as follows. If x is a function defined on some interval $[\sigma-r, \sigma + A)$, $A > 0$, then for each fixed t in $[\sigma, \sigma + A)$, x_t is a function defined on the interval $[-r, 0]$ whose values are given by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$. In other words, the graph of x_t is the graph of x on $[t-r, t]$ shifted to the interval $[-r, 0]$. To obtain a solution of (1) for $t \geq \sigma$, one specifies an initial function φ on the interval $[\sigma-r, \sigma]$ and then extends φ to $t \geq \sigma$ by relation (1).

Functional differential equations have been discussed in the literature since the early 1700's, beginning with John Bernoulli [1] and L. Euler [2]. Early investigations were usually devoted to particular equations to which were applied specialized techniques. Also, due to the difficulties that were being encountered in the development of a theory for ordinary differential equations, an extensive theory for functional differential equations was naturally postponed. One of the most prominent proponents of a more systematic development of the subject was Volterra [3,4] who pointed out very clearly the applications to the theory of viscoelastic materials and the interaction of biological species. Since Volterra, other applications have arisen in various aspects of biology, medicine, econometrics, number theory and problems of feedback control. Also it is hard to visualize an adaptive control system which would not use in a significant manner a part of its past history and a model for such systems might be a functional differential equation.

These extensive applications have led to a rapid development of the theory during the past twenty years and, as a consequence, a few books are now available on the subject (Mishkis [5], Pinney [6], Krasovskii [7], Bellman and Cooke [8], Halanay [9]). El'sgol'tz [10] also has two chapters devoted to differential difference equations. The book of Minorsky [11] contains material on differential difference equations and an excellent discussion of specific applications. Hahn [12] includes a section on the use of Lyapunov functions for a discussion of stability.

Even though system (1) is obviously a problem in an infinite dimensional space, there has been some reluctance to attempt to discuss the problem in this context. In fact, most of the papers in the literature attempt to gain as much information as possible about the solutions of (1) by considering (1) as defining curves in an n-dimensional vector space.

Krasovskii [7] was the first person to indicate the importance of studying (1) from the point of view that the solutions define curves in a function space. Krasovskii was interested in the stability properties of solutions of (1). In earlier works, El'sgol'tz [10] had shown that sufficient conditions for the stability of a solution of (1) obtained by an application of Lyapunov functions (functions on a n-dimensional vector space) could be stated in a manner completely analogous to those for ordinary differential equations. El'sgol'tz also shows by simple examples that it is impossible to prove the converse of Lyapunov's theorems in this context. By interpreting the solutions of (1) as curves in a function space, Krasovskii proved that the converse theorems of Lyapunov were valid if Lyapunov functionals were employed rather than Lyapunov functions.

The fundamental work of Krasovskii provided the direction and encouragement for the development of a qualitative or geometric theory of functional differential equations. Of course, after someone has first pointed out such a fundamental idea, it is so natural that one wonders why the theory did not always proceed in that direction. In fact, the state of any system at time t should be that part of the system which uniquely determines the behavior of the system for all time greater than t. In this context, the state of a system described by (1) should be the function x_t and not the vector $x(t)$. Of course, this implies that the orbits of the system will take place in a function space, but the results that are obtained and the basic understanding that is achieved seem to outweigh the complications introduced by working in a function space. In a sense, such an approach to a discussion of (1) implies that all aspects of (1) will have to be discussed anew, even linear equations for which t does not appear explicitly. But, fortunately, we shall see that the reinvestigation of even these simple systems leads to a much better understanding of them and also indicates methods of attack for common problems as well as the posing of new problems for more complicated systems.

This report is a discussion of some of the aspects of functional differential equations which are of particular interest to the author. Many important areas of current investigation are not mentioned at all. Also, the discussion concerns only retarded equations and the reason for this is that the theory for the other types is not as well developed and is in a rapid state of flux.

Let us now be more specific. Our function space will always be taken to be the space $C = C([-r, 0], R^n)$ of continuous functions mapping the interval $[-r, 0]$ into an n-dimensional real or complex vector space R^n. For any φ in C, we define $\|\varphi\|$ by the relation

$$\|\varphi\| = \max_{-r \leq \theta \leq 0} |\varphi(\theta)|$$

where $|x|$ is the norm of a vector x in R^n.

Suppose a is a given real number (we allow $a = -\infty$), Ω is an open subset of C and $f(t, \varphi)$ is defined for $t \geq a$, φ in Ω. If σ is a given real number in $[a, \infty)$(if $a = -\infty$, $[a, \infty) = (-\infty, \infty)$) and φ is a continuous function on $[\sigma-r, \sigma]$ with φ_σ in Ω, then we say $x = x(\sigma, \varphi)$ is a <u>solution of (1) with initial value φ at σ</u> if x is defined and continuous on $[\sigma-r, \sigma + A)$ for some $A > 0$, coincides with φ on $[\sigma-r, \sigma]$, x_t is in Ω and x satisfies (1) for $\sigma \leq t < \sigma + A$.

The following results are not difficult to prove and, in fact, the proofs can be supplied by following the methods in Coddington and Levinson [13] for ordinary differential equations or an application of well known fixed point theorems. If $f(t, \varphi)$ is continuous for $t \geq a$, φ in Ω, then there exists a solution of (1) with initial value φ at σ for any σ in $[a, \infty)$ and any φ such that φ_σ is in Ω. If, in addition, for any bounded closed subset B of Ω, there exists a continuous function $K(t) \geq 0$, $t \geq a$, such that $|f(t, \varphi)| \leq K(t)$ for all $t \geq a$, φ in B, then every solution of (1) can be continued in t until the boundary of B is reached. If, in addition, $f(t, \varphi)$ is locally Lipschitzian in φ, then for any $\sigma \geq a$ and φ such that φ_σ is in Ω, there is only one solution $x(\sigma, \varphi)$ of (1) with initial value φ at σ and $x_t(\sigma, \varphi)$ depends continuously upon σ, φ and t.

2. General properties of solutions. In the following, we shall always assume that $f(t, \varphi)$ is continuous for $t \geq a$, φ in Ω and for any bounded subset B of Ω, there exists a continuous function $K(t)$, $t \geq a$, such that $|f(t, \varphi)| \leq K(t)$ for $t \geq a$, φ in B. Also, we shall suppose that the solution $x = x(\sigma, \varphi)$ of (1) with initial value φ at σ exists for all $t \geq \sigma$, $x_t(\sigma, \varphi)$ depends continuously upon t, σ, φ, and a uniqueness property holds.

If $x = x(\sigma, \varphi)$ is a solution of (1) with initial value φ at σ, we define a trajectory of (1) through (σ, φ_σ) as the set of points in $[\sigma, \infty) \times \Omega$ given by $\{(t, x_t(\sigma, \varphi)), t \geq \sigma\}$. If (1) is autonomous, that is, $f(t, \varphi)$ is independent of t, then we choose $\sigma = 0$ and designate the solution by $x = x(\varphi)$. In the autonomous case, we have the important semigroup property

$$(2) \qquad x_{t+\tau}(\varphi) = x_t(x_\tau(\varphi)), \quad t, \tau \geq 0.$$

In the autonomous case, we define the <u>path or orbit of (1) through φ</u> as the set of points in C given by $\cup_{t\geq 0} x_t(\varphi)$.

Some words of caution are in order. First of all, a trajectory of eq. (1) through a given point (σ, φ_σ) is only defined for $t \geq \sigma$. For a given (σ, φ_σ), there may not exist a trajectory to the left of σ. In fact, if there is a trajectory through (σ, φ_σ) defined for an $s < \sigma$, then $\varphi(\theta)$ would have to be differentiable on an interval $\alpha \leq \theta \leq \sigma$ for some $\alpha < \sigma$.

Secondly, the uniqueness property only holds for $t \geq \sigma$. Therefore, this does not exclude the possibility that two distinct trajectories of (1) coincide after some finite time. As an example, consider the equation

$$\dot{x}(t) = -\alpha x(t-1)[1-x^2(t)], \tag{3}$$

which has the solution $x(t) = 1$ for all t. If C_0 is the set of functions φ in C with $\varphi(0) = 1$, then the uniqueness of solutions to the right implies that the solution $x(\varphi)$ satisfies $x_t(\varphi)$ equal to the constant function 1 for $t \geq 1$. Consequently, the corresponding trajectories and paths of distinct solutions coincide after one unit of time. If we consider the mapping which takes (σ, φ_σ) into $(t, x_t(\sigma, \varphi))$, then this example shows that this mapping will not in general be 1-1.

Even if this mapping is 1-1, it will not be a homeomorphism if $r > 0$. In fact, the hypotheses assumed on the function f in (1) imply that if φ belongs to a closed ball in C then the solution $x_t(\sigma, \varphi)$ belongs to a compact subset of C for $t \geq \sigma + r$. This follows from the fact that the functions $\{x_t(\sigma, \varphi)\}$ are uniformly bounded together with their first derivatives for any $t \geq \sigma + r$.

A function x^* defined and continuous on $(-\infty, a]$ is said to be a solution of (1) on $(-\infty, a]$ if for every σ in $(-\infty, a]$ the solution $x(\sigma, x^*_\sigma)$ of (1) with initial value x^*_σ at σ exists on $[\sigma, a]$ and $x_t(\sigma, x^*_\sigma) = x^*_t$ for t in $[\sigma, a]$. An element ψ of C is in $\Omega(\sigma, \varphi)$, the ω-limit set of a trajectory through (σ, φ_σ), it there is a sequence of nonnegative real numbers t_n, $t_n \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\sigma,\varphi)-\psi\| \to 0$ as $n \to \infty$ where $x(\sigma, \varphi)$ is the solution of (1) with initial function φ at σ. The set $A(\sigma, \varphi)$, the α-limit set of a trajectory through (σ, φ_σ), is defined in a similar manner except with negative values of t. If (1) is autonomous, we will omit σ and write $\Omega(\varphi)$ and $A(\varphi)$. A set M in C is called an invariant set if for any φ in M, there exists a function x^*, depending on φ, defined on $(-\infty, \infty)$, x^*_t in M for t in $(-\infty, \infty)$, $x^*_0 = \varphi$, such that, for every σ in $(-\infty, \infty)$, the solution $x(\sigma, x^*_\sigma)$ of eq. (1) with initial value x_σ at σ satisfies $x_t(\sigma, x^*_\sigma) = x^*_t$ for $t \geq \sigma$. Notice that to any element of an invariant set there corresponds a solution of (1) which must be defined on $(-\infty, \infty)$.

The above definition of an invariant set may at first glance not seem to be appropriate since trajectories are in general defined only on a semiaxis, but the definition is justified by the following lemma.

Lemma. If φ is such that the solution $x(\sigma, \varphi)$ is defined and bounded on $[\sigma-r, \infty)$, then $\Omega(\sigma, \varphi)$ is a nonempty compact connected set and $\mathrm{dist}(x_t(\sigma, \varphi), \Omega(\sigma, \varphi)) \to 0$ as $t \to \infty$. If φ is such that there is a solution $x(\sigma, \varphi)$ of (1) defined and bounded on $(-\infty, \sigma]$, $x_\sigma(\sigma, \varphi) = \varphi$, then $A(\sigma, \varphi)$ is a nonempty, compact, connected set and $\mathrm{dist}(x_t(\sigma, \varphi), \Omega(\varphi)) \to 0$ as $t \to -\infty$. If (1) is autonomous, then $A(\sigma, \varphi)$, $\Omega(\sigma, \varphi)$ are also invariant sets.

The proof of this lemma follows along the same lines as the proof of the same facts in ordinary differential equations (see Hale [14]). The invariance property of the ω- and α-limit sets of an autonomous system have been extended in a convenient manner by Miller [15] to periodic and almost periodic systems. We also remark that these same properties can be extended to the case in which the retardation interval is infinite if the compact open topology is used on C (see [14] for the autonomous case). For the application of these concepts to stability, see the paper of LaSalle in these proceedings and also [14].

3. Some geometric properties of linear autonomous equations and the variation of constant formula. In a few words, one could say that the ultimate goal in the qualitative theory of differential equations is a characterization of those classes of equations whose trajectories have similar topological properties. Even for ordinary differential equations, such a characterization is in its infancy for dimensions greater than 2. As a consequence, one is forced to discuss the properties of trajectories in a neighborhood of some set and, in particular, in a neighborhood of simple invariant sets.

The simplest invariant set of (1) is a constant solution (equilibrium or critical point) and a study of the trajectories near this soultion leads naturally to linear functional differential equations. However, in this section, we wish only to discuss autonomous linear equations; that is, equations of the form

$$\dot{x}(t) = L(x_t) \overset{\text{def}}{\equiv} \int_{-r}^{0} [d\eta(\theta)]x(t+\theta) \tag{4}$$

where η is an $n \times n$ matrix function whose elements have bounded variation on $[-r, 0]$.

The characteristic equation of (4) is

(5) $$\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta)$$

and to any λ satisfying (5), there exists a solution of (4), $x(t) = e^{\lambda t}b$, $\Delta(\lambda)b = 0$ defined on $(-\infty, \infty)$. Also, if λ is a root of (5) of multiplicity $m(\lambda)$, then there are exactly m linearly independent solutions of (4) of the form $p(t)e^{\lambda t}$ where $p(t)$ is a polynomial in t. A natural problem to investigate is the possibility of expanding any solution of (4) in terms of the basic functions associated with each root of (5). This problem has been studied in detail and the reader may consult Bellman and Cooke [8] for specific results and references. However, since our aim is to determine as much information as possible about the solutions of (4) without resorting to an expansion of the solution, we proceed in a different direction. The theory which is summarized below may be found in Shimanov [16], Hale [17].

If $x = x(\varphi)$ is the solution of (1) with initial value φ at 0, then we define the operator $T(t)$, $t \geq 0$, on C, by the relation

(6) $$T(t)\varphi = x_t(\varphi).$$

The operators $T(t)$, $t \geq 0$, are a strongly continuous semigroup of bounded linear operators on C with $T(0) = I$, the identity. For any $t \geq r$, $T(t)$ is also compact. This latter property leads to a great simiplification in the theory but many of the properties described below do not depend upon compactness. This fact seems to indicate that much of the theory will be applicable to linear equations with hereditary effects involving the first derivative of x. Since these results are not yet complete, we confine out attention to (4).

The properties of $T(t)$ permit one to define the infinitesimal generator A of T(t) as

(7)
$$A\varphi(\theta) = \begin{cases} \dot{\varphi}(\theta) \quad , \quad -r \leqq \theta \leqq 0 \\ \dot{\varphi}(0) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta) \end{cases}$$

The spectrum of A, $\sigma(A)$, coincides with those values of λ which satisfy (5).

The properties of T(t) (compactness is not needed) imply that the point spectrum of T(t), $P\sigma(T(t))$ is given by the relation

(8)
$$P\sigma(T(t)) = \{e^{\lambda t}, \lambda \text{ in } \sigma(A)\}.$$

If λ is in $\sigma(A)$, then $M_\lambda(A)$ denotes the generalized eigenspace of λ ; that is, the maximal set of φ in C which are annihilated by powers of $A - \lambda I$. The subspace $M_\lambda(A)$ is finite dimensional, is invariant under T(t) and if $\Phi_\lambda = (\varphi_1,\ldots,\varphi_d)$ is a basis for $M_\lambda(A)$, then

(9)
$$T(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t} \quad , \quad \Phi(\theta) = \Phi(0)e^{B_\lambda \theta} \, , \, -r \leqq \theta \leqq 0,$$

where the only eigenvalue of B_λ is λ. Relation (9) implies that on $M_\lambda(A)$, the solutions of (4) behave as an ordinary differential equation and can be defined on $(-\infty, \infty)$. Also, one easily shows that the solutions of (4) on $M_\lambda(A)$ coincide with those solutions of (4) of the form $p(t)e^{\lambda t}$ where p(t) is a polynomial in t and λ is a root of (5). Therefore, the construction of Φ can be performed as follows. Let $p_1(t)e^{\lambda t},\ldots,p_d(t)e^{\lambda t}$ be a basis for the solutions of (4) of the form $p(t)e^{\lambda t}$ where p(t) is a polynomial in t and λ is a root of (5) of multiplicity d. If we define $\varphi_j(\theta) = p_j(\theta)e^{\lambda\theta}$, $-r \leqq \theta \leqq 0$, $j = 1,2,\ldots,d$, $\Phi_\lambda = (\varphi_1,\ldots,\varphi_d)$, then Φ_λ is a basis for $M_\lambda(A)$ and the matrix B_λ is determined by the relation $\Phi_\lambda(\theta) = \Phi_\lambda(0)\exp(B_\lambda\theta)$, $-r \leqq \theta \leqq 0$.

Our next goal is to find a subspace of C which is invariant under $T(t)$ and complementary to $M_\lambda(A)$.

The author has been informed by K. Meyer that the existence of such a complementary subspace follows by an application of some general results on bounded linear operators and the theory of semigroups of transformations. However, in some applications, it seems necessary to have an explicit representation of this complementary subspace. Also, such an explicit representation allows one to interpret results in language familiar to ordinary differential equationists. We now give a recipe for finding this subspace and thereby obtain an explicit coordinate system in C.

This is accomplished by means of the bilinear form

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(\xi - \theta)[d\eta(\theta)]\varphi(\xi)d\xi \tag{10}$$

defined for all φ in C and ψ in $C^* = C([-r, 0], R^{n*})$, where R^{n*} is the n-dimensional space of row vectors. The equation "adjoint" to (4) relative to the bilinear form (10) is defined as

$$\dot{y}(s) = -\int_{-r}^{0} y(s-\theta)[d\eta(\theta)], \tag{11}$$

and its associated characteristic equation is obtained by finding conditions on λ which will ensure a solution of (11) of the form $ce^{-\lambda s}$; that is,

$$\det \Delta'(\lambda) = 0, \quad (\Delta' \text{ is the transpose of } \Delta).$$

Let $q_1(s)e^{-\lambda s}, \ldots, q_d(s)e^{-\lambda s}$, be a basis for the solutions of eq. (11)

of the form $q(s)e^{-\lambda s}$, where $q(s)$ is a polynomial in s and λ is a root of (5) of multiplicity d. If we define $\psi_j(\theta) = q_j(\theta)e^{-\lambda\theta}$, $0 \leq \theta \leq r$, $\Psi_\lambda = \text{col}\,(\psi_1,\ldots,\psi_n)$, $(\Psi_\lambda,\Phi_\lambda) = (\psi_i,\varphi_j)$, $i,j = 1, 2,\ldots,d$, then one can show (see [6],[7]) that $(\Psi_\lambda, \Phi_\lambda)$ is a nonsingular matrix, which without loss in generality can be taken to be the identity.

If Ψ_λ, Φ_λ are chosen as above; and if we define

$$
\begin{aligned}
P &= \{\varphi \text{ in } C : \varphi = \Phi_\lambda b \text{ for some constant vector } b\} \\
Q &= \{\varphi \text{ in } C : (\Psi_\lambda, \varphi) = 0\}
\end{aligned} \tag{12}
$$

then every element φ in C can be uniquely decomposed as

$$
\begin{aligned}
&\varphi = \varphi^P + \varphi^Q, \\
&\varphi^P = \Phi_\lambda b\ ,\ b = (\Psi_\lambda, \varphi)\ ,\ \varphi^Q \text{ in } Q.
\end{aligned} \tag{13}
$$

Furthermore, the subspace Q is invariant under the action of $T(t)$.

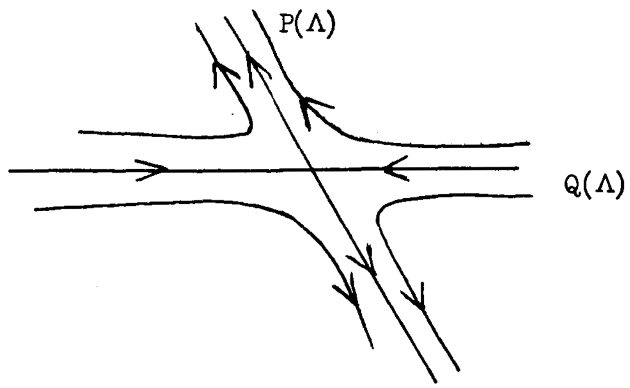
The above results do not use the compactness of $T(t)$ but only the fact that the dimension of $M_\lambda(A)$ is finite. If $T(t)$ is compact for $t \geq r$, then for any $\gamma > 0$, there are only a finite number of roots of (5) with $\text{Re}\,\lambda \geq \gamma$. If $\Lambda = \Lambda(\gamma) = \{\lambda : \text{Re}\lambda \geq \gamma,\ \det \Delta(\lambda) = 0\}$, then the above process can be carried out for each λ in Λ and one obtains a decomposition of C as follows

$$
\begin{aligned}
&\varphi = \varphi^{P(\Lambda)} + \varphi^{Q(\Lambda)} \\
&P(\Lambda) = \{\varphi \text{ in } C : \varphi = \Phi_\Lambda b\} \\
&Q(\Lambda) = \{\varphi \text{ in } C : (\Psi_\Lambda, \varphi) = 0\} \\
&\Phi_\Lambda = (\Phi_{\lambda_1},\ldots,\Phi_{\lambda_s})\ ,\ \Psi_\Lambda = \text{col}(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_s}) \\
&\Phi_\Lambda(\theta) = \Phi_\Lambda(0)e^{B_\Lambda\theta}\ ,\ -r \leq \theta \leq 0\ ,
\end{aligned} \tag{14}
$$

where Φ_{λ_j} , Ψ_{λ_j} are bases for the solutions of (4) and (11) of the type described above for λ_j in Λ and the spectrum of B_Λ is precisely Λ. The spaces $P(\Lambda)$, $Q(\Lambda)$ are both invariant under $T(t)$ and there exist $\epsilon > 0$, $K > 0$ such that

$$(15) \qquad \|T(t)\varphi\| \leq Ke^{(\gamma - \epsilon)t}\|\varphi\| , \quad \varphi \text{ in } Q(\Lambda).$$

The decomposition (14) and property (15) give us a very clear picture of the behavior of the solutions of a linear autonomous eq. (4). Also, we can now define concepts which would not have been very meaningful in R^n. For example, we can say that system (4) has a saddle point at 0 if no roots of (5) lie on the imaginary axis. This definition coincides with the definition for ordinary differential equations and the trajectories in C behave in essentially the same manner as they do for an ordinary differential equation. To see this, simply let $\Lambda = \{\lambda : \text{Re } \lambda > 0 , \det \Delta(\lambda) = 0\}$ and observe from relation (15) that the solutions on $Q(\Lambda)$ approach zero as $t \to \infty$ for $\gamma = 0$. The solutions of $P(\Lambda)$ are defined on $(-\infty, \infty)$ and approach zero as $t \to -\infty$. The picture is indicated below.



A natural problem to investigate is the preservation of the saddle point property when (4) contains some nonlinear perturbations of order higher than the first near $\varphi = 0$. The basic tool for this investigation is the variation of constants formula which has already been supplied by Bellman and Cooke [8]. In fact, if X_0 is the $n \times n$ matrix function defined on $[-r, 0]$ by

$$X_0(\theta) = \begin{cases} 0 \quad , & -r \leq \theta < 0 \\ I \quad , & \theta = 0 \end{cases}$$

then the solution of the equation

$$\dot{x}(t) = L(x_t) + N(t, x_t) \tag{16}$$

with the initial value φ at σ is given by

$$x_t(\theta) = [T(t-\sigma)\varphi_\sigma](\theta) + \int_\sigma^t [T(t-\tau)X_0](\theta)N(\tau, x_\tau)d\tau \ , \ -r \leq \theta \leq 0.$$

For simplicity in the notation we will write this as

$$x_t = T(t-\sigma)\varphi_\sigma + \int_\sigma^t T(t-\tau)X_0 N(\tau, x_\tau)d\tau \ . \tag{17}$$

If the space C is decomposed as in relations (14), then eq. (17) can be written as

$$\begin{aligned} x_t^P &= T(t-\sigma)\varphi_\sigma^P + \int_\sigma^t T(t-\tau)X_0^P \, N(\tau, x_\tau)d\tau \\ x_t^Q &= T(t-\sigma)\varphi_\sigma^Q + \int_\sigma^t T(t-\tau)X_0^Q \, N(\tau, x_\tau)d\tau \\ x_t &= x_t^P + x_t^Q \end{aligned} \tag{18}$$

where for simplicity we have let $P = P(\Lambda)$, $Q = Q(\Lambda)$. Since P is a finite dimensional space spanned by $\Phi = \Phi(\Lambda)$, the first equation in (18) can be written in a form resembling an ordinary differential equation. In fact, eqs. (18) are equivalent to

$$
\begin{aligned}
&x_t = \Phi y(t) + x_t^Q \\
(19) \qquad &\dot{y}(t) = B\, y(t) + \Psi(0)N(t,\, x_t) \\
&x_t^Q = T(t-\sigma)\varphi_\sigma + \int_\sigma^t T(t-\tau)X_0^Q\, N(\tau, x_\tau)d\tau \ .
\end{aligned}
$$

4. Applications of the preceding theory.

4.1. Behavior near an equilibrium point of an autonomous equation.

Suppose no roots of (5) lie on the imaginary axis and the space C is decomposed as in (14) with $\Lambda = \{\lambda : \mathrm{Re}\,\lambda > 0,\ \det \Delta(\lambda) = 0\}$ and consider the equation

$$(20) \qquad \dot{x}(t) = L(x_t) + N(x_t)$$

where $N(0) = 0$ and there exists a continuous function $\eta(\rho)$, $\rho \geq 0$, $\eta(0) = 0$ such that $|N(\varphi_1) - N(\varphi_2)| \leq \eta(\rho)\|\varphi_1 - \varphi_2\|$, for $\|\varphi_1\| \leq \rho$, $\|\varphi_2\| \leq \rho$.

A discussion of the saddle point property for eq. (20) must proceed with extreme care. At first glance, one might attempt to find a specific relationship between the trajectories of eq. (20) and those of eq. (5). However, such an investigation appears to be very difficult as the following simple example indicates. Consider the equation

$$\dot{x}(t) = 2\alpha x(t) + N(x_t) \ , \ \alpha > 0 \ ,$$

where $N(\varphi)$ satisfies the above conditions. This equation is the variational equation relative to the solution $x = 1$ of eq. (3). The decomposition in C is given by

$$\varphi = \varphi^P + \varphi^Q \ , \ \varphi^P(\theta) = \varphi(0) \ , \ \varphi^Q(\theta) = \varphi(\theta) - \varphi(0), \ -r \leq \theta \leq 0 \ .$$

The subspace P is one dimensional and the trajectories of $\dot{x}(t) = 2\alpha x(t)$ behave as $e^{2\alpha t}$. On the other hand, the paths associated with trajectories with initial values on Q reach the zero function in at most r units of time. For the pertrubed equation, it seems unreasonable to suppose that so many paths would posses such a property.

The following statements are a consequence of much more general results in Hale and Perelló [18] and assert that there is a very close relationship between the inital values of solutions of eq. (20) and eq. (5) which approach zero as $t \to \infty$ or $t \to -\infty$. In the statements below p_P and p_Q are the projection operators defined by the decomposition given in relations (14).

I) Let K be the constant defined in relation (15). There exists a $\delta > 0$ such that the set S of all φ in C such that $\|\varphi^Q\| \leq \delta/2k$ and the solution $x(\varphi)$ of eq. (20) satisfies $\|x_t(\varphi)\| \leq \delta$, $t \geq 0$, is homeomorphic through p_Q to the set of φ in Q with $\|\varphi\| \leq \delta/2K$. Furthermore, the set S is tangent to Q at 0 and any solution of eq.(20) with initial value in S approaches zero as $t \to \infty$.

II) There exist constants K, δ such that the set R of all φ in C such that $\|\varphi^P\| \leq \delta/2K$ and the solution $x(\varphi)$ of eq. (20) satisfies $\|x_t(\varphi)\| \leq \delta$ for $t \leq 0$ is homeomorphic through p_P to the set of φ in P with $\|\varphi\| \leq \delta/2K$. Furthermore, the set R is tangent to P at 0 and any solution with initial value in R approaches zero as $t \to -\infty$.

III) If we extend the sets S, R of I) and II) by adjoining all the points in paths of solutions starting in S and R for positive and negative values of t, respectively, and call these extended sets S^*, R^*, then S^* is positively invariant and R^* is negatively invariant with respect to system (20).

These results are proved by analyzing in detail the integral equations in (18). Only the integral equations (18) and the behavior of the solutions on P, Q are used and not the particular manner in which the decomposition of C was made. Therefore, much of the detailed theory of the previous section is unnecessary for this problem. However, more detailed knowledge of S, R are needed in applications and then an explicit representation of P and Q is required.

If some of the roots of (4) lie on the imaginary axis, then results corresponding to the above have not been given, but the manner in which they are proved in [18] and the results of Shimanov [19] on stability in critical cases seem to indicate that the problem can be solved.

The theory of the bifurcation of an equilibrium point that occurs in ordinary differential equations (see [11]) can be extended to functional differential equations by using the integral eqs. (18). These results will appear in the Ph.D. thesis of N. Chafee from Brown University and may be briefly summarized as follows. Suppose that the functions L, N in eq. (20) depend upon a small parameter Σ, say $L = L(\varphi, \Sigma)$, $N = N(\varphi, \Sigma)$ and that $N(\varphi, \Sigma)$ satisfies the same hypotheses of smallness in φ as before. Also suppose the linear equation

$$\dot{x}(t) = L(x_t, \Sigma), \quad 0 \leqq \Sigma \leqq \Sigma_0 , \; \Sigma_0 > 0 ,$$

has two simple characteristic roots of the form $a(\Sigma) \pm ib(\Sigma)$, $a(0) = 0$, $a(\Sigma) > 0$, $0 < \Sigma \leqq \Sigma_0$, $b(\Sigma) > 0$, $0 \leqq \Sigma \leqq \Sigma_0$ and the remaining characteristic roots have negative real parts. If, in addition, the zero solution of the equation

$$\dot{x}(t) = L(x_t, 0) + N(x_t, 0)$$

is asymptotically stable, then there exists a $\Sigma_1 > 0$ such that the system

$$\dot{x}(t) = L(x_t, \Sigma) + N(x_t, \Sigma) , \quad 0 < \Sigma \leq \Sigma_1 ,$$

has a nonconstant periodic solution $x^*(t, \Sigma)$ such that $x^*(t, 0) = 0$.

4.2 Equations with a small parameter. Consider the system

$$\dot{x}(t) = L(x_t) + N(t, x_t, \epsilon) \tag{21}$$

where L is the same as in eq. (4) and $N(t, \varphi, \epsilon)$ is some smooth function of t, φ, ϵ. An interesting case in the applications is when $N(t, \varphi, 0) = 0$ (that is, eq. (21) is near to linear) and also some of the roots of eq. (5) lie on the imaginary axis. The problem is to determine as much information as possible concerning the solutions of eq. (21) for ϵ small.

If $N(t, \varphi, \epsilon) = N(t + T, \varphi, \epsilon)$ for some $T > 0$, then it is of interest to determine necessary and sufficient conditions for the existence of periodic solutions of period T. To do this, one makes use of the fact that the equation

$$\dot{x}(t) = L(x_t) + f(t) \tag{22}$$

has a T-periodic solution for a T-periodic forcing function f if and only if

$$\int_0^T y(t)f(t)dt = 0 \tag{23}$$

for all T-periodic solutions of the adjoint eq. (11) and then develops a method of successive approximations to discuss the more complicated eq.

(21). In the method of successive approximations, one usually encounters systems of the form (22) where f does not satisfy the orthogonality relations (23). On the other hand, the first approximation invloves a number of independent parameters corresponding to the number of linearly independent T-periodic solutions of the unperturbed equation $\dot{x}(t) = L(x_t)$. These independent parameters are then used to attempt to make the othogonality relations (23) hold for all approximations.

For ordinary differential equations, this process can be made rigorous by a modification of the original differential equation by some terms which involve the independent parameters. The necessary and sufficient conditions for the existence of T-periodic solutions result in conditions on the parameters which will make the modified terms vanish. The resulting equations are the so called bifurcation equations (see Cesari [20], Hale [21] for details).

For functional differential equations, the modifications necessary in eq. (21) are not so obvious and, in fact, Brownell [22] states that it is impossible. This seems to be the case for eq. (21) where the modifications would take place directly on eq. (21). However, Perelló [23] has shown that the process of Cesari and Hale can be extended to functional differential equations if the modifications are performed on the first equation in relations (18). These modifications of Perelló take place in a finite dimensional space. In this manner, Perelló determines necessary and sufficient conditions for the existence of T-periodic solutions of eq. (21) and shows by means of examples that the theory can be used.

Other more complicated oscillatory phenomena can occur in eq. (21). For ordinary differential equations, the most widely used method for a discussion of equations with a small parameter is the method of averaging of Krylov, Bogoliubov, Mitropolski, and Diliberto (see [24] and [21].) The simplest case of this method for equations of the form (21) is the following. Consider the equation

$$\dot{x}(t) = \epsilon g(t, x_t) \tag{24}$$

where $g(t, \varphi)$ is almost periodic in t uniformly with respect to φ in any compact subset of C. If

$$g_o(\varphi) = \lim_{T\to\infty} \frac{1}{T}\int_o^T g(t, \varphi)dt,$$

are the solutions of the equation

$$\dot{y}(t) = \epsilon g_o(y_t)$$

related in any manner whatsoever to the solutions of (24) for ϵ small? For a small retardation interval, namely, $r = \epsilon s$, Halanay [25] proved that the solutions of the ordinary differential equation

$$\dot{z}(t) = \epsilon g_o(z(t)) \tag{25}$$

(here $z(t)$ represents both an n-vector and a function on C whose values for each θ in $[-r, 0]$ are equal to $z(t)$) yield valuable information about the solutions of (25) for ϵ small. In fact, if (25) has an equilbrium point whose associated linear variational equation has all solutions approaching zero exponentally as $t \to \infty$, then eq. (24) has an asymptotically stable almost periodic solution for ϵ small and positive. This same result also holds true with an arbitrary retardation r. This extension as well as much more information on averaging was obtained by Hale [26] by using the decomposition (18). The important remark to be made is that the extension of the method of averaging to functional differential equations is accomplished by modifying the equations only through the first relation in eqs. (18). Results of this type seem to indicate once more the advantages of treating eq. (21) as a problem in an infinite dimensional space.

4.3 Asymptotic theory of linear systems. Consider the system

$$\dot{x}(t) = [A_o + A(t)]x(t) + [B_o + B(t)]x(t-r) \tag{26}$$

where A_o, B_o are constant matrices and $A(t)$, $B(t)$ are small in some sense for large t. If λ is a root of the characteristic equation,

$$\det [\lambda I - A_o - B_o e^{-\lambda r}] = 0 ,$$

then the system

$$\dot{x}(t) = A_o x(t) + B_o x(t-r) \tag{27}$$

has a finite number of linearly independent solutions of the form $p(t)e^{\lambda t}$ where $p(t)$ is a polynomial in t. Bellman and Cooke [27] considered (26) for x a scalar and have given conditions on $A(t)$, $B(t)$ which will ensure that (26) has solutions which are asymptotic at ∞ to the solutions of (27) of the form $p(t)e^{\lambda t}$. The basic difficulty that arises in this type of discussion is the determination of an ordinary differential equation associated with (27) which has solutions of the desired type. The next step is to show that this ordinary differential equation is a good approximation to (27) relative to the type of solutions being considered. In [27], this ordinary differential was obtained by some clever manipulations on equation (26) and, as a consequence, the degree of approximation to (27) resulted in some rather annoying terms which to be made small required some peculiar hypotheses on $A(t)$, $B(t)$. An application of the theory of section 3 leads directly to an ordinary differential equation and furthermore leads to results of the type in [27] except with weaker hypotheses on $A(t)$, $B(t)$ (see Hale [28]). As in the discussion of the method of averaging and the work of Perelló on periodic solutions, this indicates the advantages of making transformations on eq (18) directly rather than on the original functional differential equation.

5. <u>The Floquet theory</u>. Systems of the form

$$\dot{x}(t) = f(t, x_t) \quad , \quad f(t + 2\pi, \varphi) = f(t, \varphi) \; , \tag{28}$$

where $f(t, \varphi)$ is linear in φ and continuous in t, φ have received considerable attention in the last few years by Hahn [29], Stokes [30], Halanay [9] and Shimanov [31]. The immediate goal has been to extend the Floquet theory for ordinary differential equations to functional differential equations.

Following Stokes [30], if $x(\varphi)$ is a solution of (28) with initial value φ at 0, then $x_t(\varphi)$ defines a continuous, linear mapping of C into C for each fixed $t \geqq 0$. If we define the operator $T(t)$ by $T(t)\varphi = x_t(\varphi)$, then $T(t)$ is compact for $t \geqq r$ and the characteristic multipliers of (28) are defined as the elements of the point spectrum of $T(2\pi)$(the monodromy operator). To each characteristic multiplier $\rho \neq 0$ of (28), there is a solution of (28) of the form $p(t)e^{\lambda t}$ where $p(t) =$

$p(t + 2\pi)$ and $\rho = e^{\lambda 2\pi}$. Finding a basis for all solutions of (28) associated with a given characteristic multiplier $\rho = e^{\lambda 2\pi}$ yields a finite dimensional subspace P_λ of C which is invariant under $T(2\pi k)$, $k = 1,2,\ldots$. If Φ_λ is a basis for P_λ and $T(2\pi)\Phi_\lambda = \Phi_\lambda B_\lambda$, where B_λ has only the eigenvalue λ, then $T(t)\Phi_\lambda = A_\lambda(t) \exp (B_\lambda t)$ where $A_\lambda(t + 2\pi) = A_\lambda(t)$. This shows that on the generalized eigenspaces of the monodromy operator, the Floquet representation of the solutions is valid. Stokes [30] also shows that the behavior of the solutions of (28) for large values of t is determined by the characteristic multipliers.

Shimanov [31] has shown that the space C can be decomposed in a manner similar to the decomposition in section 3 for the autonomous equation. These results can therefore be used to supply the natural extension of the application in section 4.1.

The book of Halanay [9] contains an extensive discussion of periodic systems with some very interesting results which are peculiar to functional differential equations. Halanay also proved that a necessary and sufficient condition for the existence of a 2π-periodic solution of

$$\dot{x}(t) = f(t, x_t) + g(t) \quad , \quad g(t + 2\pi) = g(t)$$

is that $\int_0^{2\pi} y(t)g(t)dt = 0$ for all 2π-periodic solutions of the adjoint equation. This fact together with the decomposition of C ala Shimanov will yield an extension of the results of Perelló in section 4.2.

The next natural problem is the extent to which the Floquet representation is valid on all of C. This seems to be an extremely difficult question and part of the difficulties arise from the fact that the spectrum of the monodromy operator may contain only a finite number of points. This immediately eliminates the possibility of the expansion of a solution of an arbitrary periodic system in terms of the generalized eigenspaces (notice the contrast with autonomous equations). Hahn [29] has given some conditions on the equation (28) which are supposed to ensure that solutions can be expanded in terms of the generalized eigenspaces but I have been informed by J. Lillo that there is a crucial error in the proof of these results. Lillo has a paper in the proceedings on this question.

Halanay has proposed that the following may be the proper statement of Floquet's theorem for system (28): there exist linear operators $P(t)$, $U(t)$, where $P(t + 2\pi) = P(t)$ and $U(t)$ is a semigroup of operators associated with an autonomous linear functional differential equation such that $T(t) = P(t)U(t)$, where $T(t)$ is the linear operator associated with (28). It would be interesting to either prove or disprove this fact.

6. Behavior near and existence of periodic solutions of autonomous equations. The next order of complications in the development of a qualitative theory for functional differential equations is the behavior of trajectories near a periodic solution of an autonomous equation. Even for ordinary differential equations, this problem is not completely resolved and, therefore, it is not unexpected that complications are arising in functional differential equations. The basic tool for ordinary differential is the method of sections of Poincaré, but the fact that the mapping induced by a functional differential equation is not a homeomorphism leads to difficulties in the application of this method. Another very important tool is the introduction of a local coordinate system in a neighborhood of the closed curve generated by a periodic solution and then discussing in detail the differential equations in these

coordinates. This seems to be a natural attack on the problem for functional differential equations but, at present, no one has been able to find an appropriate coordinate system.

In spite of this, one nontrivial and interesting result has been obtained by Stokes [32] without the introduction of a local coordinate system. Suppose the system

$$\dot{x}(t) = f(x_t) \tag{29}$$

has a nonconstant periodic solution. In the space C this solution generates a closed curve Γ. Define the concept of asymptotic orbital stability with asymptotic phase in a manner completely analogous to ordinary differential equations except in the space C. Stokes proves the following result: if the linear variational equation associated with a nonconstant periodic solution of (29) has all characteristic multipliers with modulus less than one (except the obvious multiplier which is equal to 1), then the periodic solution of (29) is asymptotically orbitally stable with asymptotic phase. This result of Stokes can be applied to equations with a small parameter to assess the stability properties of periodic solutions (see Perelló [23]).

In the theory of autonomous ordinary differential equations which do not contain small parameters, one of the basic methods for determining the existence of limit cycles is to determine a subset of Euclidean space which is homeomorphic to a cell such that any solution of the equation with initial value in the subset returns to the subset in a future time. One can then use the Brouwer fixed point theorem to assert the existence of a limit cycle. In a series of interesting papers, Jones [33,34] has shown that the application of similar arguments (but, of course, in the function space C) lead to existence of nonconstant periodic solutions of the equations

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)], \quad \alpha > \pi/2 ,$$

$$\dot{x}(t) = -\alpha x(t-1)[1 - x^2(t)], \quad \alpha > \pi/2 ,$$

as well as more general equations. Many more examples of this type need to be discovered in order to begin to understand how a functional differential equation can dissipate its energy in such a way as to sustain a self-excited oscillation.

7. Other problems. In the previous pages, we have failed to mention many important areas of investigation in the field of functional differential equations. One of the most important is the theory of Lyapunov stability and the application of Lyapunov functionals. This theory is far along in its development and is actually discussed in some of the other presentations at this symposium. Some references on the subject are contained in the bibliography. However, I would like to point out in particular that one of the most interesting applications of Lyapunov functionals to functional differential equations is contained in the series of papers of Levin and Nohel (see their paper in these preceedings for references).

Another important aspect of the theory is being developed in the Seminar of El'sgol'tz at Lumumba University in Moscow. Much of this theory deals with variational problems with retardations and the discussion of the corresponding boundary value problems for the Euler-Lagrange equations (see references [10],[35],[36]). For other problems in functional differential equations, see the survey article of Halanay [37]. Finally, the paper of Cooke [38] is an extremely interesting discussion of singular perturbations.

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