TECHNICAL REPORT

on

COMPARISON OF FUEL-OPTIMAL MANEUVERS USING A MINIMUM NUMBER OF IMPULSES WITH THOSE USING THE OPTIMAL NUMBER OF IMPULSES: A SURVEY

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1.0. INTRODUCTION

In many space vehicle maneuvers, the thrust levels are sufficiently high and the thrusting periods sufficiently short when compared with the flight time, that it is useful to approximate the thrust period by an impulsive thrust (i.e., an instantaneous change in velocity with no change in position).

If this approximation is to be used, there remains the question of the number, size, point of application, and direction of application of these impulses. Since there is usually an infinite number of ways in which impulses can be used to perform any given maneuver, some rule must be adopted for selecting the one actually to be used. The most commonly-used rule is to select that maneuver which will utilize the least fuel. This selection is based on the assumption that the less fuel used, the smaller, cheaper and more reliable the system will be, and, hence, the more desirable.

While there is some merit to this assumption, it is not necessarily an infallible guide. For example, a number of recent investigations have shown that for various maneuvers, the fuel-optimal number of impulses may be rather large, perhaps running as high as six.

The weight and reliability problems associated with realizing six separate impulses motivate a re-examination of the minimum-fuel criterion. As a first step in this re-examination, it is of interest to inquire just how much fuel is saved in these multiple impulse maneuvers. Thus, if a six-impulse maneuver takes only 1% less fuel than a two-impulse maneuver, then the two-impulse approach may be preferred.

T~~his report is a collection and correlation~~ of available information relating to the amount of fuel saved by using many impulses rather than the minimum number required to accomplish the maneuver. Even a cursory examination of the literature reveals that there is no single answer to this question. There are some maneuvers for which multiple impulses present no advantage whatever. There are others for which they make possible a moderate fuel saving. There are still other cases in which the multiple-impulse option is extremely advantageous.

Accordingly, a number of different maneuvers are discussed separately in the following sections, and the available information on comparisons is reviewed for each one. Before proceeding with these separate discussions, a general space maneuver problem is formulated, and its status is reviewed. If it is ever to be possible to make general statements about the optimum number of impulses, those statements will probably be based on the theory of the general problem.

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1.1. Previous Surveys

While the subject has apparently never been reviewed from exactly the standpoint considered here, there are several publications reviewing the fields of optimal control and optimal trajectories which contain many relevant references and discussions on the subject.

Lawden (83)* presents a concise review of the status of the theory of optimal trajectories in general, and impulsive trajectories in particular at the time of publication (1963). Lawden's primary emphasis is on the general problem, necessary conditions, and structure of the solution. These are all areas to which Lawden himself made substantial contributions during the preceeding decade. Two surveys of optimal control theory by Paiewonsky (115) and Athans (4) also contain some discussion of and references for the optimal impulsive problem. Two surveys by Leitmann (87, 88) are more specifically oriented toward trajectory problems, both impulsive and non-impulsive. Edelbaum (35) and Dowlen and Seddon (30) review a number of space maneuvers and discuss useful ways of performing them. The latter work has a particularly extensive list of references.

The surveys which come closest to the present one are those by Edelbaum (34,37). These papers review the field of impulsive transfer and enumerate the results available on maneuvers which have been optimized, and the optimal number of impulses for each. Edelbaum was not primarily concerned with a comparison between these optimal and various reasonable non-optimal maneuvers.

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* Numbers in parentheses indicate references.

1.2. Statement of a General Fuel-Optimal Problem (Problem A)

Consider a system defined by the integral equations

$$1.2\text{-}1) \qquad v(t) = v(t_s) + \int_{t_s}^{t} g(\tau, r(\tau))d\tau + \int_{t_s}^{t} \alpha(\tau)dV(\tau)$$

$$1.2\text{-}2) \qquad r(t) = r(t_s) + \int_{t_s}^{t} v(\tau)d\tau$$

where v, r, g and α are three-vector functions defined on $t_1 \leq t \leq t_2$. Let $g(\tau, r(\tau))$ have two continuous derivatives with respect to all arguments. The second integral of equation (1.2-1) is a Lebesque-Stieltjes integral and $V(\tau)$ is a non-decreasing (scalar) function of bounded variation with $V(t_1) = 0$. The function $\alpha(\tau)$ is a measurable function restricted by

$$1.2\text{-}3) \qquad \|\alpha(t)\| = \sqrt{\alpha_x^2(t) + \alpha_y^2(t) + \alpha_z^2(t)} = 1 \qquad t_1 \leq t \leq t_2 \quad .$$

The boundary conditions t_1, $v(t_1)$, $r(t_1)$; t_2, $v(t_2)$, $r(t_2)$ must lie on some region $B \epsilon E^{14*}$ which is defined as the set of all values satisfying the equation

$$1.2\text{-}4) \qquad \psi(t_1,\ v(t_1),\ v(t_1),\ t_2\ v(t_2),\ r(t_2),\ \beta) = 0 \quad ,$$

where ψ is a p-vector, $p \leq 14$. β is a vector composed of the parameters specifying the details of the problem being considered. For example, in an orbit transfer, β might contain the orbital elements of the initial and final orbits.

A set of controls $\alpha(t)$, $V(t)$, $t_1 \leq t \leq t_2$ and the trajectory $v(t)$, $r(t)$, $t_1 \leq t \leq t_2$ are said to be admissible if: a) the equations

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* The maximum number of boundary conditions which can be specified is 14: initial and final values of r(t) and v(t) and the initial and final values of t.

of motion (1.2-1) and (1.2-2) are satisfied; b) $\alpha(t)$ satisfies (1.2-3); c) the boundary conditions satisfy (1.2-4).

The optimization problem to be solved is: out of all the admissible trajectories, find the one which minimizes $V(t_2)$ (if such a minimizing trajectory exists).

To relate this problem to a more familiar one, suppose that $V(t)$ is continuous everywhere and differentiable almost everywhere. Where the derivative of $V(t)$ exists, designate it by $a(t)$, so that, almost everywhere,

1.2-5) $$\frac{dV}{dt} = a(t) \quad .$$

Then Equations (1.2-1) and (1.2-2) may be differentiated in the ordinary sense, yielding

1.2-6) $$\dot{v} = g(t,r) + \alpha(t)a(t)$$

1.2-7) $$\dot{r} = v$$

or, combining the two,

1.2-8) $$\ddot{r} = g(t,r) + \alpha(t)a(t) \quad .$$

This is the equation of motion of a point mass, moving in a gravitational field $g(t,r)$ and under the influence of a thrust acceleration of magnitude $a(t)$ in the direction of the unit vector $\alpha(t)$.

If, furthermore, $\alpha(t)$ is assumed to be piecewise continuous, then one obtains a conventional control problem with the function to be minimized

1.2-9) $$V(t_2) = \int_{t_1}^{t_2} a(t)dt \quad .$$

The quantity $V(t_2)$ is frequently called the "characteristic velocity" associated with the trajectory. Minimizing it will minimize the amount of fuel used, if one is considering a typical chemical rocket mechanism and if all the fuel has the same specific impulse. The slope of V is proportional to the thrust. Thus, if V is constant, no thrust is being applied. If V has a finite slope, then a thrust of finite magnitude is indicated. If V is discontinuous at some point, then an impulsive thrust is applied at that point. The magnitude of the impulse is proportional to the magnitude of the discontinuity in V.

For subsequent discussion, the optimization problem defined above is designated as Problem A. If a solution exists, there will be a certain value of $V(t_2)$ associated with the optimal trajectory. Designate this minimum characteristic velocity by J_A and observe that it will be a function of the problem parameters represented in the vector β. The minimal characteristic velocity may be written as $J_A(\beta)$.

1.3. Necessary Conditions and Sufficient Conditions

Necessary conditions were developed for this problem by Lawden in a series of papers which are summarized in his book (83). Lawden does not make use of an integral equation representation, but, rather, differential equations. He deduces formally what will happen in the case of bounded thrust as the thrust becomes large.

Lawden and others have used this approach to draw various conclusions about the nature of the optimal trajectory, and such conclusions appear to be correct. However, there remained the question of rigorous justification of these results.

Four papers appeared almost simultaneously in 1965 dealing with optimum control problems in which impulses appeared. These were by Neustadt (110), Schmaedeke (130), Rishel (124) and Warga (149).

Of these, only Neustadt (110) considered exactly the same problem as Lawden. The others were working with more general control problems. Neustadt showed that, under proper hypotheses, Lawden's conjectures are valid. He established necessary conditions for a problem similar to that stated at the beginning of this section, though he used fixed initial conditions. This restriction is not essential, however, and could be removed with relative ease, at the cost of additional transversality conditions.

Schmaedeke (130) defined what he called "measure differential equations" which are really equations like (1.2-1) which have been formally differentiated. In fact, he made use of the integral equation version in deriving many of his results. Most of his paper deals with extending classical results in differential equations to this type of equation, e.g., existence, uniqueness, dependence on initial conditions, etc., which do not relate specifically to optimization problems. He does state an optimization problem and proves some existence theorems. He does not, however, develop necessary conditions.

Rishel (124) used an interesting transformation of the independent variable to reduce the impulsive problem to one which fits within the classical Pontryagin or variational framework. Necessary conditions can then be derived directly from the usual ones. Rishel assumed that the initial and final states were fixed, but in view of his method of procedure it is a trivial matter to remove this restriction, as he points out himself in a later paper (125).

Warga's (149) procedure is somewhat similar to Rishel's; reducing the impulsive problem to the usual problem by changing the independent variable. Warga, however, treats the problem with state variable constraints, following some of his own earlier work on that subject.

To summarize, for the general problem, necessary conditions are available which make it possible to derive the two-point boundary value problem whose solution might be the optimum trajectory. Some problems have been solved analytically using these necessary conditions (110, 125), and some existence results are available.

Certain very desirable pieces of information are missing, however. It appears, on the basis of experience with the problem, that impulsive control is always at least as good as the best continuous or mixed control. It is not, however, possible to assert this in any rigorous way, for the general case. Nor can it be established, in general, how many impulses should be used to come within some given increment of the absolute minimum of fuel. These are both questions of considerable practical significance, but they remain open at present.

Lawden argued some years ago (74) that the optimal thrust program was always impulsive. However, when he later discovered (81) an arc with intermediate thrust (neither null thrust nor infinite thrust) which satisfied many of the necessary conditions, he expressed doubts about his former conclusions.

Further analysis (70,126), especially that of Kopp and Moyer (70) developed additional necessary conditions which were not satisfied along Lawden's intermediate thrust arc. This shows that this particular arc, at least, is not an optimum. McCue and Hoy (98) also showed this numerically.

While intermediate-thrust arcs cannot be completely excluded (126,47), they have yet to be shown superior to impulsive trajectories in any problem of practical interest. There are some cases where intermediate-thrust arcs are known which give the same fuel utilization as impulsive ones, but none where they are better.

As will be mentioned later, if there is only one attracting center, it can be shown that impulsive thrusting is optimal in coplanar problems. In the general problem, however, analogous results are not available.

1.4. Statement of an n-Impulse Fuel-Optimal Problem (Problem B_n)

The general problem stated in section 1.2 is clearly a variational problem. It could encompass purely continuous thrust, purely impulsive thrust, mixed continuous and impulsive thrust, or, for that matter, more complex types of control such as "chattering" or "sliding state" control.

Suppose, however, only purely impulsive control is admitted. Further, suppose the number of impulses is selected in advance. Then the fuel-optimal problem becomes an ordinary minimization problem. This is considerably easier to treat, either mathematically or computationally.

Under these assumptions, the function V(t) may be represented as

$$V(t) = \sum_{k=1}^{n} I_k \, u(t - \tau_k) \quad , \tag{1.4-1}$$

where u(t) is the unit step function and I_k is the magnitude of the impulse occurring at time τ_k where $t_1 \leq \tau_k \leq t_2$. Using this form for V, equations (1.2-1) and (1.2-2) become

1.4-2) $$v(t) = v(t_1) + \int_{t_1}^{t} g(\tau, r(\tau))\,d\tau + \sum_{k=1}^{n} I_k\, \alpha(t_k)\, u(t-\tau_k)$$

1.4-3) $$r(t) = r(t_1) + \int_{t_1}^{t} v(\tau)\,d\tau \quad .$$

The problem is now that of selecting the 5n parameters τ_k, I_k, $\alpha(\tau_k)$ to minimize

1.4-4) $$I = V(t_2) = \sum_{k=1}^{n} I_k$$

subject to the requirement that the boundary conditions of equation (1.2-4) are met, and also that

1.4-5) $$\alpha_x^2(\tau_k) + \alpha_y^2(\tau_k) + \alpha_z^2(\tau_k) = 1 \qquad k=1,2\cdots n \quad .$$

It might be argued that there are really only 4n parameters to be selected, since the three components of each $\alpha(\tau_k)$ are not independent of each other. It takes only two parameters to specify a direction in space. However, all two-parameter specifications of a direction in space suffer from ambiguity at some particular direction, so the more definite three-parameter-plus-constraint representation has been used here.

Using this representation, the impulses are completely specified by a vector with 5n components, which is designated here by s. The vector s cannot be freely chosen. Let S be the set of all vectors s such that, if the impulses are put into equations (1.4-2) and (1.4-3), the boundary conditions (1.2-4) can be met and equations (1.4-5) will be satisfied. The value of I in equation (1.4-4) will clearly depend on s.

The problem now is to find a vector $s^* \in S$ such that

1.4-6) $$I(s^*) \leq I(s) \qquad s \in S \quad .$$

If there exists such an s^*, then designate $I(s^*)$ by J_n. This will clearly depend on the problem parameters, so the minimum characteristic velocity for this problem may be designated $J_n(\beta)$. The problem will be called Problem B_n.

This is an ordinary minimization problem: find the vector s which minimizes I(s) subject to $s \in S$. It is, however, not an elementary problem because of the method of definition of S. It is unambiguous, but S cannot, in general, be defined in terms of algebraic equalities or inequalities.

There are some problems, however, for which the definition of S can be so reduced, and these will be discussed in the next section. Almost all of these involve a single attracting center, so that the trajectory segments between impulses are Keplerian.

It would be interesting to be able to compare the results of problem B_n with those of problem A. However, rigorously certifiable solutions to problem A are available only in a small number of very simple problems. It will be necessary, then, to conduct comparisons between $J_n(\beta)$, $n > 2$ with $J_2(\beta)$, or with $J_1(\beta)$ in those cases where one-impulse maneuvers are feasible. The comparison should be made for each value of β, and where possible, this will be done. However, complete definitions of $J_n(\beta)$ are available only in a small number of cases.

2.0. A SINGLE ATTRACTING CENTER (TWO-BODY PROBLEM)

Practically all studies of minimum-fuel maneuvers have involved a Keplerian force field: a pure inverse-square field about a fixed attracting center. This is an idealization which is useful, within limits, for studying interplanetary flight when the vehicle is not appreciably influenced by planetary gravitation, and in studying maneuvers in the near vicinity of a planet.

While this idealization is of limited accuracy in some contexts, it is simply described and a number of relatively general results have been obtained. These results, classified according to the types of orbit the vehicle is in before and after the maneuver, are reviewed in the following sections.

2.1. Coplanar Time-Free Transfers

Within the class of impulsive maneuver problems in a Keplerian force field, coplanar time-free transfers have received the greatest share of attention, and results for this problem are nearly complete. This is the only type of problem within the scope of this study for which such a statement can be made.

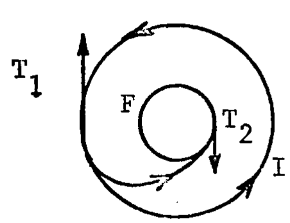
The vehicle is assumed to be initially in an orbit which is at least partially specified. The problem is to transfer to another orbit (at least partially specified) using minimum fuel. The initial and final orbit and all portions of the transfer orbit are required to lie in a single fixed plane containing the attracting center. The time required for the transfer is left completely free.

This last assumption, that of free time, immensely simplifies the analysis, but at the same time, it leads to difficulties. There are a number of cases in which the minimum-fuel maneuvers have transfer times which are either very large, or actually infinite. In those cases for which this situation is known to exist, it will be pointed out. Infinite-time trajectories are of little practical use, and their existence illustrates the danger of following a single optimization criterion too far. Even in those cases where the minimum-fuel maneuver takes an infinite time, the result is useful, however, as a lower bound on the fuel required in a practical trajectory.

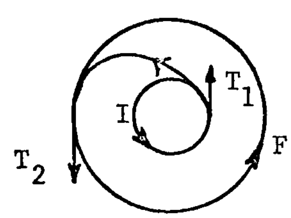
2.1.1. Circular Orbits (Hohmann Transfer)

If it is desired to transfer from one circular orbit to another of different radius but rotating in the same direction using minimum fuel, then the optimum maneuver (time-free) is now completely known.

This problem was first considered by Hohmann (56) who suggested that the optimum transfer was by means of two impulses, one applied in a tangential direction on the initial orbit, and the other in a tangential direction on the final orbit. The transfer orbit is an ellipse which is tangent to both orbits. This type of maneuver is indicated in Figure 2.1.1-1.



(a) Transfer from Outer Initial Orbit I to Inner Orbit F



(b) Transfer from Inner Initial Orbit I to Outer Orbit F

FIGURE 2.1.1-1. HOHMANN TRANSFERS

This problem has been considered by a number of investigators since Hohmann, and many results can be rigorously demonstrated. The following statements can be made:

(a) The optimal thrust program is impulsive. This has been shown by Marchal (93), Contensou (23), Winn (152), Busemann and Culp (19), and others, in some cases, for the more general problem of transfer between elliptical orbits as well.

(b) More than three impulses are not necessary to realize minimum fuel. This was shown by Ting (137) and, in a more general form, by Marchal (93). It is also consistent with Neustadt's results on neighboring orbits (109).

It is possible to realize this transfer using any number of impulses, and, in fact, it is possible to find n-impulse transfers which use the same amount of fuel as the two- or three-impulse maneuvers, but it is not possible to reduce the fuel requirement by using a number larger than three. Since the transfer cannot physically be accomplished with one impulse, except in the trivial case where the two orbits are coincident, the choice is between two impulses and three.

(c) Two-impulse Hohmann transfers are optimum if the ratio of the two orbit radii is less than 11.94. Otherwise a three-impulse transfer via infinity is optimum. It was shown by Barrar (7) that the Hohmann transfer is always the absolute optimum among two-impulse transfers. However, if the ratio between orbit radii is sufficiently large, then there exists a three-impulse transfer which uses less fuel than the Hohmann transfer (32).

The characteristic velocity ΔV_2 required to transfer from an inner orbit of radius r_1 (measured from the center of the attracting body) and circular velocity V_1 to an outer orbit with radius r_2 by means of a two-impulse transfer is [see, e.g., Ruppe (129)]:

$$2.1.1\text{-}1) \quad \Delta V_2 = V_1 \left[\frac{\frac{r_2}{r_1} - 1 + \frac{1}{\sqrt{2}}\sqrt{1 + \frac{r_2}{r_1}}}{\frac{1}{\sqrt{2}}\sqrt{\frac{r_2}{r_1}}\sqrt{1 + \frac{r_2}{r_1}}} - 1 \right] .$$

The optimum three-impulse maneuver consists of: (a) a tangential impulse T_1 on the inner orbit which raises the velocity to the parabolic escape speed; (b) an infinitesmal impulse T_2 at infinity which transfers the vehicle to a different parabola: one which has a pericenter on the second orbit; (c) a tangential impulse T_3 at the pericenter of this new parabola which reduces the velocity to the local circular velocity. This is indicated schematically in Figure 2.1.1-2.

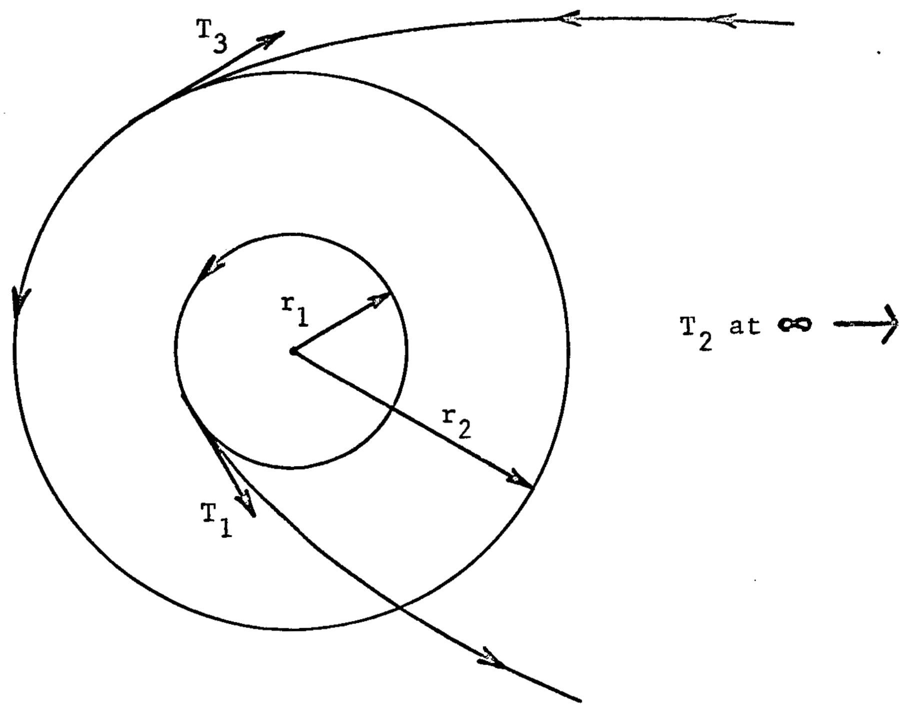


FIGURE 2.1.1-2. OPTIMUM CIRCLE-TO-CIRCLE TRANSFER USING THREE IMPULSES

The characteristic velocity required for this maneuver is

$$2.1.1\text{-}2) \quad \Delta V_3 = V_1 \ (\sqrt{2}-1) \ (1 + \sqrt{\frac{r_1}{r_2}}) .$$

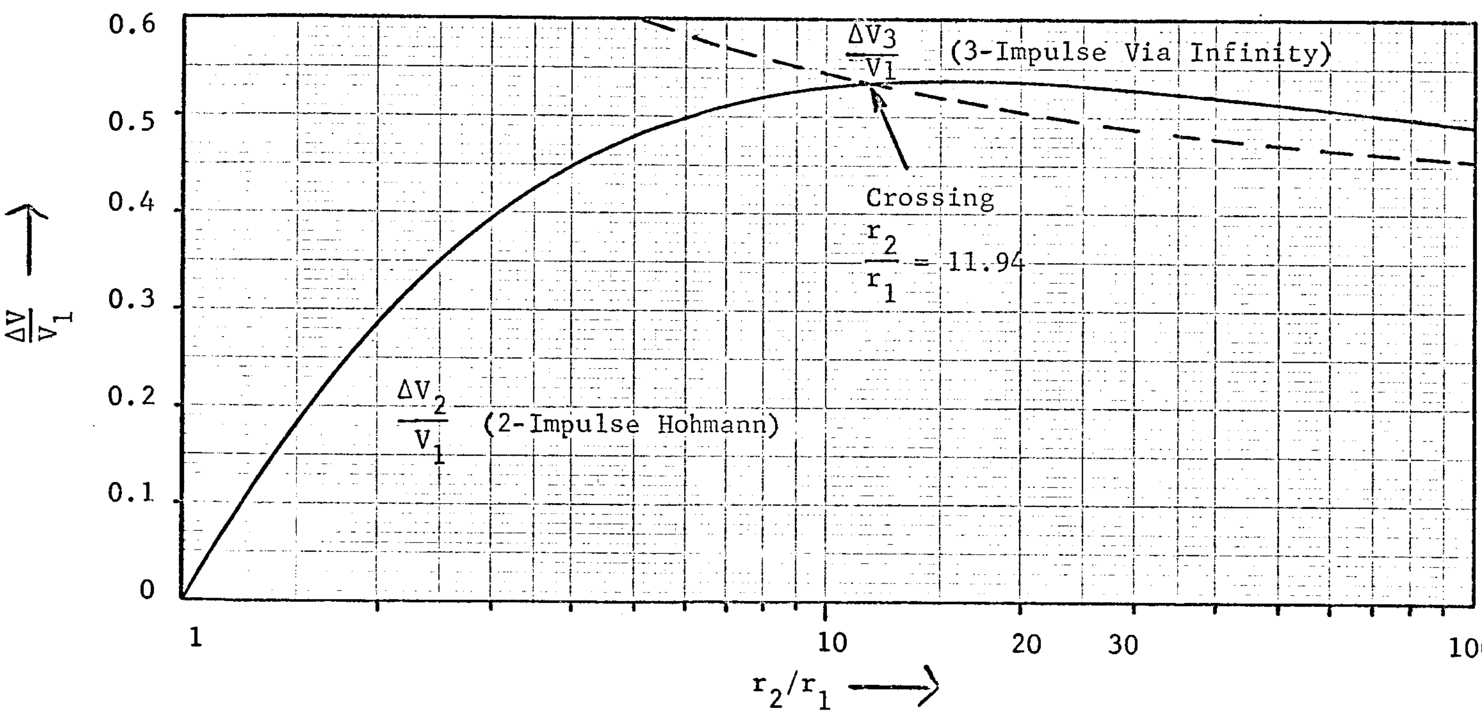


FIGURE 2.1.1-3. COMPARISON OF VELOCITY REQUIREMENTS FOR OPTIMAL 2- AND 3-IMPULSE CIRCLE-TO-CIRCLE TRANSFERS

The velocity requirements of the two maneuvers are compared in Figure 2.1.1-3.

Note that the velocity requirement for the two-impulse transfer (the solid curve) reaches a maximum at about $r_2/r_1 = 17$. For larger ratios, the two-impulse transfer actually requires less fuel than it does for $r_2/r_1 = 17$. This means it would be easier to escape the central attracting body than to transfer to an orbit 17 times farther out than the initial orbit.

The three-impulse transfer (dashed curve) becomes better than the two-impulse when r_2/r_1 reaches 11.94, and it remains better for all larger ratios. The advantage of the three-impulse maneuver over the two-impulse is, however, never very large. It does not exceed 8% for any value of r_2/r_1. Since the theoretically optimum three-impulse transfer would take an infinite amount of time, it may not be useful on a practical basis. However, modest gains are possible by applying the second impulse at a large out finite distance. For example, if T_2 is applied at a distance of only $2r_2$, 4% savings are possible.

In any event, this is one of the very small number of fuel-optimal maneuver problems which is completely solved. The form of the solution is known, and the minimum characteristic velocity is available in closed form for all values of the parameters.

2.1.1-3)

$$\frac{\Delta V}{V_1}_{\min} = \begin{cases} \dfrac{\dfrac{r_2}{r_1} - 1 + \dfrac{1}{\sqrt{2}} - \sqrt{1 + \dfrac{r_2}{r_1}}}{\sqrt{\dfrac{r_2}{2r_1}}\sqrt{1 + \dfrac{r_2}{r_1}}} - 1 & \dfrac{r_2}{r_1} \leq 11.94 \\[2em] \left(\sqrt{2} - 1\right)\left(1 + \sqrt{\dfrac{r_1}{r_2}}\right) & \dfrac{r_2}{r_1} > 11.94. \end{cases}$$

This is, of course, a particularly simple problem, depending as it does on only two parameters r_1 and r_2 (in addition to the gravitational constant which has an effect on V_1).

All of the above results have been based on the assumption that the two orbits are moving in the same sense around the attracting center. For the sake of logical completeness, it is of interest to inquire about the case of transferring from one circular orbit to another rotating in the opposite direction. This problem has apparently not been considered, perhaps because of its minimal practical interest. It seems certain that the fuel-optimal transfer would involve transferring to a parabolic orbit, reversing the direction at infinity with a new parabolic return.

2.1.2. Elliptic Orbits

The next more complex problem is that of transfer between two arbitrary elliptic orbits. In general, the solution to this problem will depend on five parameters. For example, these parameters might be taken as the two orbit eccentricities, the two orbit semimajor axes, and the angle between the two lines of apsides (see Figure 2.1.2-1 for an illustration of this angle). If both the orbits are circular, this problem degenerates to that of the preceding section. If one is circular, the apsidal orientation of the other becomes immaterial, and the problem then depends on only three parameters: the radius of the circular orbit, the semimajor axis and eccentricity of the elliptical orbit.

Before reviewing the results for some of the special cases, it is useful to consider the general results which are available, some of them rather recent:

(a) The optimal thrust program is impulsive.
This was shown by Marchal in a remarkable paper (93) which gives the most thorough treatment of this problem now available. Winn (152) shows the result in the case where one of the orbits is a circle.

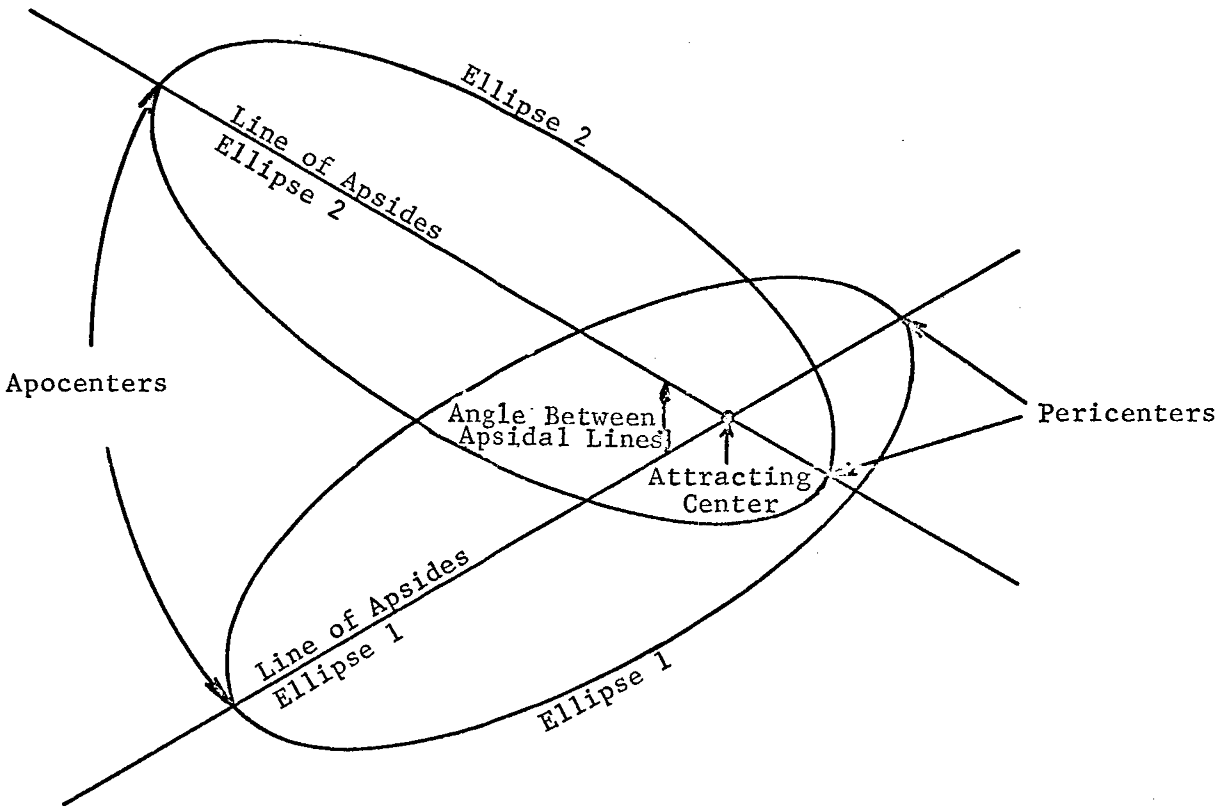


FIGURE 2.1.2-1. TWO INTERSECTING ELLIPTIC ORBITS

(b) More than three impulses are not necessary to realize minimum fuel. Ting (137) and Marchal (93) have both arrived at this result by different methods.

(c) Even if motion out of the common orbit plane is allowed, the optimal transfer orbit will lie entirely in the plane. This has been argued by a number of authors in a number of different ways, but the demonstration of Buseman and Culp (19) is perhaps the most simple and elegant.

Beyond this, not much can be said about the general case. If the two orbits intersect, then one-impulse transfers are feasible. Therefore, one-, two-, and three-impulse transfers must all be considered.

One-impulse transfers can be disposed of with relative ease. They have been studied by both Marchal (93) and Moyer (104). The number of conditions which must be met is such that one-impulse transfers will seldom be optimal.

The choice between two- and three-impulse transfers is more involved. Marchal (93) finds two possible types of three-impulse transfers. One is of the same type considered in the preceding section: (a) an accelerative impulse transferring the vehicle to a parabolic trajectory; (b) one or more infinitesmal impulses at infinity, transferring to a parabola which is tangent to the final orbit; (c) a braking impulse at the intersection of the second parabola and the final ellipse.

The second type of three-impulse transfer is one in which all three impulses are finite and applied at finite distances. The first impulse transfers the vehicle to an elliptical orbit of large, but finite apocenter. The second impulse transfers from this ellipse to another which intersects the final orbit. The third impulse is at the intersection of the second transfer ellipse and the final orbit.

Marchal (93) gives necessary conditions for the optimality of these finite three-impulse transfers, and these conditions are somewhat restrictive. They are: (a) the sum of the eccentricities of the initial and final orbits must exceed 1.712; (b) the angle between the lines of apsides of the initial and final orbits must not exceed 22°; (c) $\frac{9}{25} < \frac{P_1}{P_2} < \frac{25}{9}$ where P_1 and P_2 are the pericenter radii of the initial and final orbits, respectively. Because of the eccentricity requirement, it appears that these transfers will be somewhat unusual.

In most cases, the choice will be between two-impulse transfers (with a single intermediate transfer ellipse) and three-impulse transfers via infinity (with two intermediate transfer parabolas). If the initial and final orbits are not too dissimilar the optimum will be two-impulse. If the two are greatly dissimilar, then the optimum will be three-impulse transfer via infinity.

To give a precise statement of the conditions under which the various types of maneuver are optimal is a matter of some complexity. Marchal (93) has not succeeded in doing this completely, but he has made a significant contribution. He presents a number of results bearing on this question. For a complete review of them, the reader should consult the original paper. However, to give some idea of the type of conclusion he presents, one of the more interesting of his figures is reproduced here as Figure 2.1.2-1. This applies only to the case where the initial and final orbits do not intersect. In the case of intersecting orbits, he has a different discussion and method of presenting the data.

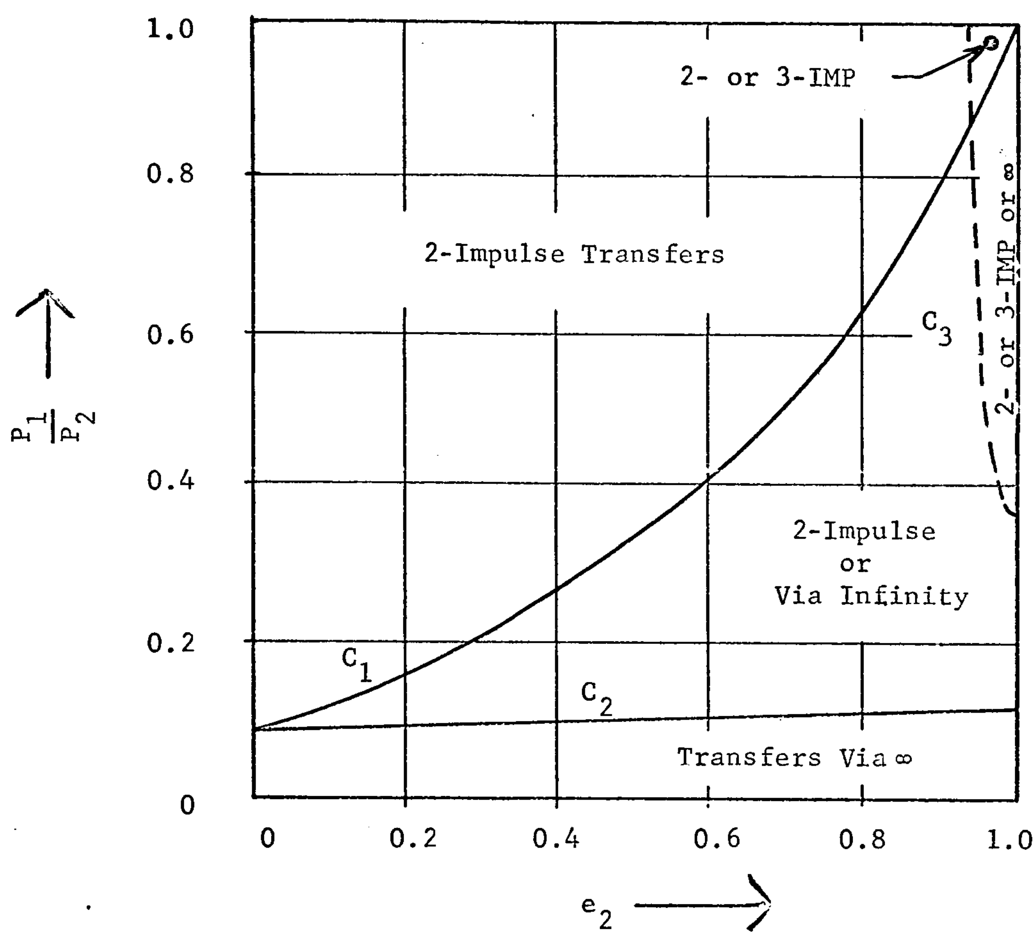


FIGURE 2.1.2-2. OPTIMAL TRANSFERS BETWEEN NON-INTERSECTING ELLIPTIC ORBITS

In Figure 2.1.2-1, P_1 and P_2 are the pericenter distances of the initial and final orbits and e_2 is the eccentricity of the final orbit. These are only three of the five parameters defining the entire problem, so it is not surprising that this plot does not permit complete determination of the optimal mode. However, for all transfer problems having P_1, P_2, and e_2, which plot into a point lying above curve C_1 and to the left of curve C_3, the optimal transfer will be two-impulse. Above C_1 and to the right of C_3, the optimal transfer could be two-impulse or finite three-impulse, depending on other parameters of the problem. Similarly, points lying below C_2 will definitely have three-impulse optimal transfers via infinity. The other regions of the plot have corresponding interpretations. This illustrates the fact that selection of the minimum-fuel maneuver in this problem is rather involved.

2.1.2.1. Contensou's Method. The general approach used by Marchal and several other recent investigators should be mentioned here, because it has led to the present satisfactory state of this problem. It appears to have been originated by Contensou (23), though a number of others have made effective use of it (19, 152, 52, 24, 103, and 14).

The general variational problem stated in Section 1.2 is difficult to handle theoretically. Contensou's approach changes this variational problem into one in which no impulses appear. The resulting problem can then be examined using the extensive theory of the calculus of variations or optimal control.

The several author's treatments differ in various ways. The method presented here is intended for illustrative purposes only. Consider the planar case, where it is desired to transfer from some initial orbit with semimajor axis a_1 and eccentricity e_1 to a final orbit with semimajor axis a_2, and eccentricity e_2, with the angle between the lines of apsides being ω. The line of apsides is assumed to point in the direction of the pericenter.

If a vehicle is moving under the influence of a Keplerian force field and its own thrust, its motion at any time may be described in terms of an "instantaneous" or "osculating" ellipse with semimajor axis a, eccentricity e and line of apsides direction ω. If no thrust were applied, then a, e, and ω would remain constant. If thrust is acting, then a, e, and ω change with time in accordance with the classical Lagrange planetary equations (see, e.g., 40, p. 451).

2.1.2.1-1) $$\frac{da}{dt} = \frac{2a^{3/2}}{\sqrt{\mu(1 - e^2)}} \left[Ue \sin f + W(1 + e \cos f) \right]$$

2.1.2.1-2) $$\frac{de}{dt} = \sqrt{\frac{a(1 - e^2)}{\mu}} \left[U \sin f + \frac{W(2 \cos f + e \cos^2 f + e)}{(1 + e \cos f)} \right]$$

2.1.2.1-3) $$\frac{d\omega}{dt} = \sqrt{\frac{a(1 - e^2)}{\mu e^2}} \left[-U \cos f + W \sin f \left(\frac{2 + e \cos f}{1 + e \cos f} \right) \right] ,$$

where

U = radial component of thrust acceleration (positive outward)

W = azimuthal component of thrust acceleration (positive forward)

f = true anomaly

μ = gravitational constant (product of universal gravitational constant and the mass of the central body).

To make a complete set of equations, it would be necessary to specify f as a function of t, and this would require the addition of another equation for an orbital element, e.g., the time of pericenter passage. Since time is immaterial in this problem, however, it will not be necessary to do this.

From equation (1.2-5), the rate of change of the characteristic velocity is

$$\frac{dV}{dt} = a = \sqrt{U^2 + W^2} \; . \tag{2.1.2.1-4}$$

Now comes the essential part of the process. Since V is a non-decreasing function of t, and since t is of no concern, it is possible to make a change of independent variable from t to V. The planetary equations then become

$$\frac{da}{dV} = \frac{2a^{3/2}}{\sqrt{\mu\ (1 - e^2)}} \cdot \left[e \sin\theta \sin f + \cos\theta\ (1 + e \cos f) \right] \tag{2.1.2.1-5}$$

$$\frac{de}{dV} = \sqrt{\frac{a(1 - e^2)}{\mu}} \left[\sin\theta \sin f + \cos\theta\ \frac{(2 \cos f + e \cos^2 f + e)}{(1 + e \cos f)} \right] \tag{2.1.2.1-6}$$

$$\frac{d\omega}{dV} = \sqrt{\frac{a(1 - e^2)}{\mu e^2}} \left[- \sin\theta \cos f + \cos\theta \sin f \left(\frac{2 + e \cos f}{1 + e \cos f} \right) \right] \tag{2.1.2.1-7}$$

where θ is the angle between the thrust direction and the location horizontal (positive when thrust is above the local horizontal). Clearly

$$\sin\theta = \frac{U}{\sqrt{U^2 + W^2}} \; ; \; \cos\theta = \frac{W}{\sqrt{U^2 + W^2}} \; . \tag{2.1.2.1-8}$$

This is now an optimal control problem of a common type. Find the controls $\theta(V)$ and $f(V)$ which take the system from $a = a_1$, $e = e_1$, and $\omega = \omega_1$ to the desired final conditions $a = a_2$, $e = e_2$, $\omega = \omega_2$ in minimum V.

In this version, there are no impulses appearing. Impulsive thrust in the original problem will appear as a finite interval of V over which f has a constant value. Continuous thrust will appear as a finite slope in the function f(V). So, despite the fact that either impulsive or continuous thrust can be represented, this transformed problem does not itself contain impulses, and standard theory can be applied.

To make this transformation, it is absolutely essential that the time of the transfer be free.

When approaching the problem in this way, it is not necessary to specify in advance whether the thrust will be impulsive, and if so, how many impulses there will be. Such assumptions were necessary in all earlier work on orbit transfer problems. The results now rest on a much better foundation.

Prior to the development of this approach, work on the general problem was either an application of numerical minimization with a specified number of impulses (68), analysis of some reasonable postulated maneuver (10), or determination of some properties of the solution (144).

2.1.2.2. Special Cases. Prior to the development of the general theory described in the preceding section, many studies were done under various restrictive hypotheses. In most of these, the assumption of impulsive thrusting was made at the outset, and, in most, the number of impulses was fixed *a priori* as well. Furthermore, in some studies, the points of application of the impulses was given also.

Despite these limitations, the work is still of interest. In many cases it can now be stated that the results are truly optimum under certain conditions. Also, some of the studies are more specific and thorough, within their limits, than the general results cited above.

2.1.2.2.1. One orbit is circular. As mentioned above, since time is free, if one orbit is circular, the orientation of the elliptic orbit is not material. Then three parameters suffice to specify the problem: the radius of the circular orbit, and the semimajor axis and eccentricity of the elliptic orbit. This can be reduced to two parameters by, for example, normalizing the problem so that the radius or the angular momentum of the circular orbit is unity. Then the optimal maneuver can be displayed in a plane as a function of only two variables.

This special case is treated, using the Contensou method, by Winn (152) and Moyer (103). Moyer gives a complete solution to the problem based, in part, on numerical integration of the state and multiplier equations. He finds only two types of minima: two-impulse transfers and transfers via infinity. The finite three-impulse transfer does not appear. This is consistent with Marchal's result that the finite three-impulse transfer cannot occur unless the sum of the eccentricities of the initial and final orbits is greater than 1.712. If one of the orbits is circular, it is not possible for this condition to be met, so the finite three-impulse optimum is not to be expected.

Moyer plots his results in terms of h, the angular momentum of the elliptic orbit (angular momentum of the circular orbit is made equal to unity) and e, the eccentricity of the elliptic orbit. There is one region in this h-e plane where the optimum transfer is by two impulses, with all thrusting done at the apsides of the transfer ellipse. The transfer ellipse is tangent to both the initial and final orbits. This is sometimes called a Hohmann transfer or "Hohmann-type" transfer, though Hohmann himself only considered the case of two circular orbits.

The remainder of Moyer's h-e plane corresponds to transfers via infinity. The two parabolas are tangent to the initial and final orbits and the finite impulses are both applied in the plane of action.

Moyer also shows that there are cases in which the two-impulse Hohmann-type transfer is not even a local minimum. This is important in evaluating earlier work which assumed that in all cases only two impulses would be used.

Winn (152) also uses Contensou's method of formulating the equations, but his argument is all analytical and proceeds somewhat differently. He does not completely solve the problem, though it appears that the complete solution could be argued from the material he presents.

The papers by Rider (122) and Munick (105) are illustrative of earlier work. Rider assumed that transfer will be along an ellipse which is tangent to both the initial and final orbits, and that two impulses will be used. The only thing to be optimized is the point on the elliptical final orbit at which the tangency occurs.

Munick (105) uses the assumption, now known to be untrue in some cases, that the Hohmann transfer is the optimum circle-to-circle transfer to show that the Hohmann-type transfer is optimum for circle-to-ellipse transfers.

Silber (132) presents extensive numerical results for the problem of transferring from an inner circular orbit to an outer elliptical orbit, using two impulses. Departure from the circular orbit is assumed to be tangential.

2.1.2.2.2. The transfer ellipse is tangent to both the initial and final orbits. If it is assumed that the transfer will be by means of two impulses with an elliptic intermediate orbit which is tangent to both initial and final orbits, then it is possible to look for the minimum-fuel transfer subject to these restrictions.

While there are cases where this type of transfer is the true optimum (e.g., the Hohmann maneuver), it is not, in general, the true optimum (144). There are cases, however, in which it is possible to show that cotangential transfers are nearly optimum (71, 134). This type of maneuver has been studied by a number of authors (e.g., Lawden (71), Smith (134), Bender (10), and Wen (150). The properties have been rather extensively investigated, both by analytical and numerical methods. In the light of current knowledge, it appears that cotangential transfers, while relatively easy to analyze, are not competitive (from the fuel standpoint) with the true optimum except in those cases where the true optimum is a two-impulse transfer.

2.1.2.2.3. Initial and final orbits are ellipses with the same line of apsides. Horner (60) and Ting (138) have studied two-impulse transfers between elliptic orbits. They found that the best orientation for the two orbits is for both ellipses to have the same line of apsides. They also find that, if this is true, the optimum two-impulse transfer is between apsides of the initial and final orbits and that the transfer ellipse has the same line of apsides as the initial and final orbits.

This means that, if transfer is to be made between two ellipses of arbitrary orientation, the amount of fuel used will not be less than that required for the same two orbits with the lines of apsides coincident.

Lawden (82) studied this problem under the assumption that two impulses would be used. Dickmanns (28) gives an interesting way of plotting the results of this problem. Winn (152) considers the co-apsidal transfer problem using the Contensou method. He confines himself to the case where the pericenter of the initial and final orbits lie on the same line, but are on opposite sides of the attracting center.

2.1.2.2.4. Only the line of apsides is to be changed. If the initial and final orbits have the same semimajor axis and eccentricity and only differ in their apsidal directions, then the optimal two-impulse transfer may be determined. Lawden (72, 82) did this, and also studied the one-impulse transfer for this case. The one-impulse transfer took considerably more fuel than the two-impulse in the case he presented.

Marchal (94) studied the same problem from a more general point of view. He states that there are two possible types of optimal transfers: (a) the two-impulse one studied by Lawden, and (b) a four-impulse transfer via infinity.

The four-impulse maneuver uses two finite impulses and two infinitesmal ones. The first (finite) impulse is used to transfer from the pericenter of the initial orbit to a parabola. At infinity, two infinitesmal impulses are used to transfer to a second parabola whose pericenter is coincident with the pericenter of the second ellipse. The final (finite) impulse is applied at this common pericenter to transfer from the second parabola to the final ellipse.

Marchal (94) does not completely resolve the question of when each of the two types of maneuvers are optimal, but he does state that if the (common) eccentricity of the initial and final orbits is less than 0.53533 then the two-impulse maneuver is better than the four-impulse maneuver.

2.1.2.2.5. Initial and final orbits are nearly tangent. In this case, special difficulties arise in numerical computation of optimal transfers. McCue and Bender (100) consider this problem and give extensive references to related work.

2.1.3. Transfers Involving Parabolic or Hyperbolic Initial or Final Orbits

2.1.3.1. Initial and Final Orbits are Both of Non-Negative Energy. One of the more unexpected results in analyzing space maneuvers is the following: if the initial and final orbits are both of non-negative energy (they are either parabolic or hyperbolic) and the force field is pure inverse-square and the time for transfer is unrestricted, then the transfer can always be accomplished (theoretically) with zero fuel.

These zero-fuel trajectories are unrealistic in that they involve either performing maneuvers infinitely distant from the attracting center, or approaching within zero distance of the attracting center or both. However, these trajectories are the solutions to the mathematical problem of minimizing fuel in an inverse-square field.

Edelbaum (34, 37) suggested a maneuver consisting of six infinitesmal impulses to transfer from one hyperbolic orbit to another. In his maneuver, four of the impulses are applied at infinity, and two are applied infinitesmally close to the attracting center. This requires, of course, an infinite time.

A more comprehensive treatment of the problem is given by Ivanshkin (63), who uses some elements of the Contensou approach together with the notion of a minimizing sequence. From the mathematical standpoint, this is a much more satisfactory way to discuss these abnormal trajectories. Rather than talking about "impulses at infinity" or "impulses at zero radius", he constructs sequences of trajectories. Each trajectory in the sequence has impulses applied farther out and closer in than the one preceding it. He then considers the characteristic velocity required by each of these trajectories, and finds the lower limit of this characteristic velocity. This limit exists, even though the trajectory corresponding to the limiting value does not exist in the usual sense. Using this method of argument, Ivanshkin (63) obtains the result stated at the beginning of this section.

The optimal maneuvers obtained by this method do not satisfy one or more constraints which would be present in real-world maneuvers. Since zero fuel is required in the absence of the constraints, it follows that the amount of fuel which will be required in an actual maneuver will depend entirely on the constraints themselves. Several authors have considered various constraints and found optimal constrained maneuvers which, of course, have non-zero fuel requirements.

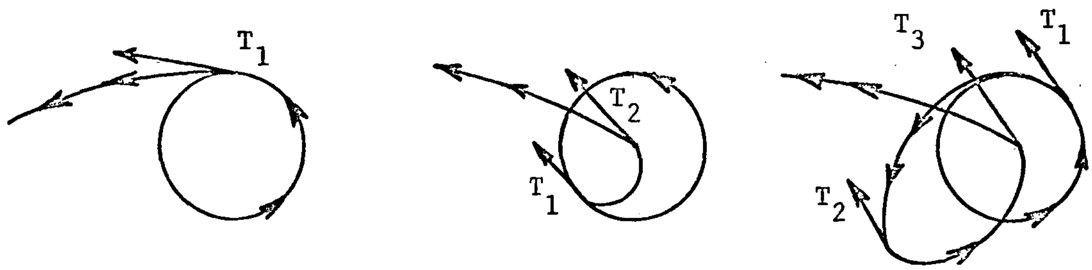
Friedlander and Harry (45) considered the problem of correcting the pericenter of a hyperbolic orbit. It was assumed that a vehicle was approaching a planet along a hyperbola whose point of closest approach to the planet was different from the desired value. In the absence of constraints, an infinitesmal impulse, applied at an infinite distance would correct the pericenter. If, however, the impulse must be applied at a finite distance, then that distance should be as large as possible. Once the maximum distance is established, then the problem is to select the impulse direction and magnitude which will accomplish the correction with minimum fuel. This is the problem solved by Friedlander and Harry.

Marchal (95) has considered the problem of transferring from one hyperbola to another with time free under the restriction that the trajectory must always remain a certain finite distance away from the attracting center. Impulses at infinity are still permitted, but impulses infinitesmally close to the center are not. He finds that the optimum thrusting program is always impulsive, that there are never more than two impulses of finite magnitude, and that there are cases where only one impulse is used. He gives a complete description of the parameter values for which various types of maneuvers are optimal.

Gobetz (50) considered the same problem from a more simplified point of view. Suppose that the hyperbolic excess velocity and direction of the asymptote is given before an encounter of a spacecraft with a planet and that the same quantities are specified afterwards. He compares the optimum one-impulse maneuver with a four-impulse maneuver (two of the impulses are infinitesmal) which was suggested by Edelbaum's (34,37) six-impulse maneuver. The same four-impulse maneuver was considered by Marchal (95). Gobetz found that either the one- or four-impulse maneuver might be optimum, depending on problem parameters. For the results presented, the difference in fuel requirements for the two maneuvers never exceeds about 20%, and in most cases it is less.

2.1.3.2. Circle-to-Hyperbola and Ellipse-to-Hyperbola Transfers. The circle-to-hyperbola problem is nearly as old as the Hohmann transfer. In 1929, Oberth (113) considered the problem of escaping from a circular orbit. He suggested that it would be advantageous to drop out of the circular orbit by decreasing velocity. Then, at the pericenter of the resulting orbit, an accelerating impulse is applied. Oberth, however, did not make a particularly thorough investigation. Lawden (73) studied the same maneuver, and showed that unless the hyperbolic excess velocity desired was rather large, it was more economical to escape directly via a single tangential accelerative impulse.

Edelbaum (32) carried the problem further by proposing a three-impulse maneuver. The first impulse is a tangential acceleration from the circular orbit, transferring the vehicle to an elliptic orbit with pericenter on the original circular orbit and with apocenter outside. When the vehicle reaches apocenter, a tangential decelerating impulse is applied, transferring to an elliptical orbit whose pericenter is as close to the center of the attracting body as possible. At the pericenter of this second elliptic orbit an accelerating impulse is applied tangentially, transferring the vehicle to the desired hyperbolic orbit. The one-, two-, and three-impulse maneuvers are indicated schematically in Figure 2.1.3.2-1.



a) One-Impulse

b) Two-Impulse (Oberth-Lawden)

c) Three-Impulse (Edelbaum)

FIGURE 2.1.3.2-1. CIRCLE-HYPERBOLA TRANSFERS

Edelbaum (32) compares all three maneuvers and finds that the three-impulse maneuver is always more economical than the two-impulse. Furthermore, if the apocenter where I_2 is applied is made sufficiently large, the three-impulse maneuver can always beat the one-impulse. The three-impulse maneuver is better as the apocenter of the elliptic orbits is increased, and as the pericenter of the second elliptic orbit is decreased.

The limiting maneuver would be to transfer from the circular orbit to a parabolic orbit. Then, at infinity, an infinitesmal impulse would transfer the vehicle to an orbit coming infinitesmally close to the attracting center. At the pericenter of this orbit, an infinitesmal impulse would transfer the vehicle to any desired hyperbolic trajectory. This last maneuver would be practically limited by the finite radius of the attracting center and its atmosphere.

Edelbaum (32) shows plots of the characteristic velocity required for all three types of maneuvers for some reasonable cases. If the attracting center can be approached rather closely, then the three-impulse maneuver can provide arbitrarily large savings in fuel if the required excess velocity is sufficiently large.

In the limit, if the attracting center can be approached arbitrarily closely, the three-impulse maneuver requires a characteristic velocity equal to that required to escape (via a parabola) from the circular orbit. The one-impulse maneuver requires this velocity and, in addition, the desired hyperbolic excess energy must be added at the same time.

These results are consistent with those of Ivanshkin (63) mentioned earlier. He considered the ellipse-to-hyperbola problem, of which the circle-to-hyperbola is a special case. Since it is possible to transfer freely among parabolic and hyperbolic trajectories, it is only necessary to transfer from the ellipse to a parabola. After reaching infinity, it will then be possible to transfer to any desired hyperbola at a cost of only infinitesmal amounts of fuel.

The optimal way to transfer from an ellipse to a parabola is by means of a single tangential impulse applied at the pericenter of the ellipse. The amount of characteristic velocity required is merely the difference between the pericenter velocity for the ellipse and the escape velocity at pericenter altitude.

2.1.4. Transfers Involving Terminals

There are a variety of transfer problems where either the initial or final state of the vehicle is not specified as lying on a given orbit. Rather, some different type of statement is made about it. For example, the initial position and initial velocity or, perhaps, only the initial position may be given. Such non-orbital specifications are sometimes called "terminals"; however, usage of this term is not uniform in the literature. In some cases, a problem involving a terminal may be thought of as an orbital transfer, but in others, it cannot.

This class of problem has received much less attention than orbital transfers, and complete results comparable to Marchal's (93) for the coplanar problem have not been obtained in any case.

Two problems of this type were stated by Vargo (142). In the first, the initial position and velocity, as well as the final position and velocity, were given. The problem was to find the two-impulse magnitudes and directions which would transfer the vehicle from the initial condition to the final condition with minimum characteristic velocity. It was assumed that there would be two impulses, one applied at each end of the transfer trajectory. The initial and final velocity vectors were required to lie in the plane defined by the initial and final positions and the attracting center, so that the entire problem was coplanar.

Vargo's second problem, the only one for which he presents any results, is a related one in which the initial and final positions are not completely specified. Only the distances from the attracting center are given. The problem is still coplanar. The initial and final velocities are specified as to magnitude and direction with respect to the local horizontal. For this problem, Vargo gives a computational method and some numerical results for one specific case. An analytical method for solving the second problem was given by Munick, McGill, and Taylor (106, 107, and 108). The complete solution in closed form is given by Horner (59).

Vargo's first problem was considered by Altman and Pistiner (1). They derived an eighth-order algebraic equation whose solution would give the solution to the problem, and obtained the solution in a special case. Pines (116) derives a similar result by an entirely different approach.

Stark (135) considered a variation of Vargo's first problem in which the initial position and velocity were given, but only the final position. The final velocity is free. This leads to a fourth-order, rather than eighth-order, algebraic equation to be solved.

Horner (61) considers a variation of Vargo's second problem. He found the solution to the problem of transfer from a terminal radial distance and velocity to an orbit. For the cases he treats, he finds the complete solution in closed form.

Ting and Pierucci (139) studied the same problem with the exception that the two impulses are allowed to have different specific impulses, and the problem is to minimize the vehicle mass change, rather than the characteristic velocity.

A related problem was considered by Leitmann (86). He required the final orbit to be circular (Ting and Pierucci allowed ellipses) and permitted an out-of-plane initial velocity component. He also allowed different specific impulses for the initial and final thrusts.

Finally, there are a variety of transfer problems associated with atmospheric reentry. The objective here is to transfer from an initial conic orbit of some kind to some terminal condition. Most studies assume a single-impulse transfer, and a number of different types of terminal specifications have been considered (3, 5, 44, and 46).

One study which deserves special mention is the recent report by Vinh and Busemann (145). They first consider several problems under the usual one-impulse assumption. Then, they consider the question of the absolute optimum, without restriction on the number of impulses and, initially, without assuming that impulsive thrust is optimal. They use the Contensou method, mentioned earlier, to show that the thrust should be impulsive and to deduce the form of the optimal maneuver in some cases.

2.2. Coplanar Rendezvous and Minimum-Time Transfers

In the problems of the preceding section, time was not restricted in any way. In effect, it was not even considered as a part of the formulation. In this section, problems are considered in which time is important to some degree.

There are several types of such problems. It is helpful to distinguish some of these: (a) time-free rendezvous (a vehicle in one orbit must either collide or rendezvous with a vehicle in another orbit, but the time required to do this is immaterial); (b) fixed-time rendezvous (a definite period is allotted for the maneuver); (c) minimum-time transfer or rendezvous; (d) fixed-time transfer. In each of these, it is possible to formulate meaningful minimum-fuel problems, and all have been studied, at least to some extent, except the last. The results available for the first three problems are reviewed in the following.

2.2.1. Time-Free Rendezvous

Suppose that a vehicle is initially on one orbit, and with some specific phase on that orbit. In other words, the position along the orbit is fixed as a function of time. The vehicle is to be transferred to another orbit and to some similarly fixed phase on the new orbit. The time required to do this is completely free. Under these rules, it is desired to accomplish the change with minimum characteristic velocity.

It has been recognized by a number of researchers that this type of rendezvous can be accomplished, at least in many cases, with the same amount of fuel as a time-free transfer between the same two orbits. It may be that a greater number of impulses is required, but the amount of fuel may be the same.

To illustrate this, consider a vehicle in a circular orbit about a planet. It is desired to move this vehicle to a rendezvous with another craft in a higher coplanar circular orbit about the same planet. If it were not for the phasing problem, the minimum-fuel maneuver would be the Hohmann transfer. A Hohmann transfer, initiated at some arbitrary time, would reach the desired final orbit, but, in general, not at the correct time.

Note, however, that the vehicle which is in the lower orbit has a rate of rotation about the attracting center which is different from that of the vehicle in the higher orbit. From this, it follows that the angle between the two radius vectors will go through all possible values if a sufficiently long period is allowed. There will be one value of this angle, such that, if a Hohmann transfer is initiated at that value, rendezvous will be achieved. The desired value of the relative orientation was found by Paiewonsky (114).

The fuel-optimum rendezvous would proceed as follows. Do nothing until the relative orientation of the two satellites has the desired value. At that time, initiate a standard Hohmann transfer. At the end of the transfer, the two vehicles will have the same position and velocity. This, of course, requires no more fuel than the simple Hohmann transfer. The only price paid is the additional waiting time.

There are, in fact, an infinite number of ways to perform this rendezvous with the same amount of fuel. The Hohmann maneuver involves raising the orbit apocenter to its final value in a single step. This could be done in any number of steps for the same fuel. For example, on departing the circular orbit, an accelerative impulse could be applied which would transfer the vehicle to an elliptic orbit whose apocenter was between the two circular orbits. At the next pericenter passage, another impulse could be applied, raising the apocenter higher, but not to the outer orbit. On the next pericenter passage, a third impulse could be applied, raising the apocenter to the desired final orbit. On reaching the desired apocenter distance, another impulse would inject the vehicle into the final orbit. If this four-impulse maneuver is started at just the right time, rendezvous would be achieved.

Van Gelder, Beltrami, and Munick (141) studied a similar problem of rendezvous when one of the orbits is circular and the other is elliptical. They consider the use of intermediate "parking" orbits which bring the two satellites into the desired phase relationship.

Billik and Roth (11) and Brunk and Flaherty (17) consider various ideas which make it possible to shorten the waiting time without increasing the fuel requirements excessively. The study of Schneider, et al, (131), has about the same approach, though the objective is to provide logic for a guidance computer.

2.2.2. Fixed-Time Rendezvous

If the motion of the two vehicles along their trajectories is uniquely determined with respect to time, and if the time period for the rendezvous maneuver is completely specified, then, in effect, the position and velocity of the maneuvering vehicle are completely specified at both ends of the time interval. A fixed time transfer between completely-specified terminals is required.

At least two impulses will be needed to perform this maneuver. If two impulses are used, then there is no minimization problem. There is, in general, only one trajectory meeting all the requirements, and that trajectory completely determines the fuel used. Determination of this trajectory is known as Lambert's problem. It is discussed by Battin (9), who gives additional references, and by Lim (89).

This two-impulse transfer is widely used in interplanetary mission planning studies to survey exhaustively a range of departure dates and flight times. This involves, however, an assumption that two-impulse transfers are optimal, and it is by no means possible to assert that this is always the case. A more general point of view is taken by Prussing (119). He considers the problem of rendezvous in a fixed time. The target vehicle is in a circular orbit and the maneuvering vehicle is close to the circular orbit in both position and velocity. The problem is restricted to the plane of the target orbit. Since all maneuvering is done in the vicinity of a circular reference orbit, Prussing linearized the equations of motion. Then, he applied Lawden's method, analyzing the behavior of the primer vector to find where the impulses are applied.

Prussing gives the results for the problem of transferring from one point on a circular orbit to another point on the same circular orbit in a fixed amount of time. The two-impulse solution has a singularity near 500° of orbital position change. Near this vicinity, Prussing shows that the use of four-impulse maneuvers eliminates the singularity completely. This is illustrated in Figure 2.2.2-1 where t is the time allowed for the maneuver, and ΔV is the characteristic velocity required.

Prussing also gives a partial definition of the circumstances under which various types of maneuvers are optimal. This is shown in Figure 2.2.2-2 where t is again the time allowed (expressed in orbital periods) and θ is the initial angular separation between the target and maneuvering vehicles. In this case, the maneuvering vehicle is initially in a circular orbit of radius different from the target orbit. The "Hohmann coast" is a two-impulse transfer with a wait. The transfer can be accomplished with a Hohmann maneuver in less time than is allowed in the problem.

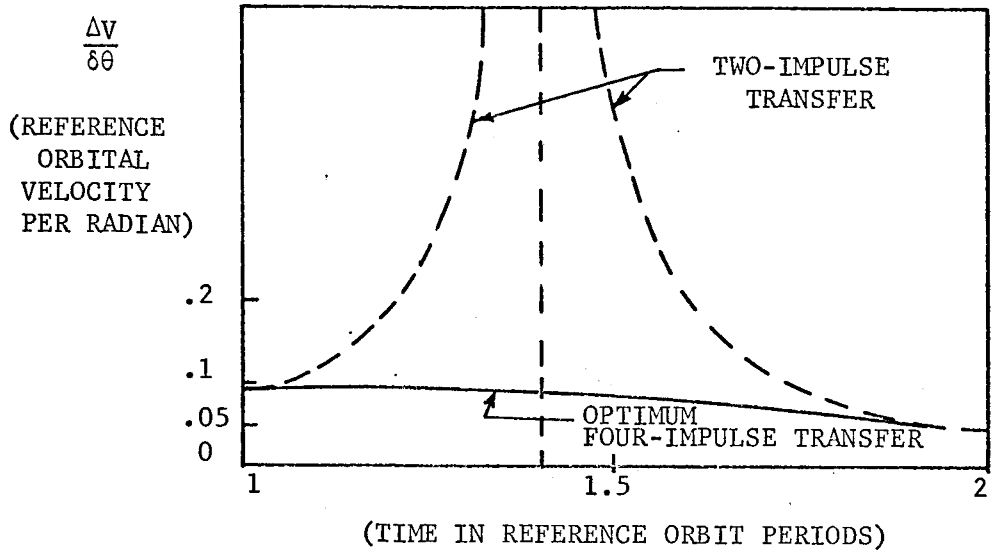


FIGURE 2.2.2-1. COMPARISON OF TWO- AND FOUR-IMPULSES FOR CIRCULAR ORBIT TRANSFER

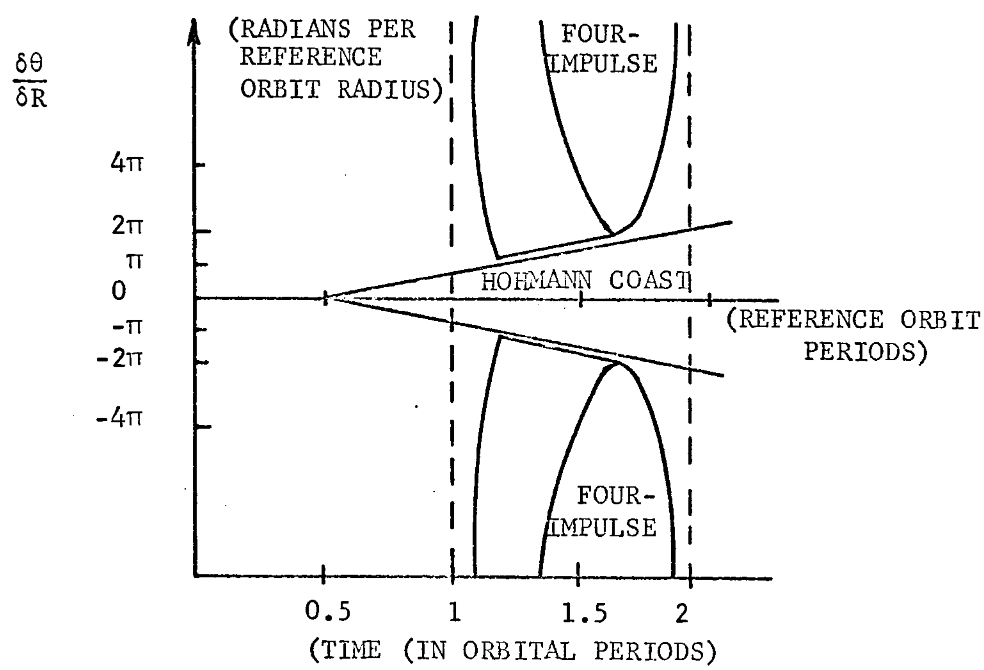


FIGURE 2.2.2-2. OPTIMAL TRANSFER MODE AS A FUNCTION OF PROBLEM PARAMETERS

These numbers of impulses are consistent with the results of Neustadt (109) and Potter and Stern (118) who showed that, for this problem, the maximum number of impulses would be four.

Since this is a simple circle-to-circle rendezvous, it seems clear that, in general, fixed time problems may be rather complex.

2.2.3. Minimum-Time Transfer

The problem of minimizing the time does not make sense unless some other constraint is applied. If an unlimited amount of fuel is available, and if it can be burned impulsively, then it should be possible to make the maneuver time as short as one pleases (so long as Newtonian mechanics apply). The most logical type of constraint to impose may be that of limiting the amount of fuel available, and minimizing the time under that restriction.

A problem of this type has been considered by Wang (148), but this appears to be the only such study reported. The scarcity of work in this area is perhaps due to the fact that minimum-fuel problems are still of major practical interest.

2.3. Non-Coplanar Time-Free Transfers

The current status of this problem is much less satisfactory than that of the coplanar case. Most of the results which have been obtained for this type of transfer have been based on the assumptions: (a) the optimal thrust program is impulsive; (b) the number of impulses and their points of application (and, sometimes, their direction) can be specified in advance. Some authors have used more than one impulse scheme and compared the fuel used by each.

Three recent papers by Winn (152), Busemann and Culp (19), and Marchal (94), promise a more comprehensive approach. All make use of three-dimensional versions of Contensou's method which can be generalized rather directly. Busemann and Culp (19) present a number of qualitative conclusions which can be derived rather easily by the variational method. Winn (152) formulates the non-coplanar problem, but does little more than begin the solution.

Marchal's treatment (94) is the most extensive. He considers several specific problems and gives some indication of the circumstances under which various maneuvers would be used. It seems probable that further results will be forthcoming using this type of analysis.

In the following paragraphs, earlier work on this problem is reviewed. This work is interesting and suggestive, despite the fact that none of the results can be affirmed to be absolute minima for the general problem. The linearized theory (109) suggests that the optimum number of impulses may go as high as five, for this problem. No studies to date have examined more than three impulses.

2.3.1. Transfers Between Non-Coplanar Circular Orbits

Perhaps the simplest problem of this class is that of rotating the plane of a circular orbit without changing its radius. Since the initial and final orbits intersect, it is possible to perform this maneuver with a single impulse.

Rider (121) suggested that it would be advantageous to use a three-impulse maneuver instead. The first impulse is applied tangentially on the initial orbit, transferring the vehicle to an elliptic orbit whose apocenter is above the circular orbit. At the apocenter of the elliptic orbit, the second impulse is applied in such a way as to rotate the plane of the elliptic orbit to the desired final plane, without changing its shape. This second elliptic orbit will have its pericenter tangent to the desired final circular orbit. This pericenter is the point of application for the final impulse. Rider (121) showed that this three-impulse maneuver is better than the single-impulse maneuver if it is desired to change the orbit plane by 49° or more. The maximum saving is about 43% in characteristic velocity if the plane-change is 90°.

Edelbaum (33) showed that it was better to do some of the plane-change at each impulse, rather than doing it all with the second. If this is done, the three-impulse maneuver becomes the optimum for all plane-change angles, though the saving for angles less than about 50° is modest. The comparison between one-impulse and three-impulse maneuvers as given by Edelbaum (34, 37) is shown in Figure 2.3.1-1.

In Figure 2.3.1-1, V_o is the circular orbit velocity (dependent on the central body and r_o), ΔV is the required characteristic velocity for the maneuver, i is the plane-change desired, r_a is the apocenter of the optimal transfer ellipse, and r_o is the radius of the circular orbit. As the plane-change angle increases, so does the apocenter of the intermediate ellipses. At about 60°, the apocenter goes to infinity and, beyond that point, all transfers are via infinity. Notice that for large plane-changes, the three-impulse maneuver becomes far better than the one-impulse.

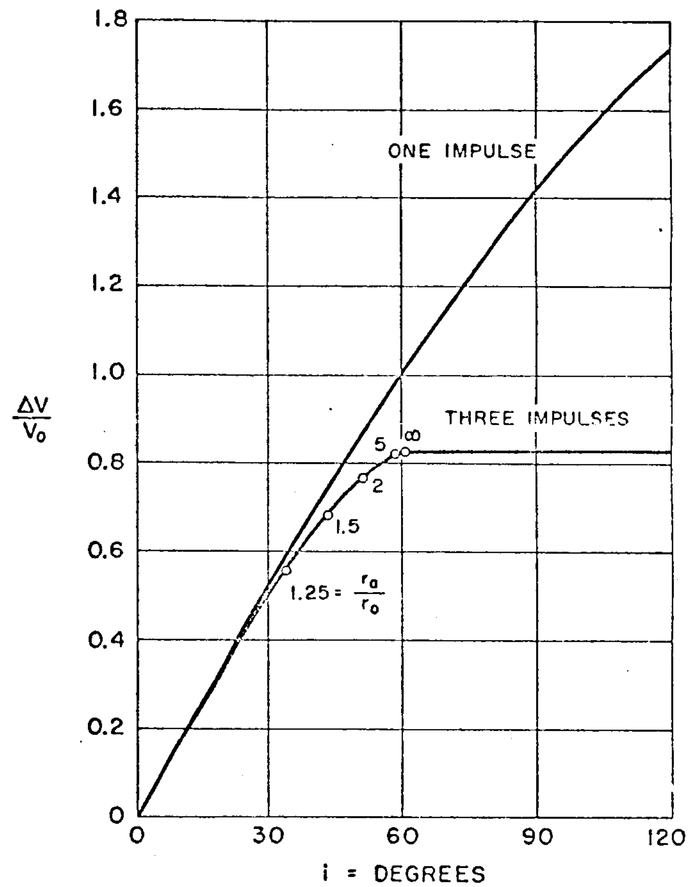


FIGURE 2.3.1-1. CHARACTERISTIC VELOCITY REQUIRED FOR CIRCULAR-ORBIT PLANE CHANGE

Wallner and Camiel (146) also considered the plane-change problem in some detail. In a later paper, Rider (123) considered a more general problem: changing the plane and also the radius of a circular orbit. He considered four types of transfers: (a) a Hohmann transfer with all the plane-change performed at the outer orbit; (b) a Hohmann transfer with part of the plane-change at each impulse; (c) a bi-elliptic transfer (apocenter outside the outer orbit) with all the plane-change at apoapse; (d) a bi-elliptic transfer with part of the plane-change performed with each impulse.

This problem can be completely described by two parameters: the angle of the plane change and the ratio of the initial and final radius. Rider (123) compares these types of transfers throughout the range of possible parameters, and shows the parameter regions within which the various types are the most advantageous.

Baker (6) considered the same problem introducing, in addition, consideration of the time required for the maneuver and the phasing or rendezvous aspects.

In a recent paper, Roth (128) studied the same problem, from about the same point of view, but with additional analysis of the minimization problems involved. Hoelker and Silber (55) gave a rather detailed analysis of a particular problem of this class: transfer from a low, non-equatorial parking orbit to an equatorial 24-hour orbit.

2.3.2. Transfers Between Non-Coplanar Elliptic Orbits

Since these maneuvers can always be accomplished with two impulses, it is natural that a substantial amount of work has been done on a two-impulse basis. The theoretical work of Eckel (31) and of Lee (84) assumes that two and only two impulses will be used. Eckel (31) develops and analyzes necessary conditions, somewhat in the spirit of Lawden (83), and reduces the problem to that of solving three algebraic equations in three unknowns. Lee (84) starts from a terminal-to-terminal problem, and derives results for the transfer case.

Numerical investigations of the same problem have been made by McCue (97), McCue and Hoy (98), and McCue and Bender (99). They use an adaptive steepest-descent procedure, and show that the functions to be minimized have rather complex behavior involving multiple minima. Incidentally, they show numerically that two-impulse maneuvers are better than the "Lawden spiral" in terms of fuel used.

Three-impulse maneuvers have been studied analytically by Hiller (53, 54) and by Niemeier (112). Hiller (53, 54) assumes the structure of two- and three-impulse maneuvers and optimizes within that structure. He finds that two-impulse, finite three-impulse and transfers via infinity all have their regions of optimality. While he does not point it out explicitly, his data appear to show that in some cases the finite three-impulse and infinity maneuvers offer substantial fuel savings over the two-impulse, perhaps amounting to 50%.

Niemeier (112) studies the problem of changing the plane of an elliptic orbit of arbitrary orientation. He assumes a certain three-impulse maneuver: (a) at apocenter of the original orbit, increase velocity to circular via a horizontal impulse; (b) when the resulting circular orbit crosses the line of intersection between the initial plane and the desired final plane, make the entire plane-change with one impulse establishing a circular orbit in the desired final plane; (c) when the vehicle reaches the apocenter of the desired final orbit, a local horizontal retro impulse will transfer to the desired final orbit.

He finds that the saving of this maneuver over the one-impulse maneuver increases with the eccentricity of the ellipse, and becomes arbitrarily large as the eccentricity approaches unity.

2.3.3. Transfers Involving Hyperbolic or Parabolic Orbits

First, consider the case where both initial and final orbits have non-negative energy. This was discussed for the coplanar case in section 2.1.3 and the situation is not materially changed here. When the trajectory goes to infinity, any plane-change can be accomplished at infinitesmal cost. If the attracting center can be approached arbitrarily closely, then the entire maneuver can be accomplished at zero cost. If the attracting center cannot be approached closely, the cost of transfer will be the same as in the coplanar case, since the plane-change is free. If it is not possible to go to infinity to perform the maneuver, the plane-change imposes an additional penalty.

There remains the problem of transferring between an elliptic and a parabolic or hyperbolic orbit. Again, as in section 2.1.3, if the attracting center can be closely approached, any ellipse-to-hyperbola transfer can be accomplished for only the amount of fuel required to escape from the pericenter of the ellipse.

Somewhat more practical maneuvers have been considered by Deerwester, McLaughlin and Wolfe (25) and by Gunther (51), without the claim that their maneuvers are absolutely optimal in any particular sense. Both consider circle-to-hyperbola transfer by either one- or two-impulse maneuvers and compare the fuel requirements of the various modes. In some cases, the two-impulse requirements are substantially less than the one-impulse.

2.3.4. Problems Involving Fixed Terminals

It is possible to define terminals in the non-coplanar case in a manner similar to those defined in section 2.1.4. Altman and Pistiner (2) and Lee (84) have studied the problem of minimum-fuel two-impulse time-free transfers between two terminals which are completely specified in terms of position and velocity. The solution hinges on the study of an eighth-degree polynomial. The report by Collins and Wallace (22) is an example of a numerical approach to this type of problem.

The problem of transferring from a fixed terminal (position and velocity given) to a circular orbit is considered by Carstens and Edelbaum (20). They assumed that two impulses would be used, and minimized the fuel requirements within that framework. They were concerned with launching a satellite into a circular orbit from a point on the Earth's surface. The launch point did not lie in the plane of the desired orbit.

Fimple (42) gives some results which, though not strictly optimal, are very suggestive of the possibilities that are offered by multiple-impulse maneuvers. He considers the problem of a planetary probe. Leaving the Earth's sphere of influence (position and velocity given), the problem is to intercept another planet (final position given but the velocity is free). This can always be done with one impulse, but, if the orbit of the target planet does not lie in the ecliptic, the one-impulse fuel requirements become large in some cases. Fimple gives velocity contours for one-impulse trajectories in the launch date-trip time plane as indicated in Figure 2.3.4-1. These results are for an Earth-Mars transfer, and the characteristic velocity has been normalized against the Earth's mean orbital speed (EMOS), 97,700 fps.

Notice the very high and steep ridge running diagonally through the figure. This ridge is a common feature of plots of this sort. These contours are plotted on the basis of fixed-time trajectories, which will be treated further in the next section. The original problem did not involve a fixed-time, so the procedure would be to select a launch date, then search along the vertical line through that launch date for the trip time which minimizes the fuel required.

To illustrate the behavior of this ridge, Fimple chooses to display a cut through the surface along a line of constant trip time of 300 days. This is shown in Figure 2.3.4-2.

If an additional impulse during midcourse is allowed, the ridge can be eliminated completely. The midcourse impulse used by Fimple cannot be affirmed as the absolute optimum one, but it is at least illustrative. Note the interesting similarity between this figure and one of Prussing's (Figure 2.2.2.-1).

If either one- or two-impulse maneuvers is allowed, whichever is better, then the contours of Figure 2.3.4-1 change into those of Figure 2.3.4-3.

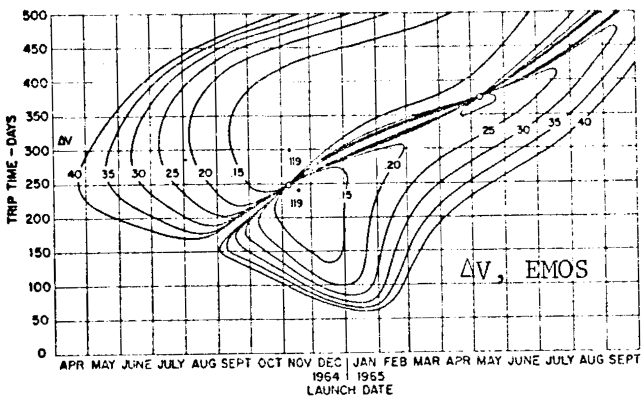


FIGURE 2.3.4-1. ONE-IMPULSE EARTH-MARS TRANSFERS

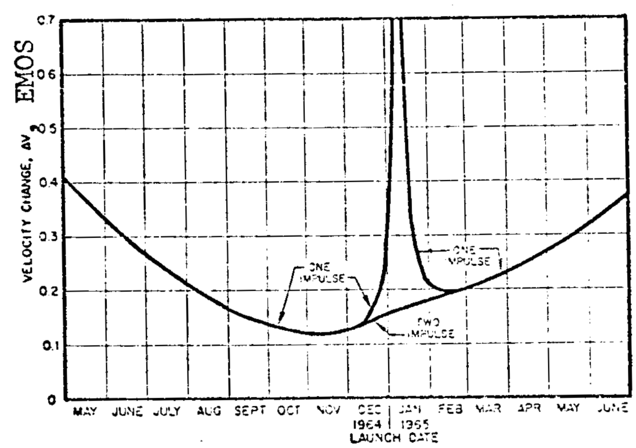


FIGURE 2.3.4-2. CHARACTERISTIC VELOCITY FOR ONE- AND TWO-IMPULSE EARTH-MARS TRANSFERS (300 DAY TRIP)

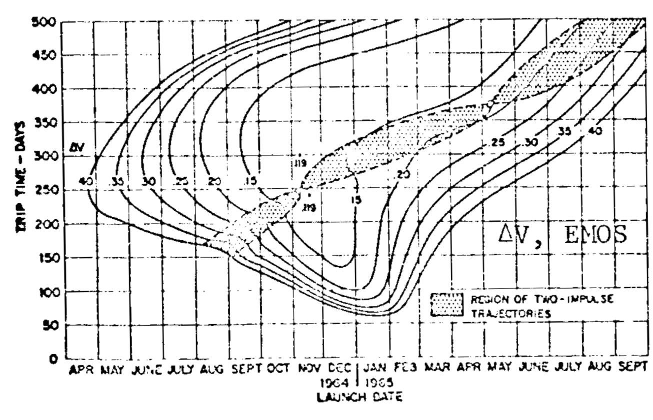


FIGURE 2.3.4-3. CONTOURS OF CONSTANT ΔV, USING EITHER ONE- OR TWO-IMPULSE MANEUVERS

Notice that the ridge is now gone and the launch opportunity is widened. In the shaded area, two-impulse maneuvers are better, and in the unshaded region, the one-impulse maneuver is better.

This is a very common situation in mission analysis, and these results suggest that multiple-impulse trajectories may have considerable practical significance. Notice that, in this example, the minimum characteristic velocity trajectory has a characteristic velocity of 0.119 EMOS whether the two-impulse maneuver is allowed or not. The absolute minimum-fuel requirement has not been decreased. However, there is a very considerable secondary benefit in widening the opportunity.

2.4. Non-Coplanar Rendezvous

Consider first the problem of time-free rendezvous. Just as in the coplanar case, this can be accomplished with the same fuel as a transfer, if the time is not restricted.

The fixed-time case is more difficult and at the same time more useful. If the time of the trajectory is fixed, the problem involves a fixed-time transfer between non-coplanar terminals (position and velocity completely specified at each end). In general, this maneuver can be accomplished with two impulses. If only two impulses are used, the trajectory is uniquely determined by the boundary conditions and there is no optimization problem involved.

Calculation of these unique two-impulse rendezvous trajectories has played an important role in interplanetary mission analyses. Several studies already mentioned have made use of this sort of computation as part of a larger study (22, 42). Many others could be mentioned, for example, Breakwell, Gillespie and Ross (12), Lee and Wilson (85), Tito (140), and Manning (92). There are many other similar studies which could be mentioned.

These studies make use of the minimal number of impulses required to do the mission, and specify enough boundary conditions on the heliocentric trajectories so that they are uniquely determined. The optimization comes from surveying the totality of solutions for all possible flight times, and selecting the one which requires least fuel.

Since multiple-impulse trajectories are arbitrarily excluded, it may be questioned whether the trajectories derived by the above process are the true optima. This question has not been answered as yet, though the tools for finding the answer seem to be available. It would be necessary to find the true optima and compare them with the two-impulse optima or one-impulse optima determined by the usual approach.

Lawden's general theory (83) is certainly applicable here, but no comprehensive application of it seems to have been made. Lion and Handelsman (90) have taken a somewhat different approach to the problem. They consider small perturbations of some non-optimum impulsive trajectory and develop necessary conditions for the addition of another impulse.

If one of these more comprehensive methods is used, it may be found that the absolute minimum fuel requirement is not decreased much compared to the usual one- or two-impulse solutions. It seems possible that those missions which have been extensively studied are rather well optimized at present.

Primary benefits of the variational multiple impulse results may be in two areas: (a) increasing mission flexibility (widening launch opportunities); (b) optimizing the more complex maneuvers which have not received much attention.

Apparently, the only actual results bearing on this question are those of Fimple (42) and Prussing (119), cited earlier. Both are limited in some respects. Prussing's work is single-plane and linearized; however, within these limits, the optimization is complete. Fimple's work is only sub-optimal. Still, perhaps these studies give some indication of what may be expected in the general case.

2.5. Maneuvers Involving Neighboring Orbits

If the initial, final and transfer orbits all lie close to one another, it is frequently possible to linearize the problem about one of the orbits. Then, the extensive theory of linear systems may be used to obtain results much more complete than is possible in the general nonlinear case.

In this event, the distinction between coplanar and non-coplanar problems is a rather minor one. It is not much more difficult to do this linearization in three dimensions than it is in two.

Edelbaum (33) gives an interesting set of formulas for the minimal characteristic velocity required to change each of the elements of a nearly circular orbit. Lawden and Long (79) consider the problem of optimally correcting an interplanetary trajectory which has deviated from the nominal. The correction is made in such a way as to intercept the target planet, though perhaps at a different point from the nominal intercept. Ribarich and Meredith (120) consider a more complicated problem of the same type, where there may be a number of constraints on the terminus of the trajectory.

Koenke (69) finds the fuel-optimal two-impulse transfer between two neighboring circular orbits under the requirement that the transfer orbit subtend a fixed angle at the attracting center. He also made numerical investigations of the range of validity of the linear results.

Eggleston (38) considered transfer from a terminal (position and velocity specified) to a circular orbit. The initial velocity vector lies in the plane of the orbit, so that the problem is coplanar. He calculated the minimum fuel required to rendezvous with a vehicle in the circular orbit. In a later study, Eggleston (39) considered the problem in three dimensions and considered the intercept problem as well. Hornby (58) considered a similar problem, except that he also studies the case where the orbit is not circular. Houboult (62) gives an introduction to rendezvous and surveys much of the earlier work in the rendezvous problem.

Two recent papers by Marec (96) and Edelbaum (36) give thorough treatments of the problem of transfer between two neighboring quasi-circular orbits when the time is free. While the methods of the two differ in many details, both use a version of the Contensou formulation, and both authors show that the transfer can always be done optimally with two impulses.

The more general linearized treatment of Neustadt (109) has already been mentioned. He considers an arbitrary problem (time-free or not) subject only to the requirement that the equations of motion can be linearized. Perhaps the most significant result derived from this study is the fact that the number of impulses required may be as large as the number of final parameters which are specified.

2.6. Penalty Due to Finite Thrust

Impulsive thrust is an idealization. Since it cannot be achieved precisely in practice, it is useful to consider the penalty for using a finite but reasonably large thrust. More specifically, the problem is that of determining how much more characteristic velocity is required on a finite-thrust maneuver than on an impulsive one.

Several authors have studied or commented on this question, and all agree that, for thrust levels consistent with chemical rocket propulsion, the excess characteristic velocity is extremely small. Typical calculations show fractions like 10^{-4}%. In view of this, interest in this question has not been especially high.

Lawden (77), Wang (147), and Robbins (127), give estimates based on an approximate consideration of the variational problem posed by finite thrust. Marchal (95) gives an order-of-magnitude estimate also, but does not state how it was obtained. Numerical investigations were carried out by Jurovics (64), McCue (102), and Willis (151). In the first two studies, optimal impulsive solutions are compared with optimum finite thrust solutions, and the differences in fuel used are very small. In any event, it seems well-established that the impulsive approximation gives an excellent estimate of the required characteristic velocity under ordinary circumstances for chemical propulsion.

3.0. MULTIPLE ATTRACTING CENTERS

3.1. Two Attracting Centers

Perhaps the simplest generalization of the problem considered in section 2.0 is that of a vehicle of negligible mass moving under the influence of two attracting centers. Lunar vehicles and solar probes are examples. Judging from the number of published papers, however, this subject has received little theoretical attention. Vargo (143) published a qualitative result on the best place to apply impulses to increase the value of Jacobi's integral. McCue and Bender (101) give a numerical method for solving the terminal-to-terminal problem (position given) when the time of flight is fixed. This is simply a two-point boundary value problem whose solutions either do not exist or are unique. There is no question of optimization. However, such a computational tool could no doubt be used in optimization studies in a manner analogous to the way terminal-to-terminal fixed time transfers are used in interplanetary optimization work.

3.2. Three Attracting Centers

Most interplanetary problems fall into this category, and these have been receiving a great deal of attention during the past several years. The three bodies involved are the departure planet (Earth in most cases), the Sun, and the target planet.

The problem of optimally transferring from a circular orbit around one planet to a circular orbit around another planet was considered from the variational point of view by Lawden (75, 76) more than a decade ago. His treatment was somewhat limited in that he assumed the locations of the (two) impulses and then constructed solutions which satisfied the the necessary conditions. In the process of this construction, he made use of what we now call "patched conic" or "sphere of influence" ideas in order to get closed-form expressions for the trajectory and the associated Lagrange multipliers. In this way, the usual difficulties of two-point boundary value problems were circumvented.

Despite these limitations, the treatment was at least a variational one, and as such, potentially more powerful than the methods which are now in common use. It is not clear to the author why interest in this approach did not continue. Since Lawden's original investigations, there has been great progress in both the theory and computation of variational problems. In view of this, it seems appropriate to revive the search for absolute optima in interplanetary problems. The principal benefit of this approach would probably lie in the discovery of trajectories using more impulses than are now fashionable.

While no absolute optima are known for interplanetary trajectories, a small number of studies have considered the use of additional impulses introduced at reasonable, though not necessarily optimal, places (29, 42, 57). Benefits are observed in some cases and not in others, but the picture is far from complete. In view of the great interest in and cost for solving interplanetary problems, it seems reasonable that truly optimal trajectories should be investigated.

These trajectories are now studied typically by means of a patched-conic method. In other words, at any given time, the vehicle is assumed to be under the influence of only one attracting center, so that the trajectory is a conic section. These conic trajectories are then patched together at somewhat arbitrary points where the vehicle is assumed to pass from the influence of one body to the influence of another. The point of application of applied impulses is usually selected in advance, as is the number of impulses to be used. Only a small number of parameters are left free to be selected, and these are usually explored in some systematic fashion to find the best (usually minimum-fuel) combination, within the framework assumed. A great amount of work has been done along these lines. Examples include (12, 85, 92, 140, and 136).

This is clearly not an unreasonable process. For some purposes, it is entirely adequate. Unfortunately, the framework is too rigid. Optimization studies performed in this manner can be carried out with respect to only a small number of parameters. Such an investigation probably would not be able to locate a six-impulse optimum trajectory, for example. Six quantities would be necessary to specify the location, direction and magnitude of each impulse, so that a total of 36 parameters might have to be found.

It cannot be asserted with any confidence that there are any interesting interplanetary problems for which the optimum trajectory has six impulses. On the other hand, it cannot be asserted with any confidence that there are not.

The question should be investigated, and it appears that a variational approach, as pioneered by Lawden would be the appropriate means. However, numerical approaches would and could now (with modern computers and techniques) play an essential role.

3.3. Four Attracting Centers

If more than two planets are involved in a maneuver, then four or more centers must be considered. Swingby trajectories (one planet is used to add energy to a vehicle on the way to another planet) are examples of this situation.

The usual mode of analysis in this case is about the same as the one used on three-center problems (see, for example, 8, 41, 43, 49, 91, 111, 136, and 117). The same opportunities exist for studying these maneuvers with variational methods.

4.0. STOCHASTIC PROBLEMS: ORBIT CORRECTIONS USING MEASUREMENTS

Up to this point, only what might be called propulsion problems have been considered. The amount of fuel to be used was a major fraction of vehicle weight. Further, it was assumed that perfect information was available and that desired maneuvers could be performed exactly. In this section, a class of minimum-fuel problems are considered for which none of these things are true. These problems are associated with orbit correction.

After launch, it is necessary, in most missions, to make one or more corrections of the trajectory. These corrections are based on measurements made either on-board or from the Earth's surface. The measurements are subject to random errors as are the orbit injection conditions (see 48 for an introduction to the problem). One can then consider the problem of using this imperfect information to bring the trajectory within some stated set of tolerances in such a way as to minimize the expected (or average) amount of fuel used. Since the errors are random, the amount of fuel needed to correct them will be random also. In general, the amount of fuel provided may have to be several times larger than the average amount needed in order to provide a high confidence that the required fuel will be available.

This problem involves a stochastic optimization process. This has been separated into two parts: (a) an optimal estimate of the vehicle position and velocity based on the noisy measurements; and (b) an optimal control program based on this estimate.

The first problem of this type was formulated by Lawden (80). He assumed that the correction would be by a finite number of impulses. Further, he assumed that, at the time of application of each impulse, there was a certain error in the trajectory, and that each impulse would be designed to eliminate completely the projected error at the end of the trajectory. He also assumed that there would be errors in execution of the desired impulses.

Based on these error sources, he found the number, spacing, magnitude and direction of the impulses which minimized the total fuel used. Unfortunately, his treatment of the statistical aspects leaves something to be desired. He gives no consideration to the second moments of any error distribution. His result indicates that if all errors have zero mean, then no fuel will be required for correction, no matter how large the variances of the errors are. In any event, Lawden formulated a meaningful problem, and included most of the elements which need to be considered.

One of the most interesting results in the field is that of Breakwell and Striebel (13). By using an argument based on Green's theorem, they show that the fuel-optimal thrusting program is continuous and not impulsive. Furthermore, there are periods of zero thrust near the beginning and end of the trajectory, separated by a period of non-zero (but finite) thrust. This is the only known instance in the entire space literature where it has been shown that continuous thrusting is superior to impulsive, other things being equal. Unfortunately, the Green's theorem argument does not readily generalize to more complex problems than the one considered by Breakwell and Striebel.

In a later report Breakwell, Rauch, and Tung (16) give a more comprehensive method based on optimal control theory and Kalman filtering (27, 65, and 66) ideas. In the same reference (16), the authors consider the problem of minimizing fuel under the requirement that the thrust be impulsive. They find that, if as many as four or five impulses are used, the fuel used approaches very closely that of the optimum (continuous thrust) case. However, if only a single impulse is used, the required fuel may approach four or five times the optimum.

Yaroshevsky and Parysheva (153) consider a similar problem except that their treatment of the measuring-error problem is much more limited than Breakwell's. They are concerned with a hyperbolic approach to a planet, and study the problem of correcting the pericenter altitude and also the problem of correcting both the pericenter altitude and the pericenter velocity. Under their assumptions, they also find that going beyond about four impulses does not save appreciable fuel. They did not calculate one-impulse requirements, but the four-impulse cases use only about half the fuel of the two-impulse ones.

Breakwell, Tung and Smith (15) show numerical application of both the continuous (optimal) and discrete (sub-optimal) correction strategies to interplanetary guidance problems. Denham and Speyer (26) considered the closely related problem of minimizing the terminal miss subject to a constraint on the amount of fuel available. Their results, however, are not nearly as complete as are Breakwell's.

Once the guidance law has been established, it may be tested, even for large perturbations, using a Monte Carlo simulation, as in References 21 and 133.

5.0. CONCLUSIONS

5.1. The General Theory

The solution to the general problem of Lawden is not known and even the form of the optimum thrust program is not known. While progress in this area is certainly desirable, it is perhaps not essential for obtaining the results of greatest practical interest.

Computational approaches, based on the theory of Rishel (124) and Warga (149) or the better-known direct method of Bryson (18) and Kelley (67) have the potential of determining optimal solutions, regardless of the number of impulses involved.

5.2. Problems Involving a Single Attracting Center

The theory of time-free transfers is nearly complete. Some details are still lacking, however, in the non-coplanar case. Of time-fixed problems, very little is known analytically and most of the computational work has involved *a priori* assumptions about the number of impulses to be used. A genuine optimal solution to the fixed-time terminal-to-terminal transfer problem would be of great practical utility. It could replace the usual Lambert's methods used in interplanetary problems. The small amount of evidence available suggests that there are cases where multiple impulse trajectories will save large amounts of fuel.

5.3. Problems Involving Multiple Attracting Centers

It is fair to say that, in this case, there is no single instance of a known, demonstrable optimal trajectory. It may be maintained, and probably correctly, that some known trajectories are rather close to the optimum, but this cannot be proved.

To get some idea of the opportunities which exist, it would be useful to use a patched-conic approach to a multiple-impulse trajectory Suppose the problem is to go into orbit around one of the outer planets and it is advantageous to use Jupiter gravity-assist. This is a four-center problem. Starting from a circular orbit around the Earth, the first task is to transfer to a hyperbolic escape orbit proceeding toward Jupiter. A three-impulse maneuver might be optimal for this purpose (32, 63). Having escaped the Earth's field, an ellipse-to-ellipse transfer problem is involved which might require as many as three impulses (93). Arriving near Jupiter it will be necessary to perform a hyperbola-to-hyperbola maneuver, which might take as many as four impulses (95). Leaving Jupiter, there is another ellipse-to-ellipse transfer with its possible three impulses (93). Arriving at the outer planet, the final maneuver is a hyperbola-to-circle transfer which might take three impulses. Putting all these together, as many as sixteen impulses could be involved. This, of course, omits the small corrective impulses which would doubtless be required prior to the approach to Jupiter and prior to the approach to the target planet. It might be practically necessary to control other aspects of the trajectory such as arrival speed or arrival time. This might require even more impulses.

While it is most unlikely that there actually are sixteen-impulse optimal trajectories, perhaps this example indicates the dangers of going too far with the patched-conic idea. It also suggests that there might be some merit in looking for optimal trajectories with numbers of impulses somewhat larger than now used.

At present, it appears that this search would have to be done numerically. The promising computational ideas were mentioned above. Naturally, it would be best to start with the simpler problems, but a start can certainly be made.

5.4. Orbit Correction

It seems well established that multiple impulses will materially reduce the amount of fuel necessary for orbit correction. It appears that four impulses would bring the fuel requirements reasonably close to the absolute minimum. The fuel thus saved will have to be balanced against the additional complexity of the multiple burns.

5.5. Combinations of Propulsion and Correction

If the primary propulsion system is used for a multiplicity of impulses, the possibility arises of doing orbit corrections at the same time as a major burn. This, in turn, raises the possibility of formulating both the propulsion and guidance problems as a single stochastic optimization problem, minimizing the total fuel required to get to the desired target with the desired accuracy.

It is offered as an opinion of the author that no benefits would be derived from this approach which would be consistent with the difficulties involved. This statement assumes that launch guidance systems will be used which are at least as accurate as current ones. If considerably poorer launch guidance were employed, then, perhaps, a combined approach might offer worthwhile fuel savings. Currently, the impulses required for guidance corrections are so much smaller than for primary propulsion that the guidance and propulsion problems can be uncoupled without appreciable penalty.

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