NASA TECHNICAL MEMORANDUM

NASA TM X-53714

NASA TM X-53714

FACILITY FORM 602

N68-20359 (ACCESSION NUMBER)	(THRU)
100 (PAGES)	1 (CODE)
TMX-53714 (NASA CR OR TMX OR AD NUMBER)	32 (CATEGORY)

ONE-DIMENSIONAL WAVE MOTION IN PRISMATIC BARS DUE TO IMPULSE LOADS WITH AND WITHOUT COULOMB DAMPING

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ABSTRACT

Several cases of wave motion without damping were solved to obtain background information which would aid in solving cases of wave motion with coulomb damping. The nonlinearities introduced by coulomb damping could be linearized so that they could be solved by using Laplace Transforms for the following cases: semi-infinite rod with a step stress impulse loading and a square wave stress impulse loading for $\tau \geqq \alpha/c$ and $\tau = \alpha/2c$; and a finite rod with a step stress impulse loading.

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Technical Memorandum X-53714

March 1, 1968

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RESEARCH AND DEVELOPMENT OPERATIONS

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DEFINITION OF SYMBOLS

Symbol	Definition
A	cross-sectional area
B_i	equation coefficients
c	wave propagation velocity, $\sqrt{E/\rho}$
c_1	inducing wave propagation velocity
c_2	induced wave propagation velocity
D_i	equation coefficient
E	modulus of elasticity
e	2.71828 ...
F	friction force per unit length
F_{st}	static friction force per unit length
f	general function
g	general function
$H(...)$	Heaviside function
K	spring constant of the rod
$\mathcal{L}$	Laplace transform operator
$\mathcal{L}^{-1}$	inverse Laplace transform operator
ℓ	rod length
m	mass
n	integer
P_{ext}	external force
p	Fourier transform domain independent variable

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
s	Laplace transform domain independent variable
$\mathrm{sgn}[\dot{u}]$	signum of $\dot{u}$, +1 when $\dot{u}$ positive or -1 when $\dot{u}$ negative
T	impulse loading force
t	time variable
$U(x,s)$	Laplace transform domain dependent variable
$U_c(x,s)$	complimentary (homogeneous) solution
$U_p(x,s)$	particular solution
$U(p,t)$	Fourier transform domain dependent variable
u	displacement
u_t	velocity, $\frac{\partial u}{\partial t}$
u_x	strain, $\frac{\partial u}{\partial x}$
v	impulse loading velocity
x	rod axial coordinate
x_o	position coordinate of wave front, ct
α	wave maximum propagation distance
ρ	mass density
τ	pulse duration
$\sum$	summation

CHAPTER I

Introduction

Although there have been numerous analyses published on wave motion from many standpoints, no publications on wave motion with coulomb damping have been uncovered. This treatise is an initial attempt to analyze wave motions with coulomb damping, using Laplace transforms as an aid in solving the partial differential equations. Wave motion without coulomb damping is presented first to establish the fundamental characteristics of wave motion. Then, the solutions which were successfully obtained of wave motion with coulomb damping are presented.

In the chapter on wave motion <u>without</u> coulomb damping, semi-infinite and finite rods are considered. For the semi-infinite rod, three types of impulse loadings are analyzed: step stress, square wave stress, and step velocity. The finite rod is analyzed for the right end boundary conditions of free and fixed. Impulse loadings of step stress, square wave stress, square wave velocity and triangular wave stress are considered for the free right end condition, and square wave stress and triangular wave stress for the fixed right end boundary condition.

In the chapter on wave motion <u>with</u> coulomb damping, only a few cases are solved because of the difficulty in obtaining solutions; however, both the semi-infinite rod and finite rod are considered.

For the semi-infinite rod, impulse loadings of step stress and square wave stress for $\tau = \alpha/2c$ and $\tau \geqq \alpha/c$ are analyzed, and for the finite rod, a step stress is analyzed.

In Chapter V the effects of friction-induced wave motion are presented for the inducing wave propagation velocities of twice, equal to, and one-half of the propagation velocity of the induced wave. In the last chapter, the advantages and disadvantages of the Fourier transform technique as applied to wave motion are presented.

CHAPTER II

Previous Work

Although almost every book dealing with dynamics, vibration, elasticity, mechanics or physics devotes some space to wave motion without damping, nothing seems to be available on wave motion with coulomb damping. Some of the works used as background information for the preparation of this analysis were Jacobsen and Ayre[1], Timoshenko[2], Burton[3], and Tong[4]. The technique of solving the wave equation using Laplace transforms was obtained from Wylie[5] and Norwacki[6]. The method of variation of parameters for solving some of the nonhomogeneous differential equations was obtained from Kells[7]. The Laplace transformations were taken from the tables and principles of Tse, Morse and Hinkle[8] and from the tables of the CRC Standard Mathematical Tables[9].

Some of the material reviewed as background information on Fourier transforms was Sneddon[10], Hildebrand[11] and Erdelyi, et al[12]. Transformation tables of Fourier cosine transforms were obtained from Sneddon[10] and Erdelyi, et al.[12].

Since no material was available on wave motion with coulomb damping, this work was generated from the collective background information of the bibliography. However, some work has been done on wave motion with various types of internal damping. Narwocki[6] and Goldsmith[13] are excellent examples of those who have performed analyses of wave motion with visco-elastic damping.

CHAPTER III

Wave Motion Without Damping

Wave motion is considered in this report only for a continuous elastic system which is covered by partial differential equations. The media are assumed to follow Hooke's law and are homogeneous and isotropic. The treatise will be limited to one-dimensional wave motion in prismatic bars where the length of the wave is large compared to the cross-sectional width. The cross-sectional planes are assumed to remain plane and for this chapter, damping is neglected.

A. Wave Equation Formulation

Consider the prismatic bar in figure 1. Let T be the force acting on the cross section at some position x, and at x + dx the force is $T + \frac{\partial T}{\partial x}\,dx$. Using Newton's law of motion, we obtain

$$\sum P_{ext} = m\ddot{u}$$

$$-T + T + \frac{\partial T}{\partial x}\,dx = \rho A\ dx\ \frac{\partial^2 u}{\partial t^2}\ , \qquad (1)$$

where ρ is the mass density, A is cross-sectional area, and u is the displacement. The force at the cross section is proportional to the strain (Hooke's law); i.e.,

$$T = AE \frac{\partial u}{\partial x} ,$$

where E is Young's Modulus of Elasticity. For a prismatic bar that is homogeneous and isotropic,

$$\frac{\partial T}{\partial x} = AE \frac{\partial^2 u}{\partial x^2}$$

and substituting into equation (1) yields

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \qquad (2)$$

where $c = \sqrt{E/\rho}$. The general solution to equation (2) can be expressed as

$$u = f(ct - x) + g(ct + x),$$

and regardless of the functions f and g, the argument $(ct \pm x)$ leads to the differential equation upon differentiating and substituting. Then, for $u = f(0)$,

$$c = \frac{x}{t} ,$$

which demonstrates that c is the wave propagation velocity.

B. Semi-Infinite Rod

This rod begins at $x = 0$ and extends in the positive x direction to infinity. It is an ideal rod used for mathematical purposes to give wave motion that is undisturbed by a boundary.

1. Step Stress Impulse Loading

The Laplace transform technique is used to obtain a solution to equation (2) because the solution for transient type impulse is desired, and these solutions are easier to obtain using Laplace transforms. To solve equation (2), two boundary conditions and two initial conditions must be known. For example, let the displacement, $u(x,t)$, and the velocity, $u_t(x,t)$, be zero at $t = 0$. Also, let a step stress impulse loading be applied to the semi-infinite rod. This means that the two boundary conditions are as follows: (1) $u_x(0,t) = \frac{-T}{AE} H(t)$ where T is the applied load, and $H(t)$ is the Heaviside Function (when the argument $t > 0$, $H(t) = 1$ and when $t < 0$, $H(t) = 0$) and (2) as $x \to \infty$, $u(x,t)$ is bounded. Taking the Laplace transformation of equation (2) with respect to t yields

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{c^2} U(x,s) + \frac{s}{c^2} u(x,0) + \frac{1}{c^2} u_t(x,0) = 0,$$

where s is the new variable. Since the initial conditions are zero, then

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{c^2} U(x,s) = 0.$$

This equation is an ordinary second order differential equation that has a solution of

$$U(x,s) = B_1 e^{\frac{s}{c}x} + B_2 e^{-\frac{s}{c}x} .$$

Since as $x \to \infty$, $u(x,t)$ is bounded, then $B_1 = 0$; therefore,

$$U(x,s) = B_2 e^{-\frac{s}{c}x}$$

$$\frac{\partial U(x,s)}{\partial x} = -B_2 \frac{s}{c} e^{-\frac{s}{c}x} .$$

Applying the other boundary condition, i.e.,

$$u_x(0,t) = \frac{-T}{AE} H(t) \quad \text{or} \quad U(0,s) = \frac{-T}{AEs} ,$$

yields

$$-\frac{T}{AEs} = -B_2 \frac{s}{c} \quad \text{or} \quad B_2 = \frac{Tc}{AEs^2}$$

$$U(x,s) = \frac{Tc}{AEs^2} e^{-\frac{x}{c}s} .$$

Taking the inverse transform gives a solution to the partial differential equation.

$$u(x,t) = \frac{Tc}{AE} (t - \frac{x}{c}) H(t - \frac{x}{c}),$$

Essential to the understanding of wave motion is the velocity and strain distribution of the rod. These can be obtained by taking the partial derivatives with respect to t and x; i.e.,

$$\frac{\partial u}{\partial t} = \frac{Tc}{AE} H(t - \frac{x}{c})$$

$$\frac{\partial u}{\partial x} = - \frac{T}{AE} H(t - \frac{x}{c}).$$

The distribution of displacement, velocity and strain for various times, t_i, is shown in figure 2. A step stress input yields a step velocity and strain but a ramp-type displacement. Also, the displacement cannot be discontinuous like the velocity and strain, because a discontinuity would indicate a break in the rod.

2. <u>Square Wave Stress Impulse Loading</u>

Let the initial conditions be zero. At t = 0, a stress is applied at x = 0 until $t = \tau$. Then the stress is zero at x = 0; i.e.,

$$u_x(0,t) = - \frac{T}{AE} [H(t) - H(t - \tau)]. \quad (3)$$

Equation (2), the differential equation to be solved, has the general solution of

$$U(x,s) = B_1 e^{\frac{s}{c}x} + B_2 e^{-\frac{s}{c}x}.$$

For a semi-infinite rod, $B_1 = 0$, and

$$U_x(x,s) = -B_2 \frac{s}{c} e^{-\frac{s}{c}x} \tag{4}$$

Taking the Laplace transform of equation (3) and substituting into (4) for x = 0 yields

$$-\frac{T}{AEs}(1 - e^{-\tau s}) = -B_2 \frac{s}{c} \quad \text{or} \quad B_2 = \frac{Tc}{AEs^2}(1 - e^{-\tau s})$$

$$U(x,s) = \frac{Tc}{AEs^2}(1 - e^{-\tau s})\, e^{-\frac{s}{c}x}$$

$$u(x,t) = \frac{Tc}{AE}\left[(t - \tfrac{x}{c})\, H(t - \tfrac{x}{c}) - (t - \tau - \tfrac{x}{c})\, H(t - \tau - \tfrac{x}{c})\right]$$

$$\frac{\partial u}{\partial t} = \frac{Tc}{AE}\left[H(t - \tfrac{x}{c}) - H(t - \tau - \tfrac{x}{c})\right]$$

$$\frac{\partial u}{\partial x} = -\frac{T}{AE}\left[H(t - \tfrac{x}{c}) - H(t - \tau - \tfrac{x}{c})\right].$$

The distribution of displacement, velocity and strain is shown in figure 3. The square wave stress input yields a square wave velocity and strain but a ramp wave front on the displacement which is constant after $t = \tau$.

3. Step Velocity Impulse Loading

For a semi-infinite rod with a step velocity at x = 0, the end condition is

$$u_t(0,t) = vH(t). \tag{5}$$

With the initial conditions zero, the solution to equation (2) is

$$U(x,s) = B_2 e^{-\frac{s}{c}x} . \tag{6}$$

Taking the Laplace transform of equation (5) with the initial conditions equal to zero and substituting into equation (6), for x = 0, we obtain

$$\frac{v}{s^2} = B_2$$

$$U(x,s) = \frac{v}{s^2} e^{-\frac{s}{c}x}$$

$$u(x,t) = v(t - \frac{x}{c})\ H(t - \frac{x}{c})$$

$$u_t(x,t) = vH(t - \frac{x}{c})$$

$$u_x(x,t) = - \frac{v}{c}\ H(t - \frac{x}{c}).$$

These distributions are shown in figure 4. It is interesting to note the simularity of the distributions for a step stress and a step velocity.

This section illustrates the characteristics of wave motion in its simplest form to form a basis for the analysis of the more complex wave motion which is to follow. An understanding of these simpler forms simplifies the understanding of the more complex and allows the individual to estimate the behavior of the more complex wave motion.

C. Finite Rod

Although the finite rod has a few of the characteristics of the semi-infinite rod, the finite rod exhibits unusual behavior at the boundaries caused by the added phenomenon of reflection.

1. Right End Boundary Condition Free

The term "free" here means that the right end is unsupported, and the stress at the end is zero unless a stress is specifically applied.

a. Step Stress Impulse Loading to the Left End

A rod of length, ℓ, has a compressive step stress applied when t = 0 and at x = 0; therefore, the boundary conditions are as follows:

$$u_x(0,t) = -\frac{T}{AE} H(t) \quad \text{or} \quad U_x(0,s) = -\frac{T}{AEs} \tag{7}$$

$$u_x(\ell,t) = 0 \quad \text{or} \quad U_x(\ell,s) = 0. \tag{8}$$

The general solution to equation (2) for zero initial condition is

$$U(x,s) = B_1 e^{\frac{s}{c}x} + B_2 e^{-\frac{s}{c}x},$$

$$U_x(x,s) = B_1 \frac{s}{c} e^{\frac{s}{c}x} - B_2 \frac{s}{c} e^{-\frac{s}{c}x}.$$

Substituting equation (7) with $x = 0$ and equation (8) with $x = \ell$ gives

$$-\frac{Tc}{AEs^2} = B_1 - B_2$$

$$0 = e^{\frac{s}{c}\ell} B_1 - e^{-\frac{s}{c}\ell} B_2 .$$

$$B_1 = \frac{Tc\, e^{-\frac{2\ell}{c}s}}{AEs^2\left(1-e^{-\frac{2\ell}{c}s}\right)} \quad \text{and} \quad B_2 = \frac{Tc}{AEs^2\left(1-e^{-\frac{2\ell}{c}s}\right)}$$

$$U(x,s) = \frac{Tc}{AEs^2\left(1-e^{-\frac{2\ell}{c}s}\right)}\left[e^{-\frac{x}{c}s} + e^{\left(-\frac{2\ell}{c}+\frac{x}{c}\right)s}\right].$$

Using the binomial expansion yields

$$\frac{1}{1-e^{-\frac{2\ell}{c}s}} = 1 + e^{-\frac{2\ell}{c}s} + e^{-\frac{4\ell}{c}s} + e^{-\frac{6\ell}{c}s} + \ldots .$$

Therefore,

$$U(x,s) = \frac{Tc}{AEs^2}\left\{\left[e^{-\frac{x}{c}s} + e^{\left(-\frac{2\ell}{c}-\frac{x}{c}\right)s} + e^{\left(-\frac{4\ell}{c}-\frac{x}{c}\right)s} + e^{\left(-\frac{6\ell}{c}-\frac{x}{c}\right)s} + \ldots\right]\right.$$

(continued on next page)

$$+ \left[e^{(-\frac{2\ell}{c} + \frac{x}{c})s} + e^{(-\frac{4\ell}{c} + \frac{x}{c})s} + e^{(-\frac{6\ell}{c} + \frac{x}{c})s} + e^{(-\frac{8\ell}{c} + \frac{x}{c})s} + \ldots \right] \Bigg\} ,$$

and taking the inverse Laplace transform yields

$$u(x,t) = \frac{Tc}{AE} \Bigg\{ \Bigg[\left(t - \frac{x}{c}\right) H \left(t - \frac{x}{c}\right) + \left(t - \frac{2\ell}{c} - \frac{x}{c}\right) H \left(t - \frac{2\ell}{c} - \frac{x}{c}\right)$$

$$+ \left(t - \frac{4\ell}{c} - \frac{x}{c}\right) H \left(t - \frac{4\ell}{c} - \frac{x}{c}\right) + \left(t - \frac{6\ell}{c} - \frac{x}{c}\right)$$

$$\cdot\ H \left(t - \frac{6\ell}{c} - \frac{x}{c}\right) + \ldots \Bigg] + \Bigg[\left(t - \frac{2\ell}{c} + \frac{x}{c}\right) H \left(t - \frac{2\ell}{c} + \frac{x}{c}\right)$$

$$+ \left(t - \frac{4\ell}{c} + \frac{x}{c}\right) H \left(t - \frac{4\ell}{c} + \frac{x}{c}\right) + \left(t - \frac{6\ell}{c} + \frac{x}{c}\right)$$

$$\cdot\ H \left(t - \frac{6\ell}{c} + \frac{x}{c}\right) + \left(t - \frac{8\ell}{c} + \frac{x}{c}\right) H \left(t - \frac{8\ell}{c} + \frac{x}{c}\right) + \ldots \Bigg] \Bigg\} . \quad (9)$$

The velocity equation is

$$u_t(x,t) = \frac{Tc}{AE} \Bigg\{ \Bigg[H \left(t - \frac{x}{c}\right) + H \left(t - \frac{2\ell}{c} - \frac{x}{c}\right) + H \left(t - \frac{4\ell}{c} - \frac{x}{c}\right)$$

$$+ H \left(t - \frac{6\ell}{c} - \frac{x}{c}\right) + \ldots \Bigg] + \Bigg[H \left(t - \frac{2\ell}{c} + \frac{x}{c}\right) + H \left(t - \frac{4\ell}{c} + \frac{x}{c}\right)$$

$$+ H \left(t - \frac{6\ell}{c} + \frac{x}{c}\right) + H \left(t - \frac{8\ell}{c} + \frac{x}{c}\right) + \ldots \Bigg] \Bigg\} . \quad (10)$$

and the strain equation is

$$u_x(x,t) = \frac{T}{AE}\left\{-\left[H\left(t - \frac{x}{c}\right) + H\left(t - \frac{2\ell}{c} - \frac{x}{c}\right) + H\left(t - \frac{4\ell}{c} - \frac{x}{c}\right)\right.\right.$$

$$\left. + H\left(t - \frac{6\ell}{c} - \frac{x}{c}\right) + \ldots\right] + \left[H\left(t - \frac{2\ell}{c} + \frac{x}{c}\right) + H\left(t - \frac{4\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left.\left. + H\left(t - \frac{6\ell}{c} + \frac{x}{c}\right) + H\left(t - \frac{8\ell}{c} + \frac{x}{c}\right) + \ldots\right]\right\}. \tag{11}$$

Shown in figure 5 is a representation of the displacement, strain and velocity distributions at time, t = 0. At t = 0, the contributions of each term of the above equations have not yet reached the rod; however, as time increases, the displacement, velocity and strain waves move in the directions of the arrows, and some of the waves immediately enter the domain of the rod. Figure 5 is used as a step to obtain figure 6. However, similar figures to figure 5 will not be shown in the solution to other types of examples to follow, but should be understood as a step that was taken but not shown.

To simplify the solutions which will follow, the equations will be written in summation notation. Equations (9), (10) and (11) in this notation would be as follows:

$$u(x,t) = \frac{Tc}{AE}\left\{\sum_{n=0}^{\infty}\left[\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right] + \sum_{n=1}^{\infty}\left[t - \frac{2n\ell}{c} + \frac{x}{c}\right]\right.$$

$$\left. \cdot\, H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right\}.$$

$$u_t(x,t) = \frac{Tc}{AE}\left\{\sum_{n=0}^{\infty}\left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right] + \sum_{n=1}^{\infty}\left[H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}.$$

$$u_x(x,t) = -\frac{T}{AE}\left\{\sum_{n=0}^{\infty}\left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right] - \sum_{n=1}^{\infty}\left[H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}.$$

It is advantageous to make some observations about figure 6 at this point. Notice that the displacement and velocity continue to increase while the strain fluctuates. The velocity at first increases from the left where the load is applied due to a compressive wave, then the velocity doubles at the right end and increases from the right due to an expansive wave. The strain reaches a constant value throughout the rod when the displacement is of a ramp shape over the length of the rod. The strain is zero over the length of the rod when the displacement is constant over the length of the rod.

b. Square Wave Stress Impulse Loading to the Left End

The square wave input is more realistic in practice than the step input, because at some time, τ, after the stress has been applied, the stress input ceases; therefore, the rod with a square wave input does not continue to increase its velocity as it did with a step input. Considering the same rod as before with zero initial conditions and

$$u_x(0,t) = -\frac{T}{AE}\left[H(t) - H(t-\tau)\right] \quad \text{or} \quad U_x(0,s) = \frac{T}{AEs}\left(e^{-\tau s} - 1\right)$$

$$u_x(\ell,t) = 0 \quad \text{or} \quad U_x(\ell,s) = 0,$$

the solution to the equation of motion (2) is

$$u = \frac{Tc}{AE}\left\{\sum_{n=0}^{\infty}\left[\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - \left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right.\right.$$

$$\left.\cdot H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right] + \sum_{n=1}^{\infty}\left[\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left.\left. - \left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right) H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}$$

with the corresponding velocity and strain distributions

$$u_t = \frac{Tc}{AE}\left\{\sum_{n=0}^{\infty}\left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right]\right.$$

$$\left. + \sum_{n=1}^{\infty}\left[H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) - H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\} \tag{12}$$

$$u_x = -\frac{T}{AE}\left\{\sum_{n=0}^{\infty}\left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right]\right.$$

$$\left. - \sum_{n=1}^{\infty}\left[H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) - H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}. \tag{13}$$

The velocity and strain distributions are shown in figures 7(a) and (b) for $\tau = \frac{\ell}{2c}$. The velocity wave travels down the rod and doubles at the right end. As the wave returns, it reduces to its original value. When it reaches the left end, the same wave motion occurs as did at the right end. The strain wave travels along the rod until it reaches the end, and then both ends of the wave go toward the middle of the wave until the wave vanishes at $c\tau/4$ from the end of the rod. After the wave vanishes, it reappears at the same position with the opposite sign, and the wave expands until it has a width of τ. When it is the width of τ, the strain wave has reached the end of the rod, and the wave moves in the opposite direction from which it initially moved. When the wave reaches the opposite end ($x = 0$), the same wave motion occurs at that end as did at the other end ($x = \ell$).

Figure 8 illustrates the displacement distribution of the square wave input of pulse duration, $\tau = \frac{\ell}{2c}$. When the wave reaches the end of the rod ($x = \ell$), the wave form changes. This is caused by the interaction of the two ramp-shaped waves meeting at the end of the rod. To give a further illustration of the displacement of the rods, the displacement versus time is shown in figure 9 for three rod positions $x = 0$, $\ell/2$ and ℓ. A rod when struck by a square wave of pulse duration, $\tau = \frac{\ell}{2c}$ does not move en masse, but only portions of the rod move at one time, like a worm crawling. The ends of the rod move in a similar manner with the end, $x = \ell$, at a phase lag of $t = \ell/c$ to the end, $x = 0$, while the middle moves only 1/2 the distance at one time, but the times occur twice as often. The reason the ends move twice as much as the middle is because a compressive wave moves down the rod tc the end, $x = \ell$, then converts to an expansive wave of the same magnitude, thereby causing a displacement twice that due to the original wave. This wave motion is not typical of nature in that nature seldom applies a stress of wavelength $\tau = \frac{\ell}{2c}$. This will be illustrated next with a square wave velocity impulse loading.

c. Square Wave Velocity Impulse Loading to the Left End

This example requires some forethought before the desired solution can be obtained. If we wish to simulate the striking of a golf ball, a baseball, or some other similar action, the duration of the square wave is important. This pulse duration is generally the time that it takes the pulse to go the length of the elastic body and return; i.e., $\tau = \frac{2\ell}{c}$. Other pulse durations may be specifically applied, but in general, they are not true simulations of nature. To illustrate this point, a solution will be obtained with a general pulse duration width of τ. The difficulty at this point is defining the boundary condition at the end where the square wave velocity is applied. When the velocity is applied, the end condition is defined as

$$u_t(0,t) = vH(t) \quad \text{or} \quad U(0,s) = \frac{v}{s^2} ,$$

where v is the applied velocity, and when the velocity is released, the end condition is

$$u_x(0,t) = 0 \quad \text{or} \quad U_x(0,s) = 0.$$

It is difficult to apply this end condition in this form; however, if the end condition could be totally defined in terms of u_x, then it could be easily applied. This can be done by obtaining the solution of a rod which is free on the right end and has a step velocity impulse loading on the left end; i.e., let the initial conditions be zero, and the boundary conditions be

$$u_t(0,t) = vH(t) \quad \text{or} \quad U(0,s) = \frac{v}{s^2}$$

$$u_x(\ell,t) = 0 \quad \text{or} \quad U_x(\ell,s) = 0.$$

The solution to the equation of motion, equation (2), is

$$u = v\left\{\sum_{n=0}^{\infty}(-1)^n\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right.$$

$$\left.-\sum_{n=1}^{\infty}(-1)^n\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right\}.$$

The corresponding velocity and strain equations are

$$u_t = v\left\{\sum_{n=0}^{\infty}(-1)^n H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - \sum_{n=1}^{\infty}(-1)^n H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right\}$$

$$u_x = -\frac{v}{c}\left\{\sum_{n=0}^{\infty}(-1)^n H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) + \sum_{n=1}^{\infty}(-1)^n H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right\}. \quad (14)$$

These are shown in figure 10. The velocity, v, travels down the rod and doubles at $x = \ell$. As the rod attains a velocity of 2v, the strain is being reduced to zero. When the wave is returned to the origin, the strain changes sign and the velocity is reduced back to v. This is because the boundary conditions require the velocity to be v at x = 0. This is as though the input is attached to the rod. If the input had not been attached to the rod, then the rod would have left the input at a velocity twice the input velocity. The fact that the strain changed signs indicated that the input was no longer pushing but was pulling.

The objective of this exercise was to obtain an end condition at x = 0 entirely in terms of u_x. Therefore, using equation (14) yields

$$u_x(0,t) = -\frac{v}{c}\left[\sum_{n=0}^{\infty}(-1)^n H\left(t - \frac{2n\ell}{c}\right) + \sum_{n=1}^{\infty}(-1)^n H\left(t - \frac{2n\ell}{c}\right)\right]$$

which is the boundary condition at x = 0. To limit this equation for $t \leqq 2\ell/c$, we eliminate the summation and the latter Heaviside function. Thus we obtain the left end boundary condition while the velocity input is applied:

$$u_x(0,t) = -\frac{v}{c} H(t).$$

To obtain the total left end boundary condition, this equation is combined with $u_x(0,t) = 0$ (after the velocity input is removed) to obtain

$$u_x(0,t) = -\frac{v}{c}\left[H(t) - H(t - \tau)\right] \quad \text{for } \tau \leqq \frac{2\ell}{c}.$$

By using this boundary condition with the right end boundary condition,

$$u_x(\ell,t) = 0,$$

and setting the initial conditions equal to zero, the solution to the equation of motion (2) is

$$u = v\left\{\sum_{n=0}^{\infty}\left[\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - \left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right.\right.$$

$$\left.\cdot H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right] + \sum_{n=1}^{\infty}\left[\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left.\left. - \left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right) H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}, \tag{15}$$

for $\tau \leqq 2\ell/c$. For $\tau = 2\ell/c$ which is generally characteristic of nature, the equation reduces to

$$u = v\left[\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) + \left(t - \frac{2\ell}{c} + \frac{x}{c}\right) H\left(t - \frac{2\ell}{c} + \frac{x}{c}\right)\right],$$

and the corresponding velocity and strain equations are

$$u_t = v\left[H\left(t - \frac{x}{c}\right) + H\left(t - \frac{2\ell}{c} + \frac{x}{c}\right)\right]$$

$$u_x = -\frac{v}{c}\left[H\left(t - \frac{x}{c}\right) - H\left(t - \frac{2\ell}{c} + \frac{x}{c}\right)\right].$$

Figure 11 shows the velocity and strain distributions. The velocity, v, and strain, -v/c, traverse the rod, simultaneously. When the wave reaches $x = \ell$, then it is reflected with the velocity equal to 2v, while the strain is reduced to zero. When the wave reaches x = 0, then the rod leaves the input at a velocity of 2v and all wave motion stops. To explain this phenomenon in terms of a golf club striking a golf ball, we would need to assume that the club was rigid and did not slow down while striking the ball. When the club strikes the ball, a wave goes across the ball bringing the velocity of the ball to the velocity of the club and putting a strain in the ball. As the wave is reflected, the strain is relieving itself by pushing against the club, thereby causing the velocity of the ball to reach twice that of the club. It is now apparent why $\tau = 2\ell/c$ is characteristic of nature.

To get further information concerning the step velocity, let $\tau = \ell/2c$, then equation (15) becomes

$$u = v \left\{ \sum_{n=0}^{\infty} \left[\left(t - \frac{2n\ell}{c} - \frac{x}{c} \right) H\left(t - \frac{2n\ell}{c} - \frac{x}{c} \right) - \left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c} \right) \right.\right.$$

$$\left. \cdot\, H \left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c} \right) \right] + \sum_{n=1}^{\infty} \left[\left(t - \frac{2n\ell}{c} + \frac{x}{c} \right) H \left(t - \frac{2n\ell}{c} + \frac{x}{c} \right) \right.$$

$$\left.\left. - \left(t - \frac{(4n+1)\ell}{2c} + \frac{x}{c} \right) H \left(t - \frac{(4n+1)\ell}{2c} + \frac{x}{c} \right) \right] \right\} ,$$

with velocity and strain equations of

$$u_t = v \left\{ \sum_{n=0}^{\infty} \left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - H\left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c}\right) \right] \right.$$

$$\left. + \sum_{n=1}^{\infty} \left[H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) - H\left(t - \frac{(4n+1)\ell}{2c} + \frac{x}{c}\right) \right] \right\}$$

$$u_x = - \frac{v}{c} \left\{ \sum_{n=0}^{\infty} \left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - H\left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c}\right) \right] \right.$$

$$\left. - \sum_{n=1}^{\infty} \left[H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) - H\left(t - \frac{(4n+1)\ell}{2c} + \frac{x}{c}\right) \right] \right\} .$$

These equations are the same as that of the square wave strain input, equations (12) and (13), with the exception of the constants in front of the equations. Therefore, figures 7(a) and (b) are illustrations of the velocity and strain distributions, and only the scales need to be changed. Figure 8 is also applicable to the displacement.

d. Triangular Wave Impulse Loading to the Left End

Consider the triangular wave shape shown in figure 12. The wave shape may be expressed by the equation

$$f(t) = T \frac{4\ell}{c} \left\{ tH(t) - 2\left(t - \frac{\ell}{4c}\right) H\left(t - \frac{\ell}{4c}\right) + \left(t - \frac{\ell}{2c}\right) \quad H\left(t - \frac{\ell}{2c}\right) \right\} .$$

Therefore, the boundary conditions are

$$u_x(0,t) = -\frac{4Tc}{AE\ell}\left\{tH(t) - 2\left(t - \frac{\ell}{4c}\right)H\left(t - \frac{\ell}{4c}\right) + \left(t - \frac{\ell}{2c}\right)H\left(t - \frac{\ell}{2c}\right)\right\}$$

$$u_x(\ell,t) = 0.$$

With the initial conditions set to zero, then the solution to the equation of motion is

$$u = \frac{2Tc^2}{AE\ell}\left\{\sum_{n=0}^{\infty}\left[\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)^2 H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - 2\left(t - \frac{(8n+1)\ell}{4c} - \frac{x}{c}\right)^2\right.\right.$$

$$\left.\cdot\, H\left(t - \frac{(8n+1)\ell}{4c} - \frac{x}{c}\right) + \left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c}\right)^2 H\left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c}\right)\right]$$

$$+\sum_{n=1}^{\infty}\left[\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)^2 H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) - 2\left(t - \frac{(8n+1)\ell}{4c} + \frac{x}{c}\right)^2\right.$$

$$\left.\left.\cdot\, H\left(t - \frac{(8n+1)\ell}{4c} + \frac{x}{c}\right) + \left(t - \frac{(4n+1)\ell}{2c} + \frac{x}{c}\right)^2 H\left(t - \frac{(4n+1)\ell}{2c} + \frac{x}{c}\right)\right]\right\},$$

with a velocity and strain of

$$u_t = \frac{4Tc^2}{AE\ell}\left\{\sum_{n=0}^{\infty}\left[\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - 2\left(t - \frac{(8n+1)\ell}{4c} - \frac{x}{c}\right)\right.\right.$$

$$\cdot\, H\left(t - \frac{(8n+1)\ell}{4c} - \frac{x}{c}\right) + \left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c}\right)H\left(t - \frac{(4n+1)\ell}{2c} - \frac{x}{c}\right)$$

(equation continued on next page)

$$+\sum_{n=1}^{\infty}\left[\left(t-\frac{2n\ell}{c}+\frac{x}{c}\right)H\left(t-\frac{2n\ell}{c}+\frac{x}{c}\right)-2\left(t-\frac{(8n+1)\ell}{4c}+\frac{x}{c}\right)\right.$$

$$\left.\cdot H\left(t-\frac{(8n+1)\ell}{4c}+\frac{x}{c}\right)+\left(t-\frac{(4n+1)\ell}{2c}+\frac{x}{c}\right)H\left(t-\frac{(4n+1)\ell}{2c}+\frac{x}{c}\right)\right]\Bigg\}.$$

$$u_x=\frac{4Tc}{AE\ell}\sum_{n=0}^{\infty}\left[-\left(t-\frac{2n\ell}{c}-\frac{x}{c}\right)H\left(t-\frac{2n\ell}{c}-\frac{x}{c}\right)+2\left(t-\frac{(8n+1)\ell}{4c}-\frac{x}{c}\right)\right.$$

$$\left.\cdot H\left(t-\frac{(8n+1)\ell}{4c}-\frac{x}{c}\right)-\left(t-\frac{(4n+1)\ell}{2c}-\frac{x}{c}\right)H\left(t-\frac{(4n+1)\ell}{2c}-\frac{x}{c}\right)\right]$$

$$+\sum_{n=1}^{\infty}\left[\left(t-\frac{2n\ell}{c}+\frac{x}{c}\right)H\left(t-\frac{2n\ell}{c}+\frac{x}{c}\right)-2\left(t-\frac{(8n+1)\ell}{4c}+\frac{x}{c}\right)\right.$$

$$\left.\cdot H\left(t-\frac{(8n+1)\ell}{4c}+\frac{x}{c}\right)+\left(t-\frac{(4n+1)\ell}{2c}+\frac{x}{c}\right)H\left(t-\frac{(4n+1)\ell}{2c}+\frac{x}{c}\right)\right]\Bigg\}.$$

The only difference between the velocity and strain equations is the factor, c, and a sign change in the first group of terms. Figures 13 and 14 reveal the effects of this difference. The interesting thing about these figures is the characteristics of the wave motion at the boundaries. The velocity doubles at the same time the strain changes sign. The displacement is shown in figure 15, and continues to increase as expected. Notice the unusual wave front and its behavior at the boundaries.

2. Right End Boundary Condition Fixed

The fixed right end boundary condition implies that the end is rigidly clamped. The left end is allowed to move while the fixed end can have no motion. Thus, the impulse loads will be applied to the left end only.

a. Square Wave Stress Impulse Loading to the Left End

Of interest in this example is the wave shape at the fixed end, $x = \ell$. In solving this example, the initial conditions will be set to zero. The boundary conditions are

$$u_x(0,t) = -\frac{T}{AE}\,[(H(t) - H(t-\tau)] \quad \text{or} \quad U_x(0,s) = -\frac{T}{AE}\left(\frac{1}{s} - \frac{e^{-\tau s}}{s}\right)$$

$$u(\ell,t) = 0 \quad \text{or} \quad U(\ell,s) = 0.$$

Using these boundary conditions, the solution to the equation of motion (2) is

$$u = \frac{Tc}{AE}\left\{\sum_{n=0}^{\infty}(-1)^n\left[\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - \left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right.\right.$$

$$\left.\cdot\, H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right] - \sum_{n=1}^{\infty}(-1)^n\left[-\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left.\left. + \left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\},$$

with the corresponding velocity and strain being

$$u_t = \frac{Tc}{AE}\left\{\sum_{n=0}^{\infty}(-1)^n\left[H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right]\right.$$

$$\left. - \sum_{n=1}^{\infty}(-1)^n\left[- H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) + H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}$$

$$u_x = \frac{T}{AE}\left\{\sum_{n=0}^{\infty}(-1)^n\left[-H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) + H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right]\right.$$

$$\left. - \sum_{n=1}^{\infty}(-1)^n\left[-H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) + H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}.$$

The velocity and strain distributions are shown in figures 16 and 17 for $\tau = \ell/2c$. At the fixed end, the velocity changes direction, and the strain doubles itself; and at the free end, the strain changes direction, and the velocity doubles itself. The displacement is shown in figure 18. Notice the double slope when the wave reaches fixed end.

b. Triangular Wave Stress Impulse Loading to Left End

Consider the triangular wave shape shown in figure 19. This wave shape may be expressed by the equation

$$f(t) = T(1/\tau)[(\tau - t)\ H(t) - (\tau - t)\ H(t - \tau)].$$

Therefore, the boundary conditions are

$$u_x(0,t) = -\frac{T}{AE\tau}\left[(\tau - t)\,H(t) - (\tau - t)\,H(t - \tau)\right]$$

$$u(\ell,t) = 0.$$

With the initial conditions set to zero, then the solution to the equation of motion (2) is

$$u = \frac{Tc}{AE\tau}\left\{\sum_{n=0}^{\infty}(-1)^n\left[\tau\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) - \frac{1}{2}\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)^2\right.\right.$$

$$\left.\cdot\, H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) + \frac{1}{2}\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)^2 H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right]$$

$$-\sum_{n=1}^{\infty}(-1)^n\left[-\tau\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) + \frac{1}{2}\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)^2\right.$$

$$\left.\left.\cdot\, H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) - \frac{1}{2}\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)^2 H\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\right]\right\}$$

with the corresponding velocity and strain

$$u_t = \frac{Tc}{AE\tau}\left\{\sum_{n=0}^{\infty}(-1)^n\left[-\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right.\right.$$

$$\left. + \left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)H\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\right]$$

$$-\sum_{n=1}^{\infty}(-1)^n\left[\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\right.$$

(equation continued on next page)

$$- \left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right) H \left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right)\Bigg]\Bigg\} .$$

$$u_x = \frac{T}{AE\tau}\Bigg\{\sum_{n=0}^{\infty}(-1)^n \Bigg[\left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right) H \left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)$$

$$- \left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right) H \left(t - \tau - \frac{2n\ell}{c} - \frac{x}{c}\right)\Bigg]$$

$$- \sum_{n=1}^{\infty}(-1)^n \Bigg[\left(t - \tau - \frac{2n\ell}{c} + \frac{x}{c}\right) H \left(t - \frac{2n\ell}{c} + \frac{x}{c}\right)\Bigg]\Bigg\} .$$

The velocity and strain distributions for $\tau = \ell/2c$ are shown in figure 20. Notice at the fixed end the way that the velocity changes directions, and the strain doubles itself. At the free end the exact opposite occurs; i.e., the strain changes direction in the same way the velocity did at the fixed end, and the velocity doubles in the same manner as did the strain at the fixed end. The displacement is shown in figure 21.

CHAPTER IV

Wave Motion with Coulomb Damping

The wave motion with coulomb damping follows the same laws as wave motion without damping, with the exception that the wave motion does not continue indefinitely but is damped out. The difficulty in analyzing wave motion with coulomb damping is maintaining the damping force in the opposite direction to the velocity and at the same time having a linear relationship. One reason for this is that in coulomb damping, the damping force is not a function of velocity but is a constant; therefore, when the velocity changes direction, no non-linear terms can exist in the damping term which will correspondingly change direction.

A. Formulation of Wave Equation with Coulomb Damping

Consider the prismatic bar of figure 22. Using Newton's Law, we obtain

$$\sum P_{ext} = m\ddot{u},$$

and substituting the forces from figure 22 yields

$$-T + T + \frac{\partial T}{\partial x}\,dx + \mathrm{sgn}[\dot{u}]\ F\,dx = m\,\frac{\partial^2 u}{\partial t^2}\ ,$$

where sgn[$\dot{u}$] is positive if $\dot{u}$ is positive and negative if $\dot{u}$ is negative. Sgn[$\dot{u}$] is a nonlinear term, and the objective will be to define a suitable substitute for sgn[$\dot{u}$] that is linear. This will not be determined until a particular problem is decided, then depending on the direction of the velocity, the appropriate substitute will be used. Since

$$T = AE \frac{\partial u}{\partial x} ,$$

then

$$AE \frac{\partial^2 u}{\partial x^2} dx - \rho A \, dx \frac{\partial^2 u}{\partial t^2} = \text{sgn}[\dot{u}] \; F \; dx$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} = \text{sgn}[\dot{u}] \frac{F}{AE} .$$

Since the wave velocity is defined by

$$c^2 = \frac{E}{\rho} ,$$

then

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \text{sgn}[\dot{u}] \frac{F}{AE} , \tag{16}$$

which is the wave equation including coulomb damping.

B. Step Stress Impulse Loading to a Semi-Infinite Rod

Like the previous chapter, Laplace transform techniques are used to solve the wave equation, and two initial conditions plus two boundary conditions need to be satisfied. The wave equation (16) must be altered to fit the particular example (see figure 23); i.e., the coulomb force per unit length, F, is applied only over the length of the bar that has motion. Therefore,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{F}{AE} H\left(t - \frac{x}{c}\right). \tag{17}$$

Taking Laplace transforms and setting the initial conditions to zero, we obtain

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{c^2} U(x,s) = \frac{F}{AEs} e^{-\frac{x}{c}s}. \tag{18}$$

The solution to the homogeneous equation is

$$U_c(x,s) = B_1 e^{\frac{s}{c}x} + B_2 e^{-\frac{s}{c}x}.$$

To obtain the particular solution, substitute

$$U_\rho(x,s) = Dxe^{-\frac{s}{c}x}$$

into the differential equation (18):

$$-2\frac{s}{c}De^{-\frac{s}{c}x} + D\frac{s^2}{c^2}xe^{-\frac{s}{c}x} - \frac{s^2}{c^2}Dxe^{-\frac{s}{c}x} = \frac{F}{AEs}e^{-\frac{s}{c}x}$$

$$D = -\frac{Fc}{2AEs^2}$$

$$U_\rho(x,s) = -\frac{Fcx}{2AEs^2} e^{-\frac{s}{c}x}$$

$$U(x,s) = U_c(x,s) + U_\rho(x,s) = B_1 e^{\frac{s}{c}x} + B_2 e^{-\frac{s}{c}x} - \frac{Fcx}{2AEs^2} e^{-\frac{s}{c}x}$$

The boundary conditions for this example are

$$u(0,t) = -\frac{T}{AE} H(t) \quad \text{or} \quad U(0,s) = -\frac{T}{AEs} .$$

As $x \to \infty$ $u(x,t)$ is bounded or $U(x,s)$ is bounded. Using these boundary conditions to solve for the constants B_1 and B_2, the solution becomes

$$U(x,s) = -\frac{Fc^2}{2AEs^3} e^{-\frac{s}{c}x} + \frac{Tc}{AEs^2} e^{-\frac{s}{c}x} - \frac{Fcx}{2AEs^2} e^{-\frac{s}{c}x} .$$

Taking the inverse transform yields

$$u(x,t) = \frac{c}{AE}\left[T(t - \frac{x}{c}) - \frac{Fc}{4}(t - \frac{x}{c})^2 - \frac{Fx}{2}(t - \frac{x}{c})\right] H(t - \frac{x}{c}).$$

Let $x_o = ct$ and $\alpha = \frac{2T}{F}$. Then,

$$u = \frac{F}{4AE}(x - x_o)(x_o + x - 2\alpha)\, H(t - \frac{x}{c}), \tag{19}$$

with the corresponding velocity and strain being

$$u_t = \frac{Fc}{2AE}(\alpha - x_o)\, H(t - \frac{x}{c}) \tag{20}$$

$$u_x = -\frac{F}{2AE}(\alpha - x)\ H(t - \frac{x}{c}). \tag{21}$$

Figure 24 shows the distribution of displacement, strain and velocity. When $ct = \alpha$, all motion ceases and remains static, and the maximum distance the wave travels down the bar is not dependent on the characteristics of the bar but on the magnitude of the input force, T, and the magnitude of the friction force per unit length, F.

Consider the static case where the force T is to the right and is balanced by $F_{st}\alpha$ to the left, where F_{st} is the static function force. Since α is defined as

$$\alpha = \frac{2T}{F}$$

where F is the dynamic friction force, then

$$F_{st}(2T/F) = T$$

$$F_{st} = \frac{F}{2}.$$

This demonstrates that the static friction force is half the dynamic friction force; therefore, when the wave is traveling down the rod and reaches α, the friction force per unit length reduces from F to F/2, instantaneously.

Also of interest in this problem is verifying the conservation of energy; i.e., the energy into the bar must be equal to the energy absorbed by the bar plus the energy loss due to friction. The energy relationship to be satisfied is the following:

Energy in = Kinetic Energy + Potential Energy + Friction Losses

$$Tu = \frac{1}{2} m\dot{u}^2 + \frac{1}{2} Ku^2 + Fu$$

$$Tu(0,t) = \frac{\rho A}{2} \int_0^{x_o} u_t^2 \, dt + \frac{AE}{2} \int_0^{x_o} u_x^2 \, dx + F \int_0^{x_o} u \, dt.$$

With the appropriate substitutions from equations (19), (20) and (21), the following results:

$$\frac{TFx_o}{4AE} (x_o - 2\alpha) = \frac{TFx_o}{4AE} (x_o - 2\alpha).$$

This verifies the conservation of energy and partially verifies the correctness of the solution to this example.

Of further interest is the conservation of momentum; i.e.,

$$\int P_{ext} \, dt = \int m \, d\dot{u}$$

$$\int_0^t (T - Fct) \, dt = \int_0^{\dot{u}} \rho A \, ct \, d\dot{u}$$

with the proper substitution from equation (19) the following results:

$$Tt - \frac{Fct^2}{2} = Tt - \frac{Fct^2}{2} .$$

This verifies the conservation of momentum and also partially verifies the correctness of the solution to this example.

C. Square Wave Stress Impulse Loading to a Semi-Infinite Rod

This example, unlike the same example of the previous section without coulomb damping, may not be solved in the entirety with one solution, but this example must be subdivided for square waves of different pulse durations; i.e., $\tau \geqq \alpha/c$ and $\tau = \alpha/2c$ which are the only two square wave durations solvable by this technique. For one of these cases, a further subdivision of time is required. This will become apparent as these examples are analyzed. These subdivisions are required because of the inability to define a friction term which is all encompassing and, at the same time, is capable of being linearized.

1. $\underline{\tau \geqq \alpha/c}$

This example is the case where a step stress is applied, and the wave travels along the rod until all motion ceases. Then at that time or some later time (depending on the value of τ), the step stress at $x = 0$ will be released. This example will need to be further subdivided: (1) For the time when the wave travels along the rod and comes to rest ($t \leqq \tau$) and (2) for the time after the step stress is released ($t > \tau$).

a. $t \leqq \tau$

This case has already been solved, and it is the step stress impulse loading of paragraph B of this section.

b. $t \geqq \tau$

This case begins where the previous case ended; therefore, the initial conditions of this case are the final conditions of the previous case. By substituting into equations (19) and (20) $t = \alpha/c$, the initial conditions are obtained; i.e.,

$$u(x,0+) = \frac{F}{4AE}(x - \alpha)^2 H(\alpha - x) \quad \text{and} \quad u_t(x,0+) = 0.$$

Since it is known that the friction force in the static condition is F/2, and that the stress will be released causing a wave motion at that point with a velocity in the opposite direction to its previous motion, then the wave equation may be written as follows for this example:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{3F}{2AE} H(t - \frac{x}{c}) + \frac{F}{2AE} H(t)\, H(\alpha - x). \qquad (22)$$

Initially, the entire rod is under the influence of the static friction force, but as time progresses and wave motion takes place, the friction force changes direction and doubles its value to meet the requirements of the wave motion. The boundary conditions at $x = 0$ has changed from the previous case, because the stress is removed at $x = 0$; therefore,

$$u_x(0,t) = 0 \quad \text{or} \quad U_x(0,s) = 0.$$

As $x \to \infty$ $u(x,t)$ is bounded or $U(x,s)$ is bounded. Taking the Laplace transform of equation (22), we obtain

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{c^2} U(x,s) + \frac{s}{c^2} u(x,0+) + \frac{1}{c^2} u_t(x,0+)$$

$$= - \frac{3F}{2AEs} e^{-\frac{x}{c} s} + \frac{F}{2AEs} H(\alpha - x).$$

Substituting the initial conditions, using variation of parameters to obtain the particular solution and using the boundary conditions to resolve the constants, we obtain the following solution:

$$u = \frac{F}{4AE} \left\{ \left[\frac{3c^2}{2} (t - \frac{x}{c})^2 + (3xc - 2\alpha c)(t - \frac{x}{c}) \right] H(t - \frac{x}{c}) \right.$$

$$\left. + (x^2 - 2\alpha x + \alpha^2) H(t) H(\alpha - x) \right\}, \tag{23}$$

with the corresponding velocity and strain being

$$u_t = \frac{Fc}{4AE} (3ct - 2\alpha) H(t - \frac{x}{c}) \tag{24}$$

$$u_x = - \frac{F}{4AE} \left\{ (3x - 2\alpha) H(t - \frac{x}{c}) - (2x - 2\alpha) H(t) H(\alpha - x) \right\}. \tag{25}$$

Figure 25 shows the displacement, velocity and strain distributions. The velocity is similar to the velocity of the first part of this example except that all motion ceases at $t = 2\alpha/3c$ rather than $t = \alpha/c$.

An energy check will be applied to this example to verify that energy is conserved. The relationship to satisfy is

Original Potential Energy = Kinetic Energy + Potential Energy

+ Friction Losses

$$\frac{1}{2} Ku_o^2 = \frac{1}{2} m\dot{u}^2 + \frac{1}{2} Ku^2 + Fu$$

$$\frac{AE}{2} \int_o^{\alpha} [u_x(x,0)]^2 \, dx = \frac{\rho A}{2} \int_o^{x_o} u_t^2 \, dx + \frac{AE}{2} \int_o^{\alpha} u_x^2 \, dx + F \int_o^{x_o} - (u - u_o) \, dx.$$

Substituting from equations (23), (24) and (25) using the proper forms, we obtain the following:

$$\frac{F^2\alpha^3}{24AE} = \frac{F^2\alpha^3}{24AE} \; .$$

This verifies the conservation of energy and partially verifies the correctness of the solution to this example. The solution may be further verified by verifying the conservation of momentum; i.e.,

$$\int P_{ext} \, dt = \int m \, d\dot{u}$$

$$\int_o^t \left[- \frac{F\alpha}{2} + \frac{3Fct}{2} \right] dt = \rho A \; c \int_o^{\dot{u}} t \; d\dot{u}.$$

With the appropriate substitution for equation (24), the following is obtained:

$$-Tt + \frac{3Fct^2}{4} = -Tt + \frac{3Fct^2}{4} \; .$$

This verifies that the solution did obey the law of conservation of momentum.

2. $\tau = \alpha/2c$

Consider the case of the semi-infinite rod with a square wave stress impulse loading where the pulse duration is $\tau = \alpha/2c$. This case does not need to be subdivided into two time domains, because the friction force term can be defined for all time and be linear at the same time. It was observed when solving this case with the friction force acting only on the forward velocity wave (i.e., only on the region between $ct - \tau \leq x \leq ct$) that a negative velocity wave was generated immediately preceding the forward wave. Therefore, it was necessary to define a friction term which would resist the negative velocity rear wave as well as the positive velocity forward wave. This is done in the differential equation below:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{F}{AE}\left[H\left(t - \frac{x}{c}\right) - 2H\left(t - \frac{\alpha}{2c} - \frac{x}{c}\right)\right].$$

The boundary conditions for a semi-infinite rod with a compressive square wave stress impulse loading are

$$u(0,t) = -\frac{T}{AE}\,[H(t) - H(t - \frac{\alpha}{2c})].$$

As $x \to \infty$ $u(x,t)$ is bounded. Using these boundary conditions and zero initial conditions, we obtain the following solution:

$$u = \frac{F\alpha^2}{2AE}\left\{\left[\frac{x_o}{\alpha} - \frac{x}{\alpha} - \frac{x_o^2}{2\alpha^2} + \frac{x^2}{2\alpha^2}\right] H\left(t - \frac{x}{c}\right) + \left[-\frac{2x_o}{\alpha} + \frac{x}{\alpha} + \frac{x_o^2}{\alpha^2} - \frac{x^2}{\alpha^2} + \frac{3}{4}\right] \cdot H\left(t - \frac{\alpha}{2c} - \frac{x}{c}\right)\right\}. \quad (26)$$

$$u_t = \frac{F\alpha c}{2AE}\left\{\left[1 - \frac{x_o}{\alpha}\right] H\left(t - \frac{x}{c}\right) + \left[-2 + \frac{2x_o}{\alpha}\right] H\left(t - \frac{\alpha}{2c} - \frac{x}{c}\right)\right\}.$$

$$u_x = \frac{F\alpha}{2AE}\left\{\left[-1 + \frac{x}{\alpha}\right] H\left(t - \frac{x}{c}\right) + \left[1 - \frac{2x}{\alpha}\right] H\left(t - \frac{\alpha}{2c} - \frac{x}{c}\right)\right\}.$$

Figure 26 shows the distribution of displacement, velocity and strain. The front velocity wave and the rear velocity wave go to zero at the same time, $t = \alpha/c$, and no motion exists after that time. Also, at $t = \alpha/c$ the strain has no discontinuity; indicating that no motion will be generated due to strain after that time. The static friction force per unit length that is required to maintain the strain is

$$F_{st} = \frac{F}{2},$$

which is the same as that for the first part of the previous example with the exception that it is in two directions in this case with the change in direction at $x = \alpha/2$.

To verify the conservation of energy in this case, the energy balance must be subdivided into two time zones because of the difficulty encountered in the required integration. Therefore, for $t \leqq \tau$, the relationship to satisfy is

$$T\, u(0,t) = \frac{\rho A}{2}\int_0^{x_o} u_t^2\, dx + \frac{AE}{2}\int_0^{x_o} u_x^2 + F\int_0^{x_o} u\, dx.$$

With the proper substitutions, this results in

$$\frac{T^2 x_o}{2AE}\left[2 - \frac{x_o}{\alpha}\right] = \frac{T^2 x_o}{2AE}\left[2 - \frac{x_o}{\alpha}\right] .$$

For $t \geqq \tau$, the relationship to satisfy is

$$T\,u(0,t) = \frac{\rho A}{2}\left[\int_{o}^{c(t-\tau)} u_t^2\,dx + \int_{c(t-\tau)}^{ct} u_t^2\,dx\right] + \frac{AE}{2}\left[\int_{o}^{c(t-\tau)} u_x^2\,dx + \int_{c(t-\tau)}^{ct} u_x^2\,dx\right]$$

$$+ F\left[\int_{o}^{c(t-\tau)} u\,dx + \int_{c(t-\tau)}^{ct} u\,dx\right]$$

and with the correct substitutions, the following result is obtained:

$$\frac{3T^2\alpha}{8AE} = \frac{3T^2\alpha}{8AE} ,$$

which is a constant, as expected, because in this time zone no energy has been put into the rod. To verify the conservation of momentum, the same division in time is required. For $t \leqq \tau$,

$$\int_{o}^{t} (T - Fct)\,dt = \int_{o}^{\dot{u}} \rho Act\,d\dot{u}$$

$$Tt - \frac{Fct^2}{2} = Tt - \frac{Fct^2}{2} .$$

For $t \geqq \tau$

$$\int_0^{\tau} (T - Fct)\, dt + \int_{\tau}^{t} (-Fc\tau)\, dt + \int_{\tau}^{t} Fc(t - \tau)\, dt = \int_0^{\dot{u}} \rho A\, c\tau\, d\dot{u}$$

$$+ \int_0^{\dot{u}} \rho A\, c(t - \tau)\, d\dot{u}.$$

$$\frac{T\alpha}{c} - F\alpha t + \frac{Fct^2}{2} = \frac{T\alpha}{c} - Fct + \frac{Fct^2}{2} .$$

With these verifications of conservation of energy and momentum, greater confidence in the solution is established.

D. Step Stress Impulse Loading to the Left End of a Finite Rod with the Right End Free

This example is a rod of length, ℓ, that has no supports at the right end and a compressive step stress is applied to the left end. As the wave traverses the rod, coulomb damping is present to retard the motion. The boundary conditions for this rod are

$$u_x(0,t) = -\frac{T}{AE} H(t) \quad \text{or} \quad U_x(0,s) = -\frac{T}{AES}$$

$$u(\ell,t) = 0 \quad \text{or} \quad U(\ell,s) = 0.$$

The equation of motion for this case is simply equation (17) which is also the equation for the semi-infinite rod with a step stress input, because in both cases the velocity never changes direction.

With zero initial conditions and the above boundary conditions, the solution to equation (17) is

$$u = \frac{1}{AE}\left\{\sum_{n=0}^{\infty}\left[T(x_o - 2n\ell - x) - \frac{F}{4}(x_o - 2n\ell - x)^2 - \frac{Fx}{2}(x_o - 2n\ell - x)\right]\right.$$

$$\cdot H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) + \sum_{n=1}^{\infty}\left[T(x_o - 2n\ell + x) - \frac{F\ell}{2}(x_o - 2n\ell + x)\right]$$

$$\cdot H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) + \sum_{n=1}^{\infty}\left[\frac{F}{4}(x_o - 2n\ell - x)^2 - \frac{F\ell}{2}(x_o - 2n\ell - x)\right.$$

$$\left.\left. + \frac{Fx}{2}(x_o - 2n\ell - x)\right] H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right\},$$

with the corresponding velocity and strain being

$$u_t = \frac{c}{AE}\left\{\sum_{n=0}^{\infty}\left[T + nF\ell - \frac{Fx_o}{2}\right] H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) + \sum\left[T - \frac{F\ell}{2}\right]\right.$$

$$\left.\cdot H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) + \sum_{n=1}^{\infty}\left[\frac{Fx_o}{2} - \frac{(2n+1)F\ell}{2}\right] H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right\}$$

$$u_x = \frac{1}{AE}\left\{\sum_{n=0}^{\infty}\left[-T + \frac{Fx}{2}\right] H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right) + \sum_{n=1}^{\infty}\left[T - \frac{F\ell}{2}\right]\right.$$

$$\left.\cdot H\left(t - \frac{2n\ell}{c} + \frac{x}{c}\right) + \sum_{n=1}^{\infty}\left[\frac{F\ell}{2} - \frac{Fx}{2}\right] H\left(t - \frac{2n\ell}{c} - \frac{x}{c}\right)\right\}.$$

To discover the limitations of these equations, we must re-examine the differential equation (17). The friction term for $t > \ell/c$

begins to apply beyond the length of the rod. While, in actuality, this poses no problem, it, nonetheless induces error into the mathematical analysis. To avoid this problem, the above equations will be limited to $t \leqq \ell/c$, and, in limiting the equations, they may be simplified to the following:

$$u = \frac{1}{AE}\left\{T(x_o - x) + \frac{F}{4}(x^2 - x_o^2)\right\} H(t - \frac{x}{c}) \tag{27}$$

$$u_t = \frac{c}{AE}\left[T - \frac{Fx_o}{2}\right] H(t - \frac{x}{c}) \tag{28}$$

$$u_x = \frac{1}{AE}\left[-T + \frac{Fx}{2}\right] H(t - \frac{x}{c}). \tag{29}$$

A new differential equation will be solved for $t > \ell/c$, viz.

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{F}{AE} H(\ell - x)\, H(t).$$

Using the same boundary conditions as before but using end conditions of previous equations as the initial conditions to this analysis, i.e.,

$$u(x, +0) = \frac{1}{AE}\left[T(\ell - x) + \frac{F}{4}(x^2 - \ell^2)\right] \text{ and } u_t(x, +0) = \frac{c}{AE}\left[T - \frac{F\ell}{2}\right],$$

we obtain the following solution:

$$u = \frac{\ell}{AE}\left\{T\left(\frac{x_o}{\ell} + 1 - \frac{x}{\ell}\right) + \frac{F\ell}{4}\left(\frac{x^2}{\ell^2} - \frac{x_o^2}{\ell^2} - 1 - \frac{2x_o}{\ell}\right)\right.$$

$$+ \sum_{n=0}^{\infty}\left[T\left(\frac{x_o}{\ell} - 2n - 1 - \frac{x}{\ell}\right) - \frac{F\ell}{2}\left(\frac{x_o}{\ell} - 2n - 1 - \frac{x}{\ell}\right)\right]$$

$$\cdot H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right) + \sum_{n=0}^{\infty}\left[T\left(\frac{x_o}{\ell} - 2n - 1 + \frac{x}{\ell}\right)\right.$$

$$\left.\left. - \frac{F\ell}{2}\left(\frac{x_o}{\ell} - 2n - 1 + \frac{x}{\ell}\right)\right] H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right\}, \tag{30}$$

with the corresponding velocity and strain being

$$u_t = \frac{c}{AE}\left\{T - \frac{Fx_o}{2} - \frac{F\ell}{2} + \sum_{n=0}^{\infty}\left[T - \frac{F\ell}{2}\right] H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left. + \sum_{n=0}^{\infty}\left[T - \frac{F\ell}{2}\right] H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right)\right\}. \tag{31}$$

$$u_x = \frac{1}{AE}\left\{-T + \frac{Fx}{2} + \sum_{n=0}^{\infty}\left[T - \frac{F\ell}{2}\right] H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left. + \sum_{n=0}^{\infty}\left[-T + \frac{F\ell}{2}\right] H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right)\right\}. \tag{32}$$

This solution is further limited in that T must be of sufficient magnitude that the velocity will never go to zero or negative at any point; therefore, $T \geq F\ell$. The distribution of the velocity for $T = F\ell$ is obtained by substituting $T = F\ell$ into equations (28) and (31). For $t \leq \ell/c$,

$$u_t = \frac{F\ell c}{AE}\left[1 - \frac{x_o}{2\ell}\right]. \tag{33}$$

For $t \geqq \ell/2$,

$$u_t = \frac{F\ell c}{AE}\left\{\frac{1}{2} - \frac{x_o}{2} + \sum_{n=0}^{\infty} [1/2]\, H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right.$$
$$\left. + \sum_{n=0}^{\infty} [1/2]\, H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right)\right\} . \tag{34}$$

Equation (33) time, t_1, starts (i.e., $t_1 = 0$) when the step stress is applied. Equation (34) time, t_2, starts (i.e., $t_2 = 0$) when the wave has traversed the length of the rod. Therefore, when $t_1 = \ell/c$, $t_2 = 0$. Figure 27 is the velocity for $T = F\ell$. After the wave has traversed the rod once, the average velocity of the rod is constant (i.e., $\bar{u}_t = \frac{Fc\ell}{2AE}$); therefore, the rod is traveling at a constant velocity while the wave motion is taking place. As t approaches $2\ell/c$, the velocity of the rod at $x = 0$ approaches zero, but at $t < 2\ell/c$, the velocity is increased by the on-coming wave. Therefore, the velocity comes very close to zero, but never quite reaches it; this illustrates that $T = F\ell$ is the critical value.

Now, consider a more typical value of T (i.e., $T > F\ell$), e.g., $T = 2F\ell$. Substituting into equations (27), (28), (29), (30), (31), and (32), results in the following, for $t \leqq \ell/c$,

$$u = \frac{F\ell^2}{AE}\left[\frac{2x_o}{\ell} - \frac{2x}{\ell} + \frac{x^2}{4\ell^2} - \frac{x_o^2}{4\ell^2}\right] H(t - \frac{x}{c})$$

$$u_t = \frac{Fc\ell}{AE}\left[2 - \frac{x_o}{2\ell}\right] H(t - \frac{x}{c})$$

$$u_x = \frac{F\ell}{AE}\left[-2 + \frac{x}{2\ell}\right] H(t - \frac{x}{c}),$$

and for $t \geqq \ell/c$,

$$u = \frac{F\ell^2}{AE}\left\{\frac{3x_o}{2\ell} + \frac{7}{4} - \frac{2x}{\ell} + \frac{x^2}{4\ell^2} - \frac{x_o^2}{4\ell^2} + \sum_{n=0}^{\infty}\left[\frac{3x_o}{2\ell} - 3n - \frac{3}{2} - \frac{3x}{2\ell}\right]\right.$$

$$\left. \cdot H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right) + \sum_{n=0}^{\infty}\left[\frac{3x_o}{2\ell} - 3n - \frac{3}{2} + \frac{3x}{2\ell}\right] H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right\}$$

$$u_t = \frac{Fc\ell}{AE}\left\{\frac{3}{2} - \frac{x_o}{2\ell} + \sum_{n=0}^{\infty}[3/2]\ H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left. + \sum_{n=0}^{\infty}[3/2]\ H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right)\right\}.$$

$$u_x = \frac{F\ell}{AE}\left\{-2 + \frac{x}{2\ell} + \sum_{n=0}^{\infty}[3/2]\ H\left(t - \frac{(2n+1)\ell}{c} + \frac{x}{c}\right)\right.$$

$$\left. + \sum_{n=0}^{\infty}[-3/2]\ H\left(t - \frac{(2n+1)\ell}{c} - \frac{x}{c}\right)\right\}$$

Figure 28 shows the displacement distribution; figure 29, the velocity; and figure 30, the strain. The velocity and displacement are ever-increasing, while the strain has set up a cyclic pattern after the wave has initially traversed the length of the rod, the height of the discontinuity of the velocity and strain remains the same, $\left|T - \frac{F\ell}{2}\right|$. The velocity at the boundaries increases each time by the discontinuity height upon reflection of the wave.

CHAPTER V

Friction-Induced Wave Motion

In this case, two rods are set together in such a manner that the wave motion in one rod induces wave motion in the other rod through friction (see figure 31). It is assumed that all cross-sectional planes remain plane, and the displacement is the same everywhere in the plane. The inducing rod will have a wave propagation velocity of c_1, and the induced rod will have a wave propagation velocity of c_2. By using equation (17), the equation of motion becomes (for the induced rod):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_2^2}\frac{\partial^2 u}{\partial t^2} = \frac{F}{AE} H(t - \frac{x}{c_1}).$$

Since these are semi-infinite rods and the induced rod has no strain at x = 0, then the boundary conditions for the induced rod are

$$u(0,t) = 0.$$

As $x \to \infty$ $u(x,t)$ is bounded. Using these boundary conditions and setting the initial conditions to zero, the solution to the equation of motion is

$$u = \frac{F\, c_1^2 c_2^2}{2AE(c_1^2 - c_2^2)} \left\{ \frac{c_2}{c_1}(t - \frac{x}{c_2})\, H(t - \frac{x}{c^2}) - (t - \frac{x}{c_1})\, H(t - \frac{x}{c_1}) \right\},$$

with the following velocity and strain being

$$u_t = \frac{F\, c_1^2 c_2^2}{AE(c_1^2 - c_2^2)} \left\{\frac{c_2}{c_1} \left(t - \frac{x}{c_2}\right) H\left(t - \frac{x}{c_2}\right) - \left(t - \frac{x}{c_1}\right) H\left(t - \frac{x}{c_1}\right)\right\}$$

$$u_x = \frac{F\, c_1 c_2^2}{AE(c_1^2 - c_2^2)} \left\{- \left(t - \frac{x}{c_2}\right) H\left(t - \frac{x}{c_2}\right) + \left(t - \frac{x}{c_1}\right) H\left(t - \frac{x}{c_1}\right)\right\}.$$

To gain an understanding of this example, three cases will be examined: subsonic, $\frac{c_2}{c_1} = \frac{1}{2}$; sonic, $\frac{c_2}{c_1} = 1$ and supe sonic, $\frac{c_2}{c_1} = 2$. Figure 32 shows the distribution of displacement, velocity and strain for these three cases. Observe the similarity in the wave shapes of the subsonic and supersonic cases, and the difference in the sonic. In the subsonic case, the wave has propagated twice the distance down the rod as does the sonic and supersonic case because, in the subsonic case, the limiting factor is the inducing rods velocity, c_1; and in the sonic and supersonic case, the limiting factor is the induced wave velocity, c_2. The displacement is greater for the cases in this order: subsonic, sonic and supersonic, because the friction force is acting over a greater portion of the rod for a longer period of time in that order. The strain is greater in the same order for the same reason.

CHAPTER VI

Fourier Transform Technique

In solving examples similar to the ones in this treatise, the Fourier transform is taken of the axial coordinate variable, x, rather than the time variable, t, as was the case in Laplace transforms. Instead of having initial conditions as associated with the Laplace transforms, the strain boundary condition at $x = 0$ and for all time is required. The initial conditions are needed to resolve the constants of integration in the same way that the boundary conditions were used in Laplace transform technique. Therefore, in solving these problems, three quantities must be known: (1) $u_x(0,t)$, (2) $u(x,0)$, and (3) $u_t(x,0)$. It now becomes apparent that the Fourier transform is limited to solutions of examples with semi-infinite rods, because there is no way to stipulate the boundary condition at $x > 0$.

A. Semi-Infinite Rod with a Step Stress Impulse Loading

For a compressive step stress impulse loading, the boundary condition at $x = 0$ is a constant; i.e.,

$$u_x(0,t) = -\frac{T}{AE} H(t).$$

Using the wave equation (2) and taking the Fourier transform yields

$$-p^2\, U(p,t) - u_x(0,t) - \frac{1}{c^2}\,\frac{\partial^2 U(p,t)}{\partial t^2} = 0$$

$$\frac{\partial^2 U(p,t)}{\partial t^2} + c^2 p^2\, U(p,t) = \frac{Tc^2}{AE}\, H(t)$$

which has a solution of

$$U(p,t) = B_1 e^{icpt} + B_2 e^{-icpt} + \frac{T}{AEp^2}\, H(t).$$

Since the initial conditions are zero; i.e.,

$$u(x, +0) = 0 \quad \text{and} \quad u_t(x, +0) = 0,$$

$$U(p,t) = \frac{T}{AE}\left\{\frac{1 - \cos\, cpt}{p^2}\right\}. \tag{35}$$

Because it is very difficult to obtain the solution by integration of the inverse Fourier transform integral and because insufficient Fourier transform tables are available, the solution will be proved by taking the Fourier transform of the known correct solution. The correct solution from chapter III is

$$u = \frac{Tc}{AE}\,(t - \frac{x}{c})\; H(t - \frac{x}{c}).$$

Substituting this equation into the transform integral yields

$$U(p,t) = \frac{T}{AE}\int_{o}^{\infty}(ct - x)\, H(t - \frac{x}{c}) \cos px\, dx$$

$$U(p,t) = \frac{T}{AE}\left\{\int_{o}^{ct} ct \cos px\, dx - \int_{o}^{ct} x \cos px\, dx\right\}.$$

Upon completing the indicated operation and some simplification of terms, we obtain

$$U(p,t) = \frac{T}{AE}\left\{\frac{1 - \cos cpt}{p^2}\right\},$$

which is identical to equation (35).

B. Semi-Infinite Rod with a Square Wave Impulse Loading, $\tau = \alpha/2c$

This example must be subdivided into two time domains: $t \leqq \alpha/2c$ and $t \geqq \alpha/2c$.

For $t \leqq \alpha/2c$, the boundary condition is

$$u_x(0,t) = -\frac{T}{AE}$$

and the initial conditions are zero. Using these conditions and equation (17) results in the following:

$$U(p,t) = \frac{1}{AE}\left\{\frac{T}{p^2}(1 - \cos cpt) + \frac{F}{2}\left(\frac{ct}{p^2}\cos cpt - \frac{\sin cpt}{p^3}\right)\right\}. \tag{36}$$

The correct solution is equation (26) when it is limited to $t \leq \alpha/2c$,

$$u = \frac{1}{AE}\left\{Tct - Tx - \frac{F\,c^2t^2}{4} + \frac{Fx^2}{4}\right\} H(t - \frac{x}{c}).$$

Taking the Fourier transform of this equation, gives the following results:

$$U(p,t) = \frac{1}{AE}\left\{\frac{T}{p^2}(1 - \cos cpt) + \frac{F}{2}\left[\frac{ct}{p^2}\cos cpt + \frac{1}{p^3}\sin cpt\right]\right\},$$

which is identical to the solution obtained in equation (36).

For $t \geq \alpha/2c$, the boundary condition is

$$u(0,t) = 0$$

and the initial conditions are the end conditions of the above case; i.e.,

$$u(x,0) = \frac{1}{AE}\left\{\frac{T\alpha}{2} - Tx - \frac{F\alpha^2}{16} + \frac{Fx^2}{4}\right\} H(\frac{\alpha}{2} - x)$$

$$u_t(x,0) = \frac{c}{AE}\left\{T - \frac{F\alpha}{4}\right\} H(\frac{\alpha}{2} - x).$$

The differential equation is based on a friction force that already exists from $\alpha/2$ to 0 and moves across the rod with the wave, and a negative friction force to resist the negative rear velocity wave which follows the forward wave, i.e.,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{F}{AE}\left[H\left(ct + \frac{\alpha}{2} - x\right) - 2H(ct - x)\right].$$

Using the aforementioned boundary and initial conditions, taking the Fourier transform, and doing much simplification of terms, we obtain the following solution:

$$U(p,t) = \frac{1}{AE}\left\{\frac{T}{2p^2}\left[2\cos cpt - \cos\left(cpt + \frac{\alpha p}{2}\right)\right] - \frac{Fct}{2p^2}\left[2\cos cpt\right.\right.$$

$$\left.\left. - \cos\left(cpt + \frac{\alpha p}{2}\right)\right] + \frac{F}{2p^3}\left[2\sin cpt - \sin\left(cpt + \frac{\alpha p}{2}\right)\right]\right\}. \quad (37)$$

Again, equation (26) is the correct solution when t is set equal to $t + \alpha/2c$, because this solution began when equation (26) was at $\alpha/2c$. Therefore, substituting $t = t + \frac{\alpha}{2c}$ into equation (26) yields

$$u = \frac{1}{AE}\left\{\left[T\left(ct + \frac{\alpha}{2} - x\right) - \frac{Fc^2t^2}{4} - \frac{F\alpha ct}{4} - \frac{F\alpha^2}{16} + \frac{Fx^2}{4}\right] H\left(ct + \frac{\alpha}{2} - x\right)\right.$$

$$\left. - \left[T(ct - x) - \frac{Fc^2t^2}{2} + \frac{Fx^2}{2}\right] H(ct - x)\right\}.$$

Taking the Fourier transform yields

$$U(p,t) = \frac{1}{AE}\left\{\frac{T}{2p^2}\left[2\cos cpt - \cos\left(cpt + \frac{\alpha p}{2}\right)\right] - \frac{Fct}{2p^2}\left[2\cos cpt\right.\right.$$

$$\left.\left. - \cos\left(cpt + \frac{\alpha p}{2}\right)\right] + \frac{F}{2p^3}\left[2\sin cpt - \sin\left(cpt + \frac{\alpha p}{2}\right)\right]\right\}$$

which is identical to equation (37).

CHAPTER VII

Summary and Conclusions

The applicability of Laplace transforms to aid in solving wave motion differential equations, excluding damping, was demonstrated in that many types of boundary conditions were readily solved. For this wave motion without damping, many interesting characteristics were exhibited, especially at the boundaries of the rod. Of special interest are figures 7, 11, 12, 17, 18 and 20. The complexity of the wave shapes at the boundaries exhibits the details of the wave motion characteristics, and from these details it becomes apparent that difficulty will be encountered in defining a friction force which will always retard the motion and at the same time be linear.

In this treatise the interesting characteristics of wave motion with coulomb damping occur along the length of the rod and not at the boundaries. Perhaps the boundaries would have proved to be more interesting if more complex solutions had been obtained. In all of these solutions, it was found that the velocity was a function of time only and not the axial distance except for the sudden change at the point of discontinuity. The strain was a function of the axial distance only except for the shifting of the point of discontinuity with time. The displacement was a function of time and axial distance.

For the semi-infinite rod with the step stress impulse loading and coulomb damping, the maximum distance that the wave would propagate along the rod ($\alpha = 2T/F$) was not a function of the characteristics of the rod, but was dependent only on the magnitude of the input force and the friction force per unit length. The static friction force per unit length that existed after all motion ceased was one-half that of the dynamic. When the step stress was removed, the maximum distance the wave propagated was $2\alpha/3$. The static friction force between 0 and $2\alpha/3$ was one-fourth the dynamic friction force, and between $2\alpha/3$ and α, it was one-half. For the same rod with a square wave impulse loading of pulse duration $\tau = \alpha/2c$, all motion ceased at α, and the static friction force was one-half the dynamic friction force.

For the finite rod with a step stress impulse loading and coulomb damping, it was found that for $T = F\ell$, the average velocity of the rod was a constant ($Fc\ell/2AE$) after the wave has traversed the rod once while wave motion continued. For $T > F\ell$, the velocity continues to increase while the strain sets up a cyclic behavior. Once the wave has traversed the rod, the discontinuity height of the velocity and strain is a constant, $|T - \frac{F\ell}{2}|$, and the velocity at the boundaries is increased by the amount of the discontinuity height $(T - \frac{F\ell}{2})$ upon reflection of the wave.

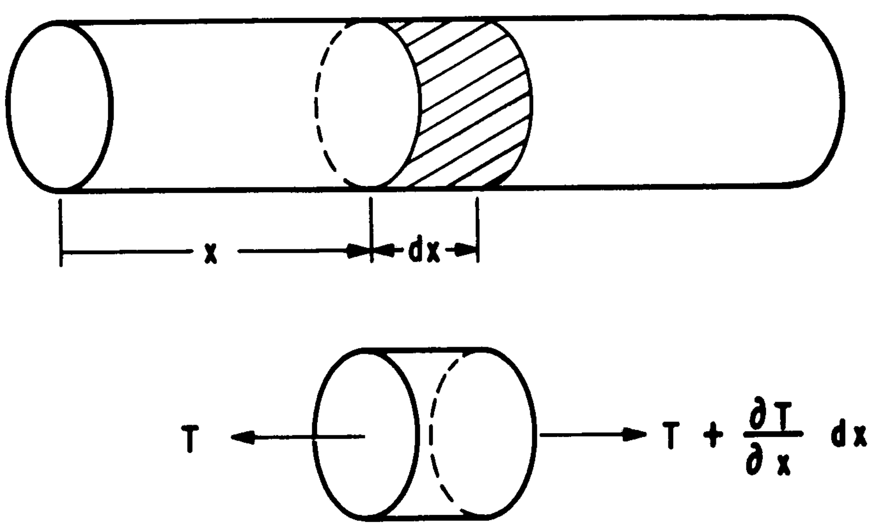


FIG. 1. WAVE MOTION FREE BODY DIAGRAM

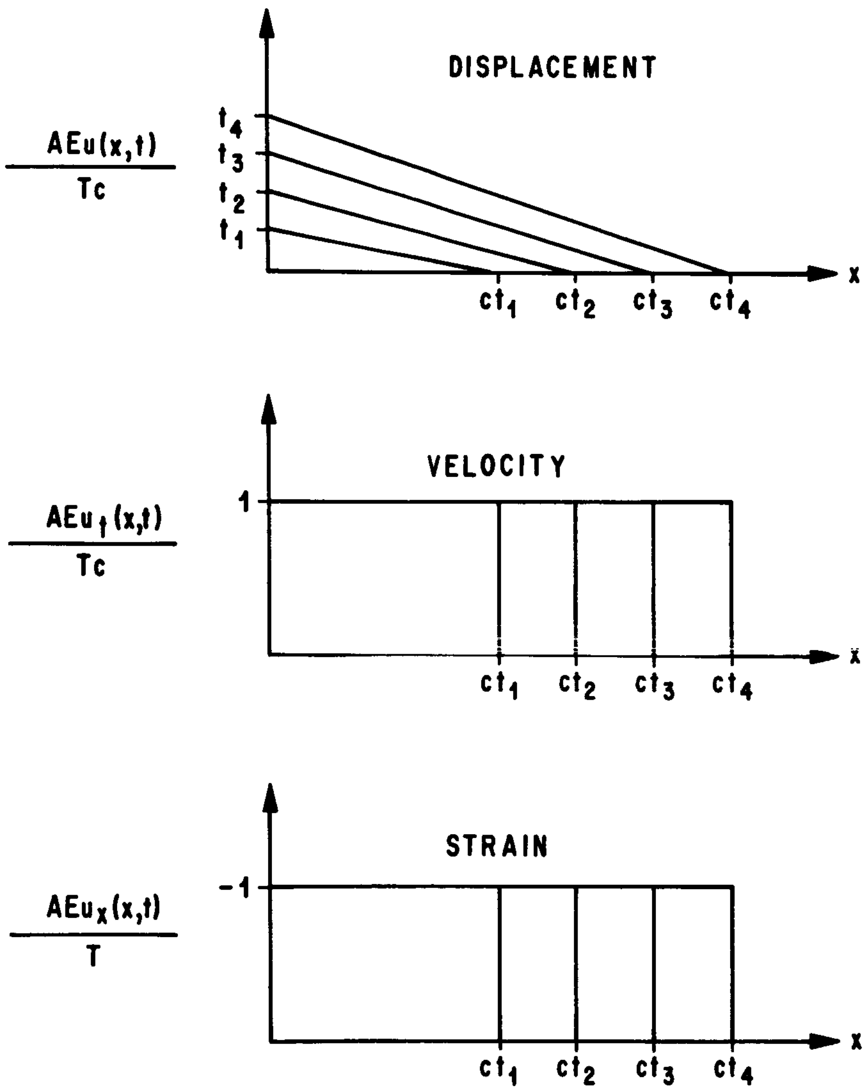


FIG. 2. DISPLACEMENT, VELOCITY AND STRAIN DISTRIBUTIONS OF A SEMI-INFINITE ROD WITH A STEP STRESS IMPULSE LOADING

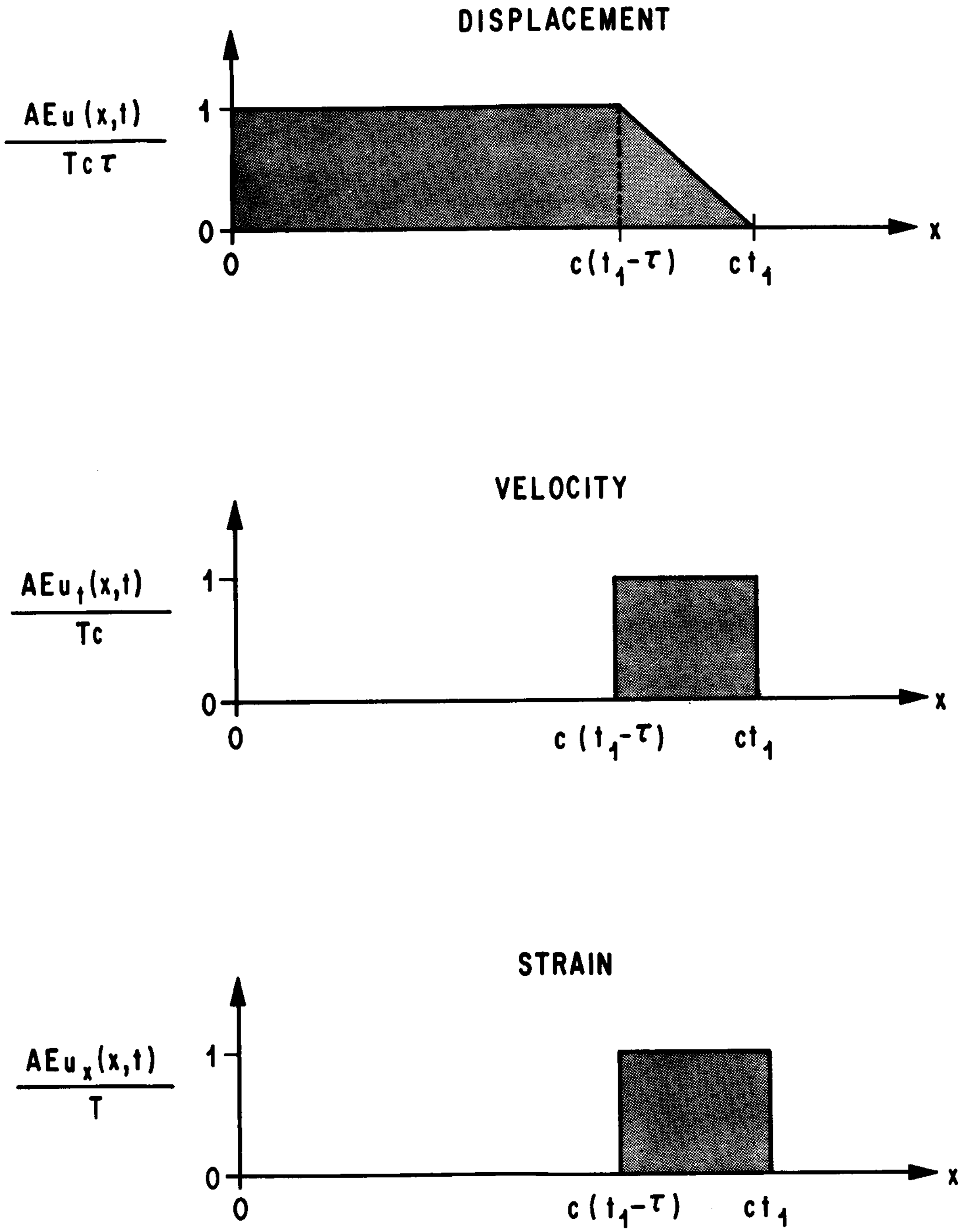


FIG. 3. DISPLACEMENT, VELOCITY AND STRAIN DISTRIBUTIONS OF A SEMI-INFINITE ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING FOR $t = t_1$.

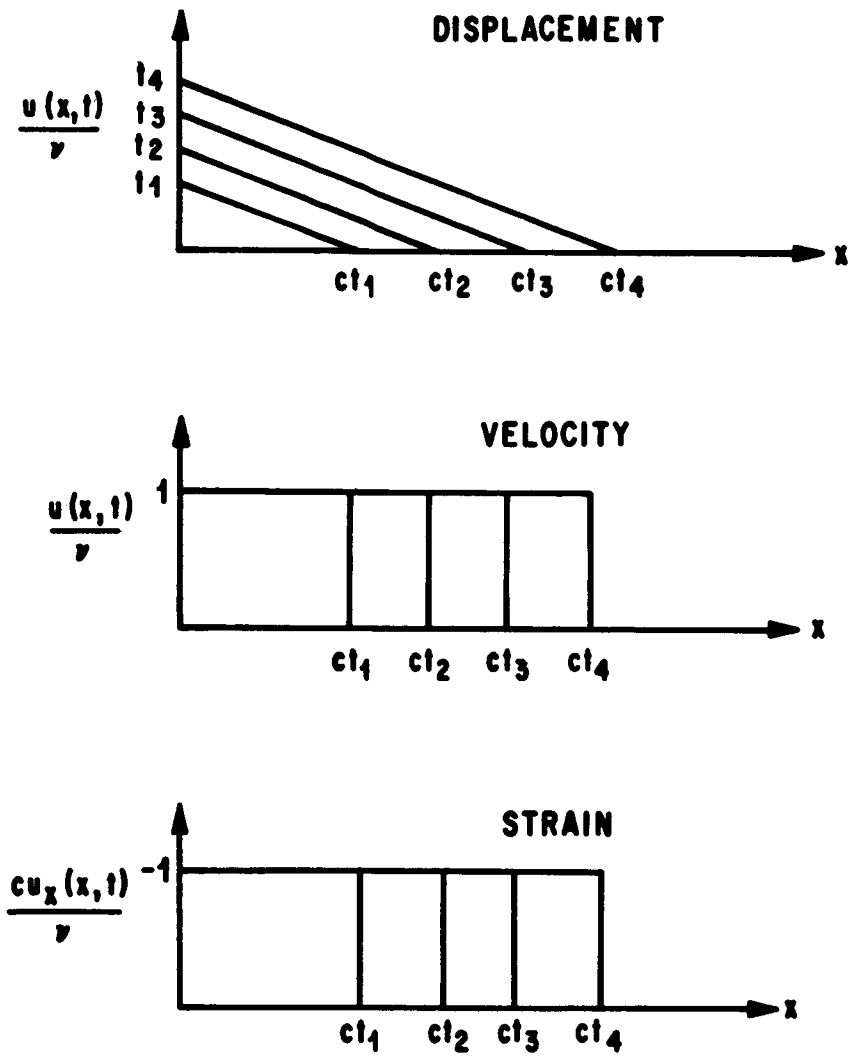


FIG. 4. DISPLACEMENT, VELOCITY AND STRESS DISTRIBUTIONS OF A SEMI-INFINITE ROD WITH A STEP VELOCITY IMPULSE LOADING

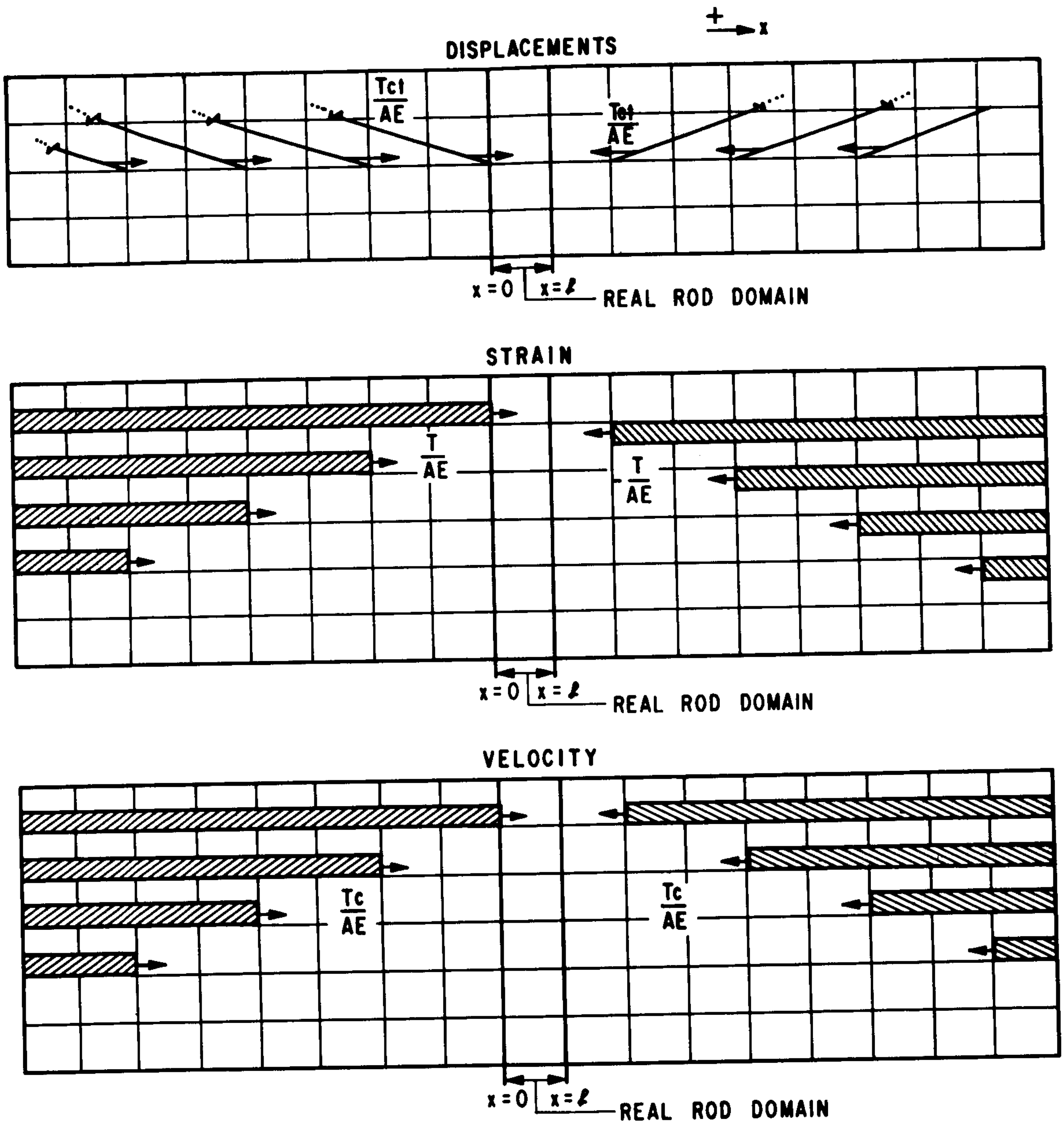


FIG. 5. MATHEMATICAL MODELS REPRESENTATION OF THE DISPLACEMENT, VELOCITY AND STRAIN OF A ROD WITH A STEP STRESS IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END AT t = 0

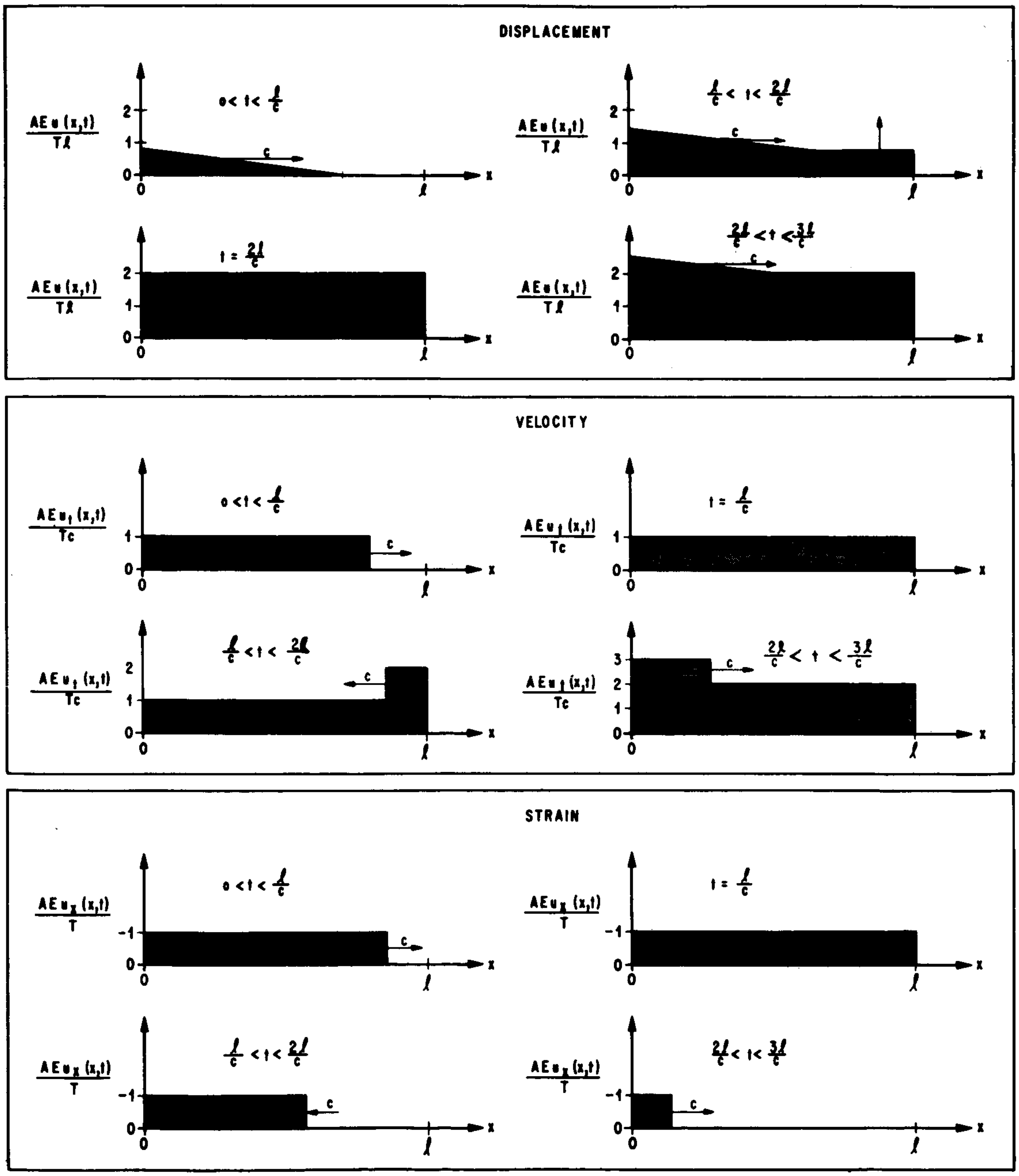


FIG. 6. DISPLACEMENT, VELOCITY AND STRAIN DISTRIBUTIONS OF A ROD WITH A STEP STRESS IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END

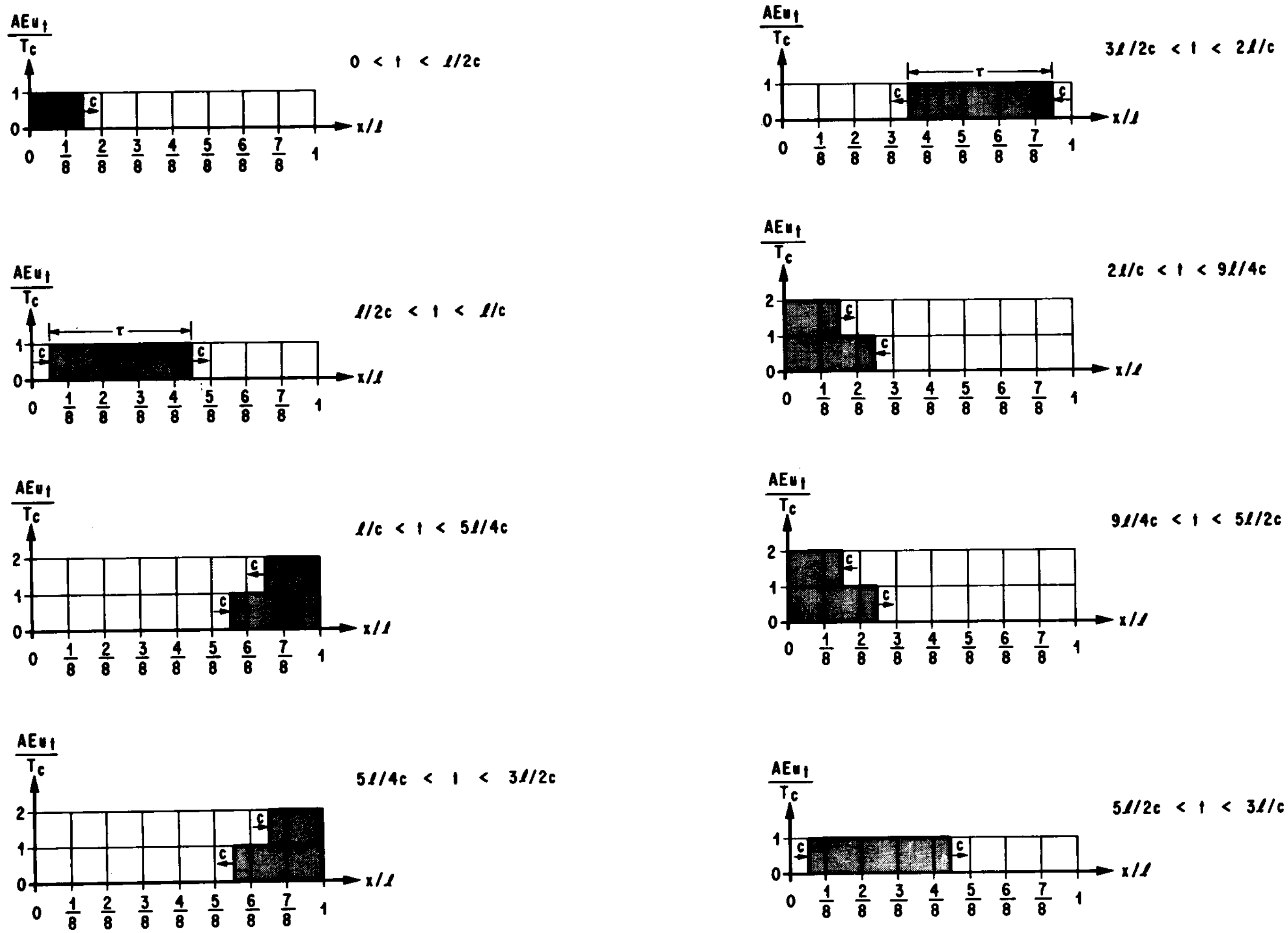


FIGURE 7 (a) VELOCITY DISTRIBUTION OF A ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END, $\tau = \ell/2c$

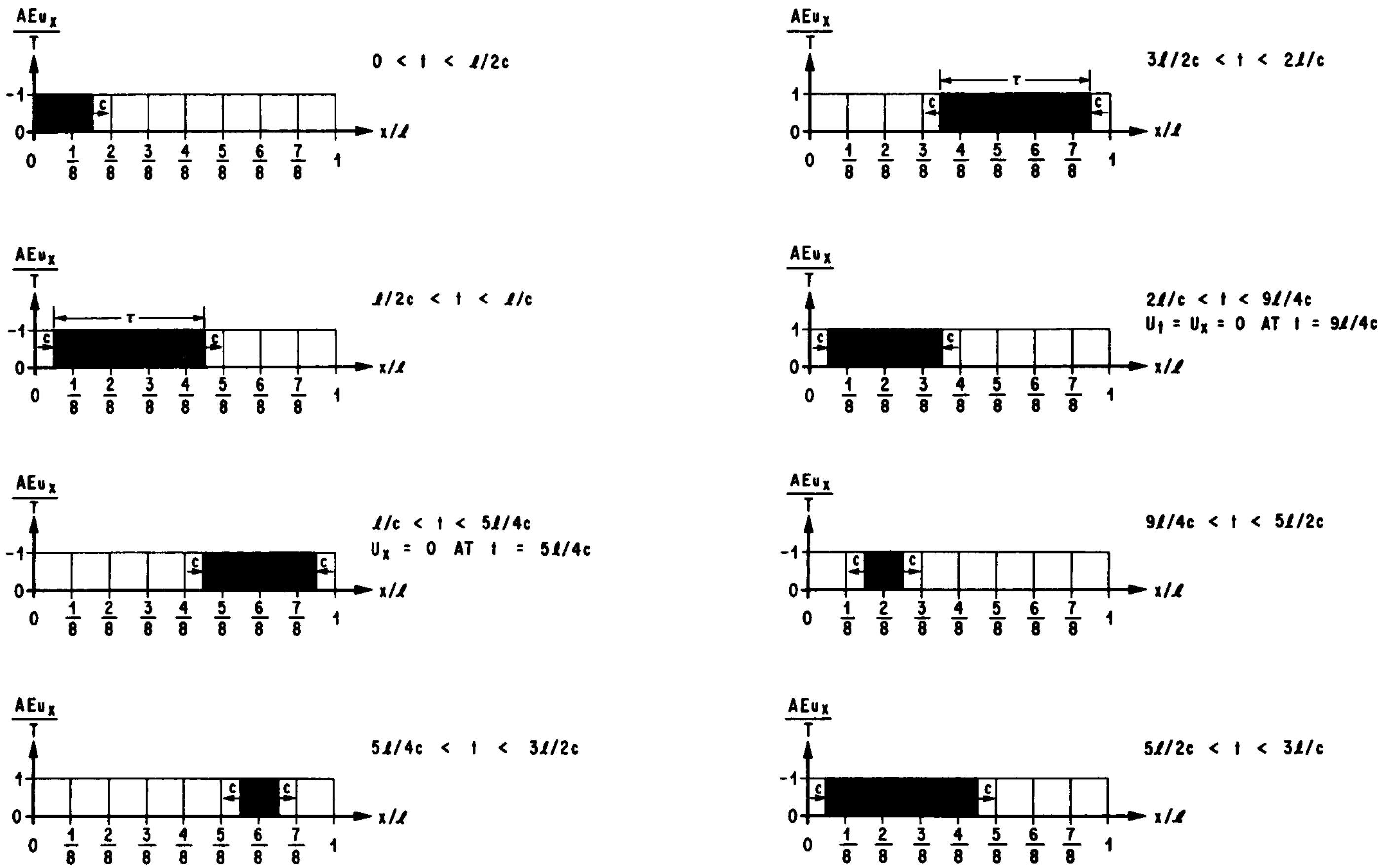


FIGURE 7 (b) STRAIN DISTRIBUTION OF A ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END, $\tau = \ell/2c$

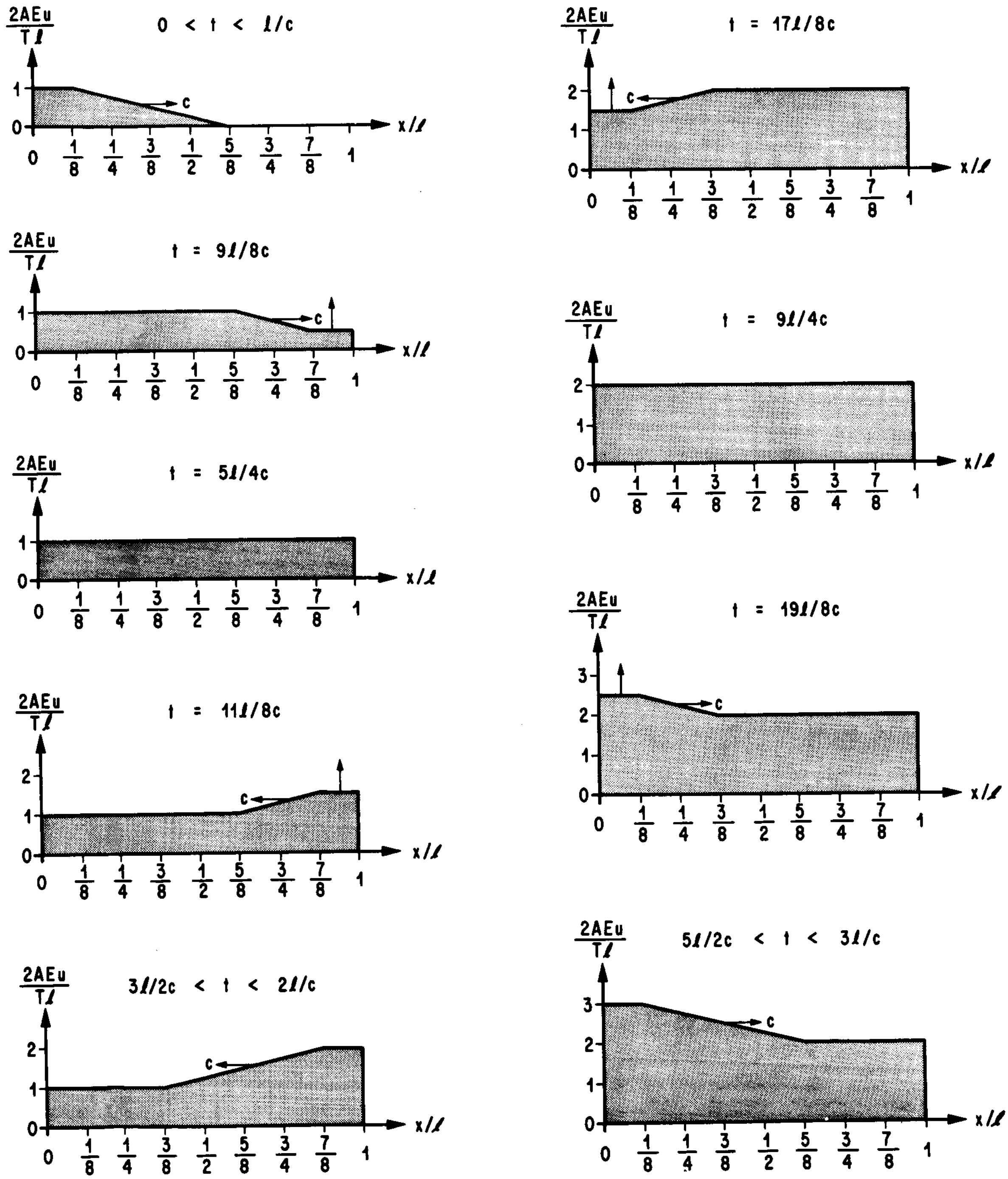


FIG. 8. DISPLACEMENT DISTRIBUTIONS OF A ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END, $\tau = l/2c$

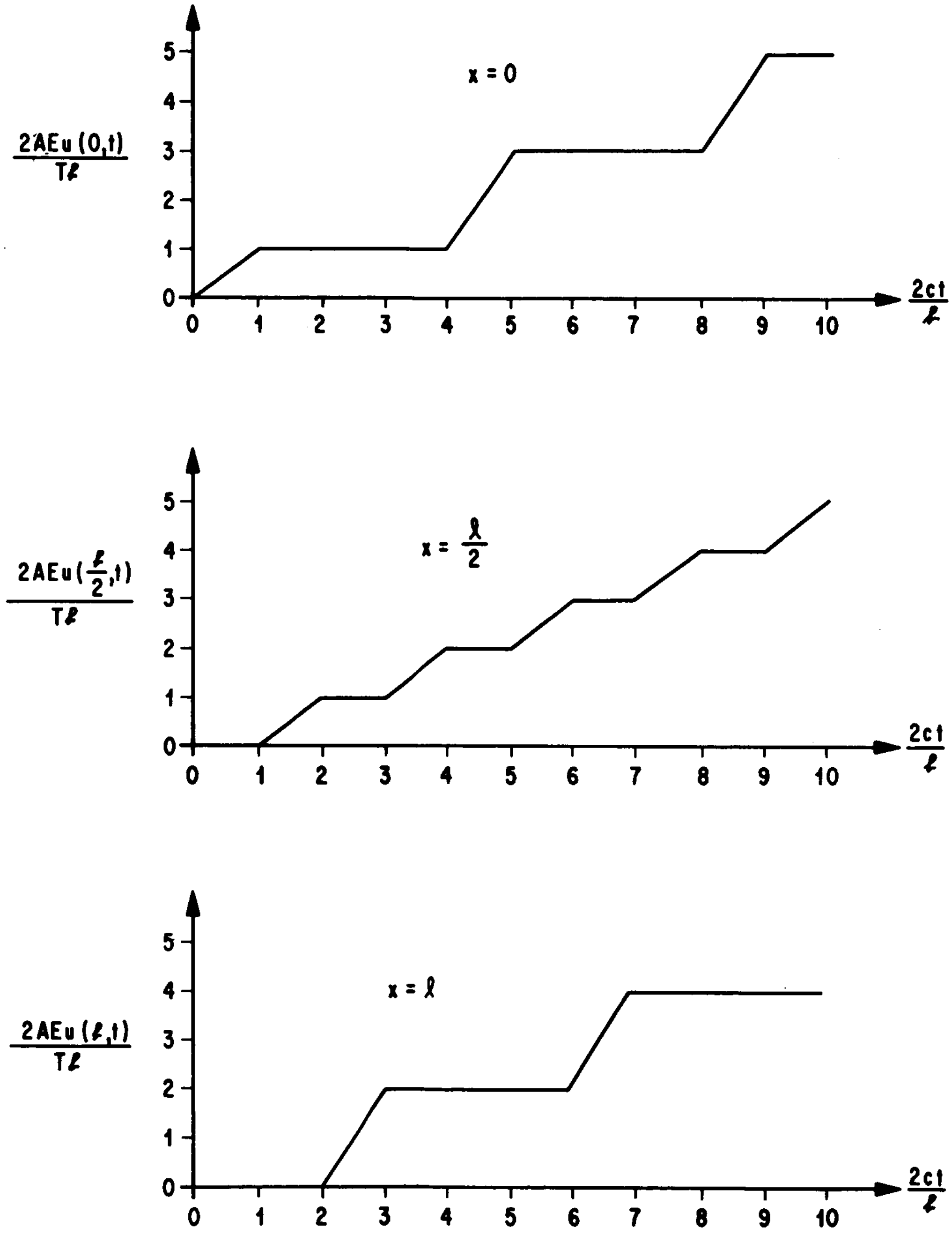


FIG. 9. DISPLACEMENT OF A ROD WITH A SQUARE WAVE STRESS OF PULSE DURATION $\tau = \frac{\ell}{2c}$ ON THE LEFT END AND FREE ON THE RIGHT END AT POSITIONS $x = 0$, $x = \frac{\ell}{2}$ AND $x = \ell$

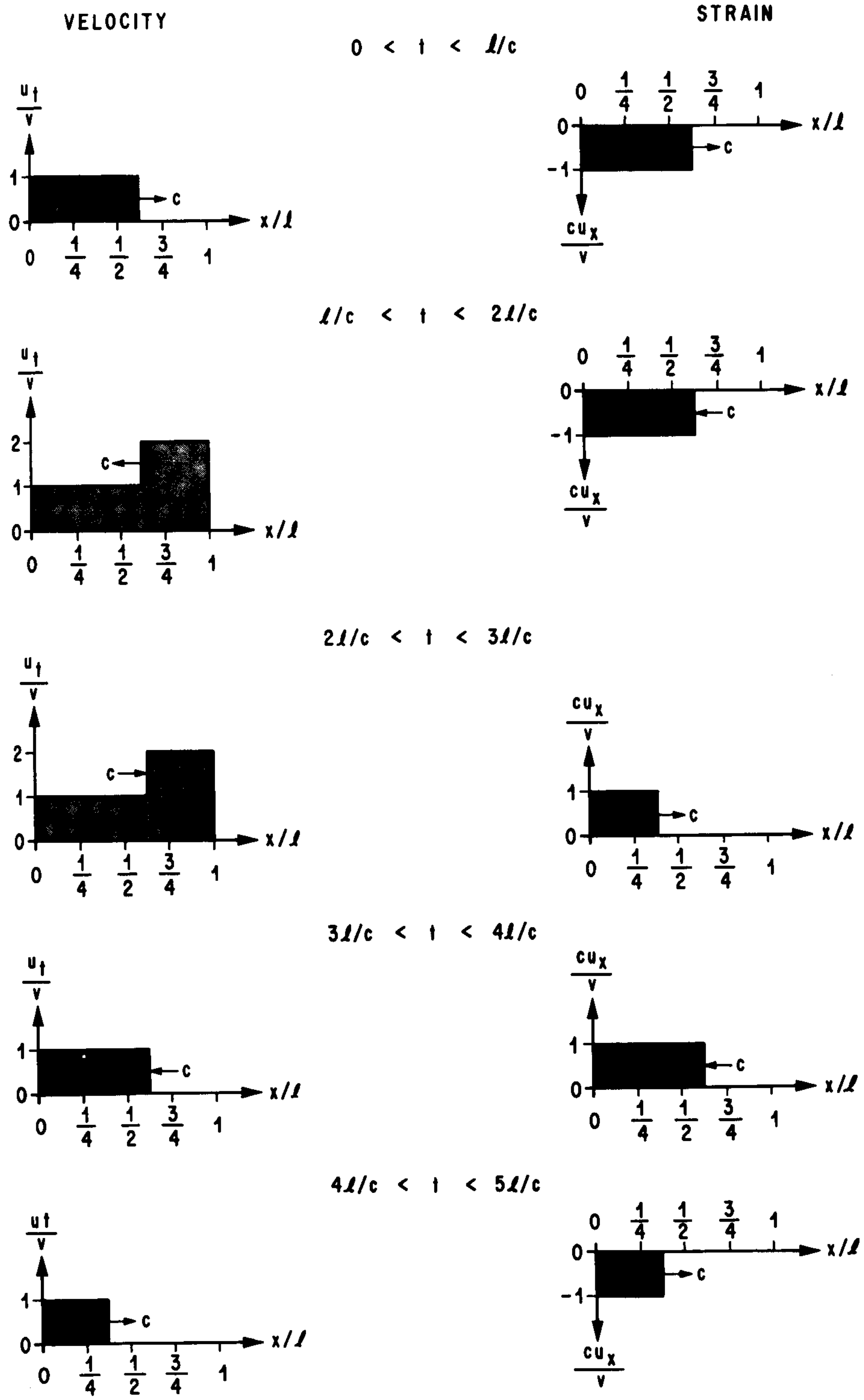


FIG. 10 VELOCITY & STRAIN DISTRIBUTIONS OF A ROD WITH AN ATTACHED STEP VELOCITY IMPULSE LOADING ON THE LEFT END & FREE ON THE RIGHT END

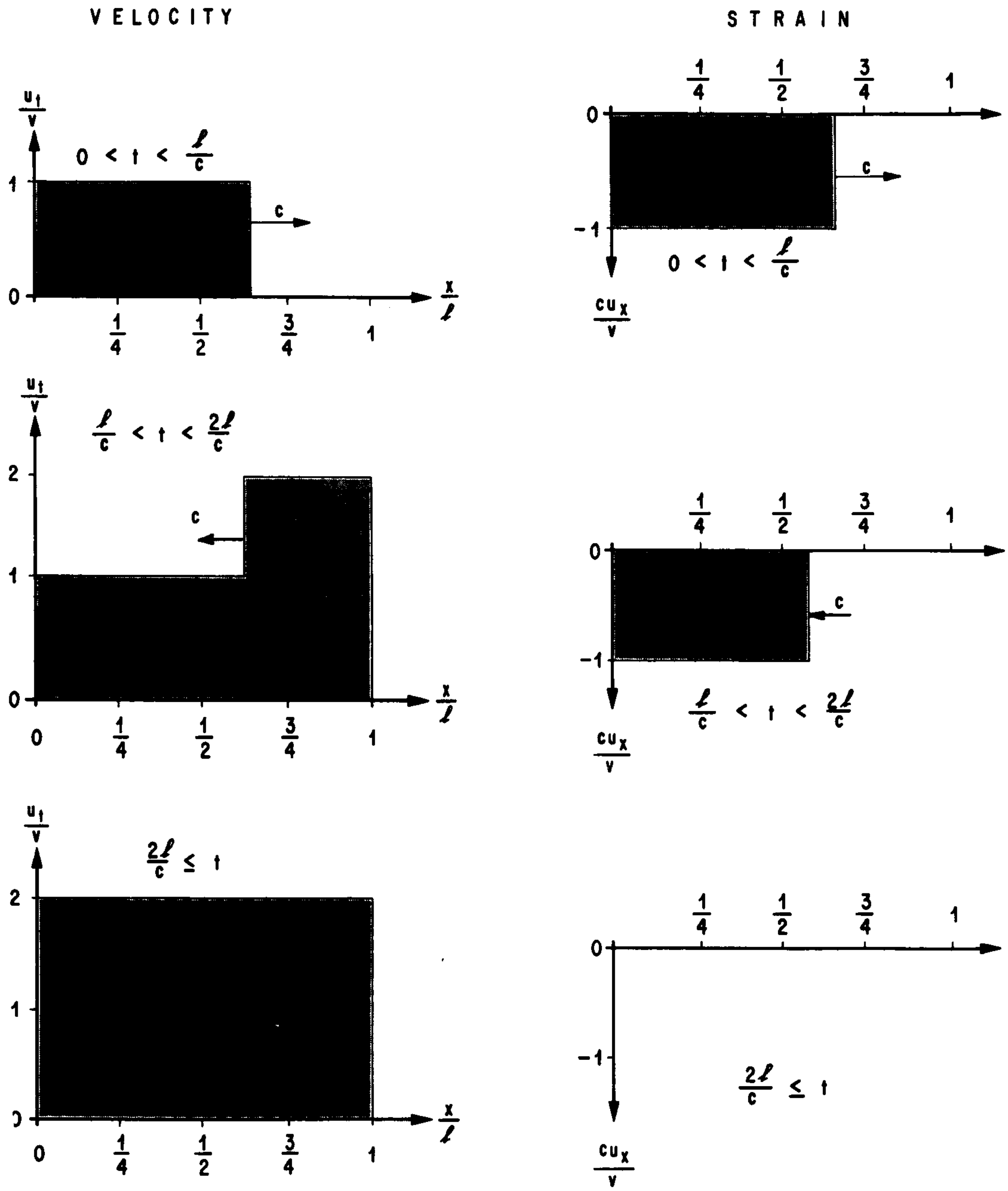


FIG. 11. VELOCITY AND STRAIN DISTRIBUTIONS FOR A ROD WITH A SQUARE WAVE VELOCITY IMPULSE LOADING $(\tau = \frac{2\ell}{c})$ ON THE LEFT END AND FREE ON THE RIGHT END

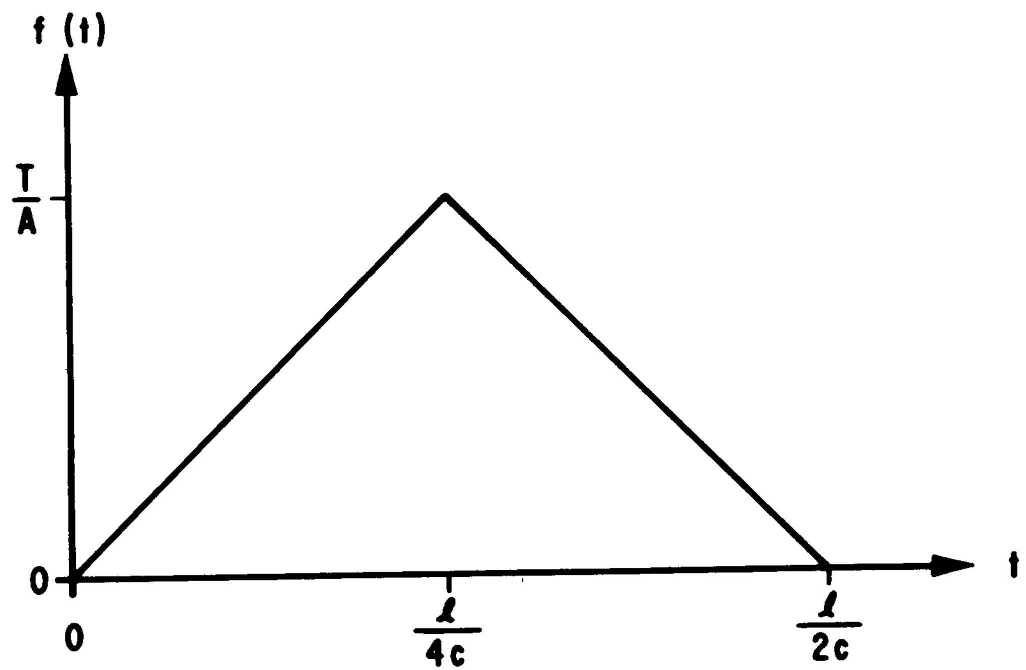


FIG. 12. TRIANGULAR STRESS WAVE IMPULSE LOADING

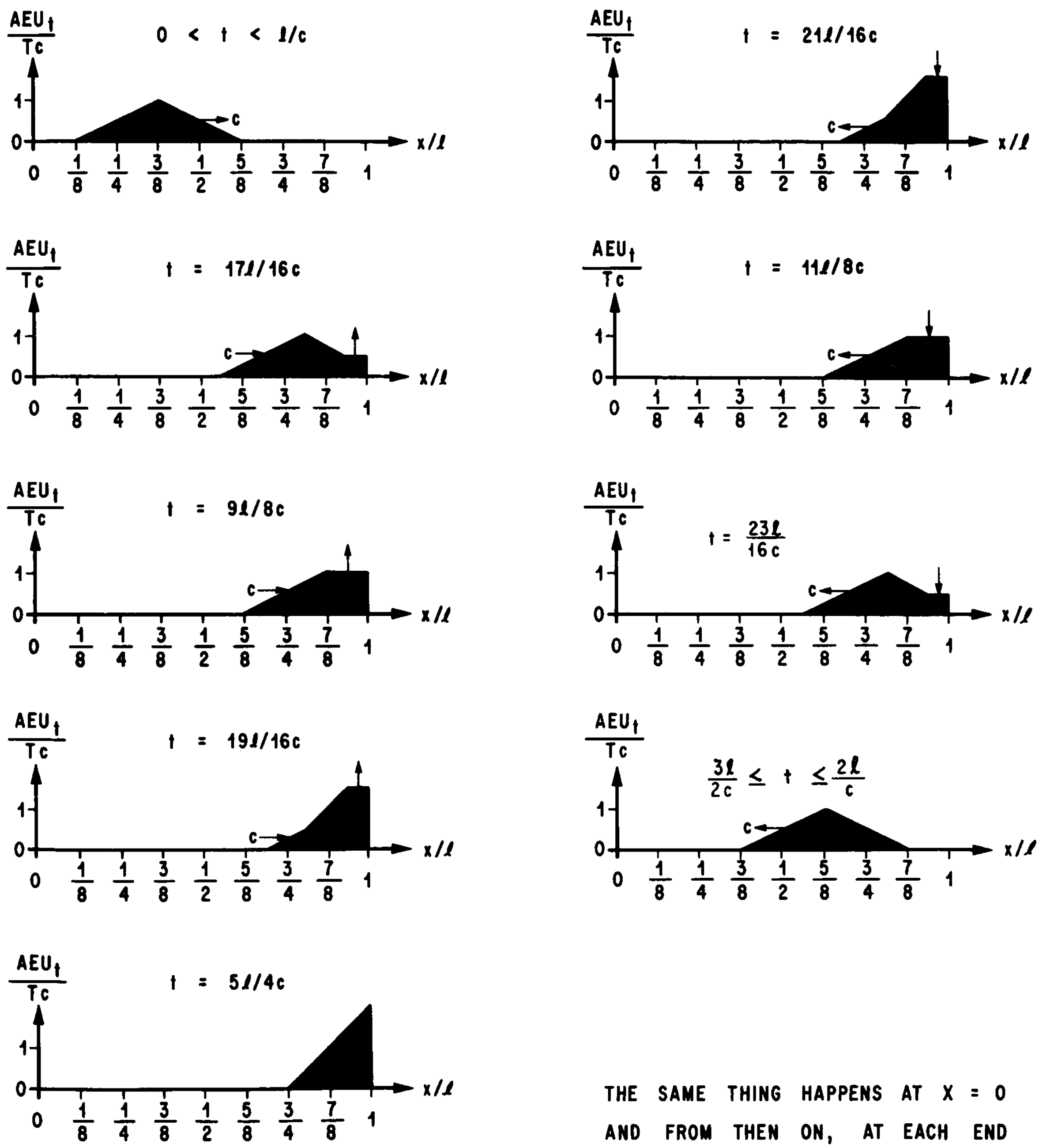


FIG. 13. VELOCITY DISTRIBUTIONS OF A ROD WITH A TRIANGULAR STRESS WAVE IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END

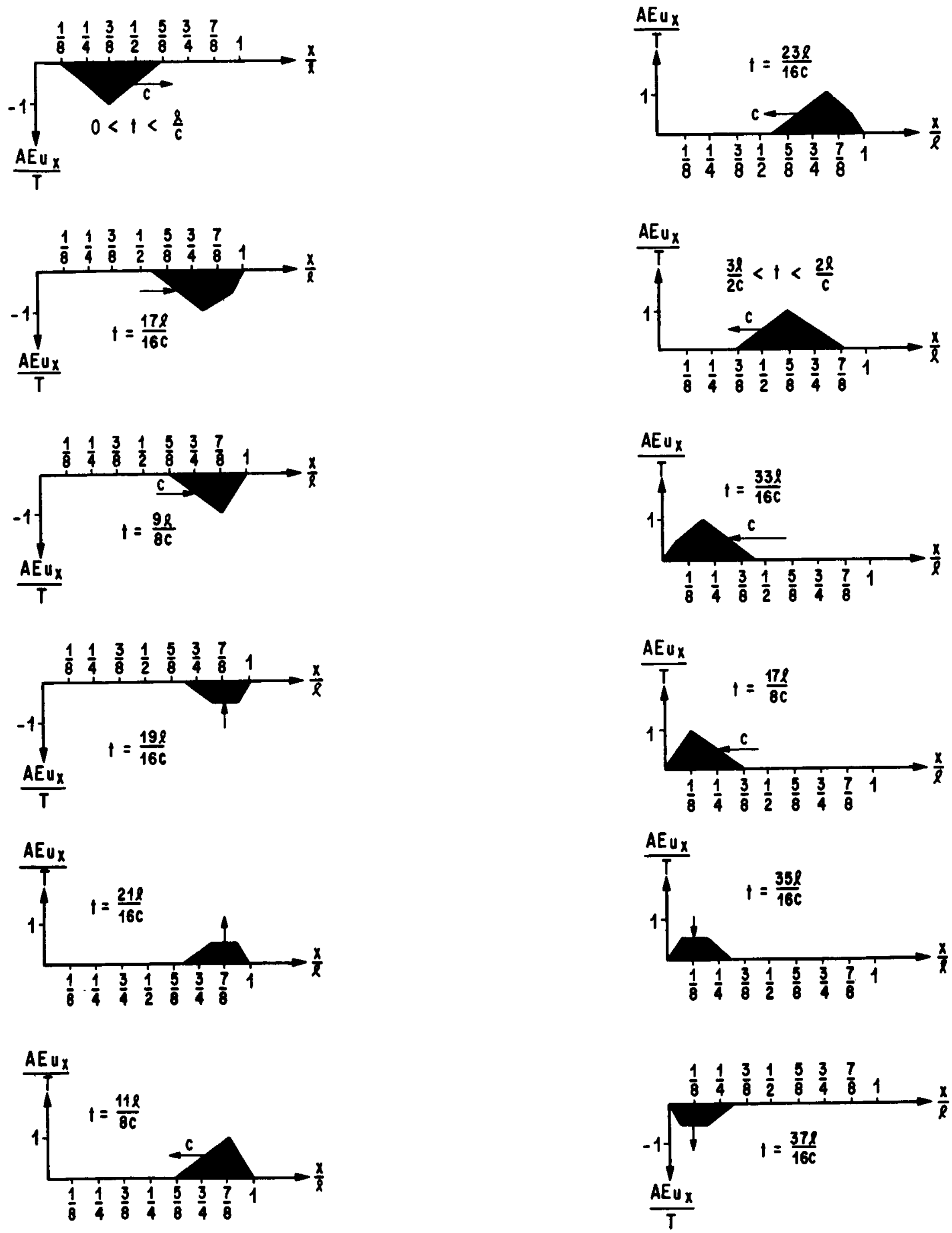


FIG. 14. STRAIN DISTRIBUTIONS OF A ROD WITH A TRIANGULAR STRESS WAVE IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END

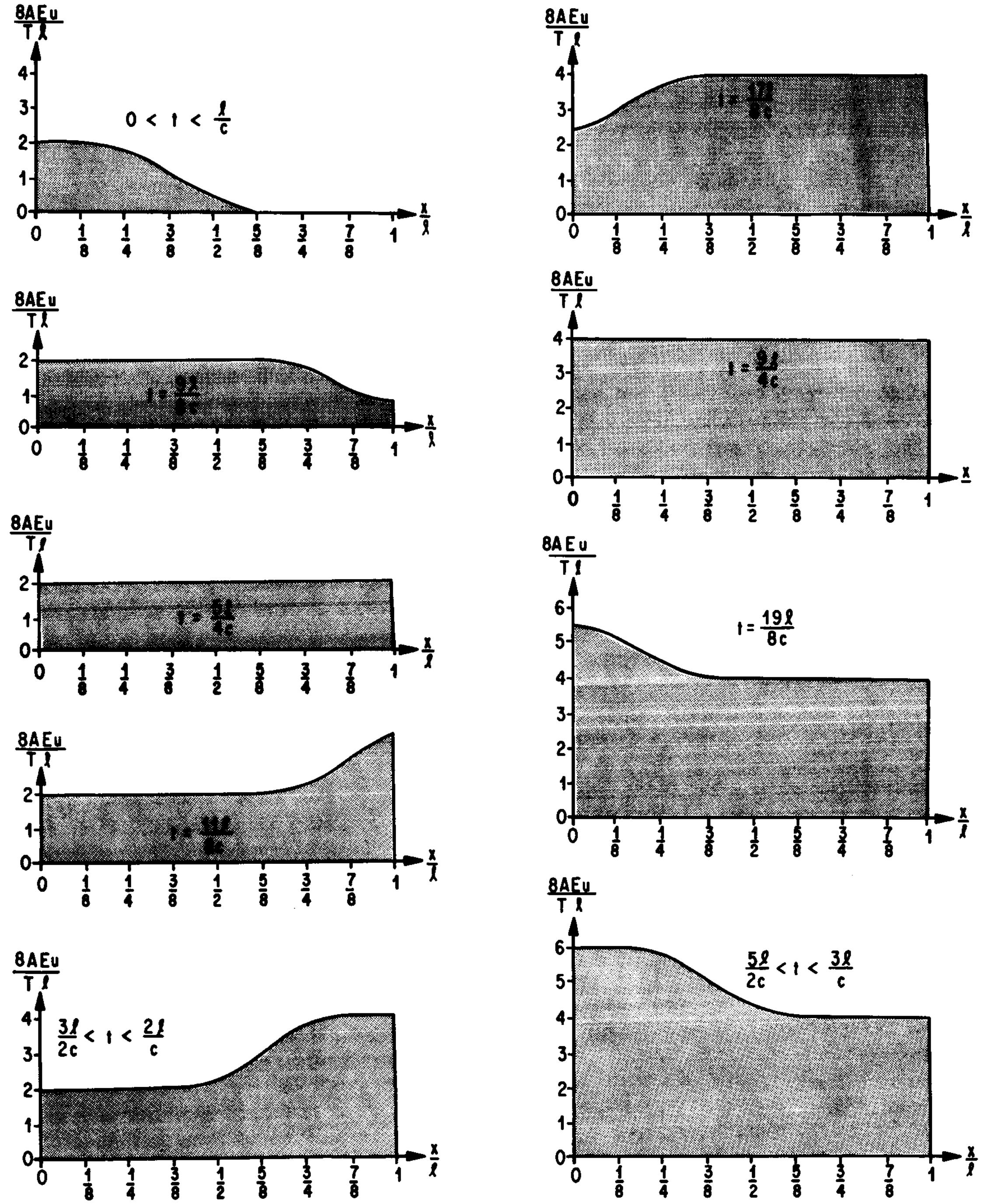


FIG. 15. DISPLACEMENT DISTRIBUTIONS OF A ROD WITH A TRIANGULAR WAVE STRESS WAVE IMPULSE LOADING ON THE LEFT END AND FREE ON THE RIGHT END

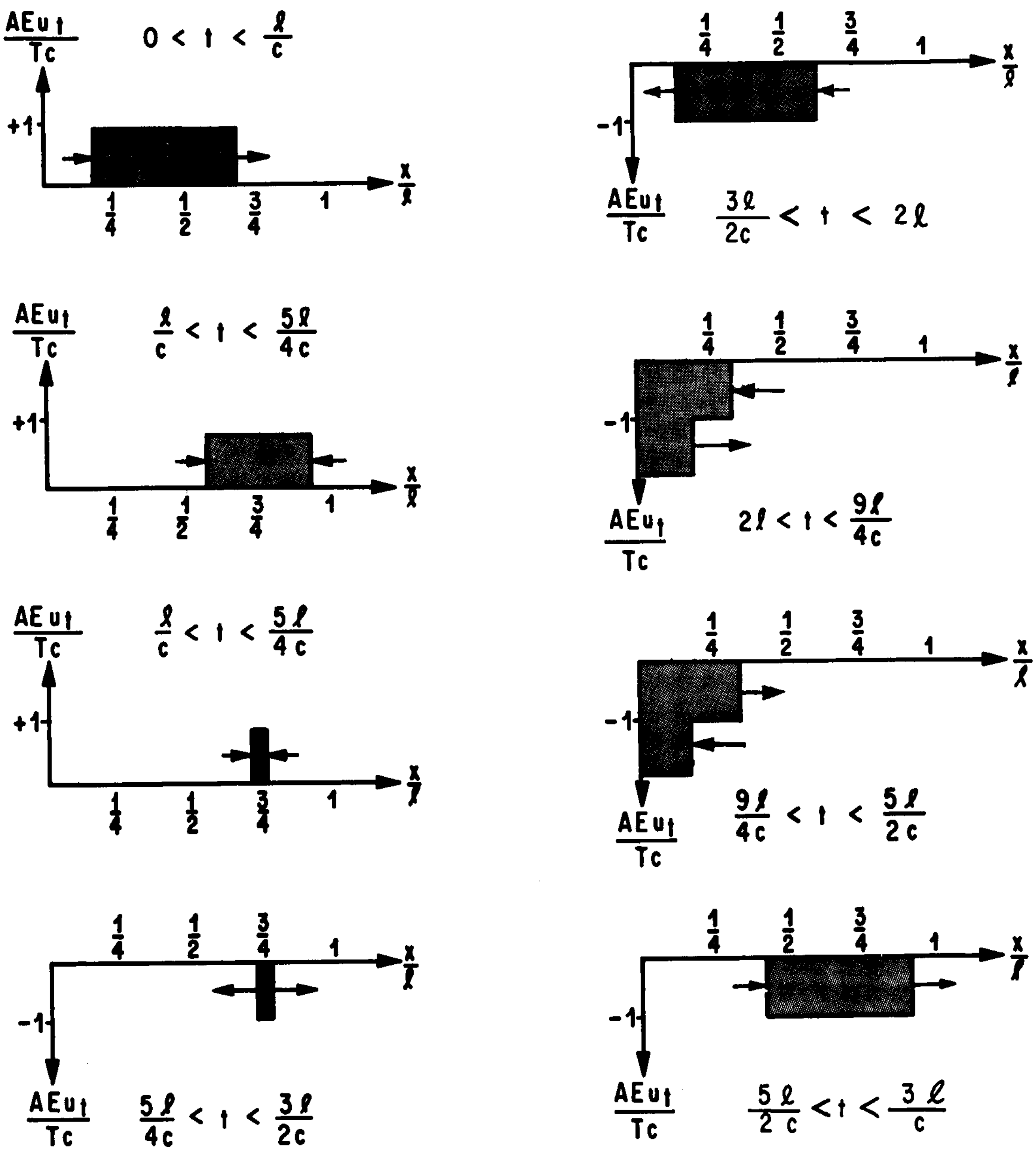


FIG. 16. VELOCITY DISTRIBUTIONS OF A ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING ($\tau = \frac{\ell}{2c}$) ON THE LEFT END AND FIXED ON THE RIGHT END

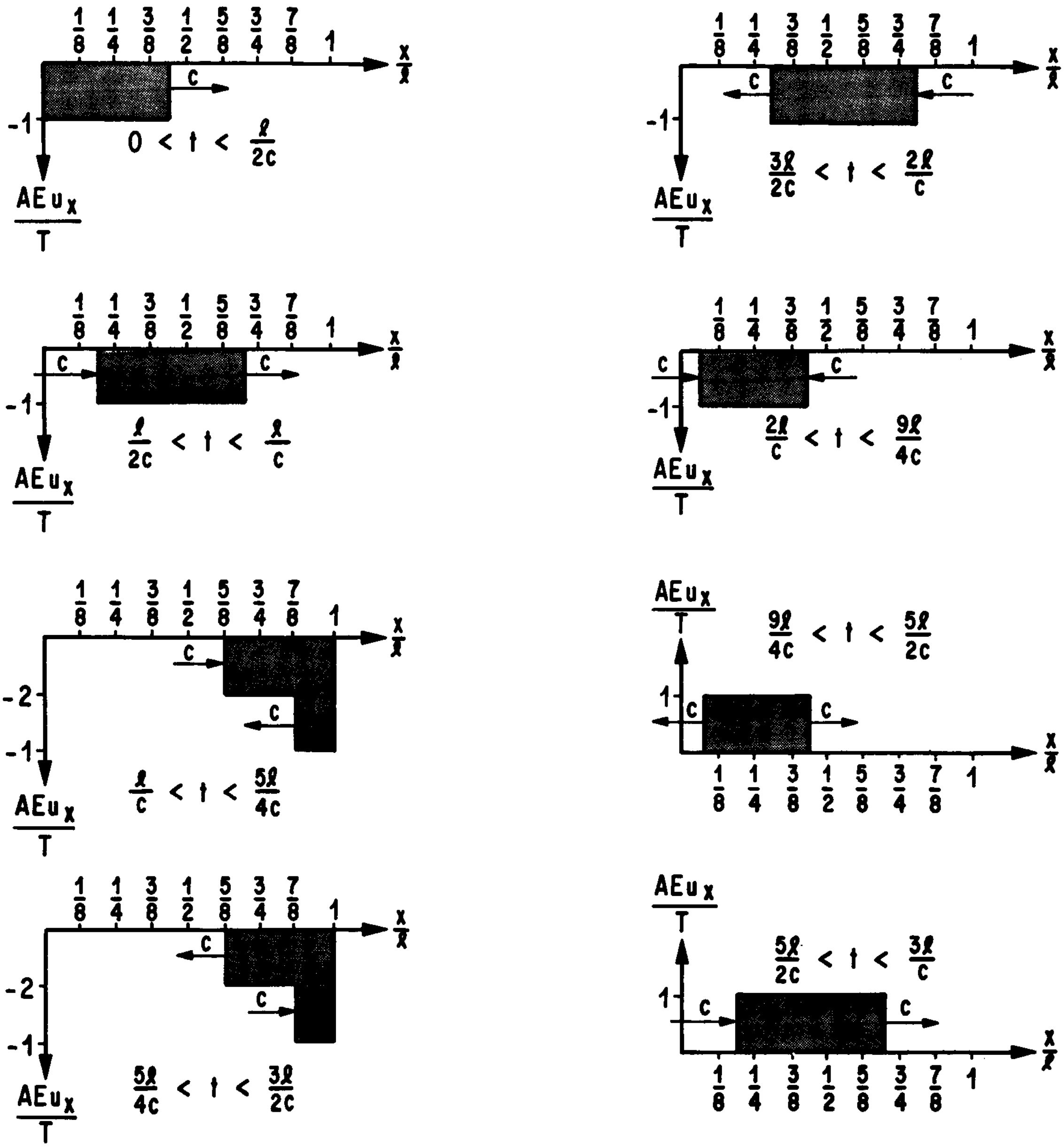


FIG. 17. STRAIN DISTRIBUTIONS OF A ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING ($\tau = \frac{\ell}{2c}$) ON THE LEFT END AND FIXED ON THE RIGHT END

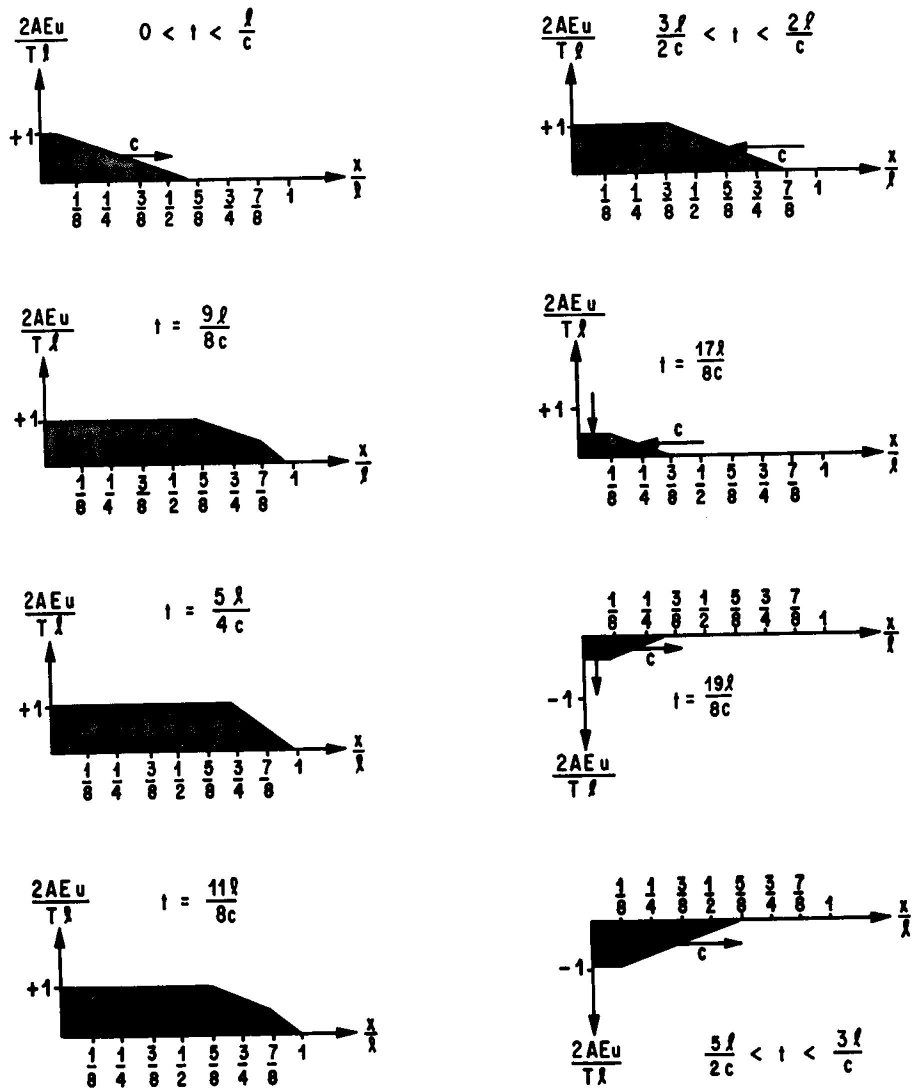


FIG. 18. DISPLACEMENT DISTRIBUTIONS OF A ROD WITH A SQUARE WAVE STRESS IMPULSE LOADING ($\tau = \frac{\ell}{2c}$) ON THE LEFT END AND FIXED ON THE RIGHT END

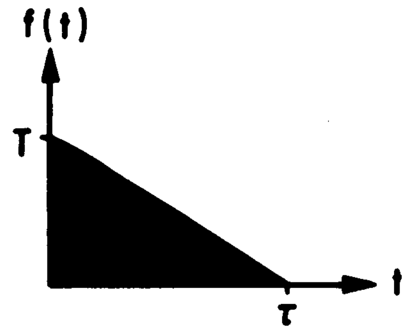


FIG. 19. TRIANGULAR IMPULSE LOADING WAVE

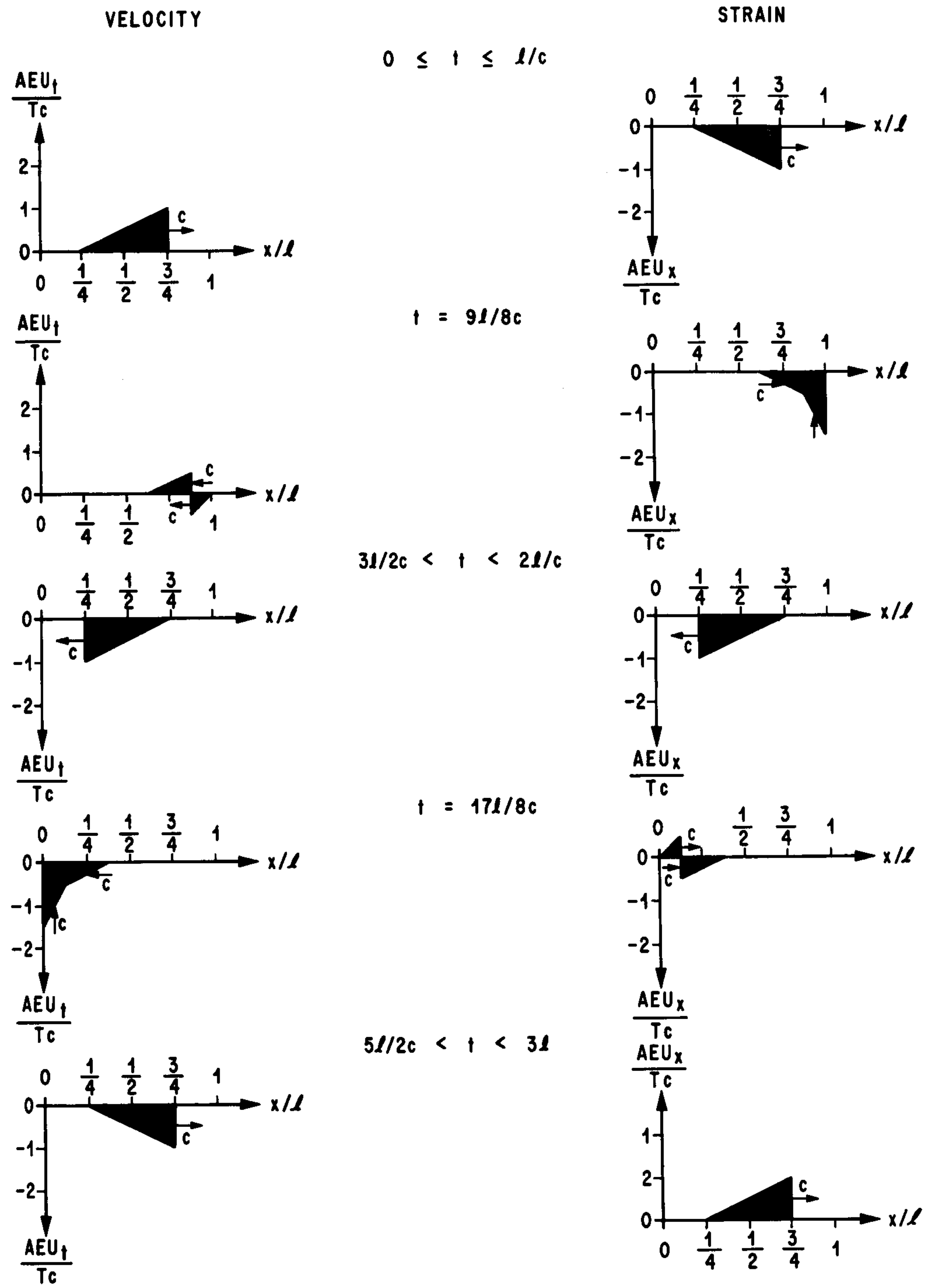


FIG. 20. VELOCITY AND STRAIN DISTRIBUTIONS OF A ROD WITH A TRIANGULAR WAVE STRESS IMPULSE LOADING ON THE LEFT END AND FIXED ON THE RIGHT END

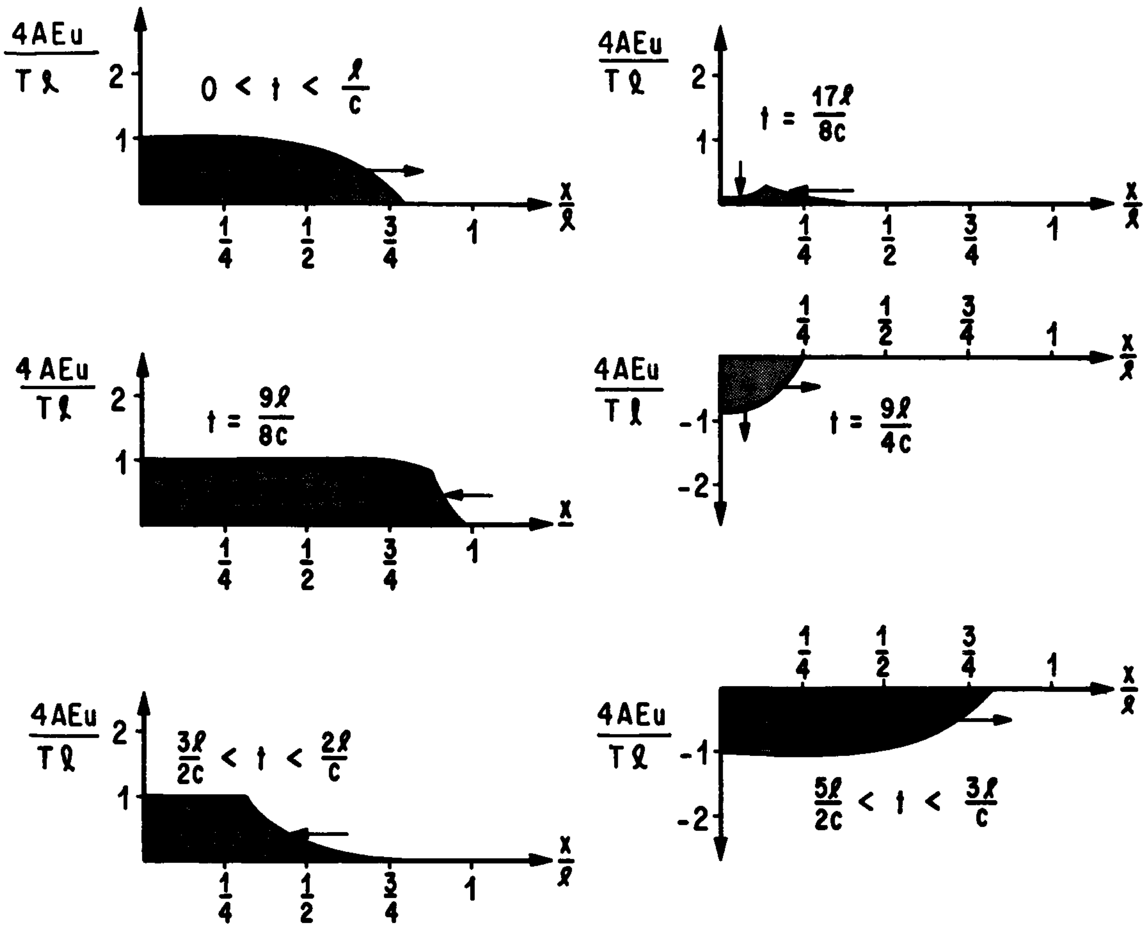


FIG. 21. DISPLACEMENT DISTRIBUTIONS OF A ROD WITH A TRIANGULAR WAVE STRESS IMPULSE LOADING ON THE LEFT END AND FIXED ON THE RIGHT END

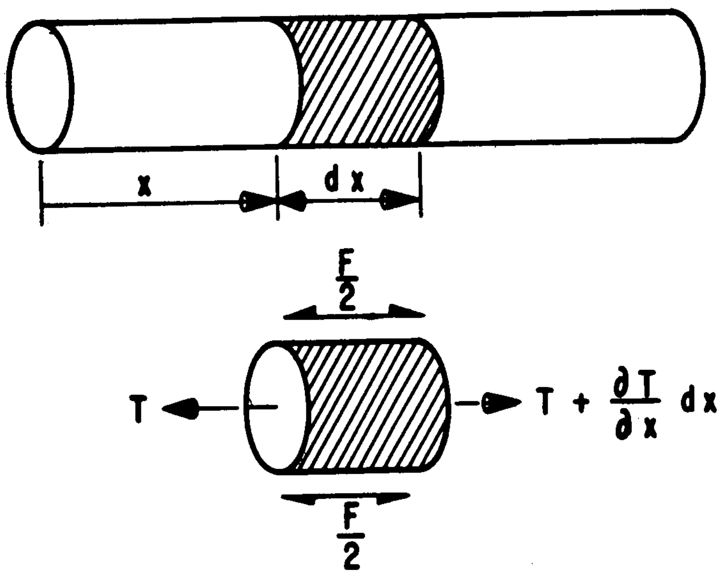


FIG. 22. WAVE MOTION WITH COULOMB DAMPING FREE BODY DIAGRAM

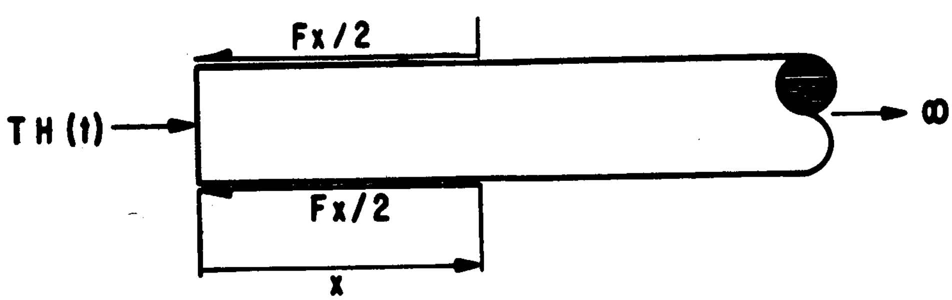


FIG. 23. WAVE MOTION WITH COULOMB DAMPING AND STEP STRESS IMPULSE LOADING

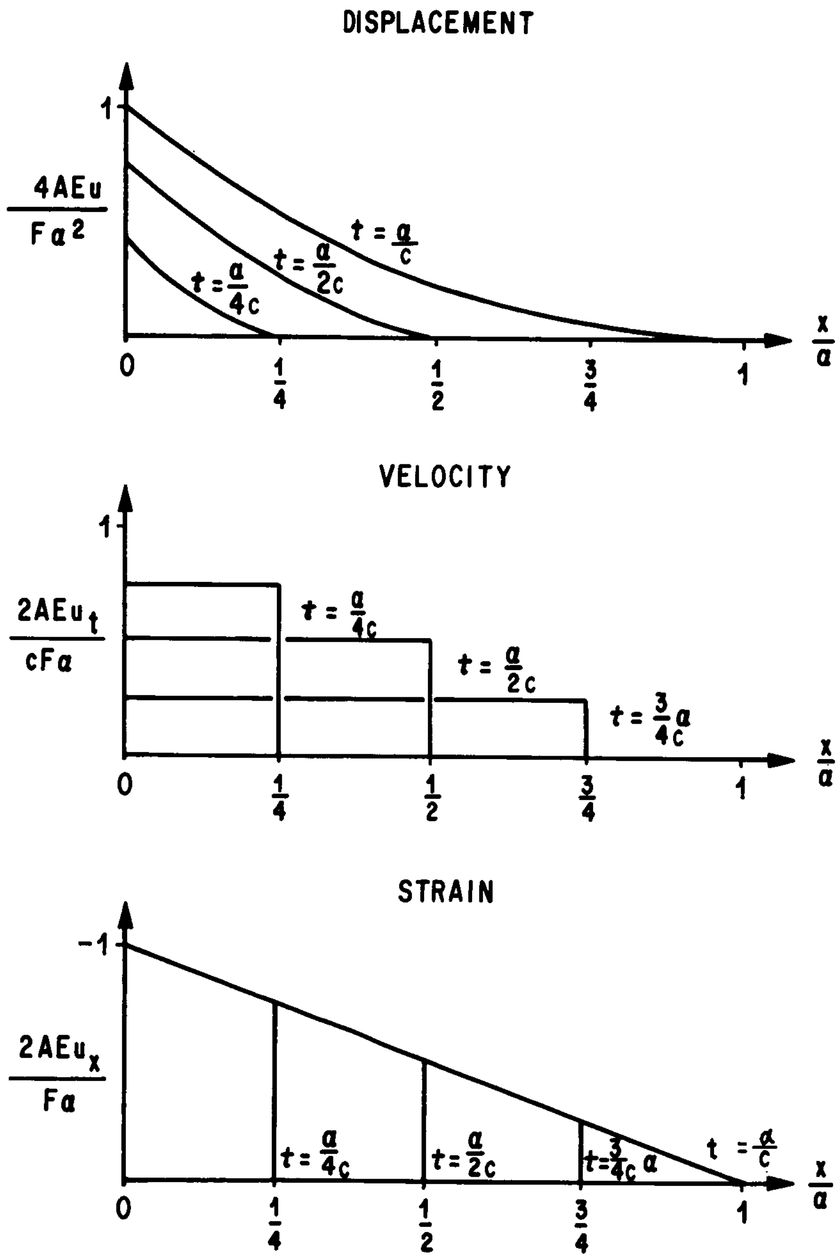


FIG. 24. DISPLACEMENT, VELOCITY AND STRAIN DISTRIBUTIONS FOR A SEMI-INFINITE ROD WITH COULOMB DAMPING AND A SQUARE WAVE STRESS IMPULSE LOADING FOR $t \geq \frac{a}{c}$ AND $t \leq \tau$

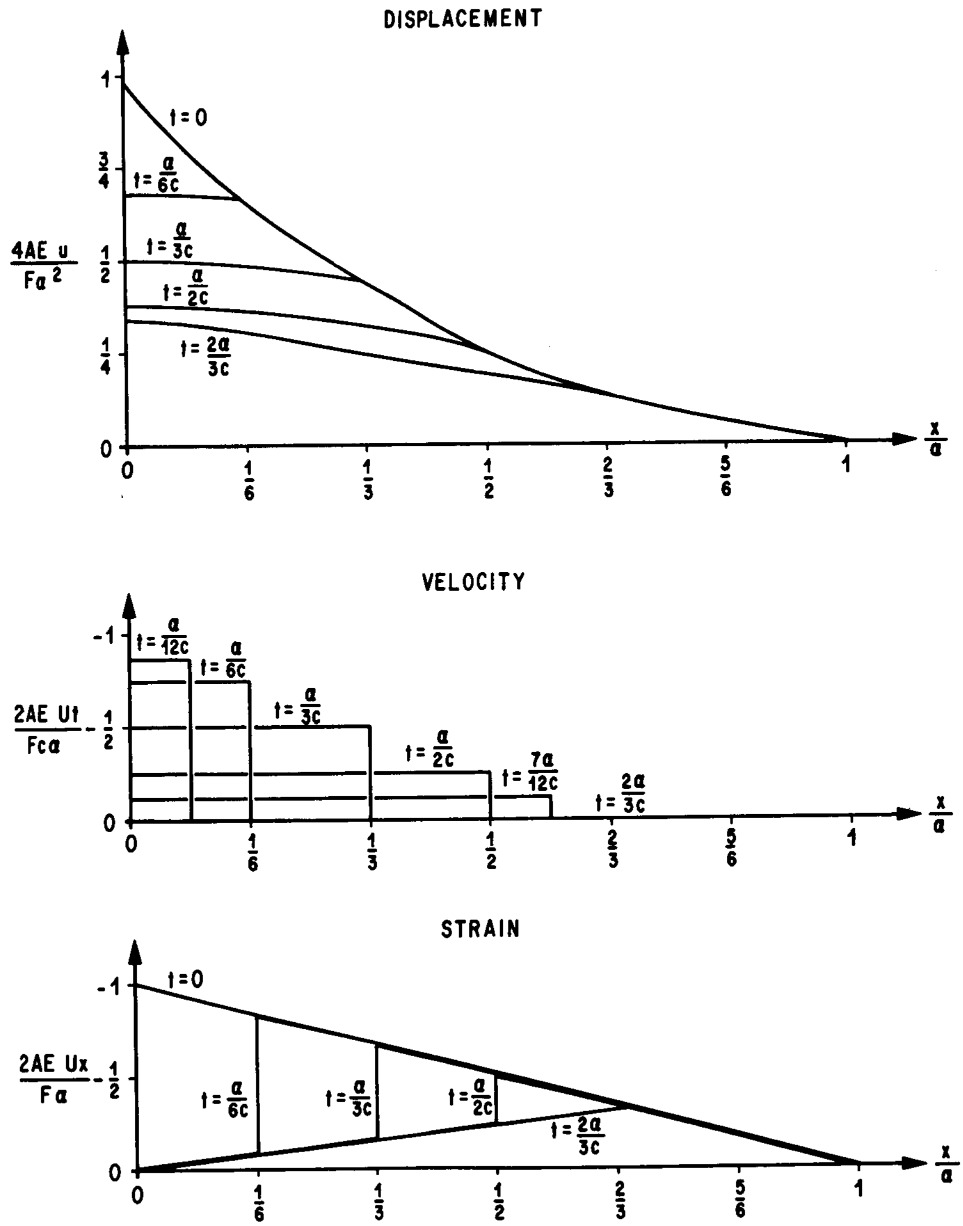


FIG. 25. DISPLACEMENT, VELOCITY AND STRAIN DISTRIBUTIONS FOR A SEMI-INFINITE ROD WITH COULOMB DAMPING AND A SQUARE WAVE STRESS IMPULSE LOADING FOR $\tau \geq \frac{a}{c}$ AND $t \geq \tau$

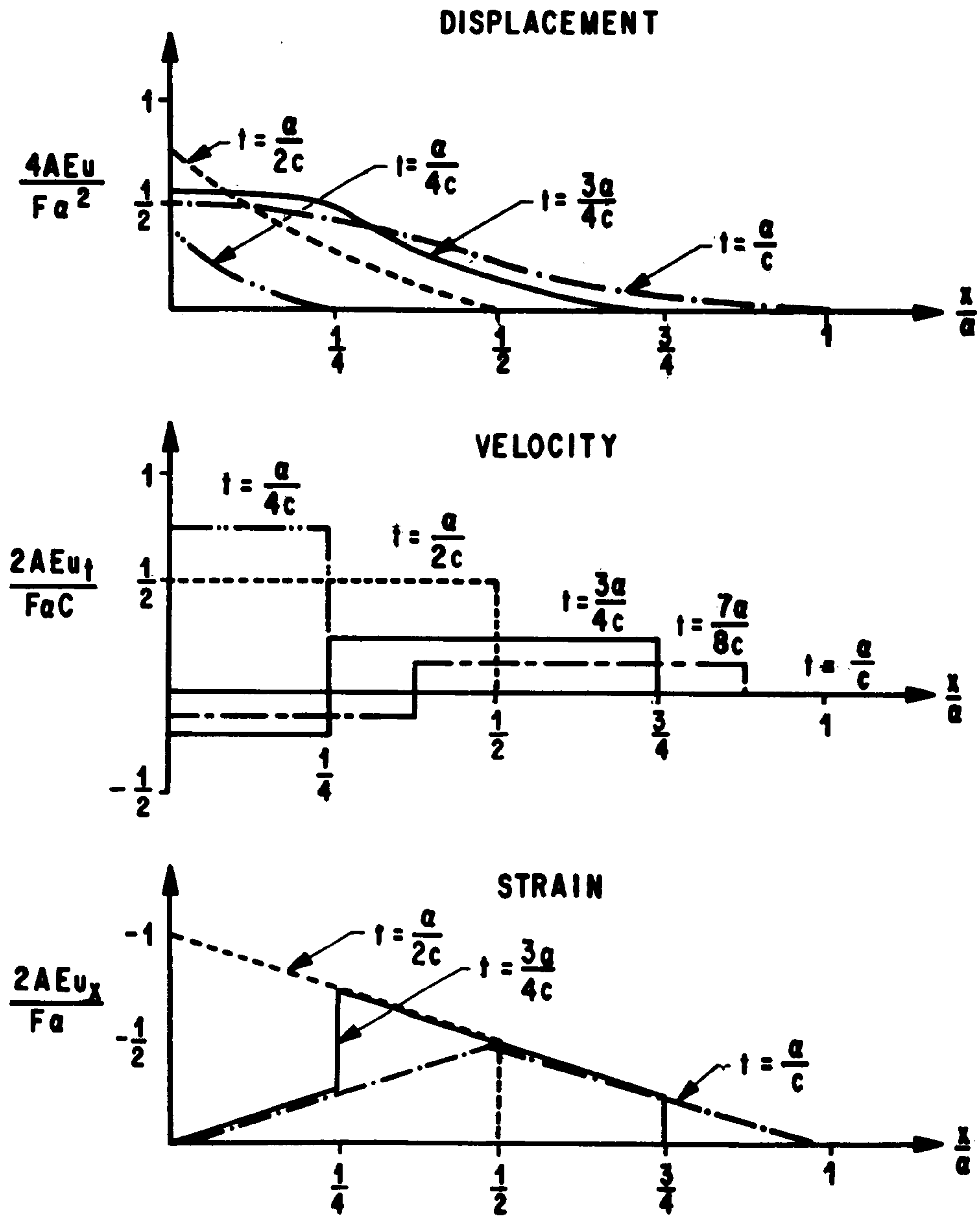


FIG. 26. DISPLACEMENT, VELOCITY AND STRAIN DISTRIBUTIONS FOR A SEMI-INFINITE ROD WITH COULOMB DAMPING AND A SQUARE WAVE STRESS IMPULSE LOADING OF T/A FOR $\tau = \frac{a}{2c}$

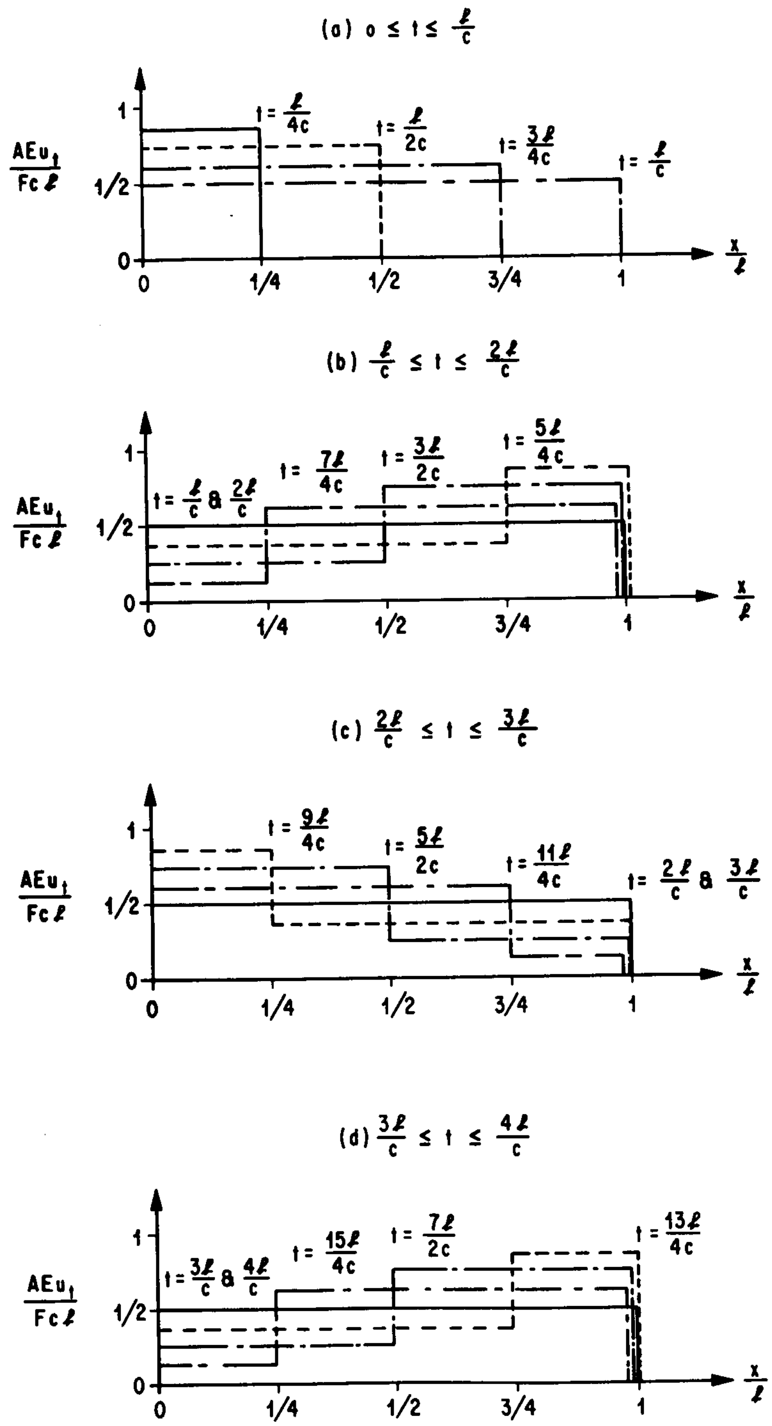


FIG. 27. VELOCITY DISTRIBUTION FOR A FINITE ROD WITH COULOMB DAMPING AND A STEP STRESS IMPULSE LOADING OF $\frac{T}{A}$ FOR T = FL.

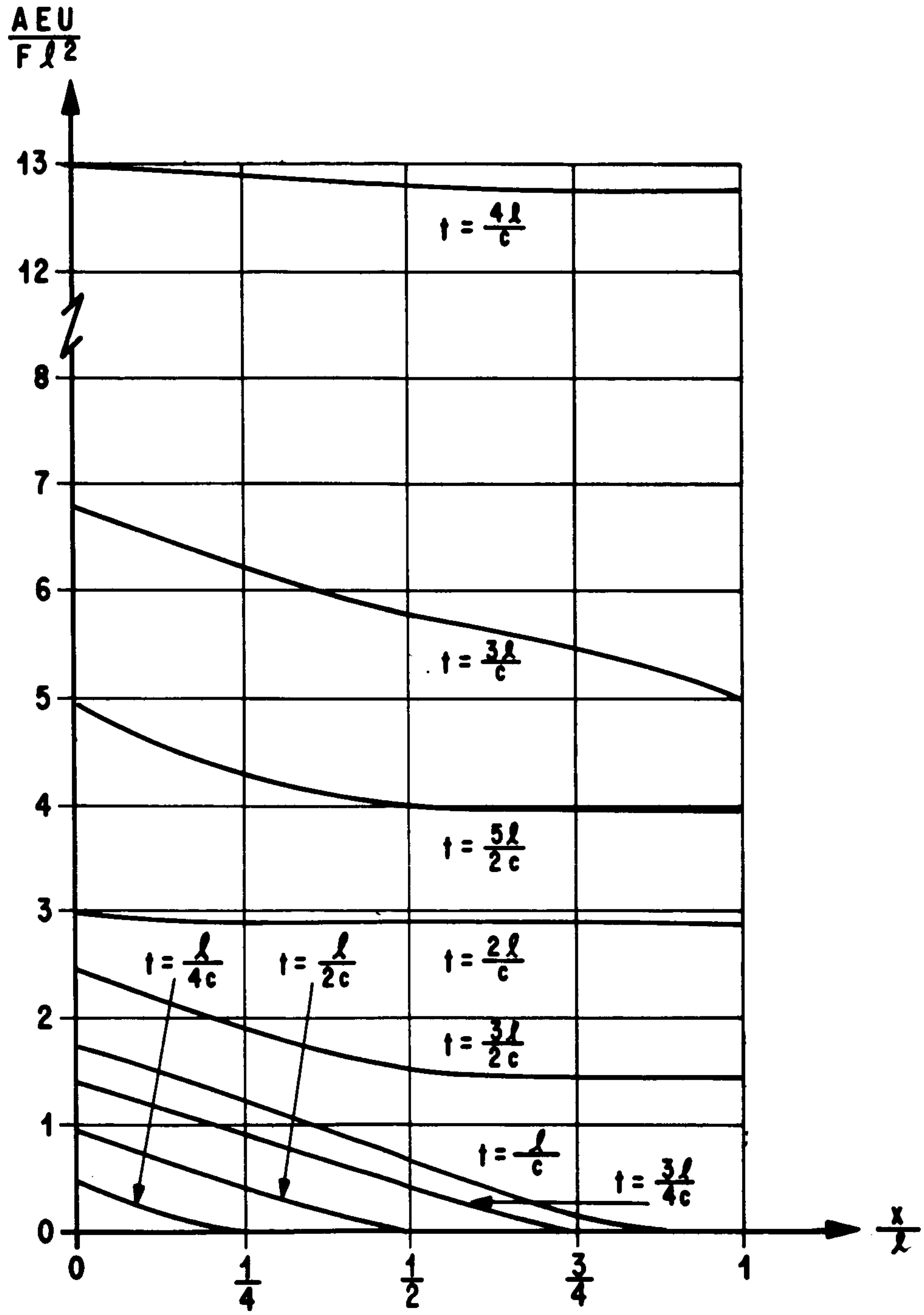


FIG. 28. DISPLACEMENT DISTRIBUTION FOR A FINITE ROD WITH COULOMB DAMPING AND A STEP STRESS IMPULSE LOADING OF $\frac{T}{A}$ FOR $T = 2F\ell$

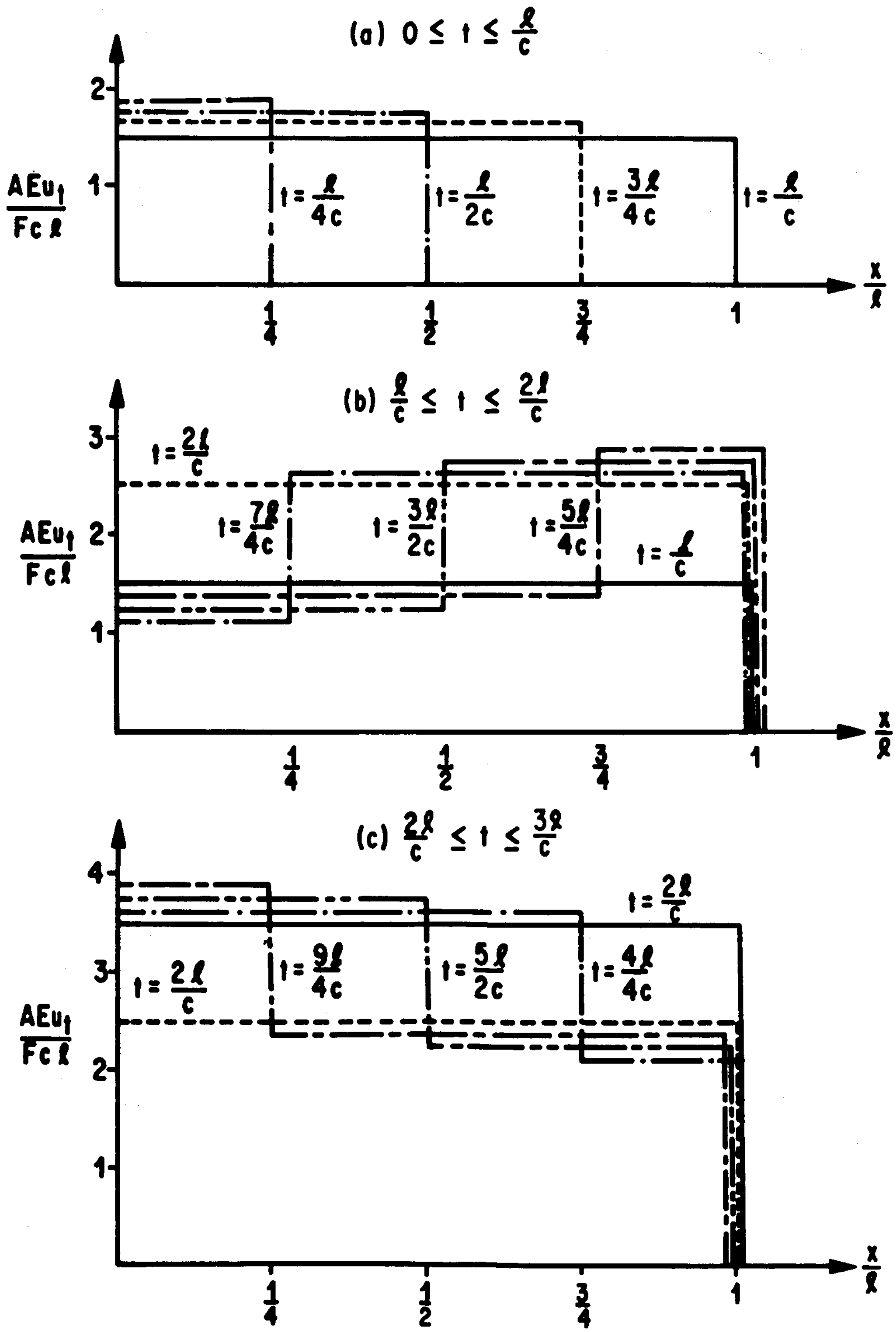
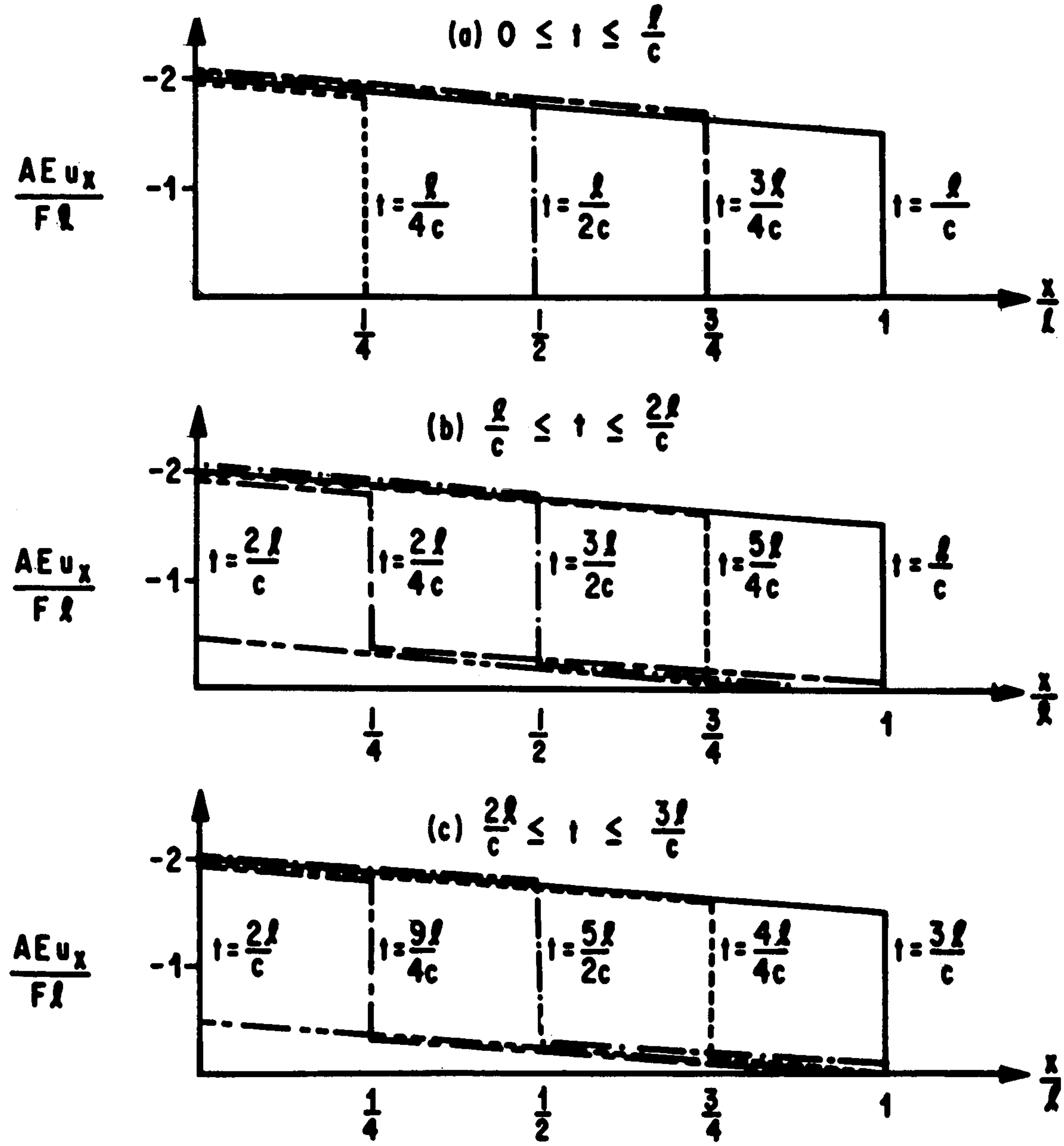


FIG. 29. VELOCITY DISTRIBUTION FOR A FINITE ROD WITH COULOMB DAMPING AND A STEP STRESS IMPULSE LOADING OF $\frac{T}{A}$ FOR $T = 2F\ell$



(d) $\frac{3\ell}{c} \le t \le \frac{4\ell}{c}$ Is the Same As $\frac{\ell}{c} \le t \le \frac{2\ell}{c}$ Above, etc.

FIG. 30. STRAIN DISTRIBUTION FOR A FINITE ROD WITH COULOMB DAMPING AND A STEP STRESS IMPULSE LOADING OF $\frac{I}{A}$ FOR $T = 2F\ell$

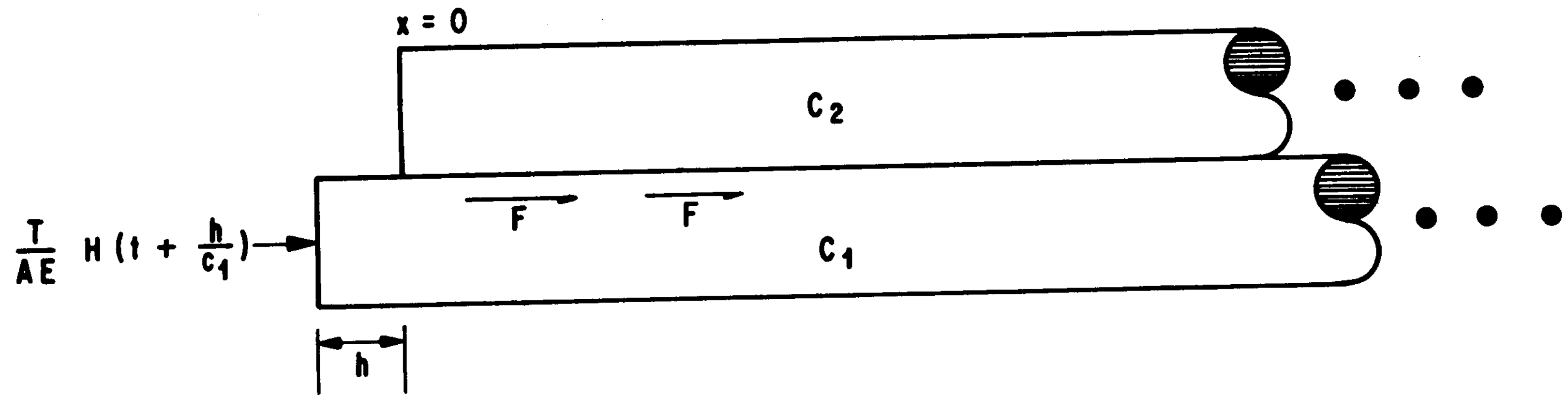


FIG. 31. FRICTION - INDUCED WAVE MOTION

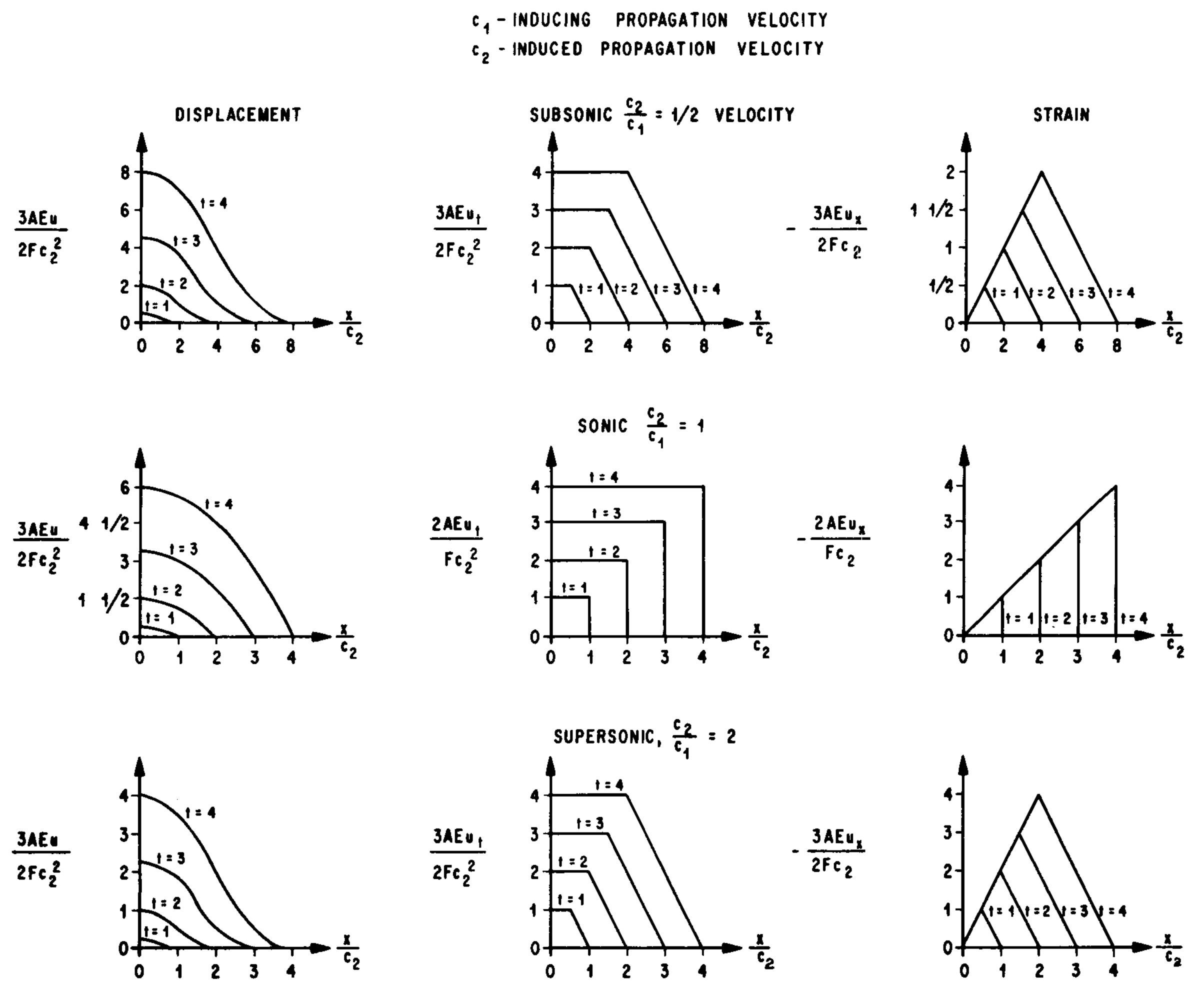


FIG. 32. DISPLACEMENT VELOCITY AND STRAIN DISTRIBUTIONS FOR FRICTION-INDUCED WAVE MOTION

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APPROVAL TM X-53714

ONE-DIMENSIONAL WAVE MOTION IN PRISMATIC BARS DUE TO IMPULSE LOADS WITH AND WITHOUT COULOMB DAMPING

by Donald Dean Tomlin, Jr.

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