TRANSPORT PHENOMENA OF GENERAL NON-EQUILIBRIUM GAS SYSTEMS

by

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ABSTRACT

The theory of gas mixtures is extended to cases where there may be large differences in the flow velocities and/or temperatures of the species in the mixture; the species are not assumed to be in local equilibrium. Transfer equations for the thirteen moments of each species are constructed relative to the species' flow velocity; the equations are closed by means of the Grad approximate velocity-space solution for the species' distribution functions. The partial collision integrals occurring in the transfer equations are then expressed as functions of a dimensionless velocity, $\vec{\varepsilon} \equiv (\vec{u}_t - \vec{u}_s)/(a_s^2 + a_t^2)^{1/2}$, the ratio of the difference in species' flow velocities to a "mixed sound speed." The integrals are evaluated for two limiting cases: (i) $|\vec{\varepsilon}| << 1$, arbitrary isotropic collision cross sections ; (ii) $|\vec{\varepsilon}| >> 1$, arbitrary inverse power interparticle force laws. A final set of exact calculations is made for the "Maxwell molecule" force law.

Various statistical collision models are next presented as possible substitutes for the Boltzmann binary collision operator, with a view towards duplicating the partial collision integrals of that operator.

Finally, transport quantities are calculated for: (a) weakly ionized gas ; (b) binary Maxwell molecule gas; (c) fully ionized gas.

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CHAPTER I

INTRODUCTION

The theory of gas mixtures has received widespread attention over the past sixty years, beginning with the pioneer work of Chapman[1] and Enskog[2] whose series solutions to the Boltzmann equation converge sufficiently rapidly when the individual species of the mixture are near complete equilibrium with a common flow velocity and temperature. The theory was extended over the ensuing years to treat nonequilibrium situations where the species are close to local equilibrium individually, but not necessarily in equilibrium with each other. Hence, closed sets of transfer equations for mass, momentum, and energy, for each species, have been developed by several workers[3-11] for a mixture of gases having separate Maxwellian velocity distributions (with separate flow velocities and/or temperatures). Exact calculations have been made for certain inverse power interparticle force laws, $f = \kappa/r^p$, namely, $p \to \infty$ ("hard spheres"), $p = 5$ ("Maxwell molecules"), $p = 3$, $p = 7/3$, and $p = 2$ (Coulomb force law); approximate results have been given for other force laws.[11] Because of the assumption of local Maxwellian distribution functions, the calculations in references [3] - [11] involve only the first five "velocity moments" of the species' distribution functions: the species' number density, flow velocity, and temperature. The situation where the species are not in local equilibrium has been investigated by several authors[12-16] using the so called "Grad[17] thirteen moment approximation" of the individual species' velocity distribution functions; here, the distribution functions depend upon the higher order velocity moments — the traceless pressure tensors and heat flow vectors, in addition to the first five moments. Thus, in this scheme the gas mixture is described by closed sets of transfer equations for mass, momentum, energy, traceless pressure, and heat flow, for each species. Small differences in flow velocities are considered in references [12] - [15], with arbitrary temperature differences assumed in [12], [14], [15]; references [12] - [14] involve general interparticle force laws, while [15] deals with a fully ionized plasma. The calculations of Everett[16] are for a fully ionized plasma (i.e. the Coulomb force law) and allow for large differences in both the species flow velocities and temperatures.

The primary goal of this dissertation is to extend the theory of gas mixtures to situations involving general interparticle force laws where the individual species are not in local equilibrium, and where differences in species flow velocities and temperatures are arbitrary; furthermore, the species are completely general in the sense that no assumption is made regarding particle mass, number density, or electric charge. The Grad thirteen moment approximation forms the basis for the calculations, which are all relative to the species flow velocities and temperatures. It is anticipated that the results will hold for a large body of non-equilibrium problems not covered by the work in references [3] - [11] (i.e. when the species distribution functions are <u>non</u>-Maxwellian) or by references [1], [2], [12] - [15] (i.e. when there exist large differences in the species flow velocities and/or temperatures). Many of the results in reference [16] can be recovered as a special case (i.e. the Coulomb force law) from the present work.

Throughout this dissertation the particles in the gas mixture are treated as ideal point centers of force (except for the case of "hard spheres" which are treated as "billiard balls" with finite spatial extensions); hence, the internal structures of particles such as positive ions and neutral atoms are completely ignored. Processes of ionization, recombination, dissociation, association, and radiation by moving charged particles are not taken into account; relativistic and quantum mechanical effects are also ignored.

The usual Boltzmann equation, with the Boltzmann binary collision operator used for the collision term, is assumed to be an adequate equation of motion for the species velocity distribution function (i.e. the "one-particle" distribution function). This of course assumes that binary collisions are of predominant importance, and does not take into account the influence of the positions of the colliding particles; hence, restrictions are placed upon the density of particles and upon the temperatures of the species.

In Chapter II a brief review of kinetic and transfer theory is presented. The species' velocity distribution function is defined, along with the macroscopic properties of the species. The aforementioned Boltzmann equation is introduced and "velocity moments" are taken to construct species transfer equations for a general gas mixture. These equations are "closed" and the calculation of the accompanying partial collision integrals is made possible by the introduction of the Grad scheme for the approximation of the species distribution functions. The validity of this approximation and the ensuing evaluation of the collision integrals are then discussed.

The partial collision integrals are evaluated in Chapter III; they are first expressed as functions of a dimensionless velocity, $\vec{\varepsilon} \equiv (\vec{u}_t - \vec{u}_s)/(a_s^2 + a_t^2)^{1/2}$, which is the ratio of the difference in species' flow velocities to a "mixed sound speed," where $a_s^2 \equiv 2KT_s/m_s$ with K denoting Boltzmann's constant and T_s, m_s the temperature and mass, respectively, of species "s". Before proceding with the evaluation, a relation between the partial momentum and random kinetic energy collision integrals is derived, which is valid for all "diffusion Mach number," $|\vec{\varepsilon}|$; this relation affords physical insight into the transfer of random kinetic energy between species. Next, the collision integrals are evaluated for two limiting ranges of $|\vec{\varepsilon}|$: (i) $|\vec{\varepsilon}| \ll 1$, arbitrary isotropic collision cross sections; (ii) $|\vec{\varepsilon}| \gg 1$, arbitrary inverse power interparticle force laws. The two sets of results are compared with respect to their dependence upon the "higher order moments" — the traceless pressure tensors and heat flow vectors; the directions of the momentum collision integrals of cases (i) and (ii) are also discussed. Next, the collision integrals are calculated exactly for the "Maxwell molecule" interparticle force law for two cases: (a) where all quantities are relative to the <u>individual species'</u> flow velocities; (b) where all quantities are relative to the <u>mixture's</u> flow velocity. From these calculations, conclusions are drawn regarding the level of accuracy of calculations (i), (ii), and those of references [12] - [14]. Finally, the dependence of the collision frequencies upon the interparticle force law and the diffusion Mach number is exhibited.

In Chapter IV a temporary digression is made from the transfer phenomena theme in order to present certain simplified kinetic models for the collision term in the equation of motion of the species' velocity distribution function. Analyzed are the Gross-Krook model[18], the Sirovich model[19], a revised form of the Sirovich model, and a model based upon a Grad-like expansion of the collision term. The ability of the models to imitate the properties of the Boltzmann binary collision operator is discussed.

The subject of transport phenomena is returned to in Chapter V, in which the traceless pressure tensors and heat flow vectors of three systems are calculated for small diffusion Mach numbers. The systems are: (I) a weakly ionized gas, with general inverse power interparticle force laws and a magnetic field of arbitrary magnitude; (II) a two-species gas composed of Maxwell molecules, with arbitrary mass and density ratios; (III) a fully ionized gas, in which the <u>total</u> (i.e. system) traceless pressure tensor is determined as a function of the <u>system's</u> flow velocity and current density.

CHAPTER II

DEVELOPMENT OF THE TRANSFER EQUATIONS AND THE GRAD THIRTEEN-MOMENT APPROXIMATION

In this chapter the basic quantities pertinent to kinetic theory are defined through the usual concept of a distribution function. All of the macroscopic quantities corresponding to a certain species of the mixture are defined relative to the distribution function and flow velocity of that species.

The Boltzmann equation is then presented, and by taking velocity moments of this equation the various macroscopic quantities are related by the resulting transfer equations.

The next section presents an approximate velocity-space solution for the species' distribution function based upon a truncated expansion in three-dimensional Hermite polynomials, commonly known as the "Grad thirteen moment approximation." In this way the transfer equations become a closed set of coupled partial differential equations, provided their collision integrals can be calculated.

2.1 The Distribution Function and Macroscopic Quantities

The material of this section can be found in references [20] - [22]. We begin by considering a general gas mixture composed of an arbitrary number of distinct species.* The velocity distribution function, or simply the <u>distribution function</u>, for species "s" is defined such that the quantity

$$F_s(\vec{x}, \vec{v}, t)d\vec{x}\, d\vec{v} \tag{2.1}$$

gives the probable number of particles of species "s" located in the volume element $d\vec{x}$ about $\vec{x}$, with velocities in the range $d\vec{v}$ about $\vec{v}$, at the time t. The differential lengths and velocities in (2.1) must be small compared with macroscopic distances and velocity intervals over which there are significant changes in the macroscopic properties of the species; at the same time, however, they must be sufficiently large so that there are a large number of particles in $d\vec{x}\, d\vec{v}$, thus allowing $F_s(\vec{x}, \vec{v}, t)$ to be a continuous function of its variables. We note from the definition that the species distribution function is non-negative, and that

$$F_s(\vec{x}, \vec{v}, t) \to 0 \quad \text{as} \quad |\vec{v}| \to \infty \ . \tag{2.2}$$

It follows as a consequence of the definition (2.1) that the <u>number density</u> for species "s" (the number of "s" particles per unit volume) is given by

$$N_s(\vec{x}, t) = \int F_s(\vec{x}, \vec{v}, t)d\vec{v} \tag{2.3}$$

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*A "species" is in general defined by its electric charge and mass.

where the integration is performed over the entire velocity space.*
The average over all velocity space, or simply the velocity average, of any quantity $\Phi(\vec{x}, \vec{v}, t)$ is defined as

$$< \Phi(\vec{x}, \vec{v}, t) >_s \equiv \frac{1}{N_s} \int F_s(\vec{x}, \vec{v}, t) \, \Phi(\vec{x}, \vec{v}, t) d\vec{v} \qquad (2.4)$$

where the subscript "s" on the average symbol "< >" indicates that the velocity averaging is to be done with respect to the species "s" distribution function. We note from (2.3) and (2.4) that if Φ is independent of the velocity, then

$$< \Phi(\vec{x}, t) >_s = \Phi(\vec{x}, t) \quad . \qquad (2.4a)$$

The average velocity of species "s" particles or the flow velocity of species "s" is

$$\vec{u}_s(\vec{x},t) \equiv < \vec{v} >_s = \frac{1}{N_s} \int F_s(\vec{x},\vec{v},t)\vec{v} \, d\vec{v} \quad . \qquad (2.5)$$

The peculiar or random velocity of species "s" particles relative to the species "s" flow velocity is

$$\vec{c}_s(\vec{x},\vec{v},t) \equiv \vec{v} - \vec{u}_s(\vec{x},t) \qquad (2.6)$$

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*Unless otherwise stated, all integrals without explicit limits are to be taken over the entire domain of the variable of integration, e.g. $\int F_s(\vec{x},\vec{v},t)d\vec{v} \equiv \iiint_{-\infty}^{\infty} F_s(\vec{x},\vec{v},t)dv_1dv_2dv_3$.

from which it immediately follows that

$$\langle \vec{c}_s \rangle_s \equiv 0 \quad . \tag{2.7}$$

The temperature of species "s" is defined in terms of the velocity average of the random kinetic energy of species "s"

$$\frac{3}{2} K \, T_s(\vec{x},t) \equiv \langle \frac{1}{2} m_s c_s^2 \rangle_s \tag{2.8}$$

where K is Boltzmann's constant, and m_s is the mass of a species "s" particle.

The j-k element of the species "s" pressure tensor, or simply the species "s" pressure tensor, is defined as

$$p_{sjk}(\vec{x},t) \equiv \langle \rho_s c_{sj} c_{sk} \rangle_s \,{}^{*} \tag{2.9}$$

where $j,k = 1,2,3$, and ρ_s is the mass density of species "s"

$$\rho_s(\vec{x},t) \equiv m_s N_s(\vec{x},t) \quad . \tag{2.10}$$

The hydrostatic, mean, or scalar pressure of species "s" is given by

$$p_s(\vec{x},t) \equiv \frac{1}{3} p_{sii}(\vec{x},t) \tag{2.11}$$

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*With any quantity such as p_{sjk} , the first subscript refers to the species; any other subscripts refer to the spatial direction. The one exception to this occurs in Section 5.4, where quantities appear without species subscripts.

where, unless otherwise stated, a repeated direction index is to be summed, e.g. $p_{sii} \equiv p_{s11} + p_{s22} + p_{s33}$. From (2.8), (2.9), and (2.11) we obtain

$$p_s = N_s K T_s \tag{2.12}$$

which is a form of the perfect gas law. The non-hydrostatic or traceless pressure tensor of species "s" is defined as

$$P_{sjk}(\vec{x},t) \equiv \langle \rho_s (c_{sj} c_{sk} - \frac{1}{3} \delta_{jk} c_s^2) \rangle_s = p_{sjk} - \delta_{jk} p_s \tag{2.13}$$

where δ_{jk} is the Kronecker delta

$$\delta_{jk} \equiv \begin{cases} 1 \ , & j=k \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{2.14}$$

The heat flow tensor of species "s" is defined as

$$q_{sijk}(\vec{x},t) \equiv \langle \rho_s c_{si} c_{sj} c_{sk} \rangle_s \ , \tag{2.15}$$

while the heat flow vector of species "s" is given by

$$\vec{q}_s(\vec{x},t) \equiv \frac{1}{2} q_{siik} \vec{a}_k = \langle \frac{1}{2} \rho_s c_s^2 \vec{c}_s \rangle \ , \tag{2.16}$$

where $\vec{a}_k$ is the unit vector in the $k^{\underline{th}}$ - direction, $k = 1,2,3$.

Finally, we define a fourth-order tensor

$$p_{shijk}(\vec{x},t) \equiv \langle \rho_s c_{sh} c_{si} c_{sj} c_{sk} \rangle_s \quad . \tag{2.17}$$

The preceding velocity average quantities are often referred to as "velocity moments," or simply "moments", of the species distribution function. It is important to observe that these moments have herein been defined relative to the species' flow velocity, $\vec{u}_s$ (see Eq. (2.6)), as opposed to analogous definitions relative to the mixture flow velocity[23], $\vec{u}$, where

$$\rho\vec{u}(\vec{x},t) \equiv \sum_s \rho_s \vec{u}_s \quad , \text{ and} \tag{2.18}$$

$$\rho(\vec{x},t) \equiv \sum_s \rho_s \tag{2.19}$$

is the mass density of the mixture.

2.2 The Boltzmann Equation and Equations of Transfer

The equation of motion for the species "s" distribution function is given by[24]

$$\frac{\partial F_s}{\partial t} + v_i \frac{\partial F_s}{\partial x_i} + \frac{f_{si}}{m_s} \frac{\partial F_s}{\partial v_i} = \left(\frac{\delta F_s}{\delta t}\right)_{\text{collisions}} = \sum_r \left(\frac{\delta F_s}{\delta t}\right)_{\substack{sr\\ \text{collisions}}} \tag{2.20}$$

where $\vec{f}_s(\vec{x},\vec{v},t)$ is the external force* acting on a species "s"

*The term "external force" refers to all forces other than those involved in collisions.

particle which is located at $\vec{x}$ with velocity $\vec{v}$ at the time t, and $(\delta F_s/\delta t)_{\substack{sr \\ \text{collisions}}}$ gives the average time rate of change of the species "s" distribution function due to collisions of species "s" particles with species "r" particles. The summation in (2.20) is over all the species in the mixture, including "r" = "s", and there is an analogous equation for each species. Equation (2.20) becomes the Boltzmann equation if the right-hand side is given by[25]

$$\Sigma(\delta F_s/\delta t)_{\substack{sr \\ \text{collisions}}} = \sum_r [\iiint(F'_s F'_{r1} - F_s F_{r1}) g b db d\varepsilon d\vec{v}_1] \tag{2.21}$$

where $\vec{g}$ is the relative velocity between a particle of species "s" and its collision partner of species "r"

$$\vec{g} \equiv \vec{v}_1 - \vec{v} \ ,^{*} \tag{2.22}$$

b is the impact parameter of the binary collision, ε is an angle specifying the collision plane (i.e. the orbit of the colliding "r" particle relative to the "s" particle), and the primes refer to post collision quantities, e.g. $F'_{r1} \equiv F_r(\vec{x},\vec{v}'_1,t)$. The integrations in (2.21) are taken over all possible impact parameters, collision plane angles, and velocities. The term in

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*The subscript "1" is used simply to distinguish the velocities of the colliding partners.

brackets, "[ ]" , in (2.21) is known as the Boltzmann binary collision operator for collisions between particles of species "s" and "r" .

If the equation of motion (2.20) could be solved for the species' distribution function, $F_s(\vec{x},\vec{v},t)$, then of course all of the previously defined velocity moments could, in principle, be directly calculated. However, when the Boltzmann binary collision operator, (2.21), is included, the set of equations for the species' distribution functions becomes a set of nonlinear integro-partial differential equations which is in general untractable. One method of circumventing the difficulty is to construct a set of transfer equations for the moments and employ some sort of truncation scheme to close the set.

If we multiply equation (2.20) by an arbitrary quantity $Q(\vec{x},\vec{v},t)d\vec{v}$ and integrate over all velocity space, we obtain a generalized transfer equation for the quantity $Q(\vec{x},\vec{v},t)$. Performing the integration term by term in the usual way[26], we find

$$\frac{\partial}{\partial t}(N_s\langle Q\rangle_s) - N_s\left\langle \frac{\partial Q}{\partial t}\right\rangle_s + \frac{\partial}{\partial x_i}(N_s\langle Qv_i\rangle_s) - N_s\left\langle v_i\frac{\partial Q}{\partial x_i}\right\rangle_s -$$

$$-\frac{N_s}{m_s}\left\langle f_{si}\frac{\partial Q}{\partial v_i}\right\rangle_s = \int Q\left(\frac{\delta F_s}{\delta t}\right)_{coll.} d\vec{v} \equiv \delta Q \qquad (2.23)$$

$$\text{where} \quad \delta Q = \sum_r \iiiint (Q' - Q)F_sF_{r1}\, gbdbd\varepsilon d\vec{v}d\vec{v}_1 \qquad (2.24)$$

when the Boltzmann binary collision operator (2.21) is used. The right-hand side of (2.23) is referred to as the collision integral for the quantity $Q(\vec{x},\vec{v},t)$. A trivial consequence of (2.24) is that for any velocity independent quantities, $\tilde{Q}(\vec{x},t)$, $\Phi(\vec{x},t)$,

$$\delta\tilde{Q}(\vec{x},t) \equiv 0 \quad , \text{ and} \tag{2.25a}$$

$$\delta[\Phi(\vec{x},t)Q(\vec{x},\vec{v},t)] = \Phi(\vec{x},t)\delta Q(\vec{x},\vec{v},t) \ . \tag{2.25b}$$

In what follows we shall be dealing with Q's which are explicit functions of the random velocity

$$Q = h(\vec{c}_s) \quad , \tag{2.26}$$

but it must be kept in mind that $\vec{c}_s \equiv \vec{v} - \vec{u}_s(\vec{x},t)$, so that

$$Q = h(\vec{c}_s) = h(\vec{v} - \vec{u}_s(\vec{x},t)) = H(\vec{x},\vec{v},t) \ . \tag{2.27}$$

Then taking $Q(\vec{x},\vec{v},t)$ to be m_s , $m_s c_{sk}$, $\frac{1}{2} m_s c_s^2$, $m_s(c_{sj}c_{sk} - \frac{1}{3}\delta_{jk}c_s^2)$, and $\frac{1}{2} m_s c_s^2 c_{sk}$ we obtain, respectively, the mass conservation equation, the momentum equation, the random kinetic energy equation (or simply the "energy" equation), the traceless pressure equation, and the heat flow equation for species "s"

mass conservation: $$\frac{D_s N_s}{Dt} + N_s \nabla . \vec{u}_s = 0 \tag{2.28a}$$

<u>momentum</u>: $$\rho_s \frac{D_s u_{sk}}{Dt} + \frac{\partial p_s}{\partial x_k} + \frac{\partial P_{sik}}{\partial x_i} - N_s[e_s(\vec{E} + \vec{u}_s \times \vec{B}) + m_s\vec{G}]_k$$

$$= \delta(m_s c_{sk}) \qquad (2.28b)$$

<u>energy</u>: $$\frac{3}{2}\frac{D_s p_s}{Dt} + \frac{5}{2} p_s \nabla.\vec{u}_s + P_{sij} \frac{\partial u_{sj}}{\partial x_i} + \nabla.\vec{q}_s = \delta(\frac{1}{2} m_s c_s^2)$$

$$(2.28c)$$

<u>traceless pressure</u>: $$\frac{D_s P_{sjk}}{Dt} + P_{sjk} \nabla.\vec{u}_s - 2\frac{e_s}{m_s}(B_i \epsilon_{ijh} P_{skh})^{\dagger} +$$

$$+2(P_{sij}\frac{\partial u_{sk}}{\partial x_i})^{\dagger\dagger} + 2p_s(\frac{\partial u_{sj}}{\partial x_k})^{\dagger\dagger} + \frac{\partial}{\partial x_i}(q_{sijk} - \frac{2}{3}\delta_{jk} q_{si})$$

$$= \delta(m_s c_{sj} c_{sk}) - \frac{2}{3}\delta_{jk}\,\delta(\frac{1}{2} m_s c_s^2) \qquad (2.28d)$$

<u>heat flow</u>: $$\frac{D_s q_{sk}}{Dt} + q_{sk} \nabla.\vec{u}_s + q_{si}\frac{\partial u_{sk}}{\partial x_i} - \frac{e_s}{m_s}(\vec{q}_s \times \vec{B})_k -$$

$$-(\frac{\partial p_s}{\partial x_j} + \frac{\partial P_{sij}}{\partial x_i})(\frac{5}{2}\frac{p_s}{\rho_s}\delta_{jk} + \frac{P_{sjk}}{\rho_s}) + q_{sijk}\frac{\partial u_{sj}}{\partial x_i} +$$

$$+\frac{1}{2}\frac{\partial}{\partial x_i} p_{sikjj} = \delta(\frac{1}{2} m_s c_s^2 c_{sk}) - \frac{1}{\rho_s}(\frac{5}{2} p_s \delta_{jk} + P_{sjk})\delta(m_s c_{sj}).$$

$$(2.28e)$$

In (2.28a–e) we have introduced the "hydrodynamic" differential operator

$$\frac{D_s}{Dt} \equiv (\frac{\partial}{\partial t} + \vec{u}_s.\nabla) \quad , \qquad (2.29)$$

the alterating tensor

$$\epsilon_{ijh} \equiv \begin{cases} +1, \text{ if } i,j,h \text{ is an even permutation of } 1,2,3 \\ -1, \text{ if } i,j,h \text{ is an odd permutation of } 1,2,3 \\ 0, \text{ otherwise} \end{cases} \qquad (2.30)$$

the symmetrized second order tensor

$$(A_{jk})^{\dagger} \equiv \frac{1}{2}(A_{jk} + A_{kj}) \quad , \qquad (2.31a)$$

and the traceless symmetrized second order tensor

$$(A_{jk})^{\dagger\dagger} \equiv \frac{1}{2}(A_{jk} + A_{kj}) - \frac{1}{3}\delta_{jk}A_{ii} \quad . \qquad (2.31b)$$

In obtaining (2.28a-e) we have employed the definitions for the moments given in Section 2.1, along with the fact that $<\vec{c}_s>_s \equiv 0$. The right-hand side of the mass conservation equation, (2.28a), is zero since we are considering only those collisions in which the mass of a given particle is unchanged, so that $Q' - Q = m_s - m_s = 0$. Finally, we have assumed the external force $\vec{f}_s$ to be given by the Lorentz force plus a gravitational force

$$\vec{f}_s = e_s(\vec{E} + \vec{v} \times \vec{B}) + m_s\vec{G} \;^{*} \qquad (2.32)$$

where e_s is the electric charge of an "s" species particle, $\vec{E}$ is the external electric field intensity, $\vec{B}$ is the external

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*Rationalized MKS units are used throughout this work.

magnetic flux density, and $\vec{G}$ is the external gravitational acceleration.

Inspection of the set of transfer equations (2.28a-e) shows that there are, in general, a total of thirteen independent scalar equations with thirteen unknown scalars N_s, $\vec{u}_s$, T_s (or p_s), P_{sjk}, $\vec{q}_s$, plus the second and third order tensors p_{sikjj}, q_{sijk}, for each species (the traceless pressure tensor P_{sjk} constitutes only five independent unknown scalars, since it is symmetric, $P_{sjk} \equiv P_{skj}$, and traceless, $P_{sii} \equiv 0$). Hence, as it stands, the set of transfer equations is not closed; however, we shall see in the next section that, with suitable approximate velocity-space solutions for the species' distribution functions, the higher order tensors can be expressed in terms of the "first" thirteen moments, thus closing the set of transfer equations. Of course, without some such knowledge of the distribution functions, we could never hope to obtain a closed set of transfer equations since, as can be seen from (2.28a-e), each succeeding equation is coupled to the next higher moment equation. The reason for this coupling can easily be seen from the equation of motion, (2.20); this equation contains the term $v_i \partial F_s/\partial x_i$, so that the transfer equation for a quantity proportional to v_k^n will involve a "higher velocity moment" proportional to $< v_k^{n+1} >_s$.

2.3 The Grad Velocity Space Approximation to the Distribution Functions

In order to express the higher order tensors p_{sikjj} , q_{sijk} , which appear in the traceless pressure and heat flow equations in terms of the first thirteen moments and therby close the set of transfer equations, we employ the Grad scheme.[17] In this scheme the distribution functions are expressed in three-dimensional Hermite polynomials, and the expansion is terminated in such a way that the distribution functions depend upon the first thirteen moments $(N_s, \vec{u}_s, T_s$ or p_s , P_{sjk} , $\vec{q}_s)$, but not upon any higher order moments ("higher" both in the sense of higher order tensors and higher degree in the random velocity). Such a termination is justified to the extent that these thirteen moments are the ones of usual interest in kinetic theory and plasma physics, plus the reasonable expectation that higher order moments should be relatively unimportant over a wide range of situations.

Following Grad, we expand the species' distribution functions about a locally Maxwellian distribution in terms of three dimensional Hermite polynomials

$$F_s = F_s^{(o)} \sum_{n=0}^{\infty} b_{si}^{(n)} H_{si}^{(n)}(\vec{\beta}_s) \tag{2.33}$$

where
$$F_s^{(o)} \equiv \frac{N_s}{\pi^{3/2} a_s^3} e^{-c_s^2/a_s^2} = \frac{N_s}{\pi^{3/2} a_s^3} e^{-\beta_s^2/2} \tag{2.34}$$

is the local Maxwellian distribution function for species "s",

$$\vec{\beta}_s \equiv \sqrt{2}\,\vec{c}_s/a_s \quad , \quad \text{and} \tag{2.35}$$

$$a_s \equiv (2K\,T_s/m_s)^{1/2} \tag{2.36}$$

is the "sound" or "thermal" speed of species "s". In the expansion (2.33), $H_{si}^{(n)}(\vec{\beta}_s)$ is an $n^{\underline{th}}$ order tensor with n indices, $i \equiv (i_1,\ldots,i_n)$, and is also an $n^{\underline{th}}$ degree polynomial in the dimensionless velocity $\vec{\beta}_s$; the coefficient $b_{si}^{(n)}$ is an $n^{\underline{th}}$ order tensor and the usual convention for summation over a repeated direction index is to be applied to the index set $i \equiv (i_1,\ldots,i_n)$.

Before proceding we should note from (2.33) that, because each component polynomial of $H_i^{(n)}$ is orthogonal to each component polynomial of $H_j^{(m)}$ with respect to the "weighting function" $F_s^{(o)}$, unless $m=n$ and $(i_1,\ldots,i_n)$ is a permutation of $(j_1,\ldots,j_n)$, the calculation of the coefficients $b_{si}^{(n)}$ is unaffected by the choice of the truncation point.[27]

The first four tensors $H_i^{(n)}$ are[27]

$$H_s^{(0)} = 1 \tag{2.37a}$$

$$H_{si}^{(1)} = \beta_{si} \tag{2.37b}$$

$$H_{sij}^{(2)} = \beta_{si}\beta_{sj} - \delta_{ij} \tag{2.37c}$$

$$H_{sijk}^{(3)} = \beta_{si}\beta_{sj}\beta_{sk} - (\beta_{si}\delta_{jk} + \beta_{sj}\delta_{ik} + \beta_{sk}\delta_{ij}) \;. \tag{2.37d}$$

Substituting (2.37a-d) into (2.33) and contracting the indices of $b^{(3)}_{sijk}\ \mathcal{H}^{(3)}_{sijk}$ so that the highest order tensor introduced into the expansion will be of second order, thus insuring the closing of the set of transfer equations (2.28a-e), we obtain

$$F_s = F_s^{(o)}[b_s^{(0)}\mathcal{H}_s^{(0)} + b_{si}^{(1)}\mathcal{H}_{si}^{(1)} + b_{sij}^{(2)}\ \mathcal{H}_{sij}^{(2)} + b_{sijj}^{(3)}\ \mathcal{H}_{sikk}^{(3)}]$$

$$= F_s^{(o)}[b_s^{(0)\prime} + b_{si}^{(1)\prime}\ \beta_{si} + b_{sij}^{(2)}\ \beta_{si}\beta_{sj} + b_{sijj}^{(3)}\beta_{si}\beta_s^2\ ]$$

or, in terms of the random velocity, $\vec{c}_s$,

$$F_s = F_s^{(o)}[b_s + A_{si}c_{si} + B_{sij}c_{si}c_{sj} + C_{si}c_{si}c_s^2]\ . \tag{2.38}$$

The evaluation of the coefficients in (2.38) is accomplished by recalling the following "constraints"

$$N_s = \int F_s d\vec{v} = \int F_s d\vec{c}_s \ ^{*} \tag{2.39a}$$

$$0 \equiv \langle \vec{c}_s \rangle_s \equiv \frac{1}{N_s} \int F_s \vec{c}_s d\vec{c}_s \tag{2.39b}$$

$$\frac{3}{2} K\ T_s \equiv \langle \frac{1}{2} m_s c_s^2 \rangle_s \equiv \frac{1}{N_s} \int F_s\ \frac{1}{2} m_s c_s^2\ d\vec{c}_s \tag{2.39c}$$

---

*Note that $\int h(\vec{x},\vec{v},t)d\vec{v} = \int h(\vec{x},\vec{v},t)d\vec{c}_s$, since $dv_i = d(c_{si}+u_{si}) = dc_{si}$ inasmuch as $u_{si} = u_{si}(\vec{x},t)$ is held constant during the integration.

$$P_{sjk} \equiv \langle \rho_s (c_{sj} c_{sk} - \frac{1}{3} \delta_{jk} c_s^2) \rangle_s \equiv \frac{1}{N_s} \int F_s \rho_s (c_{sj} c_{sk} - \frac{1}{3} \delta_{jk} c_s^2) d\vec{c}_s \tag{2.39d}$$

and

$$\vec{q}_s \equiv \langle \frac{1}{2} \rho_s c_s^2 \vec{c}_s \rangle_s \equiv \frac{1}{N_s} \int F_s \frac{1}{2} \rho_s c_s^2 \vec{c}_s d\vec{c}_s \ . \tag{2.39e}$$

Substituting (2.34), (2.38) into (2.39a-e) we obtain

$$b_s = 1 \tag{2.40a}$$

$$A_{si} = -4 q_{si} / \rho_s a_s^4 \tag{2.40b}$$

$$B_{sij} = P_{sij} / p_s a_s^2 \tag{2.40c}$$

$$C_{si} = -\frac{2}{5} \frac{A_{si}}{a_s^2} \quad , \tag{2.40d}$$

so that (2.38) becomes

$$F_s = F_s^{(o)} [1 + \frac{P_{sij}}{p_s} \frac{c_{si} c_{sj}}{a_s^2} - \frac{4 q_{si}}{\rho_s a_s^4} (1 - \frac{2}{5} \frac{c_s^2}{a_s^2}) c_{si}] \ . \tag{2.41}$$

From (2.34), (2.41) we see that, to this level of approximation, the velocity-space solution for F_s depends upon the "first" thirteen moments N_s, $\vec{u}_s$ (through $\vec{c}_s$) , T_s or p_s , P_{sjk} , and $\vec{q}_s$; hence, the solution (2.41) is often referred to as "Grad's thirteen moment approximation."

The higher order terms p_{sikjj} , q_{sijk} which occur in the traceless pressure and heat flow equations, (2.28d, e), can now be related to the first thirteen moments; substituting (2.41) into the definitions for these terms, we obtain

$$p_{sikjj} \equiv < \rho_s c_s^2 c_{si} c_{sk} >_s = \frac{p_s}{\rho_s} (7P_{sik} + 5p_s \delta_{ik}) \tag{2.42}$$

and $$q_{sijk} \equiv < \rho_s c_{si} c_{sj} c_{sk} >_s = \frac{2}{5}(q_{sk}\delta_{ij} + q_{sj}\delta_{ik} + q_{si}\delta_{jk}) \; . \tag{2.43}$$

Substituting (2.42), (2.43) into (2.28d, e) we obtain the following <u>closed set of transfer equations:</u> *

<u>conservation of mass:</u> $$\frac{D_s N_s}{Dt} + N_s \, \nabla \cdot \vec{u}_s = 0 \tag{2.44a}$$

<u>momentum:</u> $$\rho_s \frac{D_s u_{sk}}{Dt} + \frac{\partial p_s}{\partial x_k} + \frac{\partial P_{sik}}{\partial x_i} - N_s[e_s(\vec{E} + \vec{u}_s x \vec{B}) + m_s \vec{G}]_k = \delta(m_s c_{sk}) \tag{2.44b}$$

<u>energy:</u> $$\frac{3}{2} \frac{D_s p_s}{Dt} + \frac{5}{2} p_s \, \nabla \cdot \vec{u}_s + P_{sij} \frac{\partial u_{sj}}{\partial x_i} + \nabla . \vec{q}_s = \delta(\frac{1}{2} m_s c_s^2) \tag{2.44c}$$

<u>traceless pressure:</u> $$\frac{D_s P_{sjk}}{Dt} + P_{sjk} \, \nabla \cdot \vec{u}_s - 2 \frac{e_s}{m_s} (B_i \varepsilon_{ijh} P_{skh})^{\dagger} + + 2(P_{sij} \frac{\partial u_{sk}}{\partial x_i})^{\dagger\dagger} + 2p_s (\frac{\partial u_{sj}}{\partial x_k})^{\dagger\dagger} + \frac{4}{5}(\frac{\partial q_{sj}}{\partial x_k})^{\dagger\dagger} = \delta(m_s c_{sj} c_{sk}) - \frac{2}{3} \delta_{jk} \delta(\frac{1}{2} m_s c_s^2) \tag{2.44d}$$

---

* Whenever we refer to a "closed" set of transfer equations, such as (2.44a–e), it is understood that we are actually referring to the <u>system's</u> closed set of <u>13r</u> equations, where r is the number of species; a similar understanding holds for <u>a</u> phrase such as "thirteen moments" N_s, $\vec{u}_s$, T_s or p_s , P_{sjk} , $\vec{q}_s$.

heat flow: $$\frac{D_s q_{sk}}{Dt} + \frac{7}{5} q_{si}(\delta_{ik}\, \nabla\cdot\vec{u}_s + \frac{\partial u_{sk}}{\partial x_i} + \frac{2}{7}\frac{\partial u_{si}}{\partial x_k}) - \frac{e_s}{m_s}(\vec{q}_s x \vec{B})_k + \frac{P_{sik}}{\rho_s}(\frac{5}{2}\frac{\partial p_s}{\partial x_i} - \frac{7}{2}\frac{KT_s}{m_s}\frac{\partial N_s}{\partial x_i} - \frac{\partial P_{sij}}{\partial x_j}) + \frac{p_s}{\rho_s}\frac{\partial P_{sik}}{\partial x_i} + \frac{5}{2}\frac{Kp_s}{m_s}\frac{\partial T_s}{\partial x_k}$$

$$= \delta(\frac{1}{2} m_s c_s^2 c_{sk}) - \frac{1}{\rho_s}(\frac{5}{2} p_s \delta_{ik} + P_{sik})\delta(m_s c_{si}) \ . \qquad (2.44e)$$

If we had not terminated the expansion (2.33) with the contraction of the third order Hermite tensor but rather had retained the full tensor, the distribution function F_s would have contained the third order tensor q_{sijk} and a transfer equation for this tensor would have been required in order to obtain a closed set of transfer equations. The point of truncation is to a large extent arbitrary, it depending upon the degree of complexity one is willing to introduce into the analysis and upon how greatly the species distribution functions deviate from their local equilibrium forms, that is, from $F_s^{(o)}$.*

---

*A <u>system</u> is in <u>local equilibrium</u> when $(\delta F_s/\delta t)_{coll.} = 0$, for all "s" ; a <u>species</u> "s" is in <u>local equilibrium</u> when $(\delta F_s/\delta t)ss_{coll.} = 0$. It can be shown that $F_s^{(o)}$, the Maxwellian distribution, is the <u>only</u> local equilibrium distribution function,[22,28] where $F_s^{(o)}$ can depend upon $(\vec{x}, t)$.

2.4 Further Approximations to the Distribution Functions

We note that the distribution function given by (2.41) can be written in the form

$$F_s = F_s^{(o)}(1 + \phi_s) \tag{2.45}$$

where $$\phi_s \equiv \frac{P_{sij}}{p_s} \frac{c_{si}c_{sj}}{a_s^2} - \frac{4q_{si}}{\rho_s a_s^4}\left(1 - \frac{2}{5}\frac{c_s^2}{a_s^2}\right)c_{si} \; . \tag{2.46}$$

In the subsequent calculation of the collision integrals we shall encounter a double integration over the quantity

$$F_s F_{t1} d\vec{c}_s d\vec{c}_t = F_s^{(o)}F_{t1}^{(o)}(1 + \phi_s + \phi_{t1} + \phi_s\phi_{t1})d\vec{c}_s d\vec{c}_t . \tag{2.47}$$

Because of the exponential factor in $F_s^{(o)}F_{t1}^{(o)}$, $\exp[-(\frac{c_s^2}{a_s^2} + \frac{c_t^2}{a_t^2})]$, the major contribution to the integration (2.47) will stem from the region $c_s \lesssim a_s$, $c_t \lesssim a_t$. For this region we assume that $|\phi_s|$, $|\phi_t|$ are sufficiently small to allow the discarding of the term $\phi_s\phi_t$ in (2.47); that is

$$|\phi_s|, |\phi_t| \ll 1 \quad \text{for} \quad c_s \lesssim a_s \; , \; c_t \lesssim a_t \; . \tag{2.48}$$

For the condition (2.48) to be satisfied in general, each of the terms in (2.46) must be small in the same sense as (2.48); taking c_s to be of the order of a_s , we find that these terms will be small provided that

$$|P_{sij}| \ll p_s \tag{2.49}$$

and $$|q_{si}| \ll a_s p_s = p_s(2p_s/\rho_s)^{1/2} \quad . \tag{2.50}$$

Simple calculations show that, when the species "s" is in <u>local equilibrium</u>,

$$P_{sij} \doteq 0 \quad , \quad q_{si} \equiv 0 \quad , \quad \text{for } F_s = F_s^{(o)} \quad . \tag{2.51}$$

Hence, (2.49), (2.50) simply reflect the fact that F_s is "close" to its local equilibrium form, $F_s^{(o)}$.

Finally, we note that although the condition $|\phi_s| \ll 1$ cannot be made to hold for indefinitely increasing values of c_s, no matter how small the coefficients in (2.46) are, the species "s" distribution function F_s is still quite close to its local equilibrium form, $F_s^{(o)}$, due to the presence of the factor $\exp(-c_s^2/a_s^2)$ in F_s, equation (2.45) .

CHAPTER III

EVALUATION OF THE COLLISION INTEGRALS

In this chapter we shall calculate the collision integrals encountered in the construction of the transfer equations of Section 2.3. Insofar as the actual details are concerned, we shall only present the calculation of the momentum collision integral (i.e. the right hand side of the momentum equation, (2.44b)), since all the computations are extremely involved and tedious. The results for the other three collision integrals (energy, traceless pressure, and heat flow) are also presented here, but the detailed calculations are relegated to the appendices. For the most part the four evaluations follow parallel analyses and the choice of the momentum collision integral for presentation is merely one of convenience (it being the simplest to calculate).

After several intermediate steps (both in the text and in the appendices) we shall arrive at a point where the collision integrals are presented as functions of a dimensionless velocity, $\vec{\varepsilon}$, whose magnitude is sometimes referred to as the "diffusion Mach number."[10] At this point a relation between the momentum and energy collision integrals will be derived which is "exact" in the sense that it holds for all diffusion Mach number, ε .

Next, the collision integrals will be evaluated for two limiting cases: (i) $\varepsilon \ll 1$, and (ii) $\varepsilon \gg 1$. In case (i) the calculations are for interparticle force laws which are arbitrary to the extent that the collision cross sections are isotropic, i.e. dependence on the relative velocity is limited to dependence on its magnitude. In case (ii) the interparticle force laws are of the inverse power type.

The four collision integrals are next evaluated exactly for a particular inverse power interparticle force law, namely the "Maxwell molecule" force law; the results are "exact" within the limitations of the Boltzmann binary collision model.

In the last section the dependence of the collision frequencies upon the interparticle force law and the diffusion Mach number is exhibited.

3.1 The Partial Collision Integrals as Functions of the Diffusion Mach Number

To make the notation somewhat simpler we shall deal with "partial" collision integrals; that is, for any quantity $Q(\vec{x},\vec{v},t)$, the total collision integral, or simply the collision integral, for species "s" will be given by the sum of the partial collision integrals

$$\delta Q \equiv \sum_t (\delta Q)_{st} \tag{3.1}$$

where $(\delta Q)_{st}$ is the contribution to δQ due to collisions of the "s" particles with the "t" particles, and the summation is over all species, including "t" = "s" .

Using the Boltzmann binary collision operator, we have from (2.24)

$$(\delta Q)_{st} = \iiiint (Q' - Q) F_s F_{t1} g b\,db\,d\varepsilon\,d\vec{v}\,d\vec{v}_1 \tag{3.2}$$

where it will be recalled that primed quantities refer to <u>post collision values</u>, and where $F_{t1} \equiv F_t(\vec{x},\vec{v}_1,t)$. For the <u>partial momentum collision integral</u> we have

$$Q = m_s c_{sk} \tag{3.3}$$

so that (3.2) becomes

$$[\delta(m_s c_{sk})]_{st} = m_s \iiiint (c'_{sk} - c_{sk}) F_s F_{t1} g b\,db\,d\varepsilon\,d\vec{v}\,d\vec{v}_1 \ .^* \tag{3.4}$$

We now introduce the following quantities:

$$m_o \equiv m_s + m_t \tag{3.5a}$$

<u>reduced mass</u> $\mu \equiv m_s m_t / m_o$ (3.5b)

<u>center of mass velocity</u> $\vec{c}_o \equiv (m_s\vec{v} + m_t\vec{v}_1)/m_o$ (3.5c)

<u>relative velocity between colliding partners</u> $\vec{g} \equiv \vec{v}_1 - \vec{v}$. (3.5d)

---

*Note that the <u>momentum</u> collision integral and the <u>random momentum</u> collision integral are equal since $\delta(m_s c_{sk}) = \delta(m_s v_k) - \delta(m_s u_{sk}) = \delta(m_s v_k)$ by reason of (2.25a).

Solving (3.5c,d) for $\vec{v}$, recognizing that $\vec{c}_o' = \vec{c}_o$ (the center of mass velocity is unchanged in a collision, due to the conservation of momentum), and further that $\vec{u}_s' = \vec{u}_s$ (any quantity $\Phi(\vec{x},t)$ is unaffected by a collision), we obtain

$$c_{sk}' - c_{sk} = (v_k' - u_{sk}') - (v_k - u_{sk}) = (m_t/m_o)(g_k - g_k') \,, \tag{3.6}$$

so that (3.4) becomes

$$[\delta(m_s c_{sk})]_{st} = \mu \iiiint (g_k - g_k') F_s F_{t1} g b db d\varepsilon d\vec{v} d\vec{v}_1 \,. \tag{3.7}$$

In order to perform the integration over $d\varepsilon$ (recall that ε specifies the "<u>collision plane</u>," and is not to be confused with the "<u>diffusion Mach number</u>," $|\vec{\varepsilon}|$, which will appear later in this section), we must express $(g_k - g_k')$ in terms of ε; this is readily accomplished by means of a coordinate transformation. First, we suppose that $\vec{g}$ and $\vec{g}'$ refer to a rectangular coordinate system with unit vectors $\vec{a}_i$, $i = 1,2,3$; next we construct a <u>local</u> rectangular coordinate system with unit vectors $\vec{a}_x$, $\vec{a}_y$, $\vec{a}_z$, such that $\vec{g}$ lies along $\vec{a}_z$ (see Figure 1).

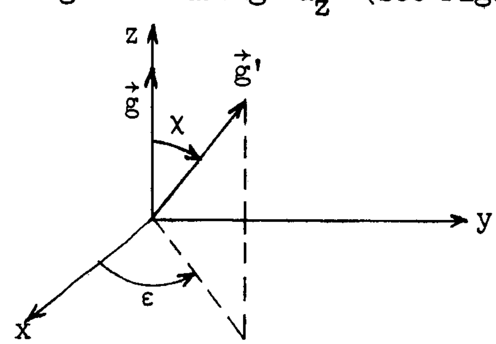

FIG. 1. LOCAL COORDINATE SYSTEM.

From Figure 1 we have

$$\vec{g} = g\vec{a}_z \ , \ \vec{g}' = g[\sin\chi(\vec{a}_x\cos\varepsilon + \vec{a}_y\sin\varepsilon) + \vec{a}_z\cos\chi]$$

where χ is the "scattering angle," and where we have noted that $g = g'$ (see reference [29]). The transformation from (1,2,3) to (g,χ,ε) coordinates then gives us

$$g_k = \vec{g}\cdot\vec{a}_k = g\alpha_{zk} \tag{3.8a}$$

$$g'_k = \vec{g}'\cdot\vec{a}_k = g[\sin\chi(\alpha_{xk}\cos\varepsilon + \alpha_{yk}\sin\varepsilon) + \alpha_{zk}\cos\chi] \ , \tag{3.8b}$$

where $k = 1,2,3$, and α_{xk} is the direction cosine between the x and k – axes, $\vec{a}_x\cdot\vec{a}_k$, etc. Substituting (3.8a,b) into (3.7) and performing the integration over $d\varepsilon$ then gives us

$$[\delta(m_s c_{sk})]_{st} = \mu \iiint 2\pi(1-\cos\chi) b\,db F_s F_{t1} g g_k d\vec{v} d\vec{v}_1 \ . \tag{3.9}$$

At this point we introduce a general collision cross section[30]

$$S^{(\ell)} \equiv 2\pi \int (1-\cos^{\ell}\chi) b\,db \ > 0 \ , \quad \ell = 1,2,\dots \ , \tag{3.10}$$

where the limits on b (the impact parameter) are usually taken to be $(0, \infty)$. Since the scattering angle χ depends upon the magnitude of the relative velocity, g (for central force laws), upon the form of the force law, and upon the impact parameter, b , we see from (3.10) that the cross section $S^{(\ell)}$ depends, for a given central force law, upon the magnitude of the relative velocity, i.e. $S^{(\ell)} = S^{(\ell)}(g)$. We shall make use of this "isotropy" of the cross sections in the calculations of Section 3.3.* Substitution of (3.10) into (3.9) gives us

$$[\delta(m_s c_{sk})]_{st} = \mu \iint F_s F_{t1} g g_k S^{(1)}(g) d\vec{v} d\vec{v}_1 \ . \tag{3.11}$$

This is a well-known result and may be found in various sources (see references [22], [31], for example).

To proceed with the integrations we now make a succession of coordinate transformations. First, from (3.5c,d) we have

$$dc_{oi} dg_i = |J| dv_i dv_{1i} \quad \text{(no summation here)}$$

$$\text{where} \quad J \equiv \det \begin{bmatrix} \dfrac{\partial c_{oi}}{\partial v_i} & \dfrac{\partial g_i}{\partial v_i} \\[2ex] \dfrac{\partial c_{oi}}{\partial v_{1i}} & \dfrac{\partial g_i}{\partial v_{1i}} \end{bmatrix} = \det \begin{bmatrix} \dfrac{m_s}{m_o} & -1 \\[2ex] \dfrac{m_t}{m_o} & +1 \end{bmatrix} = 1 \ ,$$

---

*These calculations also hold for the larger class of cross sections in which $S^{(\ell)}$ is an <u>even and symmetric function</u> of the three variables (g_1, g_2, g_3) , i.e. $S^{(\ell)}(g_1, g_2, g_3) \equiv S^{(\ell)}(\pm g_i, \pm g_j, \pm g_k)$ where i, j, k is any permutation of 1, 2, 3 ; however, we shall write $S^{(\ell)} = S^{(\ell)}(g)$ for brevity, keeping in mind that the results of Section 3.3 also hold for $S^{(\ell)}(g_1, g_2, g_3) \equiv S^{(\ell)}(\pm g_i, \pm g_j, \pm g_k)$.

so that $dc_{oi}dg_i = dv_i dv_{1i}$ (no summation here), and thus,

$$d\vec{c}_o d\vec{g} = d\vec{v}d\vec{v}_1 \quad . \tag{3.12}$$

Substitution of (3.12) into (3.11) gives us

$$[\delta(m_s c_{sk})]_{st} = \mu \iint F_s F_{t1} g g_k S^{(1)}(g) d\vec{c}_o d\vec{g} \quad . \tag{3.13}$$

Substituting the approximate velocity-space solutions for F_s, F_{t1}, (2.45), into (3.13) we obtain

$$[\delta(m_s c_{sk})]_{st} = \frac{\mu N_s N_t}{(\pi a_s a_t)^3} \iint e^{-\left(\frac{c_s^2}{a_s^2} + \frac{c_t^2}{a_t^2}\right)} (1+\phi_s+\phi_{t1}) g g_k S^{(1)}(g) d\vec{c}_o d\vec{g} \ , \tag{3.14}$$

where we have neglected the term involving $\phi_s \phi_{t1}$ in accordance with the discussion of Section 2.4. A second transformation of integration variables is now made; first, solving (3.5c,d) for $\vec{v}$, $\vec{v}_1$, we obtain

$$\vec{c}_s \equiv \vec{v} - \vec{u}_s = \vec{c}_o - (m_t/m_o)\vec{g} - \vec{u}_s \tag{3.15a}$$

and

$$\vec{c}_t \equiv \vec{v}_1 - \vec{u}_t = \vec{c}_o + (m_s/m_o)\vec{g} - \vec{u}_t \quad . \tag{3.15b}$$

Introducing the velocity

$$\tilde{\vec{c}}_o \equiv \vec{c}_o - \{\vec{g}[(m_t/m_o)a_t^2 - (m_s/m_o)a_s^2] + (\vec{u}_s a_t^2 + \vec{u}_t a_s^2)\} a_o^{-2} \ , \tag{3.16}$$

we obtain from (3.15a,b)

$$\vec{c}_s = \tilde{\vec{c}}_o - (a_s^2/a_o)(\vec{y} - \vec{\varepsilon}) \qquad (3.17a)$$

and

$$\vec{c}_t = \tilde{\vec{c}}_o + (a_t^2/a_o)(\vec{y} - \vec{\varepsilon}) \qquad (3.17b)$$

where $\vec{y} \equiv \vec{g}/a_o$ (3.18a)

$$\vec{\varepsilon} \equiv (\vec{u}_t - \vec{u}_s)/a_o \qquad (3.18b)$$

and $a_o \equiv (a_s^2 + a_t^2)^{1/2}$. (3.18c)

From (3.16), (3.18a) we have

$$d\vec{c}_o d\vec{g} = a_o^3 \, d\tilde{\vec{c}}_o \, d\vec{y} \quad , \qquad (3.19)$$

since $\vec{c}_o$ and $\vec{g}$ in (3.16) are to be treated as independent variables (see definitions (3.5c,d)). Substitution of (3.17a,b), (3.18a), (3.19) into (3.14) gives us

$$[\delta(m_s c_{sk})]_{st} = \frac{\mu N_s N_t}{\pi^3} \frac{a_o^2}{a_\mu^3} \iint d\tilde{\vec{c}}_o d\vec{y} \; y \, y_k \, S^{(1)}(a_o y) \cdot$$

$$\cdot \, e^{-\left[\frac{\tilde{c}_o^2}{a_\mu^2} + (\vec{y}-\vec{\varepsilon})^2\right]} \left\{ 1 + \frac{P_{sij}}{p_s} \frac{c_{si} c_{sj}}{a_s^2} - \frac{4q_{si}}{\rho_s a_s^4} \left(1 - \frac{2}{5} \frac{c_s^2}{a_s^2}\right) c_{si} + \right.$$

$$\left. + \frac{P_{tij}}{p_t} \frac{c_{ti} c_{tj}}{a_t^2} - \frac{4q_{ti}}{\rho_t a_t^4} \left(1 - \frac{2}{5} \frac{c_t^2}{a_t^2}\right) c_{ti} \right\} \quad , \qquad (3.20)$$

where $a_\mu \equiv a_s a_t/a_o = a_s a_t/(a_s^2 + a_t^2)^{1/2}$, (3.21)

and where we have written out the term $(1+\phi_s+\phi_{t1})$ in full (see (2.46)). The integration over $d\tilde{\vec{c}}_o$ in (3.20) can be performed directly; we have the following integrals (see Appendix A for the major steps involved):

$$\text{(i)} \qquad \int d\tilde{\vec{c}}_o \, e^{-\tilde{c}_o^2/a_\mu^2} = \pi^{3/2} a_\mu^3$$

$$\text{(ii)} \qquad \int d\tilde{\vec{c}}_o \, c_{si} \, e^{-\tilde{c}_o^2/a_\mu^2} = -\pi^{3/2} a_s^2 \, a_\mu^3 \, a_o^{-1}(y_i-\varepsilon_i)$$

$$\text{(iii)} \qquad \int d\tilde{\vec{c}}_o \, c_{si} c_{sj} \, e^{-\tilde{c}_o^2/a_\mu^2} = \pi^{3/2} \, a_\mu^3 [\tfrac{1}{2} a_\mu^2 \delta_{ij} + a_s^4 a_o^{-2}(y_i-\varepsilon_i)(y_j-\varepsilon_j)]$$

$$\text{(iv)} \qquad \int d\tilde{\vec{c}}_o \, c_s^2 c_{si} \, e^{-\tilde{c}_o^2/a_\mu^2} = -\pi^{3/2} a_\mu^3 a_s^2 a_o^{-1} [\tfrac{5}{2} a_\mu^2 + a_s^4 a_o^{-2} (\vec{y}-\vec{\varepsilon})^2](y_i-\varepsilon_i).$$

For the same integrals (i) - (iv) with "t" instead of "s" we may simply replace $(-a_s^2)$ in the results by $(+a_t^2)$ in view of the equations for $\vec{c}_s$, $\vec{c}_t$, (3.17a,b). Using (i) - (iv) to perform the $d\tilde{\vec{c}}_o$ integration in (3.20) we obtain, after considerable manipulation,

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{y}\; y\, y_k\, S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{1 +$$

$$+E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j) + R_i(y_i-\varepsilon_i)[1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\} \;, \quad (3.22)$$

where $$C_{st} \equiv \mu N_s N_t a_o^2\, \pi^{-3/2} \quad (3.22a)$$

$$E_{ij} \equiv 2a_o^{-2}\left(\frac{P_{sij}}{\rho_s} + \frac{P_{tij}}{\rho_t}\right) \quad (3.22b)$$

and $$R_i \equiv 4a_o^{-3}\left(\frac{q_{si}}{\rho_s} - \frac{q_{ti}}{\rho_t}\right) \;. \quad (3.22c)$$

Expression (3.22) gives the <u>partial momentum collision integral</u> as a function of the dimensionless velocity $\vec{\varepsilon}$, whose magnitude, ε , we shall refer to as the <u>diffusion Mach number</u>, following Morse's[10] nomenclature. As can be seen from (3.18b,c), $\vec{\varepsilon}$ is the ratio of the difference in the species' flow velocities, $(\vec{u}_t - \vec{u}_s)$, to the "mixed sound speed," $a_o = (a_s^2 + a_t^2)^{1/2}$.

The results for the partial random kinetic energy collision integral (which we shall simply call the partial energy collision integral henceforth), the partial pressure collision integral, and the partial heat flow collision integral are taken from Appendices B and C ,

<u>Energy</u>: $$[\delta(\tfrac{1}{2} m_s c_s^2)]_{st} = a_o C_{st} \int d\vec{y}\; y y_j S^{(1)}(a_o y) e^{-(\vec{y}-\varepsilon)^2} \{F_{ij}(y_i-\varepsilon_i) +$$

$$+ S_j[1 - \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2] - \tfrac{4}{5} S_i(y_i-\varepsilon_i)(y_j-\varepsilon_j) +$$

$$+(ay_j + b\varepsilon_j)[1+E_{ip}(y_i-\varepsilon_i)(y_p-\varepsilon_p)+R_i(y_i-\varepsilon_i)(1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2)]\} \quad (3.23)$$

<u>Pressure</u>: $$[\delta(m_s c_{sj} c_{sk})]_{st} = a_o C_{st} \int d\vec{y}\, y\, e^{-(\vec{y}-\vec{\varepsilon})^2}[2S^{(1)}(a_o y)y_j^{\dagger} \cdot$$

$$\cdot\{F_{ik}(y_i-\varepsilon_i)+ S_k[1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2] - \tfrac{4}{5} S_i(y_i-\varepsilon_i)(y_k-\varepsilon_k) +$$

$$+(ay_k+b\varepsilon_k)[1+E_{ip}(y_i-\varepsilon_i)(y_p-\varepsilon_p)+R_i(y_i-\varepsilon_i)(1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2)]\}^{\dagger} +$$

$$+ \frac{m_t}{2m_o} S^{(2)}(a_o y)(y^2\delta_{jk} - 3y_j y_k)\{1+E_{ip}(y_i-\varepsilon_i)(y_p-\varepsilon_p) +$$

$$+ R_i(y_i-\varepsilon_i)[1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\}] \quad (3.24)$$

<u>Heat Flow</u>: $$[\delta(\tfrac{1}{2} m_s c_s^2\, c_{sk})]_{st} = \tfrac{1}{2} a_o^2\, C_{st} \int d\vec{y}\, y\, S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2}\{ \ldots$$

$$[2y_p(ay_p+b\varepsilon_p)(ay_k+b\varepsilon_k)+ y_k((a\vec{y}+b\vec{\varepsilon})^2+ \frac{m_t^2}{m_o^2} y^2)]\cdot[1 +$$

$$+ E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j)+ R_i(y_i-\varepsilon_i)(1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2)] +$$

$$+ 2[F_{ij}(y_i-\varepsilon_i)+S_j(1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2) - \tfrac{4}{5} S_i(y_i-\varepsilon_i)(y_j-\varepsilon_j)]\cdot[2ay_j y_k +$$

$$+ 2b(y_j\varepsilon_k)^{\dagger}+(ay^2+ bx_p\varepsilon_p)\delta_{jk}] + \frac{a_\mu^2}{a_o^2}[\tfrac{5}{2} y_k + y_i G_{ik} +$$

$$+ \tfrac{5}{2} E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j)y_k + \tfrac{5}{2} R_i(y_i-\varepsilon_i)y_k(1-\tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2) +$$

$$+ \frac{1}{5} H_i(2\delta_{ik}y_j\ (y_j-\varepsilon_j) + 9y_iy_k - 7y_k\varepsilon_i - 2y_i\varepsilon_k)]\} \ +$$

$$+ \frac{1}{2}\frac{m_t}{m_o} a_o^2\ C_{st}\int d\vec{y}\ y(y^2\delta_{kp} - 3y_ky_p)S^{(2)}(a_oy)e^{-(\vec{y}-\vec{\varepsilon})^2}\{F_{ip}(y_i-\varepsilon_i) +$$

$$+ S_p[1 - \frac{2}{5}(y-\vec{\varepsilon})^2] - \frac{4}{5} S_i(y_i-\varepsilon_i)(y_p-\varepsilon_p) +$$

$$+(ay_p+b\varepsilon_p)[1+ E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j) + R_i(y_i-\varepsilon_i)(1-\frac{2}{5}(\vec{y}-\vec{\varepsilon})^2)]\} \ , \tag{3.25}$$

where
$$a \equiv (m_ta_t^2 - m_sa_s^2)/m_oa_o^2 = \frac{2K}{m_oa_o^2}(T_t - T_s) \tag{3.26a}$$

$$b \equiv a_s^2/a_o^2 \quad , \quad (a+b = m_t/m_o) \tag{3.26b}$$

$$F_{ij} \equiv (a_\mu^2/a_o^2)(\frac{P_{tij}}{p_t} - \frac{P_{sij}}{p_s}) \quad , \quad (F_{ii} \equiv 0) \tag{3.26c}$$

$$S_i \equiv -(a_\mu^2/a_o^3)(\frac{q_{si}}{p_s} + \frac{q_{ti}}{p_t}) \tag{3.26d}$$

$$G_{ij} \equiv (a_\mu^2/2)(\frac{\rho_s}{p_s^2} P_{sij} + \frac{\rho_t}{p_t^2} P_{tij}) \quad , \quad (G_{ii} \equiv 0) \tag{3.26e}$$

$$H_i \equiv (a_\mu^2/a_o)(\frac{\rho_t}{p_t^2} q_{ti} - \frac{\rho_s}{p_s^2} q_{si}) \quad , \tag{3.26f}$$

and where $y_j^\dagger A_k^\dagger = \frac{1}{2}(y_jA_k + y_kA_j)$.

It is to be noted that the partial energy collision integral (3.23) is simply one-half the trace of the partial pressure collision integral (3.24); the partial <u>traceless</u> pressure collision integral is obtained from (3.23), (3.24),

$$[\delta(m_s c_{sj} c_{sk} - \frac{1}{3}\delta_{jk} m_s c_s^2)]_{st} = [\delta(m_s c_{sj} c_{sk})]_{st} - \frac{2}{3}\delta_{jk}[\delta(\frac{1}{2} m_s c_s^2)]_{st} \ .$$

We see from inspection of the results (3.22) - (3.25) that the final integration is, in general, untractable. In order to proceed, we shall consider two limiting cases: (i) $\varepsilon << 1$, (ii) $\varepsilon >> 1$. In case (i) we consider general central force laws (i.e. isotropic collision cross sections); in case (ii) we consider general inverse power force laws. First, however, we shall derive a relationship between the partial momentum and energy collision integrals which is valid for all ε ; the result will afford us some physical insight concerning the transfer of energy between species.

3.2 Relationship Between the Partial Momentum and Energy Collision Integrals

The partial momentum collision integral is

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{y}\, y\, y_k S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{1 +$$

$$+ E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j) + R_i(y_i-\varepsilon_i)[1-\frac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\} \ . \qquad (3.27)$$

We now rewrite the partial energy collision integral (3.23) in a form which resembles (3.27),

$$[\delta(\frac{1}{2} m_s c_s^2)]_{st} = a_o b \varepsilon_k C_{st} \int d\vec{y}\, y\, y_k S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{1 +$$

$$+ E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j)+ R_i(y_i-\varepsilon_i)[1- \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\} +$$

$$+ aa_oC_{st} \int d\vec{y}\ y^3S^{(1)}(a_oy)e^{-(\vec{y}-\vec{\varepsilon})^2}\{1+E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j) +$$

$$+ R_i(y_i-\varepsilon_i)[1- \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\}+a_oC_{st} \int d\vec{y}\ y\ y_kS^{(1)}(a_oy)e^{-(\vec{y}-\vec{\varepsilon})^2}\{\cdots$$

$$F_{ik}(y_i-\varepsilon_i)+S_k[1- \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]- \tfrac{4}{5}\, S_i(y_i-\varepsilon_i)(y_k-\varepsilon_k)\}$$

$$\equiv I_1+ I_2+ I_3 \text{ , respectively.} \tag{3.28}$$

We see immediately from (3.27), (3.28), that

$$I_1 = a_ob\varepsilon_k[\delta(m_sc_{sk})]_{st} \quad . \tag{3.29}$$

We note further that I_2 may be obtained by differentiating $[\delta(m_sc_{sk})]_{st}$ with respect to ε_k ,

$$I_2 = \frac{a_oa}{2}\ \frac{\partial[\delta(m_sc_{sk})]_{st}}{\partial\varepsilon_k} + a_oa\varepsilon_k[\delta(m_sc_{sk})]_{st} +$$

$$+ \frac{a_oa}{2}\, C_{st} \int d\vec{y}\ y\ y_k\ S^{(1)}(a_oy)e^{-(\vec{y}-\vec{\varepsilon})^2}\{2E_{ik}(y_i-\varepsilon_i) +$$

$$+ R_k[1- \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]- \tfrac{4}{5}\, R_i(y_i-\varepsilon_i)(y_k-\varepsilon_k)\} \quad . \tag{3.30}$$

Substitution of (3.29), (3.30) into (3.28) gives us

$$[\delta(\tfrac{1}{2}\, m_s c_s^2)]_{st} = \frac{a_o a}{2}\, \frac{\partial[\delta(m_s c_{sk})]_{st}}{\partial \varepsilon_k} + a_o\, \frac{m_t}{m_o}\, \varepsilon_k [\delta(m_s c_{sk})]_{st} +$$

$$+\, a_o C_{st} \int d\vec{y}\; y\, y_k S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{(F_{ik}+aE_{ik})(y_i-\varepsilon_i) +$$

$$+(S_k + \tfrac{a}{2} R_k)[1 - \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2] - \tfrac{4}{5}(S_i + \tfrac{a}{2} R_i)(y_i-\varepsilon_i)(y_k-\varepsilon_k)\}\ , \tag{3.31}$$

and substituting the definitions for E_{ij}, R_i, a, F_{ij}, S_i , (3.22b,c), (3.26a,c,d), respectively, we obtain finally

$$[\delta(\tfrac{1}{2}\, m_s c_s^2)]_{st} = \underline{\frac{K}{m_o a_o}\, (T_t-T_s)\, \frac{\partial[\delta(m_s c_{sk})]_{st}}{\partial \varepsilon_k} + \frac{m_t}{m_o}\, (u_{tk} - u_{sk})[\delta(m_s c_{sk})]_{st}} +$$

$$+\, \frac{2C_{st}}{m_o a_o} \int d\vec{y}\; y\, y_k S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{(\frac{P_{tik}}{N_t} - \frac{P_{sik}}{N_s})(y_i-\varepsilon_i) +$$

$$+\, \frac{1}{a_o}\, (\frac{q_{si}}{N_s} + \frac{q_{ti}}{N_t})[\tfrac{4}{5}(y_i-\varepsilon_i)(y_k-\varepsilon_k) - \delta_{ik}(1 - \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2)]\}\ . \tag{3.32}$$

The underscored terms in (3.32) correspond to the case where the species' distribution functions F_s, F_t are Maxwellian; this is the result obtained by Tanenbaum.[11] The first underscored term is proportional to the difference in species' temperatures and represents the flow of energy between species "s" and "t" (that is, the increase or decrease of the random kinetic energy of species "s" due to collisions between the "s" and "t" particles) in the form of heat transfer (not to be confused with the heat flow vectors); the second underscored term is proportional to the difference in species flow velocities and represents the transfer of random kinetic energy between the species due to the "frictional heating" of species "s", caused by the forces acting on it which arise when the flow velocities are unequal. Since the remaining term in (3.32) vanishes identically when either $\vec{\varepsilon} = 0$ (since $P_{sii} \equiv 0$, and the integrands involving the heat flow vectors are odd for $\vec{\varepsilon} = 0$), or when the species are in local equilibrium, $F_s = F_s^{(o)}$, we may consider its contribution as an additional "frictional heating" of species "s" which arises as the species come to local equilibrium with a common flow velocity.

Finally, when $\vec{\varepsilon} = 0$, all the terms in (3.32) vanish except the first underscored term, provided $T_s \neq T_t$, since from (3.32) and (3.23),

$$\left.[\delta(\tfrac{1}{2} m_s c_s^2)]_{st}\right|_{\varepsilon=0} = \frac{K}{m_o a_o}(T_t - T_s) \left.\frac{\partial[\delta(m_s c_{sk})]_{st}}{\partial \varepsilon_k}\right|_{\varepsilon=0}$$

$$= a a_o C_{st} \int d\vec{y}\, y^3 S^{(1)}(a_o y) e^{-y^2}$$

$$= \frac{8\pi}{m_o a_o} C_{st} K(T_t - T_s) \int_0^\infty dy\, y^5 S^{(1)}(a_o y) e^{-y^2} \neq 0 \,,$$

$$\text{if } T_t \neq T_s \,. \tag{3.33}$$

Thus, even when all the species flow velocities are equal, there still exists energy transfer between species of different temperatures.

3.3 Evaluation of Collision Integrals for Small Diffusion Mach Number

In this section we shall evaluate the partial collision integrals (3.22) - (3.25) for the limiting range $\varepsilon << 1$. For convenience we bring forward the partial momentum collision integral, (3.22),

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{y}\; y\; y_k S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{1 + E_{ij}(y_i - \varepsilon_i)(y_j - \varepsilon_j) +$$

$$+ R_i(y_i - \varepsilon_i)[1 - \tfrac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\} \quad . \qquad (3.34)$$

If we consider $[\delta(m_s c_{sk})]_{st}$ as a function of the three independent variables $\varepsilon_i (i = 1,2,3)$, with all other quantities in (3.34) treated as parameters (e.g. P_{sij} , q_{si} , etc.), then for $\varepsilon << 1$ we may expand (3.34) about $\vec{\varepsilon} = 0$ using Taylor's theorem with remainder

$$[\delta(m_s c_{sk})]_{st} = [\delta(m_s c_{sk})]_{st}\Big|_{\vec{\varepsilon}=0} + \frac{\partial[\delta(m_s c_{sk})]_{st}}{\partial \varepsilon_i}\Big|_{\vec{\varepsilon}=0} \cdot \varepsilon_i +$$

$$+ \frac{\partial^2[\delta(m_s c_{sk})]_{st}}{\partial \varepsilon_i \partial \varepsilon_j}\Big|_{\vec{\varepsilon}=0} \cdot \frac{\varepsilon_i \varepsilon_j}{2!} + O(\varepsilon^3) \quad . \qquad (3.35)$$

All four partial collision integrals have been calculated following this technique; however, the steps involved are extremely tedious and lengthy and we shall not present them here. Instead, we shall work with the direct expansion of the exponential in (3.34) and retain terms up to those proportional to <u>second order in</u> ε_i $(i = 1,2,3)$. For this purpose we rewrite (3.34),

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{y}\; y\; y_k S^{(1)}(a_o y) e^{2y_p\varepsilon_p-\varepsilon^2}\; e^{-y^2}\{1 +$$

$$+E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j)+ R_i(y_i-\varepsilon_i)[1- \tfrac{2}{5}(y^2- 2y_j\varepsilon_j+\varepsilon^2)]\} \;. \tag{3.36}$$

Then expanding the factor $\exp(2y_p\varepsilon_p-\varepsilon^2)$, multiplying out, and retaining terms up to second order in ε_i , we have

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{y}\; y\; y_k S^{(1)}(a_o y) e^{-y^2}\{\overset{\circ}{1}+E_{ij}(\overset{\circ}{y_iy_j}-y_i\varepsilon_j-y_j\varepsilon_i+\overset{\circ}{\varepsilon_i\varepsilon_j}) +$$

$$+R_iy_i[1- \tfrac{2}{5}(y^2- \overset{\circ}{2y_j\varepsilon_j}+\varepsilon^2)]- R_i\varepsilon_i[\overset{\circ}{1}- \tfrac{2}{5}(\overset{\circ}{y^2}- 2y_j\varepsilon_j)] +$$

$$+2y_p\varepsilon_p+ 2E_{ij}y_iy_jy_p\varepsilon_p- \overset{\circ}{4E_{ij}y_i\varepsilon_jy_p\varepsilon_p} +$$

$$+2R_iy_iy_p\varepsilon_p[\overset{\circ}{1}- \tfrac{2}{5}(\overset{\circ}{y^2}- 2y_j\varepsilon_j)]-2R_i\varepsilon_iy_p\varepsilon_p(1- \tfrac{2}{5}\, y^2) +$$

$$+\overset{\circ}{2y_p\varepsilon_py_q\varepsilon_q}+\overset{\circ}{2y_p\varepsilon_py_q\varepsilon_qE_{ij}y_iy_j}+2y_p\varepsilon_py_q\varepsilon_qR_iy_i(1- \tfrac{2}{5}\, y^2) -$$

$$-\varepsilon^2[\overset{\circ}{1}+\overset{\circ}{E_{ij}y_iy_j}+ R_iy_i(1- \tfrac{2}{5}\, y^2)]\} \quad , \tag{3.37}$$

where the symbol "o" above a term indicates an odd integrand whose integral vanishes (recall that $\int d\vec{y}\, h(y) \equiv \iiint_{-\infty}^{\infty} dy_1 dy_2 dy_3 h(\vec{y})$). Before proceding we should note that the convergence of the series of integrals resulting from the expansion of $\exp(2y_p\varepsilon_p-\varepsilon^2)$,

$$\exp(2y_p\varepsilon_p-\varepsilon^2) = (1+ 2y_p\varepsilon_p+ \frac{4}{2!} y_p\varepsilon_p y_q\varepsilon_q+ \ldots)(1-\varepsilon^2 + \ldots) , \quad (3.38)$$

is assured by the presence of the factor $\exp(-y^2)$ in (3.36); that is, the major contribution to the complete integral (3.36) and to each of the integrals in the resulting series (3.37) comes from the neighborhood of $\vec{y} = 0$. Hence, we need not be concerned about the possible appearance of large y in the expansion (3.38). In addition, the results of the expansion (3.37) agree exactly with those found from the Taylor expansion (3.35). The remaining integrations in (3.37) are all straightforward provided the "cross section for momentum transfer," $S^{(1)}(a_o y)$, is a reasonably simple function of $a_o y$. For the moment we do not specify $S^{(1)}(a_o y)$, but following the notation of Burgers,[32] let

$$\pi^{3/2} Z(\ell,j) \equiv \int d\vec{y}\, e^{-y^2} y^{2j+1} S^{(\ell)}(a_o y) = 4\pi \int_0^{\infty} dy\, e^{-y^2} y^{2j+3} S^{(\ell)}(a_o y) > 0 \quad (3.39a)$$

$$z \equiv 1- \frac{2}{5} \frac{Z^{(1,2)}}{Z^{(1,1)}} \quad (3.39b)$$

$$z' \equiv 1- \frac{4}{35} \frac{Z^{(1,3)}}{Z^{(1,1)}} \quad (3.39c)$$

$$\zeta \equiv 1 + 5z - \frac{7}{2} z' \tag{3.39d}$$

$$\hat{z} \equiv z - z' \tag{3.39e}$$

$$z'' \equiv 1 - \frac{8}{315} \frac{Z^{(1,4)}}{Z^{(1,1)}} \tag{3.39f}$$

$$z^{(2)} \equiv 1 - \frac{2}{5} \frac{Z^{(2,2)}}{Z^{(1,1)}} \tag{3.39g}$$

$$z'^{(2)} \equiv 1 - \frac{4}{35} \frac{Z^{(2,3)}}{Z^{(1,1)}} \tag{3.39h}$$

$$\hat{z}^{(2)} \equiv z^{(2)} - z'^{(2)} \tag{3.39i}$$

$$z''^{(2)} \equiv 1 - \frac{8}{315} \frac{Z^{(2,4)}}{Z^{(1,1)}} \text{ .}^{*} \tag{3.39j}$$

From Appendix A we have, in conjunction with (3.39a),

$$\int d\vec{y}\, e^{-y^2} y^{2j-1} y_p y_q S^{(\ell)}(a_o y) = \frac{\pi^{3/2}}{3} \delta_{pq} Z^{(\ell,j)} \tag{3.40a}$$

and $$\int d\vec{y}\, e^{-y^2} y^{2j+3} y_p y_q y_r y_s S^{(\ell)}(a_o y) = \frac{\pi^{3/2}}{15}(\delta_{pq}\delta_{rs}+\delta_{pr}\delta_{qs}+\delta_{ps}\delta_{qr}) Z^{(\ell,j)} \text{ .} \tag{3.40b}$$

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*Expressions (3.39a-d) are those used by Burgers; expressions (3-39e-j) are peculiar to this dissertation. Also, Burgers attaches the subscripts "st" to the quantities (3.39a-d) as well as to the cross sections $S^{(\ell)}(a_o y)$; we do not require this distinction inasmuch as we are working with <u>partial</u> collision integrals. The quantities (3.39a-j) are tabulated in Appendix A for various interparticle force laws.

Then using (3.39a-c), (3.40a,b) to perform the remaining integrations in (3.37), we obtain, after collecting terms

$$[\delta(m_s c_{sk})]_{st} = \frac{\pi^{3/2}}{3} C_{st} Z^{(1,1)} \{2\varepsilon_k - 2z\varepsilon_i E_{ik} +$$

$$+ \frac{14}{5}(z' - 2z)\varepsilon_k \varepsilon_i R_i + [z(1-\varepsilon^2) + \frac{\varepsilon^2}{5}(7z' - 9z)]R_k\} \ . \quad (3.41)$$

Substituting the definitions for E_{ik} and R_i , (3.22b,c), we obtain

$$[\delta(m_s c_{sk})]_{st} = \mu N_s \nu_{st} \{(u_{tk} - u_{sk}) - 2z \frac{\varepsilon_i}{a_o} (\frac{P_{sik}}{\rho_s} + \frac{P_{tik}}{\rho_t}) +$$

$$+ \frac{2}{a_o^2}[(z + \frac{7}{5}(z' - 2z)\varepsilon^2)\delta_{ik} + \frac{14}{5}(z' - 2z)\varepsilon_i \varepsilon_k](\frac{q_{si}}{\rho_s} - \frac{q_{ti}}{\rho_t})\} \ ,$$

(3.42)

where we have defined a <u>collision frequency for small ε</u>

$$\nu_{st} \equiv \frac{2}{3} \pi^{3/2} \frac{C_{st}}{\mu N_s a_o} Z^{(1,1)} = \frac{2}{3} N_t a_o Z^{(1,1)} > 0 \ , \quad (3.42a)$$

with $N_s \nu_{st} = N_t \nu_{ts}$ (3.42b)

since $S_{st}^{(\ell)} \equiv S_{ts}^{(\ell)}$ (see (3.40a)). It should be mentioned that many authors work with an "<u>effective collision frequency for transfer of momentum between species "s" and "t"</u> ,

$$\nu_{st}^{M} = (m_t/m_o)\nu_{st} \tag{3.42c}$$

with

$$\rho_s\nu_{st}^{M} = \rho_t\nu_{ts}^{M} \quad . \tag{3.42d}$$

The collision frequencies (3.42a) are exhibited in Section 3.6 for various interparticle force laws. It must be emphasized that this form for the collision frequencies is only valid for the range $\varepsilon << 1$; in this range ν_{st} is, in general, <u>temperature dependent and independent of the species' flow velocities</u>. We shall see in the next section that the converse is true for the range $\varepsilon >> 1$.

The results for the other three partial collision integrals are taken from Appendices B and C,

<u>Energy</u>: $$[\delta(\tfrac{1}{2} m_s c_s^2)]_{st} = \frac{2\mu}{m_o} N_s\nu_{st}\{\frac{m_t}{2}(\vec{u}_t - \vec{u}_s)^2 + \frac{3}{2} K(T_t - T_s)(1 - \frac{5}{3} z\varepsilon^2) +$$
$$+2z \frac{K^2 T_s T_t}{\mu a_o^2} \varepsilon_i\varepsilon_j(\frac{P_{sij}}{p_s} - \frac{P_{tij}}{p_t}) + \frac{2K}{a_o^2}[(\zeta - 1)(T_t - T_s) -$$
$$-z\frac{m_o}{m_s} T_s]\varepsilon_i\varepsilon_j(\frac{P_{sij}}{\rho_s} + \frac{P_{tij}}{\rho_t}) + \frac{2K}{a_o^3}[2(1-\zeta)(T_t - T_s) + z\frac{m_o}{m_s} T_s]\varepsilon_i \cdot$$
$$(\frac{q_{si}}{\rho_s} - \frac{q_{ti}}{\rho_t}) - 6z\frac{K^2 T_s T_t}{\mu a_o^3}\varepsilon_i(\frac{q_{si}}{p_s} + \frac{q_{ti}}{p_t})\} \tag{3.43}$$

<u>Pressure</u>: $$[\delta(m_s c_{sj} c_{sk})]_{st} = 2\frac{\mu}{m_o} N_s\nu_{st}\{\frac{m_t}{4}(u_{tp} - u_{sp})(u_{tq} - u_{sq}) \cdot$$
$$\cdot [(1 + 3z^{(2)})\delta_{jp}\delta_{kq} +$$

$$+(1-z^{(2)})\delta_{pq}\delta_{jk}]+K(T_t-T_s)[(1-z\varepsilon^2)\delta_{jk}-2z\varepsilon_j\varepsilon_k] +$$

$$+\frac{2K^2T_sT_t}{\mu a_o^2}[(1-z\varepsilon^2)\delta_{jp}\delta_{kq}-2z(\varepsilon_j\varepsilon_p\delta_{kq})^{\dagger}](\frac{P_{tpq}}{p_t}-\frac{P_{spq}}{p_s}) +$$

$$+\{\delta_{jp}\delta_{kq}[(2K/a_o^2)(T_t-T_s)(1-z+\hat{z}\varepsilon^2)-(3/4)m_t(1-z^{(2)}+\hat{z}^{(2)}\varepsilon^2)] +$$

$$+2(\varepsilon_j\varepsilon_p\delta_{kq})^{\dagger}[(4K/a_o^2)(T_t-T_s)\hat{z}-2z(KT_sm_o/m_sa_o^2)-(3/2)m_t\hat{z}^{(2)}] +$$

$$+\varepsilon_p\varepsilon_q\delta_{jk}[(2K/a_o^2)(T_t-T_s)(2z-z')+m_t\hat{z}^{(2)}]\}(\frac{P_{spq}}{\rho_s}+\frac{P_{tpq}}{\rho_t}) -$$

$$-\frac{2}{5}z\frac{K^2T_sT_t}{\mu a_o^3}[18(\varepsilon_j\delta_{ki})^{\dagger}+4\varepsilon_i\delta_{jk}](\frac{q_{si}}{p_s}+\frac{q_{ti}}{p_t}) +$$

$$+\frac{2}{a_o^3}[2(\varepsilon_j\delta_{ki})^{\dagger}(\frac{m_o}{m_s}zKT_s-\frac{14}{5}K(T_t-T_s)\hat{z}+\frac{21}{20}a_o^2m_t\hat{z}^{(2)}) +$$

$$+\frac{7}{10}\varepsilon_i\delta_{jk}(4K(T_t-T_s)(z'-\frac{12}{7}z)-a_o^2m_t\hat{z}^{(2)})](\frac{q_{si}}{\rho_s}-\frac{q_{ti}}{\rho_t})\} \quad (3.44)$$

Heat Flow: $$[\delta(\tfrac{1}{2}m_sc_s^2c_{sk})]_{st}=\frac{1}{20}\frac{\mu}{m_o}N_s\nu_{st}\{25(u_{tk}-u_{sk})[4\frac{m_t}{m_o}K(T_t-T_s)z^{(2)}+$$

$$+\frac{4K^2T_sT_t}{\mu a_o^2}+\frac{4K^2}{m_oa_o^2}(T_t-T_s)^2(1-3z)+\frac{m_t^2}{m_o}a_o^2(1-z)] +$$

$$+80\frac{(K^2T_sT_t)^2}{\mu a_o^3m_sm_t}\varepsilon_j(\frac{\rho_s}{p_s^2}P_{sjk}+\frac{\rho_t}{p_t^2}P_{tjk}) +$$

$$+ 10[\frac{4K^2}{m_o a_o^3}(T_t-T_s)^2(2+19z-21z')+2\frac{K}{a_o}\frac{m_t}{m_o}(T_t-T_s)(3-4z-13z^{(2)}+14z'^{(2)})-$$

$$-\frac{20K^2}{\mu a_o^3} zT_sT_t+\frac{m_t^2}{m_o} a_o(5z-7z'-1+3z^{(2)})]\varepsilon_j(\frac{P_{sjk}}{\rho_s}+\frac{P_{tjk}}{\rho_t}) +$$

$$+ 40\frac{K^2T_sT_t}{a_o m_s m_t}[\frac{2K}{a_o^2}(T_t-T_s)(2-9z)+\frac{m_t}{2}(3+z^{(2)})]\varepsilon_j(\frac{P_{tjk}}{p_t}-\frac{P_{sjk}}{p_s}) -$$

$$-8\frac{K^2T_sT_t}{a_o^2 m_s m_t}\{5\delta_{jk}[\frac{2K}{a_o^2}(T_t-T_s)(11z-6+\frac{7}{5}\varepsilon^2(13z'-\frac{144}{7}z)) +$$

$$+ 2m_t(1-z^{(2)}+\frac{7}{10}\varepsilon^2(\hat{z}^{(2)}+\frac{18}{7}z))]+14\varepsilon_j\varepsilon_k[4\frac{K}{a_o^2}(T_t-T_s)(4z'-\frac{47}{7}z) +$$

$$+ m_t(\frac{7}{2}\hat{z}^{(2)}+4z)]\}(\frac{q_{sj}}{p_s}+\frac{q_{tj}}{p_t}) +$$

$$+ 4\{\delta_{jk}[\frac{30K^2}{m_o a_o^4}(T_t-T_s)^2(7z'-5z-2+\varepsilon^2(189z''-350z'+171z)) +$$

$$+ 10\frac{K}{a_o^2}\frac{m_t}{m_o}(T_t-T_s)(2+5z^{(2)}-7z'^{(2)}+\frac{7}{5}\varepsilon^2(17z'^{(2)}-9z''^{(2)}-8z^{(2)} +$$

$$+ 4z'-\frac{38}{7}z))+50\frac{K^2T_sT_t}{\mu a_o^4}(z+\frac{7}{5}\varepsilon^2(z'-2z))+\frac{5}{2}\frac{m_t^2}{m_o}(7z'-5z-2 +$$

$$+\frac{\varepsilon^2}{5}(63z''-98z'+45z+42\hat{z}^{(2)}))]+ 14\varepsilon_j\varepsilon_k[10\frac{K^2T_sT_t}{\mu a_o^4}(z'-2z) +$$

$$+\frac{2K^2}{m_o a_o^4}(T_t-T_s)^2(27z''-60z'+\frac{291}{7}z)+\frac{m_t}{m_o}\frac{K}{a_o^2}(T_t-T_s)(18z'-\frac{196}{7}z +$$

$$+ 29z'^{(2)} - 11z^{(2)} - 18z''^{(2)}) + \frac{m_t^2}{2m_o}(\hat{z}^{(2)} + 9z'' - 14z' + \frac{45}{7}z)]\}(\frac{q_{sj}}{\rho_s} - \frac{q_{tj}}{\rho_t}) +$$

$$+ \frac{(4K^2T_sT_t)^2}{\mu m_s m_t a_o^4}[18z\varepsilon_j\varepsilon_k - (15-19z\varepsilon^2)\delta_{jk}](\frac{\rho_s}{p_s^2}q_{sj} - \frac{\rho_t}{p_t^2}q_{tj})\} \quad . \qquad (3.45)$$

In the partial pressure collision integral, (3.44), the symmetrization $(\cdots)^\dagger$, is with respect to the non-repeated indices, j, k. The partial traceless pressure collision integral is of course given by (3.44) minus two-thirds δ_{jk} times (3.43).

The results (3.42) - (3.45) are accurate to second order in $\varepsilon_i (i = 1,2,3)$; they simplify considerably once the interparticle force law is given (i.e. once the "z's" are given). As will be seen in the subsequent calculations of traceless pressures and heat flows in Chapter V, the expressions (3.42) - (3.45) become quite manageable for certain special systems (e.g. weakly ionized gas). It should be noted that in connection with the fully ionized gas system, Everett[33] claims an accuracy of results to third order in ε_i ; however, his results are only to third order in ε_i implicitly, due to assumptions concerning the relation between ε and the electron and ion traceless pressures and heat flows. Our results are accurate to second order in ε_i explicitly.

When the species' distribution functions are Maxwellian, the traceless pressures and heat flows vanish identically, and the partial momentum and energy collision integrals (3.42), (3.43), reduce to Tanenbaum's[11] results

$$[\delta(m_s c_{sk})]_{st} = \mu N_s \nu_{st}(u_{tk} - u_{sk}) \tag{3.46}$$

$$[\delta(\tfrac{1}{2} m_s c_s^2)]_{st} = 2\,\frac{\mu}{m_o} N_s \nu_{st}\Big[\frac{m_t}{2}(\vec{u}_t - \vec{u}_s)^2 + \frac{3}{2} K(T_t - T_s)(1 - \frac{5}{3} z\varepsilon^2)\Big] \,. \tag{3.47}$$

Tanenbaum's results are in terms of the effective collision frequency for transfer of momentum, ν_{st}^{M} , given by (3.42c).

The collision integrals for a <u>simple gas</u> or the "<u>self-partial collision integrals</u>" are obtained simply by setting "t" = "s" (so that $\vec{\varepsilon} = 0$) in (3.42) - (3.45)

$$[\delta(m_s c_{sk})]_{ss} = [\delta(\tfrac{1}{2} m_s c_s^2)]_{ss} = 0 \tag{3.48}$$

$$[\delta(m_s c_{sj} c_{sk})]_{st} = -\frac{3}{10}\,\frac{Z^{(2,2)}}{Z^{(1,1)}}\,\nu_{ss} P_{sjk} \tag{3.49}$$

$$[\delta(\tfrac{1}{2} m_s c_s^2 c_{sk})]_{st} = -\frac{1}{5}\,\frac{Z^{(2,2)}}{Z^{(1,1)}}\,\nu_{ss} q_{sk} \,. \tag{3.50}$$

Finally, if we form the <u>sum over all species</u> of the <u>total</u> momentum collision integrals, we see immediately from (3.42) that $\sum_s \sum_t [\delta(m_s c_{sk})]_{st} \equiv 0$, as must be the case for <u>any</u> level of accuracy in ε since the total momentum of the system cannot change because of collisions (such a change is only possible through <u>external</u> force fields, which are not involved in the collision integrals).* However, for the sum of the total <u>random kinetic energy</u> collision integrals, we find from (3.43) that

$\sum_s \sum_t [\delta(\frac{1}{2} m_s c_s^2)]_{st} \neq 0$, in general. This is not surprising inasmuch as the total <u>random</u> kinetic energy of the system <u>can change</u> because of collisions; that is, part of the system's "ordered" kinetic energy can be transformed into random kinetic energy. As a matter of fact, for the special case of Maxwellian distribution functions, (3.47), we have

$$\sum_s \sum_t [\delta(\tfrac{1}{2} m_s c_s^2)]_{st} = \sum_s \sum_t \mu N_s \nu_{st} \frac{m_t}{m_o} (\vec{u}_t - \vec{u}_s)^2 = \sum_{s<t} \mu N_s \nu_{st} (\vec{u}_t - \vec{u}_s)^2 ,$$

$s, t = 1,2,3, \ldots$, which vanishes <u>if and only if $\vec{u}_t = \vec{u}_s$ for all s, t</u> .

3.4 <u>Evaluation of Collision Integrals for Large Diffusion Mach Number</u>.

In this section the partial collision integrals will be evaluated for the limiting case $\varepsilon \gg 1$ (i.e. at least one of the components ε_i is "much larger" than unity). The partial momentum collision integral is again

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*Note that for <u>any</u> quantity Q which is conserved in a collision, it can easily be shown from the Boltzmann binary collision operator, (2.21), that $\sum_{s,t} (\delta Q)_{st} \equiv 0$.

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{y}\; y\; y_k S^{(1)}(a_o y) e^{-(\vec{y}-\vec{\varepsilon})^2} \{1 + E_{ij}(y_i-\varepsilon_i)(y_j-\varepsilon_j) +$$

$$+ R_i(y_i-\varepsilon_i)[1-\frac{2}{5}(\vec{y}-\vec{\varepsilon})^2]\} \quad . \tag{3.51}$$

Upon introducing the transformation

$\vec{w} \equiv \vec{y} - \vec{\varepsilon}$, $d\vec{w} = d\vec{y}$ (ε finite) ,

expression (3.51) becomes

$$[\delta(m_s c_{sk})]_{st} = C_{st} \int d\vec{w} (\varepsilon^2 + 2w_p\varepsilon_p + w^2)^{1/2} (\varepsilon_k + w_k) S^{(1)}(a_o|\vec{\varepsilon}+\vec{w}|) e^{-w^2} \cdot$$

$$\cdot \{1 + E_{ij}w_i w_j + R_i w_i (1 - \frac{2}{5} w^2)\} \quad . \tag{3.52}$$

We are now in a position to expand $[\delta(m_s c_{sk})]_{st}$ in an "ε-series," in a manner analogous to that for the case of $\varepsilon << 1$; in the present case, however, we expand the term $(\varepsilon^2 + 2w_p\varepsilon_p + w^2)^{1/2}$ and retain only the <u>higher order</u> terms in ε . We see that the collision cross section in (3.52) is a function of $\vec{\varepsilon}$; hence, in order to perform the ε-expansion we must specify the functional dependence of $S^{(\ell)}$ upon ε . For this purpose we consider those cross sections $S^{(\ell)}$ which correspond to <u>inverse power interparticle force laws</u>,

$$f_{st} = \kappa_{st}/r^p \quad , \quad 2 \leq p < \infty \tag{3.53a}$$

where

$$S^{(\ell)}(g) = 2\pi(\kappa_{st}/\mu)^{-n/2} A_\ell(p) g^n \,, \quad (n \neq 0) \tag{3.53b}$$

$$S^{(\ell)}(g) = \frac{\pi}{2}\sigma^2[2-\{1+(-1)^\ell\}/(\ell+1)] \,, \text{ ("hard spheres," } p\to\infty \,, n\to 0^-) \tag{3.53c}$$

$$\text{with } n = -4/(p-1) \,.^* \tag{3.53d}$$

The dimensionless cross section $A_\ell(p)$, typically of order unity, is tabulated by Chapman and Cowling[35] for $\ell = 1,2$ for certain values of p. The quantity "σ" appearing in (3.53c) is the sum of the radii of the colliding "hard spheres." The cross sections $S^{(1)}(g)$, $S^{(2)}(g)$ are given in Table 1 for various interparticle force laws.

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*See reference [34]. Whenever we are dealing with force laws of the type (3.53a) the results of any calculation will be for $n \neq 0$; the corresponding result for "hard spheres" can be obtained by replacing $2\pi A_1(p)$ by $\pi\sigma^2$, and $2\pi A_2(p)$ by $\frac{2}{3}\pi\sigma^2$.

TABLE 1. COLLISION CROSS SECTIONS FOR VARIOUS FORCE LAWS

Force Law	n	$S^{(1)}(g)$	$S^{(2)}(g)$
Hard Spheres	0	$\pi\sigma^2$	$\frac{\pi}{2}\sigma^2$
Maxwell Molecules κ_{st}/r^5	-1	$2\pi(\kappa_{st}/\mu)^{1/2} A_1(5)g^{-1}$	$\frac{A_2(5)}{A_1(5)} S^{(1)}(g)$
κ_{st}/r^3	-2	$2\pi(\kappa_{st}/\mu)A_1(3)g^{-2}$	$\frac{A_2(3)}{A_1(3)} S^{(1)}(g)$
$\kappa_{st}/r^{7/3}$	-3	$2\pi(\kappa_{st}/\mu)^{3/2}A_1(7/3)g^{-3}$	$\frac{A_2(7/3)}{A_1(7/3)} S^{(1)}(g)$
Coulomb,* $\frac{e_s e_t}{4\pi\varepsilon_o}\frac{1}{r^2}$	-4	$2\pi(e_s e_t/4\pi\varepsilon_o\mu)^2 A_1(2)g^{-4}$	$\frac{A_2(2)}{A_1(2)} S^{(1)}(g)$

*ε_o is the permittivity of free space.

In terms of $\vec{\varepsilon}$ and $\vec{w}$ the collision cross sections (3.53b) become

$$S^{(\ell)}(a_o|\vec{\varepsilon}+\vec{w}|) = 2\pi(\kappa_{st}/\mu)^{-\frac{n}{2}} A_\ell(p)a_o^n(\varepsilon^2+2w_p\varepsilon_p+ w^2)^{n/2} \quad , \qquad (3.54)$$

so that (3.52) becomes

$$[\delta(m_s c_{sk})]_{st} = C'_{st}\varepsilon^{n+1}\int d\vec{w}\left[1+\left(\frac{2w_p\varepsilon_p}{\varepsilon^2}+\frac{w^2}{\varepsilon^2}\right)\right]^{\frac{n+1}{2}}(\varepsilon_k+w_k)e^{-w^2}\{1+E_{ij}w_iw_j +$$

$$+ R_iw_i(1-\tfrac{2}{5}w^2)\} \quad , \qquad (3.55)$$

where $C'_{st} \equiv 2\pi(\kappa_{st}/\mu)^{-n/2}\, A_1(p)a_o^n\, C_{st}$. (3.56)

We shall evaluate the integral (3.55) in such a manner that all terms of zero or higher order in ε which occur in the integrand are retained (note that the factor ε^{n+1} does not enter into this consideration). Then expanding the binomial in the integrand of (3.55), multiplying out, retaining only those terms of zero or higher order in ε , and performing the integration, we obtain

$$[\delta(m_s c_{sk})]_{st} = C'_{st}\varepsilon^{n+1}\int d\vec{w}\; e^{-w^2}\left[1+(n+1)\frac{w_p\varepsilon_p}{\varepsilon^2}+\cdots\right](\varepsilon_k+w_k)\{1 +$$

$$+ E_{ij}w_iw_j + R_iw_i(1-\tfrac{2}{5}w^2)\}$$

$$= C'_{st}\varepsilon^{n+1}\int d\vec{w}\; e^{-w^2}\{\varepsilon_k+\varepsilon_kE_{ij}w_iw_j+\varepsilon_k\overset{\circ}{R_i}w_i(1-\tfrac{2}{5}w^2) +$$

$$+ \overset{\circ}{w_k} + \overset{\circ}{E_{ij}}w_iw_jw_k + R_iw_iw_k(1-\tfrac{2}{5}w^2) +$$

$$+ (n+1)\frac{\varepsilon_p\varepsilon_k}{\varepsilon^2}[\overset{\circ}{w_p} + \overset{\circ}{E_{ij}}w_iw_jw_p + R_iw_iw_p(1-\tfrac{2}{5}w^2)]\}$$

$$\text{or,}\quad [\delta(m_s c_{sk})]_{st} = C'_{st}\, \varepsilon^{n+1}\, \pi^{3/2} \varepsilon_k \tag{3.57}$$

where terms with overscore "o" again indicate odd integrands whose integrals vanish. We note that the convergence of the series of integrals resulting from the expansion of the binomial

$$[1 + (\frac{2w_p\varepsilon_p}{\varepsilon^2} + \frac{w^2}{\varepsilon^2})]^{(n+1)/2} = 1 + \frac{(n+1)}{\varepsilon^2}(w_p\varepsilon_p + w^2/2) + \cdots \tag{3.58}$$

is assured by the fact that the major contribution to the integrals comes from a neighborhood of $w = 0$, due to the presence of the factor $\exp(-w^2)$ in the integrands; hence, we need not be concerned about the appearance of large w in the expansion (3.58), which is, in general, only valid for

$$(2\frac{w_p\varepsilon_p}{\varepsilon^2} + \frac{w^2}{\varepsilon^2})^2 < 1 .$$

Rewriting (3.57) we have

$$[\delta(m_s c_{sk})]_{st} = \mu N_s \nu_{st}(u_{tk} - u_{sk}) \tag{3.59}$$

where we have defined a <u>collision frequency for large ε</u> ,

$$\nu_{st} \equiv 2\pi(\kappa_{st}/\mu)^{-n/2} A_1(p) N_t |\vec{u}_t - \vec{u}_s|^{n+1} > 0 , \tag{3.59a}$$

$$\text{with}\quad N_s\nu_{st} = N_t\nu_{ts} \quad . \tag{3.59b}$$

The "effective collision frequency for transfer of momentum between species "s" and "t" "is (cf. (3.42c,d)

$$\nu_{st}^{M} = (m_t/m_o)\nu_{st} \tag{3.59c}$$

$$\text{with} \quad \rho_s \nu_{st}^{M} = \rho_t \nu_{ts}^{M} \quad . \tag{3.59d}$$

The collision frequency (3.59a) is exhibited in Section 3.6 for various interparticle force laws. This expression for the collision frequency is of course only valid for the range $\varepsilon >> 1$; in this range ν_{st} is, in general, dependent upon the difference in species' flow velocities and independent of species' temperatures. This is to be compared with the situation in Section 3.3 where the collision frequency was shown to be temperature dependent and independent of flow velocities (see 3.42a). This is one example of the striking difference in form of the partial collision integrals between the two extreme cases $\varepsilon << 1$ and $\varepsilon >> 1$.

We note that, according to (3.59), the partial momentum collision integral vanishes when $\varepsilon_k = 0$; this of course simply reflects the neglectance of the lower order terms in ε -- the integral does not, in general, vanish when $\varepsilon_k = 0$ (except for "Maxwell molecules," c. f. Section 3.5). Such a case, however, is pathological, and in any event we have for the vector partial momentum collision integral

$$[\delta(m_s\vec{c}_s)]_{st} = \mu N_s \nu_{st}(\vec{u}_t - \vec{u}_s) \quad , \tag{3.60}$$

which cannot vanish inasmuch as $\vec{u}_t \neq \vec{u}_s$. We note from this result that $[\delta(m_s\vec{c}_s)]_{st}$ is in the direction of $\vec{\varepsilon} \equiv (\vec{u}_t - \vec{u}_s)/a_o$, to this level of accuracy, and does not involve the traceless pressures or heat flows. This is to be contrasted with the result of Section 3.3, equation (3.42), where $[\delta(m_s\vec{c}_s)]_{st}$ is in the direction of $\vec{\varepsilon}$ only if the species' traceless pressures and heat flows vanish.

Bringing forth the results from Appendices B and C for the other three partial collision integrals we have

<u>Energy</u>:
$$[\delta(\tfrac{1}{2} m_s c_s^2)]_{st} = (\mu/m_o)N_s\nu_{st}\{m_t(\vec{u}_t-\vec{u}_s)^2 + (n+4)K(T_t-T_s) + $$
$$+ (n+1)(\varepsilon_i\varepsilon_j/\varepsilon^2)[(n+1)(m_t/2)(\frac{P_{tij}}{\rho_t} + \frac{P_{sij}}{\rho_s}) +$$
$$+ (\frac{P_{tij}}{N_t} - \frac{P_{sij}}{N_s})]\} \tag{3.61}$$

<u>Pressure</u>:
$$[\delta(m_s c_{sj} c_{sk})]_{st} = (\mu/m_o)N_s\nu_{st}\{\frac{m_t}{2}[[4-3\frac{A_2}{A_1})(u_{tj}-u_{sj})(u_{tk}-u_{sk}) +$$
$$+ \delta_{jk}\frac{A_2}{A_1}(\vec{u}_t-\vec{u}_s)^2] + 2K(T_t-T_s)[\delta_{jk} + \frac{\varepsilon_j\varepsilon_k}{\varepsilon^2}(n+1)] +$$
$$+ \frac{4K^2T_sT_t}{\mu a_o^2}[\delta_{jp}\delta_{kq} + (n+1)(\delta_{kq}\varepsilon_p\varepsilon_j/\varepsilon^2)^\dagger](\ \cdots$$

$$\cdots\left(\frac{P_{tpq}}{\rho_t}-\frac{P_{spq}}{\rho_s}\right)+\{2\delta_{jp}\delta_{kq}[2\,\frac{K}{a_o^2}\,(T_t-T_s)-\frac{3}{4}\,\frac{A_2}{A_1}\,m_t]\;+$$

$$+\,2(n+1)(\delta_{kq}\varepsilon_p\varepsilon_j/\varepsilon^2)^{+}[\,\frac{2K}{a_o^2}\,(T_t-T_s)+(1-\frac{3}{2}\,\frac{A_2}{A_1})m_t]\;+$$

$$+\,(n+1)(\varepsilon_p\varepsilon_q/\varepsilon^2)m_t[\delta_{jk}(n+3)\,\frac{A_2}{4A_1}+(n-1)(\varepsilon_j\varepsilon_k/\varepsilon^2)(1-\frac{3}{4}\,\frac{A_2}{A_1})]\}\,\cdot$$

$$\cdot\left(\frac{P_{tpq}}{\rho_t}+\frac{P_{spq}}{\rho_s}\right)\}\quad. \tag{3.62}$$

<u>Heat Flow</u>: $[\delta(\frac{1}{2}\,m_s c_s^2 c_{sk})]_{st} = (\mu/m_o^2)N_s\nu_{st}\{(u_{tk}-u_{sk})[\frac{5K^2}{a_o^2}\,(T_t-T_s)^2\;+$

$$+\,\frac{5K^2}{a_o^2}\,T_sT_t\,\frac{m_o}{\mu}+m_tK(T_t-T_s)(8+3n-\frac{A_2}{A_1}\,(n+6))+\frac{m_t^2}{4}\,a_o^2\,(8\varepsilon^2(1-\frac{A_2}{2A_1})\;+$$

$$+\,2n^2+12n+15-\frac{A_2}{A_1}\,(n+1)(n+6))]+\frac{(2K^2T_sT_t)^2}{\mu^2a_o^3}\,\varepsilon_i\left(\frac{\rho_s}{p_s^2}\,P_{sik}+\frac{\rho_t}{p_t^2}\,P_{tik}\right)\;+$$

$$+\,\{\delta_{jk}a_o\varepsilon_i[\,\frac{4K^2}{a_o^4}\,(T_t-T_s)^2+2m_t\,\frac{K}{a_o^2}\,(T_t-T_s)(n+3+\frac{n}{2}\,\frac{A_2}{A_1})\;+$$

$$+\,m_t^2\,(n+2)(1-\frac{3}{2}\,\frac{A_2}{A_1})]+a_o\varepsilon_k(\varepsilon_i\varepsilon_j/\varepsilon^2)(n+1)[4m_t\,\frac{K}{a_o^2}\,(T_t-T_s)(1-\frac{3}{4}\,\frac{A_2}{A_1})\;+$$

$$+\,m_t^2\,(n+1-\frac{n}{2}\,\frac{A_2}{A_1})]\}\left(\frac{P_{sij}}{\rho_s}+\frac{P_{tij}}{\rho_t}\right)+\frac{2K^2T_sT_t}{\mu a_o}\,\{\delta_{jk}\varepsilon_i[\frac{4K}{a_o^2}\,(T_t-T_s)\;+$$

$$+\,m_t(3+n(1+\frac{A_2}{2A_1}))]+2(n+1)\varepsilon_k(\varepsilon_i\varepsilon_j/\varepsilon^2)m_t(1-\frac{3}{4}\,\frac{A_2}{A_1})\}\left(\frac{P_{tij}}{p_t}-\frac{P_{sij}}{p_s}\right)+$$

$$+\,\frac{2}{5}\,\frac{K^2T_sT_t}{\mu a_o^2}\,\{\delta_{ik}[\frac{4K}{a_o^2}\,(T_t-T_s)(2n+17)+m_t(n+1)(n+8+(1+\frac{n}{2})\,\frac{A_2}{A_1})\;-$$

$$- 10 \frac{A_2}{A_1} m_t] + 2(n+1)(\varepsilon_i \varepsilon_k/\varepsilon^2)[18 \frac{K}{a_o^2} (T_t - T_s) + 2m_t(2n+5- \frac{7}{8} (n+6) \frac{A_2}{A_1})]\} \cdot$$

$$\cdot (\frac{q_{si}}{p_s} + \frac{q_{ti}}{p_t}) - \frac{1}{5} \{\delta_{ik}[4 \frac{K^2}{a_o^4} (T_t - T_s)^2 (2n+17) +$$

$$+ m_t \frac{K}{a_o^2} (T_t - T_s)(2(n+1)(n+8) + ((n+1)(n+2)-20) \frac{A_2}{A_1}) +$$

$$+ m_t^2 (5 + (n+1)(n+6)(1- \frac{3}{2} \frac{A_2}{A_1}))] + (n+1)(\varepsilon_i \varepsilon_k/\varepsilon^2)[36 \frac{K^2}{a_o^4} (T_t - T_s)^2 +$$

$$+ 2m_t \frac{K}{a_o^2} (T_t - T_s)(4(2n+5) + (11n+54) \frac{A_2}{2A_1}) +$$

$$+ m_t^2 (2n^2 + 10n + 7 + (2n^2 + \frac{25}{2} n + 9) \frac{A_2}{A_1})]\}(\frac{q_{si}}{\rho_s} - \frac{q_{ti}}{\rho_t}) +$$

$$+ \frac{4}{5} \frac{(K^2 T_s T_t)^2}{\mu^2 a_o^4} [\delta_{ik}(2n+17) + 9(n+1)(\varepsilon_i \varepsilon_k/\varepsilon^2)](\frac{\rho_t}{p_t^2} q_{ti} - \frac{\rho_s}{p_s^2} q_{si})\} \; . \qquad (3.63)$$

An interesting observation can be made concerning the influence of the traceless pressures and heat flows upon the partial collision integrals. Comparing the results (3.60) - (3.63) with those of Section 3.3, (3.42) - (3.45), we see that, in general, the traceless pressures and heat flows have considerably less influence upon any of the partial collision integrals for the range $\varepsilon \gg 1$ as against the range $\varepsilon \ll 1$ (this observation is of course with reference to the order of the ε-coefficients which appear with the traceless pressure and heat flow terms).* This decreased influence of the higher order velocity moments can be seen more clearly if we lower our level of accuracy by discarding all zero order terms involving the traceless pressures and heat flows; the expressions (3.60) - (3.63) are then drastically simplified to

Momentum: $$[\delta(m_s\vec{c}_s)]_{st} = \mu N_s \nu_{st}(\vec{u}_t-\vec{u}_s) \qquad (3.64)$$

Energy: $$[\delta(\tfrac{1}{2}m_s c_s^2)]_{st} = 2(\mu/m_o)N_s\nu_{st}\{\frac{m_t}{2}(\vec{u}_t-\vec{u}_s)^2+\frac{(n+4)}{2}K(T_t-T_s)\}\ . \qquad (3.65)$$

Pressure: $$[\delta(m_s c_{sj}c_{sk})]_{st} = (\mu/m_o)N_s\nu_{st}\{\frac{m_t}{2}[(4-3\frac{A_2}{A_1})(u_{tj}-u_{sj})(u_{tk}-u_{sk})+$$

$$+\delta_{jk}\frac{A_2}{A_1}(\vec{u}_t-\vec{u}_s)^2]+2K(T_t-T_s)[\delta_{jk}+(\varepsilon_j\varepsilon_k/\varepsilon^2)(n+1)]\} \qquad (3.66)$$

Heat Flow: $$[\delta(\tfrac{1}{2}m_s c_s^2 c_{sk})]_{st} = (\mu/m_o^2)N_s\nu_{st}\{(u_{tk}-u_{sk})[5\frac{K^2}{a_o^2}(T_t-T_s)^2+$$

$$+5\frac{K^2}{a_o^2}T_sT_t\frac{m_o}{\mu}+m_tK(T_t-T_s)(8+3n-(n+6)\frac{A_2}{A_1})+$$

---

*The case of "Maxwell molecules" is an exception; for this force law the partial collision integrals have the same functional form for all ε — c. f. Section 3.5.

$$+ \frac{m_t^2}{4} a_o^2(8\varepsilon^2(1- \frac{A_2}{2A_1})+2n^2+12n+15-(n+1)(n+6) \frac{A_2}{A_1})] +$$

$$+ \frac{(2K^2T_sT_t)^2}{\mu^2 a_o^3} \varepsilon_i(\frac{\rho_s}{p_s^2} P_{sik}+ \frac{\rho_t}{p_t^2} P_{tik}) + \{\delta_{jk}a_o\varepsilon_i[\frac{4K^2}{a_o^4} (T_t-T_s)^2 +$$

$$+ 2m_t \frac{K}{a_o^2} (T_t-T_s)(n + 3 + \frac{n}{2} \frac{A_2}{A_1}) \quad +$$

$$+ m_t^2(n+2)(1- \frac{3}{2} \frac{A_2}{A_1})]+ a_o\varepsilon_k(\varepsilon_i\varepsilon_j/\varepsilon^2)(n+1)[4m_t \frac{K}{a_o^2} (T_t-T_s)(1- \frac{3}{4} \frac{A_2}{A_1}) +$$

$$+ m_t^2(n+1- \frac{n}{2} \frac{A_2}{A_1})]\}(\frac{P_{sij}}{\rho_s} + \frac{P_{tij}}{\rho_t})+ \frac{2K^2T_sT_t}{\mu a_o} \{\delta_{jk}\varepsilon_i[\frac{4K}{a_o^2} (T_t-T_s) +$$

$$+ m_t(3+(1+ \frac{A_2}{2A_1})n)]+2(n+1)\varepsilon_k(\varepsilon_i\varepsilon_j/\varepsilon^2)m_t(1- \frac{3}{4} \frac{A_2}{A_1})\}(\frac{P_{tij}}{p_t} - \frac{P_{sij}}{p_s})\} \ . \tag{3.67}$$

We thus see from (3.64) - (3.67) that, to this level of approximation, the partial collision integrals, with the exception of the partial heat flow collision integral (3.67), are the results corresponding to the case where the species' distribution functions are Maxwellian (expressions (3.64), (3.65) are identical to Tanenbaum's results[11]); that is, for the limiting case $\varepsilon >> 1$, the non-Maxwellian or "non-equilibrium" (cf. Section 2.4) parts of the species distribution functions have little effect upon the partial collision integrals.

Finally, the comments at the end of Section 3.3 concerning the sums over all species of the total collision integrals apply here also; that is, $\sum_s \sum_t [\delta(m_s c_{sk})]_{st} = 0$, $\sum_s \sum_t [\delta(\frac{1}{2} m_s c_s^2)]_{st} \neq 0$, in general.

3.5 Exact Evaluation of Collision Integrals for the Maxwell Molecule Force Law

For the case where the particles obey the "Maxwell molecule" force law,

$$f_{st} = \kappa_{st}/r^5 \quad , \quad (\text{i.e.} \quad p = 5 \; , \; n = -1) \tag{3.68}$$

the partial collision integrals can be calculated exactly (within the limitations of the Boltzmann binary collision operator), without any knowledge of the species' distribution functions. In this section we shall give the details of the calculations of all four partial collision integrals, for the interparticle force law (3.68).

From (3.11) we have

$$[\delta(m_s c_{sk})]_{st} = \mu \iint F_s F_{t1} g g_k S^{(1)}(g) d\vec{v} d\vec{v}_1 \quad , \tag{3.69}$$

where, from Table 1,

$$S^{(1)}(g)\Big|_{n=-1} = 2\pi(\kappa_{st}/\mu)^{1/2} A_1(5) g^{-1} \quad , \tag{3.70}$$

so that

$$[\delta(m_s c_{sk})]_{st} = 2\pi(\kappa_{st}\mu)^{1/2} A_1(5) \iint F_s F_{t1} g_k d\vec{v} d\vec{v}_1 \ . \tag{3.71}$$

Recalling that $\vec{g} \equiv \vec{v}_1 - \vec{v}$, we then have

$$[\delta(m_s c_{sk})]_{st} = 2\pi(\kappa_{st}\mu)^{1/2} A_1(5) \iint F_s F_{t1}(v_{1k} - v_k) d\vec{v} d\vec{v}_1$$

$$\equiv 2\pi(\kappa_{st}\mu)^{1/2} A_1(5)[N_s N_t u_{tk} - N_t N_s u_{sk}] ,$$

or, $$[\delta(m_s c_{sk})]_{st} = \mu N_s \nu_{st}(u_{tk} - u_{sk}) , \qquad (3.72)$$

where $$\nu_{st} \equiv 2\pi \, A_1(5)(\kappa_{st}/\mu)^{1/2} N_t \qquad (3.72a)$$

is the collision frequency for Maxwell molecules which is independent of the flow velocities and temperatures (cf. (3.42a), (3.59a)). In obtaining (3.72) we have simply invoked the definitions for N_s, $\vec{u}_s$, (2.3), (2.5) .

For the partial pressure collision integral we have from Appendix B

$$[\delta(m_s c_{sj} c_{sk})]_{st} = \mu \iint F_s F_{t1} g\{[g_k(c_{oj} - u_{sj}) + g_j(c_{ok} - u_{sk})]S^{(1)}(g) -$$

$$- \frac{m_t}{2m_o}[3g_j g_k - g^2 \delta_{jk}]S^{(2)}(g)\} d\vec{v} d\vec{v}_1 . \qquad (3.73)$$

From Table 1,

$$S^{(2)}(g)\Big|_{n=-1} = 2\pi(\kappa_{st}/\mu)^{1/2} A_2(5) g^{-1} . \qquad (3.74)$$

Substituting (3.70) and (3.74) into (3.73) gives us

$$[\delta(m_s c_{sj} c_{sk})]_{st} = 2\pi(\kappa_{st}\mu)^{1/2}\iint F_s F_{t1}\{[g_k(c_{oj}-u_{sj})+g_j(c_{ok}-u_{sk})]A_1(5) -$$

$$- (m_t/2m_o)[3g_j g_k - g^2\delta_{jk}]A_2(5)\}d\vec{v}d\vec{v}_1 \quad . \qquad (3.75)$$

It will prove convenient to express all velocities in (3.75) in terms of the random velocities, $\vec{c}_s$, $\vec{c}_t$; we have

$$\vec{g} = \vec{v}_1 - \vec{v} = (\vec{c}_t+\vec{u}_t)-(\vec{c}_s+\vec{u}_s) = \vec{c}_t-\vec{c}_s + a_o\vec{\varepsilon} \quad , \qquad (3.76a)$$

and $$\vec{c}_o-\vec{u}_s = (m_s/m_o)(\vec{c}_s+\vec{u}_s)+(m_t/m_o)(\vec{c}_t+\vec{u}_t)-\vec{u}_s$$

$$= (m_s/m_o)\vec{c}_s+(m_t/m_o)\vec{c}_t+(m_t/m_o)a_o\vec{\varepsilon} \quad , \qquad (3.76b)$$

so that $$g_k(c_{oj}-u_{sj}) = (m_s/m_o)c_{sj}\overset{\circ}{c}_{tk}+(m_t/m_o)c_{tj}c_{tk}+$$

$$+ (m_t/m_o)a_o\overset{\circ}{c}_{tk}\varepsilon_j-(m_s/m_o)c_{sj}c_{sk} -$$

$$- (m_t/m_o)c_{sk}\overset{\circ}{c}_{tj}-(m_t/m_o)a_o\overset{\circ}{\varepsilon}_j c_{sk} +$$

$$+ (m_s/m_o)a_o\overset{\circ}{\varepsilon}_k c_{sj}+(m_t/m_o)a_o\overset{\circ}{\varepsilon}_k c_{tj} +$$

$$+ (m_t/m_o)a_o^2\varepsilon_j\varepsilon_k \quad , \qquad (3.76c)$$

and $$3g_j g_k-g^2\delta_{jk} = 3(c_{tj}c_{tk}-c_{sk}\overset{\circ}{c}_{tj}+a_o\overset{\circ}{\varepsilon}_k c_{tj}-c_{sj}\overset{\circ}{c}_{tk}+c_{sj}c_{sk} -$$

$$- a_o\overset{\circ}{\varepsilon}_k c_{sj}+a_o\overset{\circ}{\varepsilon}_j c_{tk}-a_o\overset{\circ}{\varepsilon}_j c_{sk}+a_o^2\varepsilon_j\varepsilon_k) -$$

$$- (c_t^2+c_s^2+a_o^2\varepsilon^2+2a_o\overset{o}{\varepsilon_1 c_{ti}}-2a_o\overset{o}{\varepsilon_1 c_{si}}-2\overset{o}{c_{si}c_{ti}})\delta_{jk} \tag{3.76d}$$

where terms overscored with "o" indicate integrands whose integrals over $d\vec{v}d\vec{v}_1$ vanish due to the fact that $\langle\vec{c}_s\rangle_s = \langle\vec{c}_t\rangle_t \equiv 0$. Substituting (3.76c,d) into (3.75) gives us

$$[\delta(m_s c_{sj} c_{sk})]_{st} = 2\pi(\kappa_{st}\mu)^{1/2} \iint F_s F_{t1} \{2A_1(5)[(m_t/m_o)c_{tj}c_{tk} -$$

$$- (m_s/m_o)c_{sj}c_{sk}+(m_t/m_o)a_o^2\varepsilon_j\varepsilon_k]-(m_t/2m_o)A_2(5)[3(c_{tj}c_{tk} +$$

$$+ c_{sj}c_{sk}+a_o^2\varepsilon_j\varepsilon_k)-\delta_{jk}(c_t^2+c_s^2+a_o^2\varepsilon^2)]\}d\vec{v}d\vec{v}_1 \; . \tag{3.77}$$

Performing the remaining integrations by reference to the definitions for N_s, T_s, P_{sjk}, (2.3), (2.8), (2.13), and rearranging, we obtain

$$[\delta(m_s c_{sj} c_{sk})]_{st} = (\mu/m_o)N_s\nu_{st}\{\frac{1}{2}\frac{A_2(5)}{A_1(5)}\delta_{jk}\, m_t(u_t-u_s)^2 +$$

$$+ 2[1-\frac{3}{4}\frac{A_2(5)}{A_1(5)}]m_t(u_{tj}-u_{sj})(u_{tk}-u_{sk})+\delta_{jk}2K(T_t-T_s) +$$

$$+ 2(\frac{P_{tjk}}{N_t} - \frac{P_{sjk}}{N_s}) - \frac{3}{2}\frac{A_2(5)}{A_1(5)}m_t(\frac{P_{tjk}}{\rho_t}+\frac{P_{sjk}}{\rho_s})\} \; . \tag{3.78}$$

The <u>partial energy collision integral</u> is given by one-half the trace of (3.78),

$$[\delta(\tfrac{1}{2}\,m_s c_s^2)]_{st} = 2(\mu/m_o)N_s\nu_{st}\{\frac{m_t}{2}\,(\vec{u}_t-\vec{u}_s)^2+\frac{3}{2}\,K(T_t-T_s)\}\ . \qquad (3.79)$$

For the partial heat flow collision integral we have from Appendix C

$$[\delta(\tfrac{1}{2}\,m_s c_s^2 c_{sk})]_{st} = (\mu/2)\iint F_s F_{t1} g\{[2g_i(c_{oi}-u_{si})(c_{ok}-u_{sk}) +$$
$$+ g_k(g^2(m_t/m_o)^2+(\vec{c}_o-\vec{u}_s)^2)]S^{(1)}(g) +$$
$$+ [(m_t/m_o)g^2(c_{ok}-u_{sk})-3(m_t/m_o)(c_{oi}-u_{si})g_i g_k]S^{(2)}(g)\}d\vec{v}d\vec{v}_1\ . \qquad (3.80)$$

Substituting (3.70), (3.74), (3.76a,b) into (3.80), multiplying out, and retaining only those terms which do not involve $\langle\vec{c}_s\rangle_s$, $\langle\vec{c}_t\rangle_t$ (these of course vanish), we obtain, after collecting terms

$$[\delta(\tfrac{1}{2}\,m_s c_s^2 c_{sk})]_{st} = \pi(\kappa_{st}\mu)^{1/2}\iint \frac{F_s F_{t1}}{m_o^2}\{-c_s^2 c_{sk}[(3m_s^2+m_t^2)A_1 +$$
$$+ 2m_s m_t A_2]+2c_t^2 c_{tk} m_t^2(2A_1-A_2)+c_{si}c_{sk}a_o\varepsilon_i[2(m_s-m_t)^2 A_1 +$$
$$+ (m_s m_t-3m_t^2)A_2]+4c_{ti}c_{tk}a_o m_t^2\varepsilon_i(2A_1-A_2) +$$
$$+ c_s^2 a_o\varepsilon_k[(m_t-m_s)^2 A_1+(m_t^2+3m_s m_t)A_2] +$$
$$+ 2c_t^2 m_t^2 a_o\varepsilon_k(2A_1-A_2)+2a_o^3 m_t^2\varepsilon^2\varepsilon_k(2A_1-A_2)\}d\vec{v}d\vec{v}_1\ . \qquad (3.81)$$

From the definitions for N_s, T_s, P_{sjk}, $\vec{q}_s$, (2.3), (2.8), (2.13), (2.16), the remaining integrations yield

$$[\delta(\tfrac{1}{2} m_s c_s^2 c_{sk})]_{st} = \mu N_s \nu_{st} \{\gamma(m_t/m_o)^2(\vec{u}_t-\vec{u}_s)^2(u_{tk}-u_{sk}) +$$

$$+ 5K(u_{tk}-u_{sk})[\gamma(m_t/m_o^2)(T_t-T_s)+(1/2m_s)T_s] +$$

$$+ (u_{ti}-u_{si})[2\gamma(m_t/m_o)^2(\frac{P_{sik}}{\rho_s} + \frac{P_{tik}}{\rho_t})+(\frac{m_t}{m_o}(\frac{A_2}{2A_1} -4)+1)\frac{P_{sik}}{\rho_s}] +$$

$$+ 2\gamma(m_t/m_o)^2(\frac{q_{tk}}{\rho_t} - \frac{q_{sk}}{\rho_s})+(6\,\frac{m_t}{m_o} -3-2\,\frac{m_t}{m_o}\frac{A_2}{A_1})\,\frac{q_{sk}}{\rho_s}\} \qquad (3.82)$$

where $\gamma \equiv 2-A_2(5)/A_1(5)$. (3.82a)

It is important to note that in all the preceding calculations no knowledge of the species' distribution functions has been assumed; the partial collision integrals have been evaluated using only the definitions in Section 2.1 for the first thirteen velocity moments N_s, $\vec{u}_s$, T_s (or p_s), P_{sjk}, $\vec{q}_s$. Furthermore, since the calculations are exact, the results are valid for <u>all</u> diffusion Mach number, $\varepsilon \equiv |\vec{u}_t-\vec{u}_s|/a_o$.

The highest order moments occurring in the exact results are the traceless pressures and heat flows, and these appear linearly. Since our "thirteen moment approximations" to the species' distribution functions (2.41) are accurate "up to" these moments, we expect the approximate results of sections 3.3, 3.4 to agree, for the case of Maxwell molecules, with the exact results of this section, provided only that the former are of sufficient accuracy in ε .* This is indeed the case as can easily be seen by substituting the appropriate values for the "z" integrals from Appendix A into the results for $\varepsilon \ll 1$, (3.42) - (3.45),** and $n = -1$ into the results for $\varepsilon \gg 1$, (3.60) - (3.63) .

Due to the fact that the results for Maxwell molecules are independent of the species' distribution functions, it seems appropriate to employ this force law as a test for accuracy of any calculations based upon assumptions concerning the species' distribution functions. We have just done this for the approximate results of Sections 3.3, 3.4. We shall now examine the accuracy of the calculations made by Burgers[13] and by Lyman,[14] in reference to the exact Maxwell molecule results; we shall show that their calculations do not give the correct results for Maxwell molecules.

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*The fact that the exact Maxwell molecule results contain no products of traceless pressures and/or heat flows assures us that the discarding of the $\phi_s \phi_t$ term (cf. (2.48)) in the approximate calculations will have no bearing upon the accuracy of the approximate results insofar as the Maxwell molecule force law is concerned.

**In the approximate result for the partial heat flow collision integral for $\varepsilon \ll 1$, (3.45), there is no third order term in ε as there is in the exact calculation for Maxwell molecules, (3.82); this simply reflects the level of accuracy in the approximate calculation, i.e. to <u>second</u> order in ε .

The calculations of Burgers and Lyman are somewhat similar to ours with the very important difference that they are based upon an expansion of the species' distribution function in terms of a random velocity $\vec{c}_s$ which is relative to the flow velocity of the mixture; that is,

$$F'_s = F_s^{(o)'}(1+\phi'_s) = \frac{N_s e^{-c'^2_s/a'^2_s}}{\pi^{3/2} a'^3_s}[1+\phi'_s(\vec{c}'_s)] \quad , \tag{3.83a}$$

$$\text{where} \quad \vec{c}'_s \equiv \vec{v}-\vec{u} \quad , \tag{3.83b}$$

$$\vec{u} \equiv (\sum_s \rho_s \vec{u}_s)/\sum_s \rho_s \quad , \tag{3.83c}$$

and where the "s"- species' temperature, T'_s, and all higher velocity moments are defined relative to $\vec{c}'_s$, e.g.

$$\frac{3}{2} KT'_s \equiv \left< \frac{1}{2} m_s c'^2_s \right>_s \quad . \tag{3.83d}$$

In Burgers' expansion, $a'^2_s \equiv 2KT'/m_s$, where $T' \equiv (\sum_s N_s T'_s)/\sum_s N_s$, so that only small temperature differences are considered; in Lyman's expansion, $a'^2_s \equiv 2KT'_s/m_s$.

Let us consider the exact Maxwell molecule result for the partial energy collision integral, $[\delta(\frac{1}{2} m_s c'^2_s)]_{st}$. Introducing the diffusion velocity of species "s"

$$\vec{W}_s(\vec{x},t) \equiv \vec{u}_s - \vec{u} \quad , \tag{3.84}$$

we have $\vec{c}_s' \equiv \vec{v}-\vec{u} = (\vec{v}-\vec{u}_s)+\vec{W}_s = \vec{c}_s+\vec{W}_s$ (3.85)

where $\vec{c}_s$ is the random velocity used in this dissertation, (2.6). We can then express $[\delta(\frac{1}{2} m_s c_s'^2)]_{st}$ in terms of our partial collision integrals; we find from (3.85),

$$[\delta(\frac{1}{2} m_s c_s'^2)]_{st} = [\delta(\frac{1}{2} m_s c_s^2)+\delta(\frac{1}{2} m_s W_s^2)+\delta(m_s \vec{W}_s \cdot \vec{c}_s)]_{st}$$

or, $$[\delta(\frac{1}{2} m_s c_s'^2)]_{st} = [\delta(\frac{1}{2} m_s c_s^2)]_{st} + \vec{W}_s \cdot [\delta(m_s \vec{c}_s)]_{st} \tag{3.86}$$

by reason of (2.25a,b). Substituting the exact Maxwell molecule results (3.72), (3.79) into (3.86), we obtain

$$[\delta(\frac{1}{2} m_s c_s'^2)]_{st} = 2(\mu/m_o) N_s \nu_{st} \{ \frac{m_t}{2} (\vec{u}_t-\vec{u}_s)^2 + \frac{3}{2} K(T_t-T_s) +$$

$$+ (m_o/2)\vec{W}_s \cdot (\vec{u}_t-\vec{u}_s) \} \quad . \tag{3.87}$$

The temperatures T_s, T_s' are related as follows

$$\frac{3}{2} KT_s \equiv \langle \frac{1}{2} m_s c_s^2 \rangle_s = \langle \frac{1}{2} m_s c_s'^2 + \frac{1}{2} m_s W_s^2 - m_s \vec{W}_s \cdot \vec{c}_s' \rangle_s$$

$$\equiv \frac{3}{2} KT_s' - \frac{1}{2} m_s W_s^2 \tag{3.88}$$

since $< \vec{c}_s' >_s = < \vec{c}_s + \vec{W}_s >_s = \vec{W}_s$. Substituting (3.88) into (3.87), recognizing from (3.84) that $\vec{u}_t - \vec{u}_s = \vec{W}_t - \vec{W}_s$, and collecting terms, we obtain finally

$$[\delta(\frac{1}{2} m_s c_s'^2)]_{st} = 2(\mu/m_o)N_s \nu_{st}\{ \frac{3}{2} K(T_t' - T_s') + \frac{1}{2} (m_s - m_t)\vec{W}_s \cdot \vec{W}_t\} \ . \qquad (3.89)$$

The result obtained by specializing Burger's[36] or Lyman's[37] calculation to the case of Maxwell molecules is

$$[\delta(\frac{1}{2} m_s c_s'^2)]_{st} = 2(\mu/m_o)N_s \nu_{st}\{ \frac{3}{2} K(T_t' - T_s')\} \ . \qquad (3.90)$$

Comparison of (3.89), (3.90) shows that Burger's result is missing a term proportional to

$$\vec{W}_s \cdot \vec{W}_t = (\vec{u}_s - \vec{u}) \cdot (\vec{u}_t - \vec{u}) \ . \qquad (3.91)$$

This is a "higher order" term inasmuch as the expansion (3.83a) is only valid for "small" $|\vec{u}_s - \vec{u}|$, "small" in the sense that[38]

$$|\vec{u}_s - \vec{u}| << a_s' \ . \qquad (3.92)$$

The absence of the higher order term (3.91) from Burger's result is due to the fact that $\vec{W}_s$ appears linearly in the expansion (3.83a), occurring in the non-Maxwellian or perturbation part $\phi_s'(\vec{c}_s')$, and thus because the product $\phi_s'(\vec{c}_s')\phi_t'(\vec{c}_t')$ is neglected in the calculation of the collision integrals, there can be no terms in the results higher than first order in the diffusion velocities, $\vec{W}_s$.

Similar discrepancies can of course be exhibited for the partial pressure and heat flow integrals; in the case of the partial momentum collision integrals, the diffusion velocities enter the exact Maxwell molecule result linearly so that no discrepancy arises.

We thus see that the results of the expansion (3.83a,b,c) are quite limited in accuracy compared to the results of our expansion (2.41), (2.6); we pay a price, however, in the complexity of the results.

Before leaving this section we should point out that the Maxwell molecule force law is not merely an academic one; it has been used as a realistic force model in the scattering of electrons by neutral atoms.[39]

3.6 The Collision Frequencies as Functions of the Diffusion Mach Number

In this section we shall exhibit the collision frequencies introduced in Sections 3.3, 3.4, 3.5, as explicit functions of the diffusion Mach number ε for various inverse power interparticle force laws.

Recall from Section 3.3, equation (3.42a), that for $\varepsilon \ll 1$ the collision frequency is given by

$$\nu_{st} = (2/3)N_t a_o Z^{(1,1)} \quad . \tag{3.93}$$

Substituting the expression for $Z^{(1,1)}$ from Appendix A, we have for inverse power interparticle force laws, $f_{st} = \kappa_{st}/r^p$, $n = -4/(p-1)$,

$$\nu_{st} = \begin{cases} (8/3)\sqrt{\pi}\,\Gamma(3+n/2)A_1(p)(\kappa_{st}/\mu)^{-n/2}N_t a_o^{n+1} & , \quad n \neq 0 \\ (8/3)\sqrt{\pi}\,\sigma^2 N_t a_o & , \quad n = 0 \ , \ (\text{"hard spheres"}) \ . \end{cases} \tag{3.94}$$

From Section 3.4, equation (3.59a), we have for $\varepsilon \gg 1$,

$$\nu_{st} = \begin{cases} 2\pi(\kappa_{st}/\mu)^{-n/2}A_1(p)N_t|\vec{u}_t-\vec{u}_s|^{n+1} & , \quad n \neq 0 \\ \pi\sigma^2 N_t|\vec{u}_t-\vec{u}_s| & , \quad n = 0 \ , \ (\text{"hard spheres"}) \ . \end{cases} \tag{3.95}$$

In (3.94), (3.95), the results for "hard spheres," $n = 0$, are obtained from the results for $n \neq 0$ in accordance with the footnote on page 55.

From (3.94), (3.95), we have, for <u>all</u> values of n , the <u>normalized</u> collision frequency

$$\tilde{\nu}_{st} = \begin{cases} 1 \ , & \varepsilon \ll 1 \\ \dfrac{3\sqrt{\pi}}{4\Gamma(3+n/2)}\varepsilon^{n+1} \ , & \varepsilon \gg 1 \ , \end{cases} \tag{3.96}$$

<u>where the normalization is with respect to the small diffusion Mach number collision frequency</u>, (3.94). Expression (3.96) is evaluated below for various interparticle force laws

<u>hard spheres</u>, $p \to \infty$, $n = 0$: $\tilde{\nu}_{st} = \begin{cases} 1 \ , & \varepsilon \ll 1 \\ (3\sqrt{\pi}/8)\varepsilon \ , & \varepsilon \gg 1 \end{cases}$ (3.96a)

<u>Maxwell molecules</u>,* $p=5$, $n=-1$: $\tilde{\nu}_{st} = 1$ for <u>all</u> ε (3.96b)

$p=3$, $n=-2$: $\tilde{\nu}_{st} = \begin{cases} 1 \ , & \varepsilon \ll 1 \\ (3\sqrt{\pi}/4)\varepsilon^{-1} \ , & \varepsilon \gg 1 \end{cases}$ (3.96c)

$p=7/3$, $n=-3$: $\tilde{\nu}_{st} = \begin{cases} 1 \ , & \varepsilon \ll 1 \\ (3/2)\varepsilon^{-2} \ , & \varepsilon \gg 1 \end{cases}$ (3.96d)

<u>Coulomb</u>, $p=2$, $n=-4$: $\tilde{\nu}_{st} = \begin{cases} 1 \ , & \varepsilon \ll 1 \\ (3\sqrt{\pi}/4)\varepsilon^{-3} \ , & \varepsilon \gg 1 \ . \end{cases}$ (3.96e)

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*Note that, as pointed out in Section 3.5, ν_{st} is <u>independent of flow velocities and temperatures for the case of the Maxwell</u> molecule force law.

The results (3.96a-e) are exhibited in the log-log plot of Figure 2.

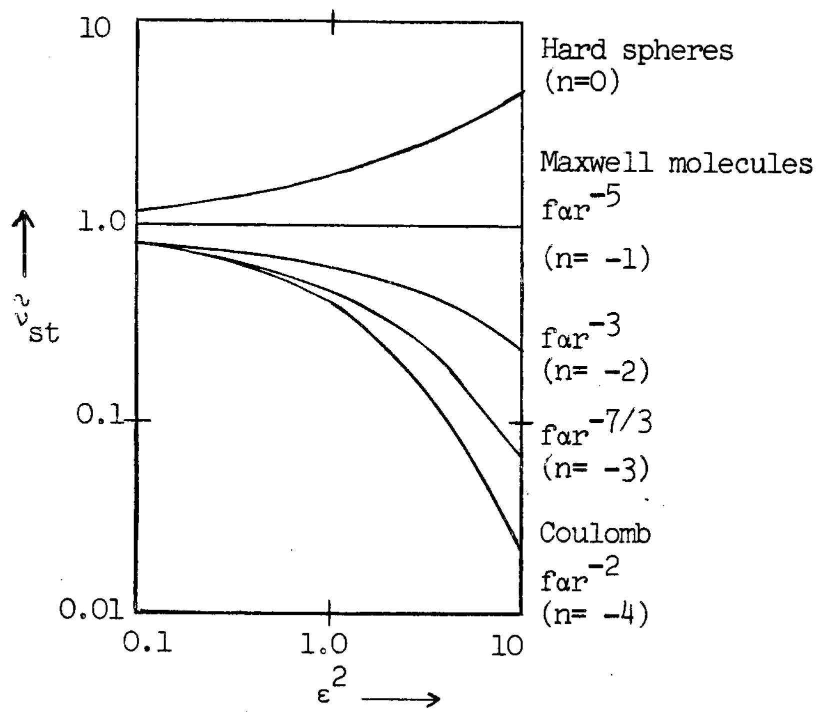

FIG. 2 COLLISION FREQUENCY AS A FUNCTION OF DIFFUSION MACH NUMBER FOR INVERSE-POWER INTERPARTICLE FORCE LAWS.

Inspection of Figure 2 shows that, in comparison with the Maxwell molecule results, $\tilde{\nu}_{st}$ either increases or decreases with increasing ε depending upon whether the power p in the force law is greater than or less than 5. The explanation for this is fairly simple; first, from (3.53b) we see that, while $S^{(1)}(g)$ decreases with increasing g for all possible values of p, $2 \leq p < \infty$, it decreases much faster for long-range (small p, large $|n|$) forces than for short. Tanenbaum[11] shows that

$$\langle g^2 \rangle = \frac{3}{2} a_o^2 (1+\varepsilon^2) ;$$

hence, in an "average sense" g increases with increasing ε. Thus, the "average" cross section falls off with increasing ε, but at a rate which increases with the range of the interparticle force law. Since the collision frequencies depend strongly upon $S^{(1)}(g)$, it follows that a plot of $\tilde{\nu}_{st}$ as a function of increasing ε, Figure 2, when compared with the Maxwell molecule results ($p=5$, $n=-1$), should increase for shorter-range forces (larger p, smaller $|n|$) and decrease for longer-range forces.

CHAPTER IV

KINETIC MODELS FOR THE COLLISION TERM $\left(\frac{\delta F}{\delta t}\right)_{\text{collisions}}$

Up to now we have employed the full Boltzmann binary collision model for the term $(\delta F/\delta t)_{\text{coll.}}$, the average time-rate of change of the distribution function due to collisions. We have seen that the calculation of the ensuing collision integrals, $\int Q(\delta F/\delta t)_{\text{coll.}}\,d\vec{v}$, has been tedious and the results quite cumbersome even after several simplifying assumptions. The origin of the complexity lies in the fact that the Boltzmann collision operator takes into account the geometry of each possible binary collision and the calculation of the accompanying collision integrals requires knowledge of the distribution functions (except for the case of Maxwell molecules).

In this chapter we present <u>kinetic models</u> (for general gas mixtures) as substitutes for the Boltzmann collision operator; the models possess relatively simple mathematical forms, but duplicate several important properties of the Boltzmann operator. The models do not involve the geometry of the individual collisions; they are essentially statistical averages over all possible collisions with the interparticle force law occurring implicitly in a phenomenological collision frequency.

Generally speaking, such models are not used for the calculation of transport quantities (e.g. coefficients of viscosity and thermal conductivity); more often, they are substituted for the Boltzmann operator in the Boltzmann equation which is then solved for the distribution function, either directly or by taking moments. However, in order to be reasonable substitutes, the models must be able to reproduce, at least in part, the properties of the Boltzmann operator. The models must conserve the species mass, total momentum, and total energy, in order to be physically acceptable; furthermore, the collision integrals of the Boltzmann operator should be reproduced as nearly as possible, including the pressure and heat flow integrals. It is with these considerations that this chapter deals; the models presented are constructed in such a manner so as to make the calculation of the partial collision integrals possible without explicit knowledge of the distribution functions.

The existing kinetic models and their collision integrals are appealing in their simplicity; however, this simplicity results in a shortcoming to a certain extent inasmuch as certain important results of the Boltzmann operator cannot be reproduced.

The first two models we shall analyze are the Gross-Krook[18] and Sirovich models[19]; we shall find that these models are in serious disagreement with regard to the partial pressure and heat flow collision integrals of the Boltzmann operator, both in the form of their results, and in the magnitude of certain terms when there are large differences in the species' masses. We shall next introduce two new models which overcome this difficulty to different degrees. The first is a revised form of the Sirovich model, while the second is based upon a velocity-space expansion of the collision term $(\delta F/\delta t)_{coll.}$ and reproduces all four partial collision integrals of the Boltzmann operator exactly.

4.1 The Gross-Krook Model

The Gross-Krook model[18] as originally presented was for a binary gas mixture; the extension to a system composed of an arbitrary number of species is straightforward. One simply lets

$$(\delta F_a/\delta t)_{coll.} = \sum_b (\delta F_a/\delta t)_{ab\ coll.} = - \sum_b \nu'_{ab}(F_a - \psi_{ab}) \tag{4.1}$$

where "a" and "b" are species subscripts, ν'_{ab} is the model's collision frequency between species "a" and "b" , taken to be velocity independent,* and

$$\psi_{ab} \equiv N_a(\pi^{-3/2} a_{ab}^{-3}) e^{-(\vec{v}-\vec{u}_{ab})^2/a^2_{ab}}, \quad a^2_{ab} \equiv 2K\, T_{ab}/m_a \tag{4.2a}$$

$$\int \psi_{ab} d\vec{v} = N_a \quad , \quad T_{aa} \equiv T_a \quad , \quad \vec{u}_{aa} \equiv \vec{u}_a \quad . \tag{4.2b}$$

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*If $\nu'_{ab} = \nu'_{ab}(\vec{v})$, then, in general, the species' mass cannot be conserved, i.e. $[\delta(m_a)]_{ab} \neq 0$.

The self-collision term is just the Krook model[40] for a simple gas

$$(\delta F_a/\delta t)_{\substack{aa \\ \text{coll.}}} = -\nu'_{aa}(F_a - F_a^{(o)}) \qquad (4.3a)$$

where $F_a^{(o)} \equiv \psi_{aa}$. (4.3b)

The parameters in the model (4.1) are the "mixed" flow velocity $\vec{u}_{ab}$, the "mixed" temperature T_{ab} , and the model's collision frequency ν'_{ab} , constituting a total of five scalar parameters. The number of parameters is thus more than sufficient to allow the model to reproduce exactly the partial momentum and energy collision integrals of the Boltzmann binary collision operator for the most general type of problem. However, in order to obtain relatively simple expressions for the parameters, we shall only require the model (4.1) to reproduce these integrals for the case where the species' distribution functions are Maxwellian for $\varepsilon << 1$ (general central force laws) and for $\varepsilon >> 1$ (general inverse power force laws); the restrictions on the distribution functions and ε will be removed for the Maxwell molecule force law. Once this is done the requirements for conservation of total momentum and energy are automatically satisfied by the model.

We see from (4.1), (4.2b) that the model automatically conserves the species' mass

$$[\delta(m_a)]_{ab} = \int m_a(\delta F_a/\delta t)_{\substack{ab\\ \text{coll.}}}\, d\vec{v} = -m_a\nu'_{ab}\int(F_a-\psi_{ab})d\vec{v}$$

$$= -m_a\nu'_{ab}(N_a - N_a) \equiv 0 \; . \tag{4.4}$$

The partial momentum and energy collision integrals of the Boltzmann operator for Maxwellian distribution functions and $\varepsilon << 1$ are given by (3.46), (3.47), respectively,

$$[\delta(m_a\vec{c}_a)]_{ab} = N_a\nu_{ab}\mu(\vec{u}_b - \vec{u}_a) \tag{4.5a}$$

$$[\delta(\tfrac{1}{2}m_a c_a^2)]_{ab} = 2N_a\nu_{ab}(\mu/m_o)[\tfrac{3}{2}K(T_b - T_a) + \frac{m_b}{2}(\vec{u}_b - \vec{u}_a)^2] \; . \tag{4.5b}$$

Note that (4.5a,b) are <u>exact for Maxwell molecules</u> (see (3.72), (3.79)). The corresponding integrals of the Gross-Krook model are

$$[\delta(m_a\vec{c}_a)]_{ab} = N_a\nu'_{ab}\, m_a(\vec{u}_{ab} - \vec{u}_a) \tag{4.6a}$$

$$[\delta(\tfrac{1}{2}m_a c_a^2)]_{ab} = N_a\nu'_{ab}[\tfrac{3}{2}K(T_{ab} - T_a) + \frac{m_a}{2}(\vec{u}_{ab} - \vec{u}_a)^2] \; . \tag{4.6b}$$

Equating (4.5a) to (4.6a), and (4.5b) to (4.6b) gives us

$$\vec{u}_{ab} = \alpha_{ab}(m_b/m_o)(\vec{u}_b - \vec{u}_a) + \vec{u}_a \tag{4.7a}$$

$$T_{ab} = T_a + 2\alpha_{ab}(\mu/m_o)(T_b - T_a) + \alpha_{ab}(2-\alpha_{ab})(\mu^2/3Km_a)(\vec{u}_b - \vec{u}_a)^2 \tag{4.7b}$$

where $\alpha_{ab} \equiv \nu_{ab}/\nu'_{ab} = \nu_{ba}/\nu'_{ba} \equiv \alpha_{ba}$, (4.8) since $N_a\nu_{ab} = N_b\nu_{ba}$ and $N_a\nu'_{ab} = N_b\nu'_{ba}$, due to the fact that the total number of collisions per unit time per unit volume of "a" particles with "b" particles is equal to the total number of collisions per unit time per unit volume of "b" particles with "a" particles.

If we identify ν'_{ab} in (4.1) with the actual collision frequency, ν_{ab} , then $\alpha_{ab} = 1$, and (4.7a,b) become, respectively,

$$\vec{u}_{ab} = (m_a\vec{u}_a + m_b\vec{u}_b)/m_o \tag{4.9a}$$

$$T_{ab} = T_a + 2(\mu/m_b)(T_b - T_a) + (\mu^2/3Km_a)(\vec{u}_b - \vec{u}_a)^2 \quad . \tag{4.9b}$$

Note that for $a = b$, we have the trivial results $\vec{u}_{ab} = \vec{u}_a$, $T_{ab} = T_a$, independent of the value of α_{aa} (see (4.7a,b)). The results (4.9a,b) agree with those of Hamel[41,42] and Morse[43] although the approaches used by these authors differ considerably from ours and involve assumptions which we have not used.*

Although this identification of collision frequencies is

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*Hamel's analysis is based exclusively upon the Maxwell molecule force law and involves the truncation of the force law range; in addition, a function of (m_a/m_b) is "determined" by finding the value of the function as $(m_a/m_b) \to 0$. Morse's work apparently involves the a priori assumption that $\vec{u}_{ab} \equiv \vec{u}_{ba}$; it is easily shown from (4.7a) that $\vec{u}_{ab} = \vec{u}_{ba}$ if and only if $\alpha_{ab} = 1$, for $a \neq b$.

appealing from an intuitive point of view, it must be emphasized that α_{ab} is essentially a "free" parameter. In any event, the Gross-Krook model given by (4.1), (4.2a,b), (4.7a,b) now reproduces the partial momentum and energy collision integrals of the Boltzmann operator for the case of Maxwellian distribution functions, $\varepsilon << 1$, and general central force laws, with the restrictions on the distribution functions and ε removed for the Maxwell molecule force law; the usual conservation laws are consequently satisfied automatically. The free parameter, α_{ab}, can be adjusted, if desired, to bring the model's partial pressure and/or heat flow collision integrals into closer agreement with those of the Boltzmann operator.

From (3.60),(3.61) we see that for $\varepsilon >> 1$ and Maxwellian distribution functions, the partial momentum and energy collision integrals have the same form as (4.5a), (4.5b), respectively, with the term $(3/2)K(T_b-T_a)$ in (4.5b) replaced by $\frac{(n+4)}{2}K(T_b-T_a)$; the collision frequencies are of course different for the two ranges $\varepsilon << 1$ and $\varepsilon >> 1$ (see (3.42a), (3.59a)). Hence, the results for $\varepsilon >> 1$ can be obtained directly from (4.7a,b), (4.8), simply by multiplying the term involving (T_b-T_a) in (4.7b) by $(n+4)/3$, and using (3.59a) for ν_{ab}.

The partial pressure collision integral of the Boltzmann operator is, for Maxwell molecules (cf. (3.78)), exactly

$$[\delta(m_a c_{aj} c_{ak})]_{ab} = \nu_{ab}\{\frac{1}{2}\frac{A_2(5)}{A_1(5)}\delta_{jk}(m_b/m_o)^2\rho_a(\vec{u}_b-\vec{u}_a)^2 +$$

$$+ 2[1-\frac{3}{4}\frac{A_2(5)}{A_1(5)}](m_b/m_o)^2\rho_a(\vec{u}_b-\vec{u}_a)_j(\vec{u}_b-\vec{u}_a)_k+2\delta_{jk}(\mu/m_o)N_a K(T_b-T_a) +$$

$$+[2-\frac{3}{2}\frac{A_2(5)}{A_1(5)}](\mu/m_o)(N_a/N_b)P_{bjk} - [2+\frac{3}{2}\frac{A_2(5)}{A_1(5)}\frac{m_b}{m_a}](\mu/m_o)P_{ajk}\} \ . \tag{4.10}$$

The corresponding integral of the Gross-Krook model is

$$[\delta(m_a c_{aj} c_{ak})]_{ab} = \nu'_{ab}[\rho_a(\vec{u}_{ab}-\vec{u}_a)_j(\vec{u}_{ab}-\vec{u}_a)_k - \delta_{jk}N_a K(T_a-T_{ab}) - P_{ajk}] \ ,$$

or substituting $\vec{u}_{ab}$, T_{ab} , ν'_{ab} , (4.7a,b),(4.8),

$$[\delta(m_a c_{aj} c_{ak})]_{ab} = \nu_{ab}[(1/3)(2-\alpha_{ab})\delta_{jk}(m_b/m_o)^2\rho_a(\vec{u}_b-\vec{u}_a)^2 +$$

$$+\alpha_{ab}(m_b/m_o)^2\rho_a(\vec{u}_b-\vec{u}_a)_j(\vec{u}_b-\vec{u}_a)_k + 2\delta_{jk}(\mu/m_o)N_a K(T_b-T_a)-(1/\alpha_{ab})P_{ajk}] \ . \tag{4.11}$$

In (4.10), (4.11) the collision frequency ν_{ab} , and hence α_{ab} , now corresponds to the Maxwell molecule force law (see (3.72a)) . Comparison of (4.10), (4.11) shows that the Gross-Krook result contains no "cross" traceless pressure term, P_{bjk} ; this could have been anticipated from the model (4.1) inasmuch as it contains no tensorial-like parameter.* The ratios of the terms involving $(\vec{u}_b-\vec{u}_a)$, (Boltzmann:Gross-Krook) are of order $(1/\alpha_{ab})$; the terms involving (T_b-T_a) agree exactly. The ratio of the coefficients of P_{ajk} are (Boltzmann: Gross-Krook)

$$[2+(3A_2/2A_1)(m_b/m_a)](\mu/m_o)\alpha_{ab} \stackrel{\sim}{\sim} 0.89\ \alpha_{ab}\ ,\ m_a \stackrel{\sim}{\sim} m_b \tag{4.12a}$$

$$\stackrel{\sim}{\sim} 1.55\ \alpha_{ab}\ ,\ m_a << m_b \tag{4.12b}$$

$$\stackrel{\sim}{\sim} 2(m_b/m_a)\alpha_{ab}\ ,\ m_a >> m_b\ . \tag{4.12c}$$

We see from (4.12c) that if we set $\alpha_{ab} = m_a/2m_b$ for agreement of the P_{ajk} terms in (4.10), (4.11), for the case $m_a >> m_b$, then the terms involving $(\vec{u}_b-\vec{u}_a)$ will be in disagreement by a factor proportional to $m_b/m_a << 1$. Furthermore, we see from (4.12b,c) that if we set $1.55\ \alpha_{ab} = 1$ for agreement between the P_{ajk} terms for the case $m_a << m_b$, then the ratio of the P_{bjk} coefficients (Boltzmann: Gross-Krook) in $[\delta(m_b c_{bj} c_{bk})]_{ba}$ will be $1.29(m_a/m_b)<<1$. Hence, we see that the integrals $[\delta(m_a c_{aj} c_{ak})]_{ab}$ can be made to agree fairly closely for the cases $m_a \stackrel{\sim}{\sim} m_b$, $m_a << m_b$, but <u>not</u> for $m_a >> m_b$; if we consider <u>both</u> partial integrals, $[\delta(m_a c_{aj} c_{ak})]_{ab}$

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*Note that for the partial pressure collision integrals to agree exactly the kinetic model would have to incorporate in addition a tensor with at least five independent elements, corresponding to the five independent elements of the partial pressure collision integrals. The Gross-Krook model is a degenerate form of Holway's[44] "ellipsoidal statistical model" which does exactly this.

and $[\delta(m_b c_{bj} c_{bk})]_{ba}$, then close agreement between the Gross-Krook and Boltzmann results is <u>only</u> possible for $m_a \stackrel{\sim}{\sim} m_b$. Inspection of (4.10), (4.11), shows that if we set

$$\alpha_{ab} = 2[1- 3A_2(5)/4A_1(5)] \stackrel{\sim}{\sim} 0.45 \quad , \quad (a \neq b) \quad , \tag{4.13a}$$

then the integrals $[\delta(m_a c_{aj} c_{ak})]_{ab}$ agree exactly for the cases (4.12a,b), apart from the missing P_{bjk} term in (4.11) and the fact that the ratio of P_{ajk} coefficients (Boltzmann: Gross-Krook) is (0.40) for $m_a \stackrel{\sim}{\sim} m_b$, and (0.70) for $m_a << m_b$.

Finally, we note that agreement between the <u>self</u>-partial pressure collision integrals $[\delta(m_a c_{aj} c_{ak})]_{aa}$ can be achieved by adjusting α_{aa} ; the result is

$$\alpha_{aa} = 4A_1(5)/3A_2(5) = 1.29 \quad , \tag{4.13b}$$

which agrees with the result Hamel[41] obtained by considering a binary system with one component a "trace species" (i.e. $N_b/N_a \to 0$). Alternately, (4.13b) gives the discrepancy between the Gross-Krook and Boltzmann results if we set $\alpha_{aa} = 1$, so that $\nu'_{aa} = \nu_{aa}$.

The partial heat flow collision integral of the Boltzmann operator is, for <u>Maxwell molecules</u> (cf. (3.82)), exactly

$$[\delta(\tfrac{1}{2} m_a c_a^2 c_{ak})]_{ab} = \nu_{ab}\{N_a\mu(\vec{u}_b-\vec{u}_a)_k \, [\gamma(m_b/m_o)^2(\vec{u}_b-\vec{u}_a)^2 \; +$$

$$+ 5\gamma(m_b/m_o^2)K(T_b-T_a)+(5/2)K\,T_a/m_a] + N_a\mu(\vec{u}_b-\vec{u}_a)_i[2\gamma(m_b/m_o)^2(\frac{P_{aik}}{\rho_a} +$$

$$+ \frac{P_{bik}}{\rho_b}) + (1+(\frac{A_2}{2A_1} - 4)(m_b/m_o))\frac{P_{aik}}{\rho_a}] + 2\gamma N_a\mu(m_b/m_o)^2(q_{bk}/\rho_b) -$$

$$- [2\gamma(m_b/m_o)^2 + 2(m_b/m_o)(-3+A_2/A_1)+3]\,(m_b/m_o)q_{ak}\} \qquad (4.14)$$

where $\gamma \equiv 2-A_2(5)/A_1(5)$. The corresponding integral of the Gross-Krook model is

$$[\delta(\frac{1}{2}\,m_a c_a^2 c_{ak})]_{ab} = \nu'_{ab}\{(N_a/2)(\vec{u}_{ab}-\vec{u}_a)_k[5K\,T_{ab} + m_a(\vec{u}_{ab}-\vec{u}_a)^2] - q_{ak}\}$$

or, substituting for $\vec{u}_{ab}$, T_{ab} , ν'_{ab} , (4.7a,b), (4.8),

$$[\delta(\frac{1}{2}\,m_a c_a^2 c_{ak})]_{ab} = \nu_{ab}\{N_a\mu(\vec{u}_b-\vec{u}_a)_k[(\alpha_{ab}/2)((5/3)(2-\alpha_{ab})+\alpha_{ab})\,\cdot$$

$$\cdot(m_b/m_o)^2(\vec{u}_b-\vec{u}_a)^2 + 5\alpha_{ab}(m_b/m_o^2)K(T_b-T_a)+(5/2)K\,T_a/m_a] - q_{ak}/\alpha_{ab}\}\;. \qquad (4.15)$$

Comparison of (4.14), (4.15) shows that the Gross-Krook result contains neither the "cross" heat flow term, q_{bk} , nor the traceless pressure terms. The ratio of the terms involving $(\vec{u}_b-\vec{u}_a)_k(\vec{u}_b-\vec{u}_a)^2$, (Boltzmann: Gross-Krook) is of order $(1/\alpha_{ab}^2)$; the ratio of the terms involving $(\vec{u}_b-\vec{u}_a)_k(T_b-T_a)$ is of order $(1/\alpha_{ab})$. The terms involving $(\vec{u}_b-\vec{u}_a)_k T_a$ are identical. The ratios of the coefficients of q_{ak} are (Boltzmann: Gross-Krook)

$$\{2\frac{m_b}{m_o}[(2-\frac{A_2(5)}{A_1(5)})\frac{m_b}{m_o}+\frac{A_2(5)}{A_1(5)}-3]+3\}\frac{m_b}{m_o}\alpha_{ab} \stackrel{\sim}{\sim} 0.76\ \alpha_{ab},\quad m_a \stackrel{\sim}{\sim} m_b \tag{4.16a}$$

$$\stackrel{\sim}{\sim} \alpha_{ab}\ ,\ m_a << m_b \tag{4.16b}$$

$$\stackrel{\sim}{\sim} 3(m_b/m_a)\alpha_{ab}\ ,\ m_a >> m_b\ . \tag{4.16c}$$

The set of ratios (4.16a,b,c) is similar to that for the comparison of the partial pressure collision integrals, (4.12a,b,c), implying an analogous conclusion. As can be seen, however, the ability of the Gross-Krook model to imitate the results of the Boltzmann operator decreases with ascending moments; this is what one would expect for a fixed number of model parameters, and is typical of kinetic models. Nevertheless, it is encouraging to note from the similar sets of ratios (4.12a,b,c) and (4.16a,b,c) that when α_{ab} is adjusted for maximum agreement of the pressure integrals, the heat flow integrals are simultaneously brought into closer agreement.

If the value of α_{ab} given by (4.13a) is used, the partial heat flow collision integral of the Gross-Krook model agrees closely with that of the Boltzmann operator for cases (4.16a,b), apart from the missing terms. If α_{aa} given by (4.13b) is substituted into the Gross-Krook integral, $[\delta(\frac{1}{2}m_a c_a^2 c_{ak})]_{aa}$, the ratio of the q_{ak} coefficients is (2/3).

Summarizing, we see that the Gross-Krook model's partial pressure collision integral, $[\delta(m_a c_{aj} c_{ak})]_{ab}$, can be made to agree fairly well with that of the Boltzmann operator for the cases $m_a \sim m_b$ and $m_a << m_b$, but <u>not</u> for the case $m_a >> m_b$. For the case where the "cross" traceless pressure term, P_{bjk}, is negligible, or when $N_a/N_b \to 0$ (see (4.10)), the agreement is excellent. (An example which satisfies <u>both</u> of these conditions is the calculation of the electron-neutral collision integrals in a weakly ionized gas). Similar observations apply to the model's partial heat flow collision integral with q_{bk} replacing P_{bjk}; the heat flow integral suffers an additional discrepancy in that it does not contain traceless pressure terms, P_{ajk}, P_{bjk}, as does the integral of the Boltzmann operator. Finally, we note that for the cases $m_a \sim m_b$ and $m_a << m_b$, the aforementioned agreement between integrals is affected only slightly by identifying ν'_{ab}, <u>the model's collision frequency</u>, with ν_{ab}, <u>the actual collision frequency</u>, i.e. setting α_{ab} equal to unity.

4.2 <u>The Sirovich Model</u>

The Sirovich model[19] extended to a multicomponent system is

$$(\delta F_a/\delta t)_{coll.} = -\nu'_{aa}(F_a - F_a^{(o)}) - (F_a^{(o)}/p_a) \sum_b \{\Lambda_{ab}\vec{c}_a \cdot (\vec{u}_a - \vec{u}_b) +$$

$$+ (1 - 2c_a^2/3a_a^2)[\chi_{ab}(\vec{u}_a - \vec{u}_b)^2 + (3/2)\xi_{ab}(T_b - T_a)]\} \qquad (4.17)$$

where $F_a^{(o)}$ and a_a^2 are given by (4.3b), (4.2a), and where ν'_{aa}

(the model's self-collision frequency), Λ_{ab}, χ_{ab}, and ξ_{ab} are assumed to be velocity-independent. The first term in (4.17) is just the Krook model for a simple gas, while the remaining terms are cross-collision terms reflecting the differences in species' flow velocities and temperatures.

As in the case of the Gross-Krook model, the Sirovich model automatically conserves the species' mass

$$[\delta(m_a)]_{ab} = -m_a \nu'_{aa}(N_a - N_a) - \frac{m_a}{p_a}\left(N_a - \frac{2}{3}\frac{N_a}{a_a^2}\frac{3}{2}a_a^2\right)[\chi_{ab}(\vec{u}_a - \vec{u}_b)^2 +$$

$$+(3/2)\xi_{ab}(T_b - T_a)] \equiv 0 \ . \qquad (4.18)$$

Following the same procedure as in the Gross-Krook model analysis, we shall require the model (4.17) to reproduce the partial momentum and energy collision integrals of the Boltzmann operator for the case where the species' distribution functions are Maxwellian for $\varepsilon \ll 1$ (general central force laws) and for $\varepsilon \gg 1$ (general inverse power force laws); again, the restrictions on the distribution functions and ε will be removed for the Maxwell molecule force law. The requirements for conservation of total momentum and energy will then be satisfied automatically.

The partial momentum and energy collision integrals of the Sirovich model are

$$[\delta(m_a\vec{c}_a)]_{ab} = \Lambda_{ab}(\vec{u}_b - \vec{u}_a) \qquad (4.19a)$$

$$[\delta(\tfrac{1}{2}\, m_a c_a^2)]_{ab} = \tfrac{3}{2}\, \xi_{ab}(T_b - T_a) + \chi_{ab}(\vec{u}_b - \vec{u}_a)^2 \ . \qquad (4.19b)$$

Equating (4.5a) to (4.19a), and (4.5b) to (4.19b), gives us

$$\Lambda_{ab} = \mu N_a \nu_{ab} \qquad (4.20a)$$

$$\xi_{ab} = 2(\mu/m_o) N_a K \nu_{ab} \qquad (4.20b)$$

$$\chi_{ab} = (\mu^2/m_a) N_a \nu_{ab} \quad , \qquad (4.20c)$$

where in obtaining (4.20b,c) we have made use of the fact that $(T_b - T_a)$ and $(\vec{u}_a - \vec{u}_b)^2$ are, in general, independent quantities.

As noted in the Gross-Krook model analysis, the results for $\varepsilon >> 1$, general inverse power force laws and Maxwellian distribution functions, can easily be obtained from the results for $\varepsilon << 1$; the only change here is that ξ_{ab} is now given by (4.20b) times $(n+4)/3$, and ν_{ab} in (4.20a,b,c) is now given by (3.59a).

The partial pressure collision integral of the Sirovich model is

$$[\delta(m_a c_{aj} c_{ak})]_{ab} = -\nu'_{aa} P_{ajk} \delta_{ab} + (2/3)\delta_{jk}[(3/2)\xi_{ab}(T_b - T_a) + \chi_{ab}(\vec{u}_a - \vec{u}_b)^2]$$

$$= -\nu'_{aa} P_{ajk} \delta_{ab} + \tfrac{2}{3}\delta_{jk}[\delta(\tfrac{1}{2}\, m_a c_a^2)]_{ab} \quad , \qquad (4.21)$$

where the first term in (4.21) appears only in self-collisions, i.e. "a" = "b" . Substituting for ξ_{ab} , χ_{ab} , (4.20b,c), we have

$$[\delta(m_a c_{aj} c_{ak})]_{ab} = -\nu'_{aa}\, P_{ajk}\delta_{ab} + (2/3)\delta_{jk} N_a \nu_{ab} \mu[(\mu/m_a)(\vec{u}_a - \vec{u}_b)^2 +$$

$$+ (3K/m_o)(T_b - T_a)] \; . \qquad (4.22)$$

Comparison with the Boltzmann result (4.10) shows that the Sirovich result (4.22) contains neither the "cross" traceless pressure term, P_{bjk} , nor the term involving $(\vec{u}_b-\vec{u}_a)_j(\vec{u}_b-\vec{u}_a)_k$. The terms in brackets, "[ ]" , in (4.22) are in very close agreement with the corresponding terms in (4.10). What is striking is the fact that the traceless pressure of species "a" , P_{ajk} , appears in the total pressure collision integral, $\delta(m_a c_{aj} c_{ak}) = \sum_b [\delta(m_a c_{aj} c_{ak})]_{ab}$, solely through self collisions (cf. first term in (4.22)); this of course is a consequence of the fact that the species "a" distribution function, F_a , appears in the Sirovich model for $(\delta F_a/\delta t)_{coll.}$ only in the self-collision term (the first term in (4.17)). To see more clearly the implication of this we write the equation for the species "a" traceless pressure (see (2.28d))

$$\frac{D_a P_{ajk}}{Dt} + \cdots = \delta(m_a c_{aj} c_{ak}) - \frac{2}{3}\,\delta_{jk}\,\delta(\tfrac{1}{2}\, m_a c_a^2) = -\nu'_{aa}\, P_{ajk} \; , \qquad (4.23)$$

where we have summed (4.21) over all species "b" . Expression (4.23) is exactly the result of the Krook model for a simple gas; clearly, the Sirovich model is inadequate for the calculation of traceless pressure for any system other than a simple gas.

The adjustment of the self-partial pressure collision integral is identical to that of the Gross-Krook model analysis

$$\nu'_{aa} = [3A_2(5)/4A_1(5)]\nu_{aa} \approx 0.775\nu_{aa} \ , \qquad (4.24)$$

which of course gives the discrepancy between the Sirovich and Boltzmann results if we were to identify ν'_{aa} with ν_{aa}.

The partial heat flow collision integral of the Sirovich model is

$$[\delta(\tfrac{1}{2} m_a c_a^2 c_{ak})]_{ab} = -\nu'_{aa}\, q_{ak}\delta_{ab} - (5K\, T_a/2m_a)\Lambda_{ab}(\vec{u}_a-\vec{u}_b)_k$$

$$= -\nu'_{aa}\, q_{ak}\delta_{ab} + (5K\, T_a/2m_a)[\delta(m_a c_{ak})]_{ab} \ . \qquad (4.25)$$

Comparison with (4.15) shows that the Sirovich result (4.25) does not contain the terms involving $(\vec{u}_b-\vec{u}_a)_k(\vec{u}_b-\vec{u}_a)^2$, $(\vec{u}_b-\vec{u}_a)_k(T_b-T_a)$ as does the Gross-Krook result. The comment regarding the P_{ajk} term in the Sirovich pressure integral (4.22) holds here for the q_{ak} term. The equation for the species "a" heat flow is (see (2.28e))

$$\frac{D_a q_{ak}}{Dt} + \ldots = \delta(\tfrac{1}{2} m_a c_a^2 c_{ak}) - \frac{1}{\rho_a}(\tfrac{5}{2} p_a \delta_{jk} + P_{ajk})\delta(m_a c_{aj})$$

$$= -\nu'_{aa} q_{ak} + (1/\rho_a) P_{ajk} \sum_b \Lambda_{ab}(\vec{u}_a-\vec{u}_b)_j =$$

$$= -\nu'_{aa} q_{ak} + P_{ajk} \sum_b (m_b/m_o)\nu_{ab}(\vec{u}_a - \vec{u}_b)_j \quad . \tag{4.26}$$

This is essentially the simple Krook model result with the additional term in (4.26) reflecting the difference in species flow velocities.

4.3 Models Based Exclusively Upon Equivalence of Collision Integrals

The forms of the Gross-Krook and Sirovich models are results of detailed physical and mathematical considerations with a view towards approximating the Boltzmann binary collision operator, (2.21); the equivalence of collision integrals is more of a secondary concern in this respect. In this section we will construct a kinetic model with the exclusive goal of reproducing exactly the partial collision integrals of the Boltzmann operator. The philosophy we assume is that, since the model imitates exactly the Boltzmann operator with regard to collision integrals, one can expect the constructed collision term $(\delta F/\delta t)_{coll.}$ to be fairly satisfactory in problems where the equation of motion (2.20) is solved directly for the distribution function (e.g. the propagation of a longitudinal sound wave in a plasma).

Before proceding, we may note that the Sirovich model (4.17) can be substantially improved, with regard to its partial pressure and heat flow collision integrals, in a very simple manner; if we replace the self-collision term in (4.17) by

$$-\nu'_{aa}(F_a-F_a^{(o)}) \rightarrow -\sum_b \nu'_{ab}(F_a-F_a^{(o)}) \quad , \tag{4.27}$$

then the partial momentum and energy collision integrals (4.19a,b) are unchanged, and the first term in the partial pressure collision integral (4.22) is replaced by

$$-\nu'_{aa}\, P_{ajk}\delta_{ab} \rightarrow -\, \nu'_{ab}\, P_{ajk} = -(\nu_{ab}/\alpha_{ab})P_{ajk} \quad . \tag{4.28}$$

In (4.27), (4.28), ν'_{ab} is the revised model's collision frequency between species "a" and "b" , taken as usual to be velocity-independent; in (4.28), $\alpha_{ab} \equiv \nu_{ab}/\nu'_{ab}$, as in the Gross-Krook model analysis. We note that the replacement indicated in (4.27) amounts to replacing the self-collision frequency ν'_{aa} by the total collision frequency for species "a" , $\sum_b \nu'_{ab}$.

The comparison of the P_{ajk} terms in the partial pressure collision integrals (Boltzmann: revised Sirovich model) is then identical to that made in the Gross-Krook model, (4.12a,b,c). However, because the "free" parameter α_{ab} occurs in the revised model's pressure integral only through the term (4.28), as contrasted with the Gross-Krook result (4.11) where α_{ab} appears in terms involving $(\vec{u}_b-\vec{u}_a)$ as well as P_{ajk} , the revised model's result agrees equally "well" (keeping in mind the missing terms) with the Boltzmann result for all three mass ratios: $m_a/m_b \stackrel{\sim}{\sim} 1$, $m_a/m_b << 1$, $m_a/m_b >> 1$. Of course, if α_{ab} is adjusted for agreement of the P_{ajk} terms in $[\delta(m_a c_{aj} c_{ak})]_{ab}$ for the case $m_a << m_b$, then the ratio of P_{bjk} coefficients in $[\delta(m_b c_{bj} c_{bk})]_{ba}$ is of the order $m_a/m_b <<1$, just as in the case of the Gross-Krook model.

The revision (4.27) results in the first term in the partial heat flow collision integral (4.25) being replaced by

$$-\nu'_{aa} q_{ak} \delta_{ab} \rightarrow -\nu'_{ab} q_{ak} = -(\nu_{ab}/\alpha_{ab}) q_{ak} \quad . \qquad (4.29)$$

The comparison of the q_{ak} terms in the partial heat flow collision integrals (Boltzmann: revised Sirovich model) is then the same as that for the P_{ajk} terms in the partial pressure collision integrals.

With a view towards reproducing exactly the four partial collision integrals of the Boltzmann operator, for the most general type of problem, we now suggest the following scheme. In analogy to the Grad expansion of the distribution functions in Section 2.3, we expand the partial collision term $(\delta F_a/\delta t)_{ab \atop \text{coll.}}$ in three-dimensional Hermite polynomials with a contraction of the third order tensor (see equation preceding (2.38))

$$(\delta F_a/\delta t)_{ab \atop \text{coll.}} = F_a^{(o)}[X^{ab} + X_i^{ab} c_{ai} + X_{ij}^{ab} c_{ai} c_{aj} + Y_i^{ab} c_a^2 c_{ai}] \quad . \qquad (4.30)$$

As it stands there are a total of thirteen independent scalar parameters in the model (4.30); this is the minimum number required for: conservation of species' mass (one required); equivalence of partial momentum collision integrals (three required); equivalence of partial pressure collision integrals, which includes equivalence of partial energy collision integrals (six required); and equivalence of partial heat flow collision integrals (three required).

Now, for brevity of notation, let M_k^{ab}, M_{jk}^{ab}, $\hat{M}_k^{ab}$ denote, respectively, the partial momentum, pressure, and heat flow collision integrals of the Boltzmann operator (in a completely general sense, with no qualifications as to diffusion Mach number, force law, etc.). The parameters in (4.30) are then evaluated by imposing the following "constraints" upon the model:

(i) conservation of species' mass:

$$0 = \int m_a(\delta F_a/\delta t)_{\substack{ab\\ \text{coll.}}} d\vec{v} = m_a \int F_a^{(o)}(X^{ab} + X_{ij}^{ab}\, c_{ai}c_{aj})d\vec{c}_a ,$$

$$\text{or} \quad 0 = X^{ab} + (a_a^2/2)X_{ii}^{ab} \quad , \quad a_a^2 \equiv 2K\, T_a/m_a \; ; \qquad (4.31a)$$

(ii) equivalence of partial momentum collision integrals:

$$M_k^{ab} = m_a \int c_{ak}(\delta F_a/\delta t)_{\substack{ab\\ \text{coll.}}} d\vec{v} = m_a \int F_a^{(o)} c_{ak}(X_i^{ab} c_{ai} + Y_i^{ab} c_a^2 c_{ai})d\vec{c}_a ,$$

$$\text{or} \quad M_k^{ab} = p_a[X_k^{ab} + (5a_a^2/2)Y_k^{ab}] \; ; \qquad (4.31b)$$

(iii) equivalence of partial pressure collision integrals:

$$M^{ab}_{jk} = m_a \int c_{aj} c_{ak} (\delta F_a/\delta t)_{\substack{ab \\ \text{coll.}}} d\vec{v} = m_a \int F^{(o)}_a c_{aj} c_{ak} (X^{ab} + X^{ab}_{pq} c_{ap} c_{aq}) d\vec{c}_a \ ,$$

$$\text{or} \quad M^{ab}_{jk} = p_a [\delta_{jk} X^{ab} + (a^2_a/2)(\delta_{jk} X^{ab}_{ii} + 2X^{ab}_{jk})] \ ; \tag{4.31c}$$

(iv) equivalence of partial heat flow collision integrals:

$$\tilde{M}^{ab}_{k} = (m_a/2) \int c^2_a c_{ak} (\delta F_a/\delta t)_{\substack{ab \\ \text{coll.}}} d\vec{v} = (m_a/2) \int F^{(o)}_a c^2_a \, c_{ak} (X^{ab}_i c_{ai} +$$

$$+ Y^{ab}_i c^2_a c_{ai}) d\vec{c}_a \ ,$$

$$\text{or} \quad \tilde{M}^{ab}_{k} = (5a^2_a p_a/4)[X^{ab}_k + (7a^2_a/2) Y^{ab}_k] \ . \tag{4.31d}$$

Note that (iii) assures the equivalence of the partial energy collision integrals. Solving (4.31a-d), we obtain

$$X^{ab} = -(1/2p_a) M^{ab}_{ii} \tag{4.32a}$$

$$X^{ab}_i = (1/p_a)[(7/2)M^{ab}_i - (2/a^2_a)\tilde{M}^{ab}_i] \tag{4.32b}$$

$$X^{ab}_{ij} = (1/p_a a^2_a) M^{ab}_{ij} \tag{4.32c}$$

$$Y^{ab}_i = (1/p_a a^2_a)[(4/5a^2_a)\tilde{M}^{ab}_i - M^{ab}_i] \ . \tag{4.32d}$$

Substituting (4.32a-d) into (4.30) gives us

$$\left(\frac{\delta F_a}{\delta t}\right)_{\substack{ab \\ coll.}} = \frac{F_a^{(o)}}{p_a}\left[-\frac{1}{2} M_{ii}^{ab} + \left(\frac{7}{2} M_i^{ab} - \frac{2\tilde{M}_i^{ab}}{a_a^2}\right)c_{ai} + M_{ij}^{ab}\frac{c_{ai}c_{aj}}{a_a^2} + \right.$$

$$\left. + \left(\frac{4}{5}\frac{\tilde{M}_i^{ab}}{a_a^2} - M_i^{ab}\right)\frac{c_a^2}{a_a^2} c_{ai}\right] \quad . \tag{4.33}$$

We note that when $F_a = F_a^{(o)}$, $F_b = F_b^{(o)}$, $\vec{u}_a = \vec{u}_b$, and $T_a = T_b$, all partial collision integrals of the Boltzmann operator vanish (since the Boltzmann operator itself vanishes for this case), so that $(\delta F_a/\delta t)_{\substack{ab \\ coll.}}$ as given by (4.33) also vanishes for this case. For a simple gas, (4.33) becomes

$$\left(\frac{\delta F_a}{\delta t}\right)_{\substack{aa \\ coll.}} = \frac{F_a^{(o)}}{p_a}\left[M_{ij}^{aa}\frac{c_{ai}c_{aj}}{a_a^2} + \frac{2\tilde{M}_i^{aa}}{a_a^2} c_{ai}\left(\frac{2}{5}\frac{c_a^2}{a_a^2} - 1\right)\right] \quad . \tag{4.34}$$

If we consider the Maxwell molecule force law in a completely linearized problem where flow velocity and temperature differences, traceless pressures, and heat flows are all first order quantities, then we have from (3.72), (3.78), (3.82), respectively,

$$M_i^{ab} = N_a \nu_{ab} \mu (\vec{u}_b - \vec{u}_a)_i \tag{4.35a}$$

$$M_{ij}^{ab} = N_a \nu_{ab} (\mu/m_o)[2\delta_{ij} K(T_b - T_a) + 0.45\frac{P_{bij}}{N_b} - 3.55(m_b/m_a)\frac{P_{aij}}{N_a}] \tag{4.35b}$$

$$\tilde{M}_i^{ab} = N_a \nu_{ab} \mu \{ \frac{5}{2} \frac{KT_a}{m_a} (\vec{u}_b - \vec{u}_a)_i + 1.94 (m_b/m_o)^2 \frac{q_{bi}}{\rho_b}$$

$$- [1.94 (m_b/m_o)^2 - 3.94 (m_b/m_o) + 3] \frac{q_{ai}}{\rho_a} \} \quad . \tag{4.35c}$$

As an example we consider the weakly ionized gas (see Section 5.2) where "a" and "b" represent electrons and neutrals, respectively; then $m_a/m_b << 1$, and $N_a/N_b \to 0$. For this case, the total collision term, $(\delta F_a/\delta t)_{coll.}$, can be approximated by $(\delta F_a/\delta t)_{ab \atop coll}$, and (4.35a-c) become, respectively,

$$M_i^{ab} = \nu_{ab} \rho_a (\vec{u}_b - \vec{u}_a)_i \tag{4.36a}$$

$$M_{ij}^{ab} = \nu_{ab} [2 \delta_{ij} \frac{\rho_a K}{m_b} (T_b - T_a) - 3.55 \, P_{aij}] \tag{4.36b}$$

$$\tilde{M}_i^{ab} = \nu_{ab} [\frac{5}{2} p_a (\vec{u}_b - \vec{u}_a)_i - q_{ai}] \, . \tag{4.36c}$$

Substitution of (4.36a,b,c) into (4.33) then gives us

$$(\frac{\delta F_a}{\delta t})_{coll.} \approx (\frac{\delta F_a}{\delta t})_{ab \atop coll.} = \nu_{ab} F_a^{(o)} \{ [2 (\vec{u}_b - \vec{u}_a)_i + 2 \frac{q_{ai}}{p_a}] \frac{c_{ai}}{a_a^2} -$$

$$-3.55 \frac{P_{aij}}{p_a} \frac{c_{ai} c_{aj}}{a_a^2} - 0.80 \frac{q_{ai}}{p_a} \frac{c_a^2}{a_a^4} c_{ai} \} \quad . \tag{4.37}$$

4.4 Comparison of Models

In order to determine the models' parameters we have required the Gross-Krook and Sirovich models to reproduce the partial momentum and energy collision integrals of the Boltzmann operator for the case where the species' distribution functions are Maxwellian for $\varepsilon \ll 1$ (general central force laws), and $\varepsilon \gg 1$ (general inverse power force laws); the restrictions on the distribution functions and ε were removed for the Maxwell molecule interparticle force law. We have seen that no matter what the parameters are, the partial pressure and heat flow collision integrals of the Boltzmann operator cannot be reproduced, even for the "simple" Maxwell molecule force law. However, the Gross-Krook model is a decidedly more accurate model in imitating the Boltzmann results than is the Sirovich model; as a matter of fact, the Sirovich model's partial traceless pressure integral was shown to vanish when "a" ≠ "b" , and to yield the simple Krook model result when "a" = "b" (see (4.21)). We shall show in Chapter V that, under certain conditions, the Gross-Krook model is sufficient for the calculation of traceless pressure and heat flow in the case of a weakly ionized gas. Of course, we have seen that even the more "accurate" of the two models, namely the Gross-Krook model, is in serious disagreement with the Boltzmann operator in regard to the partial pressure and heat flow collision integrals when $m_a \gg m_b$.

The revised Sirovich model is a substantial improvement over the Sirovich model but still does not reproduce the form of the Boltzmann results as well as the Gross-Krook model does; however, this revised model can also be shown to be adequate for a weakly ionized gas, subject to certain restrictions.

The "equivalence" model, (4.33), of course, is able to reproduce exactly all four partial collision integrals of the Boltzmann operator for the most general problem (i.e. no restrictions on the distribution functions, the force law, or ε); consequently, its parameters are, in general, quite complex. However, for linearized problems, particularly in the case of a weakly ionized gas, the parameters are considerably simplified.

Finally, we should emphasize again that the validity of these kinetic models as substitutes for the Boltzmann binary collision operator can, to a large extent, be ascertained by their ability to reproduce the partial collision integrals of that operator.

CHAPTER V

REDUCTION OF THE TRANSFER EQUATIONS -- CALCULATIONS OF THE TRACELESS PRESSURES AND HEAT FLOWS

In this chapter we return to our theme of transfer or transport phenomena. Up to now we have been concerned with the closed set of transfer equations (2.44a-e) for the thirteen moments N_s, $\vec{u}_s$, T_s or p_s, P_{sjk}, $\vec{q}_s$ ("closed" in the sense of the footnote on page 23). As they stand, these equations are, in general, untractable; we thus seek some means of simplification. One means, of course, is simply to <u>ignore</u> the traceless pressures and heat flows, i.e. <u>set</u> $P_{sjk} \equiv 0$, $\vec{q}_s \equiv 0$; this amounts to assuming Maxwellian distribution functions, $F_s = F_s^{(o)}$ (see (2.41)), and leads to the closed set of five transfer equations for the five moments N_s, $\vec{u}_s$, T_s or p_s, i.e. (2.44a-c) with $P_{sjk} \equiv 0$ and $\vec{q}_s \equiv 0$. This set of equations, as noted in Chapter I, has been considered by several authors for various force laws.

A less restrictive technique for simplifying the set of transfer equations (2.44a-e) is to <u>retain</u> the traceless pressures and heat flows in such a manner that they are expressible in terms of the <u>first five</u> moments N_s, $\vec{u}_s$, T_s or p_s. In this way the set of thirteen transfer equations is <u>reduced</u> to a set of <u>five</u> transfer equations involving the first <u>five</u> moments. The degree of difficulty involved in solving the system's transfer equations is thus considerably reduced, while at the same time a wide class of non-local equilibrium problems is admitted (i.e. the effects of "viscosity", corresponding to the traceless pressures, and "thermal conductivity", corresponding to the heat flows, are retained).

Of course, in order to express the higher order moments P_{sjk} and $\vec{q}_s$ in terms of the first five moments and thereby obtain a closed set of transfer equations in N_s, $\vec{u}_s$, T_s or p_s , we must be able to "solve" the equations for P_{sjk} and $\vec{q}_s$, (2.44d,e). We shall see in Section 5.1 that, subject to certain restrictions on the spatial and time variations of the macroscopic properties of the gas mixture, the equations (2.44d,e) reduce to coupled algebraic equations; their solution is then relatively straightforward. The resulting expressions for P_{sjk} and $\vec{q}_s$ can be substituted into the first five transfer equations (2.44a-c); the complete set of 5r such equations ($r \equiv$ number of species) then describes the gas mixture. Subject to further approximations (e.g. a completely linearized system) this set can then be solved for the first 5r moments, N_s , $\vec{u}_s$, T_s or p_s .

In this chapter we shall calculate the traceless pressures and heat flows for: (1) a weakly ionized gas, with arbitrary inverse power interparticle force laws and a magnetic field of arbitrary magnitude, and (2) a binary Maxwell molecule gas with arbitrary mass and number density ratios. Then, in the last section we shall determine the traceless pressure for the entire mixture in terms of the system's current density and flow velocity, for a fully ionized gas. All of the calculations are for small diffusion Mach number.

5.1 Reduction of the Traceless Pressure and Heat Flow Equations to Algebraic Expressions

It will be recalled that all of the calculations in Chapters II, III were for the case where the species' distribution functions were "close" to their local equilibrium forms, the Maxwellian distributions, $F_s^{(o)}$ (see Section 2.4).* For this case, of course, the traceless pressures and heat flows, P_{sjk} and $\vec{q}_s$, are "small" in the rough sense expressed by (2.48) - (2.50). Keeping this in mind, we see that the dominant terms on the left-hand sides of the traceless pressure and heat flow equations, (2.44d,e), are those which do not involve P_{sjk} or $\vec{q}_s$, with the possible exception of the terms involving the magnetic field, $\vec{B}$ (since $|\vec{B}|$ could conceivably become arbitrarily large). Let us now concentrate on the remaining terms on the left-hand sides of (2.44d,e) which do involve P_{sjk} and $\vec{q}_s$, bearing in mind that the right-hand sides (i.e. the collision integrals) also contain terms involving P_{sjk} and $\vec{q}_s$; we wish to compare these two sets of terms. For this

*The Maxwell molecule calculations of Section 3.5 were of course independent of the species' distribution functions.

purpose the P_{sjk} terms on the <u>right</u>-hand side of (2.44d) may be roughly viewed as being of the form

$$P_{sjk} \text{ terms on right-hand side of (2.44d)} \sim \nu_s P_{sjk} \;, \qquad (5.1a)$$

while the $\vec{q}_s$ terms on the <u>right</u>-hand side of (2.44e) may be viewed as

$$\vec{q}_s \text{ terms on right-hand side of (2.44e)} \sim \nu_s \vec{q}_s \;, \qquad (5.1b)$$

where ν_s is the <u>total</u> collision frequency of species "s" , $\nu_s \equiv \sum_t \nu_{st}$ (see (3.42) - (3.45)). We now limit our attention to gas mixtures in which the spatial and time variations of all macroscopic quantities are "small" in some sense; explicitly, we assume that

$$\tau_s/\overline{t} \ll 1 \quad \text{and} \quad \ell_s/\overline{x} \ll 1 \;, \qquad (5.2)$$

where $\tau_s = \nu_s^{-1}$ and $\ell_s = O(a_s \tau_s)$ are, respectively, the "<u>mean time between collisions</u>," and the "<u>mean free path</u>" of species "s" ; the quantities $\overline{t}$ and $\overline{x}$ in (5.2) are, respectively, the characteristic time and distance intervals for <u>significant</u> changes in the macroscopic properties of the mixture. It should be noted that what we are really assuming is that the <u>species' distribution</u>

functions are "slowly" varying functions of $(\vec{x},t)$, in the sense given by (5.2).

Then from (5.1a,b), (5.2), we see that the terms on the left-hand sides of (2.44d,e) which involve P_{sjk}, $\vec{q}_s$ can be neglected (except possibly for the terms involving $\vec{B}$) in comparison with the corresponding terms on the right-hand sides, which are given roughly by (5.1a,b). Hence, to this level of approximation, the traceless pressure and heat flow equations, (2.44d,e), reduce, respectively, to the following coupled algebraic equations:

$$-2\omega_{sc}\left(\frac{B_i}{|\vec{B}|}\epsilon_{ijh}P_{skh}\right)^{\dagger}+p_s f_{sjk} = \delta(m_s c_{sj}c_{sk}) - \frac{2}{3}\delta_{jk}\delta\left(\frac{1}{2}m_s c_s^2\right) \qquad (5.3)$$

$$\omega_{sc}\left(\frac{\vec{B}}{|\vec{B}|}\times\vec{q}_s\right)_k + \frac{5}{2}K\frac{p_s}{m_s}\frac{\partial T_s}{\partial x_k} = \delta\left(\frac{1}{2}m_s c_s^2 c_{sk}\right) -$$

$$-\frac{1}{\rho_s}\left(\frac{5}{2}p_s\delta_{ik}+P_{sik}\right)\delta(m_s c_{si}), ^{*} \qquad (5.4)$$

where $\omega_{sc} \equiv (e_s/m_s)|\vec{B}|$ (5.5a)

and $f_{sjk} \equiv \frac{\partial u_{sj}}{\partial x_k} + \frac{\partial u_{sk}}{\partial x_j} - \frac{2}{3}\delta_{jk}\nabla\cdot\vec{u}_s$. (5.5b)

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*The results (5.3), (5.4) can also be obtained using the so-called "transport approximation"[45] in which the assumptions (5.2) are applied directly to the Boltzmann equation (2.20), (2.21).

It will be noted from (5.3), (5.4), that we have retained the traceless pressure and heat flow terms involving the magnetic field $\vec{B}$ since, for sufficiently large $|\vec{B}|$, the "cyclotron frequency" for species "s", $|\omega_{sc}|$, can become comparable to or even greater than the total collision frequency ν_s (cf. (5.1a,b)).

Of course, there is, in general, a pair of analogous equations for each species in the mixture. The solution of the complete set of equations (i.e. all the P_{sjk}, $\vec{q}_s$, in terms of the number densities, flow velocities, and temperatures of all the species) is, in principle, straightforward, but, as might be expected, the algebra involved becomes progressively worse as the number of species increases.

5.2 The Weakly Ionized Gas with Arbitrary Cyclotron Frequency and General Interparticle Force Law

In this section we consider a weakly ionized macroscopicly neutral gas, in which the electrons and positive ions are trace species, with the electrons possessing a sufficiently high temperature so that the electron-neutral collision frequency is much larger than either the electron-electron or electron-ion collision frequencies. That is,

$$ZN_i/N_n \stackrel{\sim}{\sim} N_e/N_n << 1 \quad * \tag{5.6}$$

*A typical ratio, for ν_{en} corresponding to hard sphere force law, is $(\nu_{ee}+\nu_{ei})/\nu_{en} = O\{10^4(N_e/N_n)(3000/T_e)^2\}$; see (3.94).

so that ν_{ee} , $\nu_{ei} << \nu_{en}$, (5.7)

where "e" , "i" , "n" represent, respectively, electrons, positive ions, neutrals, and where "Z" is the charge number of the positive ions.* We wish to calculate the electron traceless pressure tensor and heat flow vector. From (5.7) we see that the total electron collision integrals may be approximated by the partial electron-neutral (en) collision integrals (see (3.42) - (3.45)) ,

$$(\delta Q)_e \approx (\delta Q)_{en} \quad . \qquad (5.8)$$

Furthermore, from (5.6) we have

$$\nu_{ne}, \nu_{ni} << \nu_{nn} \quad , \qquad (5.9)$$

so that the right-hand sides of the neutral species' traceless pressure and heat flow equations, (5.3), (5.4), can be approximated by the partial neutral-neutral (nn) collision integrals,**

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*The "n" used here is of course not to be confused with the "n" associated with inverse power interparticle force laws, $n \equiv -4/(p-1)$. (See (3.53a-d)).

**Note that we are not assuming $(\delta Q)_n \approx (\delta Q)_{nn}$, since this is obviously not true for the total momentum and energy collision integrals.

$$\delta(m_n c_{nj} c_{nk}) - \frac{2}{3}\,\delta_{jk}\,\delta(\tfrac{1}{2}\,m_n c_n^2) \approx [\delta(m_n c_{nj} c_{nk})]_{nn} - \frac{2}{3}\,\delta_{jk}[\delta(\tfrac{1}{2}\,m_n c_n^2)]_{nn}$$

$$= [\delta(m_n c_{nj} c_{nk})]_{nn} \qquad (5.10a)$$

$$\delta(\tfrac{1}{2}\,m_n c_n^2 c_{nk}) - \frac{1}{\rho_n}\,(\tfrac{5}{2}\,p_n \delta_{ik} + P_{nik})\,\delta(m_n c_{ni}) \approx [\delta(\tfrac{1}{2}\,m_n c_n^2 c_{nk})]_{nn} -$$

$$- \frac{1}{\rho_n}\,(\tfrac{5}{2}\,p_n \delta_{ik} + P_{nik})[\delta(m_n c_{ni})]_{nn}$$

$$= [\delta(\tfrac{1}{2}\,m_n c_n^2 c_{nk})]_{nn} \quad . \qquad (5.10b)$$

Then, <u>to first order in $P_{sjk}/p_s, \vec{q}_s/a_s p_s$, $\vec{\varepsilon}$</u> (cf. (2.49),(2.50)), we have from (3.42)-(3.45), (5.3), (5.4),(5.8) and (5.10a,b),

$$-2\omega_c(\varepsilon_{3ji} P_{eik})^{\dagger} + p_e f_{ejk} = -\nu_{en}\{\frac{1}{b_1}\,P_{ejk} - \frac{3}{2}\,(\frac{1}{3} + z^{(2)})\,\frac{\rho_e}{\rho_n}\,P_{njk}\}$$

$$(5.11)$$

$$p_n f_{njk} = -(3/4)(1-z_{nn}^{(2)})\,\nu_{nn} P_{njk} \qquad (5.12)$$

$$\omega_c(\vec{a}_3 \times \vec{q}_e)_k + \frac{5}{2}\,K\,\frac{p_e}{m_e}\,\frac{\partial T_e}{\partial x_k} = -\nu_{en}\{\frac{1}{b_2}\,q_{ek} - (\frac{1}{b_2} + 1 + 2z^{(2)})\,\frac{\rho_e}{\rho_n}\,q_{nk} -$$

$$-5K(\rho_e/m_n)(u_{nk}-u_{ek})[(1+z^{(2)})(T_n-T_e)-(z/2)(m_n/m_e)T_e]\}$$

$$(5.13)$$

$$(5/2)(Kp_n/m_n)\,\partial T_n/\partial x_k = -(1/2)(1-z_{nn}^{(2)})\,\nu_{nn} q_{nk} \quad , \qquad (5.14)$$

where $b_1 \equiv 2/3(1-z^{(2)})$ (5.15a)

$$b_2 \equiv [1+5z-(7/2)z']^{-1} \quad (5.15b)$$

and where all "z's" refer to (en) integrals unless otherwise specified. Finally, we have assumed a magnetic field in the $\vec{a}_3$-direction, $\vec{B} = B\vec{a}_3$, with $B \equiv |\vec{B}|$ arbitrary; ω_c is given by (we have suppressed the species subscript "e"), $\omega_c \equiv (-e/m_e)B$, $e > 0$, where e is the magnitude of the electron's charge. In obtaining the right-hand sides of (5.11), (5.13), we have assumed that $T_e/m_e >> T_n/m_n$, and have used the fact that $m_e/m_n << 1$.

We see that, to this level of approximation, the traceless pressure equations (5.11), (5.12), are completely decoupled from the heat flow equations (5.13), (5.14). We first solve for the electron traceless pressure tensor. We have from (5.11), (5.12) the following equations for the five independent elements of $\vec{P}_e$:

$$-2\alpha\, P_{e12} + P_{e11} = -n_o \tilde{f}_{11} \quad (5.16a)$$

$$2\alpha\, P_{e12} + P_{e22} = -n_o \tilde{f}_{22} \quad (5.16b)$$

$$\alpha(P_{e11} - P_{e22}) + P_{e12} = -n_o \vec{f}_{12} \quad (5.16c)$$

$$-\alpha\, P_{e23} + P_{e13} = -n_o \vec{f}_{13} \quad (5.16d)$$

$$\alpha P_{e13} + P_{e23} = -\eta_o \tilde{f}_{23} \tag{5.16e}$$

with the <u>redundant</u> equation

$$P_{e33} = -\eta_o \tilde{f}_{33} \tag{5.16f}$$

where $$\alpha \equiv b_1(\omega_c/\nu_{en}) = [2/3(1-z^{(2)})](\omega_c/\nu_{en}) \tag{5.17a}$$

$$\tilde{f}_{jk} \equiv f_{ejk} + [\frac{2(\frac{1}{3} + z^{(2)})}{(1 - z_{nn}^{(2)})}(m_e/m_n)(T_n/T_e)(\nu_{en}/\nu_{nn})]f_{njk} \quad , \tag{5.17b}$$

and where we have defined an "<u>electron viscosity for small magnetic fields</u>,"

$$\eta_o \equiv b_1(p_e/\nu_{en}) = [2/3(1-z^{(2)})](p_e/\nu_{en}) \ . \tag{5.17c}$$

The solution of (5.16a–e) is

$$P_{e11} = -\eta_o(\tilde{f}_{11} + 2\alpha\tilde{f}_{12} - 2\alpha^2\tilde{f}_{33})/(1 + 4\alpha^2) \tag{5.18a}$$

$$P_{e22} = -\eta_o(\tilde{f}_{22} - 2\alpha\tilde{f}_{12} - 2\alpha^2\tilde{f}_{33})/(1 + 4\alpha^2) \tag{5.18b}$$

$$P_{e12} = -\eta_o[\tilde{f}_{12} + \alpha(\tilde{f}_{22} - \tilde{f}_{11})]/(1 + 4\alpha^2) \tag{5.18c}$$

$$P_{e13} = -\eta_o(\tilde{f}_{13} + \alpha\tilde{f}_{23})/(1 + \alpha^2) \tag{5.18d}$$

$$P_{e23} = -\eta_o(\tilde{f}_{23} - \alpha\tilde{f}_{13})/(1 + \alpha^2) \ . \tag{5.18e}$$

We note from (5.16f), (5.17b,c), that P_{e33} <u>is independent of the magnetic field</u> (i.e. independent of ω_c) ; this simply reflects the fact that the magnetic force (recall that $\vec{B} = B\vec{a}_3$) on electrons moving in the $\vec{a}_3$-direction is zero so that the momentum carried by such electrons across a surface in the $\vec{a}_3$-direction, which is moving with the velocity $\vec{u}_e$, is not altered by the presence of the magnetic field, $\vec{B} = B\vec{a}_3$, which in turn means that P_{e33} , a <u>measure</u> of such momentum transport, is unchanged.

For <u>small</u> magnetic fields, we have from (5.18a-e)

$$P_{ejk} \to -n_o\tilde{f}_{jk} \quad \text{as} \quad |\alpha| = |[2/3(1-z^{(2)})](\omega_c/\nu_{en})| \to 0 \; . \qquad (5.19)$$

For an <u>infinitely large</u> magnetic field, we obtain

$$\overleftrightarrow{P}_e \to -n_o\tilde{f}_{33}\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{as} \quad |\alpha| \to \infty \; . \qquad (5.20)$$

The result (5.20) is not at all surprising from a mathematical view point; it could have been obtained with only a knowledge of the result (5.16f). Since the magnetic field $\vec{B} = B\vec{a}_3$ is so strong it produces the <u>only</u> preferred direction in the system, which in turn implies rotational symmetry about the $\vec{a}_3$-direction; hence, we immediately infer that $P_{e11} = P_{e22}$ and since $\overleftrightarrow{P}_e$ is always traceless, we find that $P_{e11} = P_{e22} = -\frac{1}{2} P_{e33} = \frac{1}{2} n_o \tilde{f}_{33}$, from (5.16f). Furthermore, because of the rotational symmetry about the $\vec{a}_3$-direction, we see that the distribution function $F_e(\vec{x},\vec{v},t)$ must be "isotropic" in the velocity $\vec{a}_1 v_1 + \vec{a}_2 v_2$, that is, $F_e = F_e(\vec{x}, v_1^2 + v_2^2, v_3, t)$; from this we immediately conclude that $u_{e1} = u_{e2} = 0$ (see (2.5)), so that $F_e = \tilde{F}_e(\vec{x}, c_{e1}^2 + c_{e2}^2, c_{e3}, t)$.* From the definition for P_{ejk}, (2.13), we then see that $P_{ejk} \equiv 0$ for $j \neq k$, since the integrands involving c_{e1} and c_{e2} are both odd.

We see from (5.19), (5.20) that the magnetic field has a pronounced effect upon the electron traceless pressure tensor, with $\overleftrightarrow{P}_e$ becoming diagonal for very large magnetic fields.

The results (5.18a-e) simplify for the case where the neutrals are in <u>local equilibrium</u>, $F_n = F_n^{(o)}$. The neutral traceless pressure then vanishes identically, and the $\tilde{f}_{jk}$ functions in (5.18a-e) become

$$\tilde{f}_{jk} = f_{ejk} \quad , \qquad F_n = F_n^{(o)} \quad , \tag{5.21}$$

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*This <u>type</u> of distribution function and the <u>form</u> of (5.20) appear as lowest order results for the <u>ions</u> in the Chew-Goldberger-Low[46] magnetohydrodynamic formulation of a <u>fully ionized</u>, <u>collisionless gas</u> subjected to a strong Lorentz force, $e_s(\vec{E}+\vec{v}\times\vec{B})$.

so that, for example,

$$P_{e11} = -n_o(f_{e11} + 2\alpha f_{e12} - 2\alpha^2 f_{e33})/(1+4\alpha^2) \quad , \quad F_n = F_n^{(o)} \quad . \qquad (5.22)$$

The results (5.21), (5.22), are generally valid even when F_n is not Maxwellian since the ratio of the f_{njk} coefficient in (5.17b) to the f_{ejk} coefficient is, for inverse power interparticle force laws, $(f_{st} = \kappa_{st}/r^p \ , \ n = -4/(p-1) \ , \ -4 \leq n \leq 0)$, roughly

$$\frac{m_e}{m_n}\,\frac{T_n}{T_e}\,\frac{\nu_{en}}{\nu_{nn}} \sim (m_e/m_n)^{(1-n)/2}(T_e/T_n)^{(n-1)/2} << 1 \quad \text{for } T_e \gtrsim T_n \ , \qquad (5.23)$$

so that the f_{njk} term in (5.17b) is generally negligible. In obtaining (5.23) we have used the expression for ν_{st} for $\varepsilon << 1$, (3.94), and have assumed the same force law (i.e. same n) for electron-neutral and neutral-neutral collisions ("hard spheres," for example), where the interparticle force law constant can be written as $\kappa_{st} = m_s m_t \kappa'_{st}$, with κ'_{st} independent of the masses m_s, m_t. (This latter assumption is of course not valid for the Coulomb force law, but one would hardly ascribe this force law to electron-neutral or neutral-neutral collisions.*) We see from (5.23) that

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*For other force laws where κ_{st} does not involve the masses m_s, m_t, (5.23) is replaced by $(m_e/m_n)^{1/2}\,(T_e/T_n)^{(n-1)/2} << 1$.

the f_{njk} term in (5.17b) is simply a "correction" term accounting for the finite mass, and hence, the finite mobility of the neutrals.

Expression (5.22) is the result we would obtain for Maxwell molecules using the Gross-Krook model of Section 4.1 if, in that model's partial pressure collision integral, (4.11), we consider terms to first order in ε and set $\alpha_{en} \equiv \nu_{en}/\nu'_{en} = b_1$ (compare the traceless form of (4.11) with (5.11), with $P_{njk} = 0$ in (5.11)). Furthermore, (5.22) is the exact result for Maxwell molecules using the revised Sirovich model of Section 4.3, with $\alpha_{en} \equiv \nu_{en}/\nu'_{en} = b_1$ (compare the traceless form of (4.22), (4.28) with (5.11), with $P_{njk} = 0$ in (5.11)).

Before proceding to the calculation of the electron heat flow vector we examine the influence of the interparticle force laws upon $\overleftrightarrow{P}_e$. Referring to the general results (5.17a,b,c), (5.18a-e), we see that this influence enters explicitly through the terms

$$b_1 \equiv 2/3(1-z^{(2)}) \quad \text{and} \quad 2(\tfrac{1}{3} + z^{(2)})/(1-z^{(2)}_{nn}) \quad . \tag{5.24}$$

From Appendix A we have for inverse power interparticle force laws, $f_{st} = \kappa_{st}/r^p$,

$$z^{(2)} = 1-(n+6)A_2(p)/5A_1(p) \quad , \quad n = -4/(p-1) \quad ,$$

so that $b_1 = 10A_1(p)/3(n+6)A_2(p) = \mathcal{O}(1)$ for $-4 \leq n \leq 0$ (5.25) where we have used the fact that $A_\ell(p) = O(1)$. For hard spheres $(n=0)$ and Maxwell molecules $(n=-1)$ we find, respectively, $b_1 = 5/6$, $b_1 = 0.65$. Similarly we find

$$2(\tfrac{1}{3} + z^{(2)})/(1-z_{nn}^{(2)}) = \frac{2[\frac{20}{3} -(n+6)A_2(p)/A_1(p)]}{(\hat{n}+6)A_2(\hat{p})/A_1(\hat{p})} = O(1) \qquad (5.26)$$

for $n = 0$ or -1, $\hat{n} = 0$ or -1, where $\hat{n}$, $\hat{p}$ refer to the <u>neutral-neutral</u> force law. Hence, from (5.17a,b,c), (5.18a–e), and (5.25), (5.26), we see that the interparticle force laws enter the result for $\overleftrightarrow{P}_e$ <u>essentially through the collision frequencies</u>, ν_{en}, ν_{nn}.

Returning to the heat flow equations (5.13), (5.14), we see that the electron heat flow equation can be written as

$$(\overleftrightarrow{I} + \alpha'\vec{a}_3 \times \overleftrightarrow{I})\cdot \vec{q}_e = -\lambda_o \widetilde{\nabla T} \qquad (5.27)$$

where $\overleftrightarrow{I}$ is the unit dyadic $\overleftrightarrow{I} \equiv \vec{a}_1\vec{a}_1 + \vec{a}_2\vec{a}_2 + \vec{a}_3\vec{a}_3$, and

$$\alpha' \equiv b_2(\omega_c/\nu_{en}) = [1+5z-(7/2)z']^{-1}(\omega_c/\nu_{en}) \qquad (5.28a)$$

$$\widetilde{\nabla T} \equiv \nabla T_e + \{[2(1+2z^{(2)}+1/b_2)/(1-z_{nn}^{(2)})](m_e/m_n)^2(T_n/T_e)(\nu_{en}/\nu_{nn})\}\nabla T_n +$$

$$+2(m_e^2\nu_{en}/Km_n)(\vec{u}_n-\vec{u}_e)[(1+z^{(2)})(1-T_n/T_e)+(z/2)(m_n/m_e)] , \qquad (5.28b)$$

and where we have defined an "<u>electron thermal conductivity for small magnetic fields</u>,"

$$\lambda_o \equiv (5b_2/2)(Kp_e/m_e\nu_{en}) = (5/2)[1+5z-(7/2)z']^{-1}(Kp_e/m_e\nu_{en}) . \quad (5.28c)$$

The solution of (5.27) is simply

$$\vec{q}_e = -\lambda_o(\overleftrightarrow{I} + \alpha'\vec{a}_3 x \overleftrightarrow{I})^{-1} \cdot \widetilde{\nabla T} = -\overleftrightarrow{\lambda} \cdot \widetilde{\nabla T} \quad (5.29)$$

where

$$\overleftrightarrow{\lambda} \equiv \begin{bmatrix} \lambda_\perp & \lambda_H & 0 \\ -\lambda_H & \lambda_\perp & 0 \\ 0 & 0 & \lambda_o \end{bmatrix} \quad (5.29a)$$

and $\lambda_\perp \equiv \lambda_o/(1+\alpha'^2)$, $\lambda_H \equiv \lambda_o\alpha'/(1+\alpha'^2)$. (5.29b)

The notation $\lambda_\perp$, λ_H will be explaned shortly.

We note from (5.29), (5.29a,b), that for <u>small</u> magnetic fields we have

$$\vec{q}_e \rightarrow -\lambda_o\widetilde{\nabla T} \text{ as } |\alpha'| = |[1+5z-(7/2)z']^{-1}(\omega_c/\nu_{en})| \rightarrow 0 , \quad (5.30)$$

while for infinitely large magnetic fields

$$\vec{q}_e \to -\lambda_o(\widetilde{\nabla T})_3\vec{a}_3 \quad \text{as} \quad |\alpha'| \to \infty \tag{5.31}$$

so that $\vec{q}_e$ is parallel or anti-parallel to the magnetic field, $\vec{B} = B\vec{a}_3$, for this case.

When the neutrals are in <u>local equilibrium</u>, $F_n = F_n^{(o)}$, the neutral heat flow vector vanishes identically and (5.29) becomes

$$\vec{q}_e = -\overleftrightarrow{\lambda} \cdot \{\nabla T_e + 2(m_e^2\nu_{en}/Km_h)(\vec{u}_h - \vec{u}_e)[(1+z^{(2)})(1-T_n/T_e)+(z/2)(m_h/m_e)] \ , \quad F_n = F_n^{(o)} \ . \tag{5.32}$$

Again, the result (5.32) is generally valid even when F_n is <u>not</u> Maxwellian since the ratio of the ∇T_n coefficient in (5.28b) to the ∇T_e coefficient is, for inverse power interparticle force laws, roughly

$$\left.\begin{array}{l} (m_e/m_h)^2(T_n/T_e)(\nu_{en}/\nu_{nn}) \sim (m_e/m_h)^{(3-n)/2}(T_e/T_n)^{(n-1)/2} \ll 1 \\ \qquad \text{or} \quad \sim (m_e/m_h)^{3/2}(T_e/T_n)^{(n-1)/2} \ll 1 \end{array}\right\} \text{for } T_e \gtrsim T_n \ , \tag{5.33}$$

depending upon whether or not κ_{st} involves the masses m_s, m_t (see the discussion following (5.23)); hence, the ∇T_n term in (5.28b) is generally negligible.

The result (5.32) can also be obtained (for Maxwell molecules) using the Gross-Krook model if, in that model, we set $\alpha_{en} \equiv \nu_{en}/\nu'_{en} = b_2$ (compare (5.13) with $\vec{q}_n = 0$, $1+z^{(2)} \approx 1.03$, $z = 0$, with (4.15) minus $(5/2)(KT_e/m_e)$ times (4.5a)).

When $T_e \approx T_n$, (5.32) becomes

$$\vec{q}_e = -\overleftrightarrow{\lambda}\cdot\{\nabla T_e + zm_e \frac{\nu_{en}}{K}(\vec{u}_n - \vec{u}_e)\} \ , \ T_e \approx T_n \ . \tag{5.34}$$

Finally, for Maxwell molecules, $z = 0$, and (5.34) becomes

$$\vec{q}_e = -\overleftrightarrow{\lambda}\cdot\nabla T_e \quad , \quad T_e \approx T_n \ , \text{ Maxwell molecules .} \tag{5.35}$$

This is the exact result obtained from the revised Sirovich model if, in that model, $\alpha'_{en} \equiv \nu_{en}/\nu'_{en} = b_2$ (compare (5.13) with the revised result (4.25), (4.29) minus $(5/2)(KT_e/m_e)$ times (4.5a)).

Let us return to the general result (5.29) and explain the notation used in (5.29a,b); expanding (5.29) into its component parts, we have

$$\vec{q}_e = -\lambda_o(\widetilde{\nabla T})_3\vec{a}_3 - \lambda_\perp[(\widetilde{\nabla T})_1\vec{a}_1 + (\widetilde{\nabla T})_2\vec{a}_2] - \lambda_H[(\widetilde{\nabla T})_2\vec{a}_1 - (\widetilde{\nabla T})_1\vec{a}_2] \ . \tag{5.36}$$

The first term in (5.36) is in the direction of the magnetic field, $\vec{B} = B\vec{a}_3$; we see that for the same number densities, flow velocities, and temperatures, this component of $\vec{q}_e$ is independent of the $\vec{B}$ field (i.e. independent of ω_c). This simply reflects the fact that the magnetic force on electrons moving in the direction of $\vec{B}$ is zero so that the random kinetic energy carried by such electrons across a surface in the direction of $\vec{B}$ is unchanged by the presence of the $\vec{B}$ field, which in turn implies that q_{e3} , which is a measure of such energy transport, is unchanged. The second term in (5.36) is normal to the $\vec{B}$ field ; from (5.29b) we see that it is reduced for increasing magnetic fields. In this case the non-zero magnetic force on electrons moving perpendicular to the $\vec{B}$ field tends to reduce their transport of random kinetic energy across a surface normal to the $\vec{B}$ field, so that $|q_{e\perp}|$ is reduced by the presence of a magnetic field. The last term in (5.36) is normal to both $(\widetilde{\nabla T})$ and $\vec{B} = B\vec{a}_3$; this is the so-called "Hall" heat flow, which of course vanishes as $|\alpha'| = |b_2(\omega_c/\nu_{en})| \to 0$.

As with the case of the electron traceless pressure tensor, it can easily be shown that the interparticle force laws (en , nn collisions) enter the result for $\vec{q}_e$, (5.28a-c), (5.29), (5.29a,b), essentially through the collision frequencies, ν_{en} and ν_{nn} , with the exception that the dominant term in the brackets, "[ ]" , in (5.28b), $(z/2)(m_n/m_e)$, vanishes for the Maxwell molecule interparticle force law (i.e. z = 0).

The results (5.22), (5.32), can be substituted into the first five transfer equations for the electron species whose variables N_e, $\vec{u}_e$, T_e or p_e , can then, in principle, be solved in terms of the neutral species' moments N_n, $\vec{u}_n$, T_n or p_n , with $\overleftrightarrow{P}_n$ and $\vec{q}_n$ given by (5.12), (5.14), respectively; alternately, one can work with the complete set of ten transfer equations (i.e. five for electrons and five for neutrals) in an attempt to solve for the ten moments N_s , $\vec{u}_s$, T_s or p_s , with "s" = e, n .

5.3 The Binary Maxwell Molecule System

In this section we shall consider a two-species gas in which the particles obey the Maxwell molecule interparticle force law. The calculations are made to first order in ε , assuming a small magnetic field (i.e. $|\omega_{sc}|$, "s" = 1, 2, is small compared to the dominant collision frequency). The essential difference from the calculations of the previous section is that here the species' traceless pressure equations are directly coupled, as are the species' heat flow equations. After obtaining the general results for species "1", we shall specialize to the following cases: (i) $m_1 \stackrel{\sim}{\sim} m_2$, (ii) $m_1 << m_2$, (iii) $m_1 >> m_2$.

To first order in ε , the traceless pressure equation for species "s" is, for Maxwell molecules, from (3.78), (3.79), (5.3),

$$p_s f_{sjk} = \sum_{t=1,2} (m_t/m_o)\rho_s \nu_{st} \{0.45 \frac{P_{tjk}}{\rho_t} - \frac{1}{N_s}(\frac{2}{m_t} + \frac{1.55}{m_s})P_{sjk}\} \text{ , s=1,2 .} \tag{5.37}$$

To the same order, the species "s" heat flow equation is, from (3.82), (3.72), (5.4),

$$\frac{5}{2} K \frac{p_s}{m_s} \frac{\partial T_s}{\partial x_k} = - \sum_{t=1,2} \nu_{st} \{4.84 (m_t/m_o)^2 (\rho_s K/m_o)(T_s - T_t)(u_{tk} - u_{sk}) +$$

$$+ (m_t/m_o)^3(u_{tj}-u_{sj})[(3.48\ \frac{m_o}{m_t} - 1.94)P_{sjk} - 1.94(\rho_s/\rho_t)P_{tjk}] -$$

$$-1.94(m_t/m_o)^3(\rho_s/\rho_t)q_{tk}+(m_t/m_o)^2(1.94\ \frac{m_t}{m_o} +3\ \frac{m_o}{m_t} -3.94)q_{sk}\} ,$$

$$s = 1,2 .$$

$$(5.38)$$

The solution of (5.37) is, for species "1",

$$P_{1jk} = -(1/D\nu_{11})\{[0.775(\nu_{22}/\nu_{12})-(N_1/N_2)(m_1/m_o)^2(1.55+2m_2/m_1)]p_1 f_{1jk}+$$

$$+ 0.45(m_2/m_o)^2(\rho_1/\rho_2)p_2 f_{2jk}\} , \qquad (5.39)$$

where $D \equiv \{[0.60+0.775(\mu/m_o)(2+1.55m_2/m_1)(\nu_{12}/\nu_{11})](\nu_{22}/\nu_{12}) +$

$$+1.55+1.20(m_1/m_2)+3.10[2+(m_1/m_2)+(m_2/m_1)](\mu/m_o)(\nu_{12}/\nu_{11})\}.$$

$$(5.39a)$$

The solution of (5.38) is, for species "1",

$$q_{1k} = -(1/D')[0.515(\nu_{22}/\nu_{12})+(N_1/N_2)(m_1/m_o)^2(1.94(m_1/m_o) +$$

$$+3(m_o/m_1)-3.94)]\cdot\{(5/2)(Kp_1/m_1\nu_{12})(\partial T_1/\partial x_k) +$$

$$+4.84(m_2/m_o)^2(\rho_1 K/m_o)(T_1-T_2)(u_{2k}-u_{1k})+(m_2/m_o)^3(u_{2j}-u_{1j}) \cdot$$

$$\cdot[(-1.94+3.48m_o/m_2)P_{1jk}-1.94(\rho_1/\rho_2)P_{2jk}]\} -$$

$$-(1.94/D')(m_2/m_o)^3(\rho_1/\rho_2)\{(5/2)(Kp_2/m_2\nu_{12})(\partial T_2/\partial x_k) +$$

$$+(m_1/m_o)^3(N_1/N_2)(u_{1j}-u_{2j})[(-1.94+3.48m_o/m_1)P_{2jk}-1.94(\rho_2/\rho_1)P_{1jk}]\},$$

(5.40)

where $D' \equiv \{[0.265(\nu_{11}/\nu_{12})-0.515(m_2/m_o)^2\Omega_2](\nu_{22}/\nu_{12}) +$

$$+ [(\mu/m_o)^2\Omega_1\Omega_2-0.515(m_1/m_o)^2\Omega_1(\nu_{11}/\nu_{12})-3.76(\mu/m_o)^3](N_1/N_2)\}$$

(5.40a)

with $\Omega_s \equiv 3.94-1.94(m_s/m_o)-3(m_o/m_s)$, $s = 1,2$. (5.40b)

The solutions for species "2" are obtained from (5.39), (5.39a), (5.40), (5.40a,b), by simply interchanging the species' subscripts. Obviously, the results obtained so far are extremely cumbersome; we shall now consider some limiting cases where a certain amount of simplification is possible.

<u>Case (i)</u>: $m_1 \sim m_2$,

$$P_{1jk} = -[(0.688+1.55\nu_{12}/\nu_{11})+(0.601+0.688\nu_{12}/\nu_{11})N_2\nu_{22}/N_1\nu_{12}]^{-1} \cdot$$

$$\cdot(1/\nu_{11})[(0.877+0.775N_2\nu_{22}/N_1\nu_{12})p_1f_{1jk}+0.112p_2f_{2jk}]$$

(5.41)

$$q_{1k} = [(N_1/N_2)+0.515(\nu_{22}/\nu_{12})][(0.645+0.437\nu_{11}/\nu_{12})(\nu_{22}/\nu_{12}) +$$

$$+(0.515+0.645\nu_{11}/\nu_{12})(N_1/N_2)]^{-1}N_1K(T_2-T_1)(u_{2k}-u_{1k})-$$

$$-[(0.39+0.265\nu_{11}/\nu_{12})(\nu_{22}/\nu_{12})+(0.515+0.39\nu_{11}/\nu_{12})(N_1/N_2)]^{-1}\cdot$$

$$\cdot\{[0.515(\nu_{22}/\nu_{12})+0.757(N_1/N_2)](5/2)(Kp_1/m_1\nu_{12})(\partial T_1/\partial x_k)+$$

$$+0.605(N_1/N_2)(Kp_2/m_2\nu_{12})(\partial T_2/\partial x_k)+0.125(u_{2j}-u_{1j})[(2.58(\nu_{22}/\nu_{12})+$$

$$+4.27(N_1/N_2))P_{1jk}-(2.69+N_2\nu_{22}/N_1\nu_{12})(N_1/N_2)^2P_{2jk}]\} \quad ; \qquad (5.42)$$

The species "2" quantities are obtained by interchanging the subscripts "1" and "2" .

Case (ii): $m_1 \ll m_2$,

$$P_{1jk} = -[1.55+3.1(\nu_{12}/\nu_{11})+(0.601+1.2\nu_{12}/\nu_{11})(\rho_2\nu_{22}/\rho_1\nu_{12})]^{-1}\cdot$$

$$\cdot(1/\nu_{11})[(2+0.775\rho_2\nu_{22}/\rho_1\nu_{12})p_1f_{1jk}+0.45p_2f_{2jk}] \qquad (5.43)$$

$$q_{1k} = [14.5(\rho_1/\rho_2)+2.50(\nu_{22}/\nu_{12})][(0.515+0.265\nu_{11}/\nu_{12})(\nu_{22}/\nu_{12})+$$

$$+(3+1.54\nu_{11}/\nu_{12})(\rho_1/\rho_2)]^{-1}(m_1/m_2)N_1K(T_2-T_1)(u_{2k}-u_{1k})-$$

$$-[(0.515+0.265\nu_{11}/\nu_{12})(\nu_{22}/\nu_{12})+(3+1.54\nu_{11}/\nu_{12})(\rho_1/\rho_2)]^{-1}\cdot$$

$$\cdot\{[0.515(\nu_{22}/\nu_{12})+3(\rho_1/\rho_2)][(5/2)(Kp_1/m_1\nu_{12})(\partial T_1/\partial x_k) +$$

$$+(u_{2j}-u_{1j})(1.54P_{1jk}-1.94(\rho_1/\rho_2)P_{2jk})]+4.85(\rho_1/\rho_2)(Kp_2/m_2\nu_{12}) \cdot$$

$$\cdot(\partial T_2/\partial x_k)\} \ . \qquad (5.44)$$

The species "2" quantities are obtained by interchanging the subscripts "1" and "2" in the results of <u>case (iii) below</u>.

<u>Case (iii)</u>: $m_1 >> m_2$,

$$P_{1jk} = -[1.55(m_1/m_2)+4(\nu_{12}/\nu_{11})+(2+0.775m_1\nu_{11}/m_2\nu_{12})(N_2\nu_{22}/N_1\nu_{11})]^{-1} \cdot$$

$$\cdot(1/\nu_{11})[(m_1/m_2)(2+N_2\nu_{22}/N_1\nu_{12})p_1f_{1jk}+0.58p_2f_{2jk}] \qquad (5.45)$$

$$q_{1k} = [2.50(\nu_{22}/\nu_{12})+4.84(N_1/N_2)][(3+0.515m_1\nu_{11}/m_2\nu_{12})(N_1/N_2) +$$

$$+(1.54+0.265m_1\nu_{11}/m_2\nu_{12})(\nu_{22}/\nu_{12})]^{-1}(m_2/m_1)N_1K(T_2-T_1)(u_{2k}-u_{1k}) -$$

$$-[(0.515(\nu_{11}/\nu_{12})+3m_2/m_1)(N_1/N_2)+(0.265(\nu_{11}/\nu_{12})+1.54m_2/m_1) \cdot$$

$$\cdot(\nu_{22}/\nu_{12})]^{-1}\{[0.515(\nu_{22}/\nu_{12})+N_1/N_2](5/2)(Kp_1/m_1\nu_{12})(\partial T_1/\partial x_k) +$$

$$+4.85(m_2/m_1)^2(N_1/N_2)(Kp_2/m_2\nu_{12})(\partial T_2/\partial x_k)+(m_2/m_1)^2(u_{2j}-u_{1j}) \cdot$$

$$\cdot[(3.48(N_1/N_2)+1.79\nu_{22}/\nu_{12})P_{1jk}-(4.94+N_2\nu_{22}/N_1\nu_{12})(N_1/N_2)^2P_{2jk}]\} \; . \tag{5.46}$$

The species "2" quantities are obtained by interchanging the subscripts "1" and "2" in the results of case (ii) above.

To continue the simplification, we consider the case where $m_1 << m_2$ and $N_1 = N_2$; then, from cases (ii) and (iii) we have

$$P_{1jk} = -(1/\nu_{11})[0.534p_1f_{1jk}+0.31(m_1/m_2)(\nu_{12}/\nu_{22})p_2f_{2jk}] \tag{5.47a}$$

$$P_{2jk} = -(1/\nu_{22})[1.29p_2f_{2jk}+0.219(m_1/m_2)p_1f_{1jk}] \tag{5.47b}$$

$$\begin{aligned} q_{1k} = {} & 2.8(m_1/m_2)(p_2-p_1)(u_{2k}-u_{1k})-(1/\nu_{11})\{2.05(Kp_1/m_1)(\partial T_1/\partial x_k) + \\ & +5.46(m_1/m_2)(\nu_{11}/\nu_{22})(Kp_2/m_2)(\partial T_2/\partial x_k)+(u_{2j}-u_{1j})[-0.475p_1f_{1jk}+ \\ & +1.25(m_1/m_2)(\nu_{11}/\nu_{22})p_2f_{2jk}]\} \end{aligned} \tag{5.48a}$$

$$\begin{aligned} q_{2k} = {} & 9.43(m_1/m_2)^2(\nu_{12}/\nu_{22})(p_2-p_1)(u_{2k}-u_{1k})-(1/\nu_{22}) \cdot \\ & \cdot\{4.85(Kp_2/m_2)(\partial T_2/\partial x_k)+5.44(m_1/m_2)^2(Kp_1/m_1)(\partial T_1/\partial x_k) + \\ & +(m_1/m_2)^2(u_{2j}-u_{1j})[8.66(\nu_{12}/\nu_{22})p_2f_{2jk}-2.7p_1f_{1jk}]\} \; . \end{aligned} \tag{5.48b}$$

If we further assume that $T_1 \approx T_2$, then the first terms in (5.48a,b) are second order terms in $(T_2 - T_1)\varepsilon_k$, and may be neglected; for this case, if we write the Maxwell molecule collision frequency as (see (3.72a))

$$\nu_{st} = 2\pi A_1(5)(\kappa_{st}/\mu)^{1/2} N_t = 2\pi A_1(5)(m_o \kappa'_{st})^{1/2} N_t \qquad (5.49)$$

where κ'_{st} does not involve m_s , m_t , then (5.47a,b), (5.48a,b), become, respectively,

$$P_{1jk} = -0.534(p_1/\nu_{11}) f_{1jk} \qquad (5.50a)$$

$$P_{2jk} = -1.29(p_2/\nu_{22}) f_{2jk} \qquad (5.50b)$$

$$q_{1k} = -[2.05(Kp_1/m_1\nu_{11})(\partial T_1/\partial x_k) - 0.475(u_{2j} - u_{1j})(p_1/\nu_{11}) f_{1jk}] \qquad (5.51a)$$

$$q_{2k} = -\{4.85(Kp_2/m_2\nu_{22})(\partial T_2/\partial x_k) + (m_1/m_2)^2 (u_{2j} - u_{1j}) \cdot$$

$$\cdot [8.66(\nu_{12}/\nu_{22}) p_2 f_{2jk} - 2.7 p_1 f_{1jk}]\} \; ;^{*} \qquad (5.51b)$$

where we have taken into account the ratios $T_2/T_1 \approx 1$, $N_1/N_2 = 1$,

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*If κ_{st} is assumed to be independent of the masses, m_s , m_t , then the results (5.50a,b), (5.51a,b) still hold, except that the $(-2.7p_1 f_{1jk})$ term in (5.51b) can be dropped.

$m_1/m_2 << 1$. It is interesting to observe that (5.50b) and the first term in (5.51b) are the results one would obtain for a simple gas of Maxwell molecules (i.e. the same coefficients of viscosity and thermal conductivity, respectively), while (5.50a) and the first term in (5.51a) are reduced from their simple gas counterparts by a factor of approximately 2.4. We thus see that, to this level of approximation, the heavy species ("2") behaves very much like a simple gas, while the light species ("1") is considerably influenced by the presence of the heavy species.

Finally, for case (ii), $m_1 << m_2$, and $N_1/N_2 \to 0$, we obtain from (5.43), (5.44), respectively,

$$P_{1jk} = -(1/\nu_{11})[0.646(\nu_{11}/\nu_{12})p_1 f_{1jk} + 0.375(\rho_1/\rho_2)(\nu_{11}/\nu_{22})p_2 f_{2jk}] \tag{5.52}$$

$$q_{1k} = 4.84(m_1/m_2)N_1 K(T_2-T_1)(u_{2k}-u_{1k}) - (1/\nu_{12})\{2.5(Kp_1/m_1)(\partial T_1/\partial x_k) +$$

$$+9.44(\nu_{12}/\nu_{22})(\rho_1/\rho_2)(Kp_2/m_2)(\partial T_2/\partial x_k) + (u_{2j}-u_{1j})[-0.995 p_1 f_{1jk} +$$

$$+1.90(\rho_1/\rho_2)(\nu_{12}/\nu_{22})p_2 f_{2jk}]\} . \tag{5.53}$$

These results agree essentially with the results of Section 5.2 for the weakly ionized gas, with "1" denoting electrons and "2" denoting neutrals, if the interparticle force laws in those results are taken to be that of Maxwell molecules and the limit $|\omega_c/\nu_{en}| \to 0$

is taken. There is, however, a slight difference; the terms involving f_{1jk} , f_{2jk} in (5.53) are not present in the corresponding result of Section 5.2, since in that calculation terms proportional to $\varepsilon_j P_{sjk}$ were discarded at the outset.

5.4 The Fully Ionized Gas

Up to now we have concerned ourselves with the properties of individual species (e.g. species traceless pressure tensors, species heat flow vectors); there are situations where one is concerned with the properties of the complete system. This is especially true in the field of magnetohydrodynamics. An important example is the study of the equation of motion for the system flow velocity (i.e. the system momentum equation); by summing the species' conservation of mass equation (2.44a) and the species' momentum equation (2.44b) over all species, and performing certain algebraic manipulations, the following system momentum equation is obtained:[22]

$$\rho \frac{Du_k}{Dt} = - \sum_s \frac{\partial}{\partial x_i} (\rho_s W_{si} W_{sk}) - \frac{\partial p}{\partial x_k} - \frac{\partial P_{ik}}{\partial x_i} + \rho_{ch} E_k + (\vec{J} \times \vec{B})_k + \rho G_k \qquad (5.54)$$

where $\rho \equiv \sum_s \rho_s$ (system mass density) (5.54a)

$$\vec{u} \equiv \sum_s \rho_s \vec{u}_s / \rho \text{ (system flow velocity)} \qquad (5.54b)$$

$$\vec{W}_s \equiv \vec{u}_s - \vec{u} \text{ (diffusion velocity of species "s")} \qquad (5.54c)$$

$$p \equiv \sum_s p_s \qquad \text{(system scalar pressure)} \qquad (5.54d)$$

$$P_{ik} \equiv \sum_s P_{sik} \qquad \text{(system traceless pressure)} \qquad (5.54e)$$

$$\rho_{ch} \equiv \sum_s e_s N_s \qquad \text{(system charge density)} \qquad (5.54f)$$

$$\vec{J} \equiv \sum_s N_s e_s \vec{u}_s \qquad \text{(system current density)} \qquad (5.54g)$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\nabla \qquad \text{(system hydrodynamic differential operator).} \qquad (5.54h)$$

If we consider only small differences in species flow velocities, and hence small diffusion velocities, the first term in (5.54) may be discarded and we obtain the linearized system momentum equation

$$\rho \frac{Du_k}{Dt} = - \frac{\partial p}{\partial x_k} - \frac{\partial P_{ik}}{\partial x_i} + \rho_{ch} E_k + (\vec{J} x \vec{B})_k + \rho G_k \; . \qquad (5.55)$$

Let us now consider a fully ionized gas (i.e. two species, electrons and positive ions with a common charge number). The question arises as to what expression to use for the total traceless pressure tensor, P_{ik} ; this in turn leads to the problem of what sort of "viscosity coefficient" to use in (5.55). Should we use some simple-gas like viscosity which is dominated by the properties of the electrons, or by the properties of the ions (as suggested by Lyman[47]), or is the viscosity coefficient of such a system in reality more complex in form. The purpose of the ensuing calculations is to answer this question.

Assuming a negligibly small magnetic field, we have to first order in $\vec{\varepsilon}$, P_{sjk}/p_s, $\vec{q}_s/a_s p_s$, the following equations for the electron and ion traceless pressures, respectively (see (3.43), (3.44), (5.3)),

$$p_e f_{ejk} = -[(\gamma_{ee}/2)\nu_{ee}+\gamma_{ei}\nu_{ei}]P_{ejk}-(\gamma_{ei}-2)(\rho_e/\rho_i)\nu_{ei}P_{ijk} \qquad (5.56)$$

$$p_i f_{ijk} = -[(\gamma_{ii}/2)\nu_{ii}+2(\rho_e/\rho_i)\nu_{ei}]P_{ijk}-[\gamma_{ei}-(4/5)-(6/5)(T_i/T_e)]\cdot$$

$$\cdot(m_e/m_i)\nu_{ei}P_{ejk} \qquad (5.57)$$

where $\gamma_{st} \equiv (3/5)A_2^{st}(2)/A_1^{st}(2)$. (5.58)

In obtaining the right-hand sides of (5.56), (5.57), we have assumed $T_e/m_e >> T_i/m_i$, and have used the fact that $m_e/m_i << 1$. The solution of (5.56), (5.57) is

$$P_{ejk} = -[(\gamma_{ei}+\gamma_{ee}\nu_{ee}/2\nu_{ei})\nu_{ei}]^{-1}\{p_e f_{ejk}+(2-\gamma_{ei})(2+\gamma_{ii}\rho_i\nu_{ii}/2\rho_e\nu_{ei})^{-1}\cdot$$

$$\cdot\, p_i f_{ijk}\} \qquad (5.59)$$

$$P_{ijk} = -[(2+\gamma_{ii}\rho_i\nu_{ii}/2\rho_e\nu_{ei})\nu_{ei}]^{-1}(\rho_i/\rho_e)\{p_i f_{ijk} +$$

$$+2(m_e/m_i)[(2/5)+(3/5)(T_i/T_e)-(\gamma_{ei}/2)] \cdot$$

$$\cdot(\gamma_{ei}+\gamma_{ee}\nu_{ee}/2\nu_{ei})^{-1}p_e f_{ejk}\} \ . \qquad (5.60)$$

Then taking into account the following collision frequency ratios (see (3.94))

$$\nu_{ii}/\nu_{ei} = \sqrt{2}\, Z^2(m_e/m_i)^{1/2}(T_e/T_i)^{3/2}A_1^{ii}(2)/A_1^{ei}(2) \qquad (5.61a)$$

$$\nu_{ee}/\nu_{ei} = (\sqrt{2}\, N_e/Z^2N_i)A_1^{ee}(2)/A_1^{ei}(2) \ , \ (Z \equiv \underline{\text{ion charge number}}) \ , \qquad (5.61b)$$

we have for the total (system) traceless pressure tensor

$$P_{jk} \equiv P_{ejk} + P_{ijk} = -(\eta_e f_{ejk} + \eta_i f_{ijk}) \qquad (5.62)$$

where $\eta_e \equiv [\gamma_{ei}+(\gamma_{ee}/2)(\nu_{ee}/\nu_{ei})]^{-1}(p_e/\nu_{ei})$ (5.62a)

$$\eta_i \equiv [2+(\gamma_{ii}/2)(\rho_i/\rho_e)(\nu_{ii}/\nu_{ei})]^{-1}(\rho_i/\rho_e)(p_i/\nu_{ei}) \ . \qquad (5.62b)$$

In obtaining (5.62), (5.62a,b), we have ignored the effects of the ratios $A_1^{ii}(2)/A_1^{ei}(2)$, $A_1^{ee}(2)/A_1^{ei}(2)$. The dimensionless cross sections $A_1^{st}(2)$ for a two-temperature ionized gas have not, to the author's knowledge, been rigorously calculated. They are usually expressed as $A_1^{st}(2) = \ell n(9N_D)$, where N_D is the number of particles in a "Debye sphere" whose radius, r_D , is taken as the finite upper limit in the integration over the impact parameter (see (3.10)) ; however, when the electrons and ions have different temperatures, the form for r_D is highly ambiguous (see references [22], [48]). Nevertheless, for the purpose of the ensuing comparisons, we can safely take $A_1^{ii}(2)/A_1^{ei}(2)$, $A_1^{ee}(2)/A_1^{ei}(2)$ to be of order unity in view of the logarithmic dependence of $A_1^{st}(2)$. Furthermore, Lyman[48] shows that

$$\gamma_{st} = (3/5)A_2^{st}(2)/A_1^{st}(2) = (3/5)(2-1/\ln\Lambda_{st}) \tag{5.63}$$

where Λ_{st} , the ratio of the Debye radius to the impact parameter for 90° scattering, is sufficiently large for a broad class of problems so that (5.63) becomes

$$\gamma_{st} = \mathcal{O}(1) \; . \tag{5.64}$$

From the results (5.61a,b), (5.62a,b), and (5.64) we have for the "viscosity coefficients" η_e , η_i ,

$$\eta_e = [\mathcal{O}(1)](p_e/\nu_{ei}) \tag{5.65a}$$

$$\eta_i = \{\mathcal{O}[(m_i/m_e)^{1/2}(T_i/T_e)^{5/2}]\}(p_e/\nu_{ei}) \; . \tag{5.65b}$$

Returning to our result (5.62), we wish to express P_{jk} in terms of the <u>system's flow velocity</u>, $\vec{u}$. We have from (5.54b), (5.54g), respectively,

$$m_e N_e \vec{u}_e + m_i N_i \vec{u}_i = (m_e N_e + m_i N_i)\vec{u} \tag{5.66a}$$

$$-eN_e \vec{u}_e + ZeN_i \vec{u}_i = \vec{J} \,. \tag{5.66b}$$

Solving (5.66a,b) for $\vec{u}_e$, $\vec{u}_i$, we obtain the <u>exact</u> expressions

$$\vec{u}_e = [(m_e + m_i N_i/N_e)\vec{u} - (m_i/ZeN_e)\vec{J}]/(m_e + m_i/Z) \tag{5.67a}$$

$$\vec{u}_i = [(m_i + m_e N_e/N_i)(1/Z)\vec{u} + (m_e/ZeN_i)\vec{J}]/(m_e + m_i/Z) \quad . \tag{5.67b}$$

Substituting (5.67a,b) into (5.62), and using the results (5.65a,b) as an accurate indication of the relative magnitudes of η_e, η_i, we obtain

$$\begin{aligned} P_{jk} = -(\eta_e + \eta_i) f_{jk} - \eta_e \{ (Z \frac{N_i}{N_e} - 1) f_{jk} + [u_j \frac{\partial}{\partial x_k}(ZN_i/N_e) + \\ + u_k \frac{\partial}{\partial x_j}(ZN_i/N_e) - \frac{2}{3}\delta_{jk} \vec{u} \cdot \nabla (ZN_i/N_e)] - [\frac{\partial}{\partial x_k}(J_j/eN_e) + \\ + \frac{\partial}{\partial x_j}(J_k/eN_e) - \frac{2}{3}\delta_{jk} \nabla \cdot (\vec{J}/eN_e)]\} \; * \end{aligned} \tag{5.68}$$

*It should be pointed out that N_e need <u>not</u> necessarily equal ZN_i, <u>even for macroscopic charge neutrality</u>.

where $f_{jk} \equiv \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} - \frac{2}{3}\,\delta_{jk}\nabla \cdot \vec{u}$. (5.68a)

In obtaining (5.68) we have assumed the following relations:

$$\rho_e << \rho_i \tag{5.68b}$$

$$\eta_i\rho_e/ZN_i << \eta_e\rho_i/N_e \tag{5.68c}$$

$$\eta_i\rho_e \left|\frac{\partial \ln N_i}{\partial x_k}\right| << \eta_e\rho_i \left|\frac{\partial \ln N_e}{\partial x_k}\right| \quad . \tag{5.68d}$$

The assumptions (5.68b-d) are, of course, self-consistent; they are also sufficiently general to admit a large class of realistic problems.

The first term on the right-hand side of (5.68) is a traceless stress term with $(\eta_e+\eta_i)$ playing the role of a "<u>coefficient of viscosity</u>"; it is important to note that this viscosity is <u>not</u> necessarily dominated by either the "electron viscosity," η_e , or the "ion viscosity," η_i , since from (5.65a,b) we have

$$\eta \equiv \eta_e+\eta_i = \eta_e \quad , \text{ for } (m_i/m_e)^{1/2}\,(T_i/T_e)^{5/2} << 1 \ , \tag{5.69a}$$

and $\eta \equiv \eta_e+\eta_i = \eta_i$, for $(m_i/m_e)^{1/2}\,(T_i/T_e)^{5/2} >> 1$. (5.69b)

The term in brackets, "{ }" , on the right-hand side of (5.68) is a "correction term," reflecting the difference in electron and ion flow velocities since, if we set $\vec{u}_e = \vec{u}_i = \vec{u}$, then from (5.66b) we see that this term vanishes. The <u>total</u> traceless pressure is then

$$P_{jk} = -(n_e+n_i)f_{jk} \equiv - nf_{jk} \tag{5.70}$$

with n_e, n_i given by (5.62a,b), respectively; the result (5.70), of course, also follows directly from (5.62) with $\vec{u}_e = \vec{u}_i = \vec{u}$. This is the result obtained by Lyman.[47]

CHAPTER VI

CONCLUDING REMARKS

We have seen in Chapter II that, by employing the Grad thirteen moment expansion for the species' distribution functions, we can construct a closed set of 13r transfer equations which describe the gas mixture, where "r" is the number of species. Because the expansion is relative to the species' flow velocities and temperatures, these transfer equations can be expected to adequately describe systems whose species have arbitrarily large differences in flow velocities and/or temperatures, in addition to having non-Maxwellian distribution functions.

The partial collision integrals occurring in the transfer equations have been calculated for general interparticle force laws, for very small and very large diffusion Mach numbers, ε. We have seen that even in these limiting ranges the integrals are very cumbersome; considerable simplification is possible, however, for systems such as weakly and fully ionized gases, and, of course, for "Maxwell molecule" gases. We noted in Chapter III that the non-Maxwellian or "non-equilibrium" parts of the species' distribution functions have considerably less influence on the partial collision integrals for the case of large ε as compared to the case of small ε.

In Chapter IV we analyzed various simplified kinetic models and discussed their ability to imitate the Boltzmann binary collision operator. In particular, we found the results of the Gross-Krook and Sirovich models to be in serious disagreement, both in form and magnitude, with the partial pressure and heat flow collision integrals of the Boltzmann operator. A model based upon a Grad-like expansion of the collision term $(\delta F/\delta t)_{\text{collisions}}$, which reproduces the partial collision integrals of the Boltzmann operator _exactly_, was shown to be a feasible working model for certain _linearized_ systems.

We demonstrated in Chapter V that, for "slowly varying" systems, the traceless pressure and heat flow transfer equations could be approximated by _algebraic_ equations whose solutions, in terms of the first 5r moments (number densities, flow velocities, temperatures), are relatively straightforward. With these solutions the system is then described by a closed set of _5r_ transfer equations, while the effects of viscosity, corresponding to traceless pressure, and thermal conductivity, corresponding to heat flow, are retained.

In our calculations of traceless pressures and heat flows a number of interesting results were obtained. We saw in the case of a weakly ionized gas that a magnetic field has a striking effect upon the electron's traceless pressure tensor and heat flow vector, with $\overleftrightarrow{P}_e$ becoming diagonal and $\vec{q}_e$ "aligning" itself with $\vec{B}$ for infinitely large $|\vec{B}|$. In the Maxwell molecule gas calculations it was demonstrated that the heavy species tends to behave like a simple gas (at least with regard to its traceless pressure and heat flow), while the light species is decidedly influenced by the presence of the heavy species. Finally, we observed that the mixture's coefficient of viscosity for a fully ionized gas can be dominated by either the electron or the ions, depending upon the ratios of the masses and temperatures.

APPENDIX A

INTEGRALS USED IN THE CALCULATION OF THE COLLISION INTEGRALS

A.1 General Integrals

Consider the arbitrary vector variable

$$\vec{x} = \vec{a}_i x_i \quad , \quad (i = 1, 2, 3) \; , \tag{A.1}$$

and its coordinate system, Figure 3,

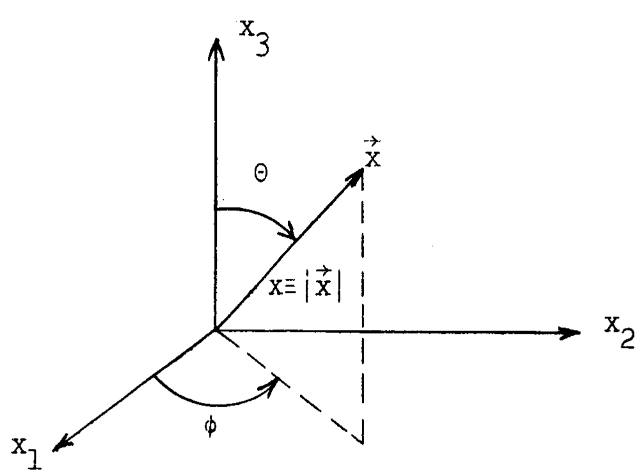

FIG. 3 ARBITRARY VECTOR VARIABLE.

where $\vec{a}_i$ is the unit vector in the i-th direction. Let $F(x)$ be any Riemann integrable function of $x \equiv |\vec{x}|$; then,

$$\int F(x)d\vec{x} \equiv \iiint_{-\infty}^{\infty} F(x)dx_1dx_2dx_3 = \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} F(x)x^2\sin\Theta d\Theta d\phi dx = 4\pi \int_0^{\infty} F(x)x^2dx \, . \tag{A.2}$$

Next, consider the integral

$$I_{2n} \equiv \int F(x)\vec{x}^{2n}d\vec{x} \tag{A.3}$$

where the 2n-th order tensor $\vec{x}^{2n}$ is given by

$$\vec{x}^{2n} \equiv \prod_{p=1}^{2n} x_{i_p} \quad , \quad n = 1, 2, 3, 4, \ldots , \quad . \tag{A.3a}$$

<u>We claim that</u>

$$I_{2n} = \frac{4\pi\vec{\delta}^{n}}{1\cdot3\cdot5\cdot\ \ldots(2n+1)} \int_{0}^{\infty} F(x)x^{2n+2}dx \tag{A.4}$$

where the 2n-th order tensor $\vec{\delta}^{n}$ is formed by taking the sum of all <u>distinct</u> products of n <u>Kronecker deltas</u>, δ_{ij} , which arise on permutating the 2n subscripts $i_1, i_2, \ldots , i_{2n}$; each term occurs <u>once</u>, e.g.

$\vec{\delta}^{1} \equiv \delta_{i_1i_2}$ $(= \delta_{ij})$, the Kronecker delta,

and $\vec{\delta}^{2} \equiv \delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}+\delta_{i_1i_4}\delta_{i_2i_3}(=\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})$.

We prove the claim (A.4) by <u>mathematical induction on n</u> .

<u>Step (i)</u>: $n=1$, $I_2 \equiv \int F(x)\vec{x}^{2}d\vec{x} = \int F(x)x_ix_jd\vec{x}$ (i.e. $i_1 \equiv i, i_2 \equiv j$, here).

We note that $\frac{\partial G(x)}{\partial x_i} = \frac{x_i}{x}\frac{dG}{dx}$. Then, letting $F(x) = \frac{1}{x}\frac{dG(x)}{dx}$, we have,

$$I_2 = \int \frac{x_i}{x}\frac{dG}{dx} x_j d\vec{x} = \int \frac{\partial G}{\partial x_i} x_j d\vec{x} = -\int G \frac{\partial x_j}{\partial x_i} d\vec{x}$$

(integrating by parts in the dx_i integration and assuming Gx_j vanishes at $x_i = \pm\infty$). Hence,

$$I_2 = -\delta_{ij} \int G d\vec{x} = -4\pi\delta_{ij} \int_0^\infty G x^2 dx \quad \text{(using A.2)}$$

$$= \frac{4\pi}{3}\delta_{ij} \int_0^\infty x^3 \frac{dG}{dx} dx$$ (integrating by parts and assuming Gx^3 vanishes at $x = 0, \infty$). Then, substituting $F(x) = \frac{1}{x}\frac{dG}{dx}$,

$$I_2 = \frac{4\pi}{3}\delta_{ij} \int_0^\infty F(x) x^4 dx$$

so that the claim (A.4) is proved for $n = 1$.

<u>Step (ii)</u>: We assume the claim (A.4) is true for <u>some</u> integer n.

<u>Step (iii)</u>: We must now show that (A.4) is true for $n+1$. From (A.3)

$$I_{2(n+1)} = I_{2n+2} \equiv \int F(x)\vec{x}^{2n+2} d\vec{x} = \int F x_j \vec{x}_{(j)}^{2n+1} d\vec{x}$$

where $\vec{x}_{(j)}^{2n+1} \equiv \frac{\vec{x}^{2n+2}}{x_j}$. Then, substituting $F(x) = \frac{1}{x}\frac{dG(x)}{dx}$ as in step (i) ,

$$I_{2n+2} = \int \frac{x_j}{x} \frac{dG}{dx} \vec{x}^{2n+1}_{(j)} d\vec{x} = \int \frac{\partial G}{\partial x_j} \vec{x}^{2n+1}_{(j)} d\vec{x}$$

$$= - \int G \frac{\partial}{\partial x_j} \vec{x}^{2n+1}_{(j)} d\vec{x}$$ (integrating by parts in the dx_j integration and assuming $G\vec{x}^{2n+1}_{(j)}$ vanishes at $x_j = \pm \infty$).

Now, $\frac{\partial}{\partial x_j} \vec{x}^{2n+1}_{(j)} = \sum_{k=i_1}^{i_{2n+1}} \frac{\vec{x}^{2n+1}_{(j)}}{x_k} \delta_{jk} = \sum_{k=i_1}^{i_{2n+1}} \delta_{jk} \vec{x}^{2n}_{(j,k)}$

where $\vec{x}^{2n}_{(j,k)} \equiv \frac{\vec{x}^{2n+2}}{x_j x_k}$. Then,

$$I_{2n+2} = - \sum_{k=i_1}^{i_{2n+1}} \delta_{jk} \int G \vec{x}^{2n}_{(j,k)} d\vec{x} = -[\sum_{k=i_1}^{i_{2n+1}} \delta_{jk} \vec{\delta}^{n}_{(j,k)}] \cdot$$

$$\cdot \frac{4\pi}{1 \cdot 3 \cdot 5 \ldots (2n+1)} \int_0^\infty G x^{2n+2} \, dx$$

<u>using the assumption of step (ii)</u>, where $\vec{\delta}^{n}_{(j,k)}$ is the 2n-th order tensor $\vec{\delta}^{n}$ which does <u>not</u> contain the subscripts "j" , "k" . Now,

$$\sum_{k=i_1}^{i_{2n+1}} \delta_{jk} \vec{\delta}^{n}_{(j,k)} \equiv \vec{\delta}^{n+1}$$ from the <u>definition</u> of $\vec{\delta}^{n}$. Hence,

$$I_{2n+2} = \frac{-4\pi \vec{\delta}^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \int_0^\infty G x^{2n+2} dx = \frac{4\pi \vec{\delta}^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \int_0^\infty x^{2n+3} \frac{dG}{dx} \, dx$$

(integrating by parts and assuming Gx^{2n+3} vanishes at $x = 0, \infty$) , or

$$I_{2(n+1)} = \frac{4\pi\delta^{n+1}}{1\cdot 3\cdot 5 \cdots [2(n+1)+1]} \int_0^\infty F\, x^{2(n+1)+2}\, dx$$

so that (A.4) is true for $n+1$ if it is true for some integer n . This completes the proof.

A.2 The "z" Integrals

For inverse power interparticle force laws $f_{st} = \kappa_{st}/r^p$, $2 \leq p < \infty$, the collision cross sections are

$$S^{(\ell)}(g) = 2\pi(\kappa_{st}/\mu)^{-n/2} A_\ell(p) g^n \quad , \quad n = -4/(p-1) \quad , \quad -4 \leq n < 0$$

and $S^{(\ell)}(g) = (\pi\sigma^2/2)[2-\{1+(-1)^\ell\}/(\ell+1)]$, $p \to \infty$, $n \to 0^-$, "hard spheres," where σ is the sum of the radii of the colliding particles.

The corresponding "Maxwell-averaged collision cross sections" are

$$Z^{(\ell,j)} = (4/\sqrt{\pi}) \int e^{-y^2} y^{2j+3} S^\ell(a_o y) dy \quad , \quad (a_o y = g) \quad , \quad \text{so that}$$

$$Z^{(\ell,j)} = 4\sqrt{\pi} A_\ell(p)(\kappa_{st}/\mu)^{-n/2} a_o^n \,\Gamma(j+2+n/2) \quad , \quad n \neq 0 \quad , \qquad \text{(A.5a)}$$

and $$Z^{(\ell,j)} = \sqrt{\pi}\, \sigma^2 [2-\{1+(-1)^\ell\}/(\ell+1)]\Gamma(j+2) \quad , \quad n \to 0^- \quad ,$$

"hard spheres." (A.5b)

We then have

$$z \equiv 1-(2/5)[Z^{(1,2)}/Z^{(1,1)}] = -(1/5)(n+1) \tag{A.6a}$$

$$z' \equiv 1-(4/35)[Z^{(1,3)}/Z^{(1,1)}] = 1-(1/35)(n+6)(n+8) \tag{A.6b}$$

$$\zeta \equiv 1+5z-(7/2)z' = (7/2)[(1/35)(n+6)(n+8)-1]-n \tag{A.6c}$$

$$\hat{z} \equiv z-z' = (1/35)(n+1)(n+6) \tag{A.6d}$$

$$z'' \equiv 1-(8/315)[Z^{(1,4)}/Z^{(1,1)}] = 1-(1/315)(n+6)(n+8)(n+10) \tag{A.6e}$$

$$z^{(2)} \equiv 1-(2/5)[Z^{(2,2)}/Z^{(1,1)}] = 1-(1/5)(n+6)A_2(p)/A_1(p) \tag{A.6f}$$

$$z'^{(2)} \equiv 1-(4/35)[Z^{(2,3)}/Z^{(1,1)}] = 1-(1/35)(n+6)(n+8)A_2(p)/A_1(p) \tag{A.6g}$$

$$\hat{z}^{(2)} \equiv z^{(2)} - z'^{(2)} = (1/35)(n+1)(n+6)A_2(p)/A_1(p) \tag{A.6h}$$

$$z''^{(2)} \equiv 1-(8/315)[Z^{(2,4)}/Z^{(1,1)}] = 1-(1/315)(n+6)(n+8)(n+10)\cdot A_2(p)/A_1(p) \ . \tag{A.6i}$$

The results (A.6a-i) hold also for "hard spheres," when n is set equal to zero, and $A_2(\infty)/A_1(\infty)$ is set equal to 2/3 (c.f. (A.5a,b)); the dimensionless cross sections $A_1(p)$, $A_2(p)$ are tabulated by Chapman and Cowling[35] for various values of p .

APPENDIX B

CALCULATION OF THE PARTIAL PRESSURE AND ENERGY COLLISION INTEGRALS

B.1 The Partial Pressure and Energy Collision Integrals as Functions of the Diffusion Mach Number

Setting $Q_s = m_s c_{sj} c_{sk}$ in (3.2), we have for the partial pressure collision integral

$$[\delta(m_s c_{sj} c_{sk})]_{st} = m_s \iiiint (c'_{sj} c'_{sk} - c_{sj} c_{sk}) F_s F_{t1} g b\, db\, d\varepsilon\, d\vec{v}\, d\vec{v}_1 \ . \qquad \text{(B.1)}$$

Expressing the random velocities in terms of the center-of-mass and relative velocities, we have (cf. Eqs. (3.5c,d))

$$\vec{c}_s = \vec{c}_o - \frac{m_t}{m_o} \vec{g} - \vec{u}_s \qquad \text{(B.2a)}$$

and

$$\vec{c}'_s = \vec{c}_o - \frac{m_t}{m_o} \vec{g}' - \vec{u}_s \ , \qquad \text{(B.2b)}$$

so that $c'_{sj} c'_{sk} - c_{sj} c_{sk} = -\frac{m_t}{m_o} (c_{oj} - u_{sj})(g'_k - g_k) -$

$$-\frac{m_t}{m_o} (c_{ok} - u_{sk})(g'_j - g_j) - \left(\frac{m_t}{m_o}\right)^2 (g_j g_k - g'_j g'_k) \ . \qquad \text{(B.2c)}$$

From Sec. 3.1 we have (cf. Eqs. (3.8a,b))

$$g_k = g\alpha_{zk} \tag{B.2d}$$

and $$g_k' = g[\sin\chi(\alpha_{xk}\cos\varepsilon + \alpha_{yk}\sin\varepsilon) + \alpha_{zk}\cos\chi] \quad . \tag{B.2e}$$

Substituting (B.2d,e) into (B.2c) and performing the integration over $d\varepsilon$ we obtain

$$\int_0^{2\pi} (c_{sj}'c_{sk}' - c_{sj}c_{sk})d\varepsilon = 2\pi(1-\cos\chi)2\,\frac{m_t}{m_o}\,[g_j(c_{ok}-u_{sk})]^\dagger +$$

$$+ 2\pi(1-\cos^2\chi)\frac{1}{2}\left(\frac{m_t}{m_o}\right)^2(g^2\delta_{jk}-3g_jg_k) \quad . \tag{B.2f}$$

Substitution of (B.2f) into expression (B.1) gives us

$$[\delta(m_s c_{sj}c_{sk})]_{st} = \mu\int\int F_sF_{t1}g\{2S^{(1)}(g)[g_j(c_{ok}-u_{sk})]^\dagger +$$

$$+ \frac{m_t}{2m_o}\,S^{(2)}(g)(g^2\delta_{jk}-3g_jg_k)\}d\vec{v}d\vec{v}_1 \;, \tag{B.3}$$

where we have used the definition of the general collision cross section, $S^{(\ell)}(g) \equiv 2\pi\int(1-\cos^\ell\chi)b\,db$.

Recalling the velocity $\tilde{\vec{c}}_o$ (see (3.16)), we have

$$\vec{c}_o - \vec{u}_s = \tilde{\vec{c}}_o + a_o(a\vec{y} + b\vec{\varepsilon}) \tag{B.4}$$

where $\vec{y}$, $\vec{\varepsilon}$, a_o , a , b are given, respectively, by (3.18a-c), (3.26a,b). Then from (2.34), (2.41), (3.12), (3.16), (3.17a,b), (3.18a-c), (3.19) and (3.21), the expression (B.3) becomes

$$[\delta(m_s c_{sj} c_{sk})]_{st} = \frac{\mu N_s N_t}{\pi^3} (a_o^2/a_\mu^3) \iint d\vec{\tilde{c}}_o d\vec{y}\, e^{-[\frac{\tilde{c}_o^2}{a_\mu^2} + (\vec{y}-\vec{\varepsilon})^2]} y \cdot$$

$$\cdot \{2S^{(1)}(a_o y)[y_j(\tilde{c}_{ok}+a_o(ay_k+b\varepsilon_k))]^{\dagger} + a_o(m_t/2m_o)S^{(2)}(a_o y)(y^2\delta_{jk} -$$

$$-3y_j y_k)\}\cdot\{1+ \frac{P_{spq}}{p_s}\frac{1}{a_s^2}[\tilde{c}_{op}-ba_o(y_p-\varepsilon_p)][\tilde{c}_{oq}-ba_o(y_q-\varepsilon_q)] -$$

$$-\frac{4q_{sp}}{\rho_s a_s^4}[1-\frac{2}{5}\frac{1}{a_s^2}(\vec{\tilde{c}}_o-ba_o(\vec{y}-\vec{\varepsilon}))^2][\tilde{c}_{op}-ba_o(y_p-\varepsilon_p)] +$$

$$+\frac{P_{tpq}}{p_t}\frac{1}{a_t^2}[\tilde{c}_{op}+(1-b)a_o(y_p-\varepsilon_p)][\tilde{c}_{oq}+(1-b)a_o(y_q-\varepsilon_q)] -$$

$$-\frac{4q_{tp}}{\rho_t a_t^4}[1-\frac{2}{5}\frac{1}{a_t^2}(\vec{\tilde{c}}_o+(1-b)(\vec{y}-\vec{\varepsilon}))^2][\tilde{c}_{op}+(1-b)a_o(y_p-\varepsilon_p)]\} \quad . \quad \text{(B.5)}$$

The integration over $d\vec{\tilde{c}}_o$ can be performed directly (see (A.4) for the integrals involved); the result is given by (3.24). One-half the trace of (3.24) then gives us the partial energy collision integral, (3.23).

B.2 Small Diffusion Mach Number

Expanding the exponential in (3.24), multiplying out, retaining terms up to second order in ε_i $(i = 1,2,3)$, and performing the straightforward integrations (see (A.2), (A.4), for the required integrals), we have

$$
\begin{aligned}
[\delta(m_s c_{sj} c_{sk})]_{st} &= a_o^2 \mu N_s \nu_{st} \{a\delta_{jk} + F_{jk} + E_{jk}[a(1-z) - \tfrac{3}{4}\tfrac{m_t}{m_o}(1-z^{(2)})] + \\
&+ \tfrac{18}{5} z(\varepsilon_j S_k)^{\dagger} + \tfrac{4}{5} z\delta_{jk}\varepsilon_i S_i + \tfrac{7}{10}\delta_{jk}\varepsilon_i R_i[2a(z' - \tfrac{12}{7} z) - \\
&- \tfrac{m_t}{m_o}\hat{z}^{(2)}] + 2(\varepsilon_j R_k)^{\dagger}(\tfrac{z}{2} b - \tfrac{7}{5}\hat{z}\, a + \tfrac{21}{20}\hat{z}^{(2)}\tfrac{m_t}{m_o}) + \\
&+ 2\varepsilon_j\varepsilon_k[\tfrac{m_t}{4m_o}(1+3z^{(2)}) - za] + \varepsilon^2[\tfrac{1}{2}(1-z^{(2)})\delta_{jk}\tfrac{m_t}{m_o} - z\delta_{jk} a - \\
&- zF_{jk} + E_{jk}(\hat{z}a - \tfrac{3}{4}\hat{z}^{(2)}\tfrac{m_t}{m_o})] + \varepsilon_i\varepsilon_p E_{ip}\delta_{jk}[(2z - z')a + \hat{z}^{(2)}\tfrac{m_t}{m_o}] + \\
&+ 2(\varepsilon_j\varepsilon_p E_{kp})^{\dagger}(2\hat{z}a - zb - \tfrac{3}{2}\hat{z}^{(2)}\tfrac{m_t}{m_o}) - 2z(\varepsilon_j\varepsilon_p F_{pk})^{\dagger}\}\ ,^{*} \qquad \text{(B.6)}
\end{aligned}
$$

where the "z" integrals are listed in Appendix A.2 .

Substituting the expressions for E_{ij} , R_i , a , b , F_{ij} , S_i , (3.22b,c), (3.26a-d), respectively, we obtain the result (3.44);

---

*The terms here are listed in order of increasing powers of ε_i $(i = 1,2,3)$.

the partial energy collision integral is then given by one half the trace of (3.44), expression (3.43).

B.3 Large Diffusion Mach Number

Upon making the transformation

$$\vec{Z} \equiv \vec{y} - \vec{\varepsilon} \quad , \quad d\vec{Z} = d\vec{y} \quad , \quad (\ \varepsilon \text{ finite}) \ , \tag{B.7}$$

and substituting the collision cross sections for inverse power interparticle force laws (3.54), the expression (3.24) becomes

$$[\delta(m_s c_{sj} c_{sk})]_{st} = a_o C'_{st} \varepsilon^{n+1} \int d\vec{Z} e^{-Z^2} [1+(2 \frac{\vec{Z}\cdot\vec{\varepsilon}}{\varepsilon^2} + \frac{Z^2}{\varepsilon^2})]^{\frac{n+1}{2}} \{ \ldots$$

$$\ldots 2[(Z_j+\varepsilon_j)\{(aZ_k + \frac{m_t}{m_o}\varepsilon_k)[1+Z_i Z_p E_{ip} + Z_i R_i (1-\frac{2}{5}Z^2)] + Z_i F_{ik} +$$

$$+ S_k(1-\frac{2}{5}Z^2) - \frac{4}{5} Z_i Z_k S_i\}]^{\dagger} + \frac{1}{2}\frac{A_2(p)}{A_1(p)}\frac{m_t}{m_o}[\delta_{jk}(Z^2+\varepsilon^2+2\vec{Z}\cdot\vec{\varepsilon}) -$$

$$- 3(Z_j+\varepsilon_j)(Z_k+\varepsilon_k)][1+Z_i Z_p E_{ip} + Z_i R_i(1-\frac{2}{5}Z^2)]\} \ .^{*} \tag{B.8}$$

Expanding the binomial in (B.8), multiplying through, retaining terms of zero and higher order in ε ,** and performing the integrations (see (A.2), (A.4), for the integrals involved), we obtain

---

*For the case of hard spheres $(n\to 0)$ see footnote on page 55.

**Note that the factor ε^{n+1} is excluded from this consideration.

$$\begin{aligned}
[\delta(m_s c_{sj} c_{sk})]_{st} = {} & \pi^{3/2} a_o C'_{st} \varepsilon^{n+1} \{ a[\delta_{jk} + (\frac{n+1}{\varepsilon^2}) \varepsilon_j \varepsilon_k] + \\
& + \frac{1}{2} \frac{m_t}{m_o} \varepsilon_j \varepsilon_k [4 - 3 \frac{A_2(p)}{A_1(p)} + \underline{\frac{(n+1)}{\varepsilon^2} (4 - \frac{9}{2} \frac{A_2(p)}{A_1(p)} + n(1 - \frac{3}{4} \frac{A_2(p)}{A_1(p)}))}] + \\
& + \frac{1}{2} \frac{A_2(p)}{A_1(p)} \frac{m_t}{m_o} \delta_{jk} [\varepsilon^2 + \underline{\frac{1}{4} (n+1)(n+6)}] + F_{jk} + E_{jk} [a - \frac{3}{4} \frac{A_2(p)}{A_1(p)} \frac{m_t}{m_o}] + \\
& + \frac{(n+1)}{\varepsilon^2} [(\varepsilon_i \varepsilon_j F_{ik})^{\dagger} + (a + \frac{m_t}{m_o} (1 - \frac{3}{2} \frac{A_2(p)}{A_1(p)}))(\varepsilon_i \varepsilon_j E_{ik})^{\dagger}] + \\
& + \frac{(n+1)}{2} \frac{\varepsilon_i \varepsilon_p}{\varepsilon^2} E_{ip} \frac{m_t}{m_o} [\delta_{jk} \frac{A_2(p)}{A_1(p)} (1 + \frac{n-1}{4}) + \frac{(n-1)}{\varepsilon^2} \varepsilon_j \varepsilon_k (1 - \frac{3}{4} \frac{A_2(p)}{A_1(p)})]\} \, .
\end{aligned}$$

(B.9)

The underscored terms in (B.9) may be discarded without any serious loss in accuracy. Then substituting for E_{ij} , a , F_{ij} , (3.22b), (3.26a,c), respectively, we obtain the expression (3.62), from which the partial energy collision integral is obtained, expression (3.61).

APPENDIX C

CALCULATION OF THE PARTIAL HEAT FLOW COLLISION INTEGRAL

C.1 The Partial Heat Flow Collision Integral as a Function of the Diffusion Mach Number

Setting $Q_s = \frac{1}{2} m_s c_s^2 c_{sk}$ in (3.2), we have for the partial heat flow collision integral

$$[\delta(\frac{1}{2} m_s c_s^2 c_{sk})]_{st} = \frac{m_s}{2} \int\int\int\int (c_s'^2 c_{sk}' - c_s^2 c_{sk}) F_s F_{t1} g b db d\varepsilon d\vec{v} d\vec{v}_1 \ . \quad (C.1)$$

Following a procedure exactly parallel to that of Appendix B we express the random velocities $\vec{c}_s$ and $\vec{c}_s'$ in terms of the center-of-mass and relative velocities $\vec{c}_o$, $\vec{g}$, and $\vec{g}'$ (see (B.2a,b)), and obtain after considerable manipulation,

$$c_s'^2 c_{sk}' - c_s^2 c_{sk} = 2 \frac{m_t}{m_o} (c_{oi} - u_{si})(g_i - g_i')(c_{ok} - u_{sk}) +$$
$$+ \frac{m_t}{m_o} [(\vec{c}_o - \vec{u}_s)^2 + (\frac{m_t}{m_o})^2 g^2](g_k - g_k') + 2(\frac{m_t}{m_o})^2 (c_{oi} - u_{si})(g_i' g_k' - g_i g_k) \ . \quad (C.2)$$

Substituting the expressions for $\vec{g}$ and $\vec{g}'$, (B.2d,e), into (C.2) and integrating over $d\varepsilon$ gives us

$$\int_0^{2\pi} (c_s'^2 c_{sk}' - c_s^2 c_{sk}) d\varepsilon = 2\pi(1-\cos\chi)\frac{m_t}{m_o}\{2g_i(c_{oi}-u_{si})(c_{ok}-u_{sk}) +$$

$$+ g_k[(\vec{c}_o-\vec{u}_s)^2+(\frac{m_t}{m_o})^2 g^2]\}+2\pi(1-\cos^2\chi)(\frac{m_t}{m_o})^2(c_{oi}-u_{si})(g^2\delta_{ik}-3g_ig_k) \,. \tag{C.3}$$

Then substituting (C.3) into (C.1) and making use of the definition for the general collision cross section, we obtain

$$[\delta(\tfrac{1}{2} m_s c_s^2 c_{sk})]_{st} = (\mu/2)\iint F_s F_{t1} g\{S^{(1)}(g)[2g_i(c_{oi}-u_{si})(c_{ok}-u_{sk}) +$$

$$+ g_k((\vec{c}_o-\vec{u}_s)^2+(m_t/m_o)^2 g^2)]+(m_t/m_o)S^{(2)}(g)(c_{oi}-u_{si})(g^2\delta_{ik} -$$

$$- 3g_ig_k)\}d\vec{v}d\vec{v}_1 \,. \tag{C.4}$$

Then from (2.34), (2.41), (3.12), (3.16), (3.17a,b), (3.18a-c), (3.19), (3.21), and (B.4) we find

$$[\delta(\tfrac{1}{2} m_s c_s^2 c_{sk})]_{st} = (\mu/2)\frac{N_s N_t}{\pi^3}(a_o^2/a_\mu^3)\iint d\tilde{\vec{c}}_o d\vec{y}\, e^{-[\frac{\tilde{c}_o^2}{a_\mu^2} + (\vec{y}-\vec{\varepsilon})^2]}\, y \,\cdot$$

$$\cdot\{S^{(1)}(a_o y)[2y_i(\tilde{c}_{oi}+a_o(ay_i+b\varepsilon_i))(\tilde{c}_{ok}+a_o(ay_k+b\varepsilon_k)) +$$

$$+ y_k((\tilde{\vec{c}}_o+a_o(a\vec{y}+b\vec{\varepsilon}))^2+(m_t/m_o)^2 a_o^2 y^2)]+S^{(2)}(a_o y)[(m_t/m_o)a_o y^2(\tilde{c}_{ok} +$$

$$+ a_o(ay_k+b\varepsilon_k))-3(m_t/m_o)(\tilde{c}_{oi}+a_o(ay_i+b\varepsilon_i))a_o y_i y_k]\} \,.$$

$$\cdot\{1+\frac{P_{spq}}{p_s}\frac{1}{a_s^2}[\tilde{c}_{op}-ba_o(y_p-\varepsilon_p)][\tilde{c}_{oq}-ba_o(y_q-\varepsilon_q)]-$$

$$-\frac{4q_{sp}}{\rho_s a_s^4}[1-\frac{2}{5}\frac{1}{a_s^2}(\tilde{\vec{c}}_o-ba_o(\vec{y}-\vec{\varepsilon}))^2][\tilde{c}_{op}-ba_o(y_p-\varepsilon_p)]+$$

$$+\frac{P_{tpq}}{p_t}\frac{1}{a_t^2}[\tilde{c}_{op}+(1-b)a_o(y_p-\varepsilon_p)][\tilde{c}_{oq}+(1-b)a_o(y_q-\varepsilon_q)]-$$

$$-\frac{4q_{tp}}{\rho_t a_t^4}[1-\frac{2}{5}\frac{1}{a_t^2}(\tilde{\vec{c}}_o+(1-b)(\vec{y}-\vec{\varepsilon}))^2][\tilde{c}_{op}+(1-b)a_o(y_p-\varepsilon_p)]\}\ . \qquad (C.5)$$

The $d\tilde{\vec{c}}_o$ integration can be directly performed (see (A.4) for the required integrals); the result is the expression (3.25).

C.2 Small Diffusion Mach Number

Expanding the exponentials in (3.25), multiplying through, retaining terms up to second order in ε_i $(i = 1,2,3)$, and performing the integrations (see (A.2), (A.4), for the required integrals), we obtain (after considerable effort)

$$[\delta(\frac{1}{2}m_s c_s^2 c_{sk})]_{st} = \frac{3}{4}a_o^3\,\mu N_s \nu_{st}\{\frac{5}{3}\varepsilon_k[a^2(1-3z)+2\frac{m_t}{m_o}az^{(2)}+\frac{a_\mu^2}{a_o^2}+$$

$$+\frac{m_t^2}{m_o^2}(1-z)]+\frac{2}{3}S_k\{a[11z-6+\frac{7}{5}\varepsilon^2(13z'-\frac{144}{7}z)]+$$

$$+\frac{m_t}{m_o}[2(1-z^{(2)})+\frac{7}{5}\varepsilon^2(\hat{z}^{(2)}+\frac{18}{7}z)]\}+\frac{28}{15}\varepsilon_k\varepsilon_j S_j[2a(4z'-\frac{47}{7}z)+$$

$$+\frac{m_t}{m_o}(\frac{7}{2}\hat{z}^{(2)}+4z)]+R_k\{\frac{a^2}{2}[7z'-5z-2+\frac{\varepsilon^2}{15}(189z''-350z'+171z)]+$$

$$+ \frac{1}{30}\frac{m_t^2}{m_o^2}[5(7z'-5z-2)+\varepsilon^2(63z''-98z'+45z+42\hat{z}^{(2)})] +$$

$$+ \frac{5}{6}\;\frac{a_\mu^2}{a_o^2}[z+\frac{7}{5}\;\varepsilon^2(z'-2z)]+\frac{1}{15}\frac{m_t}{m_o}\,a[5(2+5z^{(2)}-7z'^{(2)}) +$$

$$+ 7\varepsilon^2(17z'^{(2)}-9z''^{(2)}-8z^{(2)}+4z'-\frac{38}{7}\;z)]\} +$$

$$+ \frac{14}{15}\;\varepsilon_k\varepsilon_j R_j[a^2(\frac{27}{2}\;z''-30z'+\frac{291}{14}\;z)+\frac{m_t}{m_o}\,a\;(9z'-\frac{98}{7}\;z+\frac{29}{2}\;z'^{(2)} -$$

$$- \frac{11}{2}\;z^{(2)}-9z''^{(2)})+\frac{5}{2}\;\frac{a_\mu^2}{a_o^2}\;(z'-2z)+\frac{1}{2}\;\frac{m_t^2}{m_o^2}\;(\hat{z}^{(2)}+9z''-14z'+\frac{45}{7}\;z)] +$$

$$+ \frac{a_\mu^2}{a_o^2}\;[H_k(1-\frac{19}{15}\;z\varepsilon^2)-\frac{18}{15}\;z\varepsilon_k\varepsilon_j H_j]+\frac{2}{3}\;\frac{a_\mu^2}{a_o^2}\;\varepsilon_j G_{jk} +$$

$$+ \frac{1}{3}\;\varepsilon_j E_{jk}[a^2(2+19z-21z')+\frac{m_t}{m_o}\;a(3-4z-13z^{(2)}+14z'^{(2)}) -$$

$$- 5z\;\frac{a_\mu^2}{a_o^2}+\frac{m_t^2}{m_o^2}\;(5z-7z'+3z^{(2)}-1)] +$$

$$+ \frac{2}{3}\;\varepsilon_j F_{jk}[a(2-9z)+\frac{1}{2}\;\frac{m_t}{m_o}\;(3+z^{(2)})]\} \quad . \tag{C.6}$$

Substituting for E_{ij} , R_i , a , F_{ij} , S_i , G_{ij} , H_i,(3.22b,c), (3.26a,c-f), respectively, we obtain after considerable manipulation the expression (3.45).

C.3 Large Diffusion Mach Number

We have from (3.25), upon making the transformation (B.7), for inverse power interparticle force laws

$$[\delta(\frac{1}{2} m_s c_s^2 c_{sk})]_{st} = \frac{a_o^2}{2} C'_{st} \varepsilon^{n+1} \int d\vec{Z} e^{-Z^2} [1+(\frac{2\vec{Z}\cdot\vec{\varepsilon}}{\varepsilon^2}+\frac{Z^2}{\varepsilon^2})]^{\frac{n+1}{2}} \cdot$$

$$\cdot\{2(Z_p+\varepsilon_p)(aZ_p+\frac{m_t}{m_o}\varepsilon_p)(aZ_k+\frac{m_t}{m_o}\varepsilon_k)+(Z_k+\varepsilon_k)[Z^2(a^2+\frac{m_t^2}{m_o^2})+2\frac{m_t^2}{m_o^2}\varepsilon^2 +$$

$$+ 2(\frac{m_t}{m_o} a+\frac{m_t^2}{m_o^2})Z_p\varepsilon_p]\}\cdot\{1+Z_iZ_jE_{ij}+Z_iR_i(1-\frac{2}{5}Z^2)\} +$$

$$+ 2(Z_p+\varepsilon_p)(aZ_p+\frac{m_t}{m_o}\varepsilon_p)[Z_iF_{ik}+S_k(1-\frac{2}{5}Z^2)-\frac{4}{5}S_iZ_iZ_k] +$$

$$+ 2[(Z_p+\varepsilon_p)(aZ_k+\frac{m_t}{m_o}\varepsilon_k)+(Z_k+\varepsilon_k)(aZ_p+\frac{m_t}{m_o}\varepsilon_p)]\cdot[Z_iF_{ip} +$$

$$+ S_p(1-\frac{2}{5}Z^2)-\frac{4}{5}S_iZ_iZ_p]+\frac{5}{2}\frac{a_\mu^2}{a_o^2}(Z_k+\varepsilon_k)+\frac{a_\mu^2}{a_o^2}(Z_i+\varepsilon_i)G_{ik} +$$

$$+ \frac{5}{2}\frac{a_\mu^2}{a_o^2}Z_iZ_jE_{ij}(Z_k+\varepsilon_k)+\frac{5}{2}\frac{a_\mu^2}{a_o^2}Z_iR_i(Z_k+\varepsilon_k)-\frac{a_\mu^2}{a_o^2}Z_iR_i(Z_k+\varepsilon_k)Z^2 +$$

$$+ \frac{1}{5}\frac{a_\mu^2}{a_o^2}H_i(9Z_iZ_k+7Z_i\varepsilon_k+2Z_k\varepsilon_i)+\frac{2}{5}\frac{a_\mu^2}{a_o^2}H_kZ_j(Z_j+\varepsilon_j)\} +$$

$$+ \frac{a_o^2}{2} C'_{st} \frac{A_2(p)}{A_1(p)} \frac{m_t}{m_o} \varepsilon^{n+1} \int d\vec{Z} e^{-Z^2} [1+(2\frac{\vec{Z}\cdot\vec{\varepsilon}}{\varepsilon^2}+\frac{Z^2}{\varepsilon^2})]^{(n+1)/2} \cdot$$

$$\cdot\{(\vec{Z}+\vec{\varepsilon})^2[(aZ_k+\frac{m_t}{m_o}\varepsilon_k)(1+Z_iZ_jE_{ij}+Z_iR_i(1-\frac{2}{5}Z^2)) +$$

$$+ Z_i F_{ik} + S_k(1-\frac{2}{5}Z^2) - \frac{4}{5}S_i Z_i Z_k] - 3(Z_k+\varepsilon_k)(Z_p+\varepsilon_p)[(aZ_p +$$

$$+\frac{m_t}{m_o}\varepsilon_p)(1+Z_i Z_j E_{ij} + Z_i R_i(1-\frac{2}{5}Z^2)) + Z_i F_{ip} + S_p(1-\frac{2}{5}Z^2) - \frac{4}{5}S_i Z_i Z_p]\} \; . \tag{C.7}$$

Expanding the binomials in (C.7), multiplying through, retaining terms of zero and higher order in ε,* and performing the integrations (see (A.2), (A.4), for the integrals involved), we obtain, after substituting E_{ij} , R_i , a , F_{ij} , S_i , G_{ij} , H_i , the expression (3.63).

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*Again, the factor ε^{n+1} is excluded from this consideration.

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