THE SCATTERING OF CHARGED PARTICLES

IN A WEAKLY UNSTABLE PLASMA[*]

by

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ABSTRACT

The mean-square deflection of suprathermal test particles from a weakly unstable electron plasma is calculated. The instability is assumed to have been driven by a tenuous beam of energetic electrons. The random phase approximation expression for the spectral density is used and the electric field energy density of the turbulent spectrum is estimated by the quasi-linear theory. It is shown that the mean-square scattering angle can be increased by more than an order of magnitude by a suitable beam density or drift velocity. This enhances the potential usefulness of charged particle scattering as a diagnostic tool for turbulent laboratory plasmas.

I. INTRODUCTION

It has recently been proposed that the scattering of energetic charged particles could provide a probe of the integrated spectrum of the plasma electric field auto-correlation function.[1,2] In a quiescent stable plasma where the scattering is due to the collective thermal fluctuations in the microscopic electric field, the deflection of test electrons may be barely observable for presently attainable laboratory plasmas.[1] It is also pointed out in Reference 2 that, while it is possible to induce enhanced plasma fluctuations by the introduction of a current, this does not increase the scattering significantly as long as the current is kept within the limits of linear stability. This paper deals with the possibilities of scattering from a weakly unstable plasma.

In the paragraphs that follow, the formal expression for the mean-square deflection of a charged particle beam is set down. It involves an integration over the spectral density of the turbulent electric field, something which is not provided in closed form by any existing theory. An expression valid for the weakly unstable "bump-on-the-tail" situation has been derived by the quasi-linear theory[4] for the one-dimensional case. A three-dimensional version of the quasi-linear theory has been given,[5] but apparently does

not lead to an expression for the spectral density without consid-
erable numerical work. Even then, it may not be time-independent
for the initial-value problem considered. It is essential for the
present calculation that we have a simple estimate for the three-
dimensional spectral density. We therefore make a (necessarily
rough) estimate for it, using the one-dimensional, quasi-linear
theory as a guide. This estimate represents the only significant
uncertainty in the calculation.

II. CALCULATION OF THE MEAN-SQUARE DEFLECTION

As shown in Reference 1, the mean-square deflection for a test particle of velocity $\vec{V}_o$, charge-to-mass ratio q/m, which traverses a length L of spatially uniform plasma, is

$$\langle (\Delta\theta)^2 \rangle = \frac{2\pi}{m^2} \frac{q^2 L}{V_o^3} \int d\vec{k} \, \langle S_\perp \rangle_{\vec{k}, \, -\vec{k} \cdot \vec{V}_o} \, . \tag{1}$$

In a geometry in which $\vec{V}_o = \hat{e}_x V_o$, $\langle S_\perp \rangle_{\vec{k}\omega}$ is just the sum of the $\hat{e}_y \hat{e}_y$ and $\hat{e}_z \hat{e}_z$ components of the spectral density tensor $\langle \overset{\leftrightarrow}{S} \rangle_{\vec{k}\omega}$.

In an unstable situation where the electrostatic field has a turbulent spectrum which does not vary appreciably in a time L/V_o, we may show that,[3] within the random phase approximation,

$$\langle \overset{\leftrightarrow}{S} \rangle_{\vec{k}\omega} = 8\pi \frac{\vec{k}\vec{k}}{k^2} \, \Sigma_\ell \, \mathcal{E}_\ell(\vec{k}) \, \delta(\omega - \omega_\ell(\vec{k})) \, . \tag{2}$$

Here, $\mathcal{E}_\ell(\vec{k})$ is the electrostatic field energy per unit volume per unit wave number for waves of type ℓ. For example, ℓ might label electron plasma oscillations or ion acoustic waves. $\omega_\ell(\vec{k})$ is the oscillation frequency for the ℓth type of wave. We shall assume that the $\mathcal{E}_\ell(\vec{k})$ are time-independent over the times of interest.

We shall consider the turbulent spectrum to have grown out of an electron plasma with a tenuous flux of energetic electrons, with a uniform, positive background:

$$f(\vec{v}) = \frac{\beta}{(2\pi V_e^2)^{3/2}} e^{-\vec{v}^2/2V_e^2} + \frac{(1 - \beta)}{(2\pi V_e^2)^{3/2}} e^{-(\vec{v} - \vec{V}_d)^2/2V_e^2} , \qquad (3)$$

with the drift velocity $\vec{V}_d = V_d(\cos \gamma, 0, \sin \gamma)$ in Cartesian coordinates, $\beta \cong 1$, $1 - \beta \ll 1$, and $V_d > V_e$, the electron thermal speed. The analytical treatments of this "bump-on-the-tail" situation have usually been concerned with the initial value problem, though any actual experiments would almost certainly be better approximated by the boundary value problem.

We shall assume the parameters β, V_e, and V_d to have been chosen so that the assumptions of the quasi-linear theory are fulfilled--i.e., that the initial growth rates, Γ, associated with Equation (3) satisfy $\Gamma(\vec{k}) \ll \omega(\vec{k})$, where $\omega(\vec{k})$ is the corresponding oscillation frequency. For V_d only slightly exceeding the critical velocity for instability, the unstable region in $\vec{k}$-space will be small, as shown in Figure 1, and will be centered about the directions $\vec{k} = \pm \vec{V}_d$. For $V_d^2 \gg V_e^2$, we will also have

that $\omega(\vec{k}) \simeq \pm \omega_\rho$, the electron plasma frequency, for all $\vec{k}$'s. Thus,

$$\langle S_\perp \rangle_{\vec{k}, -\vec{k}} \cdot \vec{V}_o = 8\pi \sin^2 \theta \, \mathcal{E}(\vec{k}) \, \delta(\omega_\rho + k V_o \cos \theta)$$

where $\vec{k} = k(\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi)$, and

$$\langle (\Delta\theta)^2 \rangle \simeq \frac{32\pi^2 L q^2}{m^2 V_o^3} \sin^2 \gamma \int d\vec{k} \, \mathcal{E}(\vec{k}) \, \delta(\omega_\rho - k V_o \cos \gamma) \, , \quad (4)$$

where we have approximated $\cos \theta \simeq \cos \gamma$ by virtue of the narrow angular width subtended by the region of unstable $\vec{k}$. The extra factor of 2 takes into account the second unstable region for which $\theta \simeq \pi - \gamma$, and which contributes symmetrically. The integration in Equation (4) is over the region of initial positive Γ.

We now rotate the axes about the k_y-axis, such that the k_x-axis lies along $\vec{V}_d$. In the new set of coordinates, $\mathcal{E}(\vec{k}) = \mathcal{E}(\vec{k}, \cos \theta)$ only. This enables us to do the φ and k integrations in Equation (4) trivially. Furthermore, we now assume that $\mathcal{E}(\omega_\rho/V_o \cos \gamma, \cos \theta)$ falls off __parabolically__, with a maximum at $\theta = 0$, i.e.,

$$\mathcal{E}\left(\frac{\omega_\rho}{V_o \cos \gamma}, X \right) = \left(1 - \frac{X^2}{X_{max}^2} \right) \mathcal{E}\left(\frac{\omega_\rho}{V_o \cos \gamma}, 1 \right) \, , \quad (5)$$

where $X = \cos\theta - 1$ and X^2_{max} is the maximum value of $(\cos\theta - 1)^2$ under which the waves are unstable for $k = \omega_\rho / V_o \cos\gamma$. $\cos\theta$ is determined from the equation of the boundary of the unstable region, $\Gamma(k, \cos\theta) = 0$, which is

$$-e^{-k_e^2/2k^2} = \left(\frac{1-\beta}{\beta}\right)\left(1 - \frac{V_o k \cos\theta}{V_e k_e}\right) e^{-\left(1 - \frac{V_o k \cos\theta}{V_e k_e}\right)^2 k_e^2/2k^2},$$

(6)

where k_e^2 is ω_ρ^2 / V_e^2.

Finally, we have that

$$\int d\vec{k}\ \mathcal{E}(\vec{k})\ \delta(\omega_\rho - k V_o \cos\gamma)$$

$$\cong \frac{2\pi\ \omega_\rho^2}{V_o^3 \cos^3\gamma}\ \mathcal{E}\left(\frac{\omega_\rho}{V_o \cos\gamma}, 1\right) \int_{-X_{max}}^{+X_{max}} dX\left[1 - \frac{X^2}{X^2_{max}}\right]$$

$$= \frac{8\pi\ \omega_\rho^2}{3V_o^3 \cos^3\gamma}\ \mathcal{E}\left(\frac{\omega_\rho}{V_o \cos\gamma}, 1\right)\ |X_{max}|.$$

(7)

The steady state value of $\mathcal{E}(\vec{k})$ as obtained by the one-dimensional, initial-value, quasi-linear theory has been shown to be approximately valid in the three-dimensional case, provided it is interpreted as the maximum excitation of the fluctuating

field which occurs at a finite time.[b] We may therefore use the one-dimensional, quasi-linear treatment to estimate an upper bound for $\langle (\Delta\theta)^2 \rangle$.

The asymptotic electrostatic field spectrum in the one-dimensional quasi-linear theory is:[4]

$$\lim_{t \to \infty} \mathcal{E}(k, 1) = \left(\frac{\pi}{2} nm \frac{\omega_\rho^2}{k^3} \right) \int_{v_1}^{u} [F(u', \infty) - F(u', o)] \, du' \tag{8}$$

where $u = \omega_\rho/k$ and $F(u, \infty)$ is a constant determined by

$$F(u, \infty)(v_5 - v_1) = \int_{v_1}^{v_5} f(u', o) \, du' . \tag{9}$$

The various quantities are defined in the caption of Figure 2.

Equation (8) is approximated by expanding $F(u, o)$ in a Taylor's series about A for $v_1 \leq \omega_\rho/k \leq v_2$; about C for $v_2 \leq \omega_\rho/k \leq v_3$; about E for $v_4 \leq \omega_\rho/k \leq v_5$. This procedure gives to the lowest order:

$$\lim_{t \to \infty} \mathcal{E}(k, 1) = \frac{\pi}{4} \frac{mn \, \omega_\rho^2}{k^3} \mathcal{L}(u) \tag{10}$$

where

$$
\mathcal{L}(u) = \begin{cases}
\left|\dfrac{\partial F(v_1)}{\partial v_1}\right| (u - v_1)^2 \\[4ex]
\left|\dfrac{\partial F(v_1)}{\partial v_1}\right| (v_2 - v_1)^2 + \left|\dfrac{\partial F(v_3)}{\partial v_3}\right| (u - v_2)^2 \\[4ex]
\left|\dfrac{\partial F(v_1)}{\partial v_1}\right| (v_2 - v_1)^2 + \left|\dfrac{\partial F(v_3)}{\partial v_3}\right| \left[(v_3 - v_2)^2 - (u - v_3)^2\right] \\[4ex]
\left|\dfrac{\partial F(v_1)}{\partial v_1}\right| (v_2 - v_1)^2 + \left|\dfrac{\partial F(v_3)}{\partial v_3}\right| \left[(v_3 - v_2)^2 - (v_4 - v_3)^2\right] \\[2ex]
\qquad\qquad - \left|\dfrac{\partial F(v_5)}{\partial v_5}\right| (u - v_4)^2
\end{cases}
$$

corresponding to

$$
\begin{cases}
v_1 \leq u \leq v_2 \\[2ex]
v_2 \leq u \leq v_3 \\[2ex]
v_3 \leq u \leq v_4 \\[2ex]
v_4 \leq u \leq v_5
\end{cases}
\tag{11}
$$

Combining Equations (3), (4), (7), and the three-dimensional generalization of Equation (10), the resulting expression for the upper bound of the mean-square deflection is

$$\Theta^2 = \frac{\langle(\Delta\theta)^2\rangle}{\langle(\Delta\theta)^2\rangle_{th}} = \frac{2\sqrt{2}\,\pi^{5/2}}{3}\beta \frac{(k_o/k_e)}{\ln(k_o/k_e)}\,(V_o/v_e)^3$$

$$(v_3/v_e)\,|X_{max}|\,\sin^2\gamma\,\cos^2\gamma\,Z(V_o\,\cos\,\gamma)\,, \qquad (12)$$

where

$$Z(V_o\,\cos\,\gamma) = \begin{cases} |E(v_1)|\,(V_o\,\cos\,\gamma - v_1)^2/v_e^2 \\[2em] |E(v_1)|\,(v_2 - v_1)^2/v_e^2 + |E(v_3)|\,(V_o\,\cos\,\gamma \\ \hspace{8em} - v_2)^2/v_e^2 \\[2em] |E(v_1)|\,(v_2 - v_1)^2/v_e^2 + |E(v_3)|\,[(v_3 - v_2)^2 \\ \hspace{8em} - (V_o\,\cos\,\gamma - v_3)^2]/v_e^2 \\[2em] |E(v_1)|\,(v_2 - v_1)^2/v_e^2 + |E(v_3)|\,[(v_3 - v_1)^2 \\ - (v_4 - v_3)^2]/v_e^2 - |E(v_1)|\,(V_o\,\cos\,\gamma - v_4)^2/v_e^2 \end{cases}$$

corresponding to the inequalities in Equation (11) with $u = v_o \cos \gamma$ and

$$E(v_j) = -v_j/v_3 \; e^{-v_j^2/2v_e^2}$$
$$- \left(\frac{1 - \beta}{\beta} \right) \left(\frac{v_j - v_e}{v_3} \right) e^{-(v_j - v_e)^2/2v_e^2} \; .$$

k_e is the electron Debye wave number, $k_o = kT/e^2$, and $\langle (\Delta\theta)^2 \rangle_{th}$ is the mean scattering deflection due to scattering by the thermal equilibrium fluctuations[1] given by

$$\langle (\Delta\theta)^2 \rangle_{th} = \frac{16\pi \; q^2 \; Le^2 \; n}{m^2 \; V_o^4} \; \ln(k_o/k_e) \; . \tag{13}$$

The normalized deflection Θ^2 given by Equation (12) corresponds to the largest scattering angle that will be attained in a continuous stream of test particles due to the additional scattering by the turbulent fluctuating fields resulting from the instability.

III. NUMERICAL RESULTS

The normalized deflection Θ^2 is evaluated according to recipe provided by Equation (12) for a typical laboratory plasma with $\ln(k_0/k_e) \cong 10$ and assuming test electrons.

The variation of Θ^2 vs. test particle velocity and angle of incidence for different values of drift velocity $\vec{V}d$ and relative beam density $(1 - \beta)$, all chosen within the limits of validity of the quasi-linear theory, are shown in Figures 3, 4, and 5. For the case in which $V_d = 5V_e$ and $(1 - \beta) = 4 \times 10^{-4}$, where the maximum initial value of $\Gamma(0)/\omega_\rho$ is .0006, suggesting very weak instability, we see in Figure 3 that the additional scattering due to the turbulent fluctuations is very slight. For example, if $n = 10^{10}$, $V_0/V_e = 10$ corresponds to a test particle energy of about 60 ev and the thermal scattering angle is about 0.5°. The first peak in Figure 3 is equivalent to angular deflection of 0.5° so that the total deflection increases by only about a factor of two. It is interesting to note that for the case of an ion sound wave instability the increase in the angular scattering, as the instability boundary is approached, is also about a factor of two.[2] Because of the factor V_0^{-4} in $\langle(\Delta\theta)^2\rangle$, the scattering angle actually decreases as V_0 is increased. The peaks for

$V_o/V_e = 20$ and for $V_o/V_e = 50$ in Figure 5 corresponds to deflection angles of $0.2°$ and $.06°$, respectively.

Figure 4 illustrates the case $V_d = 5V_e$ and $(1 - \beta) = 4 \times 10^{-5}$ for which $\Gamma/\omega_p \cong .02$. We note that the normalized deflection is increased by more than two orders of magnitude compared with the previous case. For the same example used above, the peaks for $V_o/V_e = 10$, $V_o/V_e = 20$, and $V_o/V_e = 50$ correspond to $12°$, $5°$, $2°$ deflection angles, respectively. The increase over the thermal value is more than an order of magnitude.

When the drift velocity is raised to $10 V_e$, keeping the relative beam density at 4×10^{-4}, $(\Gamma/\omega_p \cong .03)$ results in an even larger increase Θ^2 as shown in Figure 5. The peaks for $V_o/V_e = 20$ and $V_o/V_e = 50$ curves correspond to deflection angles of $25°$ and $9°$, respectively. For such large deflections, one may assume the expression in Equation (1) becomes inaccurate, since it has been derived assuming the deflections to be small.

The order of magnitude increase in the scattering angle is, of course, not entirely unexpected since even in the quasi-linear limit the energy of turbulent fluctuations is much greater than the energy of thermal equilibrium fluctuations.[6] In all the cases considered, the peak in Θ^2 moves toward $90°$ as V_o increases. This peaking can be explained physically as follows.

The particles that suffer the largest deflections are those that move in phase with the waves which require that $k V_o \cos \gamma = \omega_{\rho e}$. Since there is only a small spread in the wave phase velocities if V_o is increased, $\cos \gamma$ must be decreased. For infinite V_o, γ is exactly $90°$.

IV. DISCUSSION AND SUMMARY

These calculations have shown that the deflection of test particles in an electron plasma with a bump-on-tail distribution may be very much larger than in the case of thermal equilibrium. This can be accomplished by either increasing the density of the beam of the energetic electrons or by raising its drift velocity. A significant improvement in the utility of this experimental technique can therefore be expected for weakly unstable plasmas. For example, energetic electrons used as test particles in plasmas with realistic dimensions will still exhibit measurable deflections, while at the same time be easily distinguishable from the plasma particles. The calculations also show that there is the usual shift in the peaks of the deflections toward $\gamma = 90^{\circ}$ as V_0 is increased. This very interesting feature has been predicted in the case of a stable current-carrying plasma[10] and could well be a common characteristic of scattering experiments in plasmas which support some kind of oscillations.

To the extent at which the assumptions of this calculation are satisfied, a measurement of deflections of test particles may also be construed as a possible experimental test of the quasi-linear theory.

We may finally conclude that if the measured scattering is drastically enhanced over the stable equilibrium value, the plasma is linearly unstable and is experiencing some sort of turbulent oscillations.

ACKNOWLEDGMENTS

REFERENCES

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FIGURE CAPTIONS

Figure 1   The unstable regions of $\vec{k}$ space. The shaded volumes
            are those which contribute to the mean-square scattering
            integral. The drift velocity $\vec{V}_d$ lies in the k_x k_z plane
            and makes an angle γ with the test particle velocity $\vec{V}_o$.
            As V_d approaches its critical value for instability
            from above, the shaded volume shrinks to zero.

Figure 2   The one-dimensional velocity distribution and spectral
            density (as a function of $u = \omega_\rho/k$) for the asymptotic
            one-dimensional, quasi-linear final state. The initial
            distribution is shown by a dotted line and the areas
            enclosed by ABC and CDE are equal.

Figure 3   The normalized mean-square deflection vs. angle of
            incident γ for the case of very weak instability with
            $V_d = 5V_e$ and $(1 - \beta) = 4 \times 10^{-4}$, $\Gamma(0)/\omega_\rho \approx .0006$.

Figure 4   The normalized mean-square deflection vs. angle of
            incidence for $(1 - \beta) = 4 \times 10^{-3}$, $V_d = 5V_e$, $\Gamma(0)/\omega_\rho$
            $\cong .02$.

Figure 5   The normalized mean-square deflection vs. angle of
            incidence for $(1 - \beta) = 4 \times 10^{-4}$, $V_d = 10V_e$,
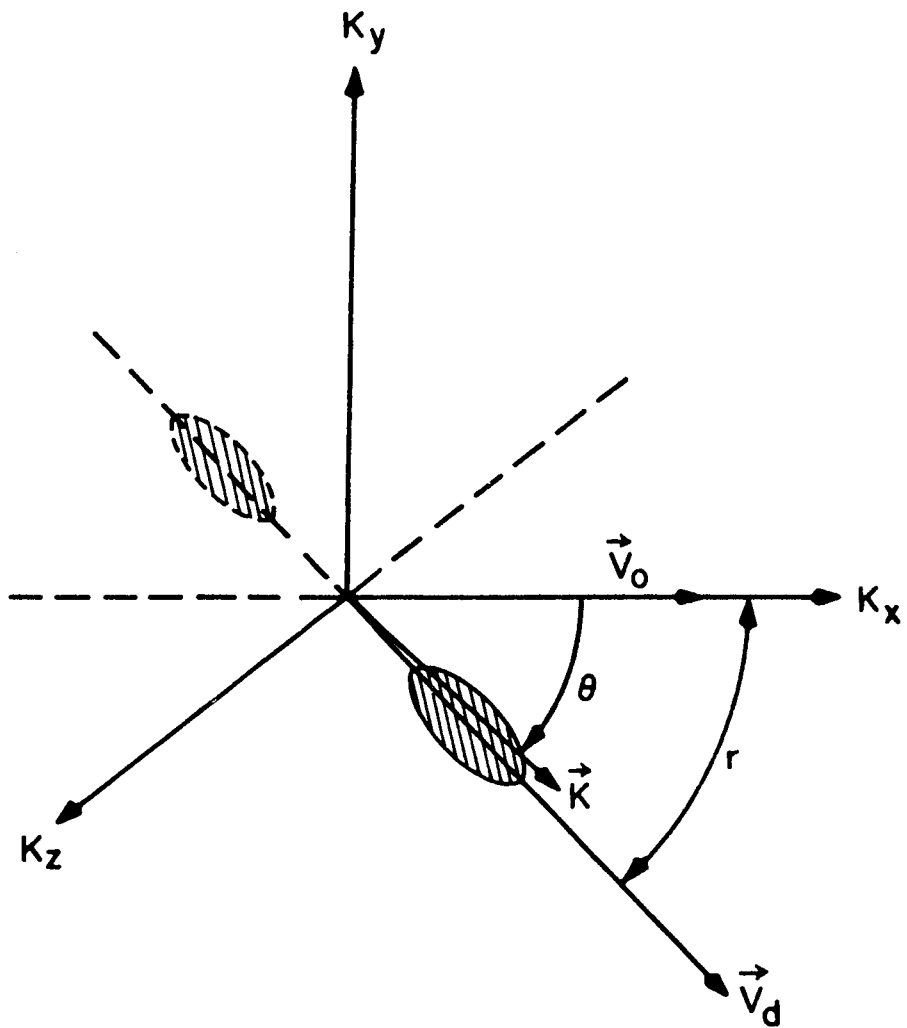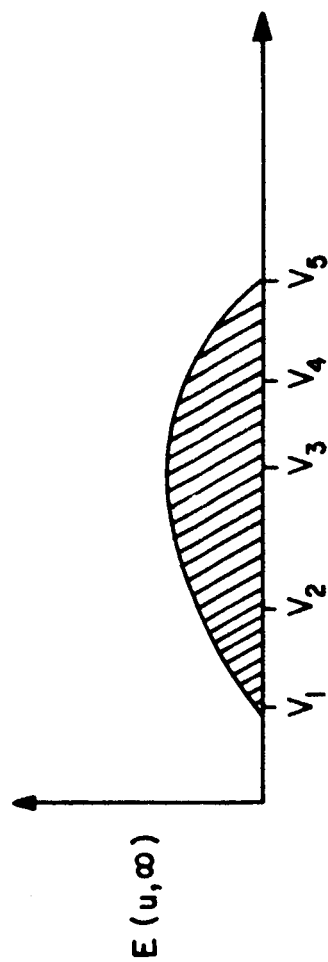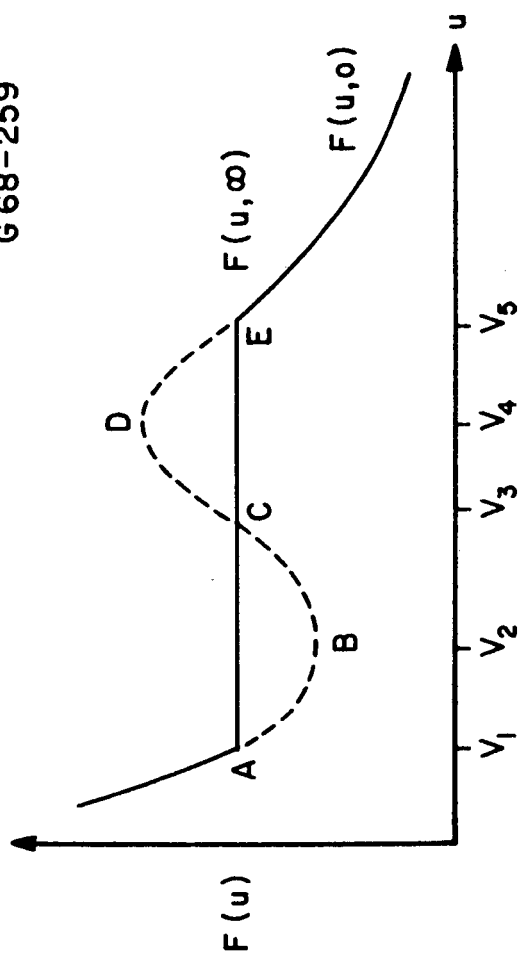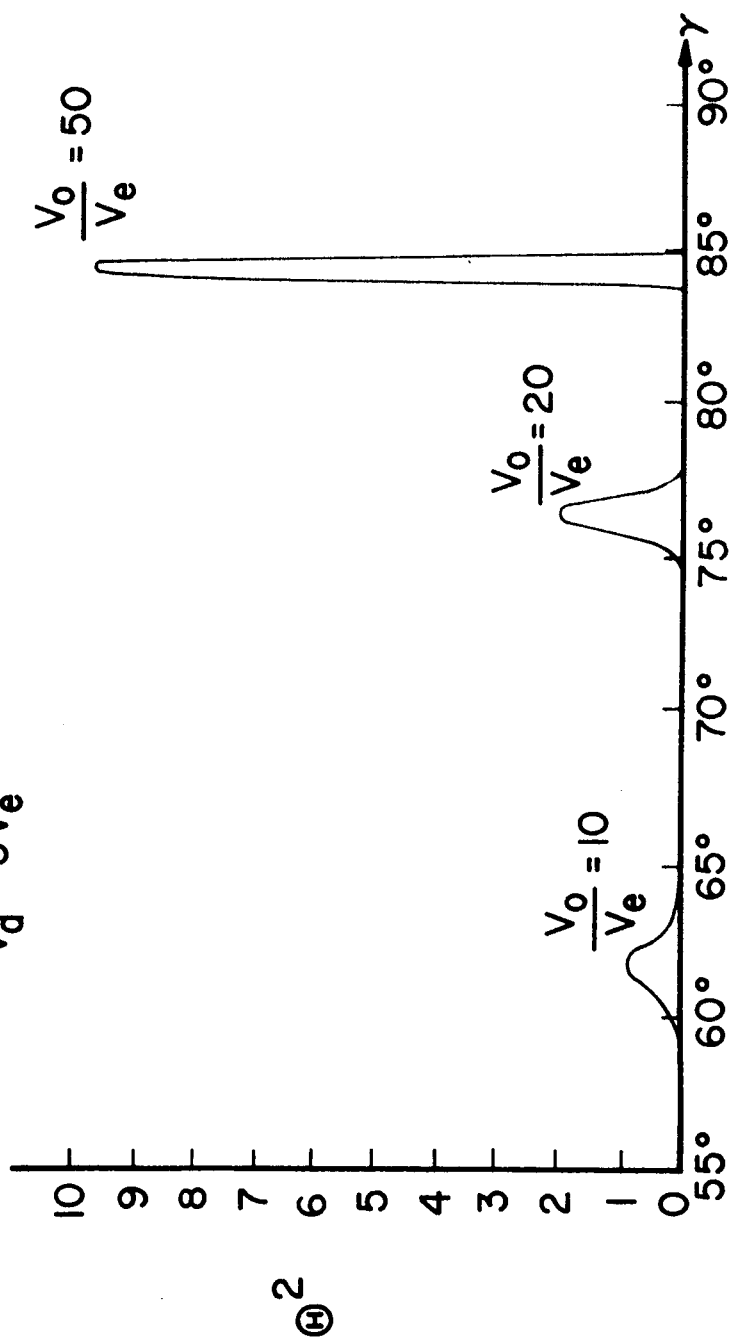 $\Gamma(0)/\omega_\rho \cong .03$.

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FIGURE I.

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FIGURE 2

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FIGURE 3
$(1-\beta)=4 \times 10^{-4}$
$V_d = 5V_e$

$\frac{V_o}{V_e} = 10$

$\frac{V_o}{V_e} = 20$

$\frac{V_o}{V_e} = 50$

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FIGURE 4
$(1-\beta) = 4 \times 10^{-3}$
$V_d = 5 V_e$

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FIGURE 5

$(1-\beta) = 4 \times 10^{-4}$

$V_\alpha = 10 \, V_e$

$\frac{V_0}{V_e} = 20$

$\frac{V_0}{V_e} = 50$

Θ^2
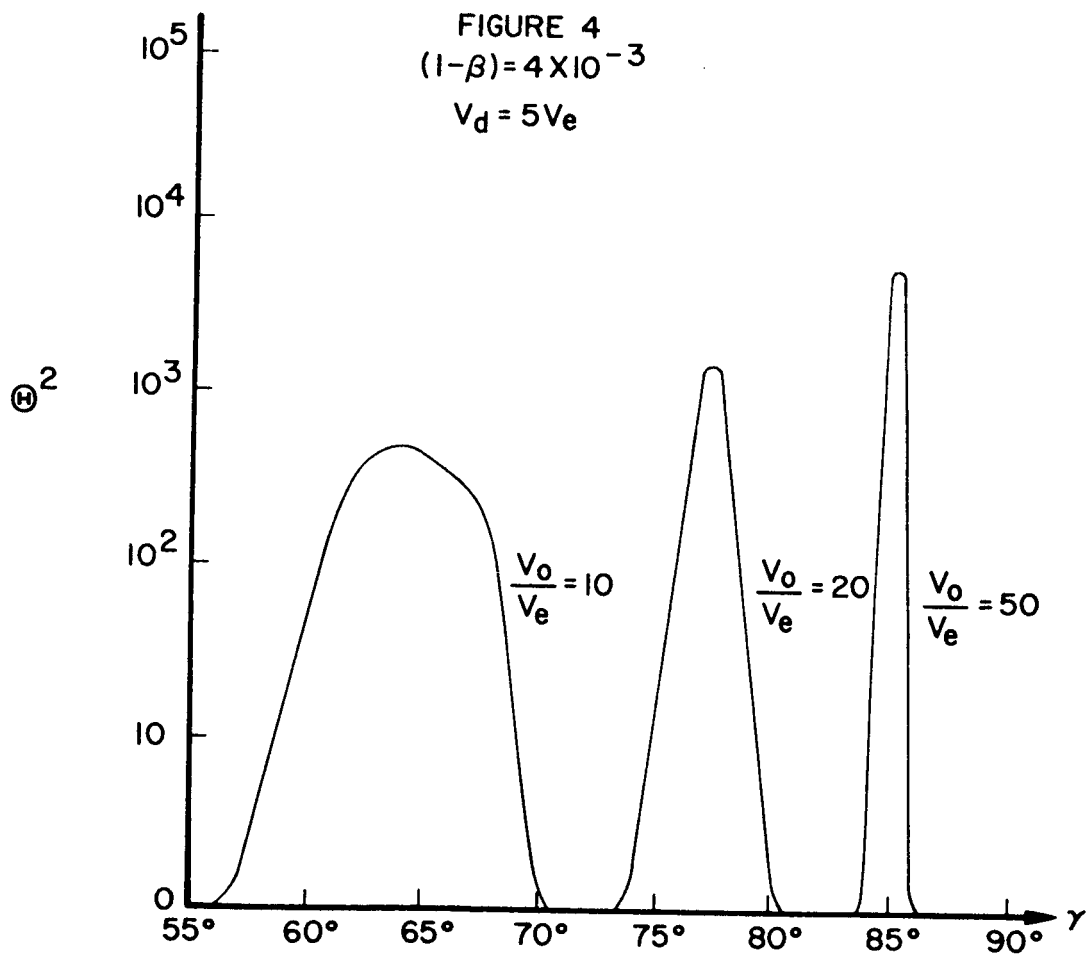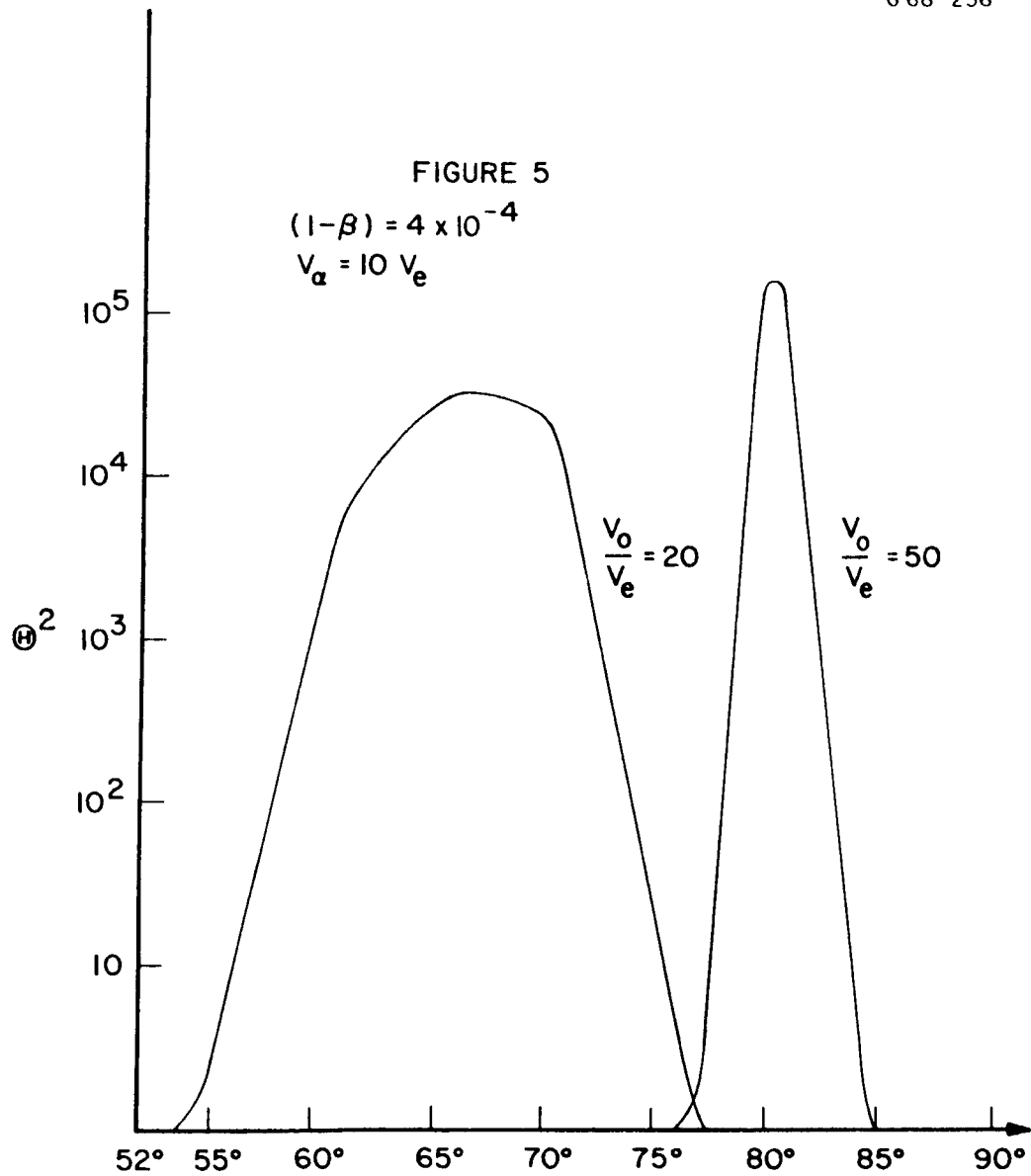
9-A

ERRATA

"The Scattering of Charged Particles in a Weakly Unstable Plasma,"
by Celso Roqúe   (U. of Iowa 68-14)

(1) Page 7, Equation (5):

Replace $\quad \mathcal{E}(\dfrac{\omega_\rho}{V_o \cos \gamma}, X)$ on the left hand side by

$$\mathcal{E}(\frac{\omega_\rho}{V_o \cos \gamma}, X + 1)$$

(2) Page 9, Second line before Equation (10):

Replace v_3 by v_4 .