PROPAGATION AND WAVE GUIDES FILLED WITH

WARM PLASMA

by

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ABSTRACT

The linearized equations describing the propagation of the normal modes in a plasma filled waveguide at microwave frequencies and in the presence of an axial constant magnetic field are derived from moments of the Boltzmann equation. Collisions are retained. For two cases where the plasma is assumed to be drifting but uniform or stationary but non-uniform in the transverse plane it is possible to completely solve for the fields by solving a set of coupled equations for the axial electric and magnetic fields and the pressure. If the plasma is assumed to be stationary and uniform these reduce to a set of coupled Helmholtz equations. Solutions for this case are considered in detail. The equations can be simplified considerably and cast into a form very similar to those used to describe wave propagation in a cold plasma. Solutions are obtained by employing an iterative technique.

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DEFINITION OF SYMBOLS

The following is a tabulation of the symbols used in this. MKS units are used throughout.

ω - radian excitation frequency

C - speed of light

ε_o - permittivity of free space = 8.854 x 10^{-12} farad/m.

μ_o - permeability of free space = 4π x 10^{-7} henries/m.

u = adiabatic electron sound speed

m - mass of the electron = 9.107 x 10^{-31} Kg.

q - charge of the electron = 1.602 x 10^{-19} coul.

B_o - magnitude of externally applied magnetic field

N_o - gross electron background number density

$\omega_o = \left(\frac{N_o q^2}{\varepsilon_o m}\right)^{1/2}$ - electron plasma frequency

$\omega_B = \frac{qB_o}{m}$ - electron cyclotron frequency

ω_{CE} - hybrid-E mode cutoff frequency

ν - collision frequency for momentum transfer

T - Kelvin temperature

$\ell_o \equiv \frac{\omega_o}{\omega}$

$\ell_B \equiv \omega_B/\omega$

$\ell_\nu \equiv \nu/\omega$

$k_o = \omega/C$

$h_o = \omega/u$

β - axial propagation wave number

$K_\nu \equiv 1 - j\ell_\nu$

$K_p \equiv 1 - \frac{\ell_o^2}{K_\nu}$

$K_B \equiv 1 - \frac{\ell_B^2}{K_\nu^2}$

$K_H \equiv K_\nu - \ell_o^2 - \frac{\ell_B^2}{K_\nu}$

$k_p^2 \equiv k_o^2 K_p - \beta^2$

$k_c^2 \equiv k_o^2 - \beta^2$

$k_p^2 \equiv h_o^2 K_H - \beta^2 K_B$

$\vec{H}_{\text{normalized}} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\ \vec{H}_{\text{actual}}$

$p = \frac{uN_o q}{\omega}\ \phi$

$\vec{v}_{\text{actual}} = \frac{\omega}{uN_o q} \sqrt{\frac{\varepsilon_o}{u_o}}\ \vec{v}_{\text{normalized}}$

$\langle e_n, w_m \rangle$ - scalar product of two vectors, e_n and w_m

CHAPTER I

INTRODUCTION

In recent years the topic of wave propagation in a gaseous plasma has been studied extensively. Most of the early work in this area was concerned with determining the dispersion characteristics of waves in an unbounded plasma. More recently, a number of bounded plasma problems have been investigated. In this work we shall be concerned only with the high frequency behavior of cylindrically bounded plasmas. The plasma will generally be anisotropic since it will be assumed that an axial steady magnetic field is applied.

Before embarking on any plasma problem it is necessary choose a model to describe the motion of the plasma constituents(generally ions, electrons and neutrals). The most satisfactory derivation of all the macroscopic plasma models is obtained by computing the moments of the Boltzmann equation. [1-2] Generally, this procedure can be carried out for each species of the plasma, the resulting equations being the so called n-fluid equations.

The process of computing moments of Boltzmann's equation yields an open set of equations since each higher moment introduces additional unknown quantities. The procedure must therefore be terminated or truncated in some manner. Two possible methods of truncation yield the cold plasma and warm plasma models.

If only the first two moments of the Boltzmann equation are retained the unknown pressure terms must be dropped and one obtains the cold plasma model. In high frequency problems the motion of the ions and neutrals is usually neglected and the ions are considered to provide simply a neutralizing stationary background for the electrons. This model is known as the Lorentz gas model. Since the term from the collision integral is still present it is generally simplified by either neglecting collisions altogether or by assuming the effect of collisions can be accounted for by introducing a collision frequency.

The collisionless Lorentz model has been used to study several classes of bounded problems. These include wave propagation on bounded plasma cylinders[3], analysis of plasma beam amplifiers[4-6] and the investigation of wave propagation in plasma filled waveguide [7-13]. This model has the advantage of being the simplest possible plasma model and hence the easiest to analyze. However, neglecting collisions can be a very poor assumption, particularly at resonances where the particles may move very rapidly. In particular, dispersion curves may be obtained from the collisionless equations that have very sharp resonances and interesting behavior, but upon the inclusion of collisions will be so highly attenuated as to become meaningless in an experiment. For this reason we shall retain collisions in all our work.

Another problem arises when the cold plasma model, with or without collisions, is used in a bounded problem. It has been shown by Sancer[14], and is later shown in this work, that the normal component of fluid velocity does not in general vanish at the walls. There is no way around this problem within the bounds of the cold plasma model and to eliminate this unphysical result we must consider the more complicated warm plasma equations.

To obtain the warm plasma equations three moments of the Boltzmann equation are retained. To close the equations the heat flow term, off-diagonal pressure terms and collision integral term are set to zero. We are left with the conservation of mass, momentum and energy equations. The linearized form of these equations, together with Maxwell's equations, provide a closed set of equations for studying bounded wave propagation. In this work we shall consider an electron gas model since we will be concerned with high frequencies. The more general and more complicated n-fluid model[15] has been used to study low frequency waveguide (or magneto-hydrodynamic waveguide) propagation[16].

The warm electron gas model has been used to study a variety of boundary value problems. Propagation along an open isotropic plasma cylinder has been studied, experimentally and theoretically, by Kolettis[17]. A study of the inhomogeneous (i.e., including sources) waveguide equations has been made by Sancer[14]. In this work Sancer considers the mathematical aspects of the linearized equations and discusses mathematically appropriate boundary conditions and possible methods of solution[19]. It is found that the warm plasma model has mathematically acceptable solutions for a number of boundary conditions on the velocity. We shall use the condition that the normal component of velocity vanish at the plasma boundary. Other possible physically acceptable boundary conditions have been discussed by Wait[21]. A formal method of solution similar to that proposed by Sancer has been outlined by Chen and Cheng[22]. This method is similar to the method used by Wang and Hopson[12] to analyze the cold collisionless bounded problem. It will be followed in Chapter 2 where we shall obtain solutions for the cold collisional model.

Solutions for wave propagation in warm, collisional anisotropic plasma have not appeared in the literature. If one attempts to solve the problem by employing the formal methods suggested by Sancer or Chen and Cheng it quickly becomes clear that a number of extremely difficult coupled transcendental equations must be solved. Besides presenting formidable numerical problems, it is felt that the basic physics is quite obscured by this approach. We shall avoid the method completely and derive a set of simpler equations by considering the coupling of the various waves and the simplifications which are evident from a coupled mode approach to the problem. These simpler equations are derived in Chapter 6 and solutions are obtained in Chapter 7 and 8.

Although the main goal of this work has been to exhibit solutions to the stationary, uniform plasma problem, it is of interest to see how the equations change when drifts or non-uniformities are present. In Chapter 3 and 4 the basic equation for these cases are derived.

CHAPTER II

SOLUTIONS FOR THE COLD, BOUNDED PLASMA

2.0 Introduction

Before considering the warm plasma we will consider the solutions for wave propagation in a waveguide filled with cold anisotropic plasma. It was pointed out in the introduction that this model is not adequate for bounded problems since we cannot impose any boundary condition on the normal component of the electron velocity. However, the equations for this model are considerably simpler than the more accurate warm plasma equations and it is of interest to compare the results of the two models. Also, we shall later (Chapter 6) derive a reduced approximate set of equations for the warm plasma equations. It will be seen that these equations are very similar to the cold plasma equations.

The method of solution employed here is very similar to that used by Wang and Hopson[12]. However, we shall include collisions in our model.

The starting equations for both models will be obtained as follows. We will consider only the small signal, linearized moment equations and Maxwell equations with no applied sources. It is assumed that the plasma is contained in a cylindrical waveguide and that a steady axially directed magnetic field is applied. In this case the field equations can be broken into equations having only transverse or axial components. From these equations it is possible to show that all the transverse field quantities and the axial velocity can be found from the axial components of the electric and magnetic fields. These axial fields thus serve as a set of potentials for the problem and are determined by solving a set of coupled Helmholtz equations. This procedure will now be illustrated.

2.1 The Linearized Cold Plasma Equations

The equations used in this analysis are the standard linearized cold plasma equations with the assumed wave variation $e^{j(\omega t-\beta z)}$. M. K. S. units are used throughout and the quantity $e^{j(\omega t-\beta z)}$ is dropped for convenience.

To simplify the equations we will normalize the magnetic field to have the dimensions of electric field.

$$\vec{H}_{\text{actual field}} \equiv \sqrt{\frac{\varepsilon_o}{\mu_o}}\ \vec{H} \tag{2.1}$$

Also define the following quantities:

$$\omega_o = \left(\frac{N_o q^2}{\varepsilon_o m}\right)^{1/2} \quad \text{; the electron plasma frequency} \tag{2.2a}$$

where N_o is the steady background electron number density, q is the charge of an electron and m is the electron mass.

$$\omega_B = \frac{qB_o}{m} \quad \text{; the electron cyclotron frequency} \tag{2.2b}$$

where B_o is the magnitude of the applied magnetic field.

$$\ell_o \equiv \frac{\omega_o}{\omega} \qquad (c)$$

$$\ell_B \equiv \frac{\omega_B}{\omega} \qquad (d) \qquad\qquad (2.2)$$

$$\ell_\nu \equiv \frac{\nu}{\omega} \qquad (e)$$

where ν is the effective collision frequency for momentum transfer.

$$k_o = \frac{\omega}{c} \qquad (2.2f)$$

where c is the speed of light in vacuum.

All vector quantities are separated into axial and transverse components, i.e.,

$$\vec{E} = \{\vec{e}(x,y) + e_z(x,y)\}e^{j(\omega t-\beta z)}$$

$$\nabla = \nabla_t - j\beta\hat{a}_z$$

$$\vec{H} = \vec{h} + \vec{h}_z \text{ , etc.}$$

The resulting set of equations is

$$\nabla_t x\ \vec{e} = -jk_o\vec{h}_z \qquad (a)$$

$$\hat{a}_z x \nabla_t e_z + j\beta\hat{a}_z x\ \vec{e} = jk_o\vec{h} \qquad (b)$$

$$\nabla_t x\ \vec{h} = jk_o\vec{e}_z + N_o q \sqrt{\frac{\mu_o}{\epsilon_o}}\ \vec{v}_z \qquad (c)$$

$$\hat{a}_z x \nabla_t h_z + j\beta \hat{a}_z x \vec{h} = -jk_o \vec{e} - N_o q \sqrt{\frac{\mu_o}{\varepsilon_o}}\, \vec{v}_t \qquad \text{(d)}$$

$$\nabla_t \cdot \vec{e} = j\beta e_z + nq/\varepsilon_o \qquad \text{(e)}$$

$$\nabla_t \cdot \vec{h} = j\beta h_z \qquad \text{(f)} \qquad (2.3)$$

$$j\omega n + N_o \nabla_t \cdot \vec{v} - j\beta N_o v_z = 0 \qquad \text{(g)}$$

$$j\omega(1-j\ell_\nu)\vec{v} = q/m\{\vec{e} + \vec{v}_t x \vec{B}_o\} \qquad \text{(h)}$$

$$j\omega(1-j\ell_\nu)v_z = (q/m)e_z \qquad \text{(i)}$$

As mentioned, to separate the equations in the above form it is necessary to assume $\vec{B}_o = B_o \hat{a}_z$. A method of solving (2.3) will now be discussed.

2.2 The Potential Equations

We now will show that a set of coupled equations can be found and solved for the axial fields e_z and h_z . It is then shown that all other quantities can be found from these quantities.

To obtain the equation for e_z operate on (2.3b) with $\hat{a}_z x$ and then with $\nabla_t \cdot$ and use (2.3c) and (2.3e) to eliminate $\nabla_t x \vec{e}$ and $\nabla_t \cdot \vec{e}$ to obtain

$$\{\nabla_t^2 + k_o^2(1 - \frac{\ell_o^2}{K_\nu}) - \beta^2\}e_z = -j \frac{\beta nq}{\varepsilon_o} \qquad (2.4a)$$

where we have defined

$K_\nu = 1 - j\ell_\nu$

Operate on (2.3d) with $\hat{a}_z x$, then with $\nabla_t \cdot$ and use (2.3a) and (2.3f) to eliminate $\nabla_t x\, \vec{e}$ and $\nabla_t \cdot \vec{h}$ to obtain

$$\{\nabla_t^2 + k_o^2 - \beta^2\} h_z = -N_o q \sqrt{\frac{\mu_o}{\varepsilon_o}}\, \hat{a}_z \cdot \nabla_t x\, \vec{v} \qquad (2.4b)$$

Equation (2.4) can be expressed entirely in terms of e_z and h_z if n and $\nabla_t x\, \vec{v}$ can be expressed in terms of these quantities. It is easily seen that this can be done by writing a set of equations including (2.3e), (2.3g), $\nabla_t x$ (2.3h) and $\nabla_t \cdot$(2.3h) as follows.

$$\begin{bmatrix} 1 & -q/\varepsilon_o & 0 & 0 \\ 0 & j\omega/N_o & 1 & 0 \\ 0 & 0 & \omega_B & j\omega K_\nu \\ -q/m & 0 & j\omega K_\nu & -\omega_B \end{bmatrix} \begin{pmatrix} \nabla_t \cdot \vec{e} \\ n \\ \nabla_t \cdot \vec{v} \\ \hat{a}_z \cdot \nabla_t x\, \vec{v} \end{pmatrix} = \begin{pmatrix} j\beta e_z \\ \frac{\beta q}{m\omega K_\nu} e_z \\ -j\frac{k_o q}{m} h_z \\ 0 \end{pmatrix} \qquad (2.5)$$

Equation (2.5) has non-trivial solutions only if the determinant of the coefficients is zero. But

$$\Delta = \frac{j\omega}{N_o} \left| \omega^2 K_\nu^2 - \omega_o^2 K_\nu - \omega_B^2 \right| \qquad (2.6)$$

If collisions were absent, $K_\nu = 1$, and Δ could be zero. With collisions present Δ will generally never vanish and solutions to (2.5) always exist.

We can now solve (2.5) for n and $\hat{a}_z \cdot \nabla_t \times \vec{v}$ and substitute the result into (2.4) to obtain;

$$\{\nabla_t^2 + k_o^2 K_p - \beta^2 (1 + \frac{\ell_o^2 \ell_B^2}{K_H K_\nu^2})\} e_z = j \frac{k_o \beta \ell_B \ell_o^2}{K_H K_\nu} h_z \qquad (a)$$

$$(2.7)$$

$$\{\nabla_t^2 + k_o^2 [K_p - \frac{\ell_o^2 \ell_B^2}{K_H K_\nu^2}] - \beta^2\} h_z = -j \frac{k_o \beta \ell_B \ell_o^2 K_p}{K_H K_\nu} e_z \qquad (b)$$

We have defined several quantities which will appear frequently throughout this work.

$$K_p \equiv 1 - \frac{\ell_o^2}{K_\nu} \qquad (a)$$

$$(2.8)$$

$$K_H \equiv K_\nu - \ell_o^2 - \frac{\ell_B^2}{K_\nu} \qquad (b)$$

Solutions to (2.7) are considered shortly. First, the equations for the other field quantities will be presented.

2.3 Equations for the Transverse Fields

The equations for the transverse fields are obtained by straightforward, but rather tedious manipulation of (2.3). Only the results are given.

Define the following quantities

$$k_p^2 \equiv k_o^2\, K_p - \beta^2 \qquad (a)$$

$$k_c^2 \equiv k_o^2 - \beta^2 \qquad (b) \qquad (2.9)$$

The transverse electric, magnetic and velocity fields are given in terms of the potentials by

$$\{k_p^4 - \ell_B^2 k_c^4\}\vec{e} = -j\beta(k_p^2 - \ell_B^2 k_c^2)\nabla_t e_z + \ell_B \ell_o^2 k_o^2 \beta \hat{a}_z x \nabla_t e_z + \ell_B \ell_o^2 k_o^3 \nabla_t h_z$$
$$+ jk_o(k_p^2 - \ell_B^2 k_c^2)\hat{a}_z x \nabla_t h_z \qquad (a)$$

$$\{k_p^4 - \ell_B^2 k_c^4\}\vec{h} = -\ell_B \ell_o^2 \beta^2 k_o \nabla_t e_z - jk_o(k_p^2 K_p - \ell_B^2 k_c^2)\hat{a}_z x \nabla_t e_z$$
$$-j\beta(k_p^2 - \ell_B^2 k_c^2)\nabla_t h_z + \beta k_o^2 \ell_o^2 \ell_B \hat{a}_z x \nabla_t h_z \qquad (b) \qquad (2.10)$$

$$\vec{v}_t = \frac{1}{[\ell_B^2 - K_\nu^2]}\left\{ \frac{jK_\nu q}{\omega m}\vec{e} - \frac{\ell_B q}{\omega m}\hat{a}_z x\, \vec{e}\right\} \qquad (c)$$

The axial velocity is given simply by

$$v_z = \frac{-jqe_z}{\omega m K_\nu} \qquad (2.11)$$

It has been pointed out several times that it is not possible to satisfy a boundary condition on $\vec{v}$ using the cold plasma model. This point is clearly illustrated by considering (2.10c). If we consider the plasma to be contained in a waveguide having perfectly conducting walls and apply the boundary conditions $\vec{E}_{tangential} = 0$ at the walls, then the normal component of $\vec{e}$ will generally not be zero. From (2.10c) it is seen that the normal component of velocity will also not be zero and is determined by $\vec{e}$.

Another consequence of using the cold plasma model is the omission of a class of acoustic modes which may be of interest. Of course, every assumption that is made in assuming immobile ions and neutrals, no heat transfer, etc., introduces approximations into the equations. The important point is that neglecting these terms may not appreciably effect the dispersion characteristics of the modes. Solutions to the cold plasma equations are sought now so that we may later compare the dispersion curves with those obtained from the warm plasma equations.

2.4 Solutions for a Circular Cylindrical Guide

Detailed solutions to (2.7) are now sought where the waveguide geometry is shown in Fig. 8.1 and the analysis is restricted to determining β for the lowest order modes. The analysis proceeds as follows. First, a transformation will be found which diagonalizes the coupled differential equations, (2.7). Then the boundary conditions $e_z|_c = \frac{dh_z}{dn}|_c = 0$ are applied and a transcendental equation from which the eigenvalues are derived is found. Solving the transcendental equation numerically gives the dispersion relation and eigenvalues.

Define the following coefficients appearing in 2.7 by

$$b_{11}^2 = k_o^2 K_p - \beta^2\left(1 + \frac{\ell_o^2 \ell_B^2}{K_H K_2^2}\right) \qquad \text{(a)}$$

$$b_{22}^2 = k_o^2\left(1 - \frac{\ell_o^2 K_p}{K_H}\right) - \beta^2 \qquad \text{(b)}$$

$$b_{12} = \frac{j\ell_o \ell_B k_o \beta}{K_H K_\nu} \qquad \text{(c)} \qquad \qquad (2.12)$$

$$b_{21} = -\, b_{12} K_p \qquad \text{(d)}$$

The coupled Helmholtz equations are now written

$$\nabla_t^2 \begin{pmatrix} e_z \\ h_z \end{pmatrix} + \begin{bmatrix} b_{11}^2 & -b_{12} \\ -b_{21} & b_{22}^2 \end{bmatrix} \begin{pmatrix} e_z \\ h_z \end{pmatrix} = [0] \qquad (2.13)$$

In matrix notation (2.13) can be written

$$[\nabla_t^2 + B] \begin{pmatrix} e_z \\ h_z \end{pmatrix} = 0 \qquad (2.14)$$

Now we construct a matrix M, such that $M^{-1}BM$ is a diagonal matrix, where M^{-1} is the inverse of M. Techniques for diagonalizing non-Hermitian matrices are discussed in Friedman, Ref. 23, and will not be elaborated on here.

To construct M we must first solve for the eigenvalues of B. The eigenvalues, λ^2, must satisfy the equation

$$\begin{vmatrix} b_{11}^2 - \lambda^2 & -b_{12} \\ -b_{21} & b_{22}^2 - \lambda^2 \end{vmatrix} = 0 \tag{2.15}$$

Expanding (2.15) and solving for λ^2 gives

$$\lambda_{1,2}^2 = \frac{(b_{11}^2+b_{22}^2) \pm \{(b_{11}^2-b_{22}^2)^2 + 4b_{12}b_{21}\}^{1/2}}{2} \tag{2.16}$$

Note that the eigenvalues cannot be determined explicitly from (2.16) since the b_{ij} contain β. To find the eigenvalues and associated values of β we must impose the boundary conditions. We can assume that the values of λ can be found and formally proceed to find β. As shown in Friedman, the columns of M are the eigenvectors associated with λ_i^2.

There are alternate ways to solve for the eigenvectors, but the one that must be used in practice should avoid any numerical difficulties caused by division by zero if, for instance, b_{12} or b_{21} should become zero. An appropriate form for M is

$$M = \begin{bmatrix} 1 & m_{12} \\ m_{21} & 1 \end{bmatrix} \tag{2.17a}$$

where $$m_{21} = \frac{b_{21}}{b_{22}^{\,2}-\lambda_1^{\,2}} \; ; \quad m_{12} = \frac{b_{12}}{b_{11}^{\,2}-\lambda_2^{\,2}} \tag{2.17b}$$

The original fields are related to a set of new field quantities, u_i, by

$$\begin{pmatrix} e_z \\ h_z \end{pmatrix} = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{2.18a}$$

where u_i satisfy the diagonalized equation

$$\{\nabla_t^{\,2}+\lambda_i^{\,2}\}u_i = 0 \tag{2.18b}$$

Now specialize the problem to cylindrical geometry and consider solutions. This geometry is later used when the warm plasma problem is analyzed and it will be of interest to compare the results obtained from the two models. The analysis is restricted to consideration of modes having no azimuthal variations.

In this case (2.18b) becomes

$$\frac{d}{dr}\, r\, \frac{du_i}{dr} + r\lambda_i^{\,2}u_i = 0 \tag{2.19}$$

The only physically allowable solution of (2.19) is a zeroth order Bessel function

$$u_1 = A_1 J_o(\lambda_1 r) \tag{2.20}$$

Note that λ_1 is not yet specified since no boundary conditions have yet been applied. The applicable boundary conditions on u_1 must be derived from the conditions $e_z = \partial h_z/\partial r = 0$ at $r = a$, where a is the waveguide radius. Using (2.18a) to relate u_1 to e_z and h_z and applying the boundary conditions gives;

$$\begin{bmatrix} J_o(\lambda_1 a) & m_{12} J_o(\lambda_2 a) \\ -m_{21}\lambda_1 J_1(\lambda_1 a) & -\lambda_2 J_1(\lambda_2 a) \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = [0] \tag{2.21}$$

Non-trivial solutions to (2.21) exist only if the determinant of the coefficients vanishes. This condition yields the equation

$$\lambda_2 J_o(\lambda_1 a) J_1(\lambda_2 a) - \lambda_1 m_{12} m_{21} J_o(\lambda_2 a) J_1(\lambda_1 a) = 0 \tag{2.22}$$

Note that λ_1 is a function of β since β is contained in (2.16). Thus to obtain the dispersion relations and the λ's it is necessary to solve (2.16) simultaneously with (2.21). Obtaining solutions to these equations is not a simple task. The numerical methods used will be discussed briefly below and then the results of the computations are presented.

When collisions are included in the cold plasma model, as they have been here, the parameters in (2.16) and (2.22) will in general be complex and the equations can not be solved graphically. To solve these equations Newton's iterative procedure, with the equations written as functions of the complex variable β, has been employed[24]. To use this method it is usually necessary to have a good approximate starting value to begin the iteration. At very high frequency the coupling becomes very small and the iteration can be started.

A computer program was written to find the dispersion relation and eigenvalues for different values of the plasma parameters, ω_o and ω_B. The solutions were divided into two classes called hybrid-E and hybrid-H modes. The term hybrid signifies that the modes are not pure E or H modes and the E or H nomenclature indicates that the modes reduce to these pure modes in the high frequency limit. Dispersion relations for some choices of parameters are shown in Figures 2.[illegible]-2.5. The frequency (normalized by $\omega_C = \sqrt{\omega_o^2 + \lambda^2}$, where ω_C is derived in the next section) and the parameters are chosen to

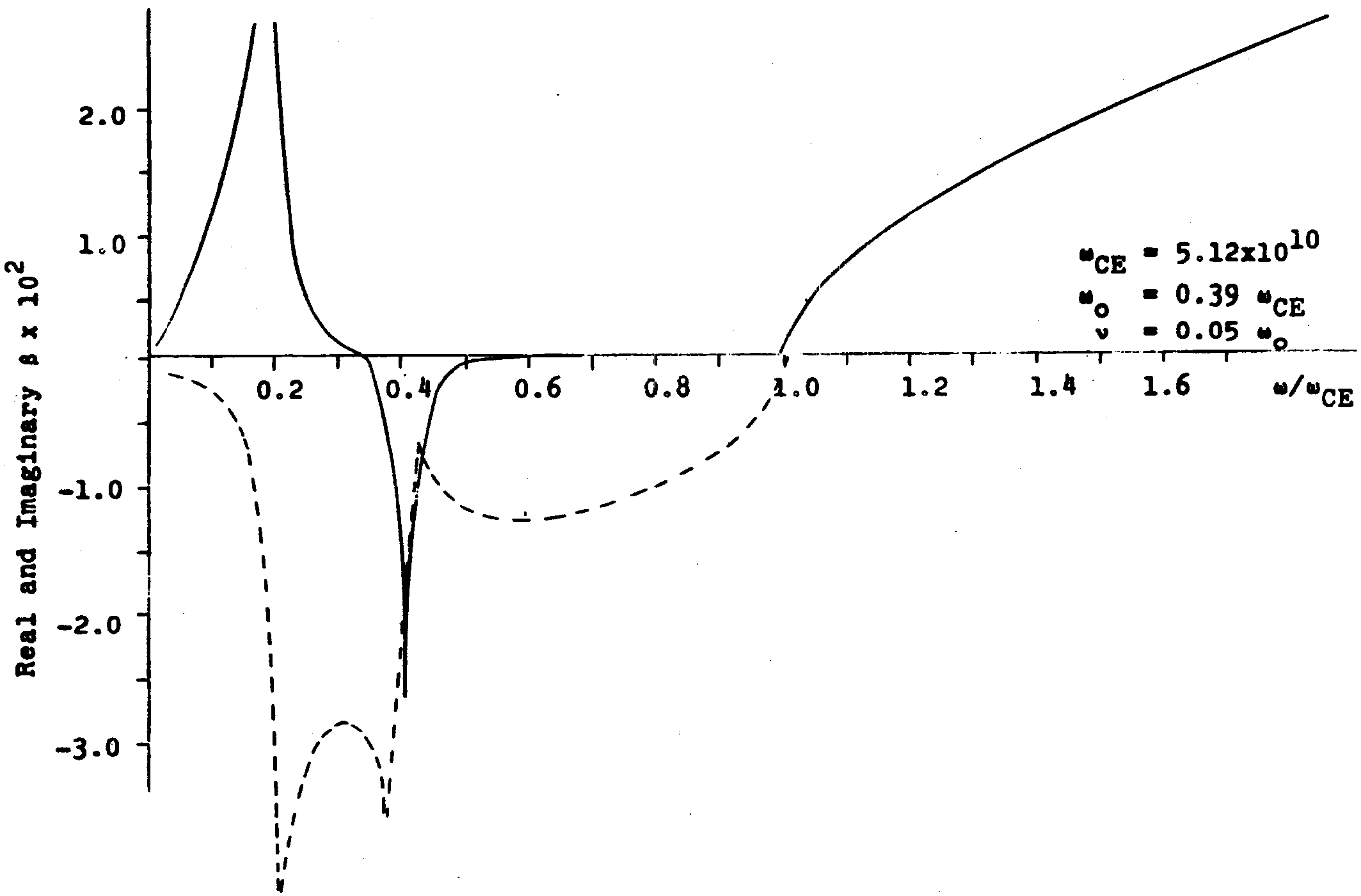

Fig. 2.1 Dispersion Curve for the Hybrid-E Mode with $\omega_B = -0.2\ \omega_{CE}$. Cold Plasma Model

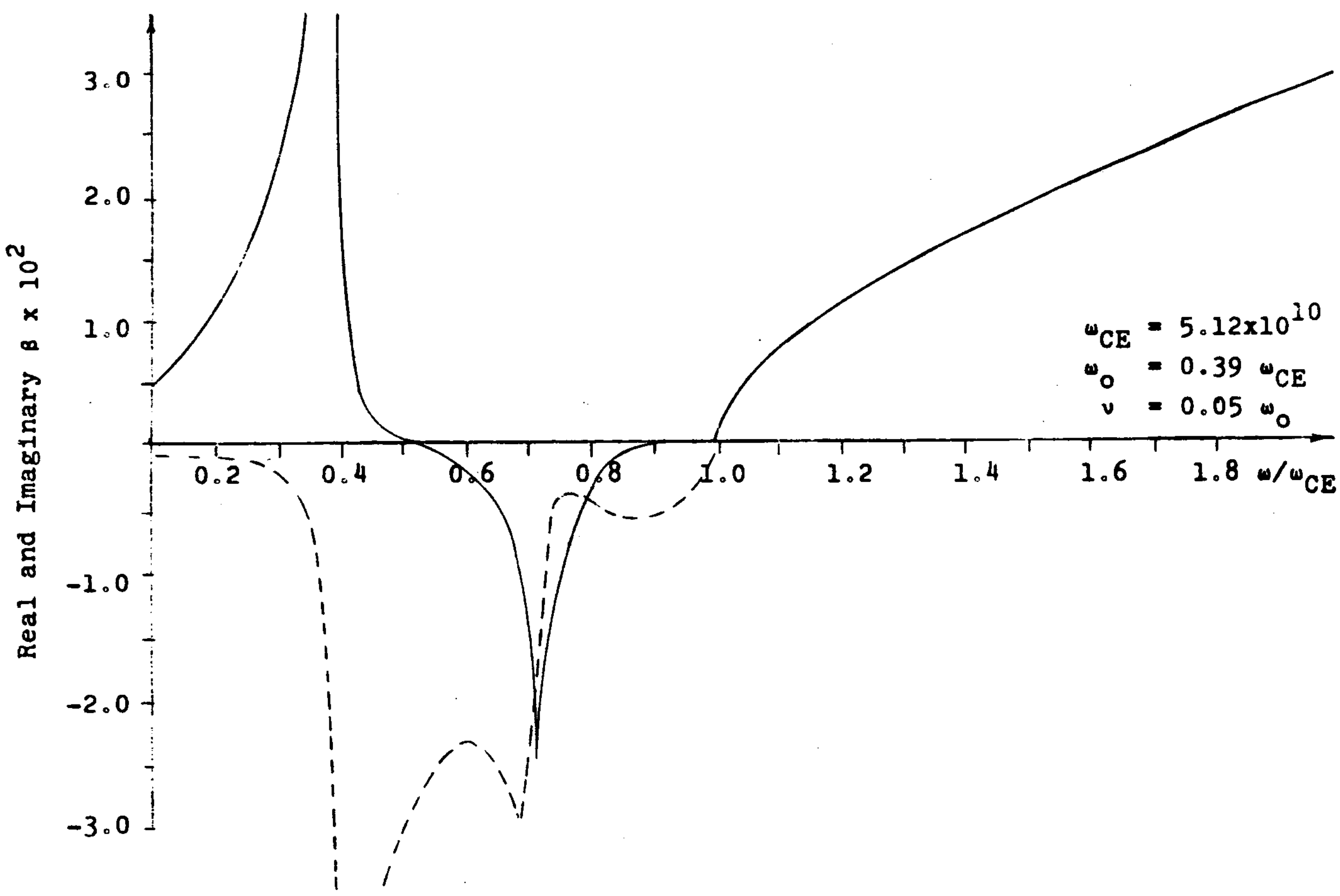

Fig. 2.2 - Dispersion Curve for the Hybrid-E Mode with $\omega_B = -3.58 \times 10^{10} = 0.7\ \omega_{CE}$. Cold Plasma Model

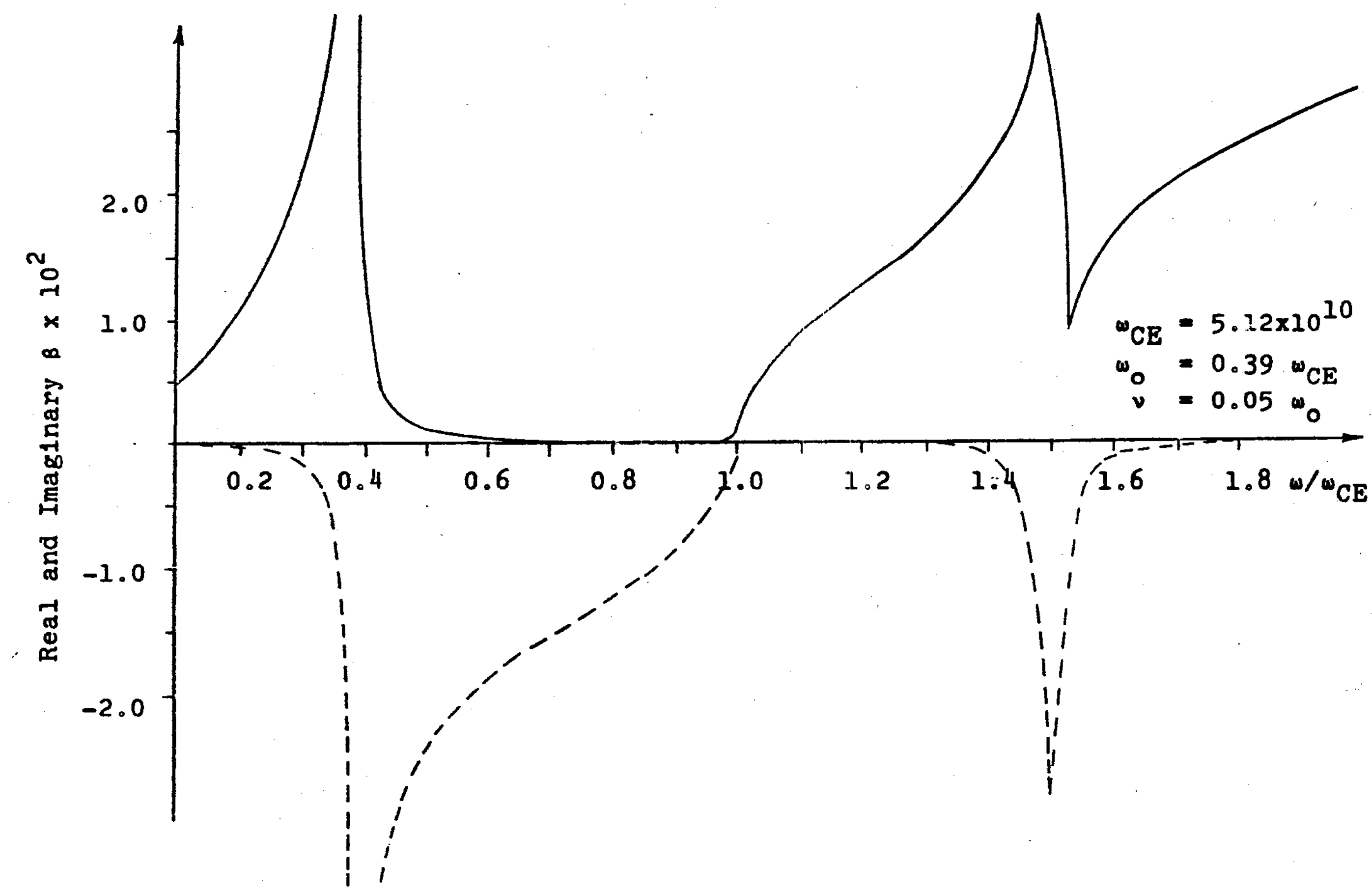

Fig. 2.3 - Dispersion Curve for the Hybrid-E Mode with $\omega_B = -1.5\ \omega_{CE} = -7.68 \times 10^{10}$. Cold Plasma Model

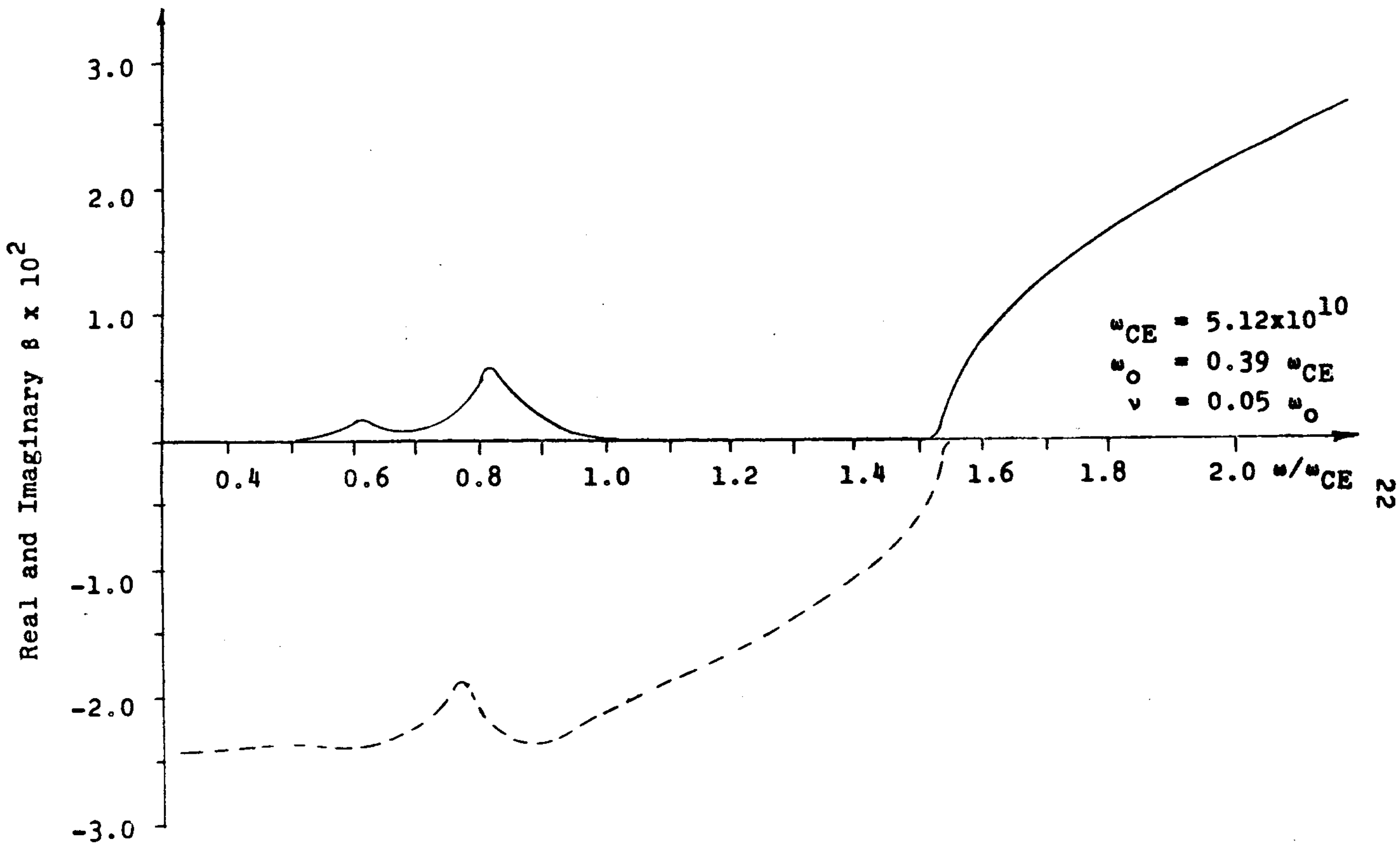

Fig. 2.4 - Dispersion Curve for the Hybrid-H Mode with $\omega_B = -3.58 \times 10^{10} = -0.7\ \omega_{CE}$. Cold Plasma Model

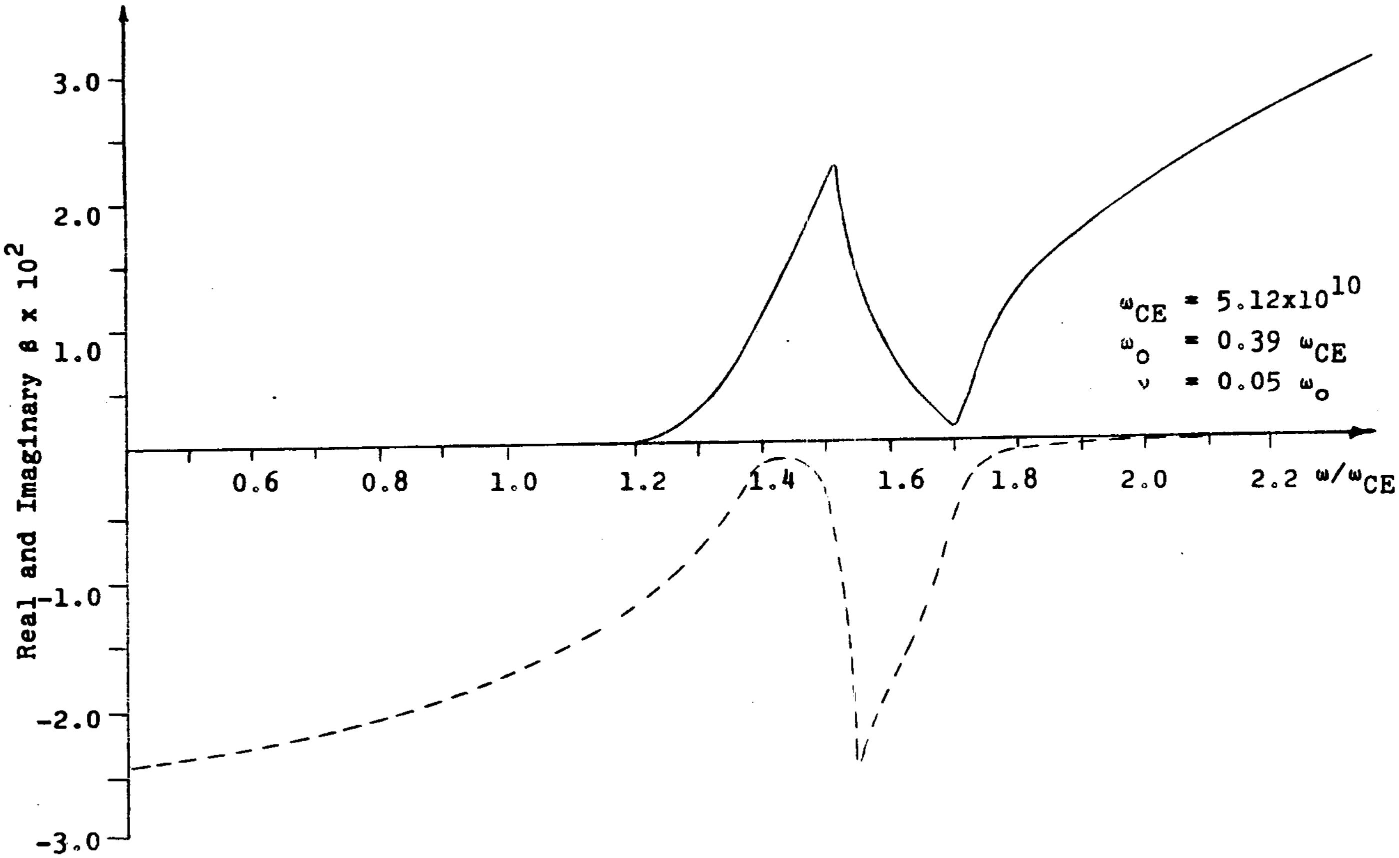

Fig. 2.5 - Dispersion Curve for the Hybrid-H Mode with $\omega_B = -7.68 \times 10^{10} = -1.5\ \omega_{CE}$. Cold Plasma Model

coincide with those used in Chapter 8 to display the warm-plasma dispersion relation so that a comparison of the results from the two models may be made.

2.5 Cutoffs, Resonances and Limiting Values for β

With the dispersion curves available it is now of interest to try to explain some of their characteristics in some limiting cases. Since strict cut-off and resonance points do not occur when β is complex (i.e., when collisions are present) we shall assume that ν is zero in the following derivations.

A. The high frequency limit

When $\omega >> \omega_o$ or ω_B, ℓ_o and ℓ_B both approach zero and K_B, K_p and K_H approach unity. Thus, at high frequencies the dispersion relation approaches that of the empty waveguide.

B. $\omega_B \to \infty$

In this limit $K_H \to - \ell_B^2$ and the right hand coupling terms in (2.7) approach zero. The equation for h_z, (2.7b), reduces to

$$\{\nabla_t^2 + k_o^2 - \beta^2\}h_z = 0 \qquad (2.23)$$

This is just the equation satisfied by the H modes in an empty waveguide. The behavior of β shown in Fig. 2.5 has exactly this behavior.

Equation (2.7a) is changed from that of the empty waveguide E modes and exhibits the behavior shown in Fig. 2.1. This is shown as follows.

In the limit of high cyclotron frequency the quantity multiplying β^2 in (2.7a) becomes

$$\left(\frac{K_H + \ell_o^2 \ell_B^2}{K_H}\right) \rightarrow \frac{(-\ell_B^2 + \ell_o^2 \ell_B^2)}{-\ell_B^2} = 1 - \ell_o^2 \qquad (2.24)$$

Equation (2.7a) thus becomes

$$\{\nabla_t^2 + K_p(k_o^2 - \beta^2)\}e_z = 0 \qquad (2.25)$$

This is an eigenvalue equation and β^2 is given in the usual way by

$$\beta = \sqrt{k_o^2 - \lambda^2/K_p} \qquad (2.26)$$

where λ^2 is the eigenvalue of (2.25). At the plasma frequency β has a pole since $K_p = 0$. Thus $\beta^2 \rightarrow -\infty$ for $\omega = \omega_o^+$ and $\beta^2 \rightarrow \infty$ for $\omega = \omega_o^-$.

C. Cut-off and Resonance Frequencies

Additional information about the dispersion curves at particular points can be obtained by writing (2.15) explicitly in terms of β. To facilitate this multiply (2.15) by K_H^2 to eliminate division by zero where K_H equals zero and define

$C_{11} = (k_o{}^2K_p - \lambda^2)K_H \qquad C_{12} = K_H b_{12}/\beta$

$C_{21} = K_H b_{21}/\beta \qquad C_{22} = k_o{}^2[K_pK_H - \ell_o{}^2\ell_B{}^2] - \lambda^2 K_H$

Expanding (2.15) in terms of the above quantities gives

$$K_H(K_H+\ell_o{}^2\ell_B{}^2)\beta^4 - \beta^2[K_HC_{11}+(K_H+\ell_o{}^2\ell_B{}^2)C_{22} - C_{12}C_{21}] + C_{11}C_{22} = 0 \qquad (2.27)$$

Cut-off frequencies occur when $\beta = 0$. But this is possible only if $C_{11}C_{22} = 0$. Note that C_{11} and C_{22} contain λ, but when β is zero equations (2.27) decouple and λ can easily be found. Setting each quantity equal to zero gives the possible cut-off frequencies. Setting C_{11} to zero and solving for ω gives

$$\omega_{CE} = \sqrt{\omega_o{}^2+\lambda^2c^2} \qquad (2.28)$$

This is the cut-off frequency that was used to normalize the frequency in the preceeding curves. The other roots of (2.27) with $\beta = 0$ are obtained from

$$C_{22} = 0 \qquad (2.29)$$

Setting C_{22} to zero gives an equation for two values of the cutoff frequency. Since this expression is quite complicated and cannot be expressed simply it will only be said that the cutoff frequencies obtained from setting C_{22} to zero are functions of the cyclotron frequency as well as the plasma frequency and that the cutoff frequencies increase with increasing ω_B. This increase of the cutoff frequency is noticed in Figures 2.4 and 2.5.

Finally, it is noted that, when $\beta = 0$, $C_{11}C_{22}$ is zero if K_H is zero. However, if K_H is set to zero before β is set to zero it is found that all terms in equation (2.27) are identically zero and one is left with the meaningless equation $0 = 0$.

The resonant conditions are found from equation (2.27) by dividing the equation by β^4, defining a new quantity $\alpha = 1/\beta$ and setting α to zero. Thus resonances can occur when

$$(K_H+\ell_o^2\ell_B^2) = (1-\ell_B^2)(1-\ell_o^2) = 0 \qquad (2.30)$$

Possible resonances then occur when the frequency equals either the plasma or cyclotron frequency. Resonant behavior is seen from the dispersion curves to occur for both modes at the cyclotron frequency, but at the plasma frequency only for the hybrid E mode.

One other case is of interest. It was stated previously that if $K_H = 0$ then (2.27) becomes an identity. To investigate this case further consider equation (2.5) and (2.6). Note that the condition $K_H = 0$ is just the condition that the determinant of the coefficients, as expressed by (2.6) with $K_\nu = 1$, vanish. To prevent the solution from blowing up it is necessary that the right-hand-side of (2.5) vanish. This can occur if h_z is zero and if either β or e_z is zero. If β vanishes we have a cutoff condition and (2.4a) has a non-zero solution for e_z. This cutoff condition is evident in the dispersion curves shown in Figures 2.1-2.3. If β is not zero then both axial fields must vanish. However, no cut-off condition is evident.

The cut-off and resonant points for the collisionless cold plasma have now to be found. For the hybrid-E modes cutoffs occur when $\omega = \sqrt{\omega_o{}^2+\lambda^2 C^2}$ or where $K_H = 0$. Resonances occur when $\omega = \omega_o$ or ω_B. For the hybrid-H modes cutoffs occur when C_{22}, as given by (2.29), is zero and resonances occur when $\omega = \omega_B$.

With the cut-off and resonant points of a particular mode known it is a relatively simple task to sketch the form of the dispersion curves. For instance a sketch of the dispersion curve shown in Fig. 2.2 is shown below.

It is helpful to have this knowledge since it enables us to check the reasonableness of results obtained by the computer Note that Fig. 2.6 is a plot of β^2 vs. ω.

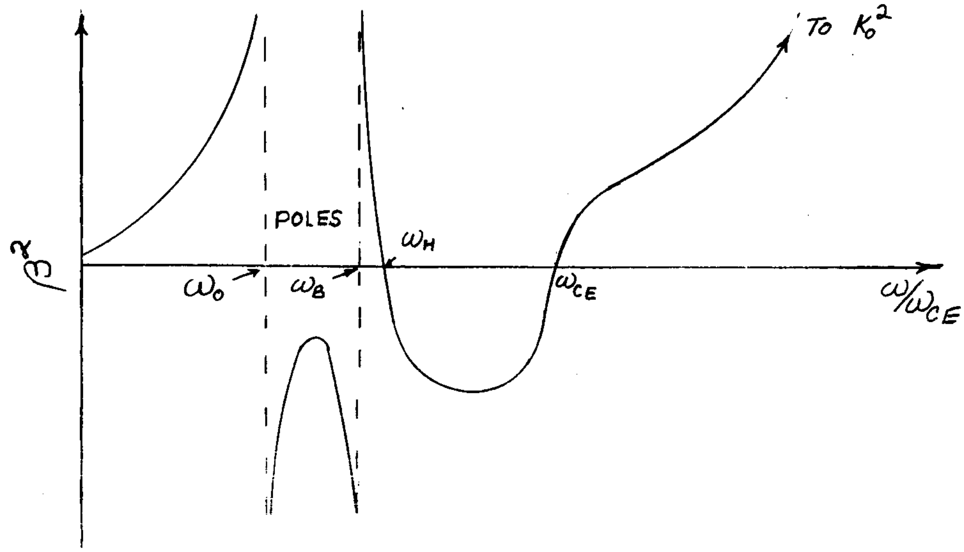

Fig. 2.6 Sketch of the Form of the Dispersion Curve with Normalized Values $\omega_B = 0.6$ and $\omega_o = 0.4$. The frequency is normalized with respect to ω_{CE}.

Finally, it should be pointed out that the preceeding analysis of the cold plasma model will be of considerable value later when the more complicated warm-plasma model is tested. It is shown in Chapter 6 that the warm plasma equation can be expressed in a form which is very close to the equations used here, if the analysis is restricted to consideration of the hybrid E and H modes. The behavior of the dispersion curves should thus be expected to exhibit a behavior which is similar to curves obtained for the cold plasma model.

CHAPTER III

THE POTENTIAL EQUATIONS FOR A WARM, ANISOTROPIC DRIFTING PLASMA

3.0 Introduction

In the last chapter it was seen that solutions to the cold plasma problem could be obtained entirely in terms of the axial field quantities. Similarly, for the warm, uniform, stationary plasma Sancer has shown [14] that solutions can be obtained by considering the axial electric and magnetic field and the pressure to be the potentials for the problem. Early in the study of this problem it occurred that such a treatment might also be possible for more general drifting and non-uniform plasmas. This is indeed the case and the potential equations for these two cases will be presented in this and the next chapter. The equations are presented for reference and no attempt will be made to solve the involved equations which will be derived.

In this chapter we consider only the linearized uniform drifting plasma. The non-linear d.c. equations are not considered.

3.1 The Basic Equations

The equations used to describe the plasma are Maxwell's equations, the equations for conservation of mass and momentum as derived from the Boltzmann equation assuming a diagonal pressure term and an effective collision frequency for transfer of momentum and the adiabatic equation of state to truncate the moments of the Boltzmann equation. Using M.K.S. units these become

$$
\begin{aligned}
&\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} && \text{(a)} \\
&\nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \vec{J} && \text{(b)} \\
&\nabla \cdot \vec{H} = 0 && \text{(c)} \\
&\nabla \cdot \vec{E} = \rho/\varepsilon_o && \text{(d)} \\
&\frac{\partial}{\partial t}(mN) + \nabla \cdot (mN\vec{V}) = 0 && \text{(e)} \\
&mN\left\{\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right\} = Nq\{\vec{E} + \vec{V} \times \vec{B}_o\} - \nabla P - Nm\nu\vec{V} && \text{(f)} \\
&PN^{-\gamma} = \text{const.} && \text{(g)}
\end{aligned}
\qquad (3.1)
$$

It is assumed that the frequency is sufficiently high that the motion of heavy particles can be neglected. Ions are assumed to provide a stationary neutralizing background for the electrons. $\vec{V}$, m , N and P are the electron fluid velocity, mass, gross number density and pressure respectively. The other symbols are the standard symbols used in Maxwell's equations. $\vec{B}_o$ is a static externally applied magnetic field and in this work is assumed to be oriented along the axis of the cylindrical waveguide.

3.2 The Normalized, Linearized Equations

The equations will now be linearized by assuming that the fluid field is composed of large static terms plus small time and space varying terms.

$$\vec{V} = \vec{V}_o + \vec{v}'(x,y)e^{j(\omega t-\beta z)} \qquad (a)$$

$$N = N_o + n(x,y)e^{j(\omega t-\beta z)} \qquad (b) \qquad (3.2)$$

$$P = P_o + p(x,y)e^{j(\omega t-\beta z)} \qquad (c)$$

$\vec{E}$ and $\vec{H}$ are assumed to be small signal terms varying as $e^{j(\omega t-\beta z)}$. The current $\vec{J}$ is assumed to be due only to motion of the plasma electrons.

$$\vec{J} = qN\vec{V} \qquad (3.3)$$

The linearized a.c. equations are thus

$$\nabla \times \vec{E} = -j\omega\mu_o\vec{H} \qquad (a)$$

$$\nabla \times \vec{H} = j\omega\varepsilon_o\vec{E} + q(N_o\vec{v}' + n\vec{V}_o) \qquad (b) \qquad (3.4)$$

$$\nabla \cdot \vec{H} = 0 \qquad (c)$$

$$\nabla \cdot \vec{E} = nq/\varepsilon_o \qquad (d)$$

$$j\omega n + \nabla \cdot \{n\vec{V}_o + N_o\vec{v}'\} = 0 \qquad (e)$$

$$j\omega N_o\vec{v}' + (N_o\vec{V}_o\cdot\nabla)\vec{v}' = \frac{q}{m} N_o\vec{e} + \frac{q}{m}\{N_o\vec{v}' \times \vec{B}_o + n\vec{V}_o \times \vec{B}_o\}$$

$$- \frac{\nabla p}{m} - N_o\nu\vec{v}' - n\nu\vec{V}_o \qquad (f)$$

$$p = (\frac{\gamma P_o}{N_o})n = (\gamma KT)n \qquad (g)$$

where K is Boltzmann's constant and T is the Kelvin temperature.

It is now convenient to normalize the variables to have the dimensions of electric field. The normalization used here has been presented in Sancer's work. [14]

$$\mathcal{H} \equiv \sqrt{\frac{\varepsilon_o}{\mu_o}}\,\vec{H} \qquad (a)$$

$$\vec{v}' \equiv \frac{\omega}{uN_oq}\sqrt{\frac{\varepsilon_o}{\mu_o}}\,\vec{v} \qquad (b)$$

$$p \equiv \frac{uN_oq}{\omega}\,\phi \qquad (c) \qquad\qquad (3.5)$$

$$n \equiv \frac{N_oq}{m\omega u}\,\phi \qquad (d)$$

u is the adiabatic electron gas sound speed.

$$u = (\frac{\gamma KT}{m})^{\frac{1}{2}} \qquad (3.6)$$

Since a steady drift is assumed to be present define the normalized term

$$\vec{W}_o \equiv \frac{q}{mu^2 h_o} \vec{V}_o \qquad \text{(a)}$$

$$\text{where} \quad h_o \equiv \frac{\omega}{u} \qquad \text{(b)} \qquad\qquad (3.7)$$

Using the above defined quantities in (3.4) gives the linearized, normalized equations

$$\nabla \times \vec{E} = -jk_o\vec{H} \qquad \text{(a)}$$

$$\nabla \times \vec{H} = jk_o\vec{E} + h_o\vec{v} + h_o\vec{W}_o\phi \qquad \text{(b)}$$

$$\nabla \cdot \vec{H} = 0 \qquad \text{(c)} \qquad\qquad (3.8)$$

$$\nabla \cdot \vec{E} = h_o \ell_o^2 \phi \qquad \text{(d)}$$

$$j\ell_o^2 k_o \phi + \nabla \cdot \vec{v} + \nabla \cdot (\vec{W}_o \phi) = 0 \qquad \text{(e)}$$

$$jh_o\vec{v} + \frac{h_o(\vec{W}_o\cdot\nabla)\vec{v}}{\ell_o^2 k_o} = \ell_o^2 k_o \vec{e} + h_o \ell_B \vec{v}\times\hat{a}_z + h_o \ell_B \phi \vec{W}_o \times \hat{a}_z - \frac{\ell_o^2 k_o}{h_o} \nabla\phi - h_o \ell_\nu \vec{v} - h_o \ell \nu \phi \vec{W}_o \qquad \text{(f)}$$

The following quantities have been defined and used in (3.8)

$$\ell_o = \frac{\omega_o}{\omega} \quad ; \quad \ell_B = \frac{\omega_B}{\omega} \qquad\qquad (3.9)$$

All other quantities have been defined previously.*

Now we will decompose the vector equations in (3.8) into equations for the transverse and axial components as was done in Chapter 2.

For convenience we define the differential operator

$$M \equiv \frac{\vec{W}_o \cdot \nabla}{\ell_o^2 k_o} \tag{3.10}$$

Equations (3.8) become

$$\begin{aligned}
&\nabla_t \times \vec{e} = -jk_o \vec{h}_z && \text{(a)}\\
&\hat{a}_z \times \nabla_t e_z + j\beta \hat{a}_z \times \vec{e} = jk_o \vec{h} && \text{(b)}\\
&\nabla_t \times \vec{h} = jk_o e_z + h_o v_z + h_o \vec{W}_z \phi && \text{(c)}\\
&\hat{a}_z \times \nabla_t h_z + j\beta \hat{a}_z \times \vec{h} = -jk_o \vec{e} - h_o \vec{v}_t - h_o \vec{W}_t \phi && \text{(d)}\\
&\nabla_t \cdot \vec{e} = j\beta e_z + h_o \ell_o^2 \phi && \text{(e)}\\
&\nabla_t \cdot \vec{h} = j\beta h_z && \text{(f)}\\
&j\ell_o^2 k_o \phi + \nabla_t \cdot \vec{v}_t - j\beta v_z + \nabla_t \cdot (\vec{W}_t \phi) - j\beta W_z \phi = 0 && \text{(g)}
\end{aligned} \tag{3.11}$$

---

*Note that a list of all definitions used in this work appears before Chapter 1.

$$jh_o\vec{v}_t + h_o M\vec{v}_t = \ell_o^2 k_o \vec{e} - h_o \ell_B \hat{a}_z \times \vec{v}_t + h_o \ell_B \phi \vec{W}_t \times \hat{a}_z$$

$$- \frac{\ell_o^2 k_o}{h_o} \nabla_t \phi - h_o \ell_\nu \vec{v}_t - h_o \ell_\nu \phi \vec{W}_t \qquad \text{(h)}$$

$$jh_o v_z + h_o M v_z = \ell_o^2 k_o e_z + j\beta \frac{\ell_o^2 k_o}{h_o} \phi - h_o \ell_\nu v_z$$

$$- h_o \ell_\nu \phi W_z \qquad \text{(i)}$$

These equations are now used to derive a set of coupled equations for e_z , h_z and ϕ .

3.3 Derivation of the Potential Equations

A study of the terms in (3.11) reveals that one term, the differential operator M in (3.11i), changes the characteristic of the potential equations from coupled Helmholtz equations to more complicated differential equations. It is still possible to manipulate the equations and find a set of coupled equations for e_z h_z and ϕ by introducing some formalism used in solving operator equations. (A discussion of the procedure is found in Friedman,[23] particularly Chapter 3).

Define the differential operator L by

$$L \equiv jh_o(1 - jM - j\ell_\nu) \qquad (3.12)$$

and assume that an inverse operator, L^{-1} , exists such that $L^{-1}L = 1$.

Using the definition we can now solve (3.11i) for v_z and then proceed to use this solution to derive the potential equations. At the end of the derivation any equation containing L^{-1} will be operated on with L to cast the equation into differential form. Thus, it is not necessary that we actually find the inverse operator. Its introduction makes the following derivations much easier than would otherwise be possible.

The solution of (3.11i) for v_z can thus be written,

$$v_z = L^{-1}\{\ell_o^2 k_o e_z + j\beta \frac{\ell_o^2 k_o}{h_o} \phi - h_o \ell_\nu \phi W_z\} \tag{3.13}$$

To derive the equations for the potential equations (3.11) are combined in such a way that all quantities except e_z , h_z and ϕ are eliminated from a particular equation. First, consider the equation for e_z .

Operate on (3.11b) with $\hat{a}_z x$ and then with $\nabla_t \cdot$ to obtain

$$\nabla_t^2 e_z + j\beta\nabla_t \cdot \vec{e} = jk_o \hat{a}_z \cdot \nabla_t x\, \vec{h} \tag{3.14}$$

Eliminating $\nabla_t \cdot \vec{e}$ and $\nabla_t x\, \vec{h}$ gives

$$\{\nabla_t^2 + k_o^2 - \beta^2\} e_z = -j\beta h_o \ell_o^2 \phi + jh_o k_o v_z + jk_o h_o W_z \phi \tag{3.15}$$

Substituting from (3.13) for v_z and operating with L gives the final equation

$$L\{\nabla_t^2 + k_o^2 - \beta^2\}e_z - k_o^2 h_o \ell_o^2 e_z = L\{-j\beta h_o \ell_o^2 \phi + jk_o h_o W_z \phi\}$$
$$-\ell_o^2 k_o^2 \beta\phi - j\ell_o^2 k_o h_o \ell_\nu \phi W_z \qquad (3.16)$$

Similarly the equation for h_z is derived by operating on (3.11d) with $\hat{a}_z x$ and $\nabla_t \cdot$ and then eliminating $\nabla_t \cdot \vec{h}$ and $\nabla_t x\ \vec{e}$ to obtain

$$\{\nabla_t^2 + k_o^2 - \beta^2\}h_z = -h_o \hat{a}_z \cdot \nabla_t x\ \vec{v}_t + h_o(\hat{a}_z x\ \vec{W}_t)\cdot\nabla_t\phi \qquad (3.17)$$

Now operate on (3.11h) with $\nabla_t x$, use (3.11g) to eliminate $\nabla_t \cdot \vec{v}$ and (3.11a) to eliminate $\nabla_t x\ \vec{e}$ to obtain,

$$L(\nabla_t x\ \vec{v}_t) = jh_o \ell_B \ell_o^2 k_o \phi - j\ell_o^2 k_o^2 h_z - j\beta h_o \ell_B W_z \phi$$
$$+ h_o \ell_\nu \vec{W}_t x\ \nabla_t \phi - j\beta h_o \ell_B v_z \qquad (3.18)$$

Using (3.18) to eliminate $\hat{a}_z \cdot \nabla_t x\ \vec{v}$ from (3.17) and substituting v_z from (3.13) gives

$$\{\nabla_t^2 + k_o^2 - \beta^2\}h_z - j\ell_o^2 k_o^2 h_o L^{-1} h_z = -h_o L^{-1}\{-j\beta h_o \ell_B L^{-1}[\ell_o^2 k_o e_z$$
$$+j\beta \frac{\ell_o^2\ k_o}{h_o}\phi - h_o \ell_\nu \phi W_z] + jh_o \ell_B \ell_o^2 k_o \phi - j\beta h_o \ell_B W_z \phi$$
$$+ h_o \ell_\nu \vec{W}_t x\ \nabla_t \phi\} + h_o(\hat{a}_z x\ \vec{W}_t)\cdot\nabla_t \phi \qquad (3.19)$$

This equation can be converted to a purely differential equation by operating with L^2 .

The equation for ϕ is the most tedious to derive and results from combining (3.11b) and $\hat{a}_z x$ (3.11h) and eliminating terms. The result is,

$$L^2\{\frac{\ell_o^2 k_o}{h_o}[(\nabla_t^2-\beta^2-\ell_o^2h_o^2)\phi - jL(h_o\phi)]-j\beta h_o\ell_\nu\phi W_z-h_o\ell_B(W_t x\hat{a}_z)\cdot\nabla_t\phi$$

$$-h_o\ell_\nu\vec{W}_t\cdot\nabla_t\phi + L[j\beta W_z\phi-W_t\cdot\nabla_t\phi]\}$$

$$+ L\{-j\ell_o^2k_oh_o^2\ell_B^2\phi + j\beta h_o^2\ell_B^2W_z\phi + j\ell_o^2h_o\ell_Bk_o^2h_z - h_o^2\ell_\nu\ell_B(\hat{a}_zxW_t)\cdot\nabla_t\phi\}$$

$$+ j\beta h_o^2\ell_B^2\ell_o^2k_oe_z-\beta^2(\frac{\ell_o^2k}{h_o})h_o^2\ell_B^2\phi - j\beta h_o^3\ell_B^2\ell_\nu\phi W_z = 0 \qquad (3.20)$$

To continue further it would now be necessary to find the transverse fields in terms of the potentials. This, in fact, can be done and the resulting equations are a set of differential equations for the transverse field quantities having a linear combination of $\nabla_t\phi$, ∇_te_z , ∇_th_z and ϕ as sources. The main interest here was to demonstrate that a coupled set of potential equations could be derived. We shall work only with the simpler drift-free equations. The potential equations and transverse field equations for this case are now presented.

3.4 The Equations for a Stationary, Uniform, Warm Plasma

Now assume that $\vec{W}_o = 0$. In this case the operator L becomes symply

$$L = jh_o(1-j\ell_\nu) \tag{3.21}$$

The potential equations simplify greatly in this case and reduce to

$$\{\nabla_t^2 + k_o^2(1 - \frac{\ell_o^2}{K_\nu}) - \beta^2\}e_z = -j\beta\ell_o^2 h_o[1 - (u/c)^2]\phi \quad \text{(a)}$$

$$\{\nabla_t^2 + k_o^2(1 - \frac{\ell_o^2}{K_\nu}) - \beta^2\}h_z = -j\,\frac{\ell_o^2\ell_B k_o\beta}{K_\nu^2}\,e_z - \frac{\ell_o^2\ell_B h_o k_o}{K_\nu}[1 - \frac{\beta^2/h_o^2}{K_\nu}]\phi \quad \text{(b)}$$

$$\{\nabla_t^2 + h_o^2 K_H - K_B\beta^2\}\phi = j\,\frac{\ell_B^2 h_o\beta}{K_\nu^2}\,e_z - \frac{\ell_B h_o k_o}{K_\nu}\,h_z \quad \text{(c)}$$

$$(3.22)$$

Equations (3.22) will be used extensively in later work.

The equations for the transverse field (with $W_o = 0$) can be derived by straightforward, but rather tedious,manipulation of (3.11) . Sancer has indicated the necessary procedure in some detail and the derivation will be omitted here. The results are

$$\{k_p^4 - \frac{\ell_B^4 k_c^4}{K_\nu^2}\}\vec{v} = -\frac{\ell_o^2 k_o\beta k_p^2}{K_\nu\,h_o}\nabla_t e_z - \frac{j\ell_o^2\ell_B k_o\beta k_c^2}{h_o\,K_\nu}\hat{a}_z \times \nabla_t e_z$$

$$- \frac{j\ell_o^2\ell_B k_o^2 k_c^2}{h_o K_\nu^2} \nabla_t h_z + \frac{\ell_o^2 k_o^2 k_p^2}{K_\nu h_o} \hat{a}_z x\ \nabla_t h_z + \frac{j\ell_o^2 k_o k_c^2 k_p^2}{K_\nu h_o^2} \nabla_t \phi$$

$$- \frac{\ell_o^2 \ell_B k_o k_c^4}{K_\nu^2 h_o^2} \hat{a}_z x\ \nabla_t \phi \tag{3.23a}$$

$$\{k_p^4 - \frac{\ell_B^2 k_c^4}{K_\nu^2}\}\vec{h} = - \frac{\ell_o^2 \ell_B k_o \beta^2}{K_\nu^2} \nabla_t e_z - jk_o\{k_p^2 K_p - \frac{\ell_B^2 k_c^2}{K_\nu^2}\}\hat{a}_z x\ \nabla_t e_z$$

$$-j\beta\{k_p^2 - \frac{\ell_B^2 k_c^2}{K_\nu^2}\}\nabla_t h_z + \frac{\ell_o^2 \ell_B k_o^2 \beta}{K_\nu^2} \hat{a}_z x\ \nabla_t h_z$$

$$+j\,\frac{\ell_B \ell_o^2 k_o \beta k_c^2}{K_\nu^2\, h_o} \nabla_t \phi - \frac{\ell_o^2 k_o \beta k_p^2}{K_\nu h_o} \hat{a}_z x\ \nabla_t \phi \tag{3.23b}$$

The transverse electric field is most simply writen

$$\vec{e} = \frac{j\nabla_t e_z}{\beta} - \frac{k_o}{\beta}\,\hat{a}_z x\ \vec{h} \tag{3.24}$$

The axial velocity is simply;

$$v_z = -\,j\,\frac{\ell_o^2 k_o}{h_o K_\nu}\,e_z + \frac{\ell_o^2 k_o \beta}{h_o^2\, K_\nu}\,\phi \tag{3.25}$$

Equations (3.22) - (3.25) completely determine all the fields. It is noted that we cannot yet proceed to solve the problem since boundary conditions on (3.22) have not been specified. These will be considered in Chapter 5 and solutions are considered in Chapters 6 - 8 .

CHAPTER IV

POTENTIAL EQUATIONS FOR A NON-UNIFORM STATIONARY WARM PLASMA

4.0 Introduction

In this chapter we shall consider one other case where the field quantities can be expressed in terms of a set of coupled potentials. Again the equations are presented for reference and to demonstrate the applicability of the technique. No solutions are attempted in this work.

It is known that laboratory plasmas are generally not uniform and in a typical discharge the number density will vary across the discharge tube. Also, it is possible to create non-uniform carrier densities in other plasma-like devices such as doped semiconductors. We shall here restrict the analysis to cases where the number density varies in the transverse plane only.

4.1 The Normalized, Linearized Equations

Since now the background number density of ions and electrons is allowed to be a function of the transverse coordinates it is necessary to retain terms arising from differentiation of the number density. The basic equations can be simplified somewhat by defining a new set of normalized variables as follows.

$$\vec{\mathcal{H}} = \sqrt{\frac{\varepsilon_o}{\mu_o}}\,\vec{H} \qquad \text{(a)}$$

$$N_o\vec{v} = \frac{\omega}{uq}\sqrt{\frac{\varepsilon_o}{\mu_o}}\,\vec{w} \qquad \text{(b)}$$

$$n = \frac{q}{m\omega u}\,\phi \qquad \text{(c)} \qquad (4.1)$$

$$p = \frac{uq}{\omega}\,\phi \qquad \text{(d)}$$

$$\ell^2 \equiv \frac{(q^2/m\varepsilon_o)}{\omega^2} = \frac{\ell_o^2}{N_o} \qquad \text{(e)}$$

These variables are similar to those obtained in (3.5) but do not now contain the background number density. In particular note that we have defined one variable, $\vec{w}$, to be proportional to $N_o\vec{v}$. By doing this we eliminate a number of terms which arise from operating on $N_o\vec{v}$ with ∇ . N_o is assumed to be a known function of the transverse coordinates.

Using the normalization presented above and separating the field equations into transverse and axial components gives

$$\nabla_t \times \vec{e} = -jk_o\vec{h}_z \qquad \text{(a)}$$

$$\hat{a}_z \times \nabla_t e_z + j\beta\hat{a}_z \times \vec{e} = jk_o\vec{h} \qquad \text{(b)} \qquad (4.2)$$

$$\nabla_t \times \vec{h} = jk_o\vec{e}_z + h_o\vec{w}_z \qquad \text{(c)}$$

$$\hat{a}_z \times \nabla_t h_z + j\beta \hat{a}_z \times \vec{h} = -jk_o \vec{e} - h_o \vec{w}_t \qquad \text{(d)}$$

$$\nabla_t \cdot \vec{e} = j\beta e_z + h_o \ell^2 \phi \qquad \text{(e)}$$

$$\nabla_t \cdot \vec{h} = j\beta h_z \qquad \text{(f)}$$

$$jk_o \ell^2 \phi + \nabla_t \cdot \vec{w}_t - j\beta w_z = 0 \qquad \text{(g)}$$

$$jh_o K_\nu \vec{w}_t + h_o \ell_B \hat{a}_z \times \vec{w}_t = \ell_o^2 k_o \vec{e} - \frac{\ell^2 k_o}{h_o} \nabla_t \phi \qquad \text{(h)}$$

$$jh_o K_\nu w_z = \ell_o^2 k_o e_z + j\beta \frac{\ell^2 k_o}{h_o} \phi \qquad \text{(i)}$$

Note that the only spatially varying parameter in (4.2) is ℓ_o^2 which appears only in (4.2h) and (4.2i) .

4.2 Equations for the Potentials

The equations for the potentials will now be written. The derivation of these is exactly like the derivations in Chapter 3. However in this case it must be remembered that ℓ_o^2 is a function of position and thus gives a contribution when operated on by a differential operator. The resulting equations are

$$\{\nabla_t^2 + k_o^2(1 - \frac{\ell_o^2}{(1-j\ell_\nu)}) - \beta^2\} e_z = -jh_o \ell^2 \beta [1 - \frac{k_o^2/h_o^2}{(1-j\ell_\nu)}]\phi \qquad \text{(a)}$$

$$\{\nabla_t^2 + k_o^2(1 - \frac{\ell_o^2}{(1-j\ell_\nu)}) - \beta^2\}h_z = -j\,\frac{\ell_o^2\ell_B k_o\beta}{(1-j\ell_\nu)^2} - \frac{\ell^2\ell_B k_o}{h_o(1-j\ell_\nu)}\ \cdot$$

$$[h_o - \frac{\beta^2}{(1-j\ell_\nu)}]\phi + \frac{jk_o}{(1-j\ell_\nu)}[\hat{a}_z\cdot(\nabla_t\ell_o^2)\times\vec{e}] \qquad \text{(b)}$$

$$\{\nabla_t^2 + h_o^2[1-j\ell_\nu - \frac{\ell_B^2}{(1-j\ell_\nu)^2} - \ell_o^2] - \beta^2K_B\}\phi = j\,\frac{N_oh_o\beta}{(1-j\ell_\nu)^2}\,e_z$$

$$-\frac{\ell_Bh_oN_ok_o}{(1-j\ell_\nu)}\,h_z + h_o\nabla_tN_o\cdot\vec{e} - j\,\frac{\ell_Bh_o}{(1-j\ell_\nu)}[\hat{a}_z\cdot\nabla_tN_o\times\vec{e}] \qquad \text{(c)}$$

$$(4.3)$$

Equation (4.3a) is identical to (3.22a) if the same normalization for ϕ is used and ℓ_o^2 in (3.22a) is regarded as a function of position.

The other equations are changed by the inclusion of terms proportional to the gradient of the background number density. Note that the last two equations contain terms in $\vec{e}$, the transverse electric field, and thus are not yet closed equations. To complete the derivation for the coupled equations it is necessary to show that $\vec{e}$ can be expressed completely in terms of the potential.

4.3 The Solution for the Transverse Electric Field

It is clear that the transverse fields can be expressed in terms of the potentials. This can be seen by writing the equations

$$j\beta\hat{a}_z x\, \vec{e} - jk_o\vec{h} = -\hat{a}_z x\nabla_t e_z \qquad (a)$$

$$jk_o\, \vec{e} + j\beta\hat{a}_z x\, \vec{h} + h_o\vec{w}_t = -\hat{a}_z x\nabla_t h_z \qquad (b) \qquad (4.4)$$

$$-\ell_o^2 k_o\vec{e} + jh_o K_\nu \vec{w}_t + h_o \ell_B \hat{a}_j x\, \vec{w}_t = -\ell^2 \frac{k_o}{h_o} \nabla_t\phi \qquad (c)$$

Equation (4.4) could be written as a set of 6 equations for the components of the transverse fields and solutions can be obtained by determinants. Only the expression for $\vec{e}$ is considered here.

The solution for $\vec{e}$ is

$$k_t^4\, \vec{e} = -j\beta[k_p^2 - \frac{\ell_B^2 k_c^2}{K_\nu^2}]\nabla_t e_z + \frac{\ell_o^2 k_o^2 \ell_B \beta}{K_\nu^2}\, \hat{a}_z x\nabla_t e_z + \frac{\ell_o^2 k_o^3 \ell_B}{K_\nu^2}\nabla_t h_z$$

$$+ jk_o[k_p^2 - \frac{\ell_B^2 k_c^2}{K_\nu^2}]\hat{a}_z x\nabla_t h_z - \frac{\ell^2 k_o^2 k_p^2}{h_o K_\nu}\nabla_t\phi$$

$$- j\,\frac{\ell^2 k_o^2 \ell_B k_c^2}{h_o\, K_\nu}\, \hat{a}_z x\nabla_t\phi \qquad (4.5)$$

Here we have defined the quantities

$$k_p^2 = k_o^2\, K_p - \beta^2 \qquad (a)$$

$$k_c^2 = k_o^2 - \beta^2 \qquad (b) \qquad (4.6)$$

$$k_t^4 = k_p^4 - \frac{\ell_B^2\, k_c^4}{K_\nu^2} \qquad (c)$$

It is obvious that substitution of (4.5) into (4.3) results in a messy, but deterministic set of equations for the potentials. We will thus not do this. Some general conclusions may be made by examining (4.5).

First note that in any problem $\nabla_t N_o$ is a known (from measurements or solution of the d.c. equation) fixed vector. The inclusion of terms from $\vec{e}$ change the equations in several ways. First, the potential equations will no longer be Helmholtz equations since terms in ∇_t will be included. Also the terms in $\hat{a}_z x \nabla_t h_z$ and $\hat{a}_z x \nabla_t \phi$ effectively eliminates the possibility of solutions which vary in only one transverse direction. Note that, when the external magnetic field is absent the coupling between transverse components arising from $\hat{a}_z x \nabla_t h_z$, etc. vanishes.

Of course the extent to which the variation in number density changes the uniform results depends on the magnitude of the variations. In some cases it may be possible to use the uniform solutions to derive corrections to the non-uniform problem by employing a perturbation technique.

In the rest of this work we shall consider the plasma to be stationary and uniform. Although the equations for this case are relatively simple compared to the equations just derived, they are still quite involved and should provide some insight into the more complicated problems.

CHAPTER V

BOUNDARY CONDITIONS, CUTOFFS AND RESONANCES

5.0 Introduction

In order to obtain solutions for the potential equations it is necessary to derive a set of boundary conditions for e_z, h_z and ϕ. These boundary conditions are obtained from the expressions on the transverse fields and are derived in Section 5.1.

It was stated in Chapter 1 that solutions to the warm plasma equations would be obtained by approximate methods. This is necessary (or at least desirable) since an exact solution of the coupled equations is very difficult and also because it is possible to make some very excellent approximations in obtaining simpler approximate solutions. However, some information can easily be obtained about the approximate positions of resonances and cutoffs by considering the coupled collisionless equations in a formal manner. The location of these points will be found in Section 5.2.

5.1 Boundary Conditions for the Warm Plasma Model

The appropriate boundary conditions for use with the warm plasma model has recently been the subject of some debate. Sancer has studied mathematically acceptable boundary conditions by considering those conditions for which the warm plasma equations have unique solutions[14]. From this analysis it is found that the conditions $E_{tangential} = V_{normal} = 0$ at a perfectly conducting rigid wall are appropriate boundary conditions for the problem. They are also reasonable physical boundary conditions and will be employed here.

Wait[21] has discussed an alternate boundary condition to describe the so called "sheath collapse" condition which can be applied if it is assumed that all electrons striking the conducting surface are absorbed.

The effect of the dielectric insulating container on the behavior of propagation in an isotropic cold plasma has been examined by Trivelpiece[3] and by Clarricoats, et al[8].

In this work we use the conditions $\vec{E}_T = \vec{V}_n = 0$. To facilitate the solutions for the potentials it is desirable to cast these equations into equations on e_z, h_z and ϕ.

5.2 Boundary Conditions for the Potentials

The transverse electric field is given by (3.24).

$$\vec{e} = \frac{j\nabla_t e_z}{\beta} - \frac{k_o}{\beta}\,\hat{a}_z \times \vec{h} \tag{5.1}$$

Since we require that the tangential components of the electric field vanish at the conducting surface the axial electric field must vanish.

$$e_z\big|_c = 0 \tag{5.2}$$

The tangential component of (5.1) must also vanish at the boundary. But this implies that the normal component of $\vec{h}$ must be zero since (5.2) forces the tangential component of $\nabla_t e_z$ to be zero. The remaining boundary conditions are thus

$$\vec{h}_n = \vec{V}_n = 0 \qquad \text{at the walls} \tag{5.3}$$

(5.2) is a boundary condition for one of the potentials. We now seek conditions which can be applied to h_z and ϕ. To find these boundary conditions let us now require that (3.23) satisfy (5.3) at the boundary. The result is

$$\vec{v} \cdot \hat{n}\big|_c = 0 = -\beta k_p^{\,2} \frac{\partial e_z}{\partial n} - j\ell_B k_o k_c^{\,2} \frac{\partial h_z}{\partial n} - k_o^{\,2} k_p^{\,2} \frac{\partial h_z}{\partial \tau}$$

$$+ \frac{j k_c^{\,2} k_p^{\,2}}{h_o} \frac{\partial \phi}{\partial n} + \frac{\ell_B k_c^{\,4}}{K_\nu h_o} \frac{\partial \phi}{\partial \tau} \tag{5.4a}$$

$$\vec{h} \cdot \hat{n}\Big|_c = 0 = \frac{-\ell_o^{\,2}\ell_B k_o \beta}{K_\nu^{\,2}} \frac{\partial e_z}{\partial n} - j\left\{k_p^{\,2} - \frac{\ell_B^{\,2} k_c^{\,2}}{K_\nu^{\,2}}\right\}\frac{\partial h_z}{\partial n}$$

$$- \frac{\ell_o^{\,2}\ell_B k_o^{\,2}}{K_\nu^{\,2}} \frac{\partial h_z}{\partial \tau} + \frac{j\ell_B \ell_o^{\,2} k_o k_c^{\,2}}{K_\nu^{\,2}\, h_o} \frac{\partial \phi}{\partial n} + \frac{\ell_o^{\,2} k_o k_p^{\,2}}{K_\nu h_o} \frac{d\phi}{d\tau} \tag{5.4b}$$

In the above equations we have used the fact that $e_z\big|_c = 0$. The components of $\nabla_t e_z$, etc. have been written as normal and tangential derivatives where $\hat{n}$, $\hat{\tau}$ and $\hat{a}_z$ form an orthogonal coordinate system illustrated in Fig. 5.2.

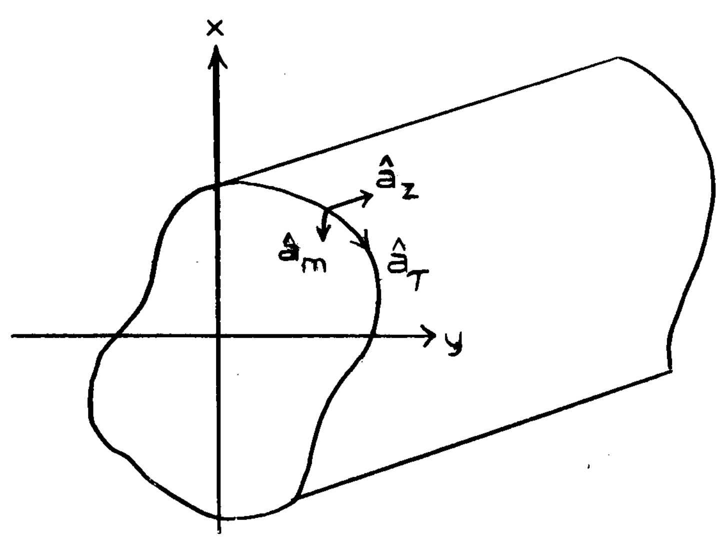

Fig. 5.1 Coordinate System for the Boundary Conditions

The rather elaborate boundary conditions, (5.4), cannot be simplified if all components of the fields are assumed to be present. However, if we assume that the field components have no tangential variations a greatly simplified set of equations can be derived. Assume that the terms involving $\frac{\partial}{\partial\tau}$ are zero, multiply (5.4a) by $\frac{-\ell_o^2 \ell_B k_o}{K_\nu^2}$, (5.4b) by k_p^2 and add. The result is

$$j\left\{\frac{\ell_o^2 \ell_B^2 k_o^2 k_c^2}{K_\nu^2} - k_p^4 + \frac{\ell_B^2 k_c^2 k_p^2}{K_\nu^2}\right\}\frac{\partial h_z}{\partial n} = 0 \tag{5.5}$$

Since the multiplying term in (5.5) is generally not zero the normal derivative of h_z must vanish at the boundary.

$$\left.\frac{\partial h_z}{\partial n}\right|_c = 0 \tag{5.6}$$

Substituting (5.6) into either (5.4a or b) gives the boundary condition on ϕ

$$\left.\frac{d\phi}{dn}\right|_c = \frac{-jh_o\beta}{k_c^2}\left.\frac{\partial e_z}{\partial n}\right|_c \tag{5.7}$$

5.3 Determination of the Cut-off and Resonance Frequencies

Now that the boundary conditions for the potentials have been determined we can proceed to consider solutions for the problem. In all of our work we shall consider only modes where the restriction that the modes have no tangential variation can hold. This excludes the rectangular waveguide and in practice restricts the analysis to the circular cylindrical and parallel plate guides. Since the circular guide is of greater practical interest we consider only this case.

Now we shall proceed in a formal manner to diagonalize the coupled Helmholtz equations. This method generally is too difficult to yield solutions for the equations, but it does yield the resonant and cutoff frequencies. Two difficulties arise when attempting to solve the equations. First, the eigenvalue equation is a cubic equation and the roots are very difficult to find (Chen and Cheng, ref. 22, have stated that the solution for the roots is "straightforward" for the collisionless case). Second, even if we can find the eigenvalues in terms of β is still necessary to solve the transcendental boundary equation for β.

The equations for the potentials are given by (3.22) and can be written

$$\nabla_t^{\,2}\begin{pmatrix} e_z \\ h_z \\ \phi \end{pmatrix} + \begin{bmatrix} a_{11} & 0 & -a_{13} \\ -a_{21} & a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{bmatrix}\begin{pmatrix} e_z \\ h_z \\ \phi \end{pmatrix} = [0] \qquad (5.8)$$

Here we have defined

$$a_{11} = k_o^2K_p - \beta^2 \qquad \text{(a)}$$

$$a_{13} = -\,j\ell_o^2\beta h_o \qquad \text{(b)}$$

$$a_{21} = \frac{-j\ell_o^2\ell_B k_o\beta}{K_\nu^2} \qquad \text{(c)}$$

$$a_{23} = \frac{-\ell_o^2\ell_B h_o k_o}{K_\nu}\left[1 - \frac{\beta^2/h_o^2}{K_\nu}\right] \qquad \text{(d)} \qquad (5.9)$$

$$a_{31} = \frac{j\ell_B^2 h_o\beta}{K_\nu^2} \qquad \text{(e)}$$

$$a_{32} = \frac{-\ell_B h_o k_o}{K_\nu} \qquad \text{(f)}$$

$$a_{33} = h_o^2K_H - K_B\beta^2 \qquad \text{(g)}$$

Define the matrix A by

$$A = [a_{ij}] \qquad (5.10)$$

Equation (5.8) can now be written

$$\{\nabla_t^2 + A\}\begin{pmatrix} e_z \\ h_z \\ \phi \end{pmatrix} = 0 \qquad (5.11)$$

To diagonalize (5.11) we must solve the characteristic equation*

$$|A - \lambda^2 I| = 0 \qquad (5.12)$$

Expanding (5.12) gives a cubic equation for λ^2.

$$\lambda^6 - \lambda^4(2a_{11} + a_{33}) + \lambda^2(a_{11}^2 + 2a_{11}a_{33} - a_{23}a_{32} - a_{13}a_{31})$$

$$- (a_{11}^2 a_{33} - a_{11}a_{23}a_{32} - a_{11}a_{13}a_{31} + a_{13}a_{21}a_{32}) = 0 \qquad (5.13)$$

Assuming that the roots of (5.13), λ_i^2, have been found we can proceed to find a transformation matrix M which diagonalizes A (the procedure used here is exactly like the one used in Chapter 2). M can be written

$$M = \begin{bmatrix} 1 & m_{12} & m_{13} \\ m_{21} & 1 & m_{23} \\ m_{31} & m_{32} & 1 \end{bmatrix} \qquad (5.14)$$

---

* Note that the matrix A is not a Hermitian matrix. Generally a non-Hermitian matrix can be expressed in a Jordan canonical form. This form is diagonal if the eigenvalues, λ, are non-degenerate. A Thorough discussion of diagonalizing non-Hermitian matrices is given by Friedman, ref. 23.

where
$$m_{12} = \frac{-a_{13}(a_{11}-\lambda_2{}^2)}{a_{21}a_{13}-a_{23}(a_{11}-\lambda_2{}^2)} \qquad \text{(a)}$$

$$m_{13} = \frac{-a_{13}}{a_{11}-\lambda_3{}^2} \qquad \text{(b)}$$

$$m_{21} = \left\{\frac{(a_{11}-\lambda_1{}^2)(a_{33}-\lambda_1{}^2)}{a_{32}a_{13}} - \frac{a_{31}}{a_{32}}\right\} \qquad \text{(c)}$$

$$m_{23} = \left\{\frac{a_{13}a_{31}}{a_{32}(a_{11}-\lambda_3{}^2)} - \frac{(a_{33}-\lambda_3{}^2)}{a_{32}}\right. \qquad \text{(d)} \quad (5.15)$$

$$m_{31} = \frac{-(a_{11}-\lambda_1{}^2)}{a_{13}} \qquad \text{(e)}$$

$$m_{32} = \frac{(a_{11}-\lambda_2{}^2)^2}{a_{21}a_{13}-a_{32}(a_{11}-\lambda_2{}^2)} \qquad \text{(f)}$$

Note that the particular form for the coefficients m_{ij} has been chosen after examining the limiting values that λ_i and m_{ij} must take in the limits $\omega_B \to 0$ or $\beta \to 0$. They have been chosen to avoid dividing by zero in any situation. This is particularly important if an attempt is made to use the above technique in a numerical problem. Computers are notoriously inaccurate when they must compute ratios of very different numbers.

Now cast (5.13) into an explicit equation for β. This is done by substituting from (5.9) for a_{ij} and regrouping in powers of β. The result is

$$K_B\beta^6 + \beta^4\{2K_BC_{11}+C_{33}+D_{32}-D_{13}\}$$

$$+ \beta^2\{2C_{11}C_{33}-D_{23}+C_{11}(K_BC_{11}+D_{32}-D_{13}) + D_3\}$$

$$+ C_{11}(C_{11}C_{33}-D_{23}) = 0 \qquad (5.16)$$

The coefficients in (5.16) are

$$C_{11} = -k_o^2K_p - \lambda^2 \qquad \text{(a)}$$

$$C_{33} = -h_o^2K_H - \lambda^2 \qquad \text{(b)}$$

$$D_{23} = \ell_o^2\ell_B^2k_o^2h_o^2 \qquad \text{(c)} \qquad (5.17)$$

$$D_{32} = \ell_o^2\ell_B^2k_o^2 \qquad \text{(d)}$$

$$D_{13} = \ell_o^2\ell_B^2h_o^2 \qquad \text{(e)}$$

$$D_3 = \ell_o^4\ell_B^2h_o^2k_o^2 \qquad \text{(f)}$$

Note that, for a given value of λ^2, three values of β^2 can be found. Equation (5.13) similarly expresses three values of λ^2 for any value of β^2. The subscript, i, has been dropped in (5.17) for this reason. Of course equations (5.13) and (5.17) are just two ways of writing the same equation and cannot be solved until boundary conditions are specified.

Thus far we have assumed that collisions were present. To compute the cut-off and resonant frequencies we now let $\nu = 0$. This must be done if true zeros and infinities of (5.17) are to occur. In our numerical work we expect that β will be small at cutoff and large at resonance.

To compute the resonance, divide (5.17) by β^6, define a new variable $\xi = 1/\beta$ and let $\xi = 0$. We obtain simply

$$K_B = 0 \qquad \text{at resonance} \tag{5.18}$$

$$\therefore \quad \omega_{\text{resonance}} = \omega_B \tag{5.19}$$

To obtain the cut-off frequencies set $\beta = 0$ in (5.17). We thus obtain

$$C_{11}(C_{11}C_{33}-D_{23}) = 0 \tag{5.20}$$

One solution to (5.20) gives a cut-off frequency identical to that obtained for a waveguided filled with a plasma dielectric. i.e.,

$$C_{11} = k_o^2 K_p + \lambda^2 = 0 \tag{5.21}$$

Solving (5.21) for the cut-off frequency gives

$$\omega_{CE} = \sqrt{\omega_o^2 + \lambda^2 c^2} \tag{5.22}$$

It will later be seen that when $\beta = 0$ the eigenvalue, λ, can easily be found since the transcendental boundary equation which determines λ decouples.

The other cut-off frequencies satisfy a more complicated equation derived from the second root of (5.20).

$$C_{11}C_{33} - D_{23} = 0 \tag{5.23a}$$

Expanding (5.23a) by using the definitions of (5.17) gives, for the cut-off,

$$\omega_C^4 - \omega_C^2\{2\omega_o^2 + \omega_B^2 + \lambda^2(u^2 + C^2)\} + \omega_o^4$$

$$+ \lambda^2\{\omega_o^2 u^2 + (\omega_o^2 + \omega_B^2)C^2\} + \lambda^2 u^2 C^2 = 0 \tag{5.23b}$$

$$\omega_C{}^2 = \frac{\{2\omega_o{}^2+\omega_B{}^2+\lambda^2(u^2+c^2)\}}{2} \tag{5.24}$$

$$\frac{\pm\{[\omega_B{}^2+\lambda^2(u^2-c^2)] + 4\ \omega_o{}^2\omega_B{}^2\}^{1/2}}{2}$$

To proceed we must determine λ. But now this is quite easily done since several of the coefficients in M vanish when $\beta = 0$. In this case M becomes

$$\underset{\beta=0}{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & m_{23} \\ 0 & m_{32} & 1 \end{bmatrix} \tag{5.25}$$

The diagonalized equations are, by definition, of the form

$$(\nabla_t{}^2+\lambda_i{}^2)u_i = 0 \tag{5.26a}$$

and solutions can be written in the form

$$u_i = A_i f_i \tag{5.26b}$$

where A_i are constants.

In terms of u_i, the original fields are given by

$$\begin{pmatrix} e_z \\ h_z \\ \phi \end{pmatrix} = M \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{5.27a}$$

Using (5.25) and (5.26b), the fields with $\beta \to 0$ are

$$e_z = A_1 f_1 \qquad \text{(a)}$$

$$h_z = A_2 f_2 + m_{23} A_3 f_3 \qquad \text{(b)} \qquad (5.27b)$$

$$\phi = A_3 f_3 + m_{32} A_2 f_2 \qquad \text{(c)}$$

Applying the boundary conditions (with = 0) gives

$$\begin{bmatrix} f_1 & 0 & 0 \\ 0 & \frac{df_2}{dn} m_{13} & \frac{df_3}{dn} \\ 0 & m_{23} \frac{df_2}{dn} & \frac{df_3}{dn} \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = [0] \qquad (5.28)$$

Setting the determinant of coefficients to zero and noting that the common multiplying term, $(1 - m_{13} m_{23})$ is generally not zero, gives

$$f_1 \Big|_c = 0 \qquad \text{(a)}$$

$$(5.29)$$

$$\frac{df_2}{dn} \; \frac{df_3}{dn} \Big|_c = 0 \qquad \text{(b)}$$

Re-examining the original equations shows that the eigenvalue determined from (5.29a) must be used in the computation of ω_{CE} in (5.22) and the eigenvalues determined from (5.29b) must be used in (5.24).

Note that the cut-off frequency, given by (5.22), is independent of the magnetic field while the other two cut-offs, as expressed by (5.24), depend on ω_B. Note also, from the dispersion equations presented in Chapter 2, that this is also the case for the cold plasma equations. The H-mode cut-off frequencies are found to increase with increasing ω_B while the E-mode cut-offs remain fixed. This behavior will be evident later when the dispersion characteristic of the modes are found.

CHAPTER VI

SIMPLIFICATION OF THE POTENTIAL EQUATIONS BY THE COUPLED MODE THEORY

6.0 Introduction

At the beginning of the work on the warm plasma filled waveguide an attempt was made to obtain solutions by employing the formal diagonalization procedure. Several difficulties were encountered and the approach was abandoned. First, it was found that explicit solutions for the cubic eigenvalue equation, (5.13), could be found only in the uninteresting cases where $\omega \gg \omega_o$ or ω_B. Thus, solutions to the eigenvalue equation had to be sought numerically. Also, a very complicated transcendental equation arises when boundary conditions are applied and this transcendental equation must be solved simultaneously with the eigenvalue equations. One of the eigenvalues is usually quite large, on the order of ω/u, and when this quantity occurs in the argument of one of the Bessel functions in the boundary equation extremely oscillatory behavior occurs and numerical solutions are very difficult to obtain. Finally, it was felt that involved numerical solutions would be of limited value since a complete analysis has to be made for every change of parameters and no simple approximate solutions are evident. Also, the physics of the wave interaction is obscured by the complicated algebra.

In the following development we shall use the fact that one of the eigenvalues is very much larger than the others to simplify the equations for the potentials. Physically this means that the potential fields consist of components which vary slowly with position and arise from the smaller eigenvalues plus a rapidly varying term arising from the large eigenvalue and due to the pressure. To clarify these ideas and those to follow consider equations (5.8) where the off-diagonal terms are written as forcing terms.*

$$
\begin{aligned}
&\{\nabla_t^2 + a_{11}^2\}\; e_z = -f_e && \text{(a)} \\
&\{\nabla_t^2 + a_{11}^2\}\; h_z = -f_h && \text{(b)} \qquad (6.0) \\
&\{\nabla_t^2 + a_{33}^2\}\; \phi = -f_\phi && \text{(c)}
\end{aligned}
$$

If the coupling vanishes the right-hand side terms of (6.0) are zero. To obtain a qualitative idea of the behavior when coupling is present assume that the coupling changes the dispersion relation very little from the uncoupled values. (This is true when the

* The notation of chapter has been changed by replacing the diagonal term a_{ii} by a^2_{ii} to indicate the square of an eigenvalue.

frequency is very high relative to the plasma frequency and cyclotron frequency but is not true in general.) If the terms on the right of (6.0) are neglected then the quantities a_{11}^2 must be the square of the eigenvalues of the differential equations. Consideration of (6.0a) with $f_e = 0$ will clarify this. Eq. (6.0a) becomes

$$[\nabla_t^2 + a_{11}^2]\, e_z = 0$$

The boundary condition, $e_z = 0$ at the boundary, determines the eigenvalues and the values of β. Let λ_n be the nth eigenvalue. Equating a_{11}^2 to λ_n^2 and using (5.9a) gives

$$\beta_n^2 = k_o^2\, K_p - \lambda_n^2$$

Now consider qualitatively the effect of the coupling. Assuming that coupling changes β very little from the above value we can compute an estimate of ϕ by substituting β into a_{33}^2 in (6.1c). a_{33}^2 now becomes

$$a_{33}^2 = h_o^2\, K_H - \beta_n^2\, K_B$$

Since h_o, (ω/u), is a very large quantity $\beta_n^2\, K_B$ has little effect on the value of a_{33}. Also, since a_{33}

determines the spatial variation of the ϕ mode it is seen that ϕ varies extremely rapidly compared to e_z.

Finally consider qualitatively the effect of the coupling of ϕ back to e_z. The equation for e_z is (6.0a). The exact effect of the rapidly varying ϕ mode coupling to the e_z equation depends on the magnitude of the coefficients. However, it will generally be expected that a very rapidly spatial varying quantity will have little effect on a mode that has much slower natural variations.

The above considerations suggest that some useful approximations may be found if we consider (6.0) to be inhomogeneous equations with the coupling terms acting as driving functions. The qualitative arguments can be expressed in a rigorous, quantitative manner by considering solutions of inhomogeneous equations in terms of Green's function solutions. A number of basic Green's function theorems are presented in appendix A. These are used in the next section to obtain a set of simpler equations for the electromagnetic waveguide modes.

6.1 Reduction of the Equations for the Electromagnetic-like Modes

Now consider the equations for the potentials which were presented in equation (5.8). For convenience they are written below.*

$$\{\nabla_t^{\,2} + a_{11}^{\,2}\}\ e_z = a_{13}\ \phi \qquad \text{(a)}$$

$$\{\nabla_t^{\,2} + a_{11}^{\,2}\}\ h_z = a_{21}e_z + a_{23}\ \phi \qquad \text{(b)} \qquad (6.1)$$

$$\{\nabla_t^{\,2} + a_{33}^{\,2}\}\ \phi = a_{31}e_z + a_{32}h_z \qquad \text{(c)}$$

The appropriate boundary conditions were derived in section 5.1 and are

$$e_z\big|_c = \frac{\partial h_z}{\partial n}\bigg|_c = 0 \qquad \text{(a)}$$

$$(5.4)$$

$$\frac{\partial h_z}{\partial n}\bigg|_c = a_4\ \frac{\partial e_z}{\partial n}\bigg|_c \qquad \text{(b)}$$

where we have defined

$$a_4 \equiv -j\ \frac{h_o\beta}{k_c^{\,2}} \qquad \text{(c)}$$

We now restrict the development to consider only solutions for which $|\beta| \ll h_o$. Later it will be seen that this is an excellent approximation, even near resonances, since collisions restrict β to a finite value, the maximum value of which is still much smaller

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* See footnote on Page 65.

than h_o. These solutions will be called the electromagnetic solutions. In this case it is noted from an examination of the coefficients in (6.1) that $a_{33}^2 \approx h_o{}^2 K_H = (\frac{\omega}{u})^2\ K_H$, generally a very large quantity. It will be seen that this fact allows equation (6.2c) to be simplified considerably. (The development that follows is related to the coupled-mode solutions for coupled equations [24]). To express the simplifications analytically it is necessary to consider the Green's function solution of (6.1c), considering the right-hand side as a driving term.

In general it is possible to express the fields in (6.1) in terms of complete sets of functions which satisfy the uncoupled Helmholz equations and appropriate boundary conditions. (The specific functions to be used later are tabulated in Appendix B.) Write these as;

$$e_z = \sum_n a_n e_n \qquad (a)$$

$$h_z = \sum_\ell b_\ell w_\ell \qquad (b) \qquad\qquad (6.2)$$

$$\phi = \sum_m c_m w_m \qquad (c)$$

$$e_{n|c} = \left.\frac{dw_K}{dn}\right|_c = 0 \qquad (d)$$

The solution of (6.2c) can now be found in terms of the Green's function for ϕ: From (A.8) we obtain

$$\phi = -a_{31}\int_{a_1}^{a_2} \sigma(x_o)G_\phi(x_o|x)e_z(x_o)dx_o$$

$$-a_{32}\int_{a_1}^{a_2} \sigma(x_o)G_\phi(x_o|x)h_z(x_o)dx_o \qquad (6.3)$$

$$+a_4 \left[p(x_o)G_\phi(x_o|x)\frac{d\,e_z}{d\,x_o}\right]_{a_1}^{a_2}$$

where $G_\phi(x_o|x)$ is the Green's function for ϕ. This Green's function has been chosen with the boundary condition $\left.\frac{dG_\phi}{dn}\right|_c = 0$ since ϕ satisfies a Neumann type of boundary condition. Assume now that e_z and h_z are expressed by uniformly convergent series as in (6.2) where the eigenfunctions satisfy equations of the Sturm-Liouville type with eigenvalues λ_n and γ_ℓ. In this case it is shown in Appendix A that the integrals in (6.3) can be evaluated and the result is;

$$\phi(x) = -a_{31}\sum_{n} \frac{a_n e_n}{\lambda_n^2 - a_{33}^2} - a_{32}\sum_{\ell} \frac{b_\ell w_\ell}{\gamma_\ell^2 - a_{33}^2}$$
$$+ a_{31}\sum_{n} \frac{a_n}{\lambda_n^2 - a_{33}^2}\left[p\, G_\phi \frac{de_z}{dx_o}\right]_{a_1}^{a_2} + a_4\left[p\, G_\phi \frac{de_z}{dx_o}\right]_{a_1}^{a_2} \tag{6.4}$$

An approximation will now be made which must later be checked and will allow (6.4) to be simplified greatly. This assumption is that the major contribution to e_z and h_z is from the first few Fourier coefficients. In particular, it is assumed that a_n and b_ℓ are very small when λ_n or γ_ℓ approach h_o. This will occur for large values of n and ℓ since, for n and ℓ small, λ and γ are on the order of k_o for waveguides and frequencies considered here. Note also that the denominator in (6.4) can never be zero since a_{33} is a complex quantity. If this assumption is true then the sums can be approximated by the first terms and a_{33}^2, since it contains h_o^2, is much larger than the eigenvalues. Equation (6.4) can then be written

$$\phi(x) = \frac{a_{31}}{a_{33}^2} e_z + \frac{a_{32}}{a_{33}^2} h_z + \left(a_4 - \frac{a_{31}}{a_{33}^2}\right)\left[p\, G_\phi \frac{de_z}{dx_o}\right]_{a_1}^{a_2} \tag{6.5}$$

To obtain (6.5) the terms λ_n^2 and γ_ℓ^2 have been dropped in the first two sums and (6.2) has been used to express the fields in terms of the sums. Note that the last term in (6.5) is a function determined by the evaluation of the Green's function at the boundary. From the results in Appendix A it is seen that the contribution from this term can be found by solving the equation

$$\left(\nabla_t^2 + a_{33}^2\right)\phi' = 0 \qquad (6.6a)$$

subject to the boundary condition

$$\left.\frac{d\phi'}{dn}\right|_c = \left[a_4 - \frac{a_{31}}{h_p^2}\right]\left.\frac{de_z}{dn}\right|_c \qquad (6.6b)$$

$\phi(x)$ can thus be written

$$\varphi(x) = \frac{a_{31}}{a_{33}^2}\, e_z + \frac{a_{32}}{a_{33}^2}\, h_z + \phi' \qquad (6.6c)$$

Now substitute (6.6c) into (6.1a) and (6.1b) and combine quantities in e_z and h_z to obtain a new set of equations for e_z, h_z and ϕ'.

$$\{\nabla_t^2 + b_{11}^2\} e_z = b_{12}h_z + b_{13}\phi' \qquad \text{(a)}$$

$$\{\nabla_t^2 + b_{22}^2\} h_z = b_{21}e_z + b_{23}\phi' \qquad \text{(b)}$$

$$\{\nabla_t^2 + h_p^2\} \phi' = 0 \qquad \text{(c)} \qquad (6.7)$$

$$e_z|_c = \frac{dh_z}{dn}\Big|_c = 0 \qquad \text{(d)}$$

$$\frac{d\phi'}{dn}\Big|_c = b_4 \frac{de_z}{dn}\Big|_c \qquad \text{(e)}$$

The coefficients are

$$b_{11}^2 = k_o^2 K_p - \beta^2 \left(1+ \frac{\ell_o^2 \ell_B^2}{K_H(1-j\ell_\nu)^2}\right) \qquad \text{(a)}$$

$$b_{22}^2 = k_o^2\left(1- \frac{\ell_o^2(1-j\ell_\nu-\ell_o^2)}{K_H(1-j\ell_\nu)}\right)-\beta^2=k_o^2\left(K_p- \frac{\ell_o^2\ell_B^2}{K_H(1-j\ell_\nu)^2}\right)-\beta^2 \qquad \text{(b)}$$

$$b_{12} = \frac{j\ell_o^2\ell_B k_o\beta}{K_H(1-j\ell_2)} \qquad \text{(c)}$$

$$b_{13} = -j\ell_o^2 h_o \beta \qquad \text{(d)}$$

$$b_{21} = -j\,\frac{\ell_o^{\,2}\ell_B k_o \beta}{K_H(1-j\ell_\nu)}\,K_p \qquad \text{(e)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.8)$$

$$b_{23} = -\,\frac{\ell_o^{\,2}\ell_B h_o k_o}{(1-j\ell_\nu)} \qquad \text{(f)}$$

$$h_p^{\,2} = h_o^{\,2}K_H - \beta^2 K_B \qquad \text{(g)*}$$

$$b_4 = -j\,\frac{h_o \beta}{k_c^{\,2}} \qquad \text{(h)}$$

In the computation of the above coefficients we have dropped ratios such as $k_o^{\,2}/h_o^{\,2} \sim (\frac{u}{c})^2$ compared to unity.

Note that the coefficients $b_{11}^{\,2}$, b_{12}, $b_{22}^{\,2}$ and b_{21} defined above are identical to the similar coupling coefficients defined by (2.10) for the cold plasma equations. If ϕ' is dropped from (6.7) we thus obtain the cold plasma equations. This fact leads to an interesting physical interpretation of the contribution of ϕ to the equations.

The pressure, through (6.6a), is seen to consist of two distinct parts. One component is due to the number density variation occurring in the cold plasma equations and not dependent on ∇p. The second component is due to

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* $h_p^{\,2}$ has been used in place of $a_{33}^{\,2}$ to show its relation to $h_o^{\,2}$, $h_p^{\,2} = a_{33}^{\,2} = h_o^{\,2}K_H - K_B\beta^2$.

the boundary conditions imposed on the velocity field and produces pressure variations which are not spatially related to e_z and h_z. The spatial behavior of this component is determined by (6.7c) and, because h_p is very large, will be a very rapidly fluctuating term.

The assumption that the rapidly fluctuating components of the pressure due to feedback from e_z and h_z was negligible will be checked later.

The elimination of volume coupling from ϕ is very convenient since it is now possible to proceed with no approximations to reduce (6.7) further and exhibit the forms of the solution in an illuminating manner.

We will now separate the equations into two sets of equations that will prove useful later when the solutions are sought.

Consider the effect of ϕ' coupling to h_z and e_z. Since ϕ' satisfies a homogeneous equation its influence upon the other fields can be found exactly through equation (A.11).

6.2 The Hybrid E Modes

The following form will prove useful later when we consider solutions that approach the E-type waveguide modes in the high frequency limit.

Denote by $e_{z\phi}$ and $h_{z\phi}$ the contribution to e_z and h_z by ϕ' alone. Thus, $h_{z\phi}$ should satisfy the equation

$$(\nabla_t^2 + b_{22}^2)\, h_{z\phi} = b_{23}\phi'$$

From (A.11), using the fact that ϕ' satisfies (6.7c), we obtain

$$h_{z\phi} = \frac{-b_{23}}{h_p^2 - b_{22}^2}\phi' + \frac{b_{23}\left[p\, G_h \frac{d\phi'}{dx_o}\right]_{a_1}^{a_2}}{h_p^2 - b_{22}^2} \qquad (6.9)$$

where G_h is the Green's function for $h_{z\phi}$. The total field h_z can thus be written in terms of the two parts of (6.9).

$$h_z = \frac{-b_{23}}{h_p^2 - b_{22}^2}\phi' + h_z' \qquad \text{(a)}$$

$$\text{where } (\nabla_t^2 + b_{22}^2)h_z' = b_{21}e_z \qquad \text{(b)} \qquad (6.10)$$

$$\frac{dh_z'}{dn}\Big|_c = \frac{b_{23}\frac{d\phi'}{dn}\Big|_c}{h_p^2 - b_{22}^2} = \frac{b_{23}b_4}{h_p^2 - b_{22}^2}\,\frac{de_z}{dn}\Big|_c \qquad \text{(c)}$$

Substituting (6.10a) into the equation for e_z gives,

$$(\nabla_t^2 + b_{11}^2)e_z = b_{12}h_z' + \left(b_{13} - \frac{b_{12}b_{23}}{h_p^2 - b_{22}^2}\right)\phi' \qquad (6.11)$$

Look now at the magnitude of the terms multiplying ϕ'.

$$b_{13} - \frac{b_{12}b_{23}}{h_p^{\,2} - b_{22}^{\,2}} \doteq -j\ell_o^{\,2} h_o \beta \left(1 - \frac{\ell_o^{\,2}\ell_B^{\,2}}{K_H(1 - j\ell_{vm})}\right)\left(\frac{u}{c}\right)^2 \sim b_{13} \tag{6.12}$$

This shows that the volume coupling from h_z to e_z due to ϕ' is on the order of the speed of sound over the speed of light squared compared to the direct coupling of ϕ' to e_z.

The set of equations that will be used to find the hybrid E modes are

$$\left(\nabla_t^{\,2} + b_{11}^{\,2}\right) e_z = b_{12} h_z' + b_{13}\phi' \qquad \text{(a)}$$

$$\left(\nabla_t^{\,2} + b_{22}^{\,2}\right) h_z' = b_{21} e_z \qquad \text{(b)}$$

$$\left(\nabla_t^{\,2} + h_p^{\,2}\right) \phi' = 0 \qquad \text{(c)}$$

$$e_z\big|_c = 0 \qquad \text{(d)} \qquad (6.13)$$

$$\frac{dh_z'}{dn}\bigg|_c = b_5 \frac{de_z}{dn}\bigg|_c \qquad \text{(e)}$$

$$\frac{d\phi'}{dn}\bigg|_c = b_4 \frac{de_z}{dn}\bigg|_c \qquad \text{(f)}$$

Here we have defined

$$b_5 = \frac{b_{23}b_4}{h_p^{\,2}-b_{22}^{\,2}} = j\,\frac{\ell_o^{\,2}\ell_B k_o \beta}{(1-j\ell_\nu)K_H k_c^{\,2}} \tag{6.14}$$

Finally, equation (A.11) can again be used to exhibit the functional form of the fields more clearly. Thus e_z will be of the form,

$$e_z = \frac{-b_{13}\phi'}{h_p^{\,2}-b_{11}^{\,2}} + e_z' \tag{6.15}$$

An equation for e_z' could be written, but is not of interest now.

Equation (6.15) shows that e_z, like h_z, is composed of a slowly spatial varying component, e_z', and a component directly proportional to ϕ'.

6.3 The Hybrid H Modes

Now we will formulate the equations in a manner particularly useful in later work when the modes which reduce to the H modes are investigated. In this case we first find a reduced equation for e_z. From (A.11), proceeding exactly as in the previous section, we find that the contribution to e_z from ϕ' in (6.7) is

$$e_{z\phi} = \frac{-b_{13}\phi'}{h_p^2-b_{11}^2} + \frac{b_{13}}{h_p^2-b_{11}^2}\left[-p\phi' \frac{dG_e}{dn}\right]_{a_1}^{a_2} \qquad (6.16)$$

The total field can be written,

$$e_z = \frac{-b_{13}\phi'}{h_p^2-b_{11}^2} + e_z' \qquad (a)$$

where

$$\{\nabla_t^2+b_{11}^2\}e_z' = b_{12}h_z \qquad (b) \qquad (6.17)$$

$$e_z'|_c = \frac{b_{13}\phi'}{h_p^2-b_{11}^2}\Big|_c \qquad (c)$$

Substitute (6.17a) into the equation for h_z to obtain

$$\{\nabla_t^2+b_{22}^2\}h_z = b_{21}e_z' + [b_{23}-\frac{b_{21}b_{13}}{h_p^2-b_{11}^2}]\phi' \qquad (6.18a)$$

The magnitude of the second term multiplying ϕ' is again very small compared to the first term. This shows that the coupling from ϕ' through e_z is considerably smaller than the direct coupling from ϕ' to h_z. Thus (6.18a) is very accurately given by

$$\{\nabla_t^2+b_{22}^2\}h_z = b_{21}e_z' + b_{23}\phi' \qquad (6.18b)$$

It is now desired to obtain a set of equations for e_z', h_z and ϕ'. To do this it is necessary that the boundary condition on ϕ' be expressed in terms of e_z'.

Take the normal derivative of (6.17a) and evaluate at the boundary.

$$\frac{de_z}{dn}\bigg|_c = \frac{-b_{13}}{h_p^{\,2}-b_{11}^{\,2}}\,\frac{d\phi'}{dn}\bigg|_c + \frac{de_z'}{dn}\bigg|_c \tag{6.19}$$

But

$$\frac{d\phi'}{dn}\bigg|_c = b_4\,\frac{de_z}{dn}\bigg|_c$$

Therefore

$$\frac{de_z}{dn}\bigg|_c \left[1 + \frac{b_{13}b_4}{h_p^{\,2}-b_{11}^{\,2}}\right] = \frac{de_z'}{dn}\bigg|_c$$

$$\frac{de_z}{dn}\bigg|_c = \left[\frac{h_p^{\,2}-b_{11}^{\,2}}{h_p^{\,2}-b_{11}^{\,2}+b_{13}b_4}\right]\frac{de_z'}{dn}\bigg|_c \tag{6.20}$$

The desired set of equations is thus,

$$\{\nabla_t^{\,2}+b_{11}^{\,2}\}e_z' = b_{12}h_z \qquad \text{(a)}$$

$$\{\nabla_t^{\,2}+b_{22}^{\,2}\}h_z = b_{21}e_z' + b_{23}\phi' \qquad \text{(b)}$$

$$\{\nabla_t^2+h_p^2\}\phi' = 0 \qquad \text{(c)}$$

(6.21)

$$e_z'|_c = b_6\phi'|_c \qquad \text{(d)}$$

$$\frac{dh_z}{dn}|_c = 0 \qquad \text{(e)}$$

$$\frac{d\phi'}{dn}|_c = b_7 \frac{de_z'}{dn}|_c \qquad \text{(f)}$$

The boundary coupling terms are

$$b_6 = \frac{b_{13}}{h_p^2-b_{11}^2} \approx \frac{-j\ell_o^2\beta}{h_oK_H} \qquad \text{(a)}$$

(6.22)

$$b_7 = b_4\left[\frac{h_p^2-b_{11}^2}{h_p^2-b_{11}^2+b_{13}b_4}\right] \approx \frac{-jh_o\beta K_H}{K_Hk_c^2-\ell_o^2\beta^2} \qquad \text{(b)}$$

Note that the boundary condition on e_z' depends on the value of ϕ' at the boundary. In the next section it will be shown that (6.21d) can be written as a mixed boundary condition on e_z'.

6.4 <u>Derivation of Mixed Boundary Condition for e_z'</u>

An examination of (6.21) shows that ϕ' depends on $\frac{de_z'}{dn}$ through the boundary condition. Since the boundary condition on e_z' involves ϕ' it will also depend on $\frac{de_z'}{dn}$ and actually is a mixed boundary condition. An examination of the solution for ϕ' will show that this is the case.

Consider the solution to (6.21c) in a circular waveguide for modes having no azimuthal variation. The solution is tabulated in appendix B, (B7), and is

$$\phi'(r) \doteq \frac{-\alpha J_o(h_p r)}{h_p J_1(h_p a)} \tag{6.22a}$$

where a is the waveguide radius and α is

$$\alpha = \frac{b_4(h_p{}^2 - b_{11}{}^2)}{h_p{}^2 - b_{11}{}^2 + b_4 b_{13}} \left.\frac{\partial e_z{}'}{\partial n}\right|_c \tag{6.22b}$$

In this case h_p is approximated by $h_o\sqrt{K_H}$ since, by hypothesis, $\beta^2 << h_o{}^2$.

Since h_o is generally very large at microwave frequencies (6.22) can be evaluated near the asymptotic expansion for the Bessel functions[25]

$$\phi'(r) \doteq \frac{-\alpha \cos(h_p r - \pi/4)}{h_p \sin(h_p a - \pi/4)}$$

$$\phi'(a) = \frac{\alpha}{h_p} \left\{ \frac{\cos h_p a + \sin h_p a}{\cos h_p a - \sin h_p a} \right\} \tag{6.23}$$

Now define $h_p a = \eta + j\xi$ where both $|\eta|$ and $|\xi|$ are much greater than unity and expand (6.23). In this case $\sinh \xi \doteq \cosh \xi \doteq e^{\xi}/2$ and (6.23) becomes

$$\phi'(a) \doteq \frac{\alpha}{h_p} \left\{ \frac{\cos\eta+\sin\eta+j(\cos\eta-\sin\eta)}{\cos\eta-\sin\eta-j(\cos\eta+\sin\eta)} \right\} \qquad (6.24)$$

Simplifying further gives

$$\phi'(a) \doteq \frac{j\alpha}{h_p} \qquad (6.25)$$

Now substitute (6.25) into (6.21f) and use (6.22b) to obtain the mixed boundary condition for e_z'.

$$e_z'|_c = \frac{j\, b_{13}b_4}{h_p(h_p^2-b_{11}^2+b_4b_{13})} \frac{de_z'}{dn}\Big|_c \qquad (6.26)$$

Even if it were not possible to use the asympototic expansion a mixed boundary condition could still be found for e_z'. In the general case it is not possible to exhibit the boundary condition quite as compactly as (6.26) since one must retain the solution for ϕ'. Generally, since h_o is so large, (6.26) will be an excellent approximation.

6.5 Relation to the Cold Plasma Model in the Zero Temperature Limit

In all the above derivations it was assumed that h_p^2 was much larger than b_{11}^2 or b_{22}^2. This assumption will almost always be valid for the electromagnetic modes, $\beta^2 \ll h_o^2$. In particular, note that as the temperature is lowered the approximation becomes better since $h_o^2 \equiv (\frac{\omega}{u})^2$ and $u^2 \propto T$. Thus, as $T \to 0$, $h_o \to \infty$.

The above equations for the warm plasma model are in a particularly useful form to consider the zero temperature limit. Recently the question whether or not the warm plasma model reduces to the cold plasma model has been the subject of some controversy. In two articles in the same volume of Electronic Letters Wait [illegible][20] claims that the warm plasma model reduces to the cold plasma model and Lee et al claim it does not.[18] In a later paper [21] Wait says "Strictly speaking, (Lee's conclusions are) quite true; however, it should be pointed out that, for any finite distance from the boundary, the transition to the cold-plasma solution is indeed uniform." An examination of the equations derived above will verify the statement. However, it will also show that the boundary term arising from the pressure mode is still extremely important. This point will now be discussed in some detail.

From the results derived in Chapter 2 it is evident that the cold plasma model yields physical inconsistences since the election velocity at the walls can be quite large. With the warm plasma model the additional boundary condition $v_n = 0$ is available and must be satisfied for any temperature. A detailed examination of the normal component of velocity will give some insight to the behavior at the boundary and allow us to look at the zero temperature limit in detail.

Using (3.5b) to recover the actual velocity from the normalized velocity in (3.30a) gives;

$$\left\{k_p^4 - \frac{\text{[illegible]}}{K_\psi^2}\right\} v_n = \frac{\sqrt{\frac{\epsilon_o}{\mu_o}}}{\epsilon_o q}\left\{\frac{\ell_o^2 k_o k_p^2}{K_\psi}\left[-\beta \frac{de_z}{dn} + j\frac{k_c^2}{h_o}\frac{d\phi}{dn}\right] - \frac{j\ell_o^2 \ell_B k_o^2 k_c^2}{K_\psi^2}\frac{dh_z}{dn}\right\} \qquad (6.27)$$

Consider (6.26) very near the boundary. We will omit the term $\frac{dh_z}{dn}$ in this discussion since it is zero precisely at the boundary. v can be written in terms of the primed fields through equations (6.19).

$$v_n\Big|_{r\to a} \sim -\beta\left[\frac{de_z'}{dn} - \frac{b_{13}}{h_p^2}\frac{d\phi'}{dn}\right] + \frac{jk_c^2}{h_o}\left[\frac{a_{31}}{h_p^2}\frac{de_z'}{dn} + \frac{a_{32}}{h_p^2}\frac{dh_z'}{dn} + \frac{d\phi'}{dn}\right] \qquad (6.28)$$

Using (6.19h) to express $\frac{dh_z'}{dn}$ in term $\frac{de_z'}{dn}$ gives

$$v_n\Big|_{r\to a} \sim \left\{-\beta + \frac{jk_c^2 a_{31}}{h_o h_p^2} + \frac{jk_c^2 a_{32}}{h_o h_p^2}\left(\frac{b_{23} b_4}{h_p^2 - b_{11}^2 + b_4 b_{13}}\right)\right\}\frac{de_z'}{dn} + \left\{\frac{\beta b_{13}}{h_p^2} + \frac{jk_c^2}{h_o}\right\}\frac{d\phi'}{dn} \qquad (6.29)$$

Now consider the magnitude of the terms multiplying $\frac{de_z'}{dn}$ in (6.28). Note that the second and third terms in the bracket are generally much smaller than β since they both are on the order of $(\frac{u}{c})\ \beta$ [illegible] $10^{-6}\beta$. Neglecting these terms we find,

$$v_n\Big|_{r\to a} \sim -\beta \frac{de_z'}{dn} + \{\beta \frac{b_{13}}{h_p^2} + j \frac{k_c^2}{h_o}\} \frac{d\phi'}{dn} \qquad (6.30)$$

In essence the above is just another derivation of the boundary condition (6.19i). However, the approximation introduced shows up more clearly. Some objection might be raised to neglecting the terms that are dropped in deriving the boundary equation. However, in the numerical work required to solve such equations one seldom requires that the remaining quantities approach zero to this limit since the effect on the dispersion curve or eigenvalues being sought is generally negligible. i.e., practically we are only interested in β or the field quantities to the first few significant figures.

Now consider the low temperature limit. As $T \to 0$ the boundary condition on e_z' becomes, from (6.[illegible] $e_z'|_c = 0$. $\frac{de_z'}{dn}|_c$ will not be zero and the contribution from ϕ' must just balance $-\beta \frac{de_z'}{dn}$. Since the quantities multiplying $\frac{d\phi'}{dn}$ approach zero as $1/h_o$, $\frac{d\phi'}{dn}$ must

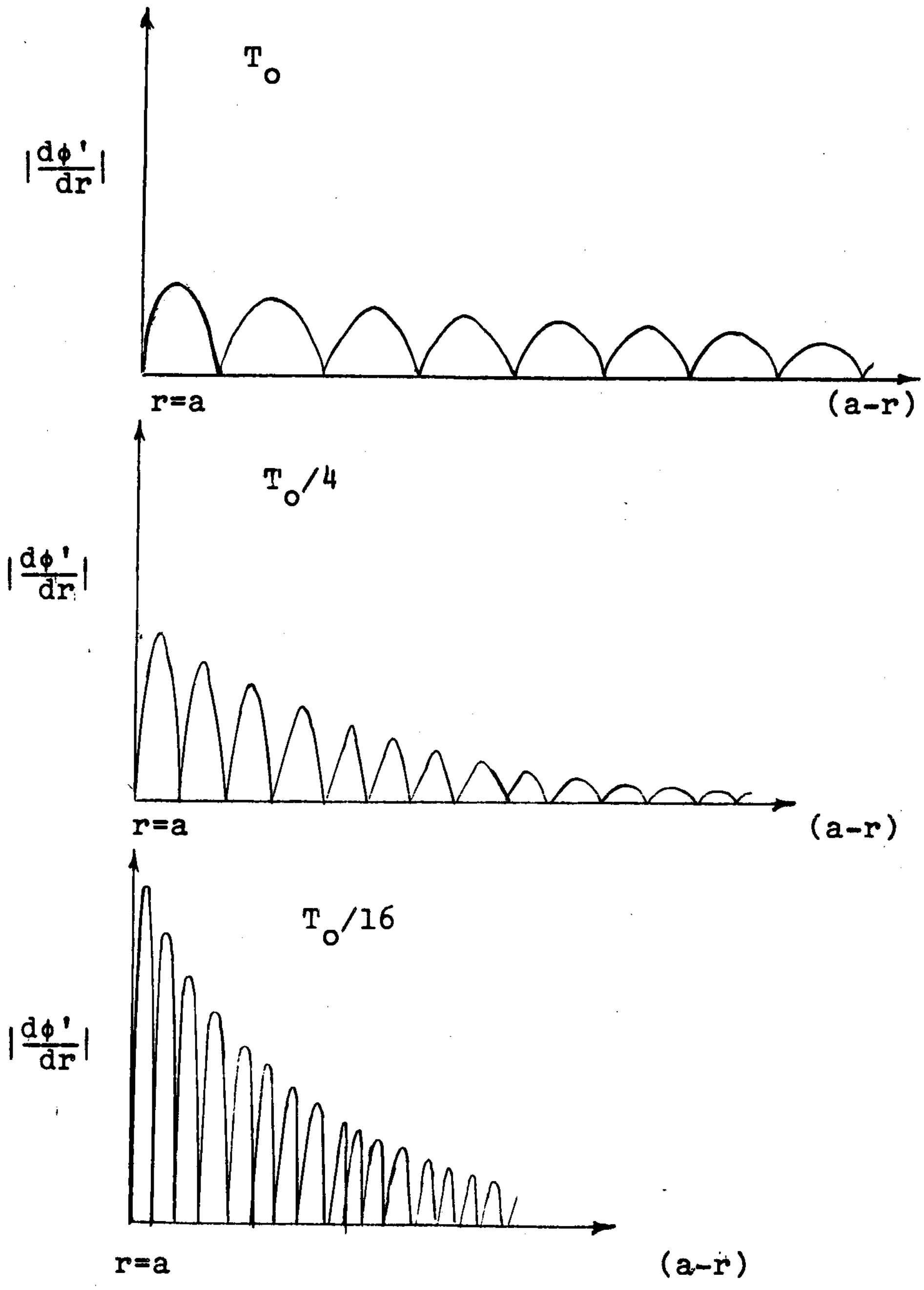

Fig. 6.1 - Sketch of $|\frac{d\phi'}{dr}|$ Near the Waveguide Boundary for Different Temperatures

approach infinity in the same limit. The solution for ϕ' obtained for the circular waveguide and given by (6.21) has exactly this behavior. It is illustrative to plot $\frac{d\phi'}{dn}$ for different values of temperature. The magnitude of this function is shown in Fig. 6.1 for decreasing values of T and with collisions sufficiently large that the exponential and oscillatory behavior are comparable.

Note that for $T\to 0$, $\frac{d\phi'}{dr}$ approaches a function very similar to the delta function. Even in this case, where the influence of the pressure is absent a short distance from the boundary, right at the boundary the normal velocity must vanish. Thus it must be said that the warm and cold plasma models do not correspond and that the inclusion of the pressure term resolves the non-physical difficulties arising from the cold plasma model.

In resolving the difficulty in the cold plasma model we have caused another problem. In the original fluid equations we neglected the term, $(\vec{V}\cdot\nabla)\vec{V}$. For very small temperatures this term may now not be negligible. The actual value of the non-linear term has to be investigated for a particular problem to see if $(\vec{V}\cdot\nabla)\vec{V}$ is truly negligible compared to $\omega\vec{V}$.

To sum up the discussion we can say that generally it is expected that use of the cold plasma model or zero temperature limit of the warm plasma model in bounded problems will cause physical inconsistencees to occur. We will henceforth assume that the temperature of the plasma is not zero.

Finally, note that in all the derivatives in this chapter we have assumed that h_o was a very large number by comparison with other terms with which it might be combined. In particular, $|\beta| << h_o$. This last restriction limits the discussion to what we have called the electromagnetic modes. It will be seen when solutions are found that the assumption is very well justified.

Another set of modes, the electro-accoustio solution, exist where $\beta \sim h_o$. It would be possible to derive a reduced set of equations for the case similar to (6.19). However it is generally just as simple to work with the original equations. These modes will be discussed in the next chapter.

CHAPTER VII

APPROXIMATE SOLUTIONS FOR THE NORMAL MODES

7.0 Introduction

At this point a procedure similar to that used in Chapter 2 could be followed to obtain solutions for the normal modes. Such a procedure has been outlined by Sancer and by Chen and Chen. Instead, a coupled-mode approach will be pursued. This procedure is discussed briefly in Friedman[2[illegible]] and is quite similar to the Feenberg iteration procedure discussed by Morse and Feshbach.[27]

Although the cases considered in this work have been for a uniform plasma, the non-uniform situation is of interest. The method of solution by transformation of the differential equations to a diagonal form is applicable only when the transformation matrix, M^{-1}, commutes with the differential operators. This is not the case when the plasma is non-uniform.

The iterative method used here is general enough to apply to the non-uniform plasma. It also has the advantage of yielding a hierarchy of approximations, the higher order approximations including more terms in the solution. An examination of the various approximations gives a good idea of the way the various modes couple and of the significance of the coupling. This procedure shows simplifications which can occur in an otherwise very complicated problem.

In the treatment to follow we will classify the solutions as 'E', 'H' and 'p' modes, the designation indicating that the solutions obtained approach these uncoupled modes in the limit of zero plasma frequency and cyclotron frequency.

The formal method of solution to be derived is in fact exact. Only after deriving the equation for the dispersion relation is it necessary to approximate the result. This approximation is necessary in order to truncate an infinite secular determinant for the dispersion relation. The accuracy with which this can be done depends on the degree to which the modes couple.

7.1 <u>The Hybrid E Modes</u>

Now consider the modes which approach the electric modes in the limit of zero coupling (or $\omega \gg \omega_o$ and ω_B).

The reduced equations derived in Chapter 6 were very useful in demonstrating the functional form of the modes. However, in the derivation to follow it is more convenient to start with (6.7).

Assume that all the fields can be expressed in complete sets of functions which satisfy the uncoupled eigenvalue equations subject to the appropriate boundary conditions. These eigenfunctions are solutions of;

$$(\nabla_t^2+\lambda_n^2)e_n = 0; \quad e_n|_c = 0 \qquad \text{(a)}$$

$$(\nabla_t^2+\gamma_\ell^2)w_\ell = 0; \quad \frac{dw_\ell}{dn}\Big|_c = 0 \qquad \text{(b)} \qquad (7.1)$$

To derive the equation for the E modes it is assumed that e_z is the source of the other fields. These fields are found and substituted back into the equation for e_z and an equation for the coefficients is found. Let the fields be given by

$$e_z = \sum_n a_n e_n \qquad \text{(a)}$$

$$h_z = \sum_\ell b_\ell w_\ell \qquad \text{(b)} \qquad (7.2)$$

$$\phi' = \sum_m C_m w_m \qquad \text{(c)}$$

Equations (6.7) are reproduced here for convenience.

$$(\nabla_t^2+b_{11}^2)e_z = b_{12}h_z + b_{13}\phi' \qquad \text{(a)}$$

$$(\nabla_t^2+b_{22}^2)h_z = b_{21}e_z + b_{23}\phi' \qquad \text{(b)}$$

$$(\nabla_t^{\,2}+h_p^{\,2})\phi' = 0 \qquad (c) \qquad (7.3)$$

$$e_z|_c = \frac{dh_z}{dn}|_c = 0 \qquad (d)$$

$$\frac{d\phi'}{dn}|_c = b_4 \frac{de_z}{dn}|_c \qquad (e)$$

First calculate ϕ'. From Appendix A, (A.3),

$$\phi' = [G_\phi(x_o/x)p(x_o)\frac{d\phi'}{dx_o}]_o^a \qquad (7.4)$$

The Green's function for this field is;

$$G_\phi = -\sum_m \frac{w_m(x_o)w_m(x)}{-\gamma_m^{\,2}+h_p^{\,2}} \qquad (7.5)$$

Assuming (7.2a) is a uniformly convergent series, the boundary condition can be written

$$\frac{d\phi'}{dn}|_c = b_4 \sum_s a_s \frac{de_n}{d_n}|_c \qquad (7.6)$$

Substituting into (7.4) gives

$$\phi'(x) = \sum_m \{\frac{-b_4 \sum_s a_s(e_s,w_m)}{-\gamma_m^{\,2}+h_p^{\,2}}\}w_m(x) \qquad (7.7)$$

The notation (e_s, w_m) is used to denote the boundary evaluation. This notation, together with the notation $\langle e_s, w_m \rangle$ for the scalar product is discussed in Appendix A.

Now compute h_z. Using (7.2b) to expand (7.3b) gives

$$\sum_{\ell} (-\gamma_\ell^2 + b_{22}^2) b_\ell w_\ell = b_{21} \sum_s a_s e_s + b_{23} \sum_m C_m w_m \qquad (7.8)$$

Taking the scalar product of (7.8) with w_ℓ gives

$$(-\gamma_\ell^2 + b_{22}^2) b_\ell = b_{21} \sum_s a_s \langle e_s, w_\ell \rangle + b_{23} C_m \delta_{\ell m} \qquad (7.9)$$

Here we have used the fact that ϕ' and h_z are expanded in the same orthonormal functions.

Substituting for C_m from (7.7) gives,

$$(-\gamma_\ell^2 + b_{22}^2) b_\ell = \sum_s a_s \{ b_{21} \langle e_s, w_\ell \rangle - \frac{b_{23} b_4 (e_s, w_\ell)}{-\gamma_\ell^2 + h_p^2} \} \qquad (7.10)$$

The expressions for h_z and ϕ' can now be substituted into (7.3a) to yield an equation for the coefficients for e_z.

$$\sum_n (-\lambda_n^2 + b_{11}^2) a_n e_n = b_{12} \sum_{\ell} b_\ell w_\ell + b_{13} \sum_m C_m w_m \qquad (7.11)$$

Taking the scalar product with e_n gives

$$(-\lambda_n^2+b_{11}^2)a_n=b_{12}\sum_{\ell} b_\ell\langle e_n,w_\ell\rangle+b_{13}\sum_{m} C_m\langle e_n,w_m\rangle \quad (7.12)$$

Substituting from (7.7) and (7.10) for b_ℓ and C_m gives,

$$(-\lambda_n^2+b_{11}^2)a_n=b_{12}\sum_{\ell}\left\{\sum_{s}\frac{a_s[b_{21}\langle e_s,w_\ell\rangle-\dfrac{b_{23}b_4(e_s,w_\ell)}{-\gamma_\ell^2+h_p^2}]}{-\gamma_\ell^2+b_{22}^2}\right\}$$

$$\langle e_n,w_\ell\rangle+b_{13}\sum_{m}\left\{\frac{-b_4\sum_{s} a_s(e_s,w_m)}{-\gamma_m^2+h_p^2}\right\}\langle e_n,w_m\rangle$$

$$(7.13)$$

Interchanging summations and replacing the dummy index m by ℓ gives

$$(-\lambda_n^2+b_{11}^2)a_n = \sum_{s} a_s\ \{b_{12}b_{21}\sum_{\ell}\frac{\langle e_s,w_\ell\rangle\langle e_n,w_\ell\rangle}{-\gamma_\ell^2+b_{22}^2}$$

$$-\sum_{\ell}[\frac{b_{12}b_{23}b_4}{(-\gamma_\ell^2+b_{22}^2)(-\gamma_\ell^2+h_p^2)}+\frac{b_{13}b_4}{(-\gamma_\ell^2+h_p^2)}]$$

$$(e_s,w_\ell)\langle e_n,w_\ell\rangle\} \quad (7.14)$$

Equation (7.14) is an infinite set of equations which must be solved for the complex wave number, β, and the coefficients a_n. Note that the derivation of (7.14) is so far exact, i.e., it is not a perturbation method and no terms have been assumed to be small. However, to obtain solutions of (7.14) the infinite set of equations must be truncated. This implies that for n sufficiently large the a_n approach zero.

Morse and Feshbach [27] discuss equations similar to (7.14) and discuss solution by an iterative technique where all terms but the first are set to zero, the zeroth order value of β found, then the first four terms are retained and the zeroth order solution is used to obtain the first order β, etc. We shall use a similar technique, except that a root-searching method will also be employed. The numerical results are presented in the next chapter.

Note that if the second sum is omitted from (7.14) we obtain the spectral solutions for the cold plasma model. In fact, this was done in obtaining approximate solutions from which to proceed with the numerical solutions obtained in Chapter 2.

At this point we will digress from the problem of finding the normal modes so that we may check the reduced equations derived in Chapter 6.

By proceeding in the manner to be illustrated all of the reduced equations can be shown to yield to the original fields when the spectral representation is used for the primed fields. To clarify this we will re-derive equation (7-14), starting this time with (6.10).

$$h_z = \frac{-b_{23}}{h_p^{\,2} - b_{22}^{\,2}} \phi' + h'_z \qquad (6.10a)$$

h'_z satisfied (6.10b) and the solution is;

$$h'_z = \sum_{\ell} \{b_{21} \sum_{s} \frac{a_s \langle e_s, w_\ell \rangle}{-\gamma_\ell^{\,2} + b_{22}^{\,2}}\} w_\ell$$

$$- \sum_{\ell} \{\frac{b_{23}\ b_4}{h_p^{\,2} - b_{22}^{\,2}} \frac{\sum_{s} a_s(e_s, w_\ell)}{(-\gamma_\ell^{\,2} + b_{22}^{\,2})}\} w_\ell \qquad (6.11)$$

Substituting for ϕ' from (7.7) and combining gives,

$$h_z = \sum_{\ell} b_{21} \sum_{s} \frac{a_s \langle e_s, w_\ell \rangle}{-\gamma_\ell^{\,2} + b_{22}^{\,2}} w_\ell$$

$$+ \sum_{\ell} \{\frac{b_{23}\ b_4}{h_p^{\,2} - b_{22}^{\,2}} \sum_{s} a_s(e_s, w_\ell) [\frac{1}{-\gamma_\ell^{\,2} + h_p^{\,2}} - \frac{1}{-\gamma_\ell^{\,2} + b_{22}^{\,2}}]\} w_m \qquad (7.15)$$

Clearing the fraction gives

$$h_z = \sum_{\ell} \{ b_{21} \sum_{s} \frac{a_s \langle e_s, w_\ell \rangle}{-\gamma_\ell^2 + b_{22}^2} - \frac{b_{23} b_4 \sum_{s} a_s (e_s, w_\ell)}{-\gamma_\ell^2 + b_{22}^2)(-\gamma_\ell^2 + h_p^2} \} w_\ell \quad (7.16)$$

This result is identical to (7.14) which was derived from the original equation for h_z, (7.3b).

7.2 The Hybrid H Modes

The preceeding derivation for the coefficients in the expansion of the perturbed E modes was straight forward and involved no approximations beyond those made in deriving (6.7). This was the case because the ϕ' modes depend directly upon e_z.

In deriving an equation for the expansion coefficients of h_z it is desirable that the field e_z which couples to h_z be expressed as simply as possible. In particular, we wish to avoid having to solve an infinite set of equations for the expansion coefficients of e_z. We can accomplish this most easily by using the field e_z' derived in (6.17) to express the coupling to h_z. Further simplification is achieved, with little loss of accuracy, if the inhomogeneous boundary (6.17c) is replaced by the simpler condition, $e_z'|_c = 0$.

Again assume the fields can be expressed by the spectral representations used previously where now

$$e_z' = \sum_n a_n e_n \tag{7.17}$$

The boundary condition used here, $e_z'|_c = 0$ will be accurate if h_o is sufficiently large and if e_z' can be represented reasonably accurately by the first few terms in the series. The exact boundary condition on e_z' was discussed in Chapter 6. The accuracy of the assumption can be checked after calculations have been made and generally it is found that the assumption is excellent.

From (6.21a),

$$\sum_n \{-\gamma_n^2 + b_{11}^2\} a_n e_n = b_{12} \sum_j b_j w_j$$

$$a_n = \frac{b_{12}}{\{-\gamma_n^2 + b_{11}^2\}} \sum_j b_j \langle e_n, w_j \rangle \tag{7.18}$$

ϕ' is now given by (7.7) where b_4 is replaced by b_7.

$$\phi' = -b_7 \sum_m \left\{ \frac{\sum_n a_n (e_n, w_m)}{-\gamma_m^2 + h_p^2} \right\} w_m(x) \tag{7.19}$$

These relations are now used to compute h_z. The equation for h_z is;

$$\{\nabla_t^{\;2}+b_{22}^{\;2}\}h_z = b_{21}e_z' + b_{23}\,\phi' \qquad \text{(a)}$$

$$(7.20)$$

$$\frac{dh_z}{dn} = 0 \qquad \text{(b)}$$

In terms of the spectrums;

$$\sum_\ell \{-\gamma_\ell^{\;2}+b_{22}^{\;2}\}b_\ell w_\ell = b_{21}\sum_n a_n e_n + b_{23}\sum_m C_m w_m \qquad (7.21)$$

$$\{-\gamma_\ell^{\;2}+b_{22}^{\;2}\}b_\ell = b_{21}\sum_n a_n\langle e_n,w_\ell\rangle + b_{23}C_m\delta_{\ell m} \qquad (7.22)$$

Substituting from (7.19) and (7.20)gives

$$\{-\gamma_\ell^{\;2}+b_{22}^{\;2}\}b_\ell = b_{21}\sum_n \{\frac{b_{12}}{(-\gamma_n^{\;2}+b_{11}^{\;2})}\sum_j b_j\langle e_n,w_j\rangle\}\langle e_n,w_\ell\rangle$$

$$+ b_{23}\{\frac{-b_7\sum_n a_n(e_n,w_\ell)}{-\gamma_\ell^{\;2}+h_p^{\;2}}\} \qquad (7.23)$$

$$= b_{21}b_{12}\sum_n \{\frac{\sum_j b_j\langle e_n,w_y\rangle\ \langle e_n,w_\ell\rangle}{-\lambda_n^{\;2}+b_{11}^{\;2}}\}$$

$$- b_{23}b_7\sum_n \frac{\{\frac{b_{12}}{-\lambda_n^{\;2}+b_{11}^{\;2}}\}\sum_j b_j\langle e_n,w_j\rangle(e_n,w_\ell)}{-\gamma_\ell^{\;2}+h_p^{\;2}}$$

$$\{-\gamma_\ell^{\,2}+b_{22}^{\,2}\}b_\ell = \sum_j b_j \ \{b_{12}b_{21}\sum_n \frac{\langle e_n,w_j\rangle\ \langle e_n,w_\ell\rangle}{-\lambda_n^{\,2}+b_{11}^{\,2}}$$

$$-\ \frac{b_{12}b_{23}b_7}{-\gamma_\ell^{\,2}+h_p^{\,2}}\ \sum_n \frac{\langle e_n,w_j\rangle\,(e_n,w_\ell)}{-\lambda_n^{\,2}+b_{11}^{\,2}}\ \} \qquad (7.24)$$

Equation (7.25) is again an infinite set of algebraic equations which must be solved for the wave numbers and the coefficients. It is similar to the equations for the coefficients of e_n derived previously for the E modes except that the boundary coupling due to the pressure mode enters with a difficient coefficient. The remarks previously made about the effect of h_p on the second sum are pertinent here and the coefficient of the second sum is approximately.

$$\frac{-b_{12}b_{23}b_7}{h_p^{\,2}} \approx \frac{-\ell_o^{\,4}\ell_B^{\,2}k_o^{\,2}\beta^2}{K_H(1-j\ell_\nu)^2[K_Hk_c^{\,2}-\ell_o^{\,2}\beta^2]} \qquad (7.25)$$

Finally, solutions for the perturbed pressure modes will be derived.

7.3 <u>The Electro-Accoustic Modes</u>

The solutions considered so far have been explicitely restricted by the assumption that $|\beta| \ll h_o$. This restriction effectively omits the pressure waves which can exist in this case where the electron-gas was assumed to behave as an ideal gas. This class of solutions is now considered.

To start the development we will use equations (6.1). The potential, e_z, h_z and ϕ, can again be expressed in terms of complete sets of functions as given by (6.2). Note that e_n satisfies the boundary condition $e_n|_c = 0$ in this discussion. Here ϕ is considered to be the source function generating the other fields.

With ϕ regarded as the source we can solve for the other fields in terms of ϕ and then substitute to find a self-consistent equation for ϕ itself. Proceeding in the manner illustrated in section 7.2 and 7.3 we find

$$a_n = a_{13} \sum_j \frac{C_j \langle e_n, w_j \rangle}{(-\lambda_n^2 + a_{11}^2)} \tag{7.26}$$

where a_n and C_j are the expansion coefficients for e_z and ϕ respectively.

The expression for e_z can now be used, together with the spectral representation for ϕ, to find h_z. The result is

$$b_\ell = \frac{a_{21} \sum_n a_n \langle e_n, w_\ell \rangle + a_{23} C_j \delta_{\ell j}}{(-\gamma_\ell^2 + a_{22}^2)} \tag{7.27}$$

where $h_z = \sum_{\ell} b_{\ell} w_{\ell}$

and we have used the fact that the expansion functions, w_{ℓ}, are an orthonormal set.

Substituting the above expressions into (6.1c) and assuming $\phi = \sum_m C_m w_m$ gives the following expression for C_m.

$$\left\{(-\gamma_m{}^2+a_{33}{}^2) - \frac{a_{23}a_{32}}{(-\gamma_m{}^2+a_{22}{}^2)}\right\}C_m$$

$$= \sum_j C_j \sum_n \frac{a_{13}\langle e_n,w_j\rangle}{(-\lambda_n{}^2+a_{11}{}^2)} \left\{\left[a_{31}+\frac{a_{32}a_{21}}{(-\gamma_m{}^2+a_{22}{}^2)}\right]\langle e_n,w_m\rangle - a_4(e_n,w_m)\right\} \quad (7.28)$$

Again we obtain an infinite set of equations which must be solved for the expansion coefficients. Although at first sight (7.28) is an extremely lengthy equation an examination of the magnitude of the terms shows that it can be simplified considerably with negligible loss of accuracy. To see this note that in the limit $\omega \not\to \omega_o$ or ω_B the coupled potential equations, (6.1), decouple and (7.28) becomes simply $(-\gamma_m{}^2+a_{33}{}^2) = 0$. (i.e., a_{ij}, $i \neq j$, approach zero as $\omega \to \infty$). In this limit β^2 is given by $\beta^2 = \frac{h_o{}^2K_H-\gamma_m{}^2}{K_B}$. But since γ_m

is an eigenvalue determined only by the geometry and is generally very small compared to h_o (it is usually on the order of k_o at microwave frequencies) we can write $\beta \to h_o$ as $\omega \to \infty$.

If it is now assumed that as the frequency is lowered β does not deviate from h_o by more than a few orders of magnitude then (7.28) can be simplified considerably.* Using the assumption that $|\beta| \gg k_o$, substituting for a_{ij} explicitly in terms of the plasma parameters and comparing the magnitude of the terms, we find that, to an approximation which neglects terms of order $(U/C)^2$, (7.28) is given by

$$a_{33}{}^2 C_m \approx \sum_j C_j \sum_n \frac{a_{13}a_{31}\langle e_n,w_m\rangle \langle e_n,w_j\rangle}{(-\lambda_n{}^2+a_{11}{}^2)} \tag{7.29}$$

Actually, we need not have gone to so much trouble trying to simplify (7.28) if we were interested only in evaluating β and C_m since (7.29) is not that much simpler than (7.28) from a computational standpoint. However, the previous arguments shed considerable light on the physics of the interaction of a pressure mode

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* Note that the argument that $|\beta| \sim h_o$ implicitely assumes that β is never zero and this implies that collisions will keep $|\beta|$ quite large, even at a frequency where, for the collisionless case, β would vanish.

with the other field quantities. In particular, by considering the quantities that could be neglected we can say that the dispersion characteristics of the accoustic mode are virtually unaffected by the volume coupling with the axial magnetic field or the boundary coupling with the axial electric field. Mathematically this means that the normal ϕ mode can be very closely approximated by solving the equation

$$\nabla_t^{\,2} \begin{pmatrix} e_z \\ \phi \end{pmatrix} + \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \begin{pmatrix} e_z \\ a \end{pmatrix} = 0 \qquad (7.30a)$$

$$e_z\Big|_c = \frac{\partial \phi}{\partial m}\Big|_c = 0 \qquad (7.30b)$$

Generally, (7.30) has two possible solutions and one solution, that for which $|\beta| \ll h_o$, must be discarded since the above arguments apply only for the pressure mode. Note, however, that when $\omega_B = 0$ the differential equations for e_z and ϕ reduce to the form of (7.30). Sancer[14] has covered the details of the solution in this case.

This completes our work on deriving methods of solutions for the coupled potential equations. These methods all yield an infinite set of equations from which the coefficients must be found. In the next chapter solutions for the fields and dispersion relations are obtained for the circular cylindrical waveguide.

7.4 Relation to Perturbation Theory

It has been mentioned that the method of solution derived in the preceeding sections is exact and the ability to compute the dispersion relations and the fields rests entirely on our ability to solve the infinite set of equations for the coefficients. The method was developed as a generalization of a perturbation method suggested in Friedman [23]. Since the perturbation method is simpler and often as accurate as the more complicated method presented above it is presented here. The presentation will closely follow that given by Friedman.

To employ a general treatment consider an equation of the form

$$(L_o-\lambda)u = -\Delta Lu \qquad (7.31)$$

The equations for the modes can all be written this form if we let $L_o \to \nabla_t^2$, $\lambda = -b_{11}^2$, $u = e_z$ for E modes, $\lambda = -b_{22}^2$; $u \to h_z$ for the H modes, etc. If instead of expressing the right-hand-side of the equations in terms of the spectrum of the coupled modes, the formal Greens function representation of the coupled modes was used, then ΔL would be an integral operator and (7.31) is an integral-differential equation. This is important since an integral operator is bounded.

Now assume that ΔL is a small operator, i.e., a bounded operator with bound ϵ. Also, the homogeneous form of (7.31) is an eigenvalue equation with eigenfunction v_n and eigenvalue ν_n.

$$(L_o-\nu_n)v_n = 0 \tag{7.32}$$

To employ the perturbation method we now assume that λ is close to the nth eigenvalue ν_n and that u is close to the nth eigenfunction v_n. We put;

$$\lambda = \nu_n + \alpha_1\epsilon + \alpha_2\epsilon^2 + \ldots \qquad \text{(a)}$$
$$(7.33)$$
$$u = v_n + w_1\epsilon + w_2\epsilon^2 + \ldots \qquad \text{(b)}$$

where α_i are unknown constants and w_i are unknown functions. Substituting (7.33) into (7.32) and separating into equations having like powers of ϵ gives an infinite set of equations which can be solved for α_i and w_i. We will retain only the first power in ε.

u can be represented in terms of the spectrum of L_o.

$$u = \sum_k a_k v_k \tag{7.34}$$

Substituting (7.34) into (7.31), and utilizing the spectral representation of the operator from (7.33) gives,

$$u = \sum_k \frac{1}{\lambda - \nu_k} \langle v_k, \Delta L u \rangle \quad v_k \tag{7.35}$$

Since we have assumed that u is of the form (7.33a), the coefficient of v_n must be unity. Therefore, equating the first term in (7.35) to one gives

$$\lambda - \nu_n \sim \langle v_n, \Delta L v_n \rangle \tag{7.36}$$

Using this value of λ in (7.35) gives

$$u \sim v_n + \sum_k^{0} \frac{1}{\lambda - \nu_k} \langle v_K, \Delta L v_n \rangle \quad v_k \tag{7.37}$$

where the prime indicates that the summation is taken over all k except k = n.

The above development is essentially that given by Friedman. Now it will be applied to our problem for the case of the hybrid-E modes. The extention to the other modes is obvious.

To proceed from the formal theory just presented to the specific case of the hybrid-E modes it is necessary simply to cast the equations for e_z into the form of (7.31) and identify terms. The algebraic manipulations in section (7.1) accomplished this and the equation for the coefficients of the Fourier coefficient of e_z, (7.14) is in the form of (7.31). Examining the terms in (7.14) shows that

$$v_K,\Delta L v_n = b_{12}b_{21} \sum_{\ell} \frac{\langle e_n,w_\ell\rangle \langle e_n,w_\ell\rangle}{-\gamma_\ell^2 + b_{22}^2}$$

$$- \sum_{\ell} \left[\frac{b_{12}b_{23}b_4}{(-\gamma_\ell^2 - b_{22}^2)} + b_{13}b_4\right] \frac{(e_n,w_\ell) \langle e_n,w_\ell\rangle}{(-\gamma_\ell^2 + h_p^2)}$$

(7.38)

Upon making this identification, it is now a straight forward procedure to solve for the perturbed wave numbers and field quantities. The dispersion relation is found from (7.36) and in this case becomes

$$b_{11}^{2} - \lambda_{n}^{2} = \langle v_n, \Delta L v_n \rangle \qquad \text{(a)}$$

$$(7.39)$$

e_z is given by;

$$e_z = e_n + \sum_{k \neq n}^{\infty} \frac{1}{b_{11}^{2} - \lambda_k^{2}} \langle v_K, \Delta L v_K \rangle \; e_K \qquad \text{(b)}$$

The advantage of the previous developments of an iteration method for determining the dispersion relations and Fourier coefficients was that it was an exact solution (within the approximations concerning the magnitude of β compared to h_o or k_o) to the problem. The disadvantage is obviously that we are left with an infinite set of equations to solve and some method of transaction clearly must be employed.

The advantage of the above perturbation method is its simplicity, but the accuracy of the method depends on the assumption that the eigenvalues and fields can be expanded in a series of the form (7.33) which is characterized by a smallness parameter, ε. This implies that the actual fields and wave number should not deviate appreciably from that of the uncoupled equation if the method is to be useful. In the next chapter the iterative equations are used to obtain solutions for the dispersion relations for the various modes of propagation. Along with some of the dispersion curves, a plot of the magnitude of the Fourier coefficients is given. When the magnitude of this coefficient is considerably less than unity the perturbation method can be expected to yield accurate results. It will be seen that this is the case for a considerable portion of the spectrum, in particular, for frequencies not too close to the cyclotron frequency. If the frequency of interest is in this region then the perturbation method will give a simple and accurate solution of the problem.

CHAPTER VIII

SOLUTIONS FOR PROPAGATION IN A CIRCULAR GUIDE

8.0 Introduction

In this chapter the iterative techniques developed in Chapter 7 will be used to obtain dispersion curves for the plasma modes. The curves presented here are obtained by numerically solving equations (7.14), (7.25) and (7.29) for the hybrid E, H and p modes respectively.

The analysis of this chapter is restricted to lowest waveguide modes having no angular variation. The geometry and coordinate system used is shown in Fig. 8.1 and the normal modes used in the expansion procedure are given in Appendix B.

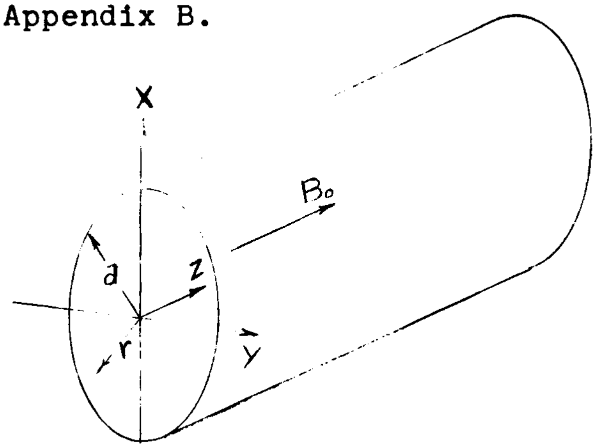

Fig. 8.1 Cylindrical Waveguide Filled with Warm Anisotropic Plasma

In order to calculate the dispersion curves and field structures for the various modes we must solve an infinite set of equations for the Fourier coefficients of the particular mode. Our ability to solve these equations depends on being able to truncate them, which in turn implies that the Fourier coefficients must fall off rapidly. Practically, it is possible to carry only the first few terms in the equations in a numerical computation since computation time increases approximately as the square of the number of coefficients. In the numerical work we have carried through the computation of three coefficients. It will later be seen that this is generally sufficient and for a large portion of the spectrum the first coefficient, corresponding to the empty waveguide mode configuration, gives an excellent description of the mode.

The iterative procedures employed are now discussed.

8.1 The Iterative Procedures

Generally, the equations that must be solved to determine the expansion coefficients and dispersion relations are of the form

$$Q_n a_n = P_{ns} a_s \qquad (8.1)$$

Two methods have been employed to solve (8.1). The first method has been discussed by Morse and Feshbach and is called the Feenberg Iteration[27]. It will be outlined below where, for concreteness, we will use (7.14) for the hybrid-E modes as a specific example.

For the E modes the terms in (8.1) are;

$$Q_n = (-\lambda_n{}^2+b_{11}{}^2) = \{-\lambda_n{}^2+k_o{}^2K_p-\beta^2(1+\frac{\ell_o{}^2\ell_B{}^2}{K_HK_\nu{}^2})\} \qquad (8.2a)$$

P_{ns} is the coefficient on the right of (7.14). It is noted that β^2 can be factored from this term so we can write in place of (8.2),

$$Q_na_n = \beta^2P'_{ns}a_s \qquad (8.3a)$$

where $\quad P_{ns} = \beta^2P'_{ns} \qquad (8.3b)$

To employ the Feenberg iteration a series of approximations are used to find higher orders of β. The value of β found is then used to compute P'_{ns} for the next order of β, etc. To illustrate, first neglect the right hand side of (8.3a) to obtain

$$\beta_{-1}^2 = \frac{k_o^{\,2}K_p-\lambda_n^{\,2}}{1+\dfrac{\ell_o^{\,2}\ell_B^{\,2}}{K_HK_\nu^{\,2}}} \tag{8.4}$$

Next, omit the off diagonal terms and using (8.4) compute the zeroth order value. Since P'_{ns} are functions of β, the value of β_{n-1} is used in the computation of β_n.

$$\beta_o^{\,2} = \frac{k_o^{\,2}K_p-\lambda_n^{\,2}}{1+\dfrac{\ell_o^{\,2}\ell_B^{\,2}}{K_HK_\nu^{\,2}}+P'_{nn}(\beta_{-1})} \tag{8.5}$$

To obtain the first order value of β let $a_n = 1$ and retain two off diagonal terms.

$$\{Q_n-P_{nn}\} = \beta^2P'_{nq}a_q$$

$$\{Q_q(\beta_o)-P_{qq}(\beta_o)\}a_q = P_{qn}(\beta_o) \tag{8.6}$$

Equation (8.6) is then solved for $\beta_1^{\,2}$.

The procedure can be continued indefinitely. As mentioned previously, the equations are carried to second order in this work.

Solutions to (8.2) will be obtained by this method only if the procedure converges. Condions for convergence are discussed by Morse and Feshbach. In practice, when the method does not converge the second order equations are used and the Poisson iteration technique is employed[28]. This method, which employees iteration on β, is much more time consuming than the first so is used only when the Feenberg method fails.

Perhaps the most striking point about the Feenberg method when actually carrying out the calculations is its simplicity and accuracy for most of the spectrum. Note that Equations (8.4)-(8.6) are simple algebraic equations which can be easily solved. The main difficulty occurs in summing the series in (7.14) etc., and in some cases, such as for the parallel plate waveguide, the series can be summed in closed form[26]. In many cases one would like to have a rapid and fairly accurate method of determining dispersion relations without tedious numerical work. The formula (8.5) is quite useful in this respect. For this reason β_o as determined from (8.5) will be plotted with some of the more accurate dispersion curves that are now presented.

A computer program, written in ALGOL, which has been used to compute the dispersion curves is presented for reference in Appendix C.

8.2 The Hybrid-E Modes

Now solutions for the modes which reduce to the E modes in an empty guide will be considered by solving (7.14) for the expansion coefficients a_n. Before discussing the solutions a simplification which can be made is discussed.

If the coefficients of the second sum in (7.14) are computed it is seen that this term is

$$S_2' = \frac{\ell_o{}^2\beta^2(e_s,w_\ell)(e_n,w_\ell)}{k_c{}^2}\sum_\ell \frac{h_o{}^2[k_p{}^2-\gamma_\ell{}^2]}{(-\gamma_\ell^2+h_p^2)(-\gamma_\ell^2+b_{22}^2)(-\gamma_\ell^2+\lambda_n^2)} \tag{8.7}$$

Here the boundary term (e_n,w_ℓ) has been factored from the sum since it does not depend on ℓ. (See Appendix B). The terms in the sum can be separated by partial fractions and simplified by using the assumption that $|h_p{}^2| \gg k_o{}^2$ or $|\beta^2|$ to give (omitting the multiplier)

$$S_2' \sim \sum_\ell \left[\frac{-h_o{}^2}{h_p{}^2(-\gamma_\ell{}^2+h_p{}^2)} + \frac{h_o{}^2(k_p{}^2-\gamma_\ell{}^2)}{h_p{}^2(-\gamma_\ell{}^2+b_{22}{}^2)(-\gamma_\ell{}^2+\lambda_n{}^2)}\right] \tag{8.8}$$

The contributions from the two terms will be greatest near resonances. With $\lambda_n \neq \gamma_\ell$ the denomenators will never become zero. (This is one advantage of including collisions. In particular, if collisions were neglected resonances with h_p could occur and would be a very sensitive function of ω and u.) Since h_p^2 is approximately $h_o^2 K_H$ and $h_o = \omega/u$ is a very large number at microwave frequencies the first term in (8.8) is negligible for small γ_ℓ and the second term is negligible for large γ_ℓ, both quantities approaching $\frac{1}{h_o^2 K_H^2}$ in absolute value in these limits.

Notice that $h_o^2/h_p^2 \gtrsim 1/K_H$. Since the temperature appears in the sound speed it is seen that the second term in (8.8) will contain no temperature dependent terms. The same argument can be made about the first sum in (7.14). Thus all the significant temperature dependence comes from the first term in (8.8).

Consider the contribution of the first term in (8.8) at a resonant point where $\gamma_\ell^2 = \text{Real } h_p^2$. This will be the dominant contribution. In this case this term contributes a value

$$S'_{21} = \frac{-1}{K_H[h_o^2 \text{Imag} K_H]} \tag{8.9}$$

Typically, h_o^2 is on the order of 10^{10} (assuming $\omega \sim 10^{10}$ and $u \sim 10^5$). Thus for any appreciable value of collision frequency the denominator in (8.9) will be a large number and the contribution of the first term in (8.8) to the sum will be very small. In fact it was found during numerical work that the effect of this term could not be seen when plotting data. In an effort to reduce computation time the sum has been omitted from the computer program and as a result no temperature dependence is shown on the dispersion curves. If one is particularly interested in the effect of temperature it is a simple matter to add the contribution of S'_{12} to the sum in the program, called S2EH in Appendix C, and investigate the temperature dependence.

With this approximation (7.14) can now be written

$$(-\lambda_n^2 K_H + b_{11}^2 K_H) a_n = \sum_s{}' a_s \left\{ \frac{\ell_o^4 \ell_B^2 k_o^2 K_p \beta^2}{(1-j\ell_v)^2} S_{1e} + \frac{\ell_o^2 \beta^2}{k_c^2} S_{2e} \right\} \qquad (8.10)$$

where

$$S_{1e} = \sum_\ell{}' \frac{(e_n, w_\ell^*)(e_s, w_\ell)}{(-\gamma_\ell^2 K_H + b_{22}^2 K_H)(-\gamma_\ell^2 + \lambda_n^2)(-\gamma_\ell^2 + \lambda_s^2)} \qquad (8.11a)$$

$$S_{2e} = \sum_\ell{}' \frac{(k_p^2 - \gamma_\ell^2)(e_s, w_\ell)(e_n, w_\ell)}{(b_{22}^2 - \gamma_\ell^2)(-\gamma_\ell^2 + \lambda_n^2)} \qquad (8.11b)$$

Equation (8.10) is solved by the method outlined in Section 8.2. The solutions are now discussed.

An attempt has been made to choose the plasma parameters for which the dispersion curves are plotted to have values which can be achieved in the laboratory. At the same time they have been chosen to exhibit the various interesting regions which occur. The plasma frequency is chosen to be 2×10^{10} which corresponds to an election gross number density $N_o = 1.26 \times 10^{17}$ particles/m^3. The electron cyclotron frequency has been taken to have values below plasma frequency, between the plasma frequency and cutoff and above cutoff. The range of magnetic field strength necessary to achieve these values is $500 < B_o < 4.5$ Kilogauss. The waveguide radius has been chosen to be 1.5 cm and gives an empty guide cutoff frequency of .76 KMC, a fairly low microwave frequency. This low value has been used since the plasma parameters can be more easily realized in this region. It is fairly difficult to achieve plasma densities much higher than the one used here or magnetic fields higher than several KG. Our main interest is in the structure of the dispersion curves. When a particular region is of interest the waveguide cutoff and cyclotron resonant locations can easily be found from (5.13b) and(5.24) and the zeroth order value of β given by (8.5) can be used to give a reasonable idea of the behavior of β.

The collision frequency has been chosen, after some experimentation, to yield curves where the resonant structure is clearly exhibited but does not vary so drastically that the behavior near cyclotron resonance is difficult to plot. High values of collision frequency wipe out the resonance completely. This behavior is illustrated by exhibiting β for one set of plasma parameters and different values of ν. The majority of the curves are plotted for $\nu = .05\ \omega_o$.

Since we are interested in the mode structure as well as the dispersion curves the second two Fourier coefficients in the field expansion are plotted. Note that since we are truncating (8.10) in three equations we effectively assume that the dispersion relation depends on the coupling of the first 3 normal modes, e_n. It was pointed out earlier that the modes can be written as a slowly spatial varying quantity and a term proportional to the pressure term ϕ'. (See (6.15)). The slowly varying term is thus represented by

$$e_z' = \sum_{n=1}^{3} a_n e_n \tag{8.12a}$$

and the rapidly varying pressure-excited term is

$$e_z'' = \frac{-b_{13}\phi'}{h_p^2 - b_{11}^2} \tag{8.12b}$$

It is implicitely assumed that the contribution from (8.12b) has negligible effect on the dispersion reduction. (This assumption is made when the term in the sum S_2' containing the temperature was dropped) However, once β is found it is easy to plot the fields since ϕ' can be found in closed form. The total axial [redacted]f(8.12a) and(8.12b), the field h_z is given by (6.10a) and ϕ' is found by solving (6.7c). The field structure for e_z, h_z and ϕ will be shown for one of the dispersion curves.

Now consider the dispersion curves for the hybrid-E modes plotted in Figures 8.4-8.8. These curves are plotted for $\omega_o = 2 \times 10^{10}$ and the cyclotron frequency is varied.

It was previously stated that a good approximation to β could be found by using (8.5) with the Feenburg iteration procedure. To illustrate this point β_o has been plotted in Figures (8.2) and (8.3) and should be compared with the more accurate accompanying curves. It is seen that β_o approximates the more accurate expression for β quite closely except for a region near cyclotron resonance. The results obtained near ω_B have been plotted as a number of disjointed points.

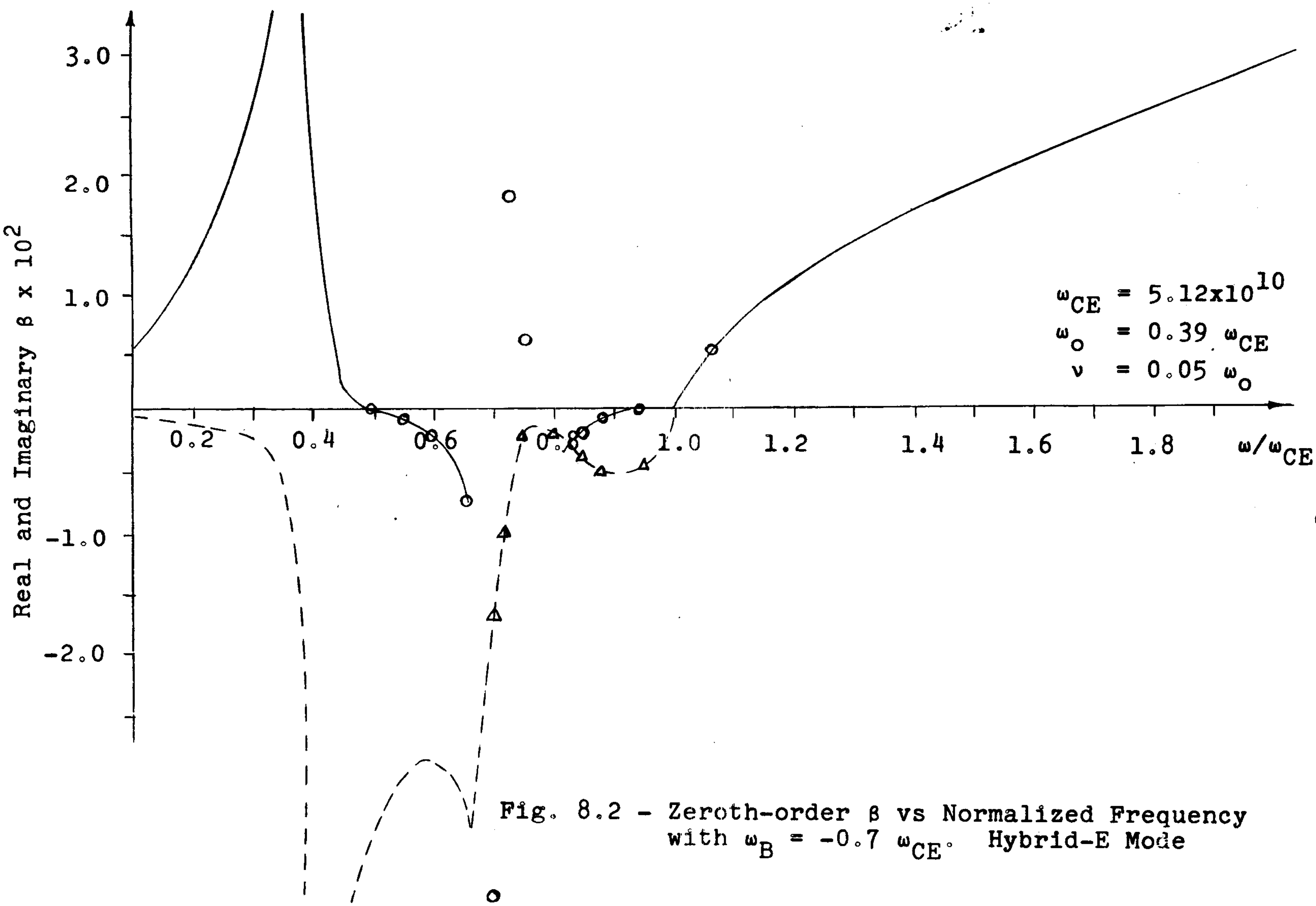

Fig. 8.2 - Zeroth-order β vs Normalized Frequency with $\omega_B = -0.7\ \omega_{CE}$. Hybrid-E Mode

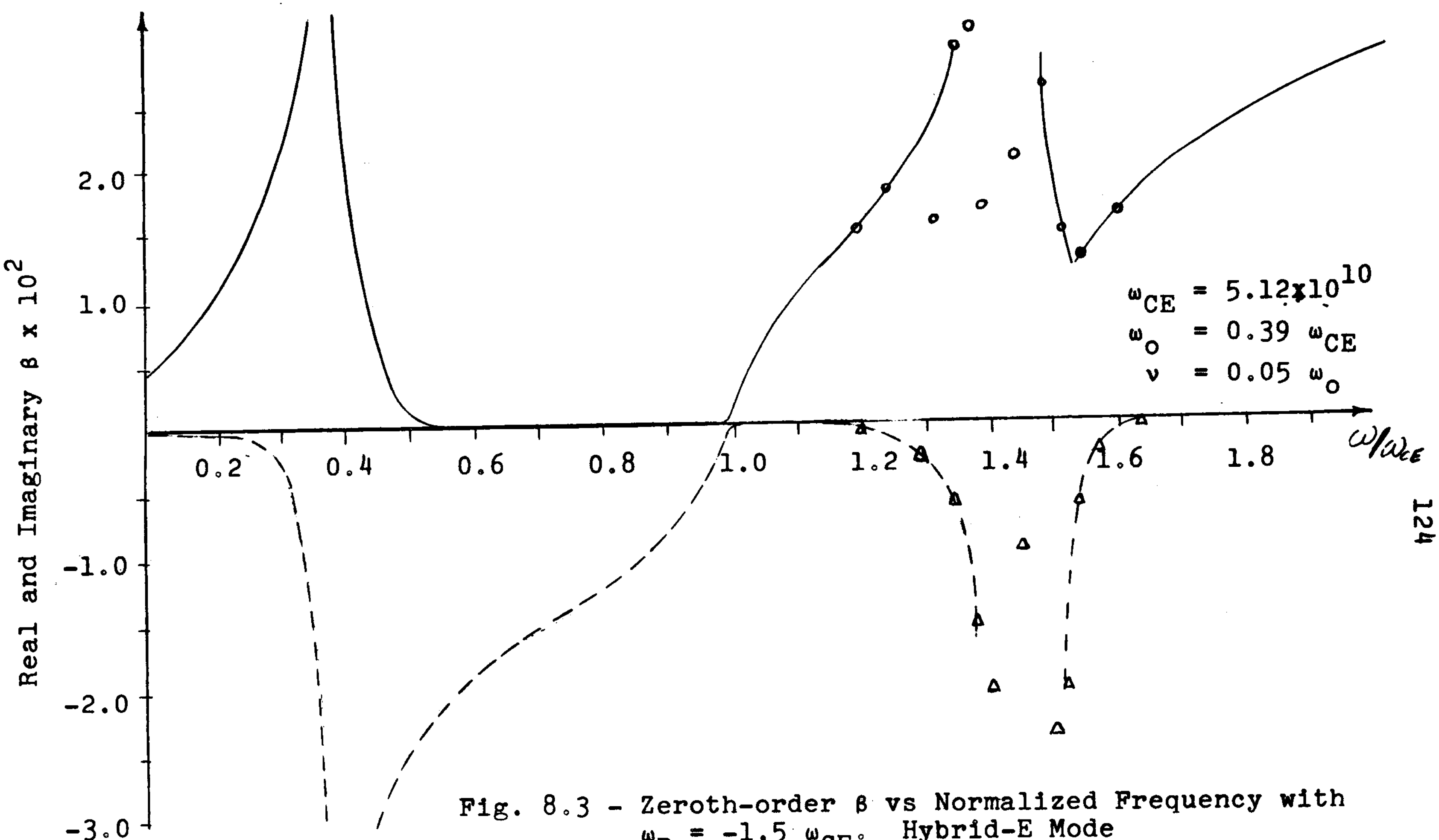

Fig. 8.3 - Zeroth-order β vs Normalized Frequency with $\omega_B = -1.5\ \omega_{CE}$. Hybrid-E Mode

From the behavior of these curves near ω_B it may be anticipated that the Feenburg iteration will not converge near the cyclotron frequency. This is the case and to find β it is necessary to employ a root searching method such as Poisson's iteration formula. The difficulty can be found by examining the magnitude of the Fourier coefficients shown with the dispersion curves. Near ω_B the higher spatial harmonics are strongly excited. These terms are neglected in computing β_o. Also, if the uncoupled electric modes are examined it is seen that they all have resonances near ω_B. Thus, near the cyclotron frequency the system of coupled equations behaves somewhat like a set of coupled oscillators, each component of which is tuned to the same frequency.

Now we digress slightly and consider the solution of a simple coupled oscillator circuit where the two tuned circuits are tuned to the same frequency. The behavior of this circuit sheds some light on the cause of some of the dispersion characteristics found from the coupled mode theory.

Consider the circuit shown in Fig. 8.4. Assume the two resonant circuits are tuned to the same frequency, ω_1. The behavior of the response curve is well known[31] and is plotted in Figure 8.5 where a is a parameter which typifies the coupling between the circuits

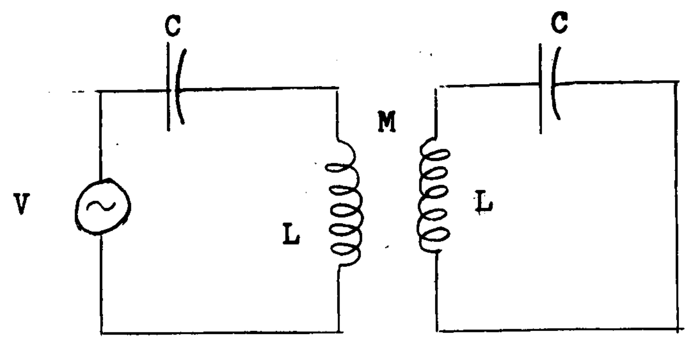

Fig. 8.4 - The Double Resonant Circuit

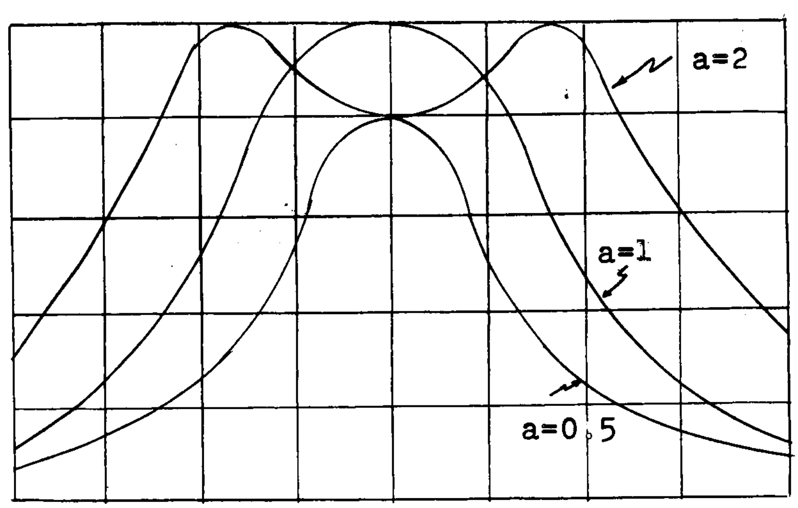

Fig. 8.5 - Response of the Double Resonant Circuit for Different Values of Coupling

$a = 1$ is the condition of critical coupling, $a > 1$ overcoupling and $a < 1$ undercoupling.

The coupling between modes in our equations is quite complicated and generally the coupled oscillator analogy is not exact since we are dealing with eigenvalue equations. However, the analogy does make plausible some of the dispersion curves that arise.

The dispersion curves for the E modes for different values of ω_B are shown in Figures 8.4-8.8. Note the double resonances that appear near the cyclotron frequency in some of the curves. Since the second harmonic, e_2, is excited quite strongly near the cyclotron frequency it appears that the presence of this mode gives use to a double resonance similar to $a = 2$ in Fig. 8.10.

The magnitudes of the higher Fourier coefficients (with $a_1 = 1$) are plotted on a semi-log scale with each dispersion curve. In each case it is noted that the higher harmonics are most strongly excited near cyclotron resonance and are excited somewhat less strongly at plasma resonance. In all cases coupling virtually vanishes near cutoff. This must be the case since it was seen previously (Chapter 5) that the equations decouple when $\beta = 0$.

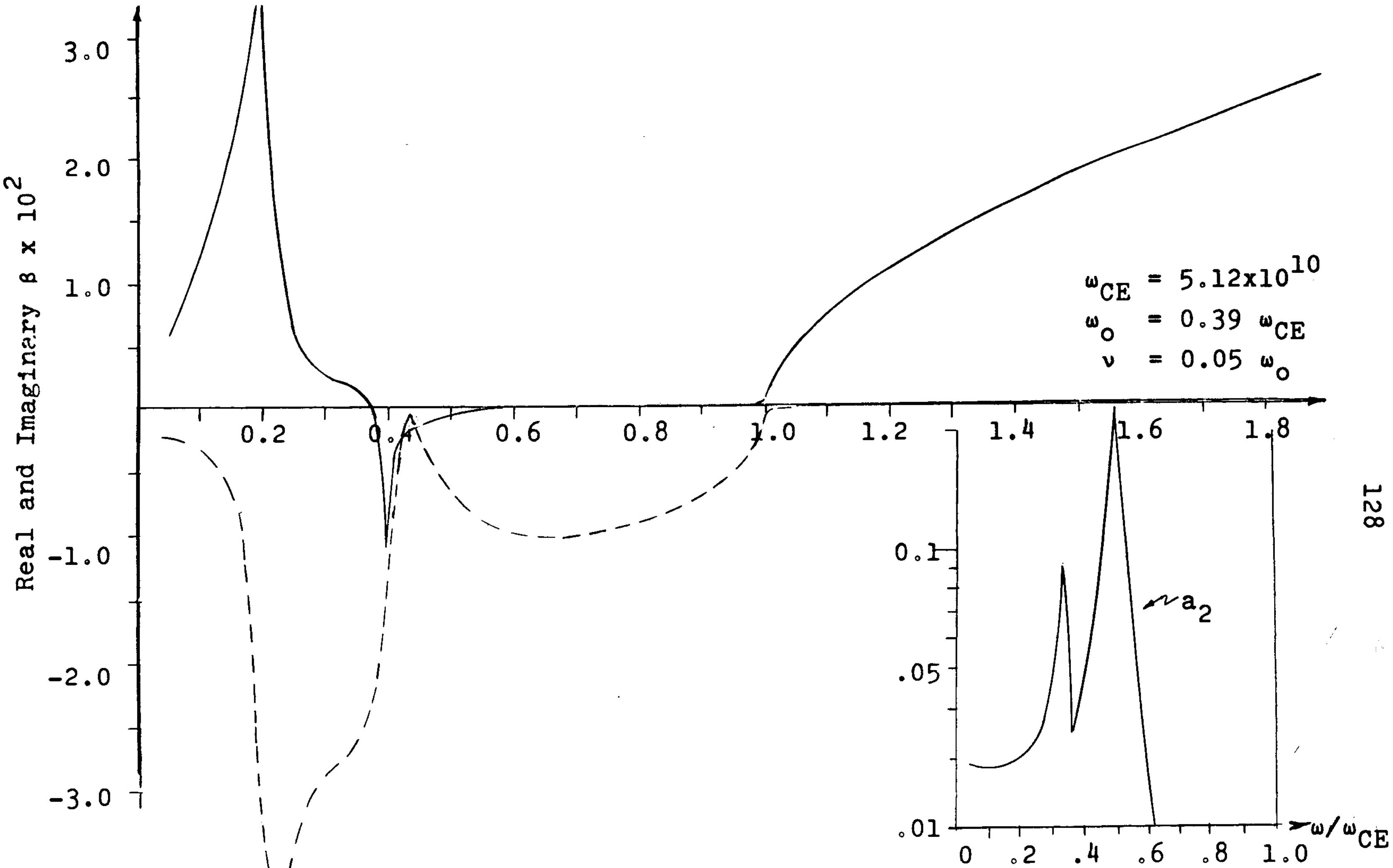

Fig. 8.6 - β and the second Fourier coefficient of e_z vs normalized ω with $\omega_B = -0.2\ \omega_{CE}$ Hybrid-E mode

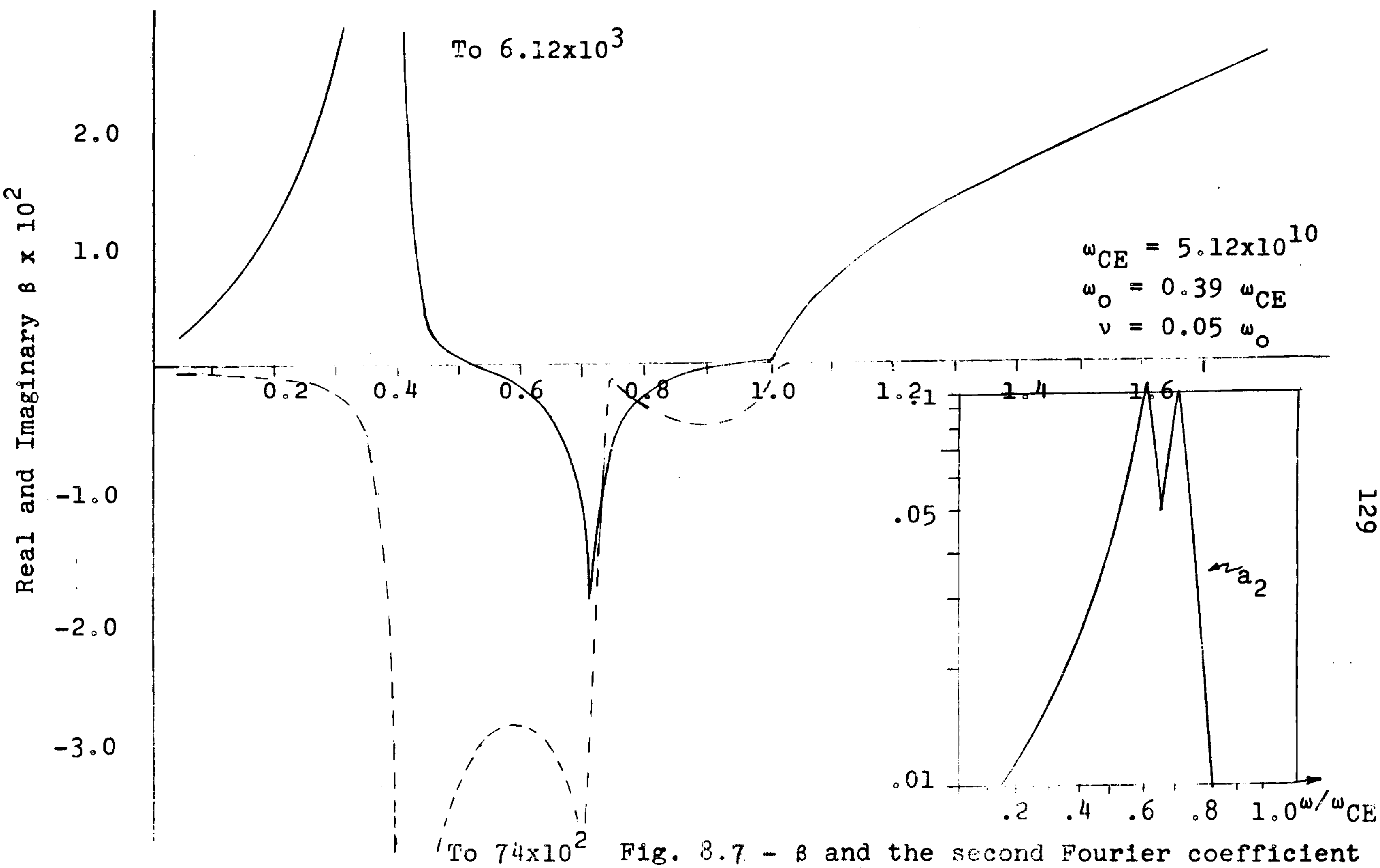

Fig. 8.7 - β and the second Fourier coefficient of e_z vs normalized ω with $\omega_B = -0.7\ \omega_{CE}$

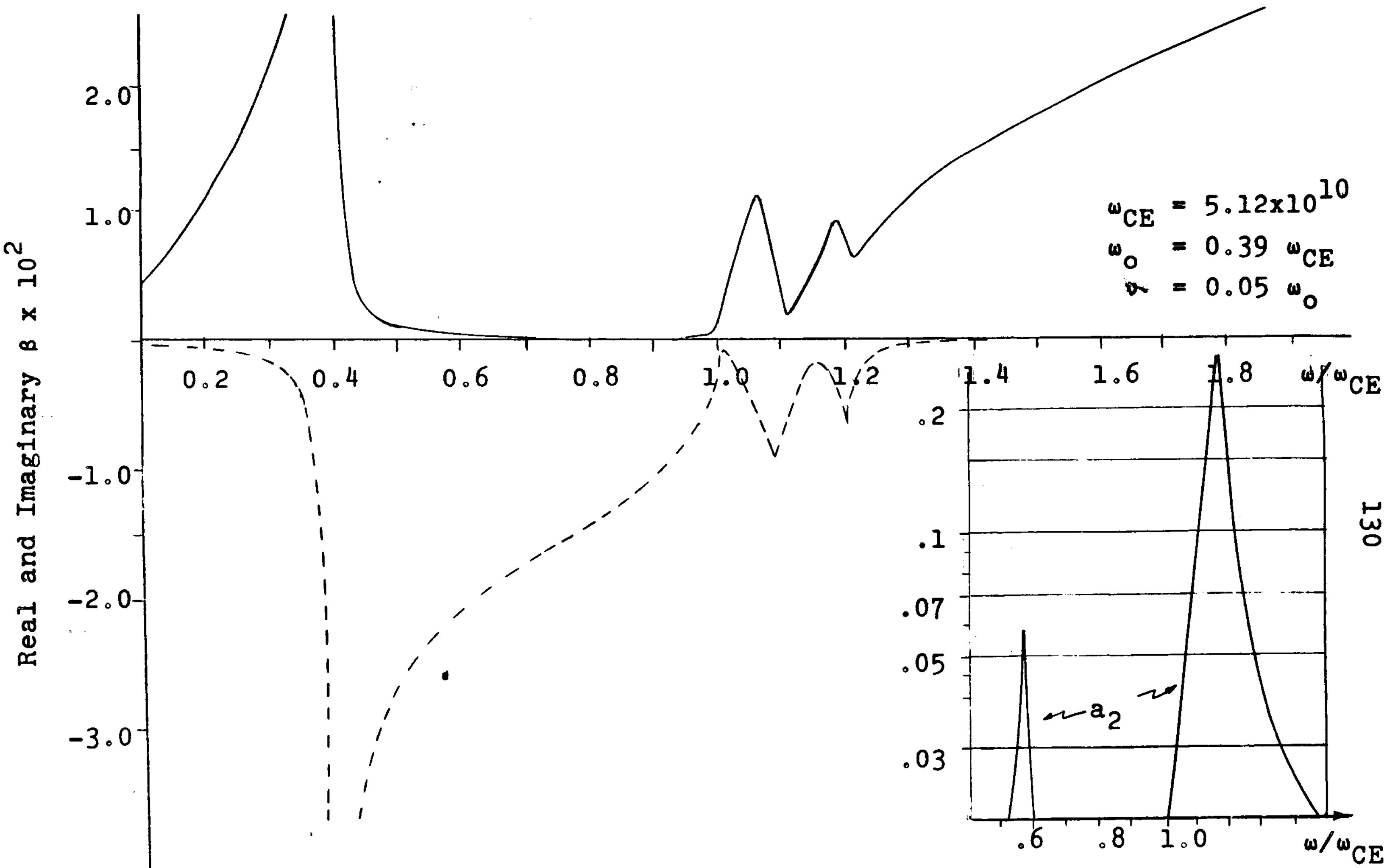

Fig. 8.8 - β and the second Fourier coefficient of $e_{\bar{z}}$ vs. normalized frequency with $\omega_B = -1.15\ \omega_{CE}$. Hybrid-E Modes

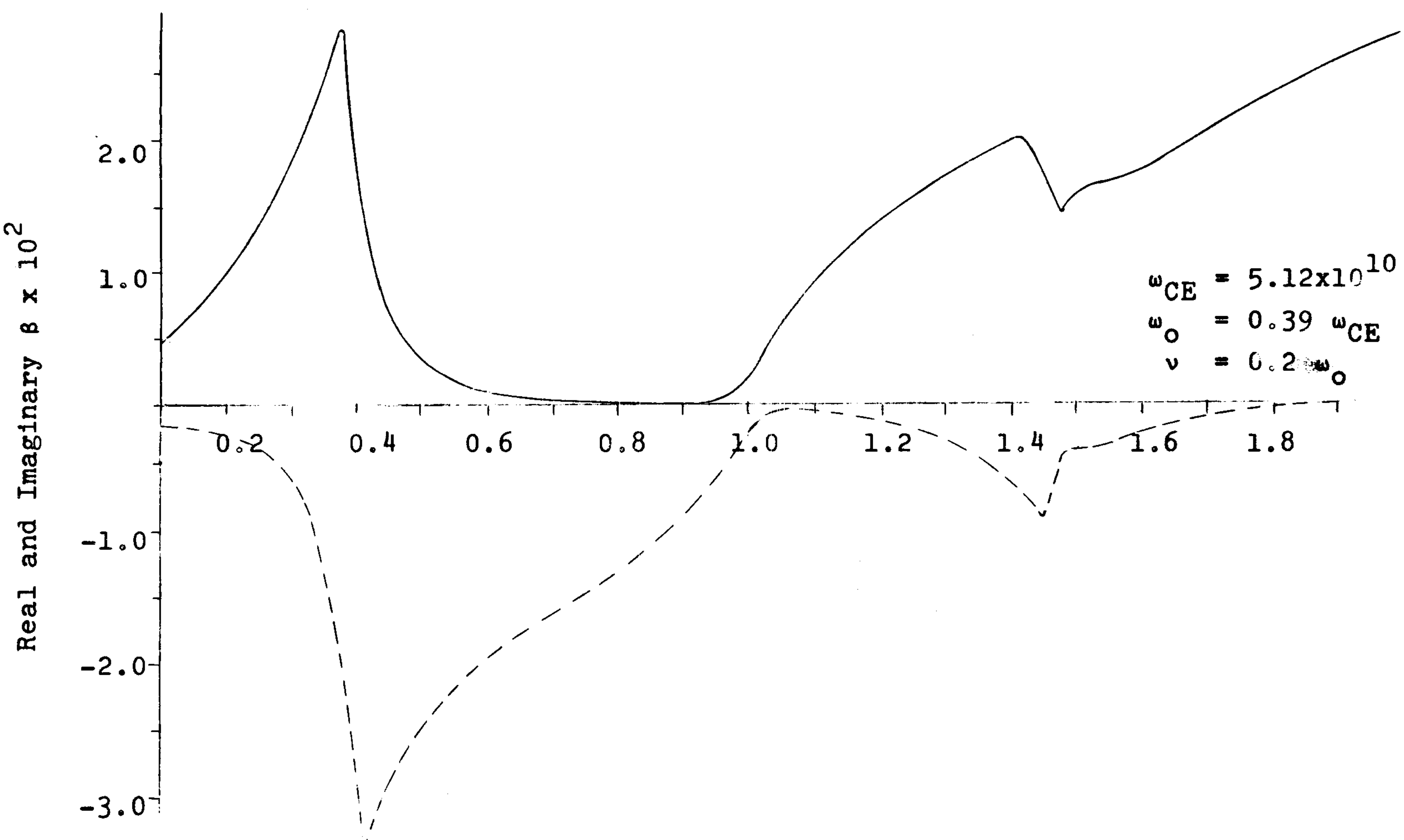

Fig. 8.9a - β vs normalized ω with $\omega_B = -1.5\ \omega_{CE}$. Hybrid-E modes

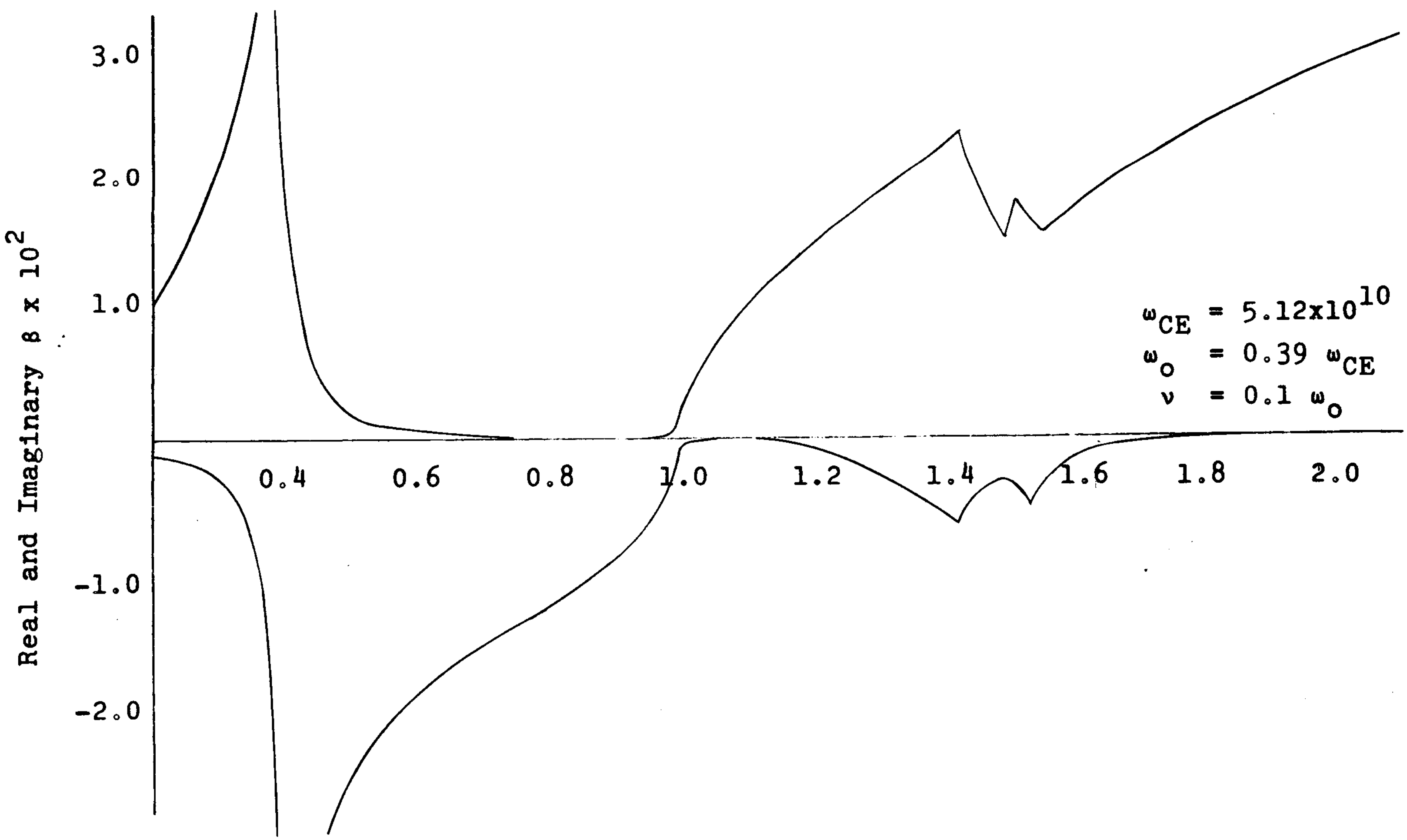

Fig. 8.9b - β vs Normalized ω with $\omega_B = -1.5\ \omega_{CE}$
Hybrid-E Mode

An interesting set of curves are obtained when the cyclotron frequency is smaller than cutoff. For an empty waveguide propagation below cutoff is, of course, not possible. In all cases it is seen that a wave can propagate below the plasma frequency, although in practice the wave may be highly damped.

The dispersion relations for $\omega_B = -1.5\ \omega_c$ has been plotted for three values of collision frequency and are shown in Figures 8.7a and 8.8. The effect of increasing collisions is to considerably lessen the cyclotron resonance, and to a lesser extent decrease magnitude of the plasma resonance.

Note also the similarity between the dispersion curves for the warm plasma model and those presented for the cold plasma model in Chapter 2. Of course, the behavior of β vs ω for the two models should not be expected to differ greatly since the equations obtained in Chapter 6 were very similar to those used in Chapter 2 to describe the cold plasma.

Once the dispersion characteristics of a particular mode have been found the functional form of the e_z, h_z and ϕ forms can be obtained by computing these fields from their spectral representations. It would be far to space-consuming to plot the field structures for all of the modes so only the e_z and h_z fields for the hybrid-E mode with $\omega_B = 1.5$ and $\nu = .05\ \omega_o$ have been plotted. The magnitudes of these fields as a function of frequency is shown in Figures 8.1[illegible]c and 8.1[illegible]d. The magnitude of the pressure mode has not been plotted since the functional form of this field is rather difficult to plot. Instead a table of values of the components of the normalized pressure is included below. To understand this table remember that, for the hybrid-E modes, ϕ was separated into two terms as expressed by (6.5). These are

$$\phi = \phi_1 + \phi' \qquad \text{(a)}$$

$$(8.13)$$

where

$$\phi_1 = \frac{a_{31}}{a_{33}{}^2} e_z + \frac{a_{32}}{a_{33}{}^2} h_z \qquad \text{(b)}$$

and ϕ' is a solution of (6.6a). The solution of 6.6a) was given by (6.22a) and in this case is well approximated by (6.23). i.e.,

$$\phi'(r) = Ae^{\xi'(r-a)} \left\{ \frac{\cos n_1 + \sin n_1 + j(\cos n_1 - \sin n_1)}{\cos n_o - \sin n_o - j(\cos n_o + \sin n_o)} \right\} \qquad (8.14)$$

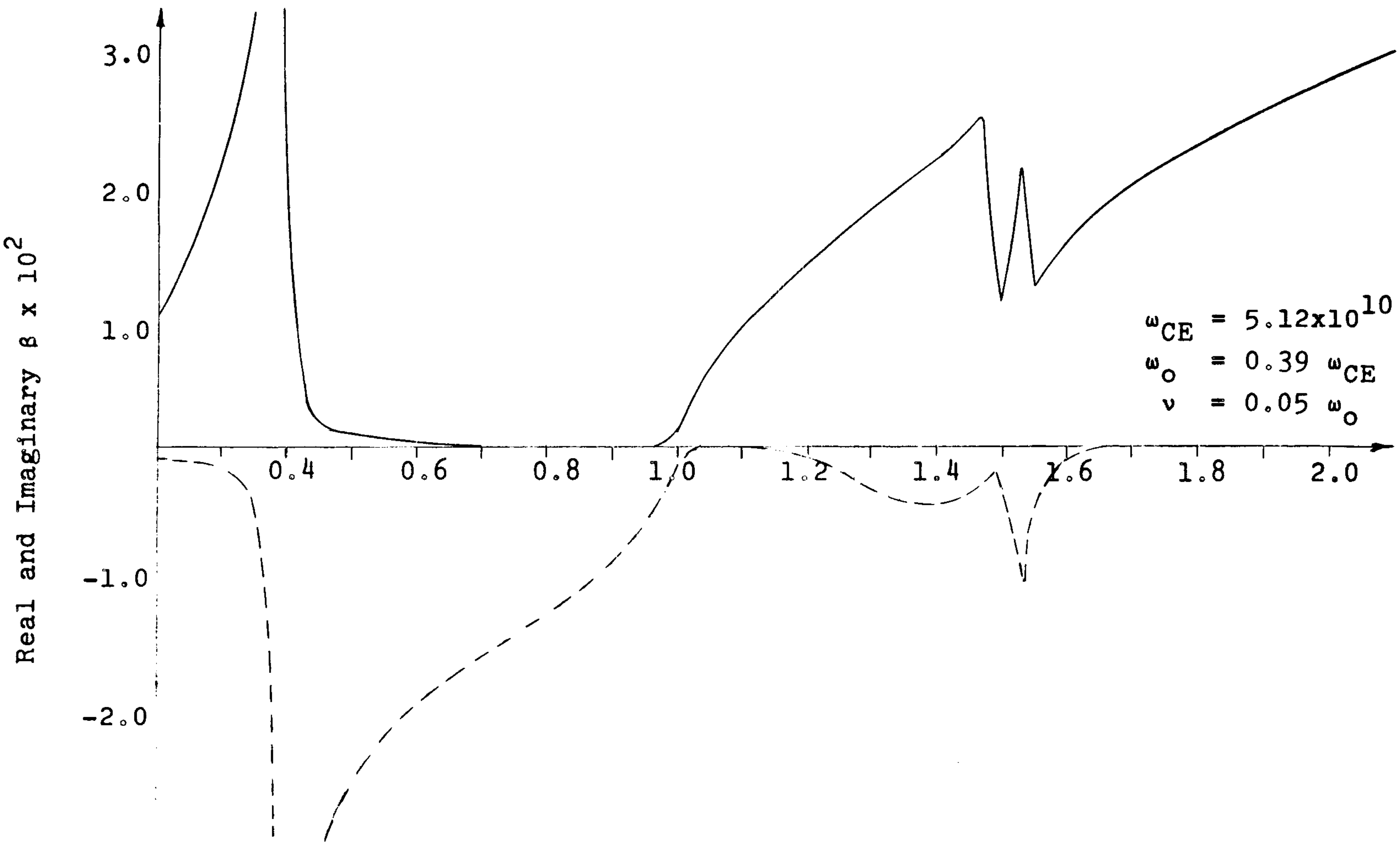

Fig. 8.10a- β vs normalized ω with $\omega_B = -1.5\ \omega_{CE}$
Hybrid-E Mode

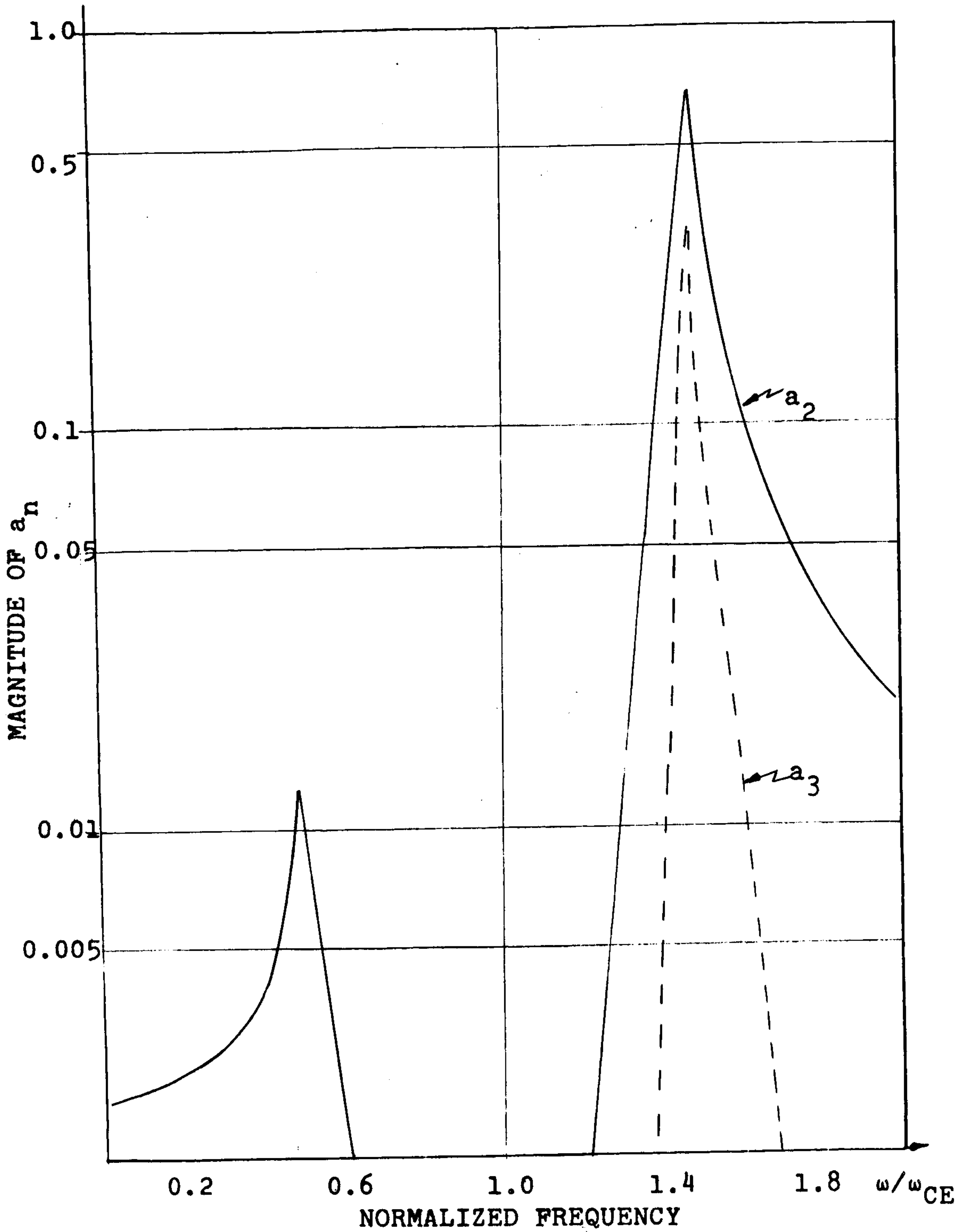

Fig.8.10b- Higher Order Expansion Coefficient for e_z for the Hybrid-E Mode. $\omega_o = 0.39\ \omega_{CE}$, $\omega_B = -1.5\ \omega_{CE}$, $\nu = 0.05\ \omega_o^o$. $a_1 = 1$.

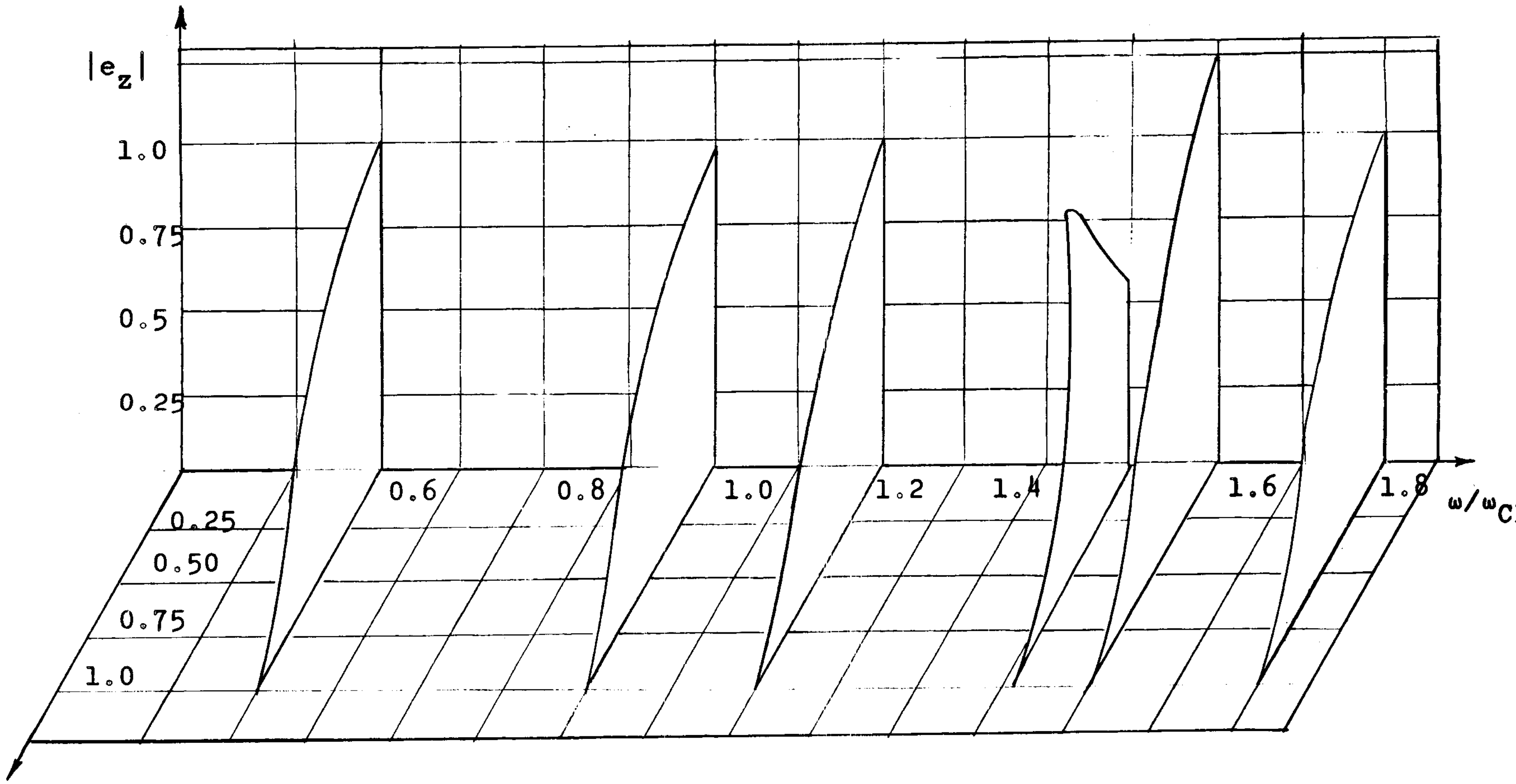

Fig. 8.10c - Functional Form of e_z vs Normalized Frequency for Hybrid-E Modes with $\omega_B = -1.5\ \omega_{CE}$ and $\nu = 0.05\ \omega_o$

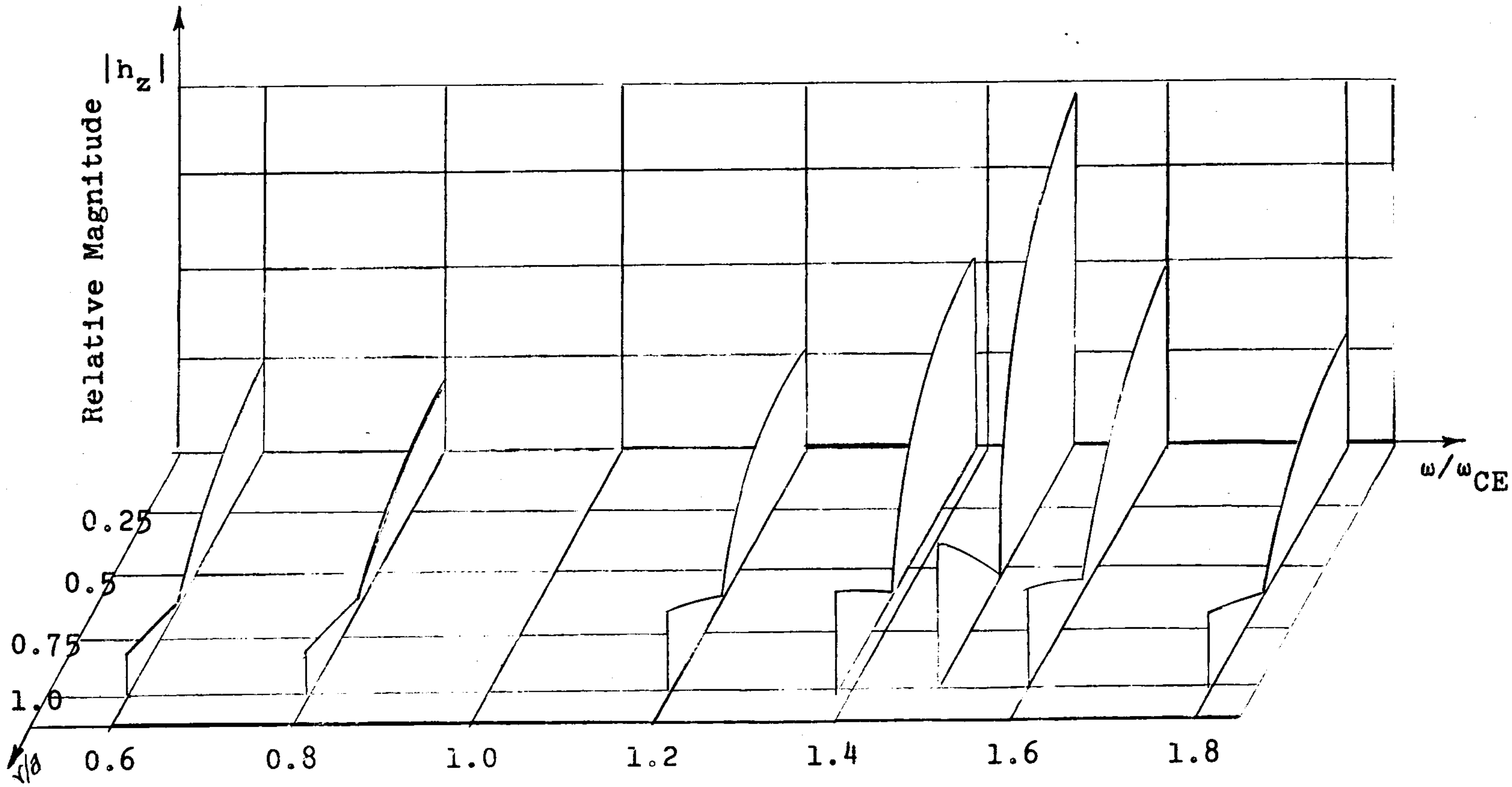

Fig.8.1[illegible]d - Functional Form of h_z vs Normalized Frequency for Hybrid-E Mode with $\omega_B = -1.5\ \omega_{CE}$ and $\nu = 0.05\ \omega_o$

ω/ω_{CE}	$\lvert\phi'\rvert$			Maximum value of $\phi_1(r)$
	$A = \frac{\alpha}{\lvert h_p \rvert}$	n_1	ξ'	
2.0	$1.7x10^2$	$2.16x10^5$	$4.1x10^3$	$7.6x10^{-4}$
1.8	$1.61x10^2$	$1.56x10^5$	$5.5x10^3$	$1.1x10^{-3}$
1.53	$5.0x10^1$	$1.9x10^4$	$4.5x10^4$	$2.6x10^{-2}$
1.36	$2.1x10^2$	$6.7x10^3$	$1.27x10^5$	$5.8x10^{-3}$
1.0	4.5	$4.6x10^3$	$2.0x10^5$	$1.22x10^{-4}$
0.8	$2.5x10^1$	$5.1x10^3$	$2.4x10^5$	$1.3x10^{-3}$
0.4	5.8	$6.7x10^3$	$2.6x10^5$	$9.4x10^{-3}$
0.2	$2.0x10^1$	$1.2x10^4$	$2.6x10^5$	$2.9x10^{-3}$

Table 8.1 - Components of ϕ as a function of Frequency for the Hybrid-E Modes with $\omega_B = -1.5\ \omega_{CE}$ and $\nu = 0.05\ \omega_o$

where we have defined $A = \frac{a}{|h_p|}$, n_1 = Real$|h_p|r$, n_o = Real$|h_p|a$, and ξ' = Imag$|h_p|$. a is defined by (6.22b).

Evidently, ϕ' is a function which behaves as a damped sinusoid. Thus a good idea of its form can be obtained by tabulating the magnitudes of the damping factor, ξ' = Imag$|h_p|$, the oscillatory factor, Real $|h_p|$ and the amplitude, A. The values of these factors along with the magnitude of ϕ_1 as a function of radius for various values of the normalized frequency are tabulated in Table 8.1. Note in particular that the magnitude of ϕ' is much larger than the magnitude of ϕ_1. This was the initial assumption used in deriving equations for the hybrid E and H modes in Chapter 6.

8.3 Dispersion Curves for the Hybrid-H and Hybrid-pressure Modes

To complete this chapter some dispersion curves for the hybrid-H and p modes are now shown. These curves were obtained by the procedure outlined in Section 8.1 and essentially the same approximations were used to simplify the sums. Equation (7.24) was used to find the coefficients of h_z and (7.28) was used to find ϕ_o.

Consider the curves shown in Figures 8.11 and 8.12. These curves were obtained from (7.24) by the iterative technique discussed previously. Note that these curves behave in a similar manner to the corresponding curves

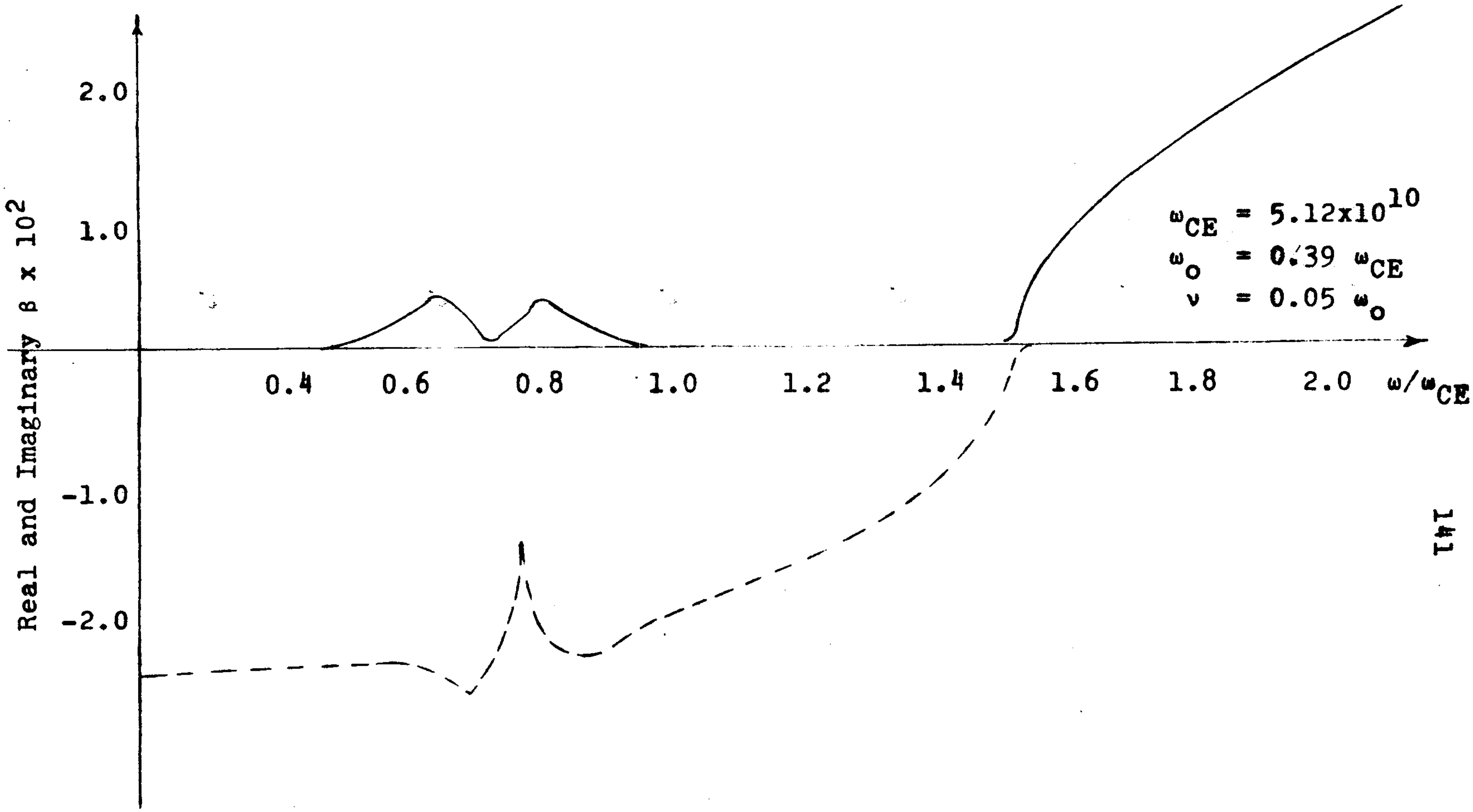

Fig. 8.11 - β vs Normalized Frequency with
$\omega_B = -0.7\ \omega_{CE}$. Hybrid-H Mode.

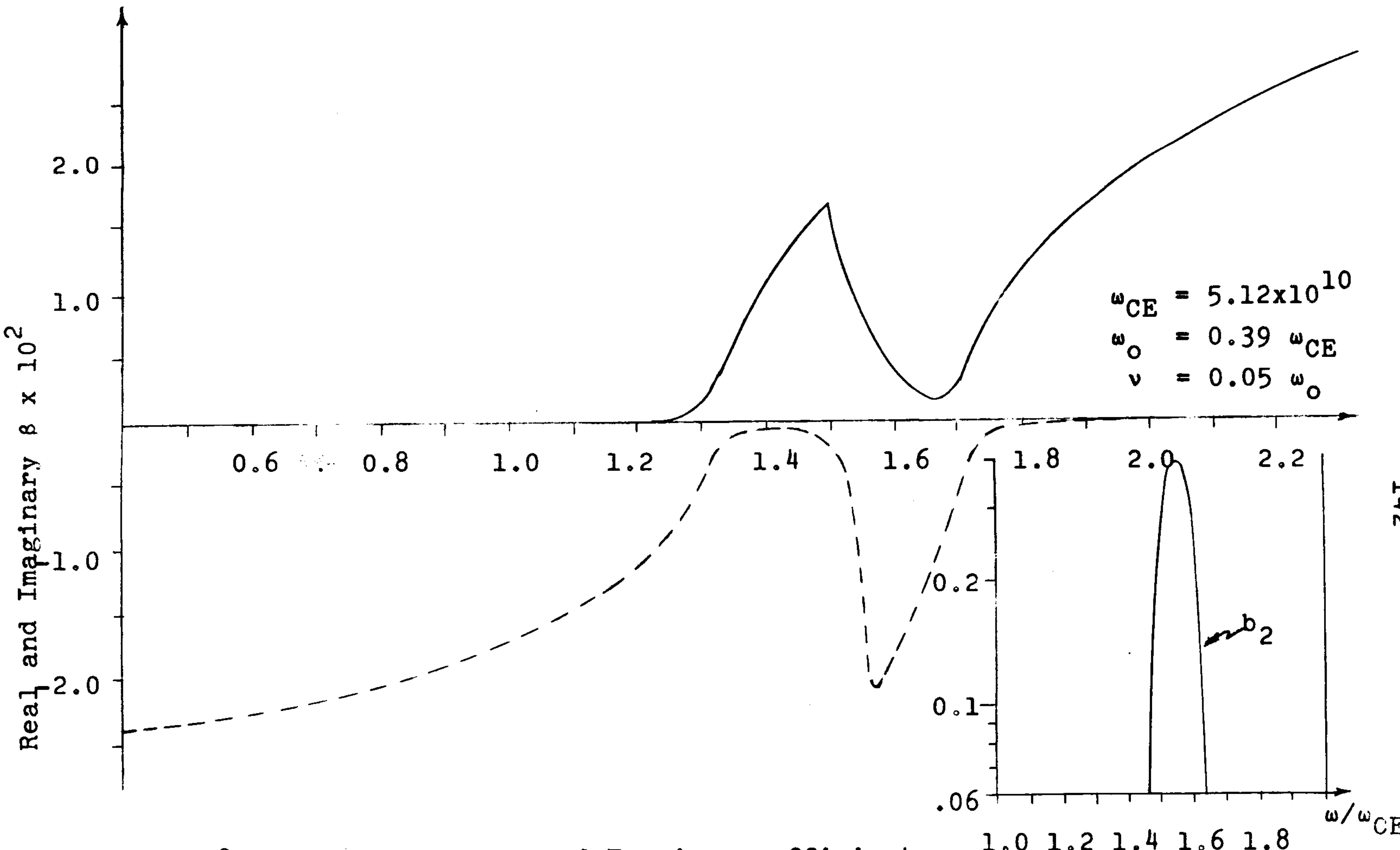

Fig. 8.13 - β and the second Fourier coefficient or h_z vs normalized frequency with $\omega_B = 1.5\ \omega_{CE}$. Hybrid-H modes.

obtained for the cold plasma model and shown in Figures 2.4 and 2.5. In particular, the cutoff is increased as ω_B increases and no resonant behavior occurs near the plasma frequency.

Finally, consider the dispersion curves for the pressure modes, Figures 8.13 and 8.14. Note that the scale has been changed here.

The pressure modes, of course, are not present when the cold plasma model is used to describe the plasma. They are introduced only if the electron gas is assumed to be compressible. The extent to which their neglect effects the solutions of the electromagnetic modes was discussed in Chapter 6 where it was seen that inclusion of the pressure effectively changed the boundary conditions on the cold plasma equations.

The other way in which inclusion of compressibility changes the solutions is, of course, the addition of the pressure mode to the solutions. In any practical experiment the question of whether a pressure mode can be seen depends to a large extent on the attenuation of the mode. Note that, for the values of parameters chosen here, the imaginary part of β is always quite large. The inclusion of collisions is seen to smooth any resonant behavior around the cyclotron frequency to such an extent

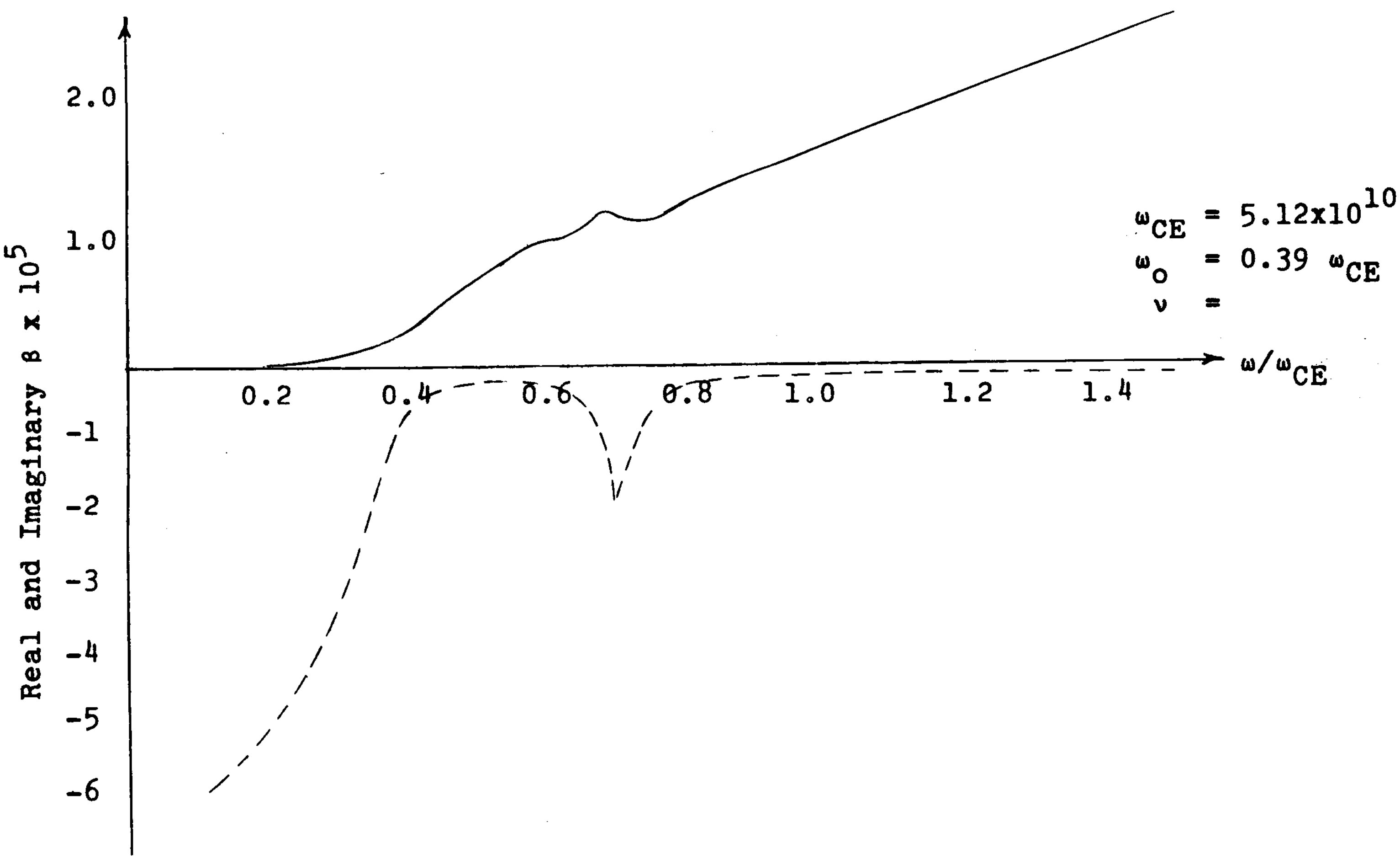

Fig. 8.13 - β vs normalized ω with $\omega_B = -0.7\ \omega_{CE}$
Hybrid-p modes

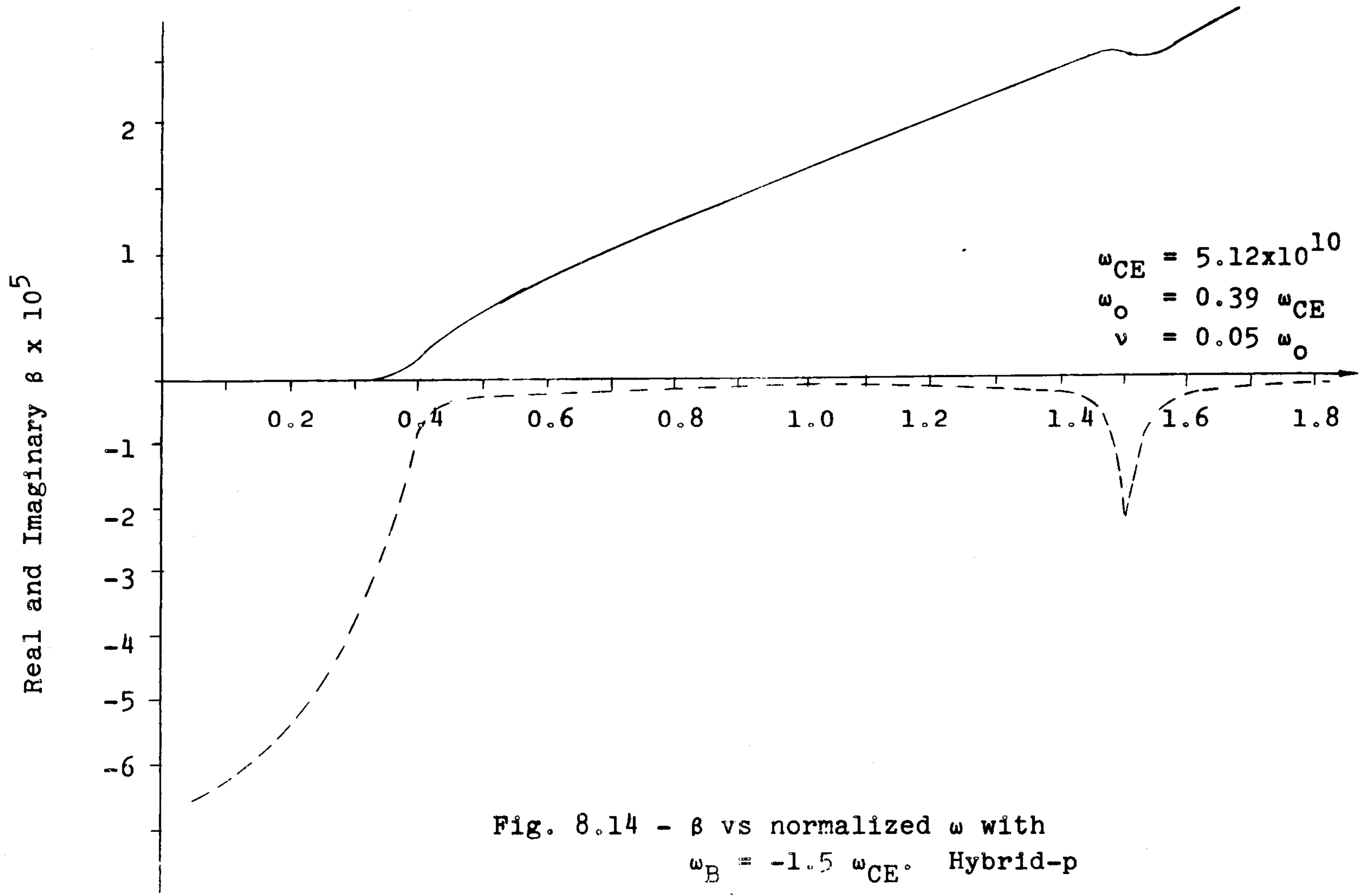

Fig. 8.14 - β vs normalized ω with $\omega_B = -1.5\ \omega_{CE}$. Hybrid-p mode

that the resonance is almost completely obliterated. The more important point is that the attenuation is so large that a pressure wave with the dispersion characteristics shown above could probably not be observed in any experiment.

CHAPTER IX

CONCLUSIONS

9.1 Summary of the Work

The main objective of this work was to obtain solutions for the normal modes of propagation in a warm bounded plasma. In Chapters 3 and 4 it was shown that, for the cases of a drifting uniform plasma and a stationary, non-uniform plasma, all the fields could be found in terms of the axial electric and magnetic fields and the pressure. These fields were called the potentials for the problem.

Generally it was found that the coupled differential equations satisfied by the potentials were too difficult to be solved unless it was assumed that the plasma was stationary and uniform. This assumption was therefore made and, in this limit, the coupled potential equations reduced to the coupled Helmholtz equations discussed by Sancer[14]. Before these equations could be solved it was necessary to find appropriate boundary conditions for e_z, h_z and ϕ. These were derived, in Chapter 5, from the physical requirements that the tangential electric fields and normal component of the velocity vanish at the boundary. In the general case it was found that the boundary conditions on h_z and ϕ were coupled, as shown by (5.4). However, if we restricted the analysis to modes having no tangential variations the boundary conditions on h_z and ϕ decouple and simplify the analysis considerably.

The solutions for the potentials was then considered in Chapter 6. The analysis was temporarily restricted to two of the three possible modes, those for which $|\beta|$ was much less than ω/u. With this assumption it was shown that the coupled equations could be greatly simplified and, in fact, were very similar to the equations that describe the potential when the cold plasma model is used to describe the plasma. It was for this reason that the cold plasma equations were presented in Chapter 2. With the equations in a form so close to the cold plasma equations it was relatively easy to examine the relation between the cold plasma equations and the warm plasma equations in the zero temperature limit. We concluded that the solutions were not identical in the limit $T \to 0$.

With the number of coupled Helmholz equations reduced from three to two we could have proceeded as in Chapter 2 to obtain solutions by diagonalizing the equations and obtaining solutions to the complicated boundary equations. Instead, an iterative method of solution was developed to obtain an exact solution of the problem in terms of an infinite set of equations for the Fourier coefficients of the fields. This method was developed in Chapter 7 for solutions of the hybrid-E, H and p modes, where the designation E, H and p indicate that the hybrid modes reduce to the pure modes in the high frequency or small coupling limit. Chapter 7 was concluded with a brief discussion of perturbation solutions for the problem.

In Chapter 8 a method for solving the infinite set of equations for the Fourier coefficients was discussed. A computer program, presented in Appendix C, was written to carry out this procedure and was found to work very well, except when the frequency was very near the cyclotron frequency. A number of dispersion curves was then shown and the field structure of the potential was presented for one value of parameters.

9.2 Discussion of the Results

Probably the most interesting and useful development in this work is the reduction of the complicated warm plasma equations to a simpler set of equations very similar in form to those used in the cold plasma analysis. With the solutions to the problem available it is possible to reevaluate the assumptions that were made in Chapter 6 when the reduced equations were derived.

These were that $|\beta|$ should be much less than $h_o K_H$ and that e_z and h_z were very accurately represented by the first few terms in their spectral representations. An examination of the dispersion curves and magnitude of the fields shows that this is indeed the case.

In retrospect it should come as no surprise that such a reduction of the equations is possible if it is remembered that the linearized fluid equations used to describe the warm plasma included just an additional moment of the Boltzman equation. However, at the beginning of this work a method of realizing this reduction was by no means obvious.

In the course of this work a number of approximations and assumptions were made in order to obtain a set of equations that could be solved. The restriction of the solutions to modes having no tangential variation was not actually necessary, but was done to simplify the analysis. The restriction that the plasma be uniform and stationary was much more essential since without this restriction the differential equations become much more complicated than the Helmholz equations we had to solve. In some cases, where drifts or non-uniformities are not large, it should be expected that solutions may omission simplifies the problem. Often it is possible to estimate the magnitude of the error involved in dropping terms and in these cases one may justify, a posteriori, the original equations. (A very complete treatment of the derivation of various moments of the Boltzman equation and the approximations inherent in their truncation can be found in Tanenbaum, Ref. 14)

In the final analysis, the degree to which a particular model accurately describes the physical phenomena must be determined by experiment. Unfortunately many of the wave propagation experiments which have been performed have been in geometries which are not easily analyzed and experimental data pertaining to this work is not available. One very successful application of the warm, uniform, isotropic plasma equations has recently been published by Kolettis. He finds that, by defining an effective plasma frequency for a non-uniform plasma column, the theory and experiment agree very closely. It is expected that the same procedure could be applied to our results. Note that, in order to compare experimental results with the theoretical analysis, it is necessary to insure that the proper mode is excited. In particular the dispersion curves for modes having no angular variation must be excited.

Finally, it is noted that the equations we have used here may apply to a wider class of physical phenomena than the gaseous plasma. The equations sometimes used to describe waves in solid state plasmas are very similar to those used here and experimentation in this field may be very rewarding.[30]

APPENDIX A

GREEN'S FUNCTIONS AND COUPLED MODE THEORY

A.1 Green's Functions

In this appendix several general relations which are useful in obtaining solutions to coupled differential equations will be derived by considering solutions in terms of appropriate Green's functions. First the solution of one dimensional inhomogeneous Sturm-Liouville equations will be reviewed.[32]

The most general inhomogeneous equation we will consider is of the form

$$\frac{d}{dx} p(x) \frac{d\Psi}{dx} + [q(x) + \lambda^2\sigma(x)]\Psi = -\sigma(x)f(x) \tag{A.1}$$

Let $G(x|x_o)$ be a Green's function which is a solution of

$$\frac{d}{dx} p \frac{dG}{dx} + (q + \lambda^2\sigma)G = -\delta(x-x_o) \tag{A.2}$$

Solutions to (A.1) can be constructed from the Green's functions as follows. Replace x by x_o , multiply (A.1) by G , (A.2) by Ψ , subtract and integrate to obtain

$$\Psi(x) = \int_0^a \sigma(x_0)G(x_0|x)f(x_0)dx_0 + \left[G(x_0|x)p(x_0)\frac{d\Psi}{dx_0} - \Psi(x_0)p(x_0)\frac{dG}{dx_0}\right]_0^a \qquad (A.3)$$

The boundary conditions on G must be chosen to eliminate any unknown quantities in the evaluation of (A.3) . Suitable conditions for different boundary conditions on Ψ are discussed in Morse and Feshbach.

A.2 The Scalar Product

The scalar product between functions satisfying Sturm-Liouville equations will be used frequently in the text. It is evaluated here for reference.

Consider the scalar product of two functions, Ψ_n and Φ_m which satisfy

$$\frac{d}{dx} p \frac{d\Psi_n}{dx} + [q + \lambda_n^2\sigma]\Psi_n = 0 \qquad (a)$$

$$\frac{d}{dx} p \frac{d\Phi_m}{dx} + [q + \gamma_m^2]\Phi_m = 0 \qquad (b) \qquad (A.4)$$

The scalar product is defined to be

$$< \Psi_n, \Phi_m > = \int_{a_1}^{a_2} \sigma\, \Psi_n\Phi_m dx \qquad (A.5)$$

Here the bracket (or bra and ket) notation is used to signify the scalar product, after Friedman[23].

Combining (A.4) as before, and integrating we obtain;

$$(\lambda_n^2 - \gamma_m^2)\int \sigma\, \Psi_n \Phi_m dx + \left[p\Phi_m \frac{d\Psi_n}{dx} - \Psi_n \frac{d\Phi_m}{dx}\right]_{a_1}^{a_2} = 0 \tag{A.6}$$

It is convenient to introduce a notation to signify the evaluation of the boundary terms. Define

$$(\Psi_n, \Phi_m) \equiv \left[p\left(\Phi_m \frac{d\Psi_n}{dx} - \Psi_n \frac{d\Phi_m}{dx}\right)\right]_{a_1}^{a_2} \tag{A.7}$$

Thus

$$< \Psi_n, \Phi_m > = \frac{(\Psi_n, \Phi_m)}{\gamma_m^2 - \lambda_n^2} \tag{A.8}$$

A.3 Coupled Wave Solutions

The above derivations are quite well known. Now consider some relations which are of particular value in obtaining solutions to coupled equations. First suppose that the function $f(x)$ appearing in (A.1) is a solution of a homogeneous Sturm-Liouville equation.

$$\frac{d}{dx} p \frac{df_o}{dx} + (q + \sigma\, \lambda_o^2) f_o = 0 \tag{A.9}$$

In this case the integral in (A.3) can be evaluated by combining (A.2) and (A.9) .

$$\int_{a_1}^{a_2} \sigma(x_o)G(x_o|x)f_o(x_o)dx_o = \frac{f_o(x)}{\lambda_o^2 - \lambda^2} - \frac{1}{\lambda_o^2-\lambda^2}\left[p\left(G\,\frac{df_o}{dx_o} - f_o\,\frac{dG}{dx_o}\right)\right]_{a_1}^{a_2} \tag{A.10}$$

$$\therefore \quad \Psi(x) = \frac{f_o(x)}{\lambda_o^2-\lambda^2} - \frac{1}{\lambda_o^2-\lambda^2}\,(f_o,G) + (\Psi, G) \tag{A.11}$$

Next, suppose that $f(x)$ in equation (A.1) can be expressed as a Fourier series of functions satisfying (A.4) .

$$f(x) = \sum_n a_n\Psi_n \tag{A.12}$$

Assuming that the series (A.12) is uniformly convergent a similar evaluation of the integral can be made. The result is

$$\Psi = \sum_n \frac{a_n\Psi_n}{\lambda_n^2-\lambda^2} - \sum_n \frac{a_n}{\lambda_n^2-\lambda^2}\,(f_n,G) + (\Psi,G) \tag{A.13}$$

APPENDIX B

EIGENFUNCTION AND BOUNDARY VALUE SOLUTION FOR CIRCULAR GEOMETRY

For reference the eigenvalue solutions and boundary value solutions for the circular waveguide are tabulated below. Also the normalization and scalar products are tabulated.

B.1 Eigenvalue Solutions

The functions e_n and w_m that are used in Chapter 7 to expand the potential field are solutions of

$$\left\{ \frac{d}{dr} r \frac{d}{dr} + \lambda_n^2 r \right\} e_n = 0, \quad e_n(a) = 0 \qquad \text{(a)}$$

$$\left\{ \frac{d}{dr} r \frac{d}{dr} + \gamma_m^2 r \right\} w_m = 0, \quad \left. \frac{dw_m}{dr} \right|_{r=a} = 0 \qquad \text{(b)} \qquad \text{(B.1)}$$

The normalized eigenmodes are

$$e_n = \frac{J_o(\lambda_n r)}{N_n} \qquad \text{(a)}$$

where $\lambda_n = p_{on}/a$ (b) (B.2)

and $N_n^2 = \dfrac{a^2 J_1^2(\lambda_n a)}{2}$ (c)

$$w_m = \frac{J_o(\gamma_m r)}{N_m} \qquad \text{(a)}$$

$$\text{where} \quad \gamma_m = p_{1m}/a \qquad \text{(b)} \qquad \text{(B.3)}$$

$$\text{and} \quad N_m^2 = \frac{a^2 J_o^2(\gamma_m a)}{2} \qquad \text{(c)}$$

For circular geometry the weighting factor used in Appendix A in the discussion of the Sturm-Liouville equation is the radial coordinate, r . The boundary quantity thus is defined as,

$$(e_n, w_m) = \left[\frac{rJ_o(\gamma_m r)}{N_n N_m} \frac{dJ_o(\gamma_n r)}{dr}\right]_o^a$$

$$= \frac{-\lambda_n a J_o(\gamma_m a) J_1(\lambda_n a)}{N_n N_m}$$

$$(e_n, w_m) = -\frac{2\lambda_n}{a} \frac{J_o(\gamma_m a)}{|J_o(\gamma_m a)|} \frac{J_1(\lambda_n a)}{|J_1(\lambda_n a)|}$$

$$(e_n, w_m) = +\frac{2\lambda_n}{a}(-1)^m(-1)^n \qquad \text{(B.4)}$$

In the above derivation the normalization factors, N_n and N_m defined in (B.2) and (B.3) have been used.

The scalar product is defined as

$$< e_n, w_m > = \int_o^a \frac{rJ_o(\lambda_n)J_o(\gamma_m)dr}{N_n N_m} \qquad \text{(B.5)}$$

B.2 Boundary Value Solutions

It is of interest to exhibit solutions to the equation

$$\frac{d}{dr}\, r\, \frac{d\Psi}{dr} + q^2 r \Psi = 0 \qquad \text{(a)}$$

$$\left.\frac{d\Psi}{dr}\right|_a = \alpha \qquad \text{(b)} \tag{B.6}$$

The solution is,

$$\Psi(r) = \frac{-\alpha J_0(qr)}{qJ_1(qa)} \tag{B.7}$$

APPENDIX C

COMPUTER PROGRAM FOR COMPUTING THE DISPERSION CURVES FOR THE WARM PLASMA

The following is an abbreviated computer program (for carrying out the Feenberg iteration procedure) written in ALGOL.* The only deviation from standard ALGOL is the use of complex variables where necessary. The program is separated into two parts, procedures and the program body. It is felt that the program is fairly self explanatory if reference is made to the explanation of the iteration procedure given in Chapter 8. A brief explanation of the meaning of the procedures will be given to aid the reader.

Procedure	Function
LN1(N1)	Computes λ_n. The first 9 eigenvalues of J_o are read into storage
L2(N2)	Computes λ_n^2
G2(M2)	Computes γ_m^2. The first 9 eigenvalues of J_1 are stored.
OD(N2)	Computes $(-1)^n$
BNDRY(NM,SL)	Computes (e_n,w_s)

---

* An excellent treatment of the use of ALGOL adequate to acquaint the reader with the language can be found in McCracken, Ref. 29.

Procedure	Function
EDOT(N,M)	Computes $\langle e_n, w_m \rangle$
S1EH(NM,SL,SKP)	Evaluates the first sum appearing in (7.14) or (7.23)
S2EH(B02,NM,SL,SKP)	Evaluates the second sums appearing in (7.14) or (7.23)
SUM3(M,J)	Evaluates the sum in (7.29)
VCALC(B02,NM,SL,SKP)	Computes the coefficient P_{nm} or P'_{nm} in Chapter 8
QNM(B02,NM,SKP)	Computes the coefficient Q_n in Chapter 8
FNBRG(B2,ODR)	Computes β^2 by the Feenberg iteration as presented in Chapter 8. ODR selects the order of the iteration. This procedure can be used to iterate on β.

SRT,XFORM and RFORM are procedures written to compute the complex square root, convert complex numbers to exponential form and take the real part of a complex number, respectively. They have not been included.

The values of the necessary plasma parameters are computed in the program body. When possible, variables have been chosen to correspond to those used in the text. For example LO $\rightarrow \ell_o$, KH $\rightarrow K_H$, etc. Since computation time increases greatly as higher order iterations are employed the first two values of β, β_{-1} and β_o are compared. If these differ appreciably β_1 is computed, etc. If this procedure does not converge the Poisson iteration is applied. In this case an indicator, ROOTEST, is computed in FNBRG and is set to 1 if the procedure converges and to 0 if it does not. In practice it was necessary to employ very small incremental steps when searching for roots near $\omega = \omega_B$. CYL is an indicator which is set to 1 for the circular waveguide computation. H is an indicator which is set to 1 when computing the hybrid-H modes and 0 when computing the hybrid-E modes. PRES is set to 1 when the hybrid-p modes are computed.

PROCEDURES

```
REAL PROCEDURE LN1(N1)  $  INTEGER N1  $
BEGIN REAL LA  $  LOCAL LABEL OUT  $
IF CYL EQL 0 THEN BEGIN LA=(N1*PI)/(2*A)  $  GO TO OUT  $
END  $  IF N1 LEQ 9 THEN BEGIN LA=LAMBDA(N1)  $
GO TO OUT  $  END  $  LA=(  (4*N1-1)*PI )/(4*A)  $
OUT.. LN1= LA  $  END OF EIGENVALUE LAMBDA  $

REAL PROCEDURE L2(N1)  $  INTEGER N1  $  BEGIN REAL LA  $
LA= LN1(N1)  $  L2= LA*LA  $
END OF EIGENVALUE LAMBDA SQUARED  $

REAL PROCEDURE G2(M1)  $  INTEGER M1  $
BEGIN REAL GA  $  LOCAL LABEL OUT  $
IF CYL EQL 0 THEN BEGIN GA=M1*PI/A  $  GO TO OUT  $
END  $  IF M1 LEQ 9 THEN BEGIN GA=GAMMA(M1)  $  GO TO OUT $
END  $ GA= (  (4*M1+1)*PI)/(4*A)  $
OUT.. G2=GA*GA  $  END OF EIGENVALUE GAMMA SQUARED  $

INTEGER PROCEDURE OD(N1)  $  INTEGER N1  $  BEGIN REAL
RL  $  INTEGER IN, IND  $  RL= IF (H+CYL) EQL 0 THEN
(N1+1)/4 ELSE N1/2  $  IN=RL  $  IND= IF ABS(RL-IN) GTR
0.25 THEN -1 ELSE 1  $  OD=IND  $  END OF SIGN INDICATOR  $

REAL PROCEDURE BNDRY(NM,SL)  $  INTEGER NM,  SL  $
BEGIN REAL TEM  $  TEM= IF NM EQL SL THEN 1.0 ELSE
OD(NM)*OD(SL)  $  BNDRY= (2*LN1(NM)/A)*TEM  $
END OF BOUNDARY PRODUCT (E,W)  *  *  *  $

REAL PROCEDURE EDOTW(N,M)  $  INTEGER N,M  $  BEGIN
REAL TEM  $  TEM= BNDRY(N,M)  $  EDOTW= TEM/( G2(M)-L2(N) ) $
END OF SCALAR PRODUCT <E,W> OR E DOT W  *  *  *  $

COMPLEX PROCEDURE S1EH(NM,SL,SKP)  $  INTEGER NM,SL,SKP  $
BEGIN COMPLEX TEM1, TEM2, SUM1, LG2  $
INTEGER K, K1  $  LOCAL LABEL  SKIPSET  $  OWN COMPLEX
ARRAY TEM(0..15)  $  OWN INTEGER INT, INC, FNL  $
IF SUMSIZE EQL 1  THEN GO TO SKIPSET  $  IF H EQL 0 THEN
BEGIN INT=CYL  $  INC=1  $  FNL= IF NM GEQ SL
THEN NM+6 ELSE SL+6  $  END  $  IF H EQL 1 THEN BEGIN
INT=1  $  INC= 2-CYL  $  FNL= IF NM GEQ SL
THEN NM+(2-CYL)*6 ELSE SL+(2-CYL)*6  $ END  $
SKIPSET.. FOR K=(INT,INC,FNL) DO BEGIN IF SKP EQL 0 THEN
BEGIN LG2= IF H EQL 0 THEN G2(K)*KH ELSE L2(K)*KH  $
TEM(K)= 1.0/(-LG2+BEH)  $  END  $
```

```
TEM1= IF H EQL 0 THEN EDOTW(SL,K)*EDOTW(NM,K) ELSE
EDOTW(K,SL)*( EDOTW(K,NM))  $  TEM2= TEM(K)*TEM1  $
SUM1= IF K EQL 0 THEN TEM2/2.0 ELSE SUM1+TEM2  $
IF SUMSIZE EQL 0 THEN BEGIN IF RFORM(TEM2) LEQ
0.0050*RFORM(SUM1) THEN BEGIN IF K1 EQL 0 THEN BEGIN
K1=1  $  FNL= K+2  $  END  $  END  $  END  $  END  $
S1EH= SUM1  $  END OF FIRST SUM FOR BOTH E AND H
MODES  *  *  $

COMPLEX PROCEDURE S2EH(B02,NM,SL,SKP)  $  COMPLEX B02  $
INTEGER NM,SL,SKP  $  BEGIN COMPLEX TEM1, TEM2, LG2, SUM2  $
INTEGER K,K1  $  LOCAL LABEL SKIPSET  $  OWN COMPLEX
ARRAY TEM(0..15)  $  OWN INTEGER INT, INC, FNL  $
IF SUMSIZE EQL  1 THEN GO TO SKIPSET  $
INT=1  $  INC=1  $  FNL=10  $  SKIPSET.. FOR
K=(INT,INC,FNL) DO BEGIN IF SKP EQL 0 THEN BEGIN
LG2= IF H EQL 0 THEN G2(K) ELSE L2(K)*KH  $  TEM(K)= IF
H EQL 0 THEN KH*( (-LG2+KOKP-B02/(-LG2*KH+BEH))
ELSE 1.0/(-LG2+BEH)  $  END  $
TEM1= IF H EQL 0 THEN BNDRY(SL,K)*EDOTW(NM,K)ELSE
BNDRY(K,NM)*EDOTW(K,SL)  $  TEM2= TEM(K)*TEM1  $
SUM2= SUM2+TEM2  $
IF SUMSIZE EQL 0 THEN BEGIN IF RFORM(TEM2) LEQ
0.0050*RFORM(SUM2) THEN BEGIN IF K1 EQL 0 THEN BEGIN
K1=1  $  FNL= K+1  $  END  $  END  $  END  $  END  $
S2EH= SUM2  $  END OF SECOND SUM FOR BOTH E AND H MODES * * $

REAL PROCEDURE SUM3(M,J)  $  INTEGER M,J  $
BEGIN REAL SUM, TEM  $  INTEGER FNL, K  $
FNL= IF M GTR J THEN M+5 ELSE J+5  $
FOR K=(1,1,FNL) DO BEGIN TEM=EDOTW(K,J)*EDOTW(K,M)  $
SUM= SUM+TEM  $  END  $  SUM3= SUM  $  END OF
SUMMATION FOR THE PERTURBED PRESSURE MODES  *  *  $

COMPLEX PROCEDURE VCALC(B02,NM,SL,SKP)  $
COMPLEX B02  $  INTEGER NM, SL, SKP $  BEGIN COMPLEX
TEM, KC2, DNM  $  OWN INTEGER MN,LS  $
OWN COMPLEX D12D21, SM1, SM2, D12D6, B12B6  $
LOCAL LABEL SAMEB2, PCALC, OUT  $  IF NM EQL SL THEN
MN=LS=50  $  IF PRES EQL 1 THEN GO TO PCALC  $
IF SKP NEQ 0 THEN GO TO SAMEB2  $
BEH= IF H EQL 0 THEN K02*(KPKH-LH)-B02*KH ELSE
KOPH-B02*(KH+LH)  $  KC2=K02-B02  $  D12D21=B12B21*B02  $
IF H EQL 1 THEN BEGIN DNM=KH*KC2-L02*B02  $
B12B6= B12B6P/DNM  $  END  $  IF H EQL 0 THEN
B12B6= L02/KC2  $  D12D6= B12B6*B02  $
SAMEB2..  IF MN NEQ NM AND LS NEQ SL THEN
SM1=S1EH(NM,SL,SKP)  $  IF COLDTEST EQL 0 THEN
SM2= S2EH(B02,NM,SL,SKP)  $  LS=NM  $  MN=SL  $
```

```
TEM= IF SKP LEQ 1 THEN B12B21*SM1+B12B6*SM2 ELSE
D12D21*SM1+D12D6*SM2  $  GO TO OUT  $
PCALC..  TEM= -LH*SUM3(NM,SL)  $  OUT..  VCALC= TEM  $
END OF COEFFICIENT V SUB NS OR SUB ML  *  *  *  S

COMPLEX PROCEDURE QNM(B02,NM,SKP)  $  COMPLEX B02  $
INTEGER NM, SKP  $  BEGIN COMPLEX TEM  $
OWN COMPLEX D11, D22  $  LOCAL LABEL PCALC, OUT  $
IF PRES EQL 1 THEN GO TO PCALC  $  IF SKP EQL 0 THEN BEGIN
IF H EQL 0 THEN D11= KOPH-B02*(KH+LH)  $  IF H EQL 1 THEN
D22= (KOPH-K02*LH) -B02*KH  $  END  $
TEM= IF H EQL 0 THEN D11-L2(NM)*KH ELSE D22-G2(NM)*KH  $
GO TO OUT  $  PCALC..TEM=KH-KB*B02  $  OUT.. QNM=TEM  $
END OF COEFFICIENT QNM IN THE PLASMA DISPERSION RELATION  $

COMPLEX PROCEDURE FNBRG(B2,ODR)  $  COMPLEX B2  $
INTEGER ODR  $  BEGIN COMPLEX DNMQ, NUMP, DNMP, B02,
BT2, DENOM  $  OWN COMPLEX NUM, DNM1  $
INTEGER SP1, SP2, STOP1  $  LOCAL LABEL OZ, ALI  $
IF PRES EQL 0 THEN BEGIN IF ODR EQL -1 THEN BEGIN
NUM= IF H EQL 0 THEN (KOKP-L2(N))*KH
ELSE K02*(KPKH-LH)-G2(N)*KH  $
DNM1= IF H EQL 0 THEN KH+LH ELSE KH  $  END  $  END  $
IF PRES EQL 1 THEN BEGIN NUM= KH  $  DENOM=KB  $  END  $
B02= BT2= B2  $  SP1= SP2= 0  $
IF PRES EQL 1 THEN BEGIN B02=B02/H02  $  BT2=BT2/H02  $  END $
STOP1=5 $ IF ODR EQL 2 THEN STOP1=1  $
IF PRES EQL 1 AND ODR LEQ 1 THEN STOP1=0  $
IF ITERATE EQL 0 THEN STOP1=0  $  ROOTEST=1  $
AP= AQ= V(N,N)= 0.0  $  SUMSIZE=0  $
OZ.. SP2=0  $  IF SP1  GTR STOP1 THEN GO TO ALI  $
IF ODR GEQ 0 THEN V(N,N)= VCALC(B02,N,N,0)  $
SUMSIZE=1  $  IF ODR GEQ 1 THEN BEGIN Q1(P)= QNM(B02,P,0)  $
V(P,P)=VCALC(B02,P,P,2)  $  V(N,P)=VCALC(B02,N,P,1)  $
V(P,N)= VCALC(B02,P,N,2)  $  IF ODR EQL 1 THEN
AP= V(P,N)/(Q1(P)-V(P,P))  $  END  $
IF ODR GEQ 2 THEN BEGIN Q1(Q)= QNM(B02,Q,1)  $
V(Q,Q)= VCALC(B02,Q,Q,2)  $  V(N,Q) = VCALC(B02,N,Q,1)  $
V(Q,N)= VCALC(B02,Q,N,2)  $  V(P,Q) = VCALC(B02,P,Q,2)  $
V(Q,P)= VCALC(B02,Q,P,2)  $  DNMQ= Q1(Q) -V(Q,Q)  $
NUMP= V(P,N) +( V(P,Q)*V(Q,N))/DNMQ  $  DNMP= Q1(P)-V(P,P)-
(V(P,Q)*V(Q,P))/DNMQ  $  AP= NUMP/DNMP  $
AQ=(V(Q,N) + V(Q,P)*AP)/DNMQ  $  END  $  IF PRES EQL 0
THEN DENOM=DNM1+V(N,N)+V(N,P)*AP+V(N,Q)*AQ  $  IF PRES
EQL 1 THEN NUM=NUM-V(N,N)-V(N,P)*AP-V(N,Q)*AQ  $
B02= NUM/DENOM  $  IF ODR EQL -1 THEN GO TO ALI  $
IF PRES EQL 1 THEN GO TO ALI  $  IF ITERATE EQL 1 THEN
BEGIN IF ODR GRQ 1 THEN WRITE(B02)  $
IF ODR EQL 5 THEN GO TO ALI  $
```

```
IF ABS(REAL(B02-BT2)) GTR 0.021*ABS(REAL(BT2)) THEN
SP2=1  $  IF ABS(IMAG(B02-BT2)) GTR 0.021*ABS(IMAG(BT2))
THEN SP2=1  $  IF SP2 EQL 1 THEN BEGIN SP1=SP1+1  $
BT2= B02  $  GO TO OZ  $  END  $
ALI.. FNBRG= IF PRES EQL 0 THEN B02 ELSE B02*H02  $
IF ITERATE EQL 1 AND SP1 GTR STOP1 THEN ROOTEST=0  $
END OF FFENBERG ITERATION OF ORDER 0 THRU 2 FOR BETA
SQUARED  $
```

PROGRAM BODY

```
C= 3&8  $ C2=C*C  $ PI=3.141590  $  P102= 1.570796  $
U=3&5  $  U2=U*U  $  A= IF CYL EQL 0 THEN 0.010 ELSE
(0.04810)/PI  $  A2=A*A  $  J= <0,1>  $
WN= A/1.41416  $  FOR N=(1,1,9) DO BEGIN READ(Q4,Q5,Q6,Q7) $
LAMBDA(N)=Q4/A  $  GAMMA(N)= Q5/A  $  NORMN(N)=WN*Q6  $
NORMM(N)= WN*Q7  $  END  $  N=S=1  $
FOR S=(4.-1,0) DO BEGIN IF S EQL 0 THEN Q4=Q5=1.0  $
FOR N=(1.1'3) DO BEGIN IF S NEQ 0 THEN READ (Q4,Q5)  $
JON(N,S)=Q4/NORMN(N)  $ JOM(N,S)= Q5/NORMM(N)  $  END  $
END  $  FOR N=(1.1'3)  DO BEGIN
DJON(N)= ((1.41416*LAMBDA(N))/A)*(D(N)  $  END  $
N=S=1  $  W0= 2&10  $  W02= W0*W0  $  WN= 0.050*W0  $
WC2= L2(N)*C2+W02  $  WCE=SQRT(WC2)  $  IF H EQL 1 THEN
BEGIN WC2= G2(N)*C2+W02  $  WC1= SQRT(WC2)  $  END  $
WB= -150*WCE  $  WB2= WB*WB  $
WCEN= IF H EQL 0 THEN 1.0 ELSE WC1/WCE  $  NORM= WCHN  $
IF H EQL 0 THEN P= IF N EQL 1 THEN (3-CYL)  ELSE
N-(2-CYL)  $  IF H EQL 1 THEN BEGIN IF N EQL 1 THEN
P= IF CYL EQL 0 THEN 0 ELSE 2  $  IF N NEQ 1THEN
P= n-1  $  END  $  IF H EQL 0 THEN Q= IF N EQL 1
THEN (5-2*CYL) ELSE N+(2-CYL)  $  IF H EQL 1 THEN
Q= IF N EQL 1 THEN 3 ELSE N+1  $  START= 2.20  $
STP=STP1=STP2=-0.10  $ STOP= 0.050  $  STP2=STP/4.0  $
FOR T=(START,STP,STOP) DO BEGIN W=T*WC  $  W2=W*W  $
STP= STP1  $  IF ABS(T-WHN) LSS0.21  THEN STP=STP2  $
IF ABS(T-WON) LSS 0.41 THEN STP=STP2  $
IF ABS(T-1.0)LSS 0.15 THEN STP- STP2  $  TN=T  $

LO= W0/W  $  L02=L 0*L 0  $ LB=WB/W  $  LB2=LB*LB  $
LN=WN/W  $  EP=1-J*LN  $  EP2-EP*EP  $  LNU=1.0/EP  $
LNU2=LNU*LNU  $  KP=1-L02*LNU  $  KB=1-LB2*LNU2  $
KH=EP-L02-LB2*LNU  $  KH2= KH*KH  $  RKH= REAL(KH)  $
RKB= REAL(KB)  $  KPKH= KP*KH  $
KO= W/C  $  KO2= KO*KO  $  KOKP= KO2*KP $ KOPH= KOKP*KH  $
HO= W/U  $  HO2= HO*HO  $  LH= L02*LB2*LNU2  $  B12B21=
(KO2*LH*L02)*KP  $  B12B6P = L02*LH*KO2*KH  $  SUMSIZE=0 $
IF PRES EQL 0 THEN BEGIN BC=FNBRG(BC,-1)  $
WRITE(BC,SRT(BC,1), TN,-1,0,0,0,0,1)  $  END  $
B20=FNBRG(BC,0)  $  WRITE(B20,SRT(B20,1),TN,0,0,0,0,0,
ROOTEST)  $  B21= IF ROOTEST EQL 0 THEN FNBRG(BC,1)
ELSE FNBRG(B20,1)  $  IF ROOTEST EQL 0 THEN B21=FNBRG(BC1,1)$
BC1=B21  $  RAP=RFORM(AP)  $  WRITE(B21,SRT(B21,1),
TN,1,RAP,0,0,ROOTEST)  $  IF ABS(REAL(B20-B21))
GTR0.09*ABS(REAL(B20)) THEN ST1=1  $  IF ABS(IMAG(B20-B21))
GTR 0.09*ABS(IMAG(B20)) THEN ST1=1  $
IF ST1 EQL 1THEN BEGIN  B22= IF ROOTEST EQL 0 THEN
FNBRG(B22,2)  ELSE FNBRG(B21,2)  $
```

```
RAP=RFORM(AP)  $  RAQ=RFORM(AQ)  $
WRITE(B22,SRT(B22,1),TN,2,RAP,RAQ,ROOTEST)  $
ST1=0  $  END  $  WRITE('1')  $
IF ST2 EQL 0 THEN B22=B21  $  ST2=1  $
ST1=0  $  Q4=ZFIELDS(B22)  $
END  $
END  $
```

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13. ABSTRACT

The linearized equations describing the propagation of the normal modes in a plasma filled waveguide at microwave frequencies and in the presence of an axial constant magnetic field are derived from moments of the Boltzmann equation. Collisions are retained. For two cases where the plasma is assumed to be drifting but uniform or stationary but non-uniform in the transverse plane it is possible to completely solve for the fields by solving a set of coupled equations for the axial electric and magnetic fields and the pressure. If the plasma is assumed to be stationary and uniform these reduce to a set of coupled Helmholtz equations. Solutions for this case are considered in detail. The equations can be simplified considerably and cast into a form very similar to those used to describe wave propagation in a cold plasma. Solutions are obtained by employing an iterative technique.

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