LEHIGH UNIVERSITY INSTITUTE OF RESEARCH

National Aeronautics and Space Administration
Grant NGR-39-007-025

Technical Report No. 3

THE USE OF EIGENFUNCTION EXPANSIONS IN THE GENERAL SOLUTION OF THREE-DIMENSIONAL CRACK PROBLEMS

by

R. J. Hartranft

G. C. Sih

February 1968

Department of Applied Mechanics
Lehigh University, Bethlehem, Pennsylvania

National Aeronautics

and

Space Administration

Grant NGR-39-007-025

Technical Report No. 3

THE USE OF EIGENFUNCTION EXPANSIONS IN THE GENERAL SOLUTION OF THREE-DIMENSIONAL CRACK PROBLEMS

by

R. J. Hartranft and G. C. Sih

Department of Applied Mechanics

Lehigh University

Bethlehem, Pennsylvania

February, 1968

THE USE OF EIGENFUNCTION EXPANSIONS IN THE GENERAL SOLUTION OF THREE-DIMENSIONAL CRACK PROBLEMS[1]

by

R. J. Hartranft[2] and G. C. Sih[3]

ABSTRACT

A method is proposed whereby, using eigenfunction expansions, the complete three-dimensional stress and displacement expressions are developed in series form for the problem of an infinite solid weakened by a plane of discontinuity or crack. A suitable coordinate system is selected so that the general solution can be expressed independently of uncertainties of both nature of the applied loads and of the stress variations in the direction parallel to the crack edge. The unbounded contributions to the stresses can be shown clearly in a finite number of terms of the series solution, and they are found to vary as the inverse square root of the distance from the crack front. Inside a small region around the crack edge, the state of affairs reduces to that of plane strain in the two-dimensional case. The results not only provide an improved under-

[1]The results communicated in this paper were obtained in the course of a research program conducted under Grant NGR-39-007-025 with the National Aeronautics and Space Administration.

[2]Assistant Professor of Mechanics, Lehigh University, Bethlehem, Pennsylvania.

[3]Professor of Mechanics, Lehigh University, Bethlehem, Pennsylvania.

standing of the three-dimensional aspects of fracture but also gain some insight into the triaxial characteristics of the crack-edge stresses interior to a thick plate.

The proposed method can also be extended to problems concerning the distribution of three-dimensional stresses around wedge-like discontinuities.

INTRODUCTION

Recent interest on the exploration of stress distribution in bodies containing flaws or cracks can be evidenced by the sizable volume of literature that has built up. Most of the work on the subject has been based on the plane theory of elasticity. It is to be expected, however, that in many instances the deviations from the two-dimensional theory, due to the three-dimensional character of the stress distribution near the crack boundary, can be of importance.

Methods for the solution of linear elasticity equations are generally dictated by the topology of the region under consideration, and developed for their intended purposes. For example, although the symbolic method of Lur'e [1][4] and the Fourier-Bessel expansion technique of Green [2] are adequate for handling thick plate problems, they are not suitable for solving three-dimensional problems with geometrically-induced singularities. Because of the lack of a systematic way of treating crack problems in three dimensions, only a few effective solutions are available and these have been restricted to a particular crack geometry under simple types of loading. A non-trivial crack geometry, which has received some attention in the past, is that of an ellipse as a surface of discontinuity, the penny-shaped crack problem of Sneddon [3] being a special case. Based on known properties of the classical potential

[4]Numbers in brackets designate References at end of paper.

functions, Green and Sneddon [4] discussed the problem of an elliptical crack whose faces are loaded normally, while Kassir and Sih [5] solved the same problem with uniform shear loads. The complementary problem of a flat crack covering the outside of an ellipse was also investigated by Kassir and Sih [6].

The three-dimensional crack problem to be considered in this paper is, in many ways, motivated by the work of Williams [7-9] and Sih et al [10-12]. It was Williams [7] who first conceived the idea of eigenfunction expansions and employed it to analyze semi-infinite crack and wedge problems in two variables. He established a series solution for the in-plane stresses

$$\sigma_r = \sum_{n=o}^{\infty} r^{\lambda_n - 1} \left[(\lambda_n+1) F_n + \frac{d^2 F_n}{d\theta^2}\right],$$

$$\sigma_\theta = \sum_{n=o}^{\infty} r^{\lambda_n - 1} \lambda_n(\lambda_n+1) F_n, \tag{1}$$

$$\tau_{r\theta} = - \sum_{n=o}^{\infty} r^{\lambda_n - 1} \lambda_n \frac{dF_n}{d\theta},$$

where

$$F_n(\theta) = a_n \sin(\lambda_n+1)\theta + b_n \cos(\lambda_n+1)\theta + c_n \sin(\lambda_n-1)\theta$$
$$+ d_n \cos(\lambda_n-1)\theta, \tag{2}$$

and the eigenvalues λ_n are obtainable from the stress and/or displacement conditions specified on the edges of the crack or wedge. The constant coefficients a_n, b_n, etc. in Williams' analysis, however, were not found until Sih and Rice [10,11] recognized that once the general form of the solution is known in r and θ, the constants in equation (2) can be determined from the Boussinesq's solution in two dimensions. In this way, a number of dissimilar (or similar) media problems involving concentrated forces applied to the surfaces of semi-infinite and finite cracks were solved. In fact, Loeber and Sih [12] have established the equivalence between the method of eigenfunction expansions and the well-known Riemann-Hilbert formulation as presented by Muskhelishvili.

It is the central purpose of this paper to incorporate a third dimension into the eigenfunction expansion method and to provide a general solution of the three-dimensional stress and displacement fields for the case of a plane crack embedded in an elastic solid. Unlike the planar problem, where homogeneous relations were obtained between the independent constants in equation (2), the three-dimensional analysis leads to recurrence relations between the coefficients, which are no longer constants, but functions of the third variable. Two individual sets of results are considered. The first set contains integer powers, say n, of r and the second set contains $n + \frac{1}{2}$ powers of r, where r is the radial distance measured from the leading edge of the crack. These results may be used to answer some of the pressing questions concerning the three-dimensional character of the stress distribution in the vicinity of the crack front.

PRELIMINARY CONSIDERATIONS

Since the chief interest in the solution to the crack problem is centered on the disturbances local to the leading edge of the crack, it is essential to select a coordinate system that will exhibit the singular character of the stresses in a natural manner. To this end, the crack problem will be formulated in circular cylindrical coordinates (r,θ,z), related to the Cartesian system according to

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

The elastic solid is assumed to be infinitely extended in the x-, y- and z-directions, and contains a crack in the form of a half-plane with the z-axis along the crack edge. The sides of the crack coincide with the surfaces $\theta = \pm\pi$, where

$$0 \le r < \infty,\ -\pi \le \theta \le \pi,\ -\infty < z < \infty.$$

Within the framework of the linear theory of elasticity, the equations of equilibrium in cylindrical coordinates must be solved for the displacement components (u_r, v_θ, w_z):

$$\frac{\partial \upsilon}{\partial r} + (1-2\nu)\left[(\nabla^2 - \frac{1}{r^2})\, u_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] = 0,$$

$$\frac{1}{r}\frac{\partial \upsilon}{\partial \theta} + (1-2\nu)\left[\left(\nabla^2 - \frac{1}{r^2}\right) v_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] = 0, \qquad (3)$$

$$\frac{\partial \upsilon}{\partial z} + (1-2\nu)\nabla^2 w_z = 0.$$

In equations (3), ν is Poisson's ratio, υ is the dilatation given by

$$\upsilon = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial w_z}{\partial z},$$

and the Laplace operator is defined as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The boundary conditions will be written in terms of the stresses, which are related to the displacements through the Hooke's law for an isotropic body:

$$\sigma_r = \lambda\upsilon + 2\mu\frac{\partial u_r}{\partial r}, \qquad \tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right),$$

$$\sigma_\theta = \lambda\upsilon + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial w_z}{\partial \theta}\right), \qquad (4)$$

$$\sigma_z = \lambda\upsilon + 2\mu\frac{\partial w_z}{\partial z}, \qquad \tau_{zr} = \mu\left(\frac{\partial w_z}{\partial r} + \frac{\partial u_r}{\partial z}\right).$$

The Lamé's coefficients are denoted by λ and μ.

CONSTRUCTION OF SERIES SOLUTION

For the solution of the half-plane crack problem, it turns out to be very convenient to expand the displacement components in the double series

$$2\mu\, u_r = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n}\, U_n^{(m)}(\theta,z;\lambda_m),$$

$$2\mu\, v_\theta = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n}\, V_n^{(m)}(\theta,z;\lambda_m), \tag{5}$$

$$2\mu\, w_z = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n}\, W_n^{(m)}(\theta,z;\lambda_m).$$

The eigenvalues λ_m (m=0,1,2,---) as powers of r are assumed to be constants, and $U_n^{(m)}$, $V_n^{(m)}$, $W_n^{(m)}$ are functions of θ and z only. From equations (4), the corresponding stress components follow immediately. They are

$$(1-2\nu)\sigma_r = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n-1} \{[(1-\nu)(\lambda_m+n)+\nu]\, U_n^{(m)} + \nu\, [\frac{\partial V_n^{(m)}}{\partial\theta} + \frac{\partial W_{n-1}^{(m)}}{\partial z}]\},$$

$$(1-2\nu)\sigma_\theta = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n-1} \{[1+\nu(\lambda_m+n-1)]\, U_n^{(m)} + (1-\nu)\frac{\partial V_n^{(m)}}{\partial\theta} + \nu\frac{\partial W_{n-1}^{(m)}}{\partial z}\},$$

$$(1-2\nu)\sigma_z = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n-1} \{\nu[(\lambda_m+n+1)\ U_n^{(m)} + \frac{\partial V_n^{(m)}}{\partial\theta}]$$

$$+ (1-\nu)\ \frac{\partial W_{n-1}^{(m)}}{\partial z}\},$$

$$2\ \tau_{r\theta} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n-1} [\frac{\partial U_n^{(m)}}{\partial\theta} + (\lambda_m+n-1)\ V_n^{(m)}],$$

$$2\ \tau_{\theta z} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n-1} [\frac{\partial V_{n-1}^{(m)}}{\partial z} + \frac{\partial W_n^{(m)}}{\partial\theta}], \qquad (6)$$

$$2\ \tau_{zr} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} r^{\lambda_m+n-1} [\frac{\partial U_{n-1}^{(m)}}{\partial z} + (\lambda_m+n)\ W_n^{(m)}].$$

where $U_n^{(m)}$, $V_n^{(m)}$, $W_n^{(m)}$ are zero for $n<0$. The required properties of differentiability have been presumed throughout the foregoing computations.

In order to gain some knowledge on the structure of the solution in θ and z, equations (5) are inserted into the displacement equations of equilibrium given by equations (3). This yields a system of three simultaneous partial differential equations in three unknown functions as

$$(1-2\nu)\ \frac{\partial^2 U_n^{(m)}}{\partial\theta^2} + 2(1-\nu)\ [(\lambda_m+n)^2-1]\ U_n^{(m)} + (\lambda_m+n-3+4\nu)\ \frac{\partial V_n^{(m)}}{\partial\theta}$$

$$= -\ (\lambda_m+n-1)\ \frac{\partial W_{n-1}^{(m)}}{\partial z} - (1-2\nu)\ \frac{\partial^2 U_{n-2}^{(m)}}{\partial z^2},$$

$$2(1-\nu) \frac{\partial^2 V_n^{(m)}}{\partial \theta^2} + (1-2\nu) [(\lambda_m+n)^2-1] V_n^{(m)} + (\lambda_m+n+3-4\nu) \frac{\partial U_n^{(m)}}{\partial \theta}$$

$$= - \frac{\partial^2 W_{n-1}^{(m)}}{\partial \theta \partial z} - (1-2\nu) \frac{\partial^2 V_{n-2}^{(m)}}{\partial z^2},$$

$$(7)$$

$$(1-2\nu) [\frac{\partial^2 W_n^{(m)}}{\partial \theta^2} + (\lambda_m+n)^2 W_n^{(m)}] = - (\lambda_m+n) \frac{\partial U_{n-1}^{(m)}}{\partial z} - \frac{\partial^2 V_{n-1}^{(m)}}{\partial \theta \partial z}$$

$$- 2(1-\nu) \frac{\partial^2 W_{n-2}^{(m)}}{\partial z^2},$$

where $m \geq 0$ and $n \geq 0$. Equations (7) may also be regarded as recurrence relations expressing $U_n^{(m)}$, $V_n^{(m)}$ and $W_n^{(m)}$ in terms of their previous values. It is clear that for each value of n, a product solution in θ and z can be found from equations (7). For n=0, the solution takes the forms

$$U_0^{(m)} = B_1^{(m)}(z) \cos(\lambda_m+1)\theta + B_2^{(m)}(z) \sin(\lambda_m+1)\theta$$

$$+ C_1^{(m)}(z) \cos(\lambda_m-1)\theta + C_2^{(m)}(z) \sin(\lambda_m-1)\theta,$$

$$V_0^{(m)} = - B_1^{(m)}(z) \sin(\lambda_m+1)\theta + B_2^{(m)}(z) \cos(\lambda_m+1)\theta$$

$$+ \frac{\lambda_m+3-4\nu}{\lambda_m-3+4\nu} [- C_1^{(m)}(z) \sin(\lambda_m-1)\theta + C_2^{(m)}(z) \cos(\lambda_m-1)\theta],$$

$$W_0^{(m)} = A_1^{(m)}(z) \cos\lambda_m\theta + A_2^{(m)}(z) \sin\lambda_m\theta \tag{8}$$

in which $A_j^{(m)}(z)$, $B_j^{(m)}(z)$ and $C_j^{(m)}(z)$ (j=1,2) are arbitrary functions of z. Similarly, solutions corresponding to the remaining values of n may be written down. These details will be deferred.

For further development of the solution, it is crucial to apply the free crack surface conditions

$$\sigma_\theta = \tau_{\theta r} = \tau_{\theta z} = 0, \text{ for } \theta = \pm \pi \tag{9}$$

so as to evaluate the eigenvalues λ_m. Without loss in generality, λ_m will be determined from the solution of equations (7) for n=0, i.e., equations (8). Making use of equations (6), (8) and (9), there results in six conditions

$$B_1^{(m)}(z) \cos(\lambda_m+1)\pi + B_2^{(m)}(z) \sin(\lambda_m+1)\pi$$

$$+ \frac{\lambda_m+1}{\lambda_m-3+4\nu} [C_1^{(m)}(z) \cos(\lambda_m-1)\pi + C_2^{(m)}(z) \sin(\lambda_m-1)\pi] = 0,$$

$$B_1^{(m)}(z) \cos(\lambda_m+1)\pi - B_2^{(m)}(z) \sin(\lambda_m+1)\pi$$

$$+ \frac{\lambda_m+1}{\lambda_m-3+4\nu} [C_1^{(m)}(z) \cos(\lambda_m-1)\pi - C_2^{(m)}(z) \sin(\lambda_m-1)\pi] = 0,$$

$$- B_1^{(m)}(z) \sin(\lambda_m+1)\pi + B_2^{(m)}(z) \cos(\lambda_m+1)\pi$$

$$+ \frac{\lambda_m-1}{\lambda_m-3+4\nu} [- C_1^{(m)}(z) \sin(\lambda_m-1)\pi + C_2^{(m)}(z) \cos(\lambda_m-1)\pi] = 0,$$

$$B_1^{(m)}(z) \sin(\lambda_m+1)\pi + B_2^{(m)}(z) \cos(\lambda_m+1)\pi$$

$$+ \frac{\lambda_m-1}{\lambda_m-3+4\nu} [C_1^{(m)}(z) \sin(\lambda_m-1)\pi + C_2^{(m)}(z) \cos(\lambda_m-1)\pi] = 0, \quad (10)$$

$$- A_1^{(m)}(z) \sin\lambda_m\pi + A_2^{(m)}(z) \cos\lambda_m\pi = 0,$$

$$A_1^{(m)}(z) \sin\lambda_m\pi + A_2^{(m)}(z) \cos\lambda_m\pi = 0,$$

to solve for the six unknown functions $A_j^{(m)}(z)$, $B_j^{(m)}(z)$ and $C_j^{(m)}(z)$ $(j=1,2)$. For a non-trivial solution, the determinant of the coefficients of these functions must vanish, and hence λ_m are found to be the roots of the characteristic-value equation

$$\sin 2\pi\lambda_m = 0$$

which renders

$$\lambda_m = \frac{m}{2}, \quad m = 0,1,2,--- \quad (11)$$

The negative values of m have been excluded in equation (11) so that the boundedness conditions of the displacements are not violated as $r\to 0$.

By virtue of equation (11), the double-series representation of each component of the displacement vector in equations (5) can be reduced to a single power series[5] in r, i.e.,

$$
\begin{aligned}
2\mu\, u_r &= \sum_{n=o}^{\infty} r^{\frac{n}{2}} f_n(\theta,z),\\
2\mu\, v_\theta &= \sum_{n=o}^{\infty} r^{\frac{n}{2}} g_n(\theta,z),\\
2\mu\, w_z &= \sum_{n=o}^{\infty} r^{\frac{n}{2}} h_n(\theta,z).
\end{aligned}
\tag{12}
$$

For the same reason, the expressions for the stresses in equations (6) may be simplified and they become

$$(1-2\nu)\sigma_r = \sum_{n=o}^{\infty} r^{\frac{n}{2}-1} \{[\tfrac{n}{2}-(\tfrac{n}{2}-1)\nu]\, f_n + \nu\, [\frac{\partial g_n}{\partial \theta} + \frac{\partial h_{n-2}}{\partial z}]\},$$

$$(1-2\nu)\sigma_\theta = \sum_{n=o}^{\infty} r^{\frac{n}{2}-1} \{[1+(\tfrac{n}{2}-1)\nu]\, f_n + (1-\nu)\, \frac{\partial g_n}{\partial \theta} + \nu\, \frac{\partial h_{n-2}}{\partial z}\},$$

$$(1-2\nu)\sigma_z = \sum_{n=o}^{\infty} r^{\frac{n}{2}-1} \{\nu[(\tfrac{n}{2}+1)\, f_n + \frac{\partial g_n}{\partial \theta}] + (1-\nu)\, \frac{\partial h_{n-2}}{\partial z}\},$$

$$2\, \tau_{r\theta} = \sum_{n=o}^{\infty} r^{\frac{n}{2}-1} [\frac{\partial f_n}{\partial \theta} + (\tfrac{n}{2}-1)\, g_n],$$

---

[5]Depending on the nature of the eigenvalues λ_m, the double-series representation of the displacement vector must, in general, be retained. This point will become evident in the analysis of the wedge problem later on.

$$2\,\tau_{\theta z} = \sum_{n=o}^{\infty} r^{\frac{n}{2}-1} \left(\frac{\partial h_n}{\partial \theta} + \frac{\partial g_{n-2}}{\partial z}\right),$$

$$2\,\tau_{zr} = \sum_{n=o}^{\infty} r^{\frac{n}{2}-1} \left(\frac{n}{2}\, h_n + \frac{\partial f_{n-2}}{\partial z}\right). \tag{13}$$

The basic unknowns f_n, g_n and h_n on the knowledge of which depends the solution of the crack problem must now be found from the linear equations in the classical theory of elasticity. This is the objective of the next section.

DETERMINATION OF DISPLACEMENT FUNCTIONS

Consider now the substitution of equations (12) into the three equations of equilibrium in terms of displacements. Assuming that the resulting expressions will hold for arbitrary values of r, it is found that

$$(1-2\nu)\,\frac{\partial^2 f_n}{\partial \theta^2} + 2(1-\nu)\left(\frac{n^2}{4}-1\right) f_n + \left(\frac{n}{2}-3+4\nu\right)\frac{\partial g_n}{\partial \theta}$$

$$= -\left(\frac{n}{2}-1\right)\frac{\partial h_{n-2}}{\partial z} - (1-2\nu)\,\frac{\partial^2 f_{n-4}}{\partial z^2},$$

$$2(1-\nu)\,\frac{\partial^2 g_n}{\partial \theta^2} + (1-2\nu)\left(\frac{n^2}{4}-1\right) g_n + \left(\frac{n}{2}+3-4\nu\right)\frac{\partial f_n}{\partial \theta}$$

$$= -\frac{\partial^2 h_{n-2}}{\partial \theta\, \partial z} - (1-2\nu)\,\frac{\partial^2 g_{n-4}}{\partial z^2},$$

$$(1-2\nu)\left(\frac{\partial^2 h_n}{\partial\theta^2} + \frac{n^2}{4} h_n\right) = -\frac{n}{2}\frac{\partial f_{n-2}}{\partial z} - \frac{\partial^2 g_{n-2}}{\partial\theta\partial z}$$

$$- 2(1-\nu)\frac{\partial^2 h_{n-4}}{\partial z^2}. \qquad (14)$$

This system of three partial differential equations is similar to that of equations (7) and can be solved for the functions f_n, g_n and h_n ($n=0,1,2,---$). The solution for n=0 is

$$f_0 = {}^{(1)}B_0 \cos\theta + {}^{(2)}B_0 \sin\theta$$

$$+ (3-4\nu)\,[{}^{(1)}C_0\,\theta\sin\theta - {}^{(2)}C_0\,\theta\cos\theta],$$

$$g_0 = -[{}^{(1)}B_0 + {}^{(1)}C_0]\sin\theta + [{}^{(2)}B_0 + {}^{(2)}C_0]\cos\theta \qquad (15)$$

$$+ (3-4\nu)\,[{}^{(1)}C_0\,\theta\cos\theta + {}^{(2)}C_0\,\theta\sin\theta],$$

$$h_0 = {}^{(1)}A_0 + {}^{(2)}A_0\,\theta,$$

while for n=2, it takes the form

$$f_2 = {}^{(1)}B_2\cos 2\theta + {}^{(2)}B_2 \sin 2\theta + {}^{(1)}C_2 - \frac{1}{4(1-\nu)}\,{}^{(2)}A_0'\,\theta,$$

$$g_2 = -{}^{(1)}B_2 \sin 2\theta + {}^{(2)}B_2\cos 2\theta + {}^{(2)}C_2, \qquad (16)$$

$$h_2 = {}^{(1)}A_2\cos\theta + {}^{(2)}A_2\sin\theta - {}^{(1)}C_0'\,\theta\sin\theta + {}^{(2)}C_0'\,\theta\cos\theta$$

In the sequel, the quantities such as $\overset{(i)}{A}_{4m}$, $\overset{(i)}{B}_{4m}$, etc. for i=1,2 are to be understood as arbitrary functions of z and primes (e.g. $\overset{(i)}{A}{}'_{4m}$) will denote differentiation with respect to z.

At this point, it is convenient to separate the unknown functions into two groups distinguished by subscripts being even and odd integers as follows:

(A) f_{4m}, f_{4m+2}, etc. for m=1,2,---.

(B) f_{4m+1}, f_{4m+3}, etc. for m=0,1,2,---.

The calculations for finding these functions involve a considerable amount of work and will not be dwelt on here. Only the final results will be given.

Group (A). To simplify the development of the recurrence relations in subsequent work, the problem will be further subdivided into two parts. The first part contains those functions of z with subscripts 4m and the second with 4m+2 for $m \geq 1$.

1. Even integers of 4m. For $m \geq 1$, it can be shown that

$$f_{4m} = \overset{(1)}{B}_{4m} \cos(2m+1)\theta + \overset{(2)}{B}_{4m} \sin(2m+1)\theta$$

$$+ (2m-3+4\nu) [\overset{(1)}{C}_{4m} \cos(2m-1)\theta + \overset{(2)}{C}_{4m} \sin(2m-1)\theta]$$

$$+ \overset{(1)}{{}_m\alpha}_{4m} \theta \sin\theta + \overset{(2)}{{}_m\alpha}_{4m} \theta \cos\theta$$

$$+ \sum_{k=o}^{m-1} [{}^{(1)}_{k}\alpha_{4m} \cos(2k+1)\theta + {}^{(2)}_{k}\alpha_{4m} \sin(2k+1)\theta],$$

$$g_{4m} = - {}^{(1)}B_{4m} \sin(2m+1)\theta + {}^{(2)}B_{4m} \cos(2m+1)\theta$$

$$+ (2m+3-4\nu) [- {}^{(1)}C_{4m} \sin(2m-1)\theta + {}^{(2)}C_{4m} \cos(2m-1)\theta]$$

$$- {}^{(1)}_{m}\beta_{4m}\, \theta \cos\theta + {}^{(2)}_{m}\beta_{4m}\, \theta \sin\theta$$

$$+ \sum_{k=o}^{m-1} [- {}^{(1)}_{k}\beta_{4m} \sin(2k+1)\theta + {}^{(2)}_{k}\beta_{4m} \cos(2k+1)\theta], \tag{17}$$

$$h_{4m} = {}^{(1)}A_{4m} \cos 2m\theta + {}^{(2)}A_{4m} \sin 2m\theta + {}^{(1)}_{m}\gamma_{4m} + {}^{(2)}_{m}\gamma_{4m}\theta$$

$$+ \sum_{k=o}^{m-2} [{}^{(1)}_{k}\gamma_{4m} \cos 2(k+1)\theta + {}^{(2)}_{k}\gamma_{4m} \sin 2(k+1)\theta].$$

The additional coefficients

$${}^{(i)}_{k}\alpha_{4p},\quad {}^{(i)}_{k}\beta_{4p},\quad {}^{(i)}_{k}\gamma_{4p},\quad i=1,2;\ p\geq 1$$

in equations (17) for the various values of k are also functions of z and they can be expressed in terms of ${}^{(i)}A_{2p}$, ${}^{(i)}B_{2p}$, etc. as given by

$$ {}^{(i)}_{0}\alpha_4 = -\frac{1}{4}\left[{}^{(i)}A'_2 + (1-2\nu)\,{}^{(i)}B''_0 + \frac{1}{4}(1-4\nu)\,{}^{(i)}C''_0\right], $$

$$ {}^{(i)}_{1}\alpha_4 = \frac{1}{4}(1-4\nu)\,{}^{(i)}C''_0, $$

$$ {}^{(i)}_{0}\beta_4 = -\,{}^{(i)}_{0}\alpha_4, \tag{18} $$

$$ {}^{(i)}_{1}\beta_4 = -\frac{1}{4}(3-4\nu)\,{}^{(i)}C''_0, $$

and,

$$ {}^{(i)}_{0}\gamma_4 = 0, $$

$$ 2(1-2\nu)\,{}^{(1)}_{1}\gamma_4 = -\,{}^{(1)}C'_2 - (1-\nu)\,{}^{(1)}A''_0,\quad 8(1-\nu)\,{}^{(2)}_{1}\gamma_4 $$

$$ = -(3-2\nu)\,{}^{(2)}A''_0. $$

The other coefficients may be obtained from the recurrence relations, where for $m \geq 2$[6]:

$$ {}_{m-2}\Delta_{4m}\,{}^{(i)}_{m-2}\alpha_{4m} = [2(1-\nu)(2m-3)^2 - (1-2\nu)(4m^2-1)]\ x $$

[6]See Part 2 of this group (A) for the terms with subscripts of the form 4m+2.

$$\times \left[(2m-1)\ {}_{m-2}\gamma'^{(i)}_{4m-2} + (1-2\nu)\ {}_{m-2}\alpha''^{(i)}_{4m-4}\right.$$

$$\left. + (1-2\nu)(2m-5+4\nu)\ C''^{(i)}_{4m-4}\right] - \left[(2m-3+4\nu)(2m-3)\right] \times$$

$$\times \left[(2m-3)\ {}_{m-2}\gamma'^{(i)}_{4m-2} + (1-2\nu)\ {}_{m-2}\beta''^{(i)}_{4m-4}\right.$$

$$\left. + (1-2\nu)(2m+1-4\nu)\ C''^{(i)}_{4m-4}\right],$$

$${}_{m-2}\Delta_{4m}\ {}_{m-2}\beta^{(i)}_{4m} = \left[(1-2\nu)(4m-3)^2 - 2(1-\nu)(4m^2-1)\right] \times$$

$$\times \left[(2m-3)\ {}_{m-2}\gamma'^{(i)}_{4m-2} + (1-2\nu)\ {}_{m-2}\beta''^{(i)}_{4m-4}\right. \tag{19}$$

$$\left. + (1-2\nu)(2m+1-4\nu)\ C''^{(i)}_{4m-4}\right] + \left[(2m+3-4\nu)(2m-3)\right] \times$$

$$\times \left[(2m-1)\ {}_{m-2}\gamma'^{(i)}_{4m-2} + (1-2\nu)\ {}_{m-2}\alpha''^{(i)}_{4m-4}\right.$$

$$\left. + (1-2\nu)(2m-5+4\nu)\ C''^{(i)}_{4m-4}\right],$$

$$2(1-2\nu)(2m-1)\ {}_{m-2}\gamma^{(i)}_{4m} = -\ m\ {}_{m-2}\alpha'^{(i)}_{4m-2} + (m-1)\ {}_{m-2}\beta'^{(i)}_{4m-2}$$

$$+ 2(1-2\nu)(2m-1)\ C'^{(i)}_{4m-2} - (1-\nu)\ A''^{(i)}_{4m-4}$$

and

$$ {}^{(i)}_{m-1}\alpha_{4m} = -\frac{1}{4m}\left[{}^{(i)}A'_{4m-2} + \left(\frac{1-2\nu}{2m-1}\right) {}^{(i)}B''_{4m-4}\right], $$

$$ {}^{(i)}_{m-1}\beta_{4m} = - {}^{(i)}_{m-1}\alpha_{4m}, \tag{20} $$

$$ {}^{(i)}_{m-1}\gamma_{4m} = 0. $$

In addition,

$$ {}_{m}\Delta_{4m} \; {}^{(i)}_{m}\alpha_{4m} = -(2m-3+4\nu)\left[{}^{(i)}_{m-1}\gamma'_{4m-2} - (1-2\nu)\, {}^{(i)}_{m-1}\beta''_{4m-4}\right] $$

$$ + [1-2(1-2\nu)(2m^2-1)][(2m-1)\, {}^{(i)}_{m-1}\gamma'_{4m-2} $$

$$ + (1-2\nu)\, {}^{(i)}_{m-1}\alpha''_{4m-4}], \tag{21} $$

$$ {}_{m}\Delta_{4m} \; {}^{(i)}_{m}\beta_{4m} = -(2m+3-4\nu)\, [(2m-1)\, {}^{(i)}_{m-1}\gamma'_{4m-2} $$

$$ + (1-2\nu)\, {}^{(i)}_{m-1}\alpha''_{4m-4}] + [1+4(1-\nu)(2m^2-1)] \times $$

$$ \times \left[{}^{(i)}_{m-1}\gamma'_{4m-2} - (1-2\nu)\, {}^{(i)}_{m-1}\beta''_{4m-4}\right], $$

$$2(1-2\nu)m^2 \, {}_{m}^{(i)}\gamma_{4m} = -m \, {}_{m-1}^{(i)}\alpha'_{4m-2} - (1-\nu) \, {}_{m-1}^{(i)}\gamma''_{4m-4}.$$

in which,

$${}_{m}\Delta_{4m} = 32(1-\nu)(1-2\nu)\, m^2(m^2-1).$$

For k=0 and m≥3, the additional coefficients are

$$
\begin{aligned}
{}_{0}\Delta_{4m} \, {}_{0}^{(i)}\alpha_{4m} = {} & [1-2(1-2\nu)(2m^2-1]\,[-2(1-2\nu)\, {}_{m}^{(i)}\alpha_{4m} \\
& + (2m-3+4\nu)\, {}_{m}^{(i)}\beta_{4m} + (2m-1)\, {}_{0}^{(i)}\gamma'_{4m-2} + (1-2\nu)\, {}_{0}^{(i)}\alpha''_{4m-4}] \\
& - [2m-3+4\nu]\,[4(1-\nu)\, {}_{m}^{(i)}\beta_{4m} + (2m+3-4\nu)\, {}_{m}^{(i)}\alpha_{4m} \\
& + {}_{m-1}^{(i)}\gamma'_{4m-2} + {}_{0}^{(i)}\gamma'_{4m-2} + (1-2\nu)\, {}_{0}^{(i)}\beta''_{4m-4}],
\end{aligned}
\qquad (22)
$$

$$
\begin{aligned}
{}_{0}\Delta_{4m} \, {}_{0}^{(i)}\beta_{4m} = {} & [2m+3-4\nu]\,[-2(1-2\nu)\, {}_{m}^{(i)}\alpha_{4m} + (2m-3+4\nu)\, {}_{m}^{(i)}\beta_{4m} \\
& + (2m-1)\, {}_{0}^{(i)}\gamma'_{4m-2} + (1-2\nu)\, {}_{0}^{(i)}\alpha''_{4m-4}] - [1+4(1-\nu)(2m^2-1]\ x \\
& x\ [4(1-\nu)\, {}_{m}^{(i)}\beta_{4m} + (2m+3-4\nu)\, {}_{m}^{(i)}\alpha_{4m} + {}_{m-1}^{(i)}\gamma'_{4m-2} \\
& + {}_{0}^{(i)}\gamma'_{4m-2} + (1-2\nu)\, {}_{0}^{(i)}\beta''_{4m-4}],
\end{aligned}
$$

$$2(1-2\nu)(m^2-1) \, {}_0^{(i)}\gamma_{4m} = - m \, {}_0^{(i)}\alpha'_{4m-2} + {}_0^{(i)}\beta'_{4m-2}$$

$$- (1-\nu) \, {}_0^{(i)}\gamma''_{4m-4}.$$

Moreover, the results for k=1,2,---, m-3 when $m \geq 4$ are

$${}_k\Delta_{4m} \, {}_k^{(i)}\alpha_{4m} = [2(1-\nu)(2k+1)^2 - (1-2\nu)(4m^2-1)] \times$$

$$\times [(2m-1) \, {}_k^{(i)}\gamma'_{4m-2} + (1-2\nu) \, {}_k^{(i)}\alpha''_{4m-4}]$$

$$- [(2m-3+4\nu)(2k+1)][(2k+1) \, {}_k^{(i)}\gamma'_{4m-2} + (1-2\nu) \, {}_k^{(i)}\beta''_{4m-4}],$$

$${}_k\Delta_{4m} \, {}_k^{(i)}\beta_{4m} = [(1-2\nu)(2k+1)^2 - 2(1-\nu)(4m^2-1)] \times \qquad (23)$$

$$[(2k+1) \, {}_k^{(i)}\gamma'_{4m-2} + (1-2\nu) \, {}_k^{(i)}\beta''_{4m-4}]$$

$$+ [(2m+3-4\nu)(2k+1)][(2m-1) \, {}_k^{(i)}\gamma'_{4m-2} + (1-2\nu) \, {}_k^{(i)}\alpha''_{4m-4}],$$

$$2(1-2\nu) \, [m^2-(k+1)^2] \, {}_k^{(i)}\gamma_{4m} = - m \, {}_k^{(i)}\alpha'_{4m-2} + (k+1) \, {}_k^{(i)}\beta'_{4m-2}$$

$$- (1-\nu) \, {}_k^{(i)}\gamma''_{4m-4},$$

The contraction ${}_k\Delta_{4m}$ stands for

$$ {}_k\Delta_{4m} = 2(1-\nu)(1-2\nu)\ [(2k+1)^2 - (2m+1)^2][(2k+1)^2 - (2m-1)^2], $$

where k=0,1,2,---, m-2.

2. Even integers of 4m+2. Following the same procedure, the functions f_{4m+2}, g_{4m+2}, etc. for $m \geq 1$ are obtained:

$$ \begin{aligned} f_{4m+2} &= {}^{(1)}B_{4m+2}\cos 2(m+1)\theta + {}^{(2)}B_{4m+2}\sin 2(m+1)\theta \\ &\quad + 2(m-1+2\nu)\ [{}^{(1)}C_{4m+2}\cos 2m\theta + {}^{(2)}C_{4m+2}\sin 2m\theta] \\ &\quad + {}^{(1)}_{m}\alpha_{4m+2} + {}^{(2)}_{m}\alpha_{4m+2}\,\theta \\ &\quad + \sum_{k=o}^{m-1} [{}^{(1)}_{k}\alpha_{4m+2}\cos 2(k+1)\theta + {}^{(2)}_{k}\alpha_{4m+2}\sin 2(k+1)\theta], \\ g_{4m+2} &= -{}^{(1)}B_{4m+2}\sin 2(m+1)\theta + {}^{(2)}B_{4m+2}\cos 2(m+1)\theta \\ &\quad + 2(m+2-2\nu)\ [-{}^{(1)}C_{4m+2}\sin 2m\theta + {}^{(2)}C_{4m+2}\cos 2m\theta] \\ &\quad + {}^{(2)}_{m}\beta_{4m+2} + \sum_{k=o}^{m-1} [-{}^{(1)}_{k}\beta_{4m+2}\sin 2(k+1)\theta \\ &\quad + {}^{(2)}_{k}\beta_{4m+2}\cos 2(k+1)\theta], \end{aligned} \qquad (24) $$

$$h_{4m+2} = {}^{(1)}A_{4m+2}\cos(2m+1)\theta + {}^{(2)}A_{4m+2}\sin(2m+1)\theta$$

$$- {}^{(1)}_{m}\gamma_{4m+2}\,\theta\sin\theta + {}^{(2)}_{m}\gamma_{4m+2}\,\theta\cos\theta$$

$$+ \sum_{k=o}^{m-1} [{}^{(1)}_{k}\gamma_{4m+2}\cos(2k+1)\theta + {}^{(2)}_{k}\gamma_{4m+2}\sin(2k+1)\theta].$$

When m=1 and k=0,1, the coefficients ${}^{(i)}_{k}\alpha_{4m+2}$, ${}^{(i)}_{k}\beta_{4m+2}$, etc. are given by

$${}^{(i)}_{0}\alpha_6 = -\frac{1}{12}[2\,{}^{(i)}A_4' + (1-2\nu)\,{}^{(i)}B_2''],$$

$$16(1-2\nu)\,{}^{(1)}_{1}\alpha_6 = {}^{(1)}A_0''' + 4\nu\,{}^{(1)}C_2''\ ,\ 16(1-\nu)\,{}^{(2)}_{1}\alpha_6 = {}^{(2)}A_0''',$$

$$ (25)$$

$${}^{(i)}_{0}\beta_6 = -\,{}^{(i)}_{0}\alpha_6,$$

$${}^{(1)}_{1}\beta_6 = 0,\ {}^{(2)}_{1}\beta_6 = -\frac{1}{8}\,{}^{(2)}C_2'',$$

and

$${}^{(i)}_{0}\gamma_6 = {}^{(i)}C_4' - \frac{1}{8}\,{}^{(i)}A_2'' + \frac{1}{8}\,{}^{(i)}B_0''' + \frac{1}{16}\,{}^{(i)}C_0''',$$

$${}^{(i)}_{1}\gamma_6 = -\frac{1}{4}\,{}^{(i)}C_0'''.$$

The first few recurrence relations are $(m \geq 2)$[7]

$$
\begin{aligned}
{}_{m-2}\Delta_{4m+2}\; {}^{(i)}_{m-2}\alpha_{4m+2} &= [8(1-\nu)(m-1)^2 - 4(1-2\nu)\, m(m+1)] \times \\
&\times [2(1-2\nu)(m-2+2\nu)\; {}^{(i)}C''_{4m-2} + 2m\; {}^{(i)}_{m-2}\gamma'_{4m} \\
&+ (1-2\nu)\; {}^{(i)}_{m-2}\alpha''_{4m-2}] - [4(m-1+2\nu)(m-1)] \times \\
&\times [2(1-2\nu)(m+1-2\nu)\; {}^{(i)}C''_{4m-2} + 2(m-1)\; {}^{(i)}_{m-2}\gamma'_{4m} \\
&+ (1-2\nu)\; {}^{(i)}_{m-2}\beta''_{4m-2}],
\end{aligned}
\qquad (26)
$$

$$
\begin{aligned}
{}_{m-2}\Delta_{4m+2}\; {}^{(i)}_{m-2}\beta_{4m+2} &= [4(1-2\nu)(m-1)^2 - 8(1-\nu)\, m(m+1)] \times \\
&\times [2(1-2\nu)(m+1-2\nu)\; {}^{(i)}C''_{4m-2} + 2(m-1)\; {}^{(i)}_{m-2}\gamma'_{4m} \\
&+ (1-2\nu)\; {}^{(i)}_{m-2}\beta''_{4m-2}] + [4(m+2-2\nu)(m-1)] \times \\
&\times [2(1-2\nu)(m-2+2\nu)\; {}^{(i)}C''_{4m-2} + 2m\; {}^{(i)}_{m-2}\gamma'_{4m} \\
&+ (1-2\nu)\; {}^{(i)}_{m-2}\alpha''_{4m-2}],
\end{aligned}
$$

[7]The terms with subscripts of the form 4m may be found in Part 1 of this group (A).

$$(1-2\nu)\,[(2m+1)^2 - (2m-3)^2]\;{}^{(i)}_{m-2}\gamma_{4m+2} = -\,(2m+1)\;{}^{(i)}_{m-2}\alpha'_{4m}$$

$$+\,(2m-3)\;{}^{(i)}_{m-2}\beta'_{4m} - 2(1-\nu)\;{}^{(i)}_{m-2}\gamma''_{4m-2}.$$

and

$${}^{(i)}_{m-1}\alpha_{4m+2} = -\,\frac{1}{2(2m+1)}\,[\,{}^{(i)}A'_{4m} + \frac{1}{2m}\,(1-2\nu)\;{}^{(i)}B''_{4m-2}],$$

$${}^{(i)}_{m-1}\beta_{4m+2} = -\;{}^{(i)}_{m-1}\alpha_{4m+2},$$

$$(27)$$

$$8(1-2\nu)m\;{}^{(i)}_{m-1}\gamma_{4m+2} = 8m(1-2\nu)\;{}^{(i)}C'_{4m} - (2m+1)\;{}^{(i)}_{m-1}\alpha'_{4m}$$

$$+\,(2m-1)\;{}^{(i)}_{m-1}\beta'_{4m} - 2(1-\nu)\;{}^{(i)}A''_{4m-2},$$

and

$$8(1-\nu)\,m(m+1)\;{}^{(i)}_{m}\alpha_{4m+2} = -\,2m\;{}^{(i)}_{m}\gamma'_{4m} - (1-2\nu)\;{}^{(i)}_{m-1}\alpha''_{4m-2},$$

$${}^{(1)}_{m}\beta_{4m+2} = 0,$$

$$(28)$$

$$4m(m+1)\;{}^{(2)}_{m}\beta_{4m+2} = -\;{}^{(2)}_{m-1}\beta''_{4m-2} + \frac{1}{4(1-\nu)m(m+1)}\,[-2m^2\;{}^{(2)}_{m}\gamma'_{4m}$$

$$+ \; (m+2-2\nu) \; {}_{m-1}^{(2)}\alpha''_{4m-2}],$$

$$4(1-2\nu)\; m(m+1) \; {}_{m}^{(i)}\gamma_{4m+2} = -\; [(2m+1) \; {}_{m}^{(i)}\alpha'_{4m} + {}_{m}^{(i)}\beta'_{4m}$$

$$+ \; 2(1-\nu) \; {}_{m-1}^{(i)}\gamma''_{4m-2}].$$

For $m \geq 3$, the terms involving $k=0$ can be written as

$$ {}_0\Delta_{4m+2} \; {}_{0}^{(i)}\alpha_{4m+2} = [8(1-\nu) - 4(1-2\nu)\; m(m+1)] \; x$$

$$x \; [2m \; {}_{0}^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_{0}^{(i)}\alpha''_{4m-2}] - [4(m-1+2\nu)] \; x$$

$$x \; [2 \; {}_{0}^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_{0}^{(i)}\beta''_{4m-2}],$$

$$\qquad (29)$$

$$ {}_0\Delta_{4m+2} \; {}_{k}^{(i)}\beta_{4m+2} = [4(m+2-2\nu)][2m \; {}_{0}^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_{0}^{(i)}\alpha''_{4m-2}]$$

$$+ \; [4(1-2\nu) - 8(1-\nu)\; m(m+1)][2 \; {}_{0}^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_{0}^{(i)}\beta''_{4m-2}],$$

$$4(1-2\nu)\; m(m+1) \; {}_{0}^{(i)}\gamma_{4m+2} = 2(1-2\nu) \; {}_{m}^{(i)}\gamma_{4m+2} - {}_{m}^{(i)}\beta'_{4m}$$

$$-\; (2m+1) \; {}_{0}^{(i)}\alpha'_{4m} + {}_{0}^{(i)}\beta'_{4m} - 2(1-\nu) \; {}_{0}^{(i)}\gamma''_{4m-2}.$$

and the remaining terms for k=1,2,---, m-3 when $m \geq 4$ are obtainable from

$$ {}_k\Delta_{4m+2} \; {}_k^{(i)}\alpha_{4m+2} = [8(1-\nu)(k+1)^2 - 4(1-2\nu)\, m(m+1)] \times $$

$$ \times \, [2m \; {}_k^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_k^{(i)}\alpha''_{4m-2}] - [4(m-1+2\nu)(k+1)] \times $$

$$ \times \, [2(k+1) \; {}_k^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_k^{(i)}\beta''_{4m-2}], $$

$$ {}_k\Delta_{4m+2} \; {}_k^{(i)}\beta_{4m+2} = [4(m+2-2\nu)(k+1)][2m \; {}_k^{(i)}\gamma'_{4m} \tag{30} $$

$$ + (1-2\nu) \; {}_k^{(i)}\alpha''_{4m-2}] + [4(1-2\nu)(k+1)^2 - 8(1-\nu)\, m(m+1)] \times $$

$$ \times \, [2(k+1) \; {}_k^{(i)}\gamma'_{4m} + (1-2\nu) \; {}_k^{(i)}\beta''_{4m-2}], $$

$$ (1-2\nu)\, [(2m+1)^2 - (2k+1)^2] \; {}_k^{(i)}\gamma_{4m+2} = -\,(2m+1) \; {}_k^{(i)}\alpha'_{4m} $$

$$ + (2k+1) \; {}_k^{(i)}\beta'_{4m} - 2(1-\nu) \; {}_k^{(i)}\gamma''_{4m-2}. $$

where

$$ {}_k\Delta_{4m+2} = 32(1-\nu)(1-2\nu)\, [(m+1)^2 - (k+1)^2][m^2 - (k+1)^2] $$

and k=0,1,2,---, m-2.

This completes the solution of f_n, g_n, h_n for values of n=0,2,4, etc. The forms of these functions for n=1,3,5, etc. remain to be found.

<u>Group (B).</u> Now, let the functions f_n, g_n, h_n with n=1,5,9, etc. and n=3,7,11, etc. be treated separately.

1. <u>Odd integers of 4m+1</u>. The satisfaction of equations (14) for $m \geq 0$ requires

$$f_{4m+1} = {}^{(1)}B_{4m+1} \cos(2m + \tfrac{3}{2})\theta + {}^{(2)}B_{4m+1} \sin(2m + \tfrac{3}{2})\theta$$

$$+ (4m-5+8\nu) [{}^{(1)}C_{4m+1} \cos(2m - \tfrac{1}{2})\theta$$

$$+ {}^{(2)}C_{4m+1} \sin(2m - \tfrac{1}{2})\theta] + \sum_{k=o}^{m} [{}^{(1)}_{k}\alpha_{4m+1} \cos(2k - \tfrac{1}{2})\theta$$

$$+ {}^{(2)}_{k}\alpha_{4m+1} \sin(2k - \tfrac{1}{2})\theta], \qquad (31)$$

$$g_{4m+1} = - {}^{(1)}B_{4m+1} \sin(2m + \tfrac{3}{2})\theta + {}^{(2)}B_{4m+1} \cos(2m + \tfrac{3}{2})\theta$$

$$+ (4m+7-8\nu) [- {}^{(1)}C_{4m+1} \sin(2m - \tfrac{1}{2})\theta$$

$$+ {}^{(2)}C_{4m+1} \cos(2m - \tfrac{1}{2})\theta]$$

$$+ \sum_{k=o}^{m} [- {}_{k}^{(1)}\beta_{4m+1} \sin(2k - \tfrac{1}{2})\theta$$

$$+ {}_{k}^{(2)}\beta_{4m+3} \cos(2k - \tfrac{1}{2})\theta],$$

$$h_{4m+1} = {}^{(1)}A_{4m+1} \cos(2m + \tfrac{1}{2})\theta + {}^{(2)}A_{4m+1} \sin(2m + \tfrac{1}{2})\theta$$

$$+ \sum_{k=o}^{m-1} [{}_{k}^{(1)}\gamma_{4m+1} \cos(2k + \tfrac{1}{2})\theta$$

$$+ {}_{k}^{(2)}\gamma_{4m+1} \sin(2k + \tfrac{1}{2})\theta],$$

and that

$$ {}_{0}^{(i)}\alpha_{1} = {}_{0}^{(i)}\beta_{1} = {}_{0}^{(i)}\gamma_{1} = 0 \qquad (32)$$

The coefficients corresponding to $m \geq 1$ are found as[8]

$${}_{m-1}\Delta_{4m+1} \, {}_{m-1}^{(i)}\alpha_{4m+1} = - [(1-2\nu)(2m + \tfrac{3}{2})(2m - \tfrac{1}{2})$$

$$- 2(1-\nu)(2m - \tfrac{5}{2})^2][(2m - \tfrac{1}{2}) \, {}_{m-1}^{(i)}\gamma'_{4m-1} + (1-2\nu) \, {}_{m-1}^{(i)}\alpha''_{4m-3}$$

$$+ (1-2\nu)(4m-9+8\nu) \, {}^{(i)}C''_{4m-3}] - [\tfrac{1}{2}(4m-5+8\nu)(2m - \tfrac{5}{2})] \times$$

[8]The terms with subscripts of the form 4m+3 are given in Part 2 of this group (B).

$$\times\ [(2m - \tfrac{5}{2})\ {}^{(i)}_{m-1}\gamma'_{4m-1} + (1-2\nu)\ {}^{(i)}_{m-1}\beta''_{4m-3}$$

$$+ (1-2\nu)(4m+3-8\nu)\ {}^{(i)}C''_{4m-3}],$$

$${}_{m-1}\Delta_{4m+1}\ {}^{(i)}_{m-1}\beta_{4m+1} = [\tfrac{1}{2}(4m+7-8\nu)(2m - \tfrac{5}{2})]\ \times$$

$$\times\ [(2m - \tfrac{1}{2})\ {}^{(i)}_{m-1}\gamma'_{4m-1} + (1-2\nu)\ {}^{(i)}_{m-1}\alpha''_{4m-3}$$

$$+ (1-2\nu)(4m-9+8\nu)\ {}^{(i)}C''_{4m-3}] - [2(1-\nu)(2m + \tfrac{3}{2})(2m - \tfrac{1}{2})$$

$$- (1-2\nu)(2m - \tfrac{5}{2})^2][(2m - \tfrac{5}{2})\ {}^{(i)}_{m-1}\gamma'_{4m-1} \tag{33}$$

$$+ (1-2\nu)\ {}^{(i)}_{m-1}\beta''_{4m-3} + (1-2\nu)(4m+3-8\nu)\ {}^{(i)}C''_{4m-3}],$$

$$4(4m-1)(1-2\nu)\ {}^{(i)}_{m-1}\gamma_{4m+1} = 8(4m-1)(1-2\nu)\ {}^{(i)}C'_{4m-1}$$

$$- 4(1-\nu)\ {}^{(i)}A''_{4m-3} - (4m+1)\ {}^{(i)}_{m-1}\alpha'_{4m-1}$$

$$+ (4m-3)\ {}^{(i)}_{m-1}\beta'_{4m-1}.$$

and

$$ {}^{(i)}_{m}\alpha_{4m+1} = -\frac{1}{4m+1}\,{}^{(i)}A'_{4m-1} - \frac{2(1-2\nu)}{16m^2-1}\,{}^{(i)}B''_{4m-3}, $$

$$ {}^{(i)}_{m}\beta_{4m+1} = -\,{}^{(i)}_{m}\alpha_{4m+1}, \qquad (34) $$

$$ {}^{(i)}_{m}\gamma_{4m+1} = 0. $$

When m≥2, the additional recurrence equations may be established:

$$ {}_{k}\Delta_{4m+1}\,{}^{(i)}_{k}\alpha_{4m+1} = -\tfrac{1}{2}(4m-5+8\nu)(2k-\tfrac{1}{2})\,[(2k-\tfrac{1}{2})\,{}^{(i)}_{k}\gamma'_{4m-1} $$

$$ +\,(1-2\nu)\,{}^{(i)}_{k}\beta''_{4m-3}] - [(1-2\nu)(2m+\tfrac{3}{2})(2m-\tfrac{1}{2}) $$

$$ -\,2(1-\nu)(2k-\tfrac{1}{2})^2][(2m-\tfrac{1}{2})\,{}^{(i)}_{k}\gamma'_{4m-1} + (1-2\nu)\,{}^{(i)}_{k}\alpha''_{4m-3}], $$

(35)

$$ {}_{k}\Delta_{4m+1}\,{}^{(i)}_{k}\beta_{4m+1} = \tfrac{1}{2}(4m+7-8\nu)(2k-\tfrac{1}{2})\,[(2m-\tfrac{1}{2})\,{}^{(i)}_{k}\gamma'_{4m-1} $$

$$ +\,(1-2\nu)\,{}^{(i)}_{k}\alpha''_{4m-3}] - [2(1-\nu)(2m+\tfrac{3}{2})(2m-\tfrac{1}{2}) $$

$$ -\,(1-2\nu)(2k-\tfrac{1}{2})^2][(2k-\tfrac{1}{2})\,{}^{(i)}_{k}\gamma'_{4m-1} + (1-2\nu)\,{}^{(i)}_{k}\beta''_{4m-3}], $$

$$(1-2\nu)\,[(2m+\tfrac{1}{2})^2-(2k+\tfrac{1}{2})^2]\;{}_k^{(i)}\gamma_{4m+1} = -\,(2m+\tfrac{1}{2})\;{}_k^{(i)}\alpha'_{4m-1}$$

$$+\,(2k+\tfrac{1}{2})\;{}_k^{(i)}\beta'_{4m-1} - 2(1-\nu)\;{}_k^{(i)}\gamma''_{4m-3}$$

which are valid for k=0,1,2,---, m-2, while the expression

$${}_k\Delta_{4m+1} = 2(1-\nu)(1-2\nu)\,[(2m+\tfrac{3}{2})^2-(2k-\tfrac{1}{2})^2]\; x$$

$$x\;[(2m-\tfrac{1}{2})^2-(2k-\tfrac{1}{2})^2]$$

applies for k=0,1,2,---, m-1.

2. <u>Odd integers of 4m+3</u>. As before, it can be verified that

$$f_{4m+3} = {}^{(1)}B_{4m+3}\cos(2m+\tfrac{5}{2})\theta + {}^{(2)}B_{4m+3}\sin(2m+\tfrac{5}{2})\theta$$

$$+\,(4m-3+8\nu)\,[{}^{(1)}C_{4m+3}\cos(2m+\tfrac{1}{2})\theta$$

$$+\,{}^{(2)}C_{4m+3}\sin(2m+\tfrac{1}{2})\theta] + \sum_{k=o}^{m}[\,{}_k^{(1)}\alpha_{4m+3}\cos(2k+\tfrac{1}{2})\theta$$

$$+\,{}_k^{(2)}\alpha_{4m+3}\sin(2k+\tfrac{1}{2})\theta],$$

$$g_{4m+3} = -\,{}^{(1)}B_{4m+3}\sin(2m+\tfrac{5}{2})\theta + {}^{(2)}B_{4m+3}\cos(2m+\tfrac{5}{2})\theta$$

$$+ (4m+9-8\nu)\,[-\,{}^{(1)}C_{4m+3}\sin(2m+\tfrac{1}{2})\theta$$

$$+ {}^{(2)}C_{4m+3}\cos(2m+\tfrac{1}{2})\theta] + \sum_{k=o}^{m}[-\,{}^{(1)}_{\ k}\beta_{4m+3}\sin(2k+\tfrac{1}{2})\theta$$

$$+ {}^{(2)}_{\ k}\beta_{4m+3}\cos(2k+\tfrac{1}{2})\theta], \qquad (36)$$

$$h_{4m+3} = {}^{(1)}A_{4m+3}\cos(2m+\tfrac{3}{2})\theta + {}^{(2)}A_{4m+3}\sin(2m+\tfrac{3}{2})\theta$$

$$+ \sum_{k=o}^{m}[{}^{(1)}_{\ k}\gamma_{4m+3}\cos(2k-\tfrac{1}{2})\theta + {}^{(2)}_{\ k}\gamma_{4m+3}\sin(2k-\tfrac{1}{2})\theta].$$

satisfy equations (14) provided

$$-\,{}^{(i)}_{\ 0}\alpha_3 = {}^{(i)}_{\ 0}\beta_3 = \tfrac{1}{3}\,{}^{(i)}A_1', \quad {}^{(i)}_{\ 0}\gamma_3 = 2\,{}^{(i)}C_1' \qquad (37)$$

Omitting the details, the results for $m \geq 1$ are[9]

$${}_{m-1}\Delta_{4m+3}\ {}^{(i)}_{m-1}\alpha_{4m+3} = -\,[(1-2\nu)(2m+\tfrac{5}{2})(2m+\tfrac{1}{2})$$

---

[9] The terms with subscripts of the form 4m+1 are given in Part 1 of the group (B).

$$- 2(1-\nu)(2m - \tfrac{3}{2})^2][(1-2\nu)(4m-7+8\nu) \; {}^{(i)}C''_{4m-1}$$

$$+ (2m + \tfrac{1}{2}) \; {}^{(i)}_{m-1}\gamma'_{4m+1} + (1-2\nu) \; {}^{(i)}_{m-1}\alpha''_{4m-1}]$$

$$- \tfrac{1}{2}(4m-3+8\nu)(2m - \tfrac{3}{2}) \; [(1-2\nu)(4m+5-8\nu) \; {}^{(i)}C''_{4m-1}$$

$$+ (2m - \tfrac{3}{2}) \; {}^{(i)}_{m-1}\gamma'_{4m+1} + (1-2\nu) \; {}^{(i)}_{m-1}\beta''_{4m-1}],$$

$${}_{m-1}\Delta_{4m+3} \; {}^{(i)}_{m-1}\beta_{4m+3} = - [2(1-\nu)(2m + \tfrac{5}{2})(2m + \tfrac{1}{2}) \tag{38}$$

$$- (1-2\nu)(2m - \tfrac{3}{2})^2][(1-2\nu)(4m+5-8\nu) \; {}^{(i)}C''_{4m-1}$$

$$+ (2m - \tfrac{3}{2}) \; {}^{(i)}_{m-1}\gamma'_{4m+1} + (1-2\nu) \; {}^{(i)}_{m-1}\beta''_{4m-1}]$$

$$+ \tfrac{1}{2}(4m+9-8\nu)(2m - \tfrac{3}{2}) \; [(1-2\nu)(4m-7+8\nu) \; {}^{(i)}C''_{4m-1}$$

$$+ (2m + \tfrac{1}{2}) \; {}^{(i)}_{m-1}\gamma'_{4m+1} + (1-2\nu) \; {}^{(i)}_{m-1}\alpha''_{4m-1}],$$

$$(1-2\nu)[(2m + \tfrac{3}{2})^2 - (2m - \tfrac{5}{2})^2] \; {}^{(i)}_{m-1}\gamma_{4m+3}$$

$$= - (2m + \frac{3}{2}) \; {}^{(i)}_{m-1}\alpha'_{4m+1} + (2m - \frac{5}{2}) \; {}^{(i)}_{m-1}\beta'_{4m+1}$$

$$- 2(1-\nu) \; {}^{(i)}_{m-1}\gamma''_{4m-1}.$$

and

$${}^{(i)}_{m}\alpha_{4m+3} = - \frac{1}{4m+3} [{}^{(i)}A'_{4m+1} + \frac{2(1-2\nu)}{4m+1} {}^{(i)}B''_{4m-1}],$$

$${}^{(i)}_{m}\beta_{4m+3} = - {}^{(i)}_{m}\alpha_{4m+3},$$

$$4(4m+1)(1-2\nu) \; {}^{(i)}_{m}\gamma_{4m+3} = 8(4m+1)(1-2\nu) \; {}^{(i)}C'_{4m+1} \tag{39}$$

$$- 4(1-\nu) \; {}^{(i)}A''_{4m-1} - (4m+3) \; {}^{(i)}_{m}\alpha'_{4m+1}$$

$$+ (4m-1) \; {}^{(i)}_{m}\beta'_{4m+1}.$$

Finally, those terms for k=0,1,2,---, m-2 when $m \geq 2$ are determined:

$${}_{k}\Delta_{4m+3} \; {}^{(i)}_{k}\alpha_{4m+3} = - \frac{1}{2}(4m-3+8\nu)(2k + \frac{1}{2}) \; [2k + \frac{1}{2}) \; {}^{(i)}_{k}\gamma'_{4m+1}$$

$$+ (1-2\nu)\ {}_{k}^{(i)}\beta''_{4m-1}] - [(1-2\nu)(2m + \tfrac{5}{2})(2m + \tfrac{1}{2})$$

$$- 2(1-\nu)(2k + \tfrac{1}{2})^2][(2m + \tfrac{1}{2})\ {}_{k}^{(i)}\gamma'_{4m+1}$$

$$+ (1-2\nu)\ {}_{k}^{(i)}\alpha''_{4m-1}],$$

$$_{k}\Delta_{4m+3}\ {}_{k}^{(i)}\beta_{4m+3} = \tfrac{1}{2}(4m+9-8\nu)(2k + \tfrac{1}{2})\ [(2m + \tfrac{1}{2})\ {}_{k}^{(i)}\gamma'_{4m+1}$$

$$+ (1-2\nu)\ {}_{k}^{(i)}\alpha''_{4m-1}] - [2(1-\nu)(2m + \tfrac{5}{2})(2m + \tfrac{1}{2})$$

$$- (1-2\nu)(2k + \tfrac{1}{2})^2][(2k + \tfrac{1}{2})\ {}_{k}^{(i)}\gamma'_{4m+1}$$

$$+ (1-2\nu)\ {}_{k}^{(i)}\beta''_{4m-1}], \qquad (40)$$

$$(1-2\nu)\ [(2m + \tfrac{3}{2})^2 - (2k - \tfrac{1}{2})^2]\ {}_{k}^{(i)}\gamma_{4m+3}$$

$$= - (2m + \tfrac{3}{2})\ {}_{k}^{(i)}\alpha'_{4m+1} + (2k - \tfrac{1}{2})\ {}_{k}^{(i)}\beta'_{4m+1}$$

$$- 2(1-\nu)\ {}_{k}^{(i)}\gamma''_{4m-1}.$$

where

$$ {}_{k}\Delta_{4m+3} = 2(1-\nu)(1-2\nu)\ [(2m + \tfrac{5}{2})^2 - (2k + \tfrac{1}{2})^2]\ x $$

$$ x\ [(2m + \tfrac{1}{2})^2 - (2k + \tfrac{1}{2})^2], $$

for k=0,1,2,---, m-1.

Equations (17) through (40) constitute a general solution of the three-dimensional equations of linear elasticity for the half-plane crack problem. The functions f_n, g_n, h_n (n=0,1,2,---) may be put into equations (12) and (13) for the determination of displacements and stresses at all interesting points both near to and far from the edge of the crack.

THREE-DIMENSIONAL STRESS DISTRIBUTION

Having established the recurrence relations for the displacement functions, it is in order to calculate the three-dimensional stresses for each value of n in equations (13). The following notation will be adopted:

$$ \sigma_r = \sum_{n=o}^{\infty} (\sigma_r)_n,\ \sigma_\theta = \sum_{n=o}^{\infty} (\sigma_\theta)_n,\ \text{etc.} $$

First, consider the case of n=0, where f_0, g_0 and h_0 are given by equations (15). Upon substitution of the appropriate stress components into the free crack surface boundary conditions in equation (9), there results

$$ {}^{(2)}A_0 = {}^{(i)}C_0 = 0, \quad i=1,2 \tag{41} $$

and hence equations (15) are simplified accordingly. These considerations lead to the vanishing of all stresses for n=0, i.e.,

$$ (\sigma_r)_o = (\sigma_\theta)_o = --- = (\tau_{zr})_o = 0 \tag{42} $$

For the purpose of illustrating the use of the recurrence formulas, stress solutions for n=1,2,3,4 will be worked out.

<u>Even integers of n</u>.

1. <u>n=2</u>. From equation (9), it is not difficult to show that

$$ {}^{(2)}A_2 = -{}^{(2)}B_0', \quad {}^{(2)}B_2 = 0, \quad {}^{(1)}C_2 = (1-2\nu)\,{}^{(1)}B_2 - \nu\,{}^{(1)}A_0', \tag{43} $$

and thus

$$ f_2 = {}^{(1)}B_2 \cos 2\theta + (1-2\nu)\,{}^{(1)}B_2 - \nu\,{}^{(1)}A_0', $$

$$ g_2 = -{}^{(1)}B_2 \sin 2\theta + {}^{(2)}C_2, $$

$$ h_2 = {}^{(1)}A_2 \cos\theta - {}^{(2)}B_0' \sin\theta. $$

It is apparent from equations (13) that for n=2, the stresses are independent of r:

$$
\begin{aligned}
(\sigma_r)_2 &= {}^{(1)}B_2\,(1 + \cos 2\theta), \\
(\sigma_\theta)_2 &= {}^{(1)}B_2\,(1 - \cos 2\theta), \\
(\sigma_z)_2 &= 2\nu\;{}^{(1)}B_2 + (1+\nu)\;{}^{(1)}A_0', \\
(\tau_{r\theta})_2 &= -\;{}^{(1)}B_2\,\sin 2\theta, \\
(\tau_{\theta z})_2 &= -\,\tfrac{1}{2}\,[{}^{(1)}A_2 + {}^{(1)}B_0']\,\sin\theta, \\
(\tau_{zr})_2 &= \tfrac{1}{2}\,[{}^{(1)}A_2 + {}^{(1)}B_0']\,\cos\theta.
\end{aligned}
\tag{44}
$$

2. <u>n=4</u>. Now, applying equations (41) and (43), the quantities in equations (18) become

$$
\begin{aligned}
-\;{}^{(1)}_{0}\alpha_4 &= {}^{(1)}_{0}\beta_4 = \tfrac{1}{4}\,[{}^{(1)}A_2' + (1-2\nu)\;{}^{(1)}B_0''], \\
-\;{}^{(2)}_{0}\alpha_4 &= {}^{(2)}_{0}\beta_4 = -\,\tfrac{1}{2}\,\nu\;{}^{(2)}B_0'', \\
{}^{(i)}_{1}\alpha_4 &= {}^{(i)}_{1}\beta_4 = 0,
\end{aligned}
\tag{45}
$$

and

$$ {}^{(i)}_{0}\gamma_4 = 0, $$

$$ {}^{(1)}_{1}\gamma_4 = -\frac{1}{2}\left[{}^{(1)}B_2' + {}^{(1)}A_0''\right], \quad {}^{(2)}_{1}\gamma_4 = 0. $$

These coefficients determine f_4, g_4 and h_4 in equations (17) which in turn give $(\sigma_\theta)_4$, $(\tau_{r\theta})_4$ and $(\tau_{\theta z})_4$. The conditions in equation (9) are thus satisfied by taking

$$ {}^{(2)}A_4 = -\frac{1}{2}\,{}^{(2)}C_2', \quad {}^{(1)}C_4 = -\frac{1}{3}\,{}^{(1)}B_4 $$

$$ -\frac{1}{12}\left[{}^{(1)}A_2' + {}^{(1)}B_0''\right], \quad {}^{(2)}C_4 = -\,{}^{(2)}B_4. \qquad (46) $$

This leads to

$$ f_4 = {}^{(1)}B_4\left[\cos 3\theta + \frac{1}{3}(1-4\nu)\cos\theta\right] $$

$$ + {}^{(2)}B_4\left[\sin 3\theta + (1-4\nu)\sin\theta\right] - \frac{1}{6}\left[(1+2\nu)\,{}^{(1)}A_2' \right. $$

$$ \left. + (1-\nu)\,{}^{(1)}B_0''\right]\cos\theta + \frac{1}{2}\nu\,{}^{(2)}B_0''\sin\theta, $$

$$ g_4 = -\,{}^{(1)}B_4\left[\sin 3\theta - \frac{1}{3}(5-4\nu)\sin\theta\right] $$

$$+ \overset{(2)}{B}_4 \, [\cos 3\theta - (5-4\nu) \cos\theta] + \frac{1}{6} [(1-2\nu) \overset{(1)}{A}{}_2'$$

$$+ (1+\nu) \overset{(1)}{B}{}_0''] \sin\theta - \frac{1}{2} \nu \overset{(2)}{B}{}_0'' \cos\theta,$$

$$h_4 = \overset{(1)}{A}_4 \cos 2\theta - \frac{1}{2} \overset{(2)}{C}{}_2' \sin 2\theta - \frac{1}{2} [\overset{(1)}{B}{}_2' + \overset{(1)}{A}{}_0''].$$

and therefore

$$r^{-1}(\sigma_r)_4 = 2 \overset{(1)}{B}_4 (\cos 3\theta + \frac{1}{3} \cos\theta)$$

$$+ 2 \overset{(2)}{B}_4 (\sin 3\theta + \sin\theta) - \frac{1}{3} [\overset{(1)}{A}{}_2' + \overset{(1)}{B}{}_0''] \cos\theta,$$

$$r^{-1}(\sigma_\theta)_4 = - 2 \overset{(1)}{B}_4 (\cos 3\theta - \cos\theta)$$

$$- 2 \overset{(2)}{B}_4 (\sin 3\theta - 3\sin\theta),$$

$$r^{-1}(\sigma_z)_4 = 8\nu \, [\frac{1}{3} \overset{(1)}{B}_4 \cos\theta + \overset{(2)}{B}_4 \sin\theta]$$

$$+ \frac{1}{3} [(3+2\nu) \overset{(1)}{A}{}_2' - \nu \overset{(1)}{B}{}_0''] \cos\theta$$

$$- (1+\nu) \overset{(2)}{B}{}_0'' \sin\theta,$$

$$r^{-1}(\tau_{r\theta})_4 = -2 \overset{(1)}{B}_4 (\sin 3\theta - \tfrac{1}{3} \sin\theta)$$

$$+ 2 \overset{(2)}{B}_4 (\cos 3\theta - \cos\theta)$$

$$+ \tfrac{1}{3} [\overset{(1)}{A}{}'_2 + \overset{(1)}{B}{}''_0] \sin\theta,$$

$$r^{-1}(\tau_{\theta z})_4 = -[\overset{(1)}{A}_4 + \tfrac{1}{2} \overset{(1)}{B}{}'_2] \sin 2\theta + \tfrac{1}{2} \overset{(2)}{C}{}'_2 (1 - \cos 2\theta), \tag{47}$$

$$r^{-1}(\tau_{zr})_4 = [\overset{(1)}{A}_4 + \tfrac{1}{2} \overset{(1)}{B}{}'_2] \cos 2\theta - \tfrac{1}{2} \overset{(2)}{C}{}'_2 \sin 2\theta$$

$$- \nu \overset{(1)}{B}{}'_2 - \frac{1+\nu}{2} \overset{(1)}{A}{}''_0.$$

It is clear that all terms containing positive integer powers of r remain finite as $r \to 0$.

<u>Odd integers of n</u>.

1. <u>n=1</u>. This portion of the solution will give rise to singularities in the stresses. The vanishing of the stresses at $\theta = \pm \pi$ renders

$$\overset{(1)}{A}_1 = 0, \quad \overset{(1)}{C}_1 = \overset{(1)}{B}_1, \quad \overset{(2)}{C}_1 = -\tfrac{1}{3} \overset{(2)}{B}_1, \tag{48}$$

from which it can be deduced that

$$f_1 = {}^{(1)}B_1 \left[\cos\frac{3\theta}{2} - (5-8\nu)\cos\frac{\theta}{2}\right]$$

$$+ {}^{(2)}B_1 \left[\sin\frac{3\theta}{2} - \frac{1}{3}(5-8\nu)\sin\frac{\theta}{2}\right],$$

$$g_1 = -{}^{(1)}B_1 \left[\sin\frac{3\theta}{2} - (7-8\nu)\sin\frac{\theta}{2}\right]$$

$$+ {}^{(2)}B_1 \left[\cos\frac{3\theta}{2} - \frac{1}{3}(7-8\nu)\cos\frac{\theta}{2}\right],$$

$$h_1 = {}^{(2)}A_1 \sin\frac{\theta}{2}.$$

The stresses, which become unbounded at $r=0$, are found to be

$$r^{1/2}(\sigma_r)_1 = \frac{1}{2}\,{}^{(1)}B_1 \left(\cos\frac{3\theta}{2} - 5\cos\frac{\theta}{2}\right)$$

$$+ \frac{1}{2}\,{}^{(2)}B_1 \left(\sin\frac{3\theta}{2} - \frac{5}{3}\sin\frac{\theta}{2}\right),$$

$$r^{1/2}(\sigma_\theta)_1 = -\frac{1}{2}\,{}^{(1)}B_1 \left(\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}\right)$$

$$- \frac{1}{2}\,{}^{(2)}B_1 \left(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right),$$

$$r^{1/2}(\sigma_z)_1 = -4\nu\left[{}^{(1)}B_1 \cos\frac{\theta}{2} + \frac{1}{3}\,{}^{(2)}B_1 \sin\frac{\theta}{2}\right],$$

$$r^{1/2}(\tau_{r\theta})_1 = -\frac{1}{2}\,{}^{(1)}B_1\left(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right)$$

$$+\frac{1}{2}\,{}^{(2)}B_1\left(\cos\frac{3\theta}{2} + \frac{1}{3}\cos\frac{\theta}{2}\right),$$

$$r^{1/2}(\tau_{\theta z})_1 = \frac{1}{4}\,{}^{(2)}A_1\cos\frac{\theta}{2}, \qquad (49)$$

$$r^{1/2}(\tau_{zr})_1 = \frac{1}{4}\,{}^{(2)}A_1\sin\frac{\theta}{2}.$$

2. <u>n=3</u>. The relations necessary to meet the conditions in equation (9) are

$${}^{(1)}A_3 = -\frac{2}{3}(7-8\nu)\,{}^{(1)}B_1', \quad {}^{(1)}C_3 = -\,{}^{(1)}B_3,$$

$${}^{(2)}C_3 = -\frac{1}{5}\,{}^{(2)}B_3 - \frac{1}{15}\,{}^{(2)}A_1' \qquad (50)$$

which may be used to give

$$f_3 = {}^{(1)}B_3\left[\cos\frac{5\theta}{2} + (3-8\nu)\cos\frac{\theta}{2}\right]$$

$$+ {}^{(2)}B_3\left[\sin\frac{5\theta}{2} + \frac{1}{5}(3-8\nu)\sin\frac{\theta}{2}\right]$$

$$-\frac{2}{15}(1+4\nu)\,{}^{(2)}A_1'\sin\frac{\theta}{2},$$

$$g_3 = - {}^{(1)}B_3 \left[\sin\frac{5\theta}{2} - (9-8\nu)\sin\frac{\theta}{2}\right]$$

$$+ {}^{(2)}B_3 \left[\cos\frac{5\theta}{2} - \frac{1}{5}(9-8\nu)\cos\frac{\theta}{2}\right]$$

$$- \frac{4}{15}(1-2\nu)\, {}^{(2)}A_1' \cos\frac{\theta}{2},$$

$$h_3 = 2\, {}^{(1)}B_1' \left[-\frac{1}{3}(7-8\nu)\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right]$$

$$+ {}^{(2)}A_3 \sin\frac{3\theta}{2} + \frac{2}{3}\, {}^{(2)}B_1' \sin\frac{\theta}{2},$$

and the stresses become

$$r^{-1/2}(\sigma_r)_3 = \frac{3}{2}\, {}^{(1)}B_3 \left(\cos\frac{5\theta}{2} + 3\cos\frac{\theta}{2}\right)$$

$$+ \frac{3}{2}\, {}^{(2)}B_3 \left(\sin\frac{5\theta}{2} + \frac{3}{5}\sin\frac{\theta}{2}\right) - \frac{1}{5}\, {}^{(2)}A_1' \sin\frac{\theta}{2},$$

$$r^{-1/2}(\sigma_\theta)_3 = -\frac{3}{2}\, {}^{(1)}B_3 \left(\cos\frac{5\theta}{2} - 5\cos\frac{\theta}{2}\right)$$

$$- \frac{3}{2}\, {}^{(2)}B_3 \left(\sin\frac{5\theta}{2} - \sin\frac{\theta}{2}\right),$$

$$r^{-1/2}(\sigma_z)_3 = 12\nu\ [{}^{(1)}B_3 \cos\frac{\theta}{2} + \frac{1}{5}\ {}^{(2)}B_3 \sin\frac{\theta}{2}]$$

$$+ \frac{1}{5}\ (5+4\nu)\ {}^{(2)}A_1' \sin\frac{\theta}{2},$$

$$r^{-1/2}(\tau_{r\theta})_3 = -\frac{3}{2}\ {}^{(1)}B_3\ (\sin\frac{5\theta}{2} - \sin\frac{\theta}{2})$$

$$+ \frac{3}{2}\ {}^{(2)}B_3\ (\cos\frac{5\theta}{2} - \frac{1}{5}\cos\frac{\theta}{2})$$

$$- \frac{1}{10}\ {}^{(2)}A_1' \cos\frac{\theta}{2}, \qquad (51)$$

$$r^{-1/2}(\tau_{\theta z})_3 = \frac{3}{4}\ {}^{(2)}A_3 \cos\frac{3\theta}{2} + (3-4\nu)\ {}^{(1)}B_1'\ (\sin\frac{3\theta}{2} + \sin\frac{\theta}{2})$$

$$+ \frac{1}{2}\ {}^{(2)}B_1'\ [\cos\frac{3\theta}{2} - \frac{2}{3}\ (3-4\nu)\cos\frac{\theta}{2}],$$

$$r^{-1/2}(\tau_{zr})_3 = -\ {}^{(1)}B_1'\ [(3-4\nu)\cos\frac{3\theta}{2} + (1-4\nu)\cos\frac{\theta}{2}]$$

$$+ \frac{3}{4}\ {}^{(2)}A_3 \sin\frac{3\theta}{2} + \frac{1}{2}\ {}^{(2)}B_1'\ [\sin\frac{3\theta}{2}$$

$$- \frac{2}{3}\ (1-4\nu)\sin\frac{\theta}{2}].$$

The same procedure may be followed for the development of the recurrence relations[10] for the stress components.

[10] Because of space limitation, these relationships will not be given here, but will be reported in a subsequent communication.

The coefficients

$$\overset{(i)}{A}_n, \quad \overset{(i)}{B}_n, \quad i=1,2; \; n=1,2,---$$

in equations (44), (47), (49) and (51) depend on the variable z and they are to be determined from the loading conditions of the problem under consideration.

Of particular interest is the behavior of the stress solution in the limit as the distance r tends to zero. The terms that approach high, mathematically infinite, values at r=0 are those shown in equations (49). In contrast to the two-dimensional solution, all six components of the stress tensor are present in the immediate vicinity of the crack front, and they all vary as the inverse square root of the radial distance r. The coefficients $\overset{(i)}{A}_1$, $\overset{(i)}{B}_1$ (i=1,2), which vary along the leading edge of the crack, may be physically interpretated as quantities that reflect the redistribution of the three-dimensional stresses in a solid due to the introduction of a plane crack. In fact, equations (49) represent the most general stress state near the crack border and include the two-dimensional problems of plane extension [8] and longitudinal shear [13] as special cases:

1. Plane extension.

If the coefficients in equations (49) are independent of z and such that

$$ {}^{(2)}A_1 = 0, \quad {}^{(1)}B_1 = -\frac{k_1}{2\sqrt{2}} = \text{const.}, \quad {}^{(2)}B_1 = \frac{3k_2}{2\sqrt{2}} = \text{const.} $$

then the two-dimensional problem of plane strain is recovered. The parameters k_1 and k_2 [14] are known, respectively, as the crack-tip stress-intensity factors for loads applied symmetrically and skew-symmetrically with respect to the line crack.

2. Longitudinal Shear.

On the other hand, if

$$ {}^{(2)}A_1 = 2\sqrt{2}\ k_3 = \text{const.}, \quad {}^{(i)}B_1 = 0, \ i=1,2 $$

the stress state in equations (49) represent that of a crack under anti-plane shear deformation and k_3 is the corresponding crack-tip stress-intensity factor.

It should be noted in passing that equations (49) also display the three-dimensional character of the singular stresses interior to a thick plate containing a through crack. On the surface layers, where the crack penetrates through the plate, singularities of order different from $r^{-1/2}$ may exist. Nevertheless, the present analysis does indicate that the crack problem solution of Hartranft and Sih [15], based on the Reissner's theory of plate bending, indeed possesses the correct functional relationship of the in-plane stresses to r and θ for values of $|z|<h/2$, where z is the thickness coordinate and h the plate thickness. For further discussion of the thickness problem, the reader may refer to the work of Sih et al [16].

WEDGE PROBLEMS IN THREE-DIMENSIONS

As mentioned earlier, the eigenfunction expansion method outlined here may also be used for examining the three-dimensional stress distribution near a V-shaped notch in an infinite solid. Similar to the two-dimensional case [7], unbounded stresses may occur for certain notch angles depending on the boundary conditions specified on the radial edges of the notch. The three possible combinations of boundary conditions are

1. Free-Free.

$$\sigma_\theta = \tau_{\theta r} = \tau_{\theta z} = 0, \text{ at } \theta = \pm \alpha \qquad (52)$$

2. Clamped-Clamped.

$$u_r = v_\theta = w_z = 0, \text{ at } \theta = \pm \alpha \qquad (53)$$

3. Clamped-Free.

$$\begin{aligned} \sigma_\theta = \tau_{\theta r} = \tau_{\theta z} = 0, \text{ at } \theta = + \alpha \\ u_r = v_\theta = w_z = 0, \text{ at } \theta = - \alpha \end{aligned} \qquad (54)$$

The boundaries of the notch or wedge are described by

$$0 \leq r < \infty, \quad -\alpha \leq \theta \leq \alpha, \quad -\infty < z < \infty.$$

To fix ideas, consider the case of free-free edges. The six boundary conditions in equations (52) yield the same expressions as those in equations (10) provided π is replaced by α. However, there results four sets of eigenvalues $\lambda_m^{(i)}$ (i=1, 2,---, 4):

$$\sin 2\lambda_m^{(1)} \alpha = \lambda_m^{(1)} \sin 2\alpha,$$

$$\sin 2\lambda_m^{(2)} \alpha = - \lambda_m^{(2)} \sin 2\alpha,$$

$$\lambda_m^{(3)} = m\left(\frac{\pi}{\alpha}\right), \tag{55}$$

$$\lambda_m^{(4)} = \left(m + \frac{1}{2}\right) \frac{\pi}{\alpha}.$$

where m=0,1,2,---. Note that the first two eigenequations in equations (55) coincide with those obtained by Williams [7] for the planar analysis, while the eigenvalues $\lambda_m^{(3)}$ and $\lambda_m^{(4)}$ correspond to those found by Sih [13] for the problem of a sector cylinder under longitudinal shear. In the three-dimensional case, the four sets of $\lambda_m^{(i)}$ occur simultaneously. It follows that the components of the displacement vector are given by

$$2\mu\, u_r = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \sum_{i=1}^{4} r^{\lambda_m^{(i)}+n}\, U_n^{(m)}(\theta,z;\, \lambda_m^{(i)}),$$

$$2\mu\, v_\theta = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \sum_{i=1}^{4} r^{\lambda_m^{(i)}+n}\, V_n^{(m)}(\theta,z;\, \lambda_m^{(i)}), \qquad (56)$$

$$2\mu\, w_z = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \sum_{i=1}^{4} r^{\lambda_m^{(i)}+n}\, W_n^{(m)}(\theta,z;\, \lambda_m^{(i)}).$$

For this problem, the stress components in equations (6) for each of the four eigenvalues $\lambda_m^{(i)}$ must be summed in i from one to four as it was done in equations (56) for the displacements.

The cases of clamped-clamped and clamped-free edges warrant no further comments, since they may be treated in the same way.

CONCLUSIONS

The eigenfunction expansion technique, used previously for analyzing two-dimensional crack and wedge problems, has been extended to the three-dimensional case. The three displacement components valid everywhere in the infinite solid weakened by a half-plane crack are derived in closed form. The pertinent steps for finding the recurrence relations of the stress components are also laid out in detail. In particular, the singular behavior of the three-dimensional stress field near a straight-edged crack is exhibited for the first time.

It is anticipated that further exploitation of the present investigation will provide a knowledge of the structure of the stress coefficients $A_n^{(i)}$ and $B_n^{(i)}$ for problems of half-plane cracks opened by concentrated normal and shear forces.

ACKNOWLEDGEMENTS

One of the authors (George C. Sih) wishes to thank Professor M. L. Williams for many helpful discussions concerning this work, which was motivated from his earlier contribution on the eigenfunction expansion method in two-dimensions. The authors are also indebted to Mr. J. E. Srawley for his encouragement and enthusiasm in this work.

REFERENCES

1. A. I. Lur'e, "On the Theory of Thick Plates", Prikl. matem. i mekh., Vol. 6, 1942, pp. 151-168.

2. A. E. Green, "The Elastic Equilibrium of Isotropic Plates and Cylinders", Proc. of Roy. Soc. (London), Vol. 195, 1949, pp. 533-552.

3. I. N. Sneddon, "The Distribution of Stress in the Neighborhood of a Crack in an Elastic Solid", Proc. of Roy. Soc. (London), Vol. 187, 1946, pp. 229-260.

4. A. E. Green and I. N. Sneddon, "The Distribution of Stress in the Neighborhood of a Flat Elliptical Crack in an Elastic Solid", Proc. of Camb. Phil. Soc., Vol. 46, 1950, pp. 159-164.

5. M. K. Kassir and G. C. Sih, "Three-Dimensional Stress Distribution Around an Elliptical Crack under Arbitrary Loadings", J. of Appl. Mech., Vol. 33, 1966, pp. 601-611.

6. M. K. Kassir and G. C. Sih, "External Elliptical Crack in Elastic Solid", Int. J. of Fracture Mech. (in Press).

7. M. L. Williams, "Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates in Extension", J. of Appl. Mech., Vol. 18, 1952, pp. 526-528.

8. M. L. Williams, "On the Stress Distribution at the Base of a Stationary Crack", J. of Appl. Mech., Vol. 24, 1957, pp. 109-114.

9. M. L. Williams, "Surface Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates under Bending", Proc. First U.S. Nat. Congress of Appl. Mech. 1950, pp. 325-329.

10. G. C. Sih and J. R. Rice, "The Bending of Plates of Dissimilar Materials with Cracks", J. of Appl. Mech., Vol. 31, 1964, pp. 477-482.

11. J. R. Rice and G. C. Sih, "Plane Problems of Cracks in Dissimilar Materials", J. of Appl. Mech., Vol. 32, 1965, pp. 418-423.

12. J. F. Loeber and G. C. Sih, "Green's Function for Cracks in Non-Homogeneous Materials", J. of Appl. Mech., 34, 1967, pp. 240-243.

13. G. C. Sih, "Stress Distribution Near Internal Crack Tips for Longitudinal Shear Cracks", J. of Appl. Mech., Vol. 32, 1965, pp. 51-58.

14. G. C. Sih and H. Liebowitz, "Mathematical Theories of Brittle Fracture", Mathematical Fundamentals of Fracture, Vol. 2, Academic Press, New York, New York, 1968.

15. R. J. Hartranft and G. C. Sih, "Effect of Plate Thickness on the Bending Stress Distribution around Through Cracks", J. of Math. Phys. (in Press).

16. G. C. Sih, M. L. Williams and J. L. Swedlow, "Three-Dimensional Stress Distribution Near a Sharp Crack in a Plate of Finite Thickness", Air Force Materials Laboratory, Wright-Patterson Air Force Base, November 1966 (AFML-TR-66-242).