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THE STRUCTURE OF THE EQUAL TIME ANTICOMMUTATOR OF THE BARYON CURRENT IN NEUTRAL PSEUDOSCALAR MESON THEORY

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CONTENTS

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Abstract

The equal time anticommutator of the baryon current h(x) which is the source term of the Dirac equation $(i\partial\!\!\!/ - m)\ \psi = ig\gamma_5 h(x)$ is studied for the case of a neutral pseudoscalar meson interaction. A method of calculating the equal time anticommutator is given which incorporates the basic structure arising out of the Wilson hypothesis used to define the current. It is found that derivatives of $\delta^3\ (\underline{x}-\underline{x}')$ do not appear in the anticommutator. The absence of such Schwinger terms permits a theorem to be stated regarding the high energy behavior of the spectral functions $\rho_1(M^2)$ and $\rho_2(M^2)$ which characterize the baryon propagator $S_F'(p)$.

I. Introduction

Presently much interest in the equal time current algebra proposed by Gell-Mann has motivated serious investigation of the properties of equal time commutation relations involving various field theoretic models of current operators. This interest in equal time commutation relations has led us to study the structure of the anticommutator $\{h(x), \bar{h}(x')\}$[1] of the baryon current in a neutral pseudoscalar meson theory.

We make use of the hypothesis of K. Wilson[2] to define the appropriate current operator h(x) that should appear in the renormalized field equation

$$(i\partial\!\!\!/ - m)\,\psi(x) = i\,g\gamma_5 h(x)\ .$$

We then use this operator to compute the equal time anticommutator by a powerful method which has recently been applied to calculate commutators of the electromagnetic current in quantum electrodynamics.[3] We find that the anticommutator $\{h(x), \bar{h}(x')\}$ contains no derivatives of $\delta^3(\underline{x}-\underline{x}')$. By trying to understand the absence of such terms we investigate the most general properties of the vacuum expectation value $\langle 0|\{h(x), \bar{h}(x')\}|0\rangle$. This analysis leads us to the conclusion that the nonappearance of derivatives, in the equal time anticommutator implies the existence of a theorem which states that the spectral functions $\rho_1(M^2)$ and $\rho_2(M^2)$ which describe the propagator $S_F'(p)$ must approach zero in the high energy limit.

In Sec. II we define the method for calculating our equal time anticommutator. In Sec. III we outline the calculation and discuss the structure of the anticommutator. In Sec. IV we make a general analysis of the vacuum expectation value of the equal time anticommutator in terms of its spectral properties and in Sec. V we derive the theorem involving the high energy behavior of ρ_1 and ρ_2.

II. Definition of the Equal Time Anticommutator of the Wilson Current h(x)

According to the hypothesis of K. Wilson[2] the product $\psi(x)\phi(x-\xi)$ of a baryon and neutral pseudoscalar meson field in which ξ is a space-time variable near the point x, is given by the expression

$$\psi(x)\phi(x-\xi) \simeq B_1(\xi)\,\psi(x) + B_{2\mu}(\xi)\,\partial^\mu\psi(x) + E_3(\xi)\,:\psi(x)\phi(x): \qquad (2.1)$$

as $\xi \to 0$. $E_3(\xi)$, $B_1(\xi)$, $B_{2\mu}(\xi)$ are matrix functions in the spinor space of the baryon field and have singularities at the point $\xi = 0$. $:\psi\phi:$ denotes the generalized wick product and is defined solely by Eq. (2.1). If we denote $h(x;\xi)$ by $E_3^{-1}(\xi)[\psi(x)\phi(x-\xi) - B_1(\xi)\psi(x) - B_{2\mu}(\xi)\partial^\mu\psi(x)]$ then the baryon current appearing in the field equation[4]

$$(i\partial\!\!\!/ - m)\,\psi(x) = ig\gamma_5 h(x) \qquad (2.2)$$

is given by the relation

$$h(x) = \lim_{\xi\to 0} h(x\,;\,\xi) \;. \qquad (2.3)$$

Brandt has proved that Eq. (2.2) with the source current h(x) given by Eq. (2.3) is valid to all orders in the sense that the renormalized integral equations which it implies are derivable from and equivalent to renormalized perturbation theory.[5]

In addition to Eq. (2.3) we also have the relation

$$h^\dagger(x') = \lim_{\xi'\to 0} h^\dagger(x'\,;\,\xi') \;. \qquad (2.4)$$

We now compute the equal time anticommutator $\{h(x), h^\dagger(x')\}$ according to the prescription

$$\{h(x), h^\dagger(x')\} = \lim_{\substack{\xi \to 0 \\ \xi' \to 0}} \{h(x\,;\,\xi), h^\dagger(x'\,;\,\xi')\}_{\xi_0 = \xi_0' = 0} \tag{2.5}$$

where we evaluate the right hand side by using the following known equal time commutation relations for the renormalized fields ψ and ϕ

$$\begin{aligned}
&\{\psi_\alpha(\underline{x}, t), \psi_\beta^\dagger(\underline{x}', t)\} = \frac{1}{Z_2}\,\delta_{\alpha\beta}\,\delta^3(\underline{x} - \underline{x}'),\\
&\{\psi_\alpha(\underline{x}, t), \psi_\beta(\underline{x}', t)\} = \{\psi_\alpha^\dagger(\underline{x}, t), \psi_\beta^\dagger(\underline{x}', t)\} = 0\,,\\
&[\phi(\underline{x}, t), \psi_\alpha^\dagger(\underline{x}', t)] = [\phi(\underline{x}, t), \psi_\alpha(\underline{x}', t)] = 0\,,\\
&[\phi(\underline{x}, t), \phi(\underline{x}', t)] = 0\,.
\end{aligned} \tag{2.6}$$

III Structure of the Equal Time Anticommutator

If we denote the matrices $E_3^{-1}(\xi)\, B_1\,(\xi)$ and $E_3^{-1}\,(\xi)\, B_{2\mu}\,(\xi)$ by $C_1'\,(\xi)$ and $C_{2\mu}(\xi) = (C_{20}\,,\, \underline{C}_2)$ respectively then $\{h(x), h^\dagger(x')\}$ may be written as the limit

$$\lim_{\substack{\xi\to 0\\ \xi'\to 0}} \{h(x\,;\,\xi),\, h^\dagger(x'\,;\,\xi')\} \qquad \xi_0 = \xi_0' = 0 \tag{3.1}$$

$$= \lim_{\substack{\xi\to 0\\ \xi'\to 0\\ \xi''\to 0\\ \xi'''\to 0}} \frac{1}{Z_2} \Big[(Q(\xi,\, \xi',\, \xi'',\, \xi''') + S(\xi,\, \xi',\, \xi'',\, \xi''')) \,\delta^3\,(\underline{x} - \underline{x}')$$

$$+ (\underline{Q}\,(\xi,\, \xi',\, \xi'',\, \xi''') + \underline{S}\,(\xi,\, \xi',\, \xi'',\, \xi''')) \cdot \nabla_x\, \delta^3(\underline{x} - \underline{x}')$$

$$+ S'\,(\xi,\, \xi',\, \xi'',\, \xi''')\, \nabla_x^2\, \delta^3\,(\underline{x} - \underline{x}') \Big]$$

taken in a rotationally invariant manner. Q, $\underline{Q}$ are q number matrix functions and S, $\underline{S}$, S' are c number functions given by the expressions

$$Q(\xi,\, \xi',\, \xi'',\, \xi''') = E_3^{-1}(\xi) E_3^{\dagger -1}(\xi')\phi(x - \xi)\phi(x' - \xi') \tag{3.2 a}$$

$$- E_3^{-1}(\xi)\; A(\xi'')\; D^{-1}(\xi'')\; C_{20}^{\dagger}\, \phi(x' - \xi'')\; \phi(x - \xi)$$

$$- C_{20}(\xi)\; H^{-1}(\xi''')\; E(\xi''')\; E_3^{\dagger -1}(\xi')\; \phi(x - \xi''')\; \phi(x' - \xi')$$

$$+ C_{20}(\xi)\; H^{-1}(\xi''')\; E(\xi''')\; A(\xi'')\; D^{-1}(\xi'')\; C_{20}^{\dagger}(\xi')\; \phi(x - \xi''')\; \phi(x' - \xi'')$$

$$- E_3^{-1}(\xi)\; C_1'^{\dagger}(\xi')\; \phi(x - \xi) + E_3^{-1}(\xi)\; B(\xi'')\; D^{-1}(\xi'')\; C_{20}^{\dagger}(\xi')\; \phi(x - \xi)$$

$$- C_1'\,(\xi) E_3^{\dagger -1}(\xi')\; \phi(x' - \xi') + C_1'\,(\xi)\; A(\xi'') D^{-1}(\xi'')\; C_{20}^{\dagger}(\xi')\; \phi(x' - \xi'')$$

$$- C_{20} H^{-1}(\xi'')F(\xi''')E_3^{\dagger -1}(\xi')\phi(x' - \xi')$$

$$+ C_{20}(\xi)H^{-1}(\xi''')E(\xi''')C_1'^{\dagger}(\xi')\phi(x - \xi''')$$

$$+ C_{20}(\xi)H^{-1}(\xi''')E(\xi''')B(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')\phi(x - \xi''')$$

$$+ C_{20}(\xi)H^{-1}(\xi''')F(\xi''')A(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')\phi(x' - \xi'')$$

$$S(\xi, \xi', \xi'', \xi''') = C_1'(\xi)C_1'^{\dagger}(\xi') + C_1'(\xi)B(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi') \tag{3.2 b}$$

$$+ C_{20}(\xi)H^{-1}(\xi''')F(\xi''')C_1'^{\dagger}(\xi') + C_{20}(\xi)H^{-1}(\xi''')F(\xi''')B(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')$$

$$\underline{Q}(\xi, \xi', \xi'', \xi''') = E_3^{-1}(\xi)\underline{C}(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')\phi(x - \xi) \tag{3.2 c}$$

$$- E_3^{-1}(\xi)\underline{C}_2^{\dagger}(\xi')\phi(x - \xi) - C_{20}(\xi)H^{-1}(\xi''')\underline{G}(\xi''')E_3^{-1}(\xi')\phi(x' - \xi')$$

$$+ \underline{C}_2(\xi)E_3^{\dagger -1}\phi(x' - \xi') + C_{20}(\xi)H^{-1}(\xi''')E(\xi''')\underline{C}_2^{\dagger}(\xi')\phi(x - \xi''')$$

$$- \underline{C}_2(\xi)A(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')\phi(x' - \xi'')$$

$$- C_{20}(\xi)H^{-1}(\xi''')E(\xi''')\underline{C}(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')\phi(x - \xi'')$$

$$+ C_{20}(\xi)H^{-1}(\xi''')\underline{G}(\xi''')A(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')\phi(x' - \xi'')$$

$$\underline{S}(\xi, \xi', \xi'', \xi''') = - C_1'(\xi)\underline{C}(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi') \tag{3.2 d}$$

$$+ C_1'(\xi)\underline{C}_2^{\dagger}(\xi') + C_{20}(\xi)H^{-1}(\xi''')\underline{G}(\xi''')C_1'^{\dagger}(\xi')$$

$$- \underline{C}_2(\xi)C_1'^{\dagger}(\xi') + C_{20}(\xi)H^{-1}(\xi''')F(\xi''')\underline{C}_2^{\dagger}(\xi')$$

$$- \underline{C}_2(\xi)B(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi') - C_{20}(\xi)H^{-1}(\xi''')F(\xi''')\underline{C}(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')$$

$$+ C_{20}(\xi)H^{-1}(\xi''')\underline{G}(\xi''')B(\xi'')D^{-1}(\xi'')C_{20}^{\dagger}(\xi')$$

$$S'(\xi, \xi', \xi'', \xi''') = \frac{1}{3}\Big[- \underline{C_2}(\xi)\cdot \underline{C_2}^\dagger(\xi') + C_{20}(\xi)\, H^{-1}(\xi''')\, \underline{G}(\xi''')\cdot \underline{C_2}^\dagger(\xi') \qquad (3.2\text{ e})$$

$$+ \underline{C_2}(\xi)\cdot \underline{C}(\xi'')\, D^{-1}(\xi'')\, C_{20}^\dagger(\xi')$$

$$- C_{20}(\xi)\, H^{-1}(\xi''')\underline{G}(\xi''')\cdot \underline{C}(\xi'')\, D^{-1}(\xi'')\, C_{20}^\dagger(\xi')\Big]$$

and A, B, $\underline{C}$, D, E, F, $\underline{G}$, H are the following functions of E_3^{-1}, C_1' and $C_{2\mu}$

$$A = g E_3^{\dagger -1}\, \gamma_5 \gamma_0$$

$$B = i\ [m\gamma_0 + i g C_1'^\dagger \gamma_5 \gamma_0]$$

$$\underline{C} = [\underline{\gamma}\gamma_0 + g \underline{C_2}^\dagger\, \gamma_5\gamma_0] \qquad (3.3)$$

$$D = [1 + g C_{20}^\dagger \gamma_5 \gamma_0]$$

$$E = A^\dagger,\ F = B^\dagger,\ \underline{G} = \underline{C}^\dagger,\ H = D^\dagger\ .$$

The reason for this rather complicated structure is due to the presence of the time derivative $\dot{\psi}$ which appears in the expression for h(x) and $h^\dagger$ (x'). We have computed all commutators involving $\dot{\psi}$ by making use of the field equation (2.2) and its adjoint.

In order to simplify Eq. (3.1) we shall relate the subtraction coefficients E_3^{-1}, C_1' at $\xi = 0$ to the renormalization parameters Z_1, Z_2, Z and $\delta m = m - m_0$ by the following method. We consider the unrenormalized field equation

$$(\, i\not\partial - m_0)\, \psi_0 = i\, g_0 \gamma_5 \psi_0 \phi_0 \qquad (3.4)$$

and rewrite it as usual in terms of the renormalized fields ψ and ϕ, the physical mass m and the renormalized coupling constant $g = z_1^{-1} Z_2 Z^{1/2} g_0$. We find Eq. (3.4) to assume the form

$$(i\partial\!\!\!/ - m)\,\psi = i\,\frac{Z_1}{Z_2}\,g\gamma_5\psi\phi - \delta m\psi\ . \tag{3.5}$$

On comparing Eq. (3.5) with Eq. (2.2) we deduce the relations

$$E_3^{-1}(0) = \frac{Z_1}{Z_2}\,.\,1$$

$$C_1'(0) = -\,i\,\frac{\delta m}{g}\,\gamma_5\ . \tag{3.6a}$$

Owing to the existence of the derivative term $C_{2\mu}\partial^\mu\psi$ appearing in h we must further introduce a new subtraction constant C_2 defined by the condition

$$C_{2\mu}(0) = (C_2\gamma_5\gamma_0,\ -C_2\gamma_5\underline{\gamma}) \tag{3.6b}$$

where C_2 is in general a complex number. We also note that Eq. (3.6b) expresses the only form $C_{2\mu}(0)$ may assume consistent with the requirement of rotational invariance. Using Eq. (3.6) to evaluate Eq. (3.3), the matrices A, B, $\underline{C}$, D, E, F, $\underline{G}$, H are found to have the form

$$A = g\,\frac{Z_1}{Z_2}\,\gamma_5\gamma_0\ ,\quad E = -\,g\,\frac{Z_1}{Z_2}\,\gamma_5\gamma_0\ ,$$

$$B = i\,m_0\gamma_0\ ,\quad F = -\,i\,m_0\gamma_0\ ,$$

$$\underline{C} = (1 + g\,C_2^*)\,\underline{\gamma}\gamma_0\ ,\quad \underline{G} = (1 + g\,C_2)\,\underline{\gamma}\gamma_0\ , \tag{3.7}$$

$$D = (1 + g\,C_2^*)\ ,\quad H = (1 + g\,C_2)\ .$$

By employing Eq. (3.7) we are formally able to evaluate the terms Q, S, $\underline{Q}$, $\underline{S}$, S' which appear on the right hand side of Eq. (3.1) in the limit $\begin{matrix} \xi \to 0 \\ \xi' \to 0 \\ \xi'' \to 0 \\ \xi''' \to 0 \end{matrix}$. We find that the functions $\underline{Q}$, $\underline{S}$ and S' vanish, which means that the equal time anticommutator $\{h(x), h^{\dagger}(x')\}$ contains no derivatives of $\delta^3(\underline{x} - \underline{x}')$, or so called Schwinger terms. This situation is to be contrasted with the result established for equal time commutators of the electromagnetic current in electrodynamics in which the existence of Schwinger terms has been demonstrated explicitly.[3]

We shall now investigate the most general conditions relating to the properties of $\{h(x), h^{\dagger}(x')\}$ that can lead to our result.

IV. The Spectral Representation of the Vacuum Expectation Value $\langle 0 | \{h(x), \bar{h}(x')\} | 0 \rangle$

We have shown that the equal time anticommutator is given by the expression

$$\{h(x), \bar{h}(x')\} = \frac{1}{Z_2}\Big[M_1\ \delta^3(\underline{x} - \underline{x}')\ :\ \phi^2(x)\ : \\ + M_2\ \delta^3(\underline{x} - \underline{x}')\ \phi(x) + M_3\ \delta^3(\underline{x} - \underline{x}')\Big] \tag{4.1}$$

where M_1, M_2, M_3 are the matrices

$$M_1 = \left[\left(\frac{Z_1}{Z_2}\right)^2 \Big/ (1 + g\,C_2)\,(1 + g\,C_2^*)\right]\gamma_0 \ , \tag{4.2}$$

$$M_2 = i\,\frac{Z_1}{Z_2}\ \frac{[2m_0 g\,|C_2|^2 + m_0(C_2^* + C_2) + \delta m(C_2 - C_2^*)]}{(1 + gC_2)\,(1 + gC_2^*)}\cdot \gamma_5\gamma_0 \ ,$$

$$M_3 = \left[\left[\frac{\delta m}{g}\right]^2 + \frac{\delta m}{g}\,m\,(C_2 + C_2^*) + m^2\,|C_2|^2 + \left(\frac{Z_1}{Z_2}\right)^2 J(0)\right] \Big/ {(1 + g\,C_2)\,(1 + g\,C_2^*)}\cdot\gamma_0$$

and $J(0)$ is related to the meson spectral function $\rho(\mu^2)$ by the equation

$$J(\xi) = i\int_0^\infty d\mu^2\ \rho(\mu^2)\ \Delta_F\,(\xi\ ;\ \mu^2)\ . \tag{4.3}$$

Now let us take Eq. (4.1) between the vacuum state. By the properties of the Wick product $:\phi^2(x):$ and because $\phi(x)$ is a pseudoscalar we find the vacuum expectation value to be of the form

$$\langle 0|\ \{h(x),\ \bar{h}(x')\}\ |0\rangle = \frac{1}{Z_2}\,M_3\ \delta^3(\underline{x} - \underline{x}') \tag{4.4}$$

Consider the vacuum expectation value in general, that is, before any equal time limit is taken. If we use Eq. (2.2) and its adjoint the general expression for the vacuum expectation value may be written as

$$\langle 0| \{h(x), \bar{h}(x')\} |0\rangle$$

$$= \frac{1}{g^2} \gamma_5 (i\overrightarrow{\partial\!\!\!/}_x - m) \langle 0| \{\psi(x), \bar{\psi}(x')\} |0\rangle (i\overleftarrow{\partial\!\!\!/}_{x'} + m) \gamma_5 \tag{4.5}$$

where $\langle 0| \{\psi(x), \bar{\psi}(x')\} |0\rangle$ is just the vacuum expectation value of the anti-commutator of the underlying baryon field and satisfies the well known spectral representation

$$i \langle 0| \{\psi(x), \bar{\psi}(x')\} |0\rangle \tag{4.6}$$

$$= S(x - x' ; m) - \frac{1}{Z_2} \int_{m_1^2}^{\infty} dM^2 [i \rho_1 (M^2) \partial\!\!\!/_x + \rho_2 (M^2)] \Delta(x - x' ; M)$$

in which $\Delta (x - x'; M)$ is the function

$$\Delta(x - x' ; M) = - i \int \frac{d^3k}{(2\pi)^3 2 \omega_k} (e^{-ik \cdot (x-x')} - e^{ik \cdot (x-x')})$$

with k_0 related to ω_k by $k_0 = \omega_k = \sqrt{\underline{k}^2 + M^2}$. ρ_1 and ρ_2 are scalar functions which have the well established properties [6]

$$\rho_1 (M^2) \text{ and } \rho_2 (M^2) \text{ are real }, \tag{i}$$

$$\rho_1 (M^2) \geq 0 , \tag{ii}$$

$$M\rho_1 (M^2) - \rho_2 (M^2) \geq 0 \tag{iii}$$

and the spectral integral which appears in Eq. (4.6) begins at m_1, the threshold of the continuum spectrum. On calculating the right hand side of Eq. (4.5) we find that $\langle 0| \{h(x), \bar{h}(0)\} |0\rangle$ becomes

$$\langle 0| \{h(x), \bar{h}(0)\} |0\rangle$$

$$= \frac{1}{Z_2} \cdot \frac{1}{g^2} \int_{a_1}^{\infty} da \Bigg[\rho_1(a) \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \Big[(a+c) \not{k} \, [e^{-ik\cdot x} + e^{ik\cdot x}]$$

$$- 2\sqrt{ac}\, [e^{-ik\cdot x} - e^{+ik\cdot x}] \Big] \tag{4.7}$$

$$- \rho_2(a) \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \Big[2\sqrt{c}\, \not{k} \, (e^{-ik\cdot x} + e^{ik\cdot x}) + (a+c)\,(e^{-ik\cdot x} - e^{ik\cdot x}) \Big] \Bigg]$$

with the variable change $M^2 = a$, $a_1 = m_1^2$ and $c = m^2$.

Now extreme caution must be exercised in computing the equal time limit! We do this by regarding $\langle 0| \{h(x), \bar{h}(0)\} |0\rangle$ as an improper function or distribution which we shall denote by $F(x, t)$. We therefore introduce a sequence of testing functions $\phi(x)\, f_n(t)$ such that the limit

$$f_n(t) \xrightarrow{n\to\infty} \delta(t)$$

holds and define $\langle F(x), \phi(\underline{x})\, f_n(t)\rangle$ by the relation

$$\langle F(x), \phi(\underline{x})\, f_n(t), \rangle = \int d^4x\, F(x)\, \phi(\underline{x})\, f_n(t) \,. \tag{4.8}$$

Then the equal time limit $F(x, 0)$ will be given by the prescription

$$\langle F(x, 0), \phi(\underline{x})\rangle = \lim_{n\to\infty} \langle F(x), \phi(\underline{x})\, f_n(t)\rangle \,. \tag{4.9}$$

To facilitate the calculation of $F(\underline{x}, 0)$ we introduce the Fourier transform $\hat{\phi}(\underline{p})\,\hat{f}_n(p_0)$ of $\phi(\underline{x})\,f_n(t)$ by the usual expression

$$\phi(\underline{x})\; f_n(t) = \frac{1}{(2\pi)^4}\int \hat{\phi}(\underline{p})\;\hat{f}_n(p_0)\; e^{-ip\cdot x}\, d^4p\;. \tag{4.10}$$

$\langle F(x), \phi(\underline{x})\, f_n(t)\rangle$ may then be written as

$$\frac{1}{(2\pi)^4}\int \hat{G}(k)\;\hat{\phi}(-\underline{k})\; f_n(-k_0)\; d^4k$$

where $\hat{G}(k)$ is the fourier transform of $F(x)$. Further introducing the functions $\pi_1^i(a)$ $(i = 1, 1/2, 0)$, $\pi_2^i(a)$ $(i = 0, 1)$ and $\hat{f}_n^e(p_0)$, $\hat{f}_n^0(p_0)$ by the relations

$$\pi_1^1 = a\rho_1(a),\; \pi_1^{1/2} = \sqrt{a}\;\rho_1(a),\; \pi_1^0 = \rho_1(a)\;,$$

$$\pi_2^0 = \rho_2(a),\; \pi_2^1(a) = a\;\rho_2(a)\;, \tag{4.11a}$$

and

$$\hat{f}_n^e(p_0) = \frac{f_n(p_0) + f_n(-p_0)}{2}\;,$$

$$\hat{f}_n^0(p_0) = \frac{f_n(p_0) - f_n(-p_0)}{2}\;, \tag{4.11b}$$

and employing Eq. (4.7) we have the useful result

$$\langle F(x),\; \phi(\underline{x})\; f_n(t)\rangle$$

$$= \frac{1}{Z_2}\cdot\frac{1}{g^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\left[\int_{a_1}^{\infty} da\left([\pi_1^1(a) + c\pi_1^0(a) - 2\sqrt{c}\;\pi_2^0(a)]\left(\frac{\partial}{\partial a}\right)^r \hat{f}_n^e(a^{1/2})\;\gamma_0\right.\right.$$

$$+ [2\sqrt{c}\,\pi_1^{1/2}(a) + \pi_2^1(a) + c_2\pi_2^0(a)]\left(\frac{\partial}{\partial a}\right)^r [\,\hat{f}_n^{\,0}(a^{1/2})a^{-1/2}]\Big)\ \frac{1}{(2\pi)^3}\int d^3k\,(-\underline{k}^2)^r\,\hat{\phi}(-k)$$

$$-\int_{a_1} da\left([\pi_1^1(a) + c\pi_1^0(a) - 2\sqrt{c}\,\pi_2^0(a)]\left(\frac{\partial}{\partial a}\right)^r [\hat{f}_n^{\,0}(a^{1/2})\,a^{-1/2}]\right)\ \mathrm{x}$$

$$\left.\frac{1}{(2\pi)^3}\int d^3k\,(-\underline{k}^2)^r\ \underline{k}\cdot\underline{\gamma}\,\hat{\phi}(-\underline{k})\right]. \tag{4.12}$$

From Eq. (4.12) we secure the desired equal time expression for $F(\underline{x}, 0)$ namely,

$$F(\underline{x},\ 0) = \langle 0|\ \{h(\underline{x},\ 0),\ \bar{h}(0)\}\ |0\rangle$$

$$= \frac{1}{Z_2}\cdot\frac{1}{g^2}\sum_{r=0}^{\infty}\Big[(K_{r1}^1 + cK_{r1}^0 - 2\sqrt{c}\,K_{r2}^0)\ \gamma_0(\nabla^2)^r\,\delta^3(\underline{x})$$

$$+\ i\,(\tilde{K}_{r1}^1 + c\tilde{K}_{r1}^0 - 2\sqrt{c}\,\tilde{K}_{r2}^0)\ (\nabla^2)^r\,\underline{\gamma}\cdot\nabla\delta^3(\underline{x}) \tag{4.13}$$

$$+\ (2\sqrt{c}\,\tilde{K}_{r1}^{1/2} + \tilde{K}_{r2}^1 + c\tilde{K}_{r2}^0)\ (\nabla^2)^r\ \delta^3(\underline{x})\Big]$$

where K_{rj}^i, $\tilde{K}_{rj}^1$ have the form

$$K_{r1}^i = \lim_{n\to\infty}\frac{(-1)^r}{r!}\int_{a_1}^{\infty} da\ \pi_1^i(a)\left(\frac{\partial}{\partial a}\right)^r\left(\hat{f}_n^{\,e}\ (a^{1/2})\right)$$

$$\tilde{K}_{r1}^i = \lim_{n\to\infty}\frac{(-1)^r}{r!}\int_{a_1}^{\infty} da\ \pi_1^i(a)\left(\frac{\partial}{\partial a}\right)^r\ [\hat{f}_n^{\,0}\ (a^{1/2})\ a^{-1/2}]$$

$$i = 1,\ 1/2,\ 0$$

$$K_{r2}^j = \lim_{n\to\infty}\frac{(-1)^r}{r!}\int_{a_1}^{\infty} da\ \pi_2^j(a)\left(\frac{\partial}{\partial a}\right)^r\left(\hat{f}_n^{\,e}\ (a^{1/2})\right)$$

$$\tilde{K}_{r2}^j = \lim_{n\to\infty}\frac{(-1)^r}{r!}\int_{a_1}^{\infty} da\ \pi_2^j(a)\left(\frac{\partial}{\partial a}\right)^r\ [\hat{f}_n\ (a^{1/2})\ a^{-1/2}]$$

$$j = 0,\ 1$$

If we choose a sequence $f_n(t)$ symmetric in t and if we demand that Eq. (4.13) be consistent with Eq. (4.4) then the following condition

$$K^1_{r1} + cK^0_{r1} - 2\sqrt{c}\,K^0_{r2} = 0 \tag{4.14}$$

must hold for all $r > 0$. Eq. (4.14) implies the very important result that the integral

$$I = \int_{a_1}^{\infty} [(a + c)\,\rho_1(a) - 2\sqrt{c}\,\rho_2(a)]\,da$$

must be finite.

V. A High Energy Theorem

We now use the fact that I is finite to show that the spectral functions $(a+c)\rho_1$ and ρ_2 approach zero as $a = M^2$ tends to infinity. The most general behavior of ρ_1 and ρ_2 compatible with the results of the previous section is given by functions of the form

$$\begin{aligned} (a + c)\, \rho_1 &= D(a) + f(a) \\ 2\sqrt{c}\, \rho_2 &= D(a) + g(a) \end{aligned} \tag{5.1}$$

where D(a) approaches a non-zero constant or infinity and both f(a) and g(a) tend to zero as a approaches infinity. The difference $h(a) = f - g$ approaches zero and is an integrable function giving the finite expression I. In this situation we have the following result

$$2\sqrt{c}\, \rho_2 \rightarrow (a + c)\, \rho_1 > 0 \text{ as } a \rightarrow \infty .$$

If we recall that $c = m^2 < M^2 = a$ then the assumed behavior of ρ_1 and ρ_2 (Eq. (5.1)) implies the condition

$$M\, \rho_1(M^2) - \rho_2(M^2) < 0$$

which is a contradiction! This inconsistency arises from the fact that D(a) was assumed to approach a non-zero constant or infinity. Therefore D(a) must be zero and hence the functions $(M^2 + m^2)\, \rho_1(M^2)$ and $\rho_2(M^2)$ approach zero as M^2 tends to infinity.

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References and Footnotes

1. Whenever we write $\{h(x), \bar{h}(x')\}$ we shall always mean $x_0 = x_0'$ unless explicitly stated otherwise as in Sec. IV.

2. K. Wilson, unpublished.

3. R. A. Brandt, University of Maryland Technical Report No. 727, August 1967 (to be published).

4. We adhere to the following notation of Bjorken and Drell

$$g_{00} = - g_{11} = - g_{22} = - g_{33} = 1, \ \underline{\gamma} = \{\gamma^i\} \ ,$$

$$\partial\!\!\!/ = \gamma_0 \frac{\partial}{\partial t} + \underline{\gamma}\cdot\nabla \, , \ \gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \, ,$$

$$\gamma_0^\dagger = \gamma_0, \ \underline{\gamma}^\dagger = -\underline{\gamma}, \ \{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta} \ .$$

5. R. A. Brandt, Annals of Physics: 44, 221-265 (1967).

6. Bjorken and Drell, Relativistic Quantum Fields, McGraw-Hill Book Company, N.Y. (1965).