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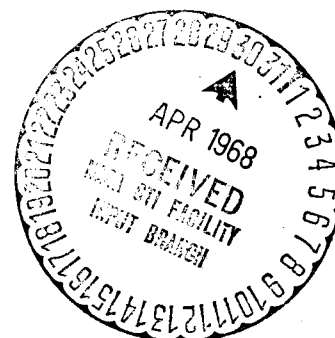
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On the local structure of hyperbolic points in Banach spaces.

by J. Palis

It has been proved by Hartman [2,3], answering a question raised by Peixoto, that a diffeomorphism or a flow on $\mathbb{R}^n$ with a hyperbolic fixed point is locally topologically equivalent to its linear part. This fact, basic for the generic theory of flows and diffeomorphisms was also proved independently by Grobman [1].

In this paper we extend this theorem for Banach spaces.

We shall make use of a key idea presented by Moser in [5], where he gave a new proof of the structural stability of the Anosov diffeomorphisms (see also [4]). As in the finite dimensional case, we do not need to assume the existence of invariant manifolds associated to the fixed point.

I wish to thank M. Hirsch, I. Kupka and M. Shub for helpful conversations.*

2. Preliminaries.

Definition 1. Let E be a Banach space and $L: E \rightarrow E$ be a linear map. L is called hyperbolic if its spectrum is disjoint from the unit circle S^1 in the complex plane.

Definition 2. Let $0 \in E$ and V be a neighborhood of 0 in E . A diffeomorphism $f: V \rightarrow E$ has 0 as a hyperbolic fixed point if $(Df)_0$ is hyperbolic.

* After the completion of this paper, I received a manuscript by C. Pugh proving the same result.

Lemma 1. Let L be a hyperbolic linear isomorphism in E, E with norm $\| \cdot \|_1$. There exist subspaces E_s and E_u , invariant under L , and a norm $\| \cdot \|_2$ equivalent to $\| \cdot \|_1$ in E such that $E = E_s \oplus E_u$, $\|L/E_s\|_2 < 1$ and $\|L^{-1}/E_u\|_2 < 1$.

Proof. The existence of E_s and E_u is given by showing that

$$P = -1/2\pi i \int_{S^1} R_\gamma d\gamma$$

is a projection map. R_γ is the resolvent transformation for L (see [6] for more details). Define $E_s = \{z \in E \mid P(z) = z\}$ and $E_u = \{z \in E \mid P(z) = 0\}$. The spectrum of $L_s = L/E_s$ has radius $r_s < 1$. So there exists a positive integer p and a real number a , $r_s < a < 1$, such that $\|L_s^p\|_1 < a^p$. From the convergence of $\sum_{n=0}^{\infty} a^{-pn} \|L_s^{pn}\|_1$ we get the convergence of $\sum_{n=0}^{\infty} a^{-n} \|L_s^n\|_1$. Set on E_s :

$$\|v\|_2 = \sum_{n=0}^{\infty} a^{-n} \|L_s^n\|_1$$

Clearly $\| \cdot \|_2$ is equivalent to $\| \cdot \|_1$ and $\|L_s\|_2 < a$. The procedure for $L_u = L/E_u$ is similar. The lemma is proved.

Lemma 2. Let $\phi: V(0) \rightarrow E$ be a C^1 map, $\phi(0) = 0$ and $(D\phi)_0 = 0$.

Given any $\epsilon > 0$, ϕ restricted to a smaller neighborhood of 0 can be extended to all of E , the extension being bounded and having a global Lipschitz constant less than ϵ .

Proof. From the continuity of $D\phi$, we can find $\rho > 0$ so that $\|(D\phi)_z\| < \epsilon/3$ for $\|z\| < \rho$. Let α be a real C^∞ function satisfying $0 \leq \alpha(t) \leq 1$, $\alpha(t) = 1$ for $|t| < 1/2$, $\alpha(t) = 0$ for $|t| \geq 1$ and $|\alpha| \leq 2$. Define

$$\begin{aligned}\tilde{\phi}(z) &= \alpha(\|z\|/\rho) \phi(z) \quad \text{for } \|z\| \leq \rho \\ \tilde{\phi}(z) &= 0 \quad \text{for } \|z\| > \rho\end{aligned}$$

Then $\tilde{\phi}(z) = \phi(z)$ for $\|z\| \leq \rho/2$ and $\tilde{\phi}$ has Lipschitz constant less than ϵ . The lemma is proved.

3. The main result

Theorem 1. Let $0 \in E$ and V be a neighborhood of 0 in E . Let $f: V \rightarrow E$ be a C^1 diffeomorphism having 0 as a hyperbolic fixed point. There exist a neighborhood U of 0 in V and a homeomorphism $h: U \rightarrow E$ such that $hL(z) = fh(z)$ for all $z \in U$, where $L = (Df)_0$.

Proof. Let $f = L + \phi$ be the local expression of f . Extending ϕ to $\tilde{\phi}$ as in lemma 2, we shall look for a homeomorphism $h: E \rightarrow E$ satisfying $hL = (L + \tilde{\phi})h$. Let $C_b^0(E)$ denote the Banach space of continuous bounded functions from E into E with the sup norm. Writting $h = \text{Id} + v$, where Id means the identity map in E , the above conjugacy becomes:

$$(\text{Id} + v)L = (L + \tilde{\Phi})(\text{Id} + v)$$

or equivalently

$$vL - Lv = \tilde{\Phi}(\text{Id} + v) \quad (1)$$

We now show that (1) has a unique solution in $C_b^0(E)$, if the Lipschitz constant of $\tilde{\Phi}$ is small enough.

Consider the operator $\mathcal{L}: C_b^0(E) \rightarrow C_b^0(E)$ defined by $\mathcal{L}(v) = vL - Lv$. $\mathcal{L}$ is an isomorphism. This is a consequence of the fact that the operator $\mathcal{L}^*: C_b^0(E) \rightarrow C_b^0(E)$, defined by $\mathcal{L}^*(v) = LvL^{-1}$ is hyperbolic. To see this we write $\mathcal{L}_s^* = \mathcal{L}^*/C_b^0(E, E_s)$ and $\mathcal{L}_u^* = \mathcal{L}^*/C_b^0(E, E_u)$ and notice that $\mathcal{L}_s^*$ and $(\mathcal{L}_u^*)^{-1}$ have norm less than one, since from lemma 1 we may assume that this is the case for the norms of L_s and L_u^{-1} . Thus $\mathcal{L}^* = \mathcal{L}_s^* \oplus \mathcal{L}_u^*$ is hyperbolic.

To solve the equation (1) is equivalent to find a fixed point for the map $\mu: C_b^0(E) \rightarrow C_b^0(E)$ defined by $\mu(v) = \mathcal{L}^{-1}\tilde{\Phi}(\text{Id} + v)$. If the Lipschitz constant of $\tilde{\Phi}$ is less than ϵ and $\max(\|L_s\|, \|L_u^{-1}\|) < a < 1$ we get

$$\|\mu(v) - \mu(w)\| \leq \epsilon(a-1)^{-1}\|v-w\|.$$

Thus taking $\epsilon(a-1)^{-1} < 1$ we have, by the Contraction Principle, a unique solution $v \in C_0^b$ for (1).

Reversing the argument we get a unique solution for the equation:

$$(\text{Id} + u)(L + \tilde{\Phi}) = L(\text{Id} + u) \quad (2)$$

Finally, we notice that from (1) and (2) we have

$$(\text{Id} + u)(\text{Id} + v)L = L(\text{Id} + u)(\text{Id} + v)$$

and from the uniqueness above $(\text{Id} + u)(\text{Id} + v) = \text{Id}$. Therefore $\text{Id} + v$ is a homeomorphism and the theorem is proved.

4. Local Stability.

Theorem 2. Let f be a local C^1 diffeomorphism as in Theorem 1. There exists a neighborhood N of f , C^1 topology, such that for each $g \in N$ there is a local homeomorphism in E satisfying $gh = hf$.

Proof. It is a consequence of Theorem 1 and the following two lemmas:

Lemma 3. Let f be as in Theorem 1. Then there exist a neighborhood N of f , C^1 topology, a neighborhood U of 0 in E and a continuous map $\gamma: N \rightarrow U$ such that $\gamma(T) = 0$ and for each $g \in N$ $g(\gamma(g)) = \gamma(g)$.

Proof. Consider the map $\psi: C_b^1(V, E) \times V \rightarrow E$ defined by $\psi(g, x) =$

$g(x)-x$. Clearly $\psi(f,0) = 0$ and $(\partial\psi/\partial x)_{(f,0)} = (Df)_0 - \text{Id}$ is an isomorphism for $(Df)_0$ is hyperbolic. The lemma is then proved, using the Implicit Function Theorem.

Lemma 4. In the Banach space of linear transformations of E into E the set of hyperbolic maps is open.

Proof. From the fact that the set of linear isomorphisms is open and S^1 is compact.

5. The flow case. We now extend Theorem 1 to the case of flows in Banach spaces. Let φ_t be a C^1 flow in E having 0 as a critical point. φ_t is hyperbolic at 0 if the induced diffeomorphism at time $t = 1$, φ_1 , has 0 as a hyperbolic fixed point. If this is the case, we claim the existence of a local homeomorphism H taking trajectories of φ_t into those of L_t , L_t being the associated linear flow. Using a device appearing in [7], we set

$$H = \int_0^1 L_{-t} h \varphi_t dt$$

where $L_{-1} h \varphi_1 = h$ as in Theorem 1. We verify now that $L_{-s} H \varphi_s = H$ for all $s \in \mathbb{R}$. It is enough to consider $s \in [0,1]$. $L_{-s} H \varphi_s = L_{-s} (\int_0^1 L_{-t} h \varphi_t dt) \varphi_s$ or $L_{-s} H \varphi_s = \int_0^1 L_{-s-t} h \varphi_{t+s} dt$. Taking $u = t+s-1$, the left hand side becomes $\int_{-1+s}^s L_{-u-1} h \varphi_{u+1} du = \int_{-1+s}^0 L_{-u} L_{-1} h \varphi_u du + \int_0^s L_{-u-1} h \varphi_{u+1} du$. And finally, making $v = u+1$ in the second summand

we get

$$\int_0^s L_{-u} h \varphi_u du + \int_s^1 L_{-v} h \varphi_v dv = H.$$

In particular, $L_{-1} h \varphi_1 = H$ and from the uniqueness of solution of this equation in $C^0(E)$ at finite distance from the identity map, we have $H = h$. The assertion is proved.

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