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REMARKS ON IMPERFECTIONS OF AXIALLY LOADED CYLINDERS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

A simple hypothesis is offered as an explanation for the large discrepancy between theory and experiment of axially compressed cylindrical shells. It is suggested that the very low experimental buckling loads (when buckling does not occur at the ends) are caused by local "flat spots." A critical size of imperfection is given and the buckled mode shape is discussed. These arguments are also applied to other common shell buckling problems.

INTRODUCTION

The following simple ideas are offered on the problem of the buckling of an axially compressed cylindrical shell. These qualitative ideas hopefully will give some further clues to the explanation of the discrepancy between the theoretical and experimental buckling loads of a cylinder.

It is recognized by most engineers concerned with this problem that imperfections of many types cause the lowering of the buckling load (see ref. 1 for a history of this problem). The remarks presented herein pertain only to imperfections in the shell geometry rather than those associated with boundary conditions, loading, material properties, etc. These remarks apply to conventionally manufactured, moderately long cylinders in which the typical experimental buckling load is well below the theoretical loads (say, 10-40 percent of the theoretical load where $O(a/h) = 1000$).

NOTATION

a	radius of cylinder
b	imperfection dimension
D	bending rigidity
E	Young's modulus
h	shell thickness
N_x	axial stress resultant

N_θ circumferential stress resultant

P_{cr} critical load, positive in compression

R radius of sphere

δ deviation in radius

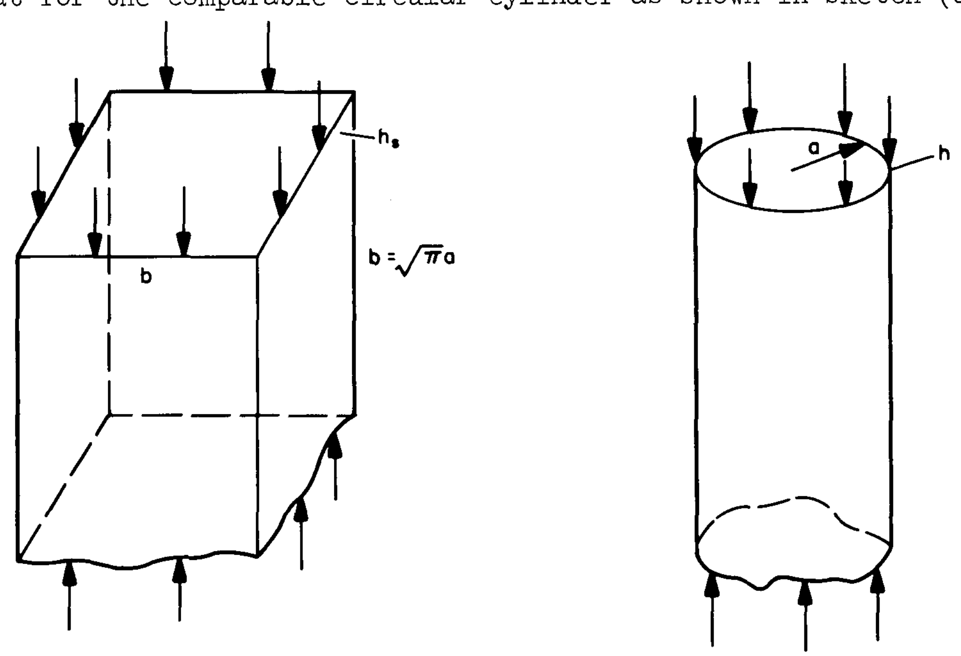
ν Poisson's ratio

σ_{cr} critical stress, positive in compression

DISCUSSION OF IMPERFECTIONS

Axially Loaded Cylinders

The first point concerns an idea that occurs to designers who are familiar with the typical experimental spread of buckling loads. This idea is that it might be better to design a cylinder with a noncircular cross-sectional shape having a lower theoretical buckling load if that load could be reliably predicted so that the design load could be above that for a circular cylinder. The correspondence between experiment and theory is quite good for the buckling of flat plates, so the simplest shape to look at is a square cylinder. Let us choose the square dimensions such that the cross-sectional area is the same as that for the comparable circular cylinder as shown in sketch (a).



Sketch (a)

The buckling load of the square cylinder is given by (ref. 2, p. 355)

$$P_{cr} = \frac{4\pi^2 E h_s^3 (4b)}{12(1 - \nu^2) b^2} \approx 8.2 \frac{E h_s^3}{a}$$

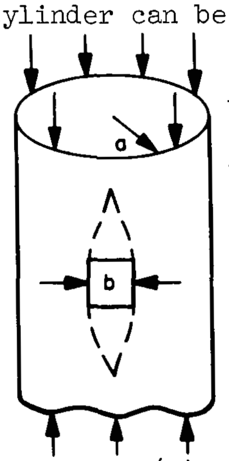
The buckling load for the circular cylinder is

$$\bar{P}_{cr} = 0.6 \frac{Eh}{a} (2\pi a h) \approx 3.8 \, Eh^2$$

To have the same volume of material for both cylinders the plate thickness is different; therefore, $h_s = (\sqrt{\pi}/2)h$. The ratio of the buckling loads becomes

$$\frac{P_{cr}}{\bar{P}_{cr}} \approx 2 \frac{h}{a}$$

If $h/a = 1/1000$, then the square cylinder buckles at 1/500 of the classical circular cylindrical load. Therefore, even though the behavior of the square cylinder can be reliably predicted, the buckling load is far below that for a circular cylinder with even large imperfections. In fact, the difference in loads is so drastic that one might wonder if the buckling load of a circular cylinder might be reduced if any small part of the cylinder were flat. Consider a cylinder that has a "flat spot" which is assumed to be square (see sketch (b)). The edges of the flat spot are subject to elastic constraints on both displacement and rotation so, for convenience, the edges are assumed to be simply supported. The applied load is assumed to be uniformly distributed over the flat spot before buckling so the buckling stress of the plate is



Sketch (b)

$$\sigma_{cr} = \frac{4\pi^2 D}{h b^2} \approx 3.6 \frac{Eh^2}{b^2}$$

If this critical stress is greater than $0.6(Eh/a)$ (the critical stress for a perfect cylinder), the imperfection would not be expected to affect the buckling load of the cylinder. However, if it is less than $0.6(Eh/a)$, then at least local buckling would be expected below the theoretical load. Therefore, let

$$3.6 \frac{Eh^2}{b_{cr}^2} = 0.6 \frac{Eh}{a}$$

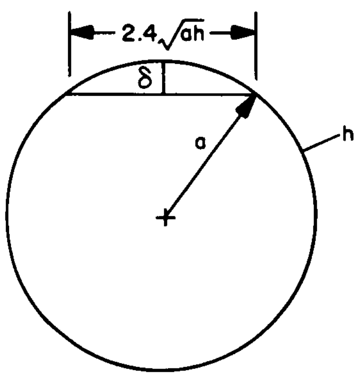
which gives

$$b_{cr} \approx 2.4 \sqrt{ah}$$

Therefore, if this critical dimension of the flat spot, b_{cr}, is about equal to the characteristic length of the shell (the length required for axisymmetric bending effects to damp out), local buckling can occur. Since $\sigma_{cr} \propto 1/b^2$ a small increase in the size of the flat spot would allow local buckling to occur at loads far below the theoretical load. Of course, one wonders whether a plate of such small dimensions is within the thin plate theory. The ratio b/h for the plate is $b/h = (2.4\sqrt{ah})/h = 2.4\sqrt{a/h} \approx 75$ for $a/h = 1000$. This value is well within the range of thin plate theory.

If the shell has only one large flat spot, local buckling could occur without causing a general collapse of the cylinder. This has been suggested in reference 3 which gives limited experimental results for laboratory-produced cylinders with single imperfections. Although these lowered the buckling loads, the larger imperfections caused local instability before general instability occurred. However, if a number of similar imperfections are scattered over the shell or if the local buckling is propagated by the local disruption of stress and deflection, then the overall buckling load of the cylinder is greatly reduced.

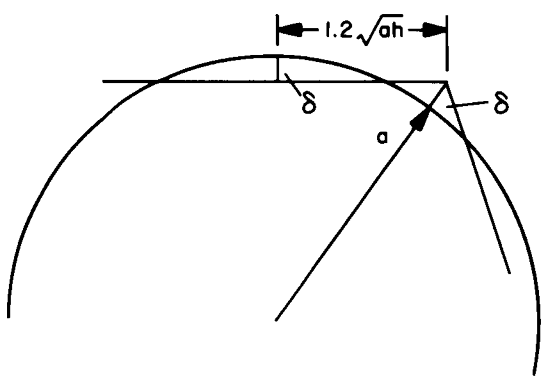
Although the flat spot has been assumed to be a simply supported square plate, the same order of magnitude of critical size would be obtained for many somewhat similar shapes and edge conditions. (The appendix gives an estimate of the buckling stress of a diamond shaped plate.) To gain a better feeling of the critical size, consider a circular cross section with an imperfection as shown in sketch (c). The radial deviation δ is given by



Sketch (c)

$$\delta = a - \sqrt{a^2 - (1.2\sqrt{ah})^2} \approx 0.7h$$

Consider the less favorable imperfection shown in sketch (d). Here, the radial deviation δ varies in an oscillating manner and is given by



Sketch (d)

$$\delta = a - \sqrt{(a + \delta)^2 - 1.44ah}$$

or

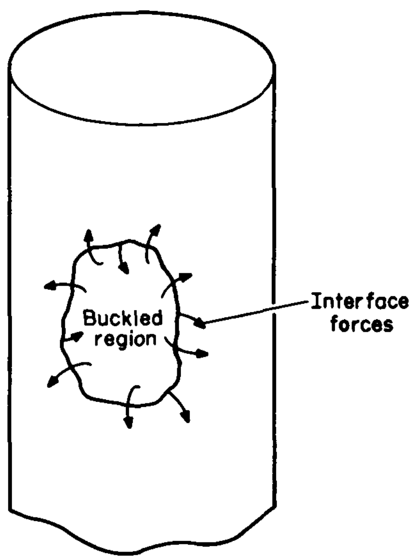
$$\delta \approx a - (a + \delta)\left[1 - \frac{0.72ah}{(a + \delta)^2}\right] \approx -\delta + 0.7h$$

Therefore

$$\delta \approx 0.35h$$

This shape, with the sharp corners, is unrealistic so a reasonable statement might be that the radius should be accurate within, say, $\pm h/2$ to avoid the possibility of a critical flat spot.

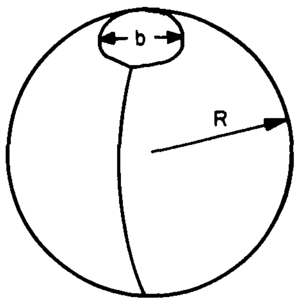
To offer a seemingly reasonable hypothesis for the overall buckling process, let us consider the loading process of two cylinders. The first is a perfect circular cylinder; the second cylinder has some region that is perfect and has flat spots of a critical size in the remaining area. In a perfect region or cylinder the imperfections are assumed to be of a negligible size and to have an insignificant effect on the behavior of the region or cylinder under the particular loading being investigated. According to the classical theory, as the first cylinder is uniformly loaded, it undergoes uniform axial shortening and stress-free radial expansion (away from the ends). When the classical buckling stress $(0.6(Eh/a))$ is reached, an equilibrium configuration in which the deflection varies sinusoidly in a checkerboard fashion can exist. This configuration extends over the entire shell. For the second cylinder, the theory for cylindrical shells can only describe the behavior of the perfect portion of the shell. As the cylinder is loaded, assume that the imperfect region buckles at a load well below the classical load. At this load, isolate the buckled region and postulate what happens to the remaining part of the cylinder (see sketch (e)). Since the uniform compressive stress throughout the cylinder is much less than $0.6(Eh/a)$, cylindrical shell theory says that a sinusoidally varying deflection pattern cannot be in equilibrium and, in fact, the displacements and stresses will decay exponentially as the distance from the buckled region increases (see ref. 4, ch. 5). However, the buckled region may propagate if the disruption of stress and deflection or inertia forces near the buckled region cause local instability; or a new buckled region may form in a different part of the shell if, for instance, a small increase in load causes other flat spots to become of critical size. The main point here is that there is no reason to expect the theoretical mode shape at loads below the classical load and, in fact, there is no reason to expect it even at loads close to the theoretical load if the buckling starts locally rather than simultaneously over the entire shell. Indeed, something would appear to be wrong with the theory if the classical mode shape were obtained experimentally at loads much less than the classical load. It seems to the author, therefore, that there are reasons to expect the mode shape to be much harder to correlate experimentally with theory than the buckling load.



Sketch (e)

Applications to Other Shell Problems

If these ideas are valid, they should apply to other shell problems as well. For the externally pressurized spherical shell (sketch (f)), consider a flat spot to be a simply supported circular plate. Its buckling stress is



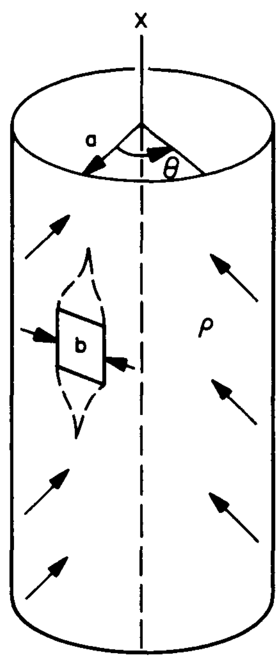
Sketch (f)

given by (ref. 2, p. 391)

$$\sigma_{cr} = (4)\ \frac{4.2D}{hb^2} = 1.54\ \frac{Eh^2}{b^2}$$

Equating this to the critical stress of a sphere ($\sigma_{cr} = 0.6(Eh/R)$), we get

$$b_{cr} \approx 1.6\sqrt{Rh}$$



Sketch (g)

So, one sees that the sphere is at least as sensitive to a flat spot as the axially loaded cylinder. Next, consider an externally pressurized long cylinder (sketch (g)) where, before buckling,

$$N_x = 0\ , \quad N_\theta = pa$$

Consider the flat spot to be a simply supported plate with critical stress

$$\sigma_{cr} = 3.6\ \frac{Eh^2}{b^2}$$

The critical stress for the cylinder is (ref. 4, p. 434)

$$\sigma_{xcr} = 0.27\ \frac{Eh^2}{a^2}$$

so

$$b \approx 3.7a$$

Thus under external pressure, the cylinder is not sensitive to flat spots since the critical size is of the same order as the size of the shell. This insensitivity agrees with the fact that classical theory and experiment are in relatively good agreement for this loading condition (ref. 5).

The flat spot of a cylinder subjected to torsion is assumed to be a simply supported square plate under pure shear. Its critical stress is (ref. 2, p. 382)

$$\tau_{cr} \approx 8.4\ \frac{Eh^2}{b^2}$$

For the shell (ref. 2, p. 504)

$$\tau_{cr} \approx 0.24E\left(\frac{h}{a}\right)^{3/2}$$

So,

$$b \approx 6h \left(\frac{a}{h}\right)^{3/4} = 6a \left(\frac{h}{a}\right)^{1/4}$$

For

$$\frac{h}{a} \approx \frac{1}{1000}, \qquad b \approx a$$

Therefore, long cylinders subjected to torsion are not sensitive to this type of imperfection. Classical theory for this case also agrees quite well with experiment (ref. 2, p. 506).

CONCLUDING REMARKS

A plausible reason has been offered as to why certain types of shells under particular types of loading buckle at loads far below those predicted by theory. The hypothesis is that conventionally manufactured shells contain a distribution of small regions of essentially zero curvature which can behave as flat plates. For some types of shells and loading, these regions can be quite small and still initiate local buckling at loads far below the theoretical load. In other words, the curved surface of a shell compared to a flat surface causes an enormous increase in load-carrying capability for certain types of shells and loading conditions but a small deviation in curvature for these types causes a large reduction in the load carrying capability. It was shown that the radius of an axially loaded cylinder should be accurate to within about $\pm h/2$ to avoid the possibility of a critical sized flat spot. It is interesting that reference 6 shows test results in which the buckling load is about 80 percent of the classical value and the accuracy of the radius is given to be $\pm h/2$.

The axially loaded cylinder and externally pressurized sphere are very sensitive to flat spots compared to the other loading conditions considered. This corresponds to the fact that the correlation between experiment and theory is worse for the axially loaded cylinder and pressurized sphere. One way to show whether or not the flat spots are the only cause of the drastic reduction in load would be to test some cylinders for which the radius was accurate within $\pm h/2$ under various types of imperfections in the boundary conditions, loading, etc., and see whether or not experimental buckling loads drastically below the classical loads could be obtained.

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National Aeronautics and Space Administration
Moffett Field, Calif., 94035, Jan. 16, 1968
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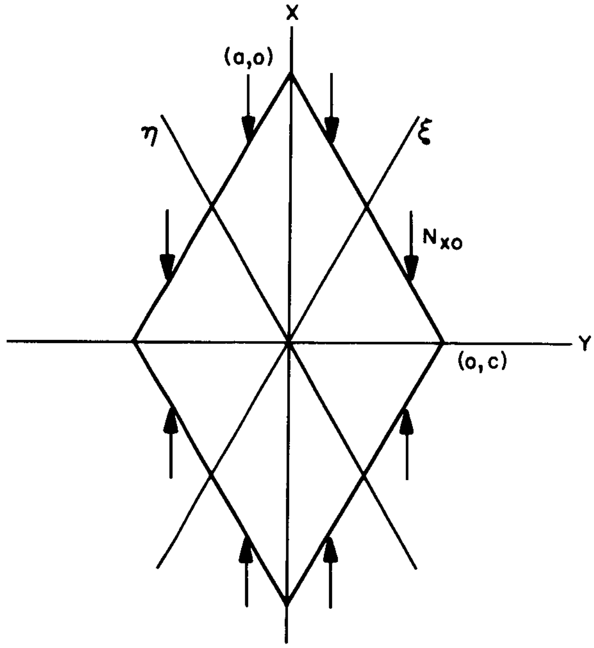
APPENDIX

BUCKLING OF A DIAMOND-SHAPED PLATE

An approximate buckling load of the diamond-shaped plate shown below can be obtained by using the method of minimum potential energy. The plate is assumed to be under uniaxial compression and the edges are simply supported. A convenient coordinate system is

$$\xi = \frac{y}{c} + \frac{x}{a}, \qquad \eta = \frac{y}{c} - \frac{x}{a}$$



Sketch (h)

The boundaries are then given by

$$\xi = \pm 1 \quad \text{and} \quad \eta = \pm 1$$

The total potential energy expression in Cartesian coordinates is

$$U + V = \frac{D}{2}\iint [w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2] dA - \frac{N_{xo}}{2}\iint \left(\frac{\partial w}{\partial x}\right)^2 dA$$

For the new coordinates,

$$ds^2 = dx^2 + dy^2 = \frac{a^2 + c^2}{4}(d\xi^2 + d\eta^2) + \frac{c^2 - a^2}{2}(d\xi\, d\eta)$$

and

$$dA = ds\big|_{d\xi = o}\, ds\big|_{d\eta = o} \sin 2\alpha = \frac{ac}{2}\, d\xi\, d\eta$$

The total potential now becomes

$$U + V = 4\frac{D}{a^3 c^3}\int_{-1}^{1}\int_{-1}^{1}\Big\{(a^4 + c^4)[(w_{,\xi\xi} + w_{,\eta\eta})^2 + 4w_{,\xi\eta}^2] + 4(a^4 - c^4)[w_{,\xi\eta}(w_{,\xi\xi} + w_{,\eta\eta})]$$

$$+ 2a^2c^2(w_{,\xi\xi} - w_{,\eta\eta})^2 + 8\nu a^2 c^2(w_{,\xi\xi} w_{,\eta\eta} - w_{,\xi\eta}^2)\Big\}\, d\xi\, d\eta$$

$$- \frac{N_{xo} c}{4a}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial w}{\partial \xi} - \frac{\partial w}{\partial \eta}\right)^2 d\xi\, d\eta$$

A form for w that satisfies the displacement constraints ($w = 0$, $\xi = \pm 1$, and $\eta = \pm 1$) is

$$w = \sum_{n=1}^{\infty} w_n (\sin^2 n\xi - \sin^2 n)(\sin^2 n\eta - \sin^2 n)$$

The single summation results from the mode shape being symmetrical. Substituting this into the total potential and performing the integration gives

$$(U+V) = \frac{Da}{2c^3} \left\{ \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \sum_{n=1}^{\infty} \frac{4 w_m w_n m^2 n^2}{(m^2-n^2)^2} \left[4m^2n^2(1+\beta^4)(\alpha_m - \alpha_n)^2 - (1+\beta^2)^2 (m^2-n^2)^2 \alpha_m \alpha_n \right.\right.$$

$$\left. + (1+\beta^2)^2 (m^2-n^2)(m^2\alpha_m - n^2\alpha_n)(\cos 2m + \cos 2n + 4 \sin n \sin^2 m - 1) \right]$$

$$\left. + \sum_{m=1}^{\infty} 2 w_m^2 m^4 [3(1-\bar{\alpha}_m)(1+\beta^2)^2 - 4\beta^2(1-\bar{\alpha}_m)^2 - 4(1+\beta^2)^2(1+\bar{\alpha}_m)\sin^2 m \cos^2 m] \right\}$$

$$- \frac{N_{xo} c}{2a} \left\{ \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \sum_{n=1}^{\infty} \frac{w_m w_n m^2 n^2}{(m^2-n^2)^2} (\alpha_m - \alpha_n)[n^2\alpha_m - m^2\alpha_n + (m^2-n^2)(\cos 2m + \cos 2n \right.$$

$$\left. - 1 + 4 \sin^2 m \sin^2 n)] + \sum_{m=1}^{\infty} w_m^2 m^2 \left[\frac{3}{4} (1-\bar{\alpha}_m)^2 - 2(1-\bar{\alpha}_m) \sin^2 m \cos^2 m \right] \right\}$$

where

$$\alpha_m = \frac{\sin 2m \cos 2n}{2m}, \quad \alpha_n = \frac{\sin 2n \cos 2m}{2n}, \quad \bar{\alpha}_m = \frac{\sin 2m \cos 2m}{2m}$$

$$\beta = \frac{c}{a}$$

Since we know that

$$\delta(U + V) = 0$$

we can obtain a set of algebraic equations to determine w_m. The series was truncated after 1, 2, and 3 terms to get some feeling of the convergence although it is recognized that this procedure is somewhat dangerous since the fact that answers, for different number of terms, are close together may just indicate the convergence is very slow rather than that the answers are nearly correct. The following table gives the numerical results.

β	$\left(\frac{(2c)^2}{\pi^2 D}\right)(N_{xo})$ critical		
	One term	Two terms	Three terms
0.75	9.86	8.98	8.82
1.0	8.46	7.59	7.55
1.25	9.28	8.35	8.30

So, taking $\beta = 1.0$,

$$\sigma_{cr} \approx \frac{7.5\pi^2 D}{(2c)^2 h}$$

The critical size, $b = 2c$, for an axially loaded cylinder is given by

$$b \approx 3.3\sqrt{ah}$$

Therefore, the critical size is seen to be of the same order of magnitude as a square flat spot for the case of an axially loaded cylinder.

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