TRANSMISSION CHARACTERISTICS OF CONICAL SHELLS UNDER LATERAL EXCITATIONS

by

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ABSTRACT

A theoretical and experimental study is presented for determining the steady-state frequency response of laterally excited truncated cones and cylinders in a low to moderate frequency range. The theoretical results are formulated in terms of forward transmission matrices, although relationships are reviewed which allow application of the results for impedance or admittance formulations as well. Expressions for the 4 × 4 matrices of transmission for coupled bending-shear displacements are developed by means of the membrane theory of thin shells. The results are then applied to the case of laterally excited shells which support a rigid top mass. Input and transfer pseudoimpedances are calculated and compared with experimental observations for 30° and 15° cones and a cylinder. Overall comparison of the results indicates that the membrane theory provides a reasonable approximation for determining response characteristics, but some definite deficiencies remain unexplained. A digital computer program is included for computing all required results.
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NOMENCLATURE

a  major base radius of cone
\( \bar{A}, \bar{B}, \bar{C}, \bar{D} \)  submatrices of forward transmission matrix
b  minor base radius of cone
E  shell elastic modulus
\( \bar{F}_1, \bar{F}_2 \)  generalized input and output force vectors, respectively
G  shell shear modulus
h  shell wall thickness
I  mass moment of inertia of supported mass about axis through center of mass
M  net moment on a cross section
\( N_s, N_\theta, N_{s\theta} \)  membrane stress resultants
Q  net lateral shearing force on a cross section
s, \( \theta \)  coordinates of point on cone surface (s is dimensional)
\( s_1(1 - \gamma) \)  slant length of truncated cone
u, v, w  local axial, tangential, and radial displacements of shell middle surface--these displacements lie parallel and perpendicular to the shell generating surface.
U, V, W  integrated axial, lateral, and radial displacements, respectively
\( U^*, V^*, W^* \)  net axial, lateral, and radial displacements, respectively
\( \bar{V}_1, \bar{V}_2 \)  generalized input and output velocity vectors, respectively
x, \( \theta \)  nondimensional coordinates of point cylinder surface
(\( ' \))  denotes a velocity for indicated displacement
\( \alpha \)  semivertex angle for cones
\( a_{ij} \) elements of forward transmission matrix for velocities

\( \beta_{ij} \) elements of forward transmission matrix for displacements

\( \gamma \) geometric ratio for truncated cones

\( \delta \) offset of center of supported mass from plane including upper weld circle

\( \Psi \) angular rotation of a cross section

\( \nu \) Poisson's ratio for shell

\( \xi \) nondimensional cone coordinate \( (\xi = \frac{s}{a} \sin \alpha) \)

\( \rho \) shell mass density

\( \phi_{ij}, \gamma_{ij} \) elements of conversion matrices

\( \omega \) circular frequency

\( \Omega \) nondimensional frequency parameter
INTRODUCTION

Within the last several years, steady-state frequency response methods have become increasingly popular for the description of dynamic behavior of linear mechanical systems. Impedance, admittance (mobility), and rearward and forward transmission methods all fall into this category of analysis. Basically, all these methods are similar in that they envision a mechanical component which possesses terminals that identify the position and direction of all external forces applied to the component and the corresponding velocities resulting from the application of those forces. The component is considered to be a "black box" whose character is determined in terms of the behavior at the accessible terminals. The methods can be applied to components that have either discrete or distributed properties, and complex systems can readily be synthesized into elements which can represent various structural, acoustical, electrical, etc., components.

Because of the nature of multi-degree of freedom and distributed components of complex systems, the above methods lend themselves immediately to matrix notation. Thus, a component can be described in terms of its impedance, admittance (mobility), or forward or rearward transmission matrix, whose elements are usually frequency dependent. The use of truncated conical or cylindrical shells as components of a complex structure such as a space vehicle falls into this category.

The purpose of the present research program has been to determine the frequency dependent matrices which describe truncated conical and cylindrical shells so that they can be used as components for the application
of the above methods to the analysis of complex mechanical space vehicle systems. This has been achieved in terms of transmission matrices for axisymmetric modes in the first half of the program, and the results of this work have already been reported¹.

The second half of the present program has dealt with lateral bending responses of truncated cones and cylinders so that more complicated matrix representations result from the additional variables required to describe the complete response. The purpose of this report is to present the work accomplished under this final phase of the program. Again the characteristics of truncated conical and cylindrical shells have been developed in terms of transmission (specifically forward transmission) matrices. However, in order to facilitate using the results in analyses incorporating impedance or mobility methods, we begin with a discussion of the relationship between the various methods that have been mentioned. At the same time, a summary of the most recently accepted definitions utilized in the various methods is presented.

The approach used for the determination of the transmission matrices of truncated conical and cylindrical shells subject to lateral bending is similar to that used in the earlier work for longitudinal excitation¹. Membrane theory of thin shells is used to derive expressions for the elements of a $4 \times 4$ transmission matrix. For the case of truncated cones, the governing equations must be integrated numerically, while, for the cylinder, the governing equations are integrated directly; but extensive numerical computations are still required to obtain the matrix
elements. A computer program has been developed for this purpose and is included in the results. Experimental results from measurements of pseudoimpedances of several specimens are then compared with predicted results.

It should be mentioned that the present analysis is basically the same as that which has been reported in an earlier progress report. However, the notation and arrangement of the analysis have been changed considerably so that they correspond with results and definitions reported in the recent work described in the next section.

RELATIONSHIPS BETWEEN IMPEDANCE, ADMITTANCE, AND TRANSMISSION MATRICES

Definitions of the various aspects of the several methods of response analysis, as well as the relationships between the matrices which characterize the components used for each method, have recently been described quite vividly by Rubin. For convenience, a brief review of these descriptions will be presented here; however, the referenced papers should be consulted for complete details. We emphasize that these relationships then allow conversion of the results to be presented for cones and cylinders from transmission to either impedance or admittance matrices. Although Rubin has described analyses which employ rectangular matrices, here we will consider only square matrices.

We introduce the notation:
as generalized force and velocity vectors, respectively, applied at the input to a "black box" component. The generalized forces $\mathcal{F}_{ij}$ can be forces, moments, etc., while the velocities $v_{ij}$ can be translational, rotational, etc., velocities. Each is understood to represent a steady-state complex vector. Correspondingly, at the output terminals, we have

$\overline{F}_2 = \begin{bmatrix} \mathcal{F}_{11} \\ \vdots \\ \mathcal{F}_{2n} \end{bmatrix}, \quad \overline{V}_2 = \begin{bmatrix} v_{11} \\ \vdots \\ v_{2n} \end{bmatrix}$

Now, with these definitions, the following partitioned matrix forms can be introduced:

$\begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} = \begin{bmatrix} H & G \\ G^T & E \end{bmatrix} \begin{bmatrix} \overline{F}_1 \\ \overline{F}_2 \end{bmatrix}$ \hspace{1cm} (Admittance)

$\begin{bmatrix} \overline{F}_1 \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} S & R \\ R^T & Q \end{bmatrix} \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix}$ \hspace{1cm} (Impedance)

$\begin{bmatrix} \overline{F}_2 \\ \overline{V}_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \overline{F}_1 \\ \overline{V}_1 \end{bmatrix}$ \hspace{1cm} (Rearward Transmission)
These forms are identical to Eqs. (20-23) of Reference 4, except for the bars on the notation of $\bar{F}_1$, $\bar{F}_2$, $\bar{V}_1$ and $\bar{V}_2$. For each case, the partitioned matrix in brackets represents an admittance, impedance, rearward transmission, or forward transmission matrix, respectively, all for the same "black box" component. We note one caution in the use of the forms. The force $\bar{F}_2$ used in the transmission matrices is the negative of the $\bar{F}_2$ in the admittance and impedance forms. For transmission matrices, $\bar{F}_2$ is the force applied by the output terminal 2 of the "black box," while, for admittance and impedance matrices, $\bar{F}_2$ is the force applied to terminal 2 of the "black box."

The relationships between the various forms above are shown in Figure 1 which has been taken directly from Reference 4. Row 1) shows the transformation from the admittance matrix to the other three forms. Similarly, rows 2), 3), and 4) show transformations beginning with the forward transmission, rearward transmission, and impedance matrices, respectively. All matrices on the indicated rows are square and have order n. The great utility of the transformations in Figure 1 becomes immediately obvious. The characteristic matrix of any component can be determined in terms of the form which is the most convenient, but can then be transformed to any of the other formulations.
Note

In present Notation

\[ T = [\alpha_{ij}] \]

Figure 1. Relationships Among Admittance, Transmission, And Impedance Matrices
Mechanical impedance and admittance methods have been utilized for a longer period of time than the others, and, as a result, more literature is available on these methods. Reference lists of much of this literature are given in early reports of the present program\textsuperscript{1,2}. A very lucid description of applying mechanical impedance and admittance methods has been given by O'Hara\textsuperscript{5}. He is particularly careful in pointing out the proper methods that must be utilized in measuring impedances and mobilities, which are the various elements of the impedance and admittance matrices, respectively. Diagonal elements are referred to as driving point, direct, or self-impedances or admittances, while off-diagonal elements are called transfer, cross, or mutual impedances or admittances. He further defines pseudoimpedance as the ratio of force input at some point of a structure to the velocity at some point. If the points are the same, driving point pseudoimpedance results, while, if the points are different, a transfer pseudoimpedance results.*

Additional recent examples of applying the impedance method in complex structures have been reported by On\textsuperscript{6}, while Rubin\textsuperscript{3,4} has described the use of all of the above methods. Rubin emphasizes the utility of the transmission matrix methods whereby the transmission matrix of a complex structure is formulated simply by multiplying in tandem all the matrices of the individual components—a procedure which is well suited to digital computation. Thus, the results of the present work,

*Note that with this definition, driving point and transfer pseudoimpedances were determined for cones and cylinders under longitudinal excitation in our earlier work, rather than driving point and transfer impedances as was indicated.
which are formulated in terms of forward transmission matrices, can be transformed to whatever method may be preferred, while the details of applying these methods to complex structures may be obtained from the references cited.

FORWARD TRANSMISSION MATRICES FOR COUPLED BENDING-SHEAR VIBRATIONS OF TRUNCATED CONICAL SHELLS

General Discussion

During the lateral vibration of a beam-type structural element, there will be, in general, a bending moment and a shearing force transmitted through each cross section, and the element will exhibit a coupled bending and shearing deformation. Due to this coupling, the structural element cannot be adequately described by a set of four-pole parameters as in the simpler case of longitudinal or torsional vibrations of the element. In general, for lateral vibrations of a linear, elastic, beam-type element, there are four boundary force or velocity variables at each terminal. These quantities are transmitted through the structural element by a linear matrix equation:

\[
\begin{pmatrix}
\dot{Q}_1 \\
M_1 \\
\dot{V}_1 \\
\dot{\Psi}_1
\end{pmatrix} = [a_{ij}] \begin{pmatrix}
\dot{Q}_2 \\
M_2 \\
\dot{V}_2 \\
\dot{\Psi}_2
\end{pmatrix}
\]

\begin{equation}
\text{where } V \text{ denotes the lateral velocity of a cross section, } \dot{\Psi} \text{ the angular velocity, } M \text{ the bending moment, and } Q \text{ the shearing force; the subscript 1}
\end{equation}
refers to the input terminal, and the subscript 2 refers to the output
terminal (Fig. 2).

The \(4 \times 4\) matrix \([a_{ij}]\) is the forward transmission matrix of the structural component. Note from Eq. (6) that

\[
\begin{align*}
\bar{F}_1 &= \begin{bmatrix} Q_1 \\ M_1 \end{bmatrix}, \quad \bar{V}_1 = \begin{bmatrix} \dot{Q}_1 \\ \dot{M}_1 \end{bmatrix} \\
\bar{F}_2 &= \begin{bmatrix} Q_2 \\ M_2 \end{bmatrix}, \quad \bar{V}_2 = \begin{bmatrix} \dot{Q}_2 \\ \dot{M}_2 \end{bmatrix}
\end{align*}
\]

It may also be noted that in Figure 2, for convenience, the sign convention on \(\bar{F}_2\) is taken as positive generalized forces for \(Q_2\) and \(M_2\) applied to the output terminal 2 by the load. The sixteen elements \(a_{ij}\), \(i, j = 1, 2, 3, 4\), are, in general, frequency-dependent complex quantities, but are not all independent from each other. In this case, it can be found that only ten of the sixteen elements are independent.

**Derivations of Transmission Matrices for Truncated Conical Shells**

We shall assume that for thin conical shells having a small semi-vertex angle \(\alpha\) and an input frequency below a certain limiting value, the beam vibrations may be satisfactorily governed by the membrane theory of shells. Thus, referring to Figure 2 for the coordinate system, there are three equations of motion:

\[
\frac{\partial N_s}{\partial s} + \frac{N_s}{s} + \frac{1}{s \sin \alpha} \frac{\partial N_s \theta}{\partial \theta} - \frac{N_\theta}{s} = \rho h \frac{\partial^2 u}{\partial t^2} \tag{8a}
\]
Figure 2. Coordinate System
\[
\frac{\partial N_s \theta}{\partial s} + \frac{2}{s} N_s \theta + \frac{1}{s \sin \alpha} \frac{\partial N \theta}{\partial \theta} = \rho h \frac{\partial^2 \nu}{\partial t^2} \\
- \frac{1}{s \tan \alpha} N \theta = \rho h \frac{\partial^2 w}{\partial t^2}
\]

and three stress resultant-displacement relations:

\[
N_s = \frac{E h}{1 - \nu^2} \left[ \frac{\partial u}{\partial s} + \nu \left( \frac{1}{s \sin \alpha} \frac{\partial v}{\partial \theta} + \frac{u}{s} + \frac{w}{s \tan \alpha} \right) \right]
\]
\[
N \theta = \frac{E h}{1 - \nu^2} \left[ \frac{1}{s \sin \alpha} \frac{\partial v}{\partial \theta} + \frac{u}{s} + \frac{w}{s \tan \alpha} + \nu \frac{\partial u}{\partial s} \right]
\]
\[
N_s \theta = \frac{E h}{2(1 + \nu)} \left[ \frac{\partial v}{\partial s} + \frac{1}{s \sin \alpha} \frac{\partial u}{\partial \theta} - \frac{v}{s} \right]
\]

For lateral beam vibrations, we are interested in the following integrated quantities:

\[
M = -s^2 \sin^2 \alpha \cos \alpha \int_0^{2\pi} N_s \cos \theta \, d\theta
\]
\[
Q = -s \sin \alpha \int_0^{2\pi} N_s \theta \sin \theta \, d\theta + s \sin^2 \alpha \int_0^{2\pi} N_s \cos \theta \, d\theta
\]

along with

\[
U = -\frac{1}{\pi} \int_0^{2\pi} u \cos \alpha \cos \theta \, d\theta
\]
\[
V = -\frac{1}{\pi} \int_0^{2\pi} v \sin \theta \, d\theta
\]
\[
W = -\frac{1}{\pi} \int_0^{2\pi} w \cos \alpha \cos \theta \, d\theta
\]
These definitions were originally given for a cylindrical shell by Simmonds\textsuperscript{7}, and the directions correspond to those for the input end in Figure 2. Equations (10) are a generalization of Simmonds' definitions to the case of a cone, while the displacement components given by Eqs. (11) have been defined here to allow maximum simplicity in the governing equations for the cone. It should be noted that the net displacements are given by

\begin{align*}
U^* &= U - W \tan \alpha \\
V^* &= V \\
W^* &= W + U \tan \alpha
\end{align*}

(12a, 12b, 12c)

where Eq. (12b) follows directly from Simmonds' definitions and applies as well for the case of a cone. The net rotation $\dot{\Psi}$ of a cross section will be introduced only later, for the resulting equations, which must be integrated numerically, are simpler in form when the variables (11) are utilized.

The governing equations will now be written in terms of the preceding definitions. We eliminate $N_\theta$ from Eqs. (8a), (8b), and (9b) by using Eq. (8c) and assuming harmonic oscillations in time. Then, by multiplying Eqs. (8a), (9a), (9b) by $\cos \theta$ and Eqs. (8b), (9c) by $\sin \theta$ and integrating over the circumference $\theta = 0$ to $2\pi$ (note that $\theta$-derivative terms must be integrated by parts), we obtain the following equations governing the lateral beam vibrations of conical shells:

\begin{equation}
\frac{dM}{ds} = \pi \rho h \omega^2 \sin^2 \alpha (W \tan \alpha - U) s^2 - Q \cos \alpha
\end{equation}

(13a)
\[
\frac{dQ}{ds} = \pi \rho \omega^2 s \sin \alpha (W + U \tan \alpha - V) \quad (13b)
\]

\[
\frac{dU}{ds} = \frac{1}{\pi \rho H \sin^2 \alpha s^2} M \frac{\nu \omega^2 s \sin \alpha}{E \cos \alpha} W \quad (13c)
\]

\[
\frac{dV}{ds} = \frac{V}{s} + \frac{1}{\sin \alpha \cos \alpha} \frac{U}{s} + \frac{1}{\pi \rho H \sin \alpha \cos \alpha s^2} M + \frac{1}{\pi \rho H \sin \alpha s} \frac{Q}{s} \quad (13d)
\]

\[
W = -\left(1 - \frac{\rho \omega^2 s^2 \sin^2 \alpha}{E \cos^2 \alpha}\right)^{-1} \left[V + U \tan \alpha + \frac{\nu}{\pi \rho H \sin \alpha \cos \alpha} \frac{M}{s}\right] \quad (13e)
\]

For convenience, we now introduce the dimensionless meridional coordinate

\[
\xi = \frac{s}{s_1} = \frac{s}{a \sin \alpha}
\]

where

\[\gamma \leq \xi \leq 1\]

and the dimensionless frequency parameter

\[\Omega^2 = \rho \frac{a^2 \omega^2}{E}\]

The governing equations become

\[
\frac{dV}{d\xi} = \frac{1}{\xi \sin \alpha} \left[V \sin \alpha + \frac{U}{\cos \alpha} + \frac{Q}{\pi \rho H} + \frac{\tan \alpha}{\pi \rho H a} \frac{M}{\xi}\right] \quad (14a)
\]

\[
\frac{dU}{d\xi} = \frac{1}{\sin \alpha} \left[\frac{M}{\pi \rho H a \xi^2} - \frac{\nu \Omega^2 \xi}{\cos \alpha} W\right] \quad (14b)
\]

\[
\frac{dM}{d\xi} = \frac{\pi \rho H \Omega^2 \xi}{\sin \alpha} (W \tan \alpha - U) - Qa \cot \alpha \quad (14c)
\]
\[ \frac{dQ}{d\xi} = \frac{\pi E_h \Omega^2 \xi}{\sin \alpha} (W + U \tan \alpha - V) \]  

(14d)

\[ W = -\left(1 - \frac{\Omega^2}{\cos^2 \alpha \xi^2}\right)^{-1} \left[ V + U \tan \alpha + \frac{v}{\pi E_h \cos \alpha} \frac{M}{\xi} \right] \]  

(14e)

This set of differential equations is in a convenient form for numerical integration. Note that the last equation is algebraic and serves to define the parametric function \( W \).

Since the boundary conditions are usually prescribed on \((Q, M, V, \Psi)\), we shall now derive a relation for the determination of the boundary values of \( U \) which appears in the differential Eqs. (14). The angle of rotation, \( \Psi \), of an arbitrary cross section is defined as

\[ \Psi = \frac{U}{a \xi} = \frac{U - W \tan \alpha}{a \xi} \]  

(15)

Elimination of \( W \) from Eqs. (14e) and (15) gives

\[ U = (1 - \Omega^2 \xi^2)^{-1} \left[ (\cos^2 \alpha - \Omega^2 \xi^2) a \xi \Psi - V \sin \alpha \cos \alpha - \frac{v \sin \alpha}{\pi E_h} \frac{M}{\xi} \right] \]  

(16)

Now, four independent numerical integrations of Eqs. (14) for the four sets of initial values at \( \xi = \gamma = b/a \),

1) \( \{Q_2, M_2, V_2, \Psi_2\} = \{1, 0, 0, 0\} \)
2) \( \{Q_2, M_2, V_2, \Psi_2\} = \{0, 1, 0, 0\} \)
3) \( \{Q_2, M_2, V_2, \Psi_2\} = \{0, 0, 1, 0\} \)
4) \( \{Q_2, M_2, V_2, \Psi_2\} = \{0, 0, 0, 1\} \)
will yield the influence coefficients $\beta_{ij}$ at the boundary $\xi = 1$:

1) $\{Q_1, M_1, V_1, \Psi_1\} = \{\beta_{11}, \beta_{21}, \beta_{31}, \beta_{41}\}$
2) $\{Q_1, M_1, V_1, \Psi_1\} = \{\beta_{12}, \beta_{22}, \beta_{32}, \beta_{42}\}$
3) $\{Q_1, M_1, V_1, \Psi_1\} = \{\beta_{13}, \beta_{23}, \beta_{33}, \beta_{43}\}$
4) $\{Q_1, M_1, V_1, \Psi_1\} = \{\beta_{14}, \beta_{24}, \beta_{34}, \beta_{44}\}$  \hspace{1cm} (18)

Note that the initial value $U_2$ at $\xi = \gamma$ should be calculated by substituting Eqs. (10) into Eq. (9):

1) $U_2 = 0$
2) $U_2 = (1 - \Omega^2 \gamma^2)^{-1} \left[ \begin{array}{c} - \nu \sin \alpha \\ \pi F h \gamma \end{array} \right]$  \hspace{1cm} (19)
3) $U_2 = (1 - \Omega^2 \gamma^2)^{-1} \left[ - \sin \alpha \cos \alpha \right]$
4) $U_2 = (1 - \Omega^2 \gamma^2)^{-1} \left[ (\cos^2 \alpha - \Omega^2 \gamma^2) a \gamma \right]$

The transmission matrix $[\beta_{ij}]$ now relates the boundary force and displacement variables as follows:

$$
\begin{bmatrix}
Q_1 \\
M_1 \\
V_1 \\
\Psi_1
\end{bmatrix}
= [\beta_{ij}]
\begin{bmatrix}
Q_2 \\
M_2 \\
V_2 \\
\Psi_2
\end{bmatrix}
$$

The conversion into $[a_{ij}]$ may be readily effected by using the relations

$$
\dot{V} = i \omega V \quad \dot{\Psi} = i \omega \Psi
$$

Thus, from Eqs. (6) and (7), we note that:
Special Case of Cylindrical Shell \( (a = 0) \)

The governing equations for a cylindrical shell may be obtained directly from Eqs. (13) by letting

\[
\alpha \rightarrow 0, \quad s \sin \alpha \rightarrow a, \quad \text{and} \quad \frac{d}{ds} \rightarrow \frac{d}{dx} = \frac{1}{a} \frac{d}{dx} \quad \text{where} \quad x = \frac{x}{a}
\]

We also note that

\[
U \rightarrow U^*, \quad V = V^*, \quad \text{and} \quad W \rightarrow W^*
\]

so that we also have

\[
\psi = \frac{U^*}{a} = \frac{U}{a}
\]

Thus, the equations become:

\[
\frac{dM}{dx} = -\pi Eh \Omega^2 a^2 \psi - Qa \quad (22a)
\]
\[
\frac{dQ}{dx} = \pi Eh\Omega^2(W - V) \quad (22b)
\]

\[
\frac{d\psi}{dx} = \frac{M}{\pi Eh} - \nu\frac{\Omega^2}{a} W \quad (22c)
\]

\[
\frac{dV}{dx} = a\psi + \frac{Q}{\pi Gh} \quad (22d)
\]

\[
W = -(1 - \Omega^2)^{-1}\left[V + \frac{\nu M}{\pi Eh}\right] \quad (22e)
\]

Equations (22) are identical to Eqs. (30) thru (34) of Simmonds\textsuperscript{7}. By eliminating \(M\) and \(Q\) from these equations, we obtain:

\[
a\frac{\partial^2 \psi}{\partial x^2} + \nu\left(\frac{\partial V}{\partial x} + \frac{\partial W}{\partial x}\right) + \frac{1 - \nu}{2}\left(\frac{\partial V}{\partial x} - a\psi\right) + (1 - \nu^2)\Omega^2 a\psi = 0
\]

\[
\frac{\partial^2 V}{\partial x^2} - a\frac{\partial \psi}{\partial x} - 2(1 + \nu)\Omega^2(W - V) = 0 \quad (23)
\]

\[
(1 - \nu^2)\Omega^2 W - \nu a\frac{\partial \psi}{\partial x} - V - W = 0
\]

which can be written as:

\[
\begin{bmatrix}
\mathcal{L}^2 - \frac{1 - \nu}{2} + (1 - \nu^2)\Omega^2 & \frac{1 + \nu}{2} \mathcal{L} & \nu \mathcal{L} \\
-\mathcal{L} & \mathcal{L}^2 + 2(1 + \nu)\Omega^2 & -2(1 + \nu)\Omega^2 \\
\nu \mathcal{L} & 1 & 1 - (1 - \nu^2)\Omega^2
\end{bmatrix}
\begin{bmatrix}
a\psi \\
V \\
W
\end{bmatrix} = 0
\]

where \(\mathcal{L} = d/dx\). By assuming solutions of the form \(e^{\lambda x}\), we obtain the characteristic equation

\[
\lambda^4 + 2P\lambda^2 - K = 0
\]
\[ P = \frac{\Omega^2}{2(1 - \Omega^2)} \left[ 5 + 2\nu - (1 + \nu)(3 - \nu)\Omega^2 \right] \]
\[ K = \frac{\Omega^2}{(1 - \Omega^2)} \left[ 1 - 2(1 + \nu)\Omega^2 \right] \left[ 2 - (1 - \nu^2)\Omega^2 \right] \]

whose solutions are
\[
(i\lambda_1)^2 = -P - \sqrt{P^2 + K} \\
\lambda_2^2 = -P + \sqrt{P^2 + K} \quad \text{for} \quad P, K > 0
\]

The case of \(P, K < 0\) will be discussed later.

The general solution to Eqs. (23), which corresponds to Eqs. (24), is
\[ V = A_1 \cos \lambda_1 x + B_1 \sin \lambda_1 x + C_1 \cosh \lambda_2 x + D_1 \sinh \lambda_2 x \]

Upon substitution of this result into Eqs. (23), we find additionally
\[ W = A_1 f_1 \cos \lambda_1 x + B_1 f_1 \sin \lambda_1 x + C_1 f_2 \cosh \lambda_2 x + D_1 f_2 \sinh \lambda_2 x \]
\[ a\Psi = A_1 g_1 \sin \lambda_1 x - B_1 g_1 \cos \lambda_1 x + C_1 g_2 \sinh \lambda_2 x + D_1 g_2 \cosh \lambda_2 x \]
where
\[
f_1 = \frac{\nu \lambda_1^2 - 1 - 2\nu(1 + \nu)\Omega^2}{1 - (1 + \nu)^2\Omega^2}, \quad f_2 = \frac{-\nu \lambda_2^2 - 1 - 2\nu(1 + \nu)\Omega^2}{1 - (1 + \nu)^2\Omega^2}
\]
\[
g_1 = \frac{(1 - \nu^2)\Omega^2 f_1 - f_1 - 1}{\nu \lambda_1}, \quad g_2 = \frac{(1 - \nu^2)\Omega^2 f_2 - f_2 - 1}{\nu \lambda_2}
\]

and upon substitution of these expressions into Eqs. (22c, d), we obtain:
\[ M = \frac{\pi a \varepsilon h}{1 - \nu^2} \left[ (g_1 \lambda_1 + \nu + \nu f_1)(A_1 \cos \lambda_1 x + B_1 \sin \lambda_1 x) \right. \]
\[ + (g_2 \lambda_2 + \nu + \nu f_2)(C_1 \cosh \lambda_2 x + D_1 \sinh \lambda_2 x) \]
\[ Q = \pi Gh \left[ (\lambda_1 + g_1)(B_1 \cos \lambda_1 x - A_1 \sin \lambda_1 x) + (\lambda_2 - g_2)(C_1 \sinh \lambda_2 x + D_1 \cosh \lambda_2 x) \right] \]

Thus, all variables have now been determined in terms of the constants \( A_1, B_1, C_1, \) and \( D_1, \) which must be determined from the boundary conditions. If we let

\[
\begin{align*}
\mu_1 &= (g_1 \lambda_1 + \nu + \nu f_1) \frac{\pi a Eh}{(1 - \nu^2)}, \\
\mu_2 &= (g_2 \lambda_2 + \nu + \nu f_2) \frac{\pi a Eh}{(1 - \nu^2)} \\
\mu_3 &= \pi Gh(\lambda_1 + g_1), \\
\mu_4 &= \pi Gh(\lambda_2 - g_2)
\end{align*}
\]

Then at \( x = 0, \) we have

\[
\begin{align*}
0 & \quad \mu_3 & \quad 0 & \quad \mu_4 \\
\mu_1 & \quad 0 & \quad \mu_2 & \quad 0 \\
1 & \quad 0 & \quad 1 & \quad 0 \\
0 & \quad -\frac{g_1}{a} & \quad 0 & \quad \frac{g_2}{a}
\end{align*}
\]

\[
\begin{pmatrix}
Q_2 \\
M_2 \\
V_2 \\
\Psi_2
\end{pmatrix}
= \begin{pmatrix}
\mu_1 & 0 & \mu_2 & 0 \\
0 & \mu_3 & 0 & \mu_4 \\
1 & 0 & 1 & 0 \\
0 & -\frac{g_1}{a} & 0 & \frac{g_2}{a}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix}
= \begin{pmatrix}
\phi_{ij}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix}
\]

And at \( x = \ell/a \)

\[
\begin{align*}
Q_1 &= \begin{pmatrix}
-\mu_3 \sin \Lambda_1 & \mu_3 \cos \Lambda_1 & \mu_4 \sinh \Lambda_2 & \mu_4 \cosh \Lambda_2 \\
\mu_1 \cos \Lambda_1 & \mu_1 \sin \Lambda_1 & \mu_2 \cosh \Lambda_2 & \mu_2 \sinh \Lambda_2 \\
\cos \Lambda_1 & \sin \Lambda_1 & \cosh \Lambda_2 & \sinh \Lambda_2 \\
\frac{g_1}{a} \sin \Lambda_1 & -\frac{g_1}{a} \cos \Lambda_1 & \frac{g_2}{a} \sinh \Lambda_2 & \frac{g_2}{a} \cosh \Lambda_2
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix}
= \begin{pmatrix}
\gamma_{ij}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix}
\]

(25)
Then, from Eqs. (20), (25) and (26), we may form the matrix equation

$$[\beta_{ij}] = [\gamma_{ij}] [\phi_{ij}]^{-1}$$

and the $a_{ij}$ are again found as in Eq. (21).

We now return to Eqs. (24) for the case

$$(i\lambda_1)^2 = -P - \sqrt{P^2 + K}$$

$$\text{for } P > 0, \ K < 0, \ \sqrt{P^2 + K} \geq 0 \quad (28)$$

It can be seen that the form of the solution will change for this case, as well as other possible combinations of $P$ and $K$, whose values are functions of frequency. Here, we will consider only the additional case of Eqs. (28) which occurs at the lowest frequency change corresponding to

$$\Omega^2 = \frac{1}{2(1 + \nu)}$$

For the case of Eqs. (28), we have

$$V = A_2 \cos \lambda_1 x + B_2 \sin \lambda_1 x + C_2 \cos \lambda_2 x + D_2 \sin \lambda_2 x$$

and, upon substitution into Eqs. (23), we find

$$W = A_2 f_1 \cos \lambda_1 x + B_2 f_1 \sin \lambda_1 x + C_2 f_2 \cos \lambda_2 x + D_2 f_2 \sin \lambda_2 x$$

$$a\Psi = A_2 g_1 \sin \lambda_1 x - B_2 g_1 \cos \lambda_1 x + C_2 g_2 \sin \lambda_2 x - D_2 g_2 \cos \lambda_2 x$$

where $f_1$, $g_1$, and $g_2$ are given as before, but

$$f_2 = \frac{\nu \lambda_2^2 - 1 - 2\nu(1 + \nu)\Omega^2}{1 - (1 + \nu)^2 \Omega^2}$$
and, similar to before, we have

\[ M = \frac{\pi a E h}{1 - \nu^2} \left[ (g_1 \lambda_1 + \nu + \nu f_1) (A_2 \cos \lambda_1 x + B_2 \sin \lambda_1 x) + (g_2 \lambda_2 + \nu + \nu f_2) (C_2 \cos \lambda_2 x + D_2 \sin \lambda_2 x) \right] \]

\[ Q = \pi G h \left[ (\lambda_1 + g_1) (B_2 \cos \lambda_1 x - A_2 \sin \lambda_1 x) + (\lambda_2 + g_2) (D_2 \cos \lambda_2 x - C_2 \sin \lambda_2 x) \right] \]

Finally, for this case, it can be seen that at \( x = 0 \):

\[ [\phi_{ij}] = \begin{bmatrix}
0 & \mu_3 & 0 & \mu_5 \\
\mu_1 & 0 & \mu_2 & 0 \\
1 & 0 & 1 & 0 \\
0 & -\frac{g_1}{a} & 0 & -\frac{g_2}{a}
\end{bmatrix} \]

(29)

where

\[ \mu_5 = \pi G h (\lambda_2 + g_2) \]

and, at \( x = l/a \),

\[ [\gamma_{ij}] = \begin{bmatrix}
-\mu_3 \sin \Lambda_1 & \mu_3 \cos \Lambda_1 & -\mu_5 \sin \Lambda_2 & \mu_5 \cos \Lambda_2 \\
\mu_1 \cos \Lambda_1 & \mu_1 \sin \Lambda_1 & \mu_2 \cos \Lambda_2 & \mu_2 \sin \Lambda_2 \\
\cos \Lambda_1 & \sin \Lambda_1 & \cos \Lambda_2 & \sin \Lambda_2 \\
\frac{g_1}{a} \sin \Lambda_1 & -\frac{g_1}{a} \cos \Lambda_1 & \frac{g_2}{a} \sin \Lambda_2 & -\frac{g_2}{a} \cos \Lambda_2
\end{bmatrix} \]

(30)

Then, Eqs. (27) and (21) are used for determining the coefficients \( a_{ij} \) for this case.
APPLICATION TO SHELL WITH RIGID TOP MASS

For a shell element which supports a rigid top mass and is excited laterally in translation only at the base, we have

\[ \Psi_1 = 0 \quad , \quad Q_2 = -\omega^2 M^* V_2 \quad , \quad M_2 + Q_2 \delta = -\omega^2 I \Psi_2 \]

where the notation is indicated in Figure 3. Combining the last two equations

\[ \Psi_1 = 0 \quad , \quad Q_2 = -\omega^2 M^* V_2 \quad , \quad M_2 = \omega^2 M^* \delta V_2 - \omega^2 I \Psi_2 \]

and, by substituting these into Eqs. (20), we have

\[
\begin{vmatrix}
Q_1 \\
M_1 \\
V_1 \\
0
\end{vmatrix} =
\begin{bmatrix}
\frac{1}{1} & \frac{1}{1} \\
\frac{1}{1} & \frac{1}{1} \\
\frac{1}{1} & \frac{1}{1} \\
\frac{1}{1} & \frac{1}{1}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\Psi_1 \\
\Psi_2
\end{bmatrix}
\]

\[ \frac{1}{1} \left[ \begin{array}{cc}
c_{11} & c_{12} \\
c_{21} & c_{22} \\
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{array} \right]
\begin{bmatrix}
V_1 \\
V_2 \\
\Psi_1 \\
\Psi_2
\end{bmatrix}
\]

Partitioning the matrices as indicated, and dividing by \( V_1 \), we have

\[
\begin{align*}
\begin{bmatrix} Q_1/V_1 \\ M_1/V_1 \\ \Psi_1/V_1 \end{bmatrix} &= [c_{ij}] \begin{bmatrix} V_2/V_1 \\ \Psi_2/V_1 \end{bmatrix} \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= [d_{ij}] \begin{bmatrix} V_2/V_1 \\ \Psi_2/V_1 \end{bmatrix}
\end{align*}
\]
30° Cone \((\alpha = 30°)\)
- \(a = 5 \text{ in.}\), \(b = 2.5 \text{ in.}\), \(s_1(1-\gamma) = 5.00 \text{ in.}\)
- \(I = 0.519 \text{ lb in sec}^2\), \(\delta = 1.08 \text{ in.}\)

15° Cone \((\alpha = 15°)\)
- \(a = 5 \text{ in.}\), \(b = 2.5 \text{ in.}\), \(s_1(1-\gamma) = 9.66 \text{ in.}\)
- \(I = 0.519 \text{ lb in sec}^2\), \(\delta = 1.08 \text{ in.}\)

Cylinder \((\alpha = 0)\)
- \(a = 5 \text{ in.}\), \(l = 15 \text{ in.}\)
- \(I = 0.547 \text{ lb in sec}^2\), \(\delta = 0.56 \text{ in.}\)

All Specimens
- \(M^*g = W_t = 32.8 \text{ lb.}\)
- \(h = 0.005 \text{ in.}\), \(E = 30 \times 10^6 \text{ psi}\)

Figure 3. Schematic Of Rigid Top Mass On Shell
Now, from Eq. (32b), we have

\[
\begin{bmatrix}
\frac{V_2}{V_1} \\
\frac{\Psi_2}{V_1}
\end{bmatrix} = [d_{ij}]^{-1} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
d_{22} \\
-d_{21}
\end{bmatrix} (d_{11}d_{22} - d_{12}d_{21})^{-1}
\]

(33)

and recalling that

\[
\dot{V} = i\omega V, \quad \dot{\Psi} = i\omega \Psi
\]

we have

\[
\begin{align*}
\left(\frac{\dot{V}_2}{\dot{V}_1}\right) &= \left(\frac{V_2}{V_1}\right) \\
\left(\frac{\dot{\Psi}_2}{\dot{\Psi}_1}\right) &= \left(\frac{\Psi_2}{V_1}\right)
\end{align*}
\]

(34a, b)

along with

\[
\begin{align*}
Q_1/\dot{V}_1 &= -\frac{i}{\omega} [c_{11}V_2/V_1 + c_{12}\Psi_2/V_1] \\
M_1/\dot{V}_1 &= -\frac{i}{\omega} [c_{21}V_2/V_1 + c_{22}\Psi_2/V_1] \\
Q_1/\dot{V}_2 &= (Q_1/\dot{V}_1) (V_1/V_2) \\
M_1/\dot{V}_2 &= (M_1/\dot{V}_1) (V_1/V_2)
\end{align*}
\]

(34c, d, e, f)

Equations (34c, d) and (34e, f) express driving point and transfer pseudo-impedances, respectively. All of Eqs. (34) hold true for cylinders as well as for truncated cones.

**EXPERIMENTAL APPARATUS AND PROCEDURE**

The apparatus which has been designed to measure the behavior of cones under lateral excitation is basically the same as that used for longitudinal excitation in the earlier part of the program so that details of most
of the apparatus can be obtained from the earlier report. Only a brief description will be given here for those parts of the system which are different from that used for longitudinal excitation. A diagram of the apparatus is shown in Figure 4, while a photograph is shown in Figure 5.

The same two cones (15° and 30°) along with the same cylinder are used for the present tests under lateral excitation, and the same terminal weights (32.8 lb) are used at the output ends. However, the specimens are now excited laterally by the use of a horizontal slip table in conjunction with the electrodynamic shaker as shown in Figure 5. The same base rings and mounting plate are again utilized, except that an alteration in the force gage arrangement is necessary. As can be seen, two vertically oriented force gages are used for measuring input moment \( M_1 \), while two horizontally oriented force gages are used to measure input force \( Q_1 \). These horizontal force gages, one on each side of the base ring, have one end bolted to the lower base plate and one end bolted to the base ring on the cone. This design allowed for essentially no cross-signals between the force and moment gages. Input velocity and output velocity and rotation are measured by means of piezoelectric accelerometers. Thus, a bare minimum of additional apparatus was necessary over that required for the earlier studies incorporating longitudinal excitation.

The procedure for experimental measurement is essentially the same as that previously utilized. A similar mass-cancellation circuit is used for nullifying the force signal resulting from the base rings, and a frequency range of 20 to 600 cps is used.
Figure 4. Diagram Of Experimental Apparatus
RESULTS AND DISCUSSION

Theoretical and experimental results are presented in the remaining figures of this report. The results are presented in terms of the absolute values of the parameters indicated in Eqs. (34) for laterally excited specimens supporting a rigid top mass. Six different parameters were utilized in order to obtain a good overall picture of the dynamic behavior of the three specimens studied. The results, as presented in eighteen different figures, can conveniently be compared if they are laid out as indicated in Figure 6. This arrangement allows a quick comparison among the six parameters for a given specimen by moving horizontally, while comparisons for geometric effects on a given type of parameter can be made by moving vertically from the 30° cone to the cylinder (top to bottom). The latter procedure will be used in the subsequent discussion. Although the absolute values of the parameters are plotted, the algebraic sign is indicated for the various branches of the curves. These algebraic signs, of course, correspond to the convention indicated in Figure 2. Experimental phase angles were found to correspond with these signs in general, except that they shifted more gradually in the vicinity of discontinuities, because of the presence of damping.

The translational velocity ratio (Figs. 7-9) and the rotational velocity ratio (Figs. 10-12) were included as parameters since they are probably the least susceptible to errors within the experimental apparatus. That is, the signals were obtained simply from piezoelectric accelerometers, and no further processing other than filtering was applied. Likewise, these
<table>
<thead>
<tr>
<th>Cone</th>
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<th>13</th>
<th>16</th>
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<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
</tr>
</tbody>
</table>

**Figure 6. Convenient Layout For Results**
Figure 7. Translational Velocity Ratio For 30° Cone And Top Mass
Figure 8. Translational Velocity Ratio For 15° Cone And Top Mass
Figure 9. Translational Velocity Ratio For Cylinder And Top Mass
Figure 10. Rotational Velocity Ratio For 30° Cone And Top Mass
Figure 11. Rotational Velocity Ratio For 15° Cone And Top Mass
Figure 12. Rotational Velocity Ratio For Cylinder And Top Mass
signals, as well as those for moments, were not influenced by the mass-
balance system. Division in the velocity ratios, as for all cases, was
performed numerically rather than electronically. A basic change in the
trend of the translational velocity ratio occurs in going from the 30° cone
(Fig. 7) to the 15° cone (Fig. 8). The 15° cone and cylinder are more
alike (Figs. 8, 9).

Overall agreement between theory and experiment is good for the
first two parameters (Figs. 7-9 and 10-12), although various discrepancies
can be observed in different areas of the frequency range. The first
resonance appears to occur at a slightly lower frequency than predicted
for all three geometries. In addition, a split peak, which probably results
from geometric defects, occurs for the first resonance in the cylinder.
This trend will be consistent throughout all the results. It may be noted
that an extraneous resonance appeared at about 75 cps in Figure 10. This
resonance does not appear in any of the other data for the 30° cone and is
probably due to extraneous motion in some part of the fixtures that did not
influence the other signals.

Force input pseudoimpedances are shown in Figures 13-15. Again
a basic change in the trend occurs in going from the 30° cone (Fig. 13) to
the 15° cone (Fig. 14). Comparison between theory and experiment is
good outside the two resonance peaks, but is poor for the range of fre-
quencies in between. The discrepancy appears to be most severe for the
cylinder, where the experimental intermediate antiresonance occurs at
a significantly lower frequency than is predicted. The source of this
Figure 13. Force Input Pseudoimpedance For 30° Cone And Top Mass
Figure 14. Force Input Pseudoimpedance For 15° Cone And Top Mass
Figure 15. Force Input Pseudoimpedance For Cylinder And Top Mass
error appears to be in the value of the net shear force $Q_1$. In order to verify the performance of the experimental system, data were retaken after the system had been disassembled and reassembled, and no essential difference in results occurred. Further, the performance of the mass-balance system was checked and rechecked. The linearity of the system was found to be good upon checking the results at various input amplitudes. Thus, the source of the discrepancy does not appear to lie in the instrumentation.

Further reflection on the location of the antiresonance in Figures 13-15 leads back to Eq. (10b). That is, the antiresonance occurs at the point where the net shear force $Q_1$ at the input becomes zero. Theoretically, this occurs in the nontrivial case where the two terms of Eq. (10b) nullify each other. It appears that in the experimental system, the actual distribution of shear forces present is different from that predicted within the intermediate frequency range. This may result from small wrinkles and eccentricities in the cylinder or may reflect the need of using a bending theory. More work is necessary to resolve this question.

Moment input pseudoimpedance is shown in Figures 16-18. Although there is a consistent change in shape in going from the 30° cone to the cylinder, the general character of the curves is unaltered. Agreement between theory and experiment appears to be worst for the 15° cone, with a local extraneous discontinuity appearing at about 250 cps. The origin of this discontinuity remains undetermined. We again emphasize that this parameter was not influenced by the mass-balance system.
Figure 16. Moment Input Pseudoimpedance For 30° Cone And Top Mass
Figure 17. Moment Input Pseudoimpedance For 15° Cone And Top Mass
Figure 18. Moment Input Pseudoimpedance For Cylinder And Top Mass
Force transfer pseudoimpedance is shown in Figures 19-21. Agreement between theory and experiment is quite good for the 30° cone, but becomes progressively worse in going to the cylinder. Again the increasing discrepancy in the position of the intermediate antiresonance appears. Figure 21 probably shows the worst overall correspondence for all the data presented. The same comments which were previously made about this discrepancy also apply here.

Moment transfer pseudoimpedance is shown in the final Figures 22-24. Agreement between theory and experiment is fair. Again the extraneous discontinuity appears in the experimental values for the 15° cone at about 250 cps. Likewise, increased discrepancy occurs above 400 cps.

Several possible sources of error in the experiments have already been mentioned but were continuously checked. An additional source, however, is inability to maintain a perfect zero rotational input ($\Psi_1 = 0$) at the base of the specimens. The sensitivity of the results to small amplitudes of this parameter would need to be determined.

Overall agreement in all results appears to be good. However, several definite discrepancies are present for part of the frequency range. Whether or not the use of a bending theory would prove more accurate, or whether consideration must be given to cylinder imperfections, remains to be determined. Nevertheless, the membrane theory does appear to provide at least a good approximation for predicting impedance and transmission characteristics of cones and cylinders.
Figure 19. Force Transfer Pseudoimpedance For 30° Cone And Top Mass
Figure 20. Force Transfer Pseudoimpedance For 15° Cone And Top Mass
Figure 21. Force Transfer Pseudoimpedance For Cylinder And Top Mass
Figure 22. Moment Transfer Pseudoimpedance For 30° Cone And Top Mass
Figure 23. Moment Transfer Pseudoimpedance For 15° Cone And Top Mass
Figure 24. Moment Transfer Pseudoimpedance For Cylinder And Top Mass
ACKNOWLEDGMENTS

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REFERENCES


APPENDIX

LISTING OF COMPUTER PROGRAM AND FORMAT OF INPUT DATA CARDS
## INPUT DATA DESCRIPTION
### (CONE)

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<td>SB</td>
<td>b</td>
<td>in.</td>
<td>Minor base radius.</td>
</tr>
<tr>
<td></td>
<td>ALPHA</td>
<td></td>
<td>deg.</td>
<td>Semivertex angle.</td>
</tr>
<tr>
<td></td>
<td>H</td>
<td>h</td>
<td>in.</td>
<td>Wall thickness of conical shell.</td>
</tr>
<tr>
<td>3</td>
<td>ENU</td>
<td>ν</td>
<td></td>
<td>Poisson's ratio.</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>E</td>
<td>psi</td>
<td>Young's modulus.</td>
</tr>
<tr>
<td></td>
<td>RHO</td>
<td>ρ</td>
<td>lb-sec²/in⁴</td>
<td>Mass density.</td>
</tr>
<tr>
<td>4</td>
<td>WT</td>
<td>Wₜ</td>
<td>lb.</td>
<td>Weight of attached mass.</td>
</tr>
<tr>
<td></td>
<td>AI</td>
<td>I</td>
<td>lb-in-sec²</td>
<td>Moment of inertia of supported mass.</td>
</tr>
<tr>
<td></td>
<td>DEL</td>
<td>δ</td>
<td>in.</td>
<td>Offset of center of supported mass.</td>
</tr>
<tr>
<td>5</td>
<td>FRQᵢ</td>
<td>fᵢ</td>
<td>cps</td>
<td>Initial frequency.</td>
</tr>
<tr>
<td></td>
<td>FRQXᵢ</td>
<td>Δfᵢ</td>
<td>cps</td>
<td>Frequency increment.</td>
</tr>
<tr>
<td></td>
<td>FRQNᵢ</td>
<td>fₙ</td>
<td>cps</td>
<td>Final frequency.</td>
</tr>
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</table>
PROGRAM OUTPUT

Printed Output

1. All input data except $I$ and $\omega$.

2. Frequency $f$ in cps and rad/sec.

3. Characteristic transfer matrix $[\alpha_{ij}]$ and transfer matrix $[\beta_{ij}]$.

4. Translational and rotational velocity ratios $(\dot{V}_2/\dot{V}_1)$ and $(\dot{\psi}_2/\dot{V}_1)$.

5. Force input and moment input pseudo impedances $ZQ_{11}$ and $ZM_{11}$.

6. Force transfer and moment transfer pseudo impedances $ZQ_{12}$ and $ZM_{12}$.

7. If $\Omega > \cos (\alpha)$, an error message will be printed and the program will continue.
Subprogram Used

In addition to the main program, the following function subprogram was used.

1. RKLDEQ, computes the solution of n first-order ordinary differential equations by the Runge-Kutta-Gill fourth-order method.
GO TO 1000
40 CONTINUE
GO TO 2000
17 PRINT 90, W
550 FORMAT (1H ,3X,E11.3,4X,25HNEAR OR ABOVE SINGULARITY)
GO TO 45
END
FORTRAN STATEMENT

N FOR $[f]_i, [\Delta f]_i, [f]_m$ (FORMAT IS)

3

$a, b, \alpha, \text{AND } h$ (FORMAT 4F10.0)

5.0  2.5  15.0  0.005

$\nu, \varepsilon, \text{AND } \rho$ (FORMAT F10.0, 2E10.2)

0.3  3.00+07  7.35-04

$w, I, \text{AND } \delta$ (FORMAT 3F10.0)

32.8  0.519  1.08

$[f]_i, [\Delta f]_i, [f]_m$ (FORMAT 3F10.0)

20.0  10.0  30.0

40.0  20.0  100.0

200.0  100.0  300.0
**INPUT DATA DESCRIPTION**  
*(CYLINDER)*

<table>
<thead>
<tr>
<th>CARD NO.</th>
<th>FORTRAN SYMBOL</th>
<th>VARIABLE NAME</th>
<th>UNITS</th>
<th>DEFINITION</th>
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<tr>
<td>1</td>
<td>N</td>
<td>n</td>
<td></td>
<td>Number of frequency sets.</td>
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<tr>
<td>2</td>
<td>SA</td>
<td>a</td>
<td>in.</td>
<td>Radius of cylindrical shell.</td>
</tr>
<tr>
<td></td>
<td>SL</td>
<td>l</td>
<td>in.</td>
<td>Length of cylindrical shell.</td>
</tr>
<tr>
<td></td>
<td>H</td>
<td>h</td>
<td>in.</td>
<td>Wall thickness of cylindrical shell.</td>
</tr>
<tr>
<td>3</td>
<td>ENU</td>
<td>ν</td>
<td></td>
<td>Poisson's ratio.</td>
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<tr>
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<td>WT</td>
<td>W_t</td>
<td>lb.</td>
<td>Weight of attached mass.</td>
</tr>
<tr>
<td></td>
<td>AI</td>
<td>I</td>
<td>lb-in-sec²</td>
<td>Moment of inertia of supported mass.</td>
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<tr>
<td></td>
<td>DEL</td>
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<td>in.</td>
<td>Offset of center of supported mass.</td>
</tr>
<tr>
<td>5</td>
<td>F_i</td>
<td>f_i</td>
<td>cps</td>
<td>Initial frequency.</td>
</tr>
<tr>
<td></td>
<td>FDX_i</td>
<td>Δf_i</td>
<td>cps</td>
<td>Frequency increment.</td>
</tr>
<tr>
<td></td>
<td>FN_i</td>
<td>f_n</td>
<td>cps</td>
<td>Final frequency.</td>
</tr>
</tbody>
</table>
PROGRAM OUTPUT

Printed Output

1. All input data except I and $\sigma$.
2. Frequency $f$ in cps and rad/sec.
3. Characteristic transfer matrix $[\zeta_{ij}]$ and transfer matrix $[\beta_{ij}]$.
4. Translational and rotational velocity ratios ($\hat{V}_2/\hat{V}_1$) and ($\hat{\psi}_2/\hat{\psi}_1$).
5. Force input and moment input pseudo impedances ZQ11 and ZM11.
6. Force transfer and moment transfer pseudo impedances ZQ12 and ZM12.
7. If $\Omega > 0.99$, an error message will be printed and the program will continue.
PROGRAM NOTES
(CYLINDER)

Subprogram Used

In addition to the main program, the following subroutine sub-program was used.

1. MATINV, computes the inverse of a real matrix.
PROGRAM CONEIMP
PROJECT 02, 03, 04
C CDR 3600 FORTRAN
DIMENSION Y(4), F(4)
DIMENSION FRQ(20), FRQX(20), FRON(20)
DIMENSION V4(4), U4(4), HM4(4), O(4)
DATA (PI=3.14159265), (C1=386.9), (C2=1.45329256-0.02), (ERH=1.5)
200 READ 200, N
200 FORMAT (155)       IF (E(N, 460), 85
STOP
80 *** GEOMETRIC PARAMETERS
80 READ 200, A, ASH, ALPHAH
200 FORMAT (4-1.0,10)
*** MATERIAL PARAMETERS
READ 210, V, Hu, E, Rho
210 FORMAT (1.0, 2E10, 2)
*** RIGID MASS
READ 215, M, A, D, F
215 FORMAT (3F10.0)
HMS = M / A
ALFR = C2 * ALPHA
HAN = CRXP(ALFR)
BNS = CFNP(ALFR)
G = E / (2.0 * 1.10 * ENU))
W0 = SORTF(CFRHO)/A
GAM = S/A
WS = WMDCN
FS = WS(2.0 * PI)
*** FREQUENCY RANGE
READ 220, (FRQ(I), FRQX(I), FRON(I), I=1, N)
220 FORMAT (3F10.0)
PRINT 300
300 FORMAT (1RF, 5X, 51H CONICAL SHELL LATERAL IMPEDANCE PROGRAM - D.D., SWR 3500
1KANA/VgAH THIS PROGRAM CALCULATES (4X4) TRANSMISSION MATRICES BETASW 3600
2T(I, J) AND F(I, J) FOR CONICAL SHELLS UNDER LATERAL EXCITATION SWR 3700
3ONS, ASOF66H CALCULATES INPUT AND TRANSFER SPECTRAL IMPEDANCES WHEN SWR 3800
A AN ARBITRARY 24H MASS M IS ATTACHED TO THE OUTPUT TERMINAL 2) SWR 3900
PRINT 305, ALPHA, A, ENU, S, RHO, A, HMS
305 FORMAT (1RF, 10X, 20H GEOMETRIC PARAMETERS, 25H, 19H MATERIAL PARAMETERS, SWR 4100
1/26H SHAPED-THX ANGLE ALPHAH = F7.3, 8H DEGREES, 6X, 20YOUNG MODULUS SWR 4200
2US E = E10.3, 4H PS1/26H MAJOR BASE RADIUS A = F7.3, 7H INCHES SWR 4300
35X, 20X POISSON'S RATIO NU = E10.3/26H MINOR BASE RADIUS B = SWR 4400
4F7.3, 7H INCHES, 7X, 20XMISS DENSITY RHO = E10.3, 17H LB(SEC)**2/IN**3 SWR 4500
5**4/10H THICKNESS12X, 4H = F7.3, 7H INCHES, 7X, 4H MASS, 12X, 4H SWR 4600
6E, E10.3, 24H ENERGY(SEC)**2/IN**3 SWR 4700
PRINT 320
320 FORMAT (1RF, 3X, 17H FREQUENCY ( CPS )) SWR 4900
PRINT 325, (FRQ(I), FRQX(I), FRON(I), I=1, N) SWR 5000
325 FORMAT (5X, 2F10.2, 18H, 1H, 18H, 1H) SWR 5100
PRINT 310, FS, WS SWR 5200
310 FORMAT (1RF, 23X, 33H FREQUENCY SINGULARITY - FS = F8.1/48X, SWR 5300
19WOME غال = F8.1 SWR 5400
PRINT 330 SWR 5500
330 FORMAT (*3RF, 5X, 18WOME غال = 14X, (((ALPHA(I, J), IM=1, 4, I=1, 4)* SWR 5600
1 27X, (((HEM(I, J), IM=1, 4, I=1, 4, I=1, 4, I=1, 4)) SWR 5700
DO 40 IM=1, N SWR 5800
40 FREQ = FRON(I) SWR 5900
1000 W = 2.0 * P1 * FREQ SWR 6000
WPS = W ** W SWR 6100
OMEGA = W / W0 SWR 6200
USN = (MEGA*OMEGA)
UX = (1.0/SH/A)*100.0
V1(1) = BM(2) = Q(1) = 1.0
V1(2) = V(4) = 0.
U(3) = -((XNU*DCN)/(1.0-OSQ*GAM*GAM))
U(4) = ((XNU*DCN=OSQ*GAM*GAM)*A/GAM)/(1.0-OSQ*GAM*GAM)
U(2) = -((XNU*DSN)/(PI*E*H*A*DCN))/((1.0-OSQ*GAM*GAM))
U(1) = 0.
WM(1) = WM(3) = BM(4) = 0.
Q(2) = Q(3) = Q(4) = 0.
IF (XNU=0.0) PRINT 150, WNU=0.0
450 FORMAT (1H,2F10.1,4X,25HNEAR OR ABOVE SINGULARITY)
GC 10 50
10 CONTINUE
10 DO 45 J=1,4
X = GAM
Y(1) = V(J)
Y(2) = UX(J)
Y(3) = BM(J)
Y(4) = V(J)
WCAP = -(Y(1)+Y(2))*TAN*(((ENU*Y(3))/((PI*E*H*A*DCN)*X)))/(1.0-
1 (((OSQ*X*X)/(DCN*DCN))
K = 0
20 WCAP = -(Y(1)+Y(2))*TAN*(((ENU*Y(3))/((PI*E*H*A*DCN)*X)))/(1.0-
1 (((OSQ*X*X)/(DCN*DCN)))
F(1) = Y(1)*DSN*Y(2)*DCN*Y(4)/(PI*G*H)*(TAN*Y(3))/(PI*G*H*
1 A(X)))/(X/DSN)
F(2) = ((Y(3)/(PI*E*H*A*X*X)))-(((ENU*OSQ*X*WSQ)/DCN))/DSN
F(3) = ((PI*E*H*A*X*X)/DSN)*(WCAP*TAN*Y(2))
1 Y(4)=A*(DCN*DSN)
F(4) = PI*H*OSQ*(X/DSN)*(WCAP*Y(1)+Y(2))*TAN)
S = RKL.UEN (4,Y,F,X,DN,K)
IF (S=0.0) 25,20,30
25 STOP
30 CONTINUE
IF (X=1.0) 20,50,50
50 CONTINUE
GO TO (70,72,73,75) J
70 H11 = Y(4) $ B41 = (Y(2)-WCAP*TAN)/(A*X) $ B31 = Y(1) $ B21 = Y(3) $B11 = Y(5)
GO TO 45
72 H12 = Y(4) $ B42 = (Y(2)-WCAP*TAN)/(A*X) $ B32 = Y(1) $ B22 = Y(3) $B12 = Y(5)
GO TO 45
73 H13 = Y(4) $ B43 = (Y(2)-WCAP*TAN)/(A*X) $ B33 = Y(1) $ B23 = Y(3) $B13 = Y(5)
GO TO 45
75 H14 = Y(4) $ B44 = (Y(2)-WCAP*TAN)/(A*X) $ B34 = Y(1) $ B24 = Y(3) $B14 = Y(5)
45 CONTINUE
A11 = H11 $ A12 = H12 $ A21 = H21 $ A22 = H22
A31 = W*H31 $ A41 = W*H41 $ A32 = W*H32 $ A42 = W*H42
A13 = -D15/W $ A23 = -H23/W $ A14 = -B14/W $ A24 = -B24/W
A33 = H33 $ A34 = H34 $ A43 = H43 $ A44 = H44
TM1 = HMS*WSQ
TM2 = A1*WSQ
TM3 = TM1=DEL
C11 = H13+TM3*H12-TM1*H11
C12 = a14-TM2+b12
C21 = H23+TM3*H22-TM1*H21
C22 = H24-TM2+b22
D11 = H33+TM3*H32-TM1*H31
D12 = H34-TM2+b32
D21 = H43+TM3*H42-TM1*H41
D22 = H44-TM2+b42
SM 6300
SM 6400
SM 6500
SM 6600
SM 6700
SM 6800
SM 6900
SM 7000
SM 7100
SM 7200
SM 7300
SM 7400
SM 7500
SM 7600
SM 7700
SM 7800
SM 7900
SM 8000
SM 8100
SM 8200
SM 8300
SM 8400
SM 8500
SM 8600
SM 8700
SM 8800
SM 8900
SM 9000
SM 9100
SM 9200
SM 9300
SM 9400
SM 9500
SM 9600
SM 9700
SM 9800
SM 9900
SM 10000
SM 10500
SM 10600
SM 10700
SM 10800
SM 10900
SM 11000
SM 11100
SM 11200
SM 11300
SM 11400
SM 11500
SM 11600
SM 11700
SM 11800
SM 11900
SM 12000
SM 12100
SM 12200
SM 12300
SM 12400
TM4 = U11*22-D12*B21
V2V1 = U22/TM4
P2V1 = -U21/TM4
ZG11 = (C11+V2V1*C12*P2V1)/W
ZM11 = (U21*V2V1*C22*P2V1)/W
ZQ12 = ZQ11/V2V1
ZM12 = ZM11/V2V1
PRINT 335, FREU, W, A11, A12, A13, A14, R11, P12, R13, R14
335 FORMAT (F4.1, F10.1, 4E12.3, 5X, 4E12.3)
prints A21, A22, A23, A24, B21, B22, B23, B24, A31, A32, A33, A34, B31,
340 FORMAT (1W, 4E12.3, 5X, 4E12.3)
PRINT 370, V2V1, P2V1, ZM11, ZQ11, ZM12, ZQ12
50 1F (FR = W + H24(1)) 35, 40, 4
35 FREW = FR3*FR2*X(1)
GO TO 1000
GO TO 2000
END