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NASA CR-1040

NASA CR-1040

GPO PRICE $ ______

CFSTI PRICE(S) $ ______

Hard copy (HC) 3.00

Microfiche (MF) .65

ff 653 July 65

# NON-LINEAR ORBIT DETERMINATION METHODS


*by Ali Hasan Nayfeh*

*Prepared by*
HELIODYNE CORPORATION
Van Nuys, Calif.
*for Langley Research Center*


FACILITY FORM 602

(ACCESSION NUMBER) (THRU)

(PAGES) (CODE)

(NASA CR OR TMX OR AD NUMBER) (CATEGORY)


NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1968

NASA CR-1040

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By Ali Hasan Nayfeh



Prepared under Contract No. NAS 1-6690 by
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# NON-LINEAR ORBIT DETERMINATION METHODS

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## SUMMARY

A non-linear correction method for orbit determination using range, azimuth, and elevation data is presented and justified mathematically and numerically. In contrast with conventional methods, such as the least-squares, the maximum likelihood, and the Kalman, this method holds in the non-linear, as well as in the linear regions. Moreover, this method does not require transition and normal matrices and, hence, avoids the problems associated with calculating and inverting them. The convergence and accuracy of the method have been demonstrated by calculations made on simulated orbits with varying levels of noise. The calculations demonstrate that the proposed method is non-linear and can be used as an editing procedure because it converges despite the combined effect of very bad reference orbits, extremely high levels of noise, wild data points, and critical orbits. The calculations show also that the errors of the resultant epoch state vectors are indeed very small.

The method has been extended for the determination of aerodynamic parameters such as $\beta$. Numerical calculations using range, azimuth, and elevation data show that the method converges and the errors are small irrespective of the bad initial conditions for the trajectory and for the $\beta$.

The method has been extended also to cis-lunar orbits where range and range rate are the only reliable measurements. In this case, the problem reduces to the problem of determining an orbit using range and range rate at three different times which can be taken to be close to each other. The method of quasilinearization is applied to this problem. In carrying out this method, a solution for a set of six quadratic algebraic equations needs to be obtained. The numerical solution of these sets of algebraic equations can be obtained using Kane's refinement of the Newton-Raphson procedure.

An alternative method based on quasilinearization has been presented for orbit determination in general and cis-lunar orbits in particular. Moreover, non-linear least-squares and maximum likelihood methods have been presented. However, work needs to be done to program these alternative methods and compare their numerical results.

## INTRODUCTION

Data pertaining to either the position or the velocity or both of a space vehicle can be obtained by its tracking. Irrespective of the kind of sensor being used, the data is, however, corrupted by noise. Thus, redundant data must be used for the determination of the best estimate of the orbital parameters and any other unknown parameters in the equations of motion.

To determine the best estimate of these parameters from the noisy data, many filtering and estimation methods based on the differential correction method have been devised. The most commonly used methods are the least-squares (refs. 1 and 2), the maximum likelihood (refs. 1 and 2), and the Kalman filter (refs. 2 and 3). These methods start by assuming nominal values for these orbital parameters and, hence, a nominal orbit. The orbital parameters are perturbed and the orbit is expanded in a Taylor series expansion about this nominal orbit. The deviations in the orbit as a function of time are related to the perturbations in the orbital parameters through what is called a transition matrix. This transition matrix is multiplied by another matrix, depending on the observables, to relate the deviations in the observables to the orbital parameter perturbations. The residuals (observed minus computed values) are optimized in some fashion. In the least-squares method, the best estimate minimizes the sum of the squares of the residuals. The maximum likelihood assumes that the errors are normally distributed and, hence, the best estimate maximizes the likelihood function. On the other hand, the Kalman filter differs from the maximum likelihood by assuming that the best estimate is a linear function of the previous estimate, as well as the measured data. With these criteria, linear simultaneous equations are obtained to solve for the perturbations. These equations have to be solved for the perturbations, generally by inverting the so-called normal matrices. Then the orbital parameters are corrected and a new reference orbit is obtained and the process is iteratively repeated until the iteration hopefully converges.

The iteration procedure may diverge and lead to irrelevant answers in cases where the linearity assumption is violated (refs. 4 - 6). These cases arise if the initial conditions are such that the reference orbit is not nominal, or the data is corrupted by either high levels of noise or wild points, or the normal matrix is near singular.

There are some disadvantages associated with the use of transition and normal matrices. Time and effort are needed for calculating the partial derivatives constituting the transition matrices. For example, if we want to determine the initial conditions (position and velocity vectors), we need to calculate a $6 \times 6$ matrix. Thus, we need to solve 36 first-order differential equations. In general, these equations have to be solved numerically. Hence, the use of transition matrices requires the solution of 42 first-order equations instead of 6 equations only. Moreover, the inversion of the normal matrices may blow up if some of the orbital parameters happen to be near parallel and, hence, the normal matrix near singular. Although the orbital parameters may be theoretically independent (given an infinite number of significant figures), they may become dependent and, hence, the normal matrix becomes singular because of the number of significant figures that can be obtained using a computer. Also, the inversion of matrices suffer from high cumulative round-off errors due to the large number of arithmetical operations needed.

To determine the best estimate of the orbit that passes through noisy data, a non-linear correction method was developed by Hunt and Nayfeh (ref. 7). This filtering method avoids the problems associated with conventional methods, such as the Bayes estimation and the Kalman filter, because it is a non-linear method and it does not use transition and normal matrices.

The method is based upon the idea that if the exact orbit could be determined, the residuals (defined as measured minus computed values) would represent the noise. If this noise is random, then the coefficients of any polynomial that fits through the residuals in the least-squares sense must vanish. Hence, the time axis passes through the residuals in the least-squares sense.

Thus, after determining an orbit, one usually checks how well the orbit has been determined by plotting the residuals and seeing how well they are distributed about the time axis. Rather than using the fact that the time axis must pass through the residuals in least-squares sense as a check on how well we determine the orbit, we use this fact in order to determine the orbit. Thus, we fit the residuals into a polynomial in least-squares sense and use it to correct the initial conditions of the assumed orbit so that the coefficients of the next polynomial least-squares fit to the new residuals are decreased.

In this paper we will describe the non-linear filter and then we will extend it to determine the orbit as well as any perturbative parameters. We will extend the method for the determination of cis-lunar orbits. We present an alternative method based on quasilinearization. Finally, we propose non-linear least-squares and maximum likelihood methods for orbit determination.

## A NON-LINEAR ORBIT DETERMINATION METHOD

In this section, we will first describe the non-linear orbit determination method, and then justify it mathematically as well as numerically.

### Description of Method

The following description of the method assumes that the data is given in Cartesian coordinates. If the data is given in range, azimuth, and elevation, one can either transform these into Cartesian coordinates or write the equations of motion in spherical coordinates. In the latter case, the range, azimuth, and elevation take the place of the x, y, and z coordinates. The method consists of the following steps:

Determination of a reference orbit. - The method is an iterative or recursive procedure, and, thus, we need to assume an orbit (called reference orbit) to start the calculation. To calculate the reference orbit, we assume values for the initial conditions (initial position and initial velocity vectors) and then integrate the equations of motion. Since the method is non-linear, the procedure converges to the right orbit, irrespective of how bad the initial conditions used in calculating the reference orbit were, as demonstrated numerically on pages 11 through 13. Thus, any values can be used for the initial conditions to calculate the reference orbit.

However, we can always do better than a mere guess of the initial conditions for the reference orbit, and, thus, avoid the unnecessary delaying of the convergence. For example, we start by determining a two-body orbit with an inverse square central field. This two-body orbit can be obtained using a Gaussian, Lambertian, or Gibbsian method employing, for example, two-positional data points and their separation times. Using this orbit, we can determine an epoch state vector (initial position and velocity vectors). Using this epoch state vector, we integrate the full orbital equations of motion and we take the resulting orbit as the reference orbit.

Least-squares fit of the residuals. - The residual differences between the measured discrete data points and the values obtained from the computed reference orbit at the corresponding times are formed. These residuals $\Delta\underline{r}_j$ at time $t_j$ are fitted in the least-squares sense to a linear or a parabolic time-dependent vector. For demonstration, we choose a linear vector $\Delta\underline{a}_o + \Delta\underline{b}_o t$. Thus,

$$\sum_j \left[\Delta\underline{r}_j - \left(\Delta\underline{a}_o + \Delta\underline{b}_o t_j\right)\right]^T \left[\Delta\underline{r}_j - \left(\Delta\underline{a}_o + \Delta\underline{b}_o t_j\right)\right]$$

is minimum. Minimizing with respect to $\Delta\underline{a}_o$ and $\Delta\underline{b}_o$ yields

$$\sum_j \left[\Delta\underline{r}_j - \left(\Delta\underline{a}_o + \Delta\underline{b}_o t_j\right)\right] = 0, \tag{1}$$

and

$$\sum_j t_j \left[\Delta\underline{r}_j - \left(\Delta\underline{a}_o + \Delta\underline{b}_o t_j\right)\right] = 0. \tag{2}$$

Solving these equations for $\Delta\underline{a}_o$ and $\Delta\underline{b}_o$ yields

$$\begin{bmatrix} \Delta\underline{a}_o \\ \\ \\ \Delta\underline{b}_o \end{bmatrix} = C^{-1} \begin{bmatrix} \sum\limits_j \Delta\underline{r}_j \\ \\ \\ \sum\limits_j t_j \, \Delta\underline{r}_j \end{bmatrix} \tag{3}$$

where $C^{-1}$ is the inverse of the matrix C which is given by

$$C = \begin{bmatrix} n & \sum\limits_j t_j \\ \\ \\ \sum\limits_j t_j & \sum\limits_j t_j^2 \end{bmatrix} \tag{4}$$

Epoch state vector updating. - Once $\Delta\underline{a}_o$ and $\Delta\underline{b}_o$ have been found, a new epoch state vector is formed. Thus,

$$\underline{a}_1 = \underline{a}_o + \Delta\underline{a}_o, \tag{5}$$

and

$$\underline{b}_1 = \underline{b}_o + \Delta\underline{b}_o. \tag{6}$$

Only the initial conditions were modified because a polynomial time-dependent vector is not a solution of the full orbital equations of motion. Moreover, even if it is a solution of these equations, we cannot obtain a solution by adding this polynomial to the reference orbit because the orbital equations of motion are non-linear and the principle of superposition does not hold.

Iteration. - Using the new epoch state vector obtained on page 6, we compute a new reference orbit by integrating the full orbital equations of motion. Then, we form the residuals $\Delta \underline{r}_j$ and fit them in the least-squares sense to a linear time-dependent vector $(\Delta \underline{a}_1 + \Delta \underline{b}_1 t)$. Once $\Delta \underline{a}_1$ and $\Delta \underline{b}_1$ are found, we determine a new epoch state vector (initial conditions $\underline{a}_2$ and $\underline{b}_2$) according to

$$\begin{aligned} \underline{a}_2 &= \underline{a}_1 + \Delta \underline{a}_1 \\ \underline{b}_2 &= \underline{b}_1 + \Delta \underline{b}_1 \end{aligned} \tag{7}$$

We repeat the above procedure by calculating a new orbit, computing the residuals, fitting them to a straight line time-dependent vector, and then updating the epoch state vector. The iteration is ended when $|\Delta \underline{a}_i|$ and $|\Delta \underline{b}_i|$ are less than preassigned small positive convergence numbers.

A similar procedure is followed when all the data is not available simultaneously or if the time interval is long. For long time intervals and for real time operation, the recursive formulation as described below is used. If there are $A_n$ discrete position data points available at the times $t_1$, $t_2$, . . . , $t_n$, the values of the reference orbit are computed at these (n) points only. The residuals at these (n) points are used to estimate a correction to the initially assumed position and velocity vectors and, hence, a new reference orbit. As more data points become available, the predicted values of the new reference orbit are computed at the times corresponding to these new data points, as well as at the old ones. The residuals at all of these times are used to estimate a correction to this new reference orbit. This procedure is shown schematically in Figure 1 where the position data becomes available at the rate of one more point for each reference orbit. When the first reference orbit was computed, the data points $A_1$ and $A_2$ were available, while when the second and third reference orbits were computed, data points $A_1$, $A_2$, $A_3$, and $A_1$, $A_2$, $A_3$, $A_4$, respectively, were available.

## Mathematical Proof of Convergence of Method

To prove the convergence of the method, deterministic, rather than statistical problems, are used. The difference between the two cases is that in the deterministic case, the procedure converges to the true orbit, whereas in the statistical case the procedure converges to an orbit that minimizes the sum of the squares of the residuals. However, the orbit obtained in the statistical case is the true orbit if the noise is Gaussian. If the noise does not have zero mean, the orbit converges to a biased orbit. Therefore, the bias must be removed, either before or after processing the data.

The convergence of the method can be proved in the following way. For simplicity, consider the first-order equation

$$\dot{x} = f(x, t), \tag{8}$$

where $f(x, t)$ is continuous in $[0, \tilde{t}]$ and satisfies the Lipschitz condition (ref. 9) with $M \geq 0$,

$$|f(x, t) - f(y, t)| \leq M |x - y|. \tag{9}$$

Let $x_t$ and $x_i$ be the solutions of Eq. (8) corresponding to the initial conditions $a_t$ and $a_i$, respectively, and let

$$\Delta a_i = \frac{6}{t_f^2}\left[\frac{2}{3} t_f \int_0^{t_f} \left(x_t - x_i\right) dt - \int_0^{t_f} t\left(x_t - x_i\right) dt\right], \tag{10}$$

and

$$a_{i+1} = a_i + \Delta a_i. \tag{11}$$

Then, there exists at $t_f > 0$ such at $\lim_{i\to\infty} a_i = a_t$ and hence, $x_i \to x_t$.

It is known from the theory of ordinary differential equations (refs. 8 and 9) that for every initial condition $a_i$, there exists a unique continuously differentiable solution $x_i$ in $[0, \tilde{t}]$. The solutions corresponding to the two initial conditions $a_i$ and $a_j$ satisfy the inequality (ref. 9)

$$\left|x_i - x_j\right| \leq \left|a_i - a_j\right| e^{Mt}. \tag{12}$$

From Eq. (8)

$$x_i = a_i + \int_o^t f\left(x_i, t\right) dt \tag{13}$$

and

$$x_t = a_t + \int_o^t f\left(x_t, t\right) dt \tag{14}$$

Subtracting Eq. (14) from (13) and dividing by $(a_t - a_i) \neq 0$ (if $a_t - a_i = 0$, then $x_t \equiv x_i$ and, hence, $\Delta a_i \equiv 0$) leads to

$$\varphi_i(t) = \frac{x_t - x_i}{a_t - a_i} = 1 + \frac{1}{a_t - a_i} \int_o^t \left[f\left(x_t, t\right) - f\left(x_i, t\right)\right] dt. \tag{15}$$

Combining Eqs. (9) and (15) leads to

$$\left|\varphi_i - 1\right| \leq \frac{M}{\left|a_t - a_i\right|} \int_o^t \left|x_t - x_i\right| dt. \tag{16}$$

Combining (12) and (16) gives

$$\left|\varphi_i - 1\right| \le \left(e^{Mt} - 1\right). \tag{17}$$

Therefore, the sequence of functions $\varphi_i(t)$ is equicontinuous, i.e., for every $\eta > 0$, there exists a $\delta$ given by $(e^{M\delta} - 1) = (\eta/7)$ such that $\varphi_i(t) - 1 \le (\eta/7)$ if $0 \le t \le \delta$ for all i.

From (10) and the equicontinuity property, we find that

$$\frac{\Delta a_i}{a_t - a_i} \le 1 + \eta, \tag{18}$$

and

$$\frac{\Delta a_i}{a_t - a_i} \le 1 - \eta. \tag{19}$$

Hence,

$$\left|\frac{a_t - a_{i+1}}{a_t - a_i}\right| \le \eta, \tag{20}$$

and

$$\left|a_t - a_i\right| \le \eta^i \left|a_t - a_o\right|. \tag{21}$$

Therefore, for $\eta < 1$, i.e., for $t_f < (1/M) \ln (8/7)$, $\lim_{i\to\infty} \left|a_t - a_i\right| = 0$, and, hence, $a_i \to a_t$ and $x_i \to x_t$.

## Computational Demonstration of Convergence and Accuracy

In carrying out the caluclations described herein, extreme situations were simulated to test the performance of the method. The simulated data are designed to test the non-linearity of the method, its accuracy, its operation in the presence of high levels of noise, and the capability of the method to be used as an editing procedure. In performing the calculations, a right-handed earth-centered inertial Cartesian coordinate system was chosen. The earth's gravitational field second and fourth harmonics were included. A fourth-order predictor was used which was mechanized as a general purpose multi-equation subprogram. Double precision was employed in the numerical integration routine. All other computations are in single precision.

In simulating the data, the equations of motion were integrated for two near circular orbits and one hundred sets of X, Y, Z, positions at one hundred sequential discrete times were chosen. Near-circular orbits were used because they are more critical due to the importance of non-linearity at some points in these orbits. Gaussian noise with varying standard deviations was generated. Then, positional data is obtained by adding the noise at the corresponding times to the X, Y, Z positions obtained above.

A deterministic case. - To demonstrate that the method is non-linear, the following test was conducted using deterministic data.

Positional data sets (elements in three coordinates), one second apart, were computed from a near-circular orbit (referred to as "true" orbit) with a period of approximately 90 minutes and an inclination of about 45°. The epoch state vector (initial position and velocity vectors) for this true orbit is given in the first row of Table 1.

An initial reference orbit was computed using an assumed initial velocity of zero and initial coordinates differing by 1,000,000 feet from the true values. It is worth noting that these initial guesses are much worse than might be expected in a practical situation. This was done to demonstrate the ability of the new method to handle very poor initial guesses of the reference orbit. The recursive formulation of the proposed method, as described above, was used to process this data at the rate of one more point for each new reference orbit. The updated epoch state vectors obtained from processing 2, 20, and 27 points are shown in Table 1.

Table 1 shows the rapid convergence of the initial conditions of the reference orbits to those of the true orbit. From the third row, it can be seen that, after processing only the first two points, the initial position coordinates converged exactly to those of the true orbit. As seen from the last row of Table 1, the initial velocity converged exactly to that of the true orbit after processing 27 points. Therefore, the method converges to the true orbit, irrespective of how bad the initial reference orbit is and, hence, the method is non-linear.

Statistical cases. - To show that the combined effect of very bad initial guesses and the presence of high levels of noise does not affect the convergence of the proposed method, the following tests were conducted.

Two different near-circular orbits, at an inclination of 45° and with periods of approximately 90 minutes and 24 hours (referred to here as "true" orbits), were used. The epoch state vectors for these orbits are given by the first rows of Tables 2 and 3, respectively. One hundred sequential positional data points, 10 seconds apart, were obtained by adding different levels of Gaussian noise to these true orbits. For the 90-minute orbit, we computed three cases having noise with a standard deviation of $10^2$, $10^4$, and $10^6$ feet, respectively (Table 2). For the second case of the 24 hour orbit, the noise was even higher: $10^4$, $10^6$, and $10^7$ feet. The same ridiculous guess of the initial conditions used in the previous case is used in all of these cases.

The recursive formulation of the new method was used in processing this data at the rate of one more point for each new reference orbit. The resultant epoch state vectors and the errors are shown in Tables 2 and 3. The residuals vs time are presented in Figures 2 and 3.

Tables 2 and 3 show that the epoch state vector converged to those of the true orbits with excellent accuracy. From rows 3, 5, and 7, it can be seen that the errors made in estimating the true epoch state vector were indeed very small. Figures 2 and 3 reflect the fact that the residuals are random. Therefore, the proposed method filtered the noise from the orbit very well, irrespective of how high the noise level was and irrespective of the bad guess of the initial conditions.

Editing capability. - To show that the new method can be used as an editing procedure, the following test was conducted.

Positional data sets were obtained by adding random noise with $10^7$ feet standard deviation and 25 wild data points (differing from the true orbit by $10^9$ feet) to a near-circular orbit with a period of 24 and an inclination of 45°. The same initial conditions used previously were employed. The residuals vs time are plotted in Figure 4.

Despite the combined effect of a high level of noise, the presence of wild points and the bad guess of the initial conditions, the proposed method converged, while the conventional estimation method might "blow up" since they depend on the linearity assumption. As expected, the mean of these residuals is not zero; data points lying outside predetermined dispersion limits (outliners) can easily be eliminated. Then, a new estimate is made and the process is repeated. This shows that the proposed method can be used as an editing procedure and that the method is certainly non-linear.

## DETERMINATION OF PERTURBATIVE FORCE PARAMETERS

The procedure described in the preceding section can be easily adapted to determine unknown parameters in the equations of motion in addition to the orbital elements. In principle, the problem of determining a constant parameter (A) in the equations of motion is equivalent to adding a new equation $(dA/dt) = 0$ to these equations and determining the initial conditions for the new set of equations. In the rest of this section, we will describe the extension of the procedure described earlier in order to determine the aerodynamic parameters and we will give the results of a simulated numerical case.

For simplicity, we consider the determination of constant drag, lift, and side-force parameters in addition to the orbital parameters. The equations of motion of a point mass under the influence of lift, drag, and side-forces, as well as gravitational forces in a earth-centered-inertial coordinate system are

$$\frac{d^2\vec{r}}{dt^2} = - \text{grad } U + \frac{1}{2} \rho g \; V_a^2 \left[ - \frac{1}{\beta} \vec{e}_D + \frac{1}{\eta} \vec{e}_L + \frac{1}{\zeta} \vec{e}_S \right] \tag{22}$$

where $\vec{r}$ is the position vector, U is the gravitational potential, $\rho$ is the air density, and g is the gravitational constant, $\beta = (W/C_D A)$, $\eta = (W/C_L A)$, and $\zeta = (W/C_S A)$. The aerodynamic coefficients are $C_D$, $C_L$, and $C_S$ for drag, lift, and side-force, respectively. W is the weight and A is a characteristic area of the vehicle. Since the atmosphere is assumed to rotate with the Earth, the relative velocity vector of the vehicle with respect to the atmosphere $\vec{V}_a$ is given by

$$\vec{V}_a = \frac{d\vec{r}}{dt} - \vec{\omega} x \vec{r} \tag{23}$$

where $\omega$ is the angular rotation vector of the Earth. If wind variations are included, then the relative velocity $\vec{V}_a$ is given by

$$\vec{V}_a = \frac{d\vec{r}}{dt} - \vec{\omega} x \vec{r} - \vec{V}_\omega \tag{24}$$

where $\vec{V}_\omega$ is the wind velocity. The unit vectors $e_D$, $e_L$, and $e_S$ are defined by

$$\vec{e}_D = \frac{V_a}{|\vec{V}_a|}, \quad \vec{e}_L = \frac{\vec{e}_D \; x \left( \vec{r} x \vec{e}_D \right)}{r}, \quad \vec{e}_S = \frac{\vec{r} x \vec{e}_D}{r} \tag{25}$$

To determine the aerodynamic parameters $\beta$, $\eta$, and $\zeta$ as well as the initial conditions of the trajectory from measurements that provide complete position fixes, we propose the following method.

a. We fit the first, say, ten data points to a straight line time-dependent vector $\vec{a}_o + \vec{b}_o t$ in least-squares sense. We choose $\vec{a}_o$ and $\vec{b}_o$ to be the initial position and velocity of the trajectory and guess values for $\beta$, $\eta$, and $\zeta$. Using these values for the aerodynamic parameters and the initial position and velocity, we calculate a reference trajectory by integrating Eqs. (22) - (25), and calculate the initial acceleration $\vec{c}_o$.

b. We form the residual differences between the measured discrete data points and the values obtained from the computed reference trajectory at the corresponding times. We fit these residuals in a least-squares sense to a parabolic time-dependent vector $\Delta\vec{a}_o + \Delta\vec{b}_o t + \frac{1}{2}\Delta\vec{c}_o t^2$.

c. We update the initial position, velocity, and acceleration. Thus,

$$\vec{a}_1 = \vec{a}_o + \frac{1}{2}\Delta\vec{a}_o \tag{26}$$

$$\vec{b}_1 = \vec{b}_o + \frac{1}{2}\Delta\vec{b}_o \tag{27}$$

$$\vec{c}_1 = \vec{c}_o + \frac{1}{2}\Delta\vec{c}_o \tag{28}$$

Only half of the corrections were applied in order to accelerate the convergence

d. The new initial position, velocity, and acceleration obtained in (c.) are substituted in Eqs. (22) - (25). New values for $\beta$, $\eta$, and $\zeta$ and the new initial position $\vec{a}_1$, and velocity $\vec{b}_1$, we calculate a new reference orbit and continue steps (b.) through (e.) until the change in initial position, velocity, and acceleration is smaller than some assigned convergence numbers.

The procedure described above has been programmed for the determination of the drag parameter $\beta$ (ballistic coefficient). We present here the results of a test case which has been conducted using this program. A trajectory of a vehicle (referred to here as true trajectory) with $\beta = 25$ lb/ft$^2$, $C_L = C_S = 0$ was simulated. One hundred and fifty positional data sets, one-tenth of a second apart, were computed by integrating the equations of motion using the initial conditions for position and velocity as shown in the first row of Table 1. Random noise of standard deviation of 250 feet was generated. Then, positional data is obtained by adding the generated noise at the corresponding times to the X, Y, Z positions obtained above.

The above mentioned method was used in processing the data. A very bad initial guess of 2000 lb/ft$^2$ for $\beta$ was used, and the initial guesses for position and velocity are shown in the second row of Table 1. It is worth noting that the initial guess for velocity is very bad. Figure 5 shows the rapid convergence of $\beta$. Thus, in spite of the very bad guess for the velocity and the ballistic coefficient and in spite of the presence of noise, the procedure converged to the simulated ballistic coefficient.

## CIS - LUNAR ORBITS

Since cis-lunar satellites are tracked by sensors on or near the Earth, range and range rate are the only reliable measurements. Thus, complete reliable positional coordinates are not available for cis-lunar orbits. Therefore, the objective of this section is to (a) extend the non-linear method to determine the best estimate of the orbit of a satellite using range data only, or range and range rate data only, (b) present a quasilinear method to determine cis-lunar orbits, (c) present non-linear least-squares and maximum likelihood methods for orbit determination.

### Extension of Method to Cis-Lunar Orbits

To start the computations, a reference orbit is needed. If an initial orbit is not known, an initial orbit can be determined using the non-linear method as explained previously utilizing the angle measurements as well as the range data. We choose three different times $T_1$, $T_2$, $T_3$ to be the initial time and two later times near $T_1$. We determine the range $R_o(T_i)$, range rate $\dot{R}_o(T_i)$, and $\ddot{R}_o(T_i)$ for i = 1, 2, and 3.

Case of range data only. - The extension of this method to this case consists of the following steps:

(a) Least-squares fit of the residuals: We form the residuals between the measured discrete range data points and the computed values from the reference orbit at the corresponding times. These residuals $\Delta R_j$ at times $t_j$ are fitted in the least-squares sense to the parabolic function of time $\Delta a_o + \Delta b_o t + \Delta c_o t^2$.

(b) Updating of R, $\dot{R}$, and $\ddot{R}$ at $T_i$'s: The values of R, $\dot{R}$, and $\ddot{R}$ at $T_i$, i = 1, 2, and 3 are updated according to

$$R_1(T_i) = R_o(T_i) + \Delta a_o + \Delta b_o T_i + \Delta c_o T_i^2 \tag{29}$$

$$\dot{R}_1(T_i) = \dot{R}_o(T_i) + \Delta b_o + 2\, \Delta c_o T_i \tag{30}$$

$$\ddot{R}_1(T_i) = \ddot{R}_o(T_i) + 2\, \Delta c_o \tag{31}$$

i = 1, 2, and 3.

(c) Iteration: Using the new range and range rate at the different times $T_1$, $T_2$, $T_3$, or using R, $\dot{R}$, and $\ddot{R}$ at $T_1$ and $T_2$ we calculate a new reference orbit. Then we form the residuals for the range and fit them in least-squares sense to a polynomial. We update R, $\dot{R}$, and $\ddot{R}$ at the three different times as in (b) above. Then a new reference orbit is computed and the procedure is repeated until the iterations converge.

Case of range and range rate data only. - The extention of the method to this case consists of the following steps:

(a) Least-squares fit of the residuals: We form the residuals between both the measured discrete range and range rate data points and the computed values from the reference orbit at the corresponding times. The range residuals $\Delta R_j$ at times $t_j$ are fitted in the least-squares sense to $\Delta a_o + \Delta b_o t + \Delta c_o t^2$ while the range rate residuals $\Delta \dot{R}_j$ are fitted to $\Delta \dot{a}_o + \Delta \dot{b}_o t + \Delta \dot{c}_o t^2$.

(b) Updating of R, $\dot{R}$, and $\ddot{R}$ at $T_i$'s: The values of R, $\dot{R}$, and $\ddot{R}$ at the times $T_i$, i = 1, 2, and 3 are updated according to

$$R_1(T_i) = R_o(T_i) + \Delta a_o + \Delta b_o T_i + \Delta c_o T_i^2 \tag{32}$$

$$\dot{R}_1(T_i) = \dot{R}_o(T_i) + \Delta \dot{a}_o + \Delta \dot{b}_o T_i + \Delta \dot{c}_o T_i^2 \tag{33}$$

$$\ddot{R}_1(T_i) = \ddot{R}_o(T_i) + \Delta \dot{b}_o + 2\, \Delta \dot{c}_o T_i \tag{34}$$

i = 1, 2, and 3

(c) Iteration: Using the updated range and range rates at $T_1$, $T_2$, and $T_3$, we calculate a new reference orbit. Then we form the range and range rate residuals. We fit each in the least-squares sense to a parabola. Then, we update R, $\dot{R}$, and $\ddot{R}$ at $T_1$, $T_2$, and $T_3$ as in (b) above. We continue the process from (a) through (c) until iterations converge.

Determination of an orbit using R and $\dot{R}$ at three different times or R, $\dot{R}$, and $\ddot{R}$ at two different times. - The extension of the non-linear orbit determination method to cis-lunar orbits is reduced to the problem of determining the orbit of a satellite using R and $\dot{R}$ at three distinct times or R, $\dot{R}$, and $\ddot{R}$ at two times. The times are close to each other and this fact can be used to advantage as will be later explained. Thus, the filtering problem in the cis-lunar case is reduced to a multipoint boundary-value problem; whereas the filtering problems in the cases of Earth orbits and ballistic trajectories were reduced to initial-value problems. The solution of initial value problems is straightforward and we compute an orbit using the updated state vector. In the case of multipoint boundary-value problems, the solution is not straightforward. The two dimensional problem of determining an orbit using range and range rates has been treated in reference 10, and the three-dimensional case has been treated in reference 11 using the method of quasilinearization (ref. 12). We will present here the solution presented in reference 11, and we will extend the analysis of reference 11 to apply the method of quasilinearization to the problem of determining an orbit using R, $\dot{R}$, and $\ddot{R}$ at two distinct times.

Let the coordinates of the origin with respect to the sensor be given by the known vector $\underline{\rho}(\rho_1, \rho_2, \rho_3)$, and let the coordinates of the satellite with respect to the origin be given by the vector $\underline{r}(r_1, r_2, r_3)$. Then, the coordinates of the satellite with respect to the sensor are given by the vector

$$\underline{R} = R\,\underline{e} = \underline{r} + \underline{\rho} \tag{35}$$

where R is the magnitude of $\underline{r} + \underline{\rho}$ and $\underline{e}$ is a unit vector in the direction of $\underline{r} + \underline{\rho}$. Dotting Eq. (35) by itself leads to

$$(\underline{r} + \underline{\rho}) \cdot (\underline{r} + \underline{\rho}) = R^2 \tag{36}$$

Differentiating Eq. (35) with respect to time yields

$$\underline{\dot{r}} + \underline{\dot{\rho}} = \dot{R}\,\underline{e} + R\,\underline{\dot{e}} \tag{37}$$

Since $\underline{e} \cdot \underline{e} = 1$, then $\underline{e} \cdot \underline{\dot{e}} = 0$. Hence, dotting Eq. (35) with Eq. (37) leads to

$$(\underline{\dot{r}} + \underline{\dot{\rho}}) \cdot (\underline{r} + \underline{\rho}) = R\,\dot{R}. \tag{38}$$

Differentiation of Eq. (37) yields

$$\underline{\ddot{r}} + \underline{\ddot{\rho}} = \ddot{R}\,\underline{e} + 2\,\dot{R}\,\underline{\dot{e}} + R\,\underline{\ddot{e}} \tag{39}$$

Dotting (35) and (39) leads to

$$(\ddot{\underline{r}} + \ddot{\underline{\rho}}) \cdot (\underline{r} + \underline{\rho}) = \ddot{R} R + R^2 \underline{e} \cdot \ddot{\underline{e}} \tag{40}$$

Since $\dot{\underline{e}} . \underline{e} = 0$, we get

$$e . \ddot{e} = - \dot{e} . \dot{e} \tag{41}$$

Dotting Eq. (37) by itself leads to

$$(\dot{\underline{r}} + \dot{\underline{\rho}}) \cdot (\dot{\underline{r}} + \dot{\underline{\rho}}) = \dot{R}^2 + R^2 \dot{\underline{e}} . \dot{\underline{e}} \tag{42}$$

Combining Eqs. (40) - (42) yields

$$(\ddot{\underline{r}} + \ddot{\underline{\rho}}) \cdot (\underline{r} + \underline{\rho}) + (\dot{\underline{r}} + \dot{\underline{\rho}}) \cdot (\dot{\underline{r}} + \dot{\underline{\rho}}) = \ddot{R} R + \dot{R}^2 \tag{43}$$

Equations (36), (38), and (43) evaluated at $T_1$, $T_2$, and $T_3$ constitute nine conditions for the determination of the orbit. Any six of these conditions are enough for the determination of the orbit.

The equations representing the motion of the satellite are given by

$$\ddot{\underline{r}} = \underline{f}(r, t) \tag{44}$$

Or in component form,

$$\ddot{x}_i = f_i (x_1, x_2, x_3, t), \quad i = 1, 2, \text{ and } 3 \tag{45}$$

The method of quasilinearization is an iterative procedure, let $x_i^{k+1}$ be the $(k+1)^{\underline{th}}$ iteration, then

$$x_i^{k+1} = x_i^k + \left(x_i^{k+1} - x_i^k\right) \tag{46}$$

Substituting (46) in the right-hand side of (45), expanding the functions $f_i$'s in powers of $\left(x_i^{k+1} - x_i^k\right)$, and retaining only the linear terms leads to

$$\ddot{x}_i^{k+1} = f_i\left(x_1^k, x_2^k, x_3^k, t\right) + \sum_{j=1}^{3} \left(x_j^{k+1} - x_j^k\right) \frac{\partial f_i}{\partial x_j}\left(x_1^k, x_2^k, x_3^k, t\right) \tag{47}$$

or

$$\ddot{x}_i^{k+1} = \sum_{j=1}^{3} B_{ij}^k x_j^{k+1} + \Delta_i^k \tag{48}$$

where

$$B_{ij}^k = \frac{\partial f_i}{\partial x_j}\left(x_1^k, x_2^k, x_3^k, t\right) \text{ and}$$

$$\Delta_i^k = f_i\left(x_1^k, x_2^k, x_3^k, t\right) - \sum_{j=1}^{3} B_{ij}^k x_j^k \tag{49}$$

The above equations constitute simultaneous linear inhomogeneous equations for the $x^{k+1}$ with coefficients that depend on the $k^{\underline{th}}$ iteration. If $\underline{p}$ is the particular solution and $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_6$ are six linear independent solutions of the homogeneous equations represented by (48), then the general solution of (48) can be given by

$$\underline{x} = \sum_{i=1}^{6} c_i \underline{z}_i + \underline{p} \tag{50}$$

where the $c_i$'s are arbitrary constants.

The iterative procedure consists of the following steps:

(a) $\underline{z}_i$ and $\underline{p}$ are computed by numerical integration of (48) starting with a suitable initial orbit (such as the orbit obtained using the angles also). The time needed for numerical integration can be short and it needs to be carried only from $T_1$ to $T_2$, or $T_3$.

(b) The values of $\underline{z}_i$ and $\underline{p}$ at the times $T_i$ are noted, and using these values, the expressions (50) are substituted in any six of the equations represented by (36), (38), and (43) to give equations for the $c_i$'s.

(c) These six simultaneous quadratic algebraic equations are solved for the $c_i$'s and a new orbit is obtained.

(d) Using the values of $\underline{x}$ and $\dot{\underline{x}}$, say at time $T_1$, the full equations (44) are integrated, and a check is made to see whether this orbit satisfies all of the nine conditions (36), (38), (43), to the required degree of accuracy. If not, starting with the new orbit, Eq. (48) is integrated numerically and the steps from (a) to (d) are continued until the conditions (36), (38), and (43) are satisfied to the required degree of accuracy.

Solutions of simultaneous quadratic equations. - In carrying out the quasilinearization procedure, a solution of six simultaneous quadratic equations is needed. These equations can be solved numerically starting with an initial guess for the values of $c_i$'s and linearize the equations to carry the solution in accordance with the Newton-Raphson procedure, which is a special case of quasilinearization. However, the iterations required to obtain the solutions of these sets of equations may diverge or converge to a different solution. The resultant orbit can be checked using the three redundant conditions in (36), (38), and (43), and the initial orbit obtained using the angles also. Moreover, the sum of the squares of the residuals using this orbit must be less than or equal to the sum of the squares of the residuals obtained using the previous orbit. If the iterations converged to a different orbit, a new guess will be used, and the procedure is repeated. To solve the problem of divergence, we will use the refinement proposed by Kane (ref. 13) to the Newton-Raphson procedure.

To obtain a value of the vector $\underline{x}$ that satisfies the algebraic vector equations

$$\underline{F}\,(\underline{x}) = 0 \tag{51}$$

Kane regards x as a function $x(\tau)$ of a scaler variable $\tau$ which takes on the values between zero and unity. He assumes that

$$\underline{x}\,(0) = \underline{k} \tag{52}$$

where k is the initial guess of the solution, and he requires that $\underline{x}\,(\tau)$ satisfies

$$\underline{F}\,(\underline{x}) = \underline{F}\,(k)\,(1 - \tau) \tag{53}$$

Differentiating Eq. (53) with respect to $\tau$ leads to

$$\frac{dx}{d\tau} = - \left[\underline{F}'\,(\underline{x})\right]^{-1} \underline{F}\,(\underline{k}) \tag{54}$$

where $[\underline{F}'(\underline{x})]^{-1}$ is the inverse of $\underline{F}'(\underline{x})$. Kane integrates Eq. (54) from $\tau = 0$ to $\tau = 1$. Then, $\underline{x}(1)$ is the required answer. This procedure converges irrespective of the initial conditions.

## Cis-Lunar Orbit Determination Using Quasilinearization

As an alternative method to cis-lunar orbit determination, we extend the analysis of reference 11 to apply to statistical range and range rate data. Thus, we want to determine the orbital elements that minimize

$$\sum_{i=1}^{N} w_i \left[ \left| \underline{r}_i + \underline{\rho}_i \right| - R_i \right]^2 + W_i \left[ \frac{\left( \dot{\underline{r}}_i + \dot{\underline{\rho}}_i \right) \cdot \left( \underline{r}_i + \underline{\rho}_i \right)}{R_i} - \dot{R}_i \right]^2 \tag{55}$$

where the subscript i refers to quantities evaluated at time $t_i$, and $w_i$ and $W_i$ are weighting functions.

The method is an iterative procedure and it consists of the following steps:

(a) An initial orbit is assumed.

(b) $\underline{z}_j$ and $\underline{p}$ are computed from (48) using the above initial orbit.

(c) The values of $\underline{z}_j$ and $\underline{p}$ are determined at the discrete times $t_i$' s and the expressions (50) are substituted into (55).

(d) Expression (55) is minimized with respect to the $c_i$' s.

(e) The resultant non-linear algebraic equations for the $c_i$' s are solved using Kane' s refinement of the Newton-Raphson procedure.

(f) Using $\underline{x}$ and $\dot{\underline{x}}$ at the initial time $t_1$, a new orbit is calculated.

(g) Using the orbit obtained in (f) as initial orbit, the process from (b) to (f) is repeated until iterations converge.

## A Non-Linear Maximum Likelihood Method

In this section, we present another alternative method to orbit determination in general and cis-lunar orbits in particular. The method is a non-linear maximum likelihood method (the least-squares is a special case) in contrast with the conventional methods that are based on the differential correction.

Suppose that the measurements are represented by the n-dimensional vector $a = s + n$, where s is an n-dimensional signal vector to be measured and n is an n-dimensional noise vector. The problem is to determine the orbital elements $c_1, c_2, \ldots, c_N$ using these measurements. The measurement residuals e are given by

$$e = a - s\left(c_1, c_2, \ldots, c_N\right)$$

If the covariance of the measurement error is given by the matrix R, then the loss function L is given by

$$L = e^T R^{-1} e = (a - s)^T R^{-1} (a - s) \tag{56}$$

where $e^T$ denotes the transpose of the matrix e, and $R^{-1}$ is the inverse of R. If the measurement errors are uncorrelated, then R is a diagonal matrix and L is given by

$$L = \frac{e_1^2}{R_{11}} + \frac{e_2^2}{R_{22}} + \ldots, + \frac{e_n^2}{R_{nn}} \tag{57}$$

which is a weighted sum of the squares of the residuals. The weights are inversely proportioned to the expected mean square value of the measurement error. Thus, if the mean square value of the error in a given measurement is large, then the effects of the associated measurement residual on the sum is reduced.

The maximum likelihood estimate of the orbital elements is the one that maximizes exp (-L), and hence, minimizes the loss function L. If we consider the orbital elements to be denoted by the n-dimensional vector c, then the maximum likelihood estimate is given by solution of

$$\frac{\partial s^T}{\partial c} R^{-1} a - \frac{\partial s^T}{\partial c} R^{-1} s = 0 \tag{58}$$

These are N simultaneous algebraic equations for $c_1, c_2, \ldots, c_N$ which are non-linear in general. To solve this system, we follow Kane by assuming that $c = c(\tau)$, and we let

$$\frac{\partial s^T}{\partial c} R^{-1} s - \frac{\partial s^T}{\partial c} R^{-1} a = (1 - \tau) \left[ \frac{\partial s^T}{\partial c} R^{-1} s - \frac{\partial s^T}{\partial c} R^{-1} a \right]_{c=k} \tag{59}$$

where k is the initial estimate for the orbital elements. Then, differentiation of (59) with respect to $\tau$ leads to

$$\frac{dc}{d\tau} = - \left[ \frac{\partial^2 s^T}{\partial^2 c} R^{-1} (s - a) + \frac{\partial s^T}{\partial c} R^{-1} \frac{\partial s}{\partial c} \right]^{-1} \left[ \frac{\partial s^T}{\partial c} R^{-1} (s - a) \right]_{c=k} \tag{60}$$

The system of equations, represented by (60) is solved from $\tau = 0$ to $\tau = 1$. Then, the desired orbital elements are given by $c(1)$.

## CONCLUSIONS

A method for orbit determination is presented. The convergence of the method, in case the data furnishes the complete coordinates of the object (i. e., all Cartesian or spherical coordinates) does not depend on a good initial guess to the reference orbit. In the calculations made, even though very bad initial reference orbits, extremely high levels of noise, wild data points, and critical orbits (near circular) are used, the iteration converged. Thus, the method is a non-linear technique. This is in contrast with the conventional estimation procedures, such as the least-squares, the maximum likelihood, and the Kalman filter, which depend on the linear assumption. In contrast with the conventional estimation procedures, the new method does not require transition and normal matrices and, hence, avoids the problems associated with calculating and inverting them. The calculation of the partial derivatives constituting the transition matrices, either numerically or analytically, and the inversion of the normal matrices, require an expenditure of computer time. The new method gives excellent accuracies, as can be seen from Tables 2 and 3, under the extreme conditions of the very high levels of noise and initial conditions which are far from being nominal.

The method has been extended to the determination of the best estimate of the orbit as well as any unknown parameters in the equations of motion. An extension of this method for the determination of aerodynamic parameters is presented. A numerical case for the determination of the ballistic coefficient showed that the method converges to the correct value irrespective of the bad initiai guess of the velocity and the ballistic coefficient.

The method has been extended to apply for cis-lunar orbits. The reliable observations in this case consist of range data only, or range and range rate data only. The cis-lunar orbit determination problem has been reduced to the determination of an orbit using range and range rate at three different times that can be taken to be close to each other. The method of quasilinearization can be used to determine the orbit. As a result the problem reduces further to the solution of sets of quadratic algebraic equations. A refinement of the Newton-Raphson procedure, proposed by Kane, can be used to solve these equations.

An alternative method based on quasilinearization is presented for the determination of cis-lunar orbits. Also, non-linear least-squares and non-linear maximum likelihood methods have been presented. Work still needs to be done to program these methods to obtain numerical answers and compare the results of these three methods.

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TABLE I. - RESULTS OF PROCESSING RECURSIVELY POSITIONAL SETS OF DATA POINTS, ONE SECOND APART, OBTAINED FROM A NEAR CIRCULAR ORBIT (REFERRED TO HERE AS TRUE ORBIT).

| | | Coordinates | | | Velocities | | |
|---|---|---|---|---|---|---|---|
| | | X(ft) | Y(ft) | Z(ft) | $\dot{X}$(ft/sec) | $\dot{Y}$(ft/sec) | $\dot{Z}$(ft/sec) |
| True Orbit | | 21,769,115 | 7,809,304.4 | -9,746,878 | 11,341.119 | -18,174.446 | 10,325.964 |
| Initial Reference Orbit | | 22,769,115 | 8,809,304.4 | -8,746,878 | 0 | 0 | 0 |
| Number of Data Points | 2 | 21,769,115 | 7,809,304.4 | -9,746,878 | 11,341.125 | -18,174.437 | 10,325.937 |
| | 20 | 21,769,115 | 7,809,304.4 | -9,746,878 | 11,341.118 | -18,174.445 | 10,325.963 |
| | 27 | 21,769,115 | 7,809,304.4 | -9,746,878 | 11,341.119 | -18,174.446 | 10,325.964 |

TABLE II. - RESULTANT EPOCH STATE VECTOR AND ERRORS MADE IN PROCESSING RECURSIVELY 100 POSITIONAL DATA POINTS, 10 SECONDS APART, OBTAINED BY ADDING VARYING LEVELS OF GAUSSIAN NOISE TO A NEAR CIRCULAR ORBIT (REFERRED TO AS TRUE ORBIT) WITH A PERIOD OF 90 MINUTES AND AN INCLINATION OF 45°.

| σ of noise (ft) | | Coordinates | | | Velocities | | |
|---|---|---|---|---|---|---|---|
| | | X(ft) | Y(ft) | Z(ft) | $\dot{X}$(ft/sec) | $\dot{Y}$(ft/sec) | $\dot{Z}$(ft/sec) |
| | True Orbit | 21,769,115 | 7,809,304.4 | -9,746,878.1 | 11,341.119 | -18,174.446 | 10,325.964 |
| | Initial Reference Orbit | 22,769,115 | 8,809,304.4 | -8,746,878.1 | 0 | 0 | 0 |
| $10^2$ | Resultant State Vector | 21,769,116 | 7,809,304.1 | -9,746,878.3 | 11,341.120 | -18,174.447 | 10,325.964 |
| | Error | 1 | -0.3 | -0.2 | 0.001 | -0.001 | 0.000 |
| $10^4$ | Resultant State Vector | 21,769,112 | 7,809,305.2 | -9,746,879.0 | 11,341.120 | -18,174.441 | 10,325.962 |
| | Error | -3 | 0.8 | 0.9 | 0.001 | -0.005 | -0.002 |
| $10^6$ | Resultant State Vector | 21,769,106 | 7,809,306.2 | -9,746,877.5 | 11,341.145 | -18,174.435 | 10,325.967 |
| | Error | -9 | 1.8 | -0.6 | 0.026 | -0.011 | 0.003 |

TABLE III. - RESULTANT EPOCH STATE VECTOR AND ERRORS MADE IN PROCESSING RECURSIVELY 100 POSITIONAL DATA POINTS, 10 SECONDS APART, OBTAINED BY ADDING VARYING LEVELS OF GAUSSIAN NOISE TO A NEAR CIRCULAR ORBIT (REFERRED TO AS TRUE ORBIT) WITH A PERIOD OF 24 HOURS AND AN INCLINATION OF 45°.

| $\sigma$ of noise (ft) | | Coordinates | | | Velocities | | |
|---|---|---|---|---|---|---|---|
| | | X(ft) | Y(ft) | Z(ft) | $\dot{X}$(ft/sec) | $\dot{Y}$(ft/sec) | $\dot{Z}$(ft/sec) |
| | True Orbit | 103,278,000 | 38,360,400 | -47,704,800 | 5,670.00 | -9,087.00 | 5,163.00 |
| | Initial Reference Orbit | 22,769,115 | 8,809,304.4 | - 8,746,878.1 | 0 | 0 | 0 |
| $10^4$ | Resultant State Vector | 103,278,002 | 38,360,395 | -47,704,803 | 5,670.01 | -9,087.03 | 5,162.99 |
| | Error | 2 | -5 | -3 | 0.01 | -0.03 | -0.01 |
| $10^6$ | Resultant State Vector | 103,278,011 | 38,360,408 | -47,704,799 | 5,669.98 | -9,086.99 | 5,163.02 |
| | Error | +11 | +8 | -1 | -0.02 | 0.01 | 0.02 |
| $10^7$ | Resultant State Vector | 103,278,013 | 38,360,396 | -47,704,807 | 5,670.18 | -9,086.79 | 5,162.95 |
| | Error | 13 | -4 | 7 | 0.18 | 0.21 | -0.05 |

TABLE IV. - EPOCH STATE VECTOR OF SIMULATED TRAJECTORY AND INITIAL GUESS OF EPOCH STATE VECTOR USED IN PROCESSING SIMULATED DATA.

| | Coordinates | | | Velocities | | |
|---|---|---|---|---|---|---|
| | X(ft) | Y(ft) | Z(ft) | $\dot{X}$(ft/sec) | $\dot{Y}$(ft/sec) | $\dot{Z}$(ft/sec) |
| True Trajectory | -20,733,909 | 1,404,451.6 | 3,524,294.8 | 10,238.66 | 13,862.06 | -11,651.32 |
| Initial Guess | -20,733,816 | 1,404,372.6 | 3,524,402.1 | 4,439.52 | 5,492.52 | - 5,278.37 |

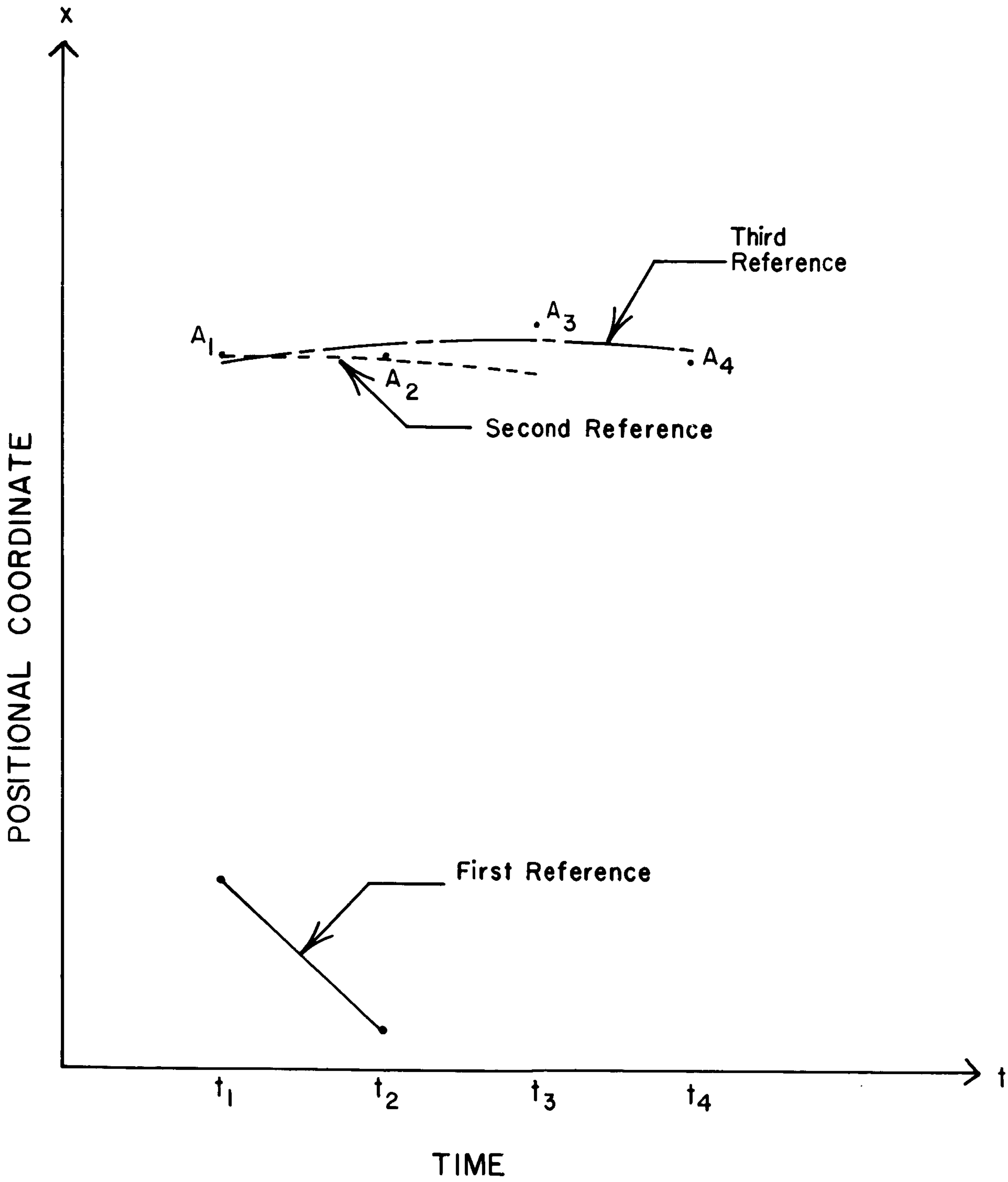


Fig. 1 A sketch that shows the recursive formulation of the method

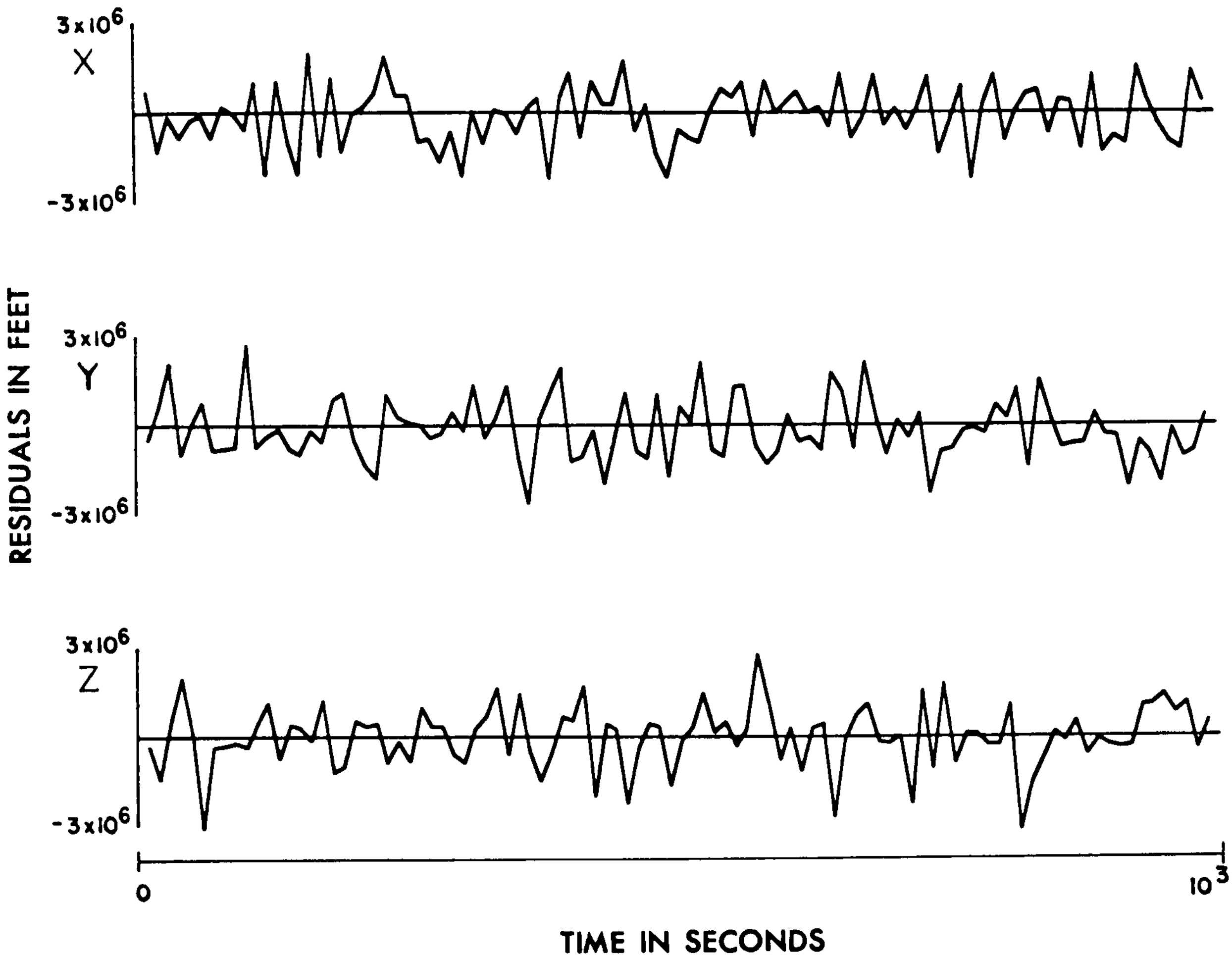


Fig. 2 Orbital Residuals vs. Time for a near circular orbit, with a period of approximately 24 hours and an inclination of about 45°, with a random noise of standard deviation of $10^6$ ft.

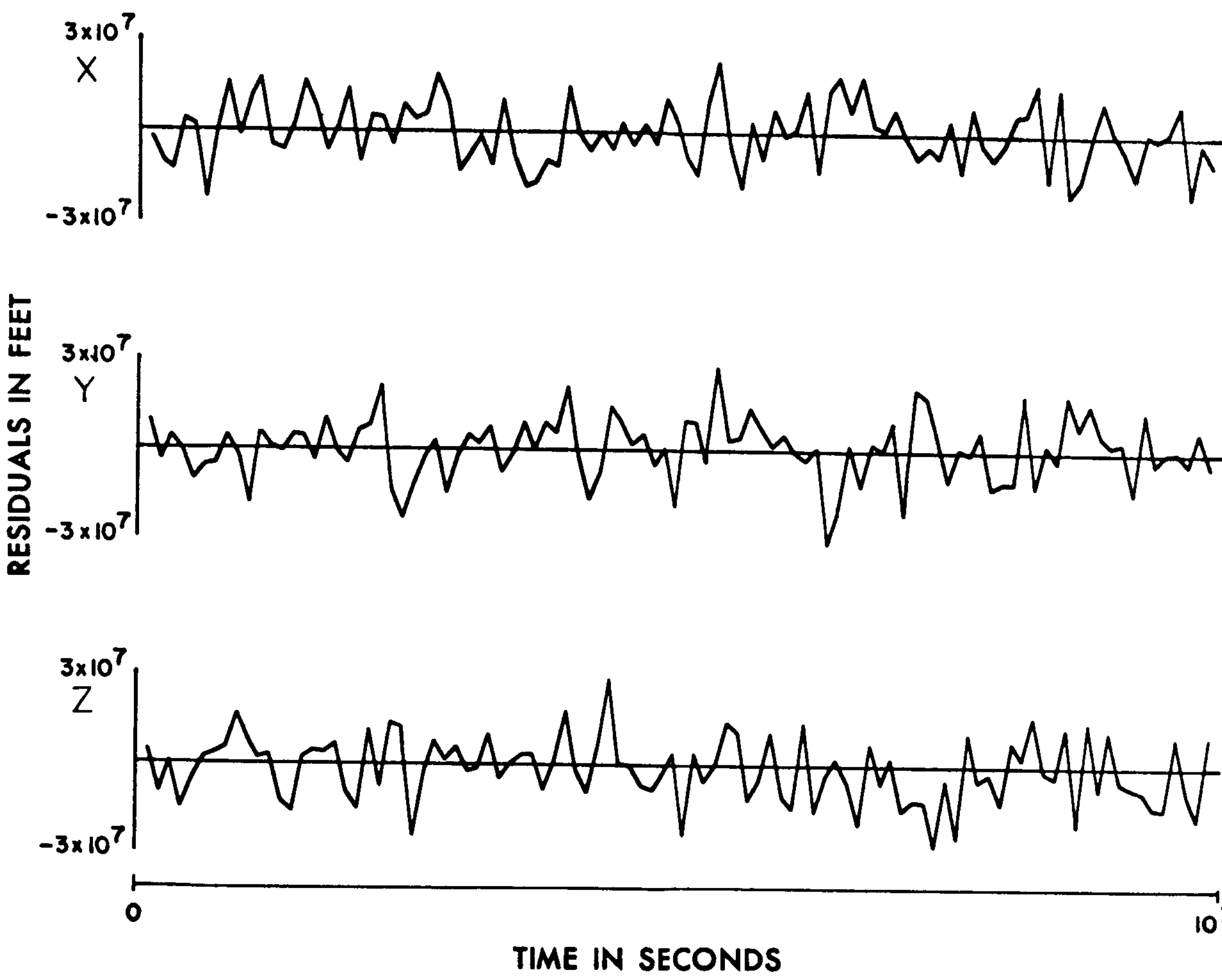


Fig. 3 Orbital Residuals vs. Time for a near circular orbit, with a period of approximately 24 hours and an inclination of about 45°, with a random noise of standard deviation of $10^7$ ft.

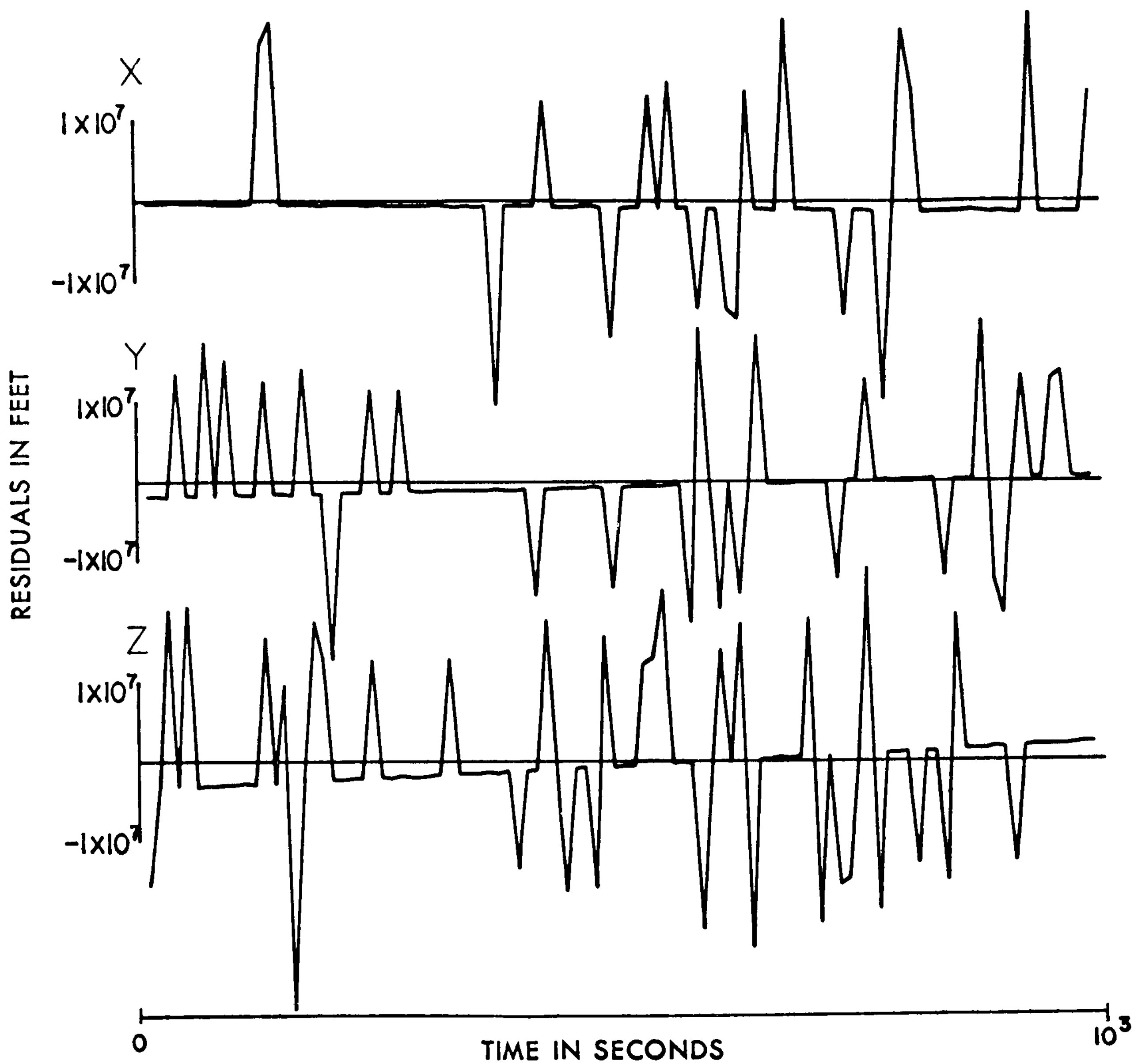


Fig. 4 Orbital Residuals vs. Time after processing data that is corrupted by noise of $10^7$ ft standard deviation and 25 wild data points. It shows that the Heliodyne Method can be used as an editing procedure.

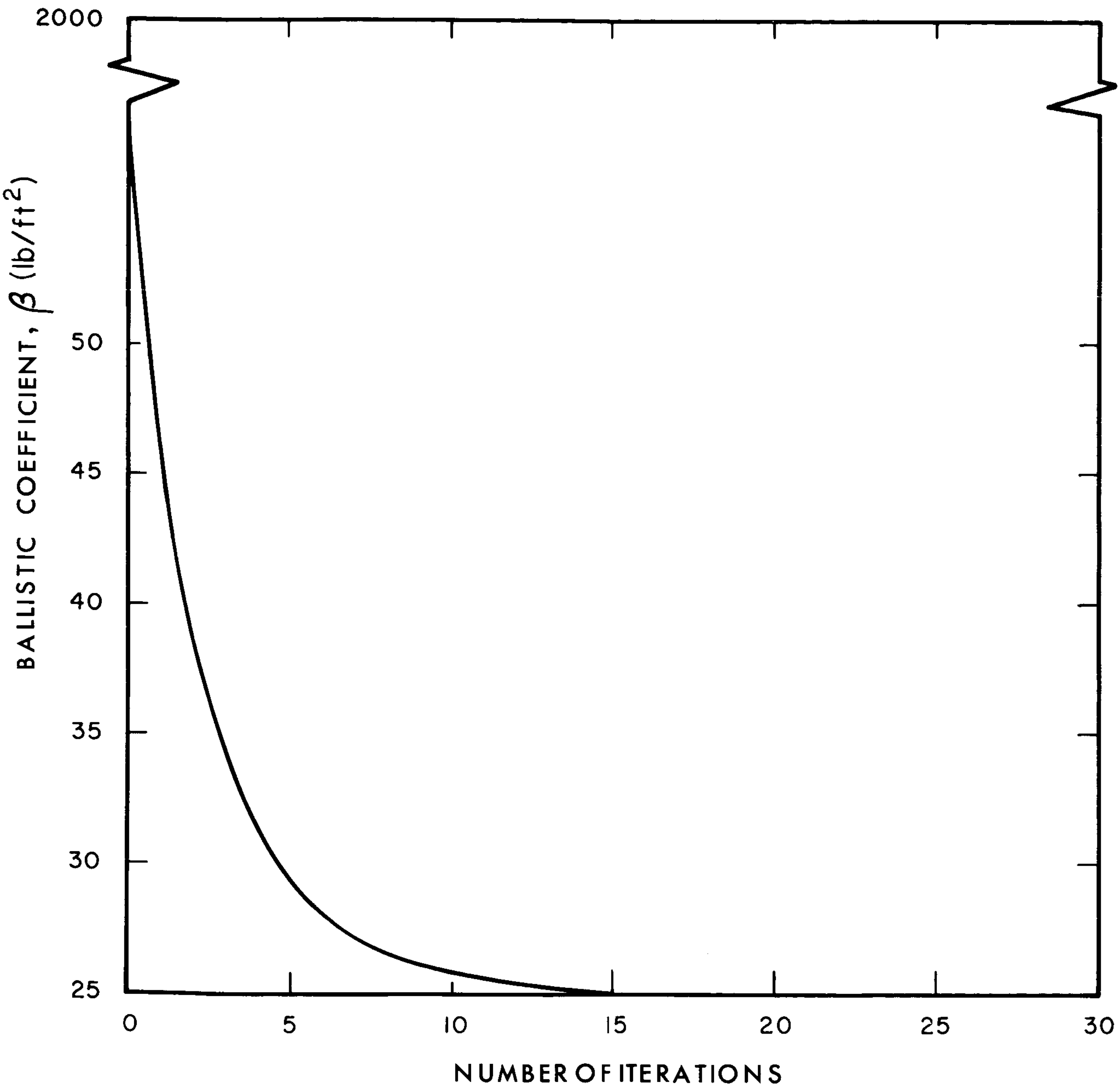


Fig. 5 Ballistic coefficient ($\beta$) vs. number of iterations for a simulated trajectory of a vehicle with $\beta$ = 25 lb/ft$^2$ and with a random noise of standard deviation of 250 ft. First guess for $\beta$ was 2000 lb/ft$^2$.

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