MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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OPTIMAL CONTROL OF LINEAR STOCHASTIC SYSTEMS WITH MEASUREMENT ERRORS

by
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ABSTRACT

A method is presented for determining optimal feedback controls for linear gaussian stochastic systems, when the system state cannot be determined without error and the cost function is nonquadratic. The method is applied to the minimum fuel spacecraft midcourse guidance problem and the form of the optimal feedback control is determined.

NOMENCLATURE

<u>English Letters</u>

b	a point on the control threshold
C	expected cost to complete the process
e	estimation error
E	expectation
f	probability density
H	measurement matrix
i	index denoting time t_i
j	dimension of the measurement vector
J	total expected cost
k	dimension of the state vector
L	incremental cost function at each time step
m	measurement vector
M	measurement history
n	index denoting time t_n
o	higher order terms
p	dimension of the control vector
P	covariance of e
q	index denoting the last time that control is applied t_q
r	pseudo measurement vector
R	pseudo measurement history
s	processed measurement information
S	variance of s
u	control vector
v	process noise
V	covariance of v
w	measurement error
W	covariance of w
x	state vector
y	partial state vector
z	partial state vector

Script Letters

$\mathcal{B}$	control threshold
$\mathcal{R}$	nonzero control set
$\mathcal{U}$	admissible control set
$\mathcal{Z}$	zero control set

Greek Letters

Φ	state transition matrix
θ	control influence matrix
ϕ	terminal cost function
ξ	dummy variable of integration
ζ	dummy variable of integration
λ	terminal weighting
∂	partial derivative

Superscripts

'	extrapolation
T	transpose
-1	inverse
*	optimal
-	conditional mean
^	minimum variance estimate

1. Introduction

A common engineering problem is the design of a feedback controller to minimize an expected cost, when the plant to be controlled is a stochastic system and its state cannot be measured without error. The controller must operate on partial information and base its action on the available measurements and the *a priori* statistics of the system. The first published results applicable to this problem seem to be the papers of Joseph and Tou [11] and Gunkel and Franklin [10], which present a separation theorem for discrete time systems. This theorem states that if the plant is linear and the cost function is the expectation of quadratic forms, then the problems of estimation and control may be solved separately. The estimator is determined by the methods of Kalman [12] and the controller is designed using the calculus of variations to minimize the cost function, under the assumption that the system is deterministic. The cascade combination of these two systems provides the optimum over-all feedback control. Florentine [9] also derives the theorem and examines the role of the estimated state as a sufficient statistic. Potter[15] extends the result to continuous systems. An extensive treatment of the general problem, without restrictions of system linearity or quadratic cost, was published by Fel'dbaum in a series of four papers [8]. He develops a method of solving these problems and illustrates the method with quadratic cost examples. Stratanovich [16] and Kushner [13] both examine the general problem and develop functional equations. Stratanovich demonstrates the solution of a bounded control problem with perfect measurements and Kushner outlines the solution for systems with measurement uncertainties. Orford [14] handles a problem with terminal cost function and bounded control, and solves a spacecraft guidance problem. Striebel [17] provides a mathematical treatment of the general problem and illustrates the method of solution for the linear gaussian case. Finally, the recent book by Aoki [1] develops an approach which is more general than Fel'dbaum's and the book illustrates the solution of many quadratic cost problems.

Linear gaussian systems are of interest because they are good models for many actual systems and they admit practical solutions to a number of interesting problems. Of particular interest here are systems for which the plant state cannot be determined without error, the cost function may be non-quadratic and the control may be required to lie in some set of admissible controls. Much of the theory involved has been published elsewhere however it is not well known by engineers interested in applications. This paper presents a tutorial exposition of the theory together with the detailed solution of a minimum fuel spacecraft guidance problem. It is hoped that the results presented will stimulate interest in applying the theory to other practical problems.

2. Problem Statement

It is assumed that the plant may be described by discrete linear equations.

$$x(n+1) = \Phi(n+1,n)x(n) + \theta(n+1,n)u(n) + v(n) \tag{2-1}$$

where

$x(n)$ = state vector of dimension k
$u(n)$ = control vector of dimension p
$\Phi(n+1,n)$ = state transition matrix (kxk)
$\theta(n+1,n)$ = control influence matrix (kxp)

The initial state $x(0)$ is a k vector of normally distributed random variables with known statistics and $v(n)$, the process disturbance, is a k vector of gaussian random variables, independent of $x(n)$ and $u(n)$, with statistics given by

$$E[v(n)] = 0$$

$$E[v(n)v^T(n)] = V(n) \qquad E[v(n)v^T(i)] = 0 \quad i \neq n \tag{2-2}$$

A set of admissible controls $\mathcal{U}(n)$ is defined so that problems involving constraints on the control may be handled. The set $\mathcal{U}(n)$ can depend upon parameters other than time. It may, for example, depend in some way on the control or measurement history. In what follows, it is only required that $\mathcal{U}(n)$ be known deterministically by the controller at time t_n.

The feedback controller has a measurement process m(n) available to it, with

$$m(n) = H(n)x(n) + w(n) \tag{2-3}$$

where

m(n) = measurement vector of dimension j
H(n) = measurement matrix (jxk)

The measurement error w(n) is a j vector of gaussian random variables with statistics given by

$$E[w(n)] = 0$$

$$E[w(n)w^T(n)] = W(n) \qquad E[w(n)w^T(i)] = 0 \quad i \neq n \tag{2-4}$$

and w(n) is independent of x(n) and v(n).

The cost to be minimized is assumed to be of the form

$$J = E\left[\sum_{n=1}^{q} L(x(n),u(n),n) + \phi(x(q+1))\right] \tag{2-5}$$

Control begins at time t_1 and the last control is applied at time t_q, with t_{q+1} a specified terminal time. L(x(n),u(n),n) is a scalar penalty at each time step and $\phi(x(q+1))$ is a scalar terminal penalty. The expectation in (2-5) is conditioned on all *a priori* information.

It is desired to find the admissible control $u(n)$, as a function of the past history of measurements up to time t_n, that will drive the plant so that the expected cost J is minimized. Note that functions $L(x(n),u(n),n)$ and $\phi(x(q+1))$ are not required to be quadratic in $x(n),u(n)$ and $x(q+1)$.

3. Estimation and Sufficient Statistics

Consider two k vectors $y(n)$ and $z(n)$, defined to satisfy the equations

$$y(n+1) = \Phi(n+1,n)y(n) + v(n) \qquad y(0) = x(0) \tag{3-1}$$

$$z(n+1) = \Phi(n+1,n)z(n) + \theta(n+1,n)u(n) \qquad z(0) = 0 \tag{3-2}$$

$$u(0) = 0$$

and evidently from (2-1)

$$x(n) = y(n) + z(n) \tag{3-3}$$

The vector $y(n)$ contains all uncertainty about the state and $z(n)$ describes the known effect of the control on the state. Also define a pseudo measurement process $r(n)$ as

$$r(n) = m(n) - H(n)z(n) \tag{3-4}$$

Since $m(n)$ is known by the controller and $z(n)$ may be calculated from the control history according to (3-2), $r(n)$ is known by the controller. With (2-3), (3-3) and (3-4), $r(n)$ may be written as

$$r(n) = H(n)y(n) + w(n) \tag{3-5}$$

Equation (3-1) describes a linear system perturbed by uncorrelated, normally distributed, random disturbances. According to (3-5) the pseudo measurement process is composed of linear combinations of the state $y(n)$ plus random measurement errors w(n). The minimum variance estimate of y(n) plays a crucial role in the determination of the optimal feedback control. Kalman [12] and Battin [2] have shown that the minimum variance estimate of y(n), given measurements r(n), can be calculated from the following recursion formulas

$$\hat{y}(n) = \hat{y}'(n)+P'(n)H^T(n)[H(n)P'(n)H^T(n)+W(n)]^{-1}[r(n)-H(n)\hat{y}'(n)]$$

$$\hat{y}'(n+1) = \Phi(n+1,n)\hat{y}(n) \qquad \hat{y}(0) = E[x(0)] \qquad (3\text{-}6)$$

$$P(n) = P'(n)-P'(n)H^T(n)[H(n)P'(n)H^T(n)+W(n)]^{-1}H(n)P'(n)$$

$$P'(n+1) = \Phi(n+1,n)P(n)\Phi^T(n+1,n)+V(n) \qquad P(0)=Cov[x(0)]$$

Define e(n) as the estimation error, so

$$e(n) = \hat{y}(n) - y(n) \qquad (3\text{-}7)$$

It is well known that

$$E[e(n)] = 0 \qquad (3\text{-}8)$$

with the covariance of e(n) identified as matrix P(n) in (3-6) and it can be readily shown that y(n) and $\hat{y}(n)$ are normally distributed, so e(n) must be normallly distributed. To investigate some additional statistical properties of the error e(n), define the history of pseudo measurements from the initial time up to time t_n as the $n \cdot j$ dimensional vector R(n) Hence

$$R^T(n) = [r^T(1),r^T(2),\ldots,r^T(n)] \qquad (3\text{-}9)$$

and it has been shown by Kalman [12], that the estimation error e(n) must be uncorrelated with the measurement history R(n).

$$E[e(n)R^T(n)] = 0 \qquad (3\text{-}10)$$

From (3-1), (3-5) and the statistics of x(0),v(n) and w(n), it is clear that r(n) and therefore R(n) must be normally distributed. Thus e(n) and R(n) are normally distributed and uncorrelated,

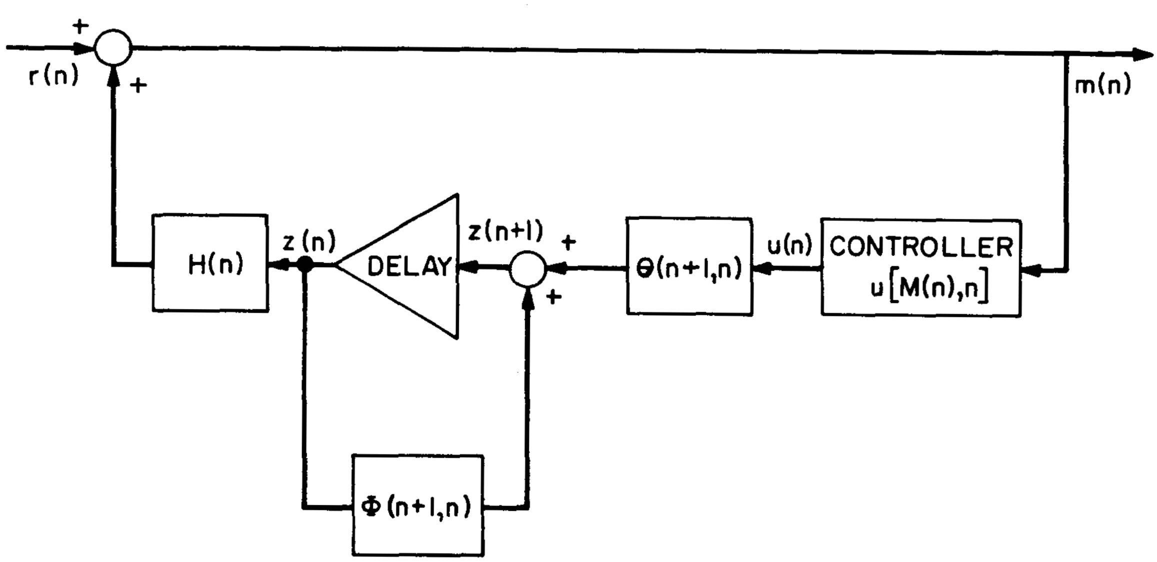

Fig 1. Calculation of m(n) from r(n)

so they must be statistically independent. Now define the history of actual measurements from the initial time up to time t_n as the $n \cdot j$ dimensional vector $M(n)$.

$$M^T(n) = [m^T(1), m^T(2), \ldots, m^T(n)] \tag{3-11}$$

Assume that some arbitrary admissible control function $u[\cdot]$, of the measurement history $M(n)$, is specified and the control at time t_n becomes

$$u(n) = u[M(n), n] \tag{3-12}$$

Note that $u[M(n),n]$ may be a <u>nonlinear</u> function of $M(n)$. Further, from (3-12), knowledge of $M(n)$ implies knowledge of the entire control history $u(1), u(2), \ldots, u(n)$. Considering (3-12), (3-11), (3-4) and (3-2) it is clear that the process $r(n)$ is a deterministic function of $M(n)$. By similar reasoning, the process $m(n)$ may be considered to be a deterministic function of $R(n)$. Figure 1 illustrates, in block diagram form, a method by which $m(n)$ could be calculated from $r(n)$.

Hence M(n) is a deterministic function of R(n) and since e(n) and R(n) are independent, it follows that e(n) and M(n) are independent. It should be emphasized that e(n) and M(n) are independent even when u[M(n),n] is nonlinear.

Consider an estimate of the state x(n) defined as $\hat{x}(n)$ and given by

$$\hat{x}(n) = \hat{y}(n) + z(n) \tag{3-13}$$

It can be shown that $\hat{x}(n)$ is the minimum variance estimate of x(n). Recursion formulas for this estimate are readily obtained from (3-13), (3-6), (3-4) and (3-2)

$$\hat{x}(n) = \hat{x}'(n)+P'(n)H^T(n)[H(n)P'(n)H^T(n)+W(n)]^{-1}[m(n)-H(n)\hat{x}'(n)] \tag{3-14}$$

$$\hat{x}'(n+1) = \Phi(n+1,n)\hat{x}(n)+\theta(n+1,n)u(n) \qquad \hat{x}(0) = E[x(0)] \tag{3-15}$$

and $P'(n)$ is obtained from the last two of Eqs. (3-6). The error in this estimate is, from (3-13), (3-7) and (3-3)

$$\hat{x}(n) - x(n) = [\hat{y}(n) + z(n)] - [y(n) + z(n)] = e(n) \tag{3-16}$$

so the error in $\hat{x}(n)$ is identical to the error in $\hat{y}(n)$. It was shown above that e(n) and M(n) are independent so the error in the estimate $\hat{x}(n)$ is independent of the measurement history M(n). Of importance here is the fact that even though x(n) and $\hat{x}(n)$ may <u>not</u> be gaussian, because the control function u[M(n),n] may be nonlinear, the error in $\hat{x}(n)$ and the measurement history M(n) are still independent.

At this point, the statistical properties of $\hat{x}(n)$ and e(n) can be utilized to obtain an expression for the posterior probability density of the state, conditioned on the measurement history. To that end, write the state as the difference between the estimate $\hat{x}(n)$ and the error e(n)

$$x(n) = \hat{x}(n) - e(n) \tag{3-17}$$

The estimate $\hat{x}(n)$ is a deterministic function of the measurements and e(n) is independent of the measurements. Also e(n) is normally distributed with zero mean and covariance P(n). Hence the posterior probability density of x(n) is

$$f_{x(n)}[\xi|M(n)]=(2\pi)^{-\frac{k}{2}}|P(n)|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}[\xi-\hat{x}(n)]^T P(n)^{-1}[\xi-\hat{x}(n)]\right\} \tag{3-18}$$

It is assumed that the error covariance matrix P(n) can be determined a priori. Since P(n) and $\hat{x}(n)$ uniquely determine the posterior state probability density and P(n) is known a priori, the estimate $\hat{x}(n)$ is a sufficient statistic for determining the posterior state probability density. In effect $\hat{x}(n)$ summarizes all posterior information about the state that is obtained by the controller from the measurement history M(n), and the posterior probability density for x(n) may be given as

$$f_{x(n)}(\xi|\hat{x}(n)) = f_{x(n)}[\xi|M(n)] \tag{3-19}$$

4. Measurement Information Statistics

In the preceding section an expression for the posterior state probability density was developed. To determine the optimal feedback control function, some additional properties of the estimate $\hat{x}(n)$ are necessary. Define a k-dimensional vector s(n) as

$$s(n) = P'(n)H^T(n)[H(n)P'(n)H^T(n)+W(n)]^{-1}[m(n)-H(n)\hat{x}'(n)] \tag{4-1}$$

so (3-14) becomes

$$\hat{x}(n) = \hat{x}'(n) + s(n) \tag{4-2}$$

Hence $\hat{x}'(n)$ is the estimated state extrapolated forward from time t_{n-1} to time t_n and s(n) represents the incremental change in the estimated state as a result of processing the measurement $m(n)$. Using (2-1), (2-3), (3-15) and (3-16), the vector $s(n)$ may be written as

$$s(n) = P'(n)H^T(n)[H(n)P'(n)H^T(n)+W(n)]^{-1}[H(n)[v(n-1) -\Phi(n,n-1)e(n-1)+w(n)] \tag{4-3}$$

so with the help of (3-6), the mean and covariance of $s(n)$ become

$$E[s(n)] = 0$$

$$E[s(n)s^T(n)] = S(n) = P'(n)H^T(n)[H(n)P'(n)H^T(n)+W(n)]^{-1}H(n)P'(n) \tag{4-5}$$

It can be readily shown, using the results above, that the $s(n)$ are gaussian random vectors, independent of each other and each $s(n)$ is independent of the entire history of the system before time t_n (i.e. the $s(n)$ are independent gaussian increments).

5. The Optimal Feedback Control Function

In this section, the properties of the sufficient statistic $\hat{x}(n)$ will be used to determine the optimal feedback control function. Consider a partially completed process at time t_n in the interval $t_1 \leq t_n \leq t_{q+1}$. Assume that some arbitrary admissible control function $u[M(i),i]$ has been used in the past and an admissible control function $u^*[M(i),i]$ is to be used in the future. Define a minimum expected value function $C^*[M(n),n]$ as follows:

> $C^*[M(n),n]$= minimum expected cost to complete the process from time t_n, given the measurement history $M(n)$, using the admissible control function $u[M(i),i]$ in the interval $t_1 \leq t_i < t_n$ and the admissible control function $u^*[M(i),i]$ in the interval $t_n \leq t_i < t_{q+1}$

By definition $u^*[M(i),i]$ is the admissible control function which, if used in the interval $t_n \leq t_i < t_{q+1}$, will produce the minimum expected cost to complete the process.

Consider C* at the terminal time t_{q+1}. The last control decision was made at time t_q, so from the definition above and (2-5).

$$C^*[M(q+1),q+1] = E[\phi(x(q+1))\,|\,M(q+1)] \tag{5-1}$$

The conditional expectation may be evaluated with the help of (3-18) and (3-19), so if $\bar{\phi}(\hat{x}(q+1))$ is defined as*

$$\bar{\phi}(x(q+1)) = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k \phi(\xi) f_{x(q+1)}(\xi\,|\,\hat{x}(q+1)) \tag{5-2}$$

then (5-1) becomes

$$C^*[M(q+1),q+1] = \bar{\phi}(\hat{x}(q+1)) \tag{5-3}$$

Since the right side of (5-3) is a function of $\hat{x}(q+1)$ only, which is itself a function of M(q+1), then without loss of generality the minimum expected value function at time t_{q+1} may be considered to be a function of $\hat{x}(q+1)$ instead of M(q+1). This important change of variables is achieved because $\hat{x}(q+1)$ is a sufficient statistic. Applying the definition at the next previous time t_q obtains an expression for C*[M(q),q]

$$C^*[M(q),q] = \min_{u(q)\in\mathcal{U}(q)} \left\{ E[L(x(q),u(q),q)+\phi(x(q+1))\,|\,M(q),u(q)]\right\} \tag{5-4}$$

If the function $\bar{L}(\hat{x}(n),u(n),n)$ is defined as

$$\bar{L}(\hat{x}(n),u(n),n) = \int_{-\infty}^{\infty} d\xi_1, \ldots \int_{-\infty}^{\infty} d\xi_k L(\xi,u(n),n) f_{x(n)}(\xi\,|\,\hat{x}(n)) \tag{5-5}$$

then by the same arguments used above

$$C^*[M(q),q] = \min_{u(q)\in\mathcal{U}(q)} \left\{ \bar{L}(\hat{x}(q),u(q),q)+E[\phi(x(q+1))\,|\,M(q),u(q)]\right\} \tag{5-6}$$

* Here and in the sequel, convergence of multiple infinite integrals is tacitly assumed.

To evaluate the second term on the right of (5-6), the posterior density of $x(q+1)$, given $M(q), u(q)$ is required. The evaluation of this expression is performed in appendix A and the result, when substituted into (5-6) yields

$$C^*[M(q),q] = \min_{u(q)\in\mathcal{U}(q)} \left\{ \bar{L}(\hat{x}(q),u(q),q) + \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k f_{s(q+1)}(\xi) \cdot \bar{\phi}(\hat{x}'(q+1)+\xi) \right\} \tag{5-7}$$

where $\hat{x}'(q+1)$ is determined by (3-15) and $f_{s(q+1)}(\xi)$ is the probability density of $s(q+1)$, determined by

$$f_{s(n)}(\xi) = (2\pi)^{-\frac{k}{2}} |S(n)|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\xi^T S(n)^{-1} \xi \right\} \tag{5-8}$$

with $S(n)$ given by (4-5).

By virtue of (3-15) and the minimization over $\mathcal{U}(q)$, the right side of (5-7) is a function of $\hat{x}(q)$ and q. Hence, without loss of generality, the minimum expected value at time t_q may be written as a function of $\hat{x}(q)$ instead of $M(q)$. If, therefore, the minimum expected value functions at times t_q and t_{q+1} are redefined as functions of $\hat{x}(q)$ and $\hat{x}(q+1)$; (5-3) and (5-7) become

$$C^*(\hat{x}(q+1),q+1) = \bar{\phi}(\hat{x}(q+1)) \tag{5-9}$$

$$C^*(\hat{x}(q),q) = \min_{u(q)\in\mathcal{U}(q)} \left\{ \bar{L}(\hat{x}(q),u(q),q) + \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k f_{s(q+1)}(\xi) \cdot C^*(\hat{x}'(q+1)+\xi, q+1) \right\} \tag{5-10}$$

By applying this reasoning to each successive backward time step, and invoking the principle of optimality [3,4,5,7] it is found that the backward recursion formula

$$C^*(\hat{x}(n),n)=\min_{u(n)\in\mathcal{U}(n)}\left\{\bar{L}(\hat{x}(n),u(n),n)+\int_{-\infty}^{\infty}d\xi_1\cdots\int_{-\infty}^{\infty}d\xi_k f_{s(n+1)}(\xi)\cdot C^*(\hat{x}'(n+1)+\xi,n+1)\right\} \qquad (5\text{-}11)$$

must be satisfied at each time step.

The error covariance matrix $P(n)$ is assumed known a priori so the posterior densities (3-18) and (5-8) are known a priori as functions of $\hat{x}(n)$ and n. Hence $\bar{L}(\hat{x}(n),u(n),n)$ and $\bar{\phi}(\hat{x}(q+1))$ can be determined a priori as functions of $\hat{x}(n)$, $u(n)$ and $\hat{x}(q+1)$ respectively. Further, it is assumed that if the control set $\mathcal{U}(n)$ depends on the control or measurement history, then $\hat{x}(n)$ is sufficient to determine $\mathcal{U}(n)$.* Under these conditions the system (5-9), (5-11) is closed and the function $C^*(\hat{x}(n),n)$ is obtained by its solution. The method of solution is essentially the Dynamic Programming procedure of Bellman [7] and as a result of the minimization in (5-11), the optimal control is determined as a function of the estimated state.

$$u^*(\hat{x}(n),n) = u^*[M(n),n] \qquad (5\text{-}12)$$

6. Midcourse Spacecraft Guidance

As a means of demonstrating the application of the method developed above, a minimum fuel spacecraft guidance problem will be solved. Because of random errors made in injecting the spacecraft into its trajectory, impulsive midcourse velocity corrections are necessary if the vehicle is to hit the target with acceptable accuracy. It is assumed that there is a reference trajectory defined which passes through the nominal point of injection and the nominal target point. Further, deviations from the nominal are assumed to be sufficiently small so that linearizations about the reference trajectory are valid. During the midcourse phase, the spacecraft is

*This is not a restriction on most practical cases because $\hat{x}(n)$ can usually be augmented by additional variables which are deterministic functions of the past and which completely determine $\mathcal{U}(n)$.

tracked by radar systems based on earth. The radars provide velocity measurements in the directions of the radius vector from Earth to the spacecraft and these measurements contain gaussian random errors. Estimates of spacecraft position and velocity are computed from this information using recursion formulas (3-14) and (3-15).

The cost to be minimized is the expected total fuel expenditure, plus a quadratic weighting on the miss distance at the target, so

$$J = E\left[\sum_{n=1}^{q} ||u(n)|| + \frac{\lambda}{2} x^T(q+1)x(q+1) \right] \tag{6-1}$$

where the terminal time t_{q+1} and correction times t_n are fixed,

$u(n)$ = Velocity correction applied at time t_n

$x(q+1)$ = Position deviation at the target

and q is the total number of velocity corrections applied. Since the cost involves only the velocity corrections and the position deviation at the target, the problem may be somewhat simplified by projecting the state forward, at each point in time, to the target. Thus, if $x(n)$ is defined as the target miss vector based on the spacecraft history up to time t_n, then $x(n)$ satisfies the recursion formula

$$x(n+1) = x(n) + \theta(n)u(n) \tag{6-2}$$

This equation implies that the target miss vector at time t_{n+1} can be altered by the application of a velocity correction at time t_n and conversely if no correction is applied, then the target miss vector is unchanged. Matrix $\theta(n)$ determines the effect of the velocity correction at time t_n on the target miss vector. $x(n)$ is the state vector of interest for the problem at hand and (6-2) determines its evolution in time.

Functions $\bar{\phi}(\hat{x}(q+1))$ and $\bar{L}(\hat{x}(n),u(n),n)$ are obtained from (5-2), (5-5) and (3-18)

$$\bar{\phi}(\hat{x}(q+1)) = \frac{\lambda}{2}\,(\hat{x}^T(q+1)\hat{x}(q+1)+\mathrm{Tr}[P(q+1)]) \tag{6-3}$$

$$\bar{L}(\hat{x}(n),u(n),n) = ||u(n)|| \tag{6-4}$$

so the minimum expected value function must satisfy

$$C^*(\hat{x}(n),n)=\min_{u(n)}\left\{||u(n)||+\int_{-\infty}^{\infty}d\xi_1..\int_{-\infty}^{\infty}d\xi_k f_{s(n+1)}(\xi)C^*(\hat{x}'(n+1)+\xi,n+1)\right\} \tag{6-5}$$

where

$$\hat{x}'(n+1) = \hat{x}(n) + \theta(n)u(n) \tag{6-6}$$

Now define the function $C^{*\prime}(\hat{x},n)$ as

$$C^{*\prime}(\hat{x},n)=\int_{-\infty}^{\infty}d\xi_1..\int_{-\infty}^{\infty}d\xi_k f_{s(n+1)}(\xi)C^*(\hat{x}+\xi,n+1) \tag{6-7}$$

and (6-5) becomes

$$C^*(\hat{x}(n),n)=\min_{u(n)}\left\{||u(n)||+C^{*\prime}(\hat{x}'(n+1),n)\right\} \tag{6-8}$$

If the $S(n)$ matrix is positive definite, then $C^{*\prime}$ has continuous second partial derivaties and (6-8) may be expanded about $\hat{x}(n)$ to produce

$$C^*(\hat{x}(n),n)=\min_{u(n)}\left\{||u(n)||+C^{*\prime}(\hat{x}(n),n)+\left[\frac{\partial C^{*\prime}}{\partial\hat{x}}(\hat{x},n)\right]_{\hat{x}=\hat{x}(n)}\theta(n)u(n)+o[||u(n)||]\right\} \tag{6-9}$$

where $\partial C^{*\prime}/\partial\hat{x}$ is a row vector of first partial derivatives (gradient). Equation (6-9) can provide some useful conditions for a minimum. If $\hat{x}(n)$ lies in the set $\mathcal{Z}(n)$ where

$$\mathcal{Z}(n) = \left\{ \hat{x} : \left\| \frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \theta(n) \right\| < 1 \right\} \tag{6-10}$$

then $u(n) = 0$ yields a local minimum on the right of (6-9). Conversely, if $\hat{x}(n)$ is an element of the set $\mathcal{R}(n)$ given by

$$\mathcal{R}(n) = \left\{ \hat{x} : \left\| \frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \theta(n) \right\| > 1 \right\} \tag{6-11}$$

then a minimum in (6-9) cannot occur for $u(n) = 0$. Assuming that $\hat{x}(n) \epsilon \mathcal{R}(n)$, the term in braces on the right of (6-8) is differentiated with respect to $u(n)$

$$\frac{\partial \{\cdot\}}{\partial u(n)} = \frac{u^T(n)}{\|u(n)\|} + \left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \right]_{\hat{x}=\hat{x}(n)+\theta(n)u(n)} \theta(n) \qquad \hat{x}(n) \epsilon \mathcal{R}(n) \tag{6-12}$$

A necessary condition for a minimum is satisfied if the direction of $u(n)$ is

$$\frac{u(n)}{\|u(n)\|} = -\theta^T(n) \left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \right]^T_{\hat{x}=\hat{x}(n)+\theta(n)u(n)} \qquad \hat{x}(n) \epsilon \mathcal{R}(n) \tag{6-13}$$

and its magnitude is such that

$$1 = \left\| \frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \theta(n) \right\|_{\hat{x}=\hat{x}(n)+\theta(n)u(n)} \qquad \hat{x}(n) \epsilon \mathcal{R}(n) \tag{6-14}$$

Equations (6-10),(6-11) and (6-14) indicate that the control must drive the estimated state to a point on the boundary between $\mathcal{Z}(n)$ and $\mathcal{R}(n)$. Define the boundary as

$$\mathcal{B}(n) = \left\{ \hat{x} : \left\| \frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \theta(n) \right\| = 1 \right\} \tag{6-15}$$

so if $b^* \epsilon \mathcal{B}(n)$ is the point to which $u(n)$ drives the estimated state, then (6-13) determines the direction of the control and b^* satisfies

$$b^* = \hat{x}(n) - \theta(n)\theta^T(n)\left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}}\right]^T_{\hat{x}=b^* \epsilon \mathcal{B}(n)} \tag{6-16}$$

Thus to each element b of $\mathcal{B}(n)$ there is associated a necessary trajectory direction given by

$$d(b,n) = -\theta(n)\theta^T(n)\left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}}\right]^T_{\hat{x}=b \epsilon \mathcal{B}(n)} \tag{6-17}$$

It can be shown [6] that these necessary conditions are also sufficient to determine the optimal control and that the optimal control is unique at each point of the $\hat{x}(n)$ space. In terms of the definitions above, the optimal control $u^*(\hat{x}(n),n)$ is given by

$$u^*(\hat{x}(n),n) = \begin{cases} 0 & \text{if } \hat{x}(n) \epsilon \mathcal{Z}(n) \cup \mathcal{B}(n) \\ \theta^{-1}(n)[b^* - \hat{x}(n)] & \text{if } \hat{x}(n) \epsilon \mathcal{R}(n) \end{cases} \tag{6-18}$$

where b^* satisfies

$$b^* = \hat{x}(n) + \rho d(b^*,n) \qquad \rho > 0,\ b^* \epsilon \mathcal{B}(n) \tag{6-19}$$

A typical solution for b^*, in the two dimensional case, is illustrated in Fig. 2. To obtain the solution, it is necessary to know the boundary $\mathcal{B}(n)$ and vectors $d(b,n)$. Knowing these, $\mathcal{B}(n)$ can be searched for the point b^* which satisfies (6-19). Boundary $\mathcal{B}(n)$ and vectors $d(b,n)$ are determined by digital computation using (6-3), (6-7) and (6-8). $\mathcal{B}(n)$ is determined by

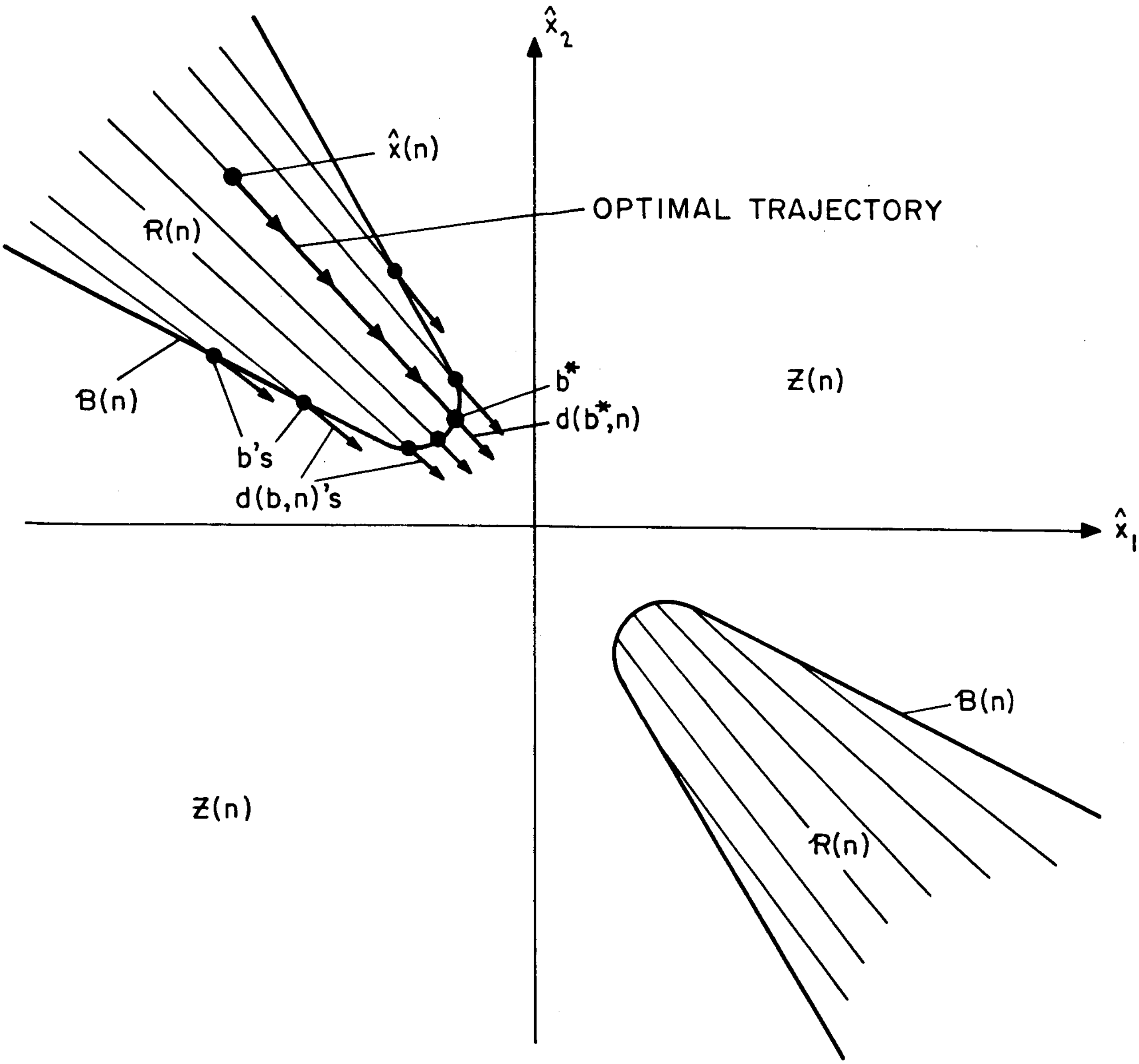

Fig. 2 Typical Solution for b*

searching the $\hat{x}$ space for points satisfying (6-15). By making the search fine enough, the points b will lie sufficiently close together to give an accurate representation of $\mathcal{B}(n)$. Vectors d(b,n) are then calculated from (6-17).

Actual numerical solutions for the two dimensional case (ignoring errors out of the plane of the reference trajectory) were obtained for a typical Earth-Mars mission [6]. Two corrections were assumed, one very early in the flight (2 hours) and the second at the Earth sphere of influence (56 hours). The situation at two hours was similar to the diagram in Fig. 2. At 56 hours the optimal control was essentially a total correction which drove the estimated miss distance to zero. A solution for the one dimensional case was obtained by Tung and Striebel [18].

7. Conclusions

It is useful to note that equations (5-9) and (5-11) correspond to a stochastic optimal control problem for which the state is a Markov process that can be measured without error. In particular, $\hat{x}(n)$ is a Markov process, s(n) is a gaussian disturbance and (5-9), (5-11) correspond to a cost function for which $\bar{L}$ is the incremental cost and $\bar{\phi}$ is the terminal cost.

For most practical problems, the solution of (5-9),(5-11) must be obtained by approximation on a digital computer. In particular, the multidimensional integral on the right of (5-11) must be evaluated. It can be shown [6] that $f_{s(n+1)}(\zeta)$ is the Green's function for a multidimensional diffusion equation. In many cases it is easier to evaluate the integral by approximating the solution of the diffusion equation, using central differences, than to work directly with quadrature formulas.

Finally, from the example problem, it is clear that the minimum fuel spacecraft control is determined by threshold surfaces in the $\hat{x}$ space. If the estimated state lies on one side of the threshold, then the optimal control drives the estimated state to the threshold. On the other side of the threshold the optimal control is zero.

8. Appendix A

The expectation of ϕ, conditioned on the measurement history $M(q)$ and control $u(q)$ is desired. The state at time t_{q+1} is

$$x(q+1) = \hat{x}'(q+1) - e(q+1) + s(q+1) \tag{A-1}$$

where $\hat{x}'(q+1)$, $e(q+1)$ and $s(q+1)$ are mutually independent. If $a(q+1)$ is defined as

$$a(q+1) = \hat{x}'(q+1) - e(q+1) \tag{A-2}$$

then the conditional density of $a(q+1)$ is

$$f_{a(q+1)}[\zeta | M(q), u(q)] = (2\pi)^{-\frac{k}{2}} |P(q+1)|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}[\zeta - x'(q+1)]^T \cdot P(q+1)^{-1}[\zeta - x'(q+1)] \right\} \tag{A-3}$$

Since $a(q+1)$ and $s(q+1)$ are independent

$$f_{x(q+1)}[\xi | M(q) u(q)] = \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k f_{s(q+1)}(\zeta) f_{a(q+1)}[\xi - \zeta | M(q), u(q)] \tag{A-4}$$

and

$$E[\phi(x(q+1)) | M(q), u(q)] = \int_{-\infty}^{\infty} d\xi_1 \ldots \int d\xi_k \phi(\xi) f_{x(q+1)}[\xi | M(q), u(q)] \tag{A-5}$$

Combining (A-4) and (A-5), reversing the order of integration and applying (5-2) yields

$$E[\phi(x(q+1))\,|M(q),u(q)] = \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k f_{s(q+1)}(\zeta)\phi(\hat{x}'(q+1)+\zeta) \qquad \text{(A-6)}$$

which is the required result.

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