ABSOLUTE STABILITY AND PARAMETER SENSITIVITY*

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ABSTRACT

This paper extends the notion of absolute stability to include the parameter variations of the linear part of the system. A simple analytic procedure is proposed to calculate the regions of absolute stability in the parameter space. Then, a parallelpiped of maximum volume is imbedded in the region to interpret its boundaries and obtain readily the information about parameter variations which do not affect the system stability.

INTRODUCTION

Stability and sensitivity are two essential properties of dynamic control systems. While stability assures a proper functioning of the system, the senistivity indicates the ability of the system to retain required performance characteristics despite changes in the operating conditions. These changes may occur due to the fact that the parameters

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of physical systems deviate from their nominal values either because of inaccuracies in the system components (time-invariant case), or because the system parameters vary in time (time-varying case). Therefore, a simultaneous consideration of stability and parameter sensitivity in system analysis is desired.

The Lur'e absolute stability concept [1] and the related criterion of Popov [2] are significant contributions to stability analysis of dynamic systems. This is mostly because the absolute stability concept is meaningful in a large class of closed-loop control systems, and the Popov criterion provides a simple procedure to conclude that kind of stability.

In the absolute stability analysis, the nonlinear characteristic is not completely specified and it should only belong to a certain defined class of functions. On the other hand, the parameters of the linear part are specified numerically. This paper proposes an absolute stability definition which will relax the conditions on the linear part and allow system parameters to deviate from their nominal values. Then, a simple analytical procedure based upon the Popov criterion is presented to determine in the parameter space the region of parameter deviations which do not violate the absolute stability.

A graphical procedure for evaluation of the absolute stability regions in the parameter plane was given in reference [3]. Under certain conditions that technique which is based upon the <u>envelope</u>

criterion can also be extended to considerations in the parameter space.

ABSOLUTE STABILITY IN THE PARAMETER SPACE

The Problem of Lur'e [1] is formulated for a class of closed-loop control systems described by the equations

$$\dot{x} = Px + q\,\phi(\sigma), \quad \sigma = r^T x \qquad (1)$$

where x, q, r are real n-vectors, P is a real $n \times n$ matrix, the pair (P,q) is completely controllable, and $\phi(\sigma)$ is a real continuous scalar function of the real scalar σ such that it belongs to the class A_κ : $\phi(0) = 0, \quad 0 < \sigma\phi(\sigma) < \kappa\sigma^2$. One asks: Is the equilibrium state of the system (1) asymptotically stable in the large for any $\phi(\sigma) \in A_\kappa$, i.e., is the system absolutely stable.

The most important solution of the Problem of Lur'e was given by Popov [2] in terms of the frequency characteristic

$$\chi(\lambda) = r^T(P - \lambda I)^{-1}q \qquad (2)$$

which is the transfer function of the linear part of the system (1) from the input ϕ to the output $(-\sigma)$, and $\lambda = \delta + j\omega$ is the complex variable. Yakobovich [4] generalized the results of Popov and proved that if $\phi(\sigma) \in A_\kappa$ and all the roots of $|P - \lambda I| = 0$ are in the half-plane $\text{Re}\,\lambda < \delta \leq 0$, and if there is a real number υ such that a Popov type inequality

$$\pi(\delta,\omega) \equiv \frac{1}{\kappa} + \mathrm{Re}(1 + j\omega\upsilon)\,\chi(\delta + j\omega) > 0, \ \forall\omega \geq 0 \tag{3}$$

is satisfied, then there exist positive constants ρ and ε such that, for any solution $x(t)$ of (1) and any $t \geq t_o$, one has $|x(t)| \leq \rho|x(t_o)| \ \exp[(\delta - \varepsilon)(t - t_o)]$.

Yakubovich [4] also treated the forced system

$$\dot{x} = Px + q\phi(\sigma) + f(t), \quad \sigma = r^T x \tag{4}$$

where $\phi(\sigma) \in A_\kappa$: $\phi(o) = 0$, $0 < \sigma\phi(\sigma) < \kappa\sigma^2$, $0 < \sigma\phi'(\sigma) < \kappa\sigma^2$, $f(t)$ is a bounded function on the interval $(-\infty, +\infty)$, and showed that a modification of (3),

$$\pi(\delta,\omega) \equiv \frac{1}{\kappa} + \mathrm{Re}\,\chi(\delta + j\omega) > 0, \ \forall\omega \geq 0 \tag{5}$$

assures that there is a unique bounded solution $x_o(t)$ of (1) on $(-\infty, +\infty)$ and that for any $x(t)$ and $t \geq t_o$, one has $|x(t) - x_o(t)| \leq \rho \ |x(t_o) - x_o(t_o)| \ \exp[(\delta - \varepsilon)(t - t_o)]$. In the same paper [4], Yakubovich treated the discontinuous functions $\phi(\sigma)$ and showed that the absolute stability is based upon the same inequalities (3) or (5).

In application of the system (1), the linear part of the system contains parameters which may deviate from their nominal values. Then, it is necessary to relax the conditions on the linear part of the system and allow these parameters to vary in some neighborhood of their nominal values while preserving the absolute stability of the system.

Let us assume that the transfer function $\chi(\lambda, p_1, p_2, \ldots, p_\ell)$ is a function of λ and ℓ parameters $(p_1, p_2, \ldots, p_\ell)$, and let us suppose that the solution $x(t, p_1, p_2, \ldots, p_\ell)$ of (1) is well-defined [5] for parameter values in a certain region R of the ℓ-dimensional euclidian space $(p_1,p_2,\ldots,p_\ell)$. Then, the definition of absolute stability for system (1) can be reformulated to include the parameter variations.

<u>The equilibrium state $x = 0$ of the system (1) is said to be absolutely stable if it is asymptotically stable in the large for any $\phi(\sigma) \in A_\kappa$ and any set $(p_1, p_2, \ldots, p_\ell) \in R$.</u>

When the system (1) is specified, one is interested to find: (a) The greatest value of κ and the largest region R; (b) A value of κ is given and the largest region R is to be determined. A graphical solution of these problems was given in [3] where the region R was determined by the envelope criterion as the largest set $\{(p_1,p_2,\ldots,p_\ell) \in R \mid \pi > 0, \ \forall \omega \geq 0 \}$.

In this paper, a simple analytical solution is presented which first yields the region R in terms of a set of algebraic inequalities involving parameters. Then, a rectangular parallelepiped of maximum volume is imbedded in the region to yield a convenient interpretation of the absolute stability region in the parameter space (this interpretation technique was proposed by George [6,7] for approximation of finite regions of asymptotic stability and linear system analysis).

Assume the transfer function of the linear part to be a rational function of the complex variable λ ,

$$\chi(\lambda, p_1, p_2, \dots, p_\ell) = \frac{\sum_{k=0}^{m} c_k \lambda^k}{\sum_{k=0}^{n} b_k \lambda^k}, \quad n > m \tag{6}$$

in which the coefficients b_k and c_k are real functions of the parameters $p_i (i = 1, 2, \dots, \ell)$. Then, let us express

$$\lambda^k = X_k + j Y_k \tag{7}$$

where $\lambda = \delta + j\omega$, and

$$\begin{aligned} X_k &= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{2\nu} \delta^{k-2\nu} \omega^{2\nu} \\ Y_k &= \sum_{\nu=1}^{k} (-1)^{\nu-1} \binom{k}{2\nu - 1} \delta^{k-2\nu+1} \omega^{2\nu-1} \end{aligned} \tag{8}$$

Functions X_k and Y_k can be easily calculated using the recurrence formulas: $X_{k+1} - 2X_1X_k + (X_1^2 + Y_1^2)X_{k-1} = 0$, $Y_{k+1} - 2X_1Y_k + (X_1^2 + Y_1^2)Y_{k-1} = 0$, $X_0 \equiv 1$, $X_1 \equiv \delta$, $Y_0 \equiv 0$, $Y_1 \equiv \omega$.

When δ is specified in an absolute stability problem, and (7), (8) are substituted in (3) or (5), one obtains

$$\pi(\omega, p_1, p_2, \dots, p_\ell) \equiv \sum_{k=0}^{2n} a_k \omega^k > 0, \forall \omega \geq 0 \tag{9}$$

where the coefficients $a_k = a_k(p_1, p_2, \dots, p_\ell)$ are real functions of the parameters. For convenience, in (9), $1/\kappa$ and υ of (3) are considered as parameters. Note that υ is not a physical parameter and only its existence is required such that $\pi > 0$, $\forall \omega \geq 0$.

From (9), one can readily conclude that the system (1), or (3), is absolutely stable if the corresponding polynomial π has no positive real roots. For this to take place, it is sufficient that the following set of algebraic inequalities

$$a_o > 0, \quad a_k \geq 0, \quad (k = 1, 2, \ldots, 2n) \tag{10}$$

is satisfied.

For example, if the transfer function

$$\chi(\lambda, p_1, p_2, p_3) = \frac{\lambda^2 + p_2\lambda + p_3}{p_1(\lambda+1)(\lambda+2)(\lambda+3)} \tag{11}$$

$\kappa = 1$, and $\delta = 0$ $(\lambda = j\omega)$ are specified, one obtains (9) as

$$\begin{aligned} \pi(\omega, p_1, p_2, p_3) \equiv p_1\omega^6 + (14p_1 - p_2 + 6)\omega^4 + (49p_1 + \\ + 11p_2 - 6p_3 - 6)\omega^2 + 36p_1 + 6p_3 \end{aligned} \tag{12}$$

Inequalities (10) are

$$\begin{aligned} 6p_1 + p_3 &\geq 0 \\ 49p_1 + 11p_2 - 6p_3 - 6 &\geq 0 \\ 14p_1 - p_2 + 6 &\geq 0 \\ p_1 &> 0 \end{aligned} \tag{13}$$

which determine the boundaries of $\overline{R}$.

Inequalities (10) specify a region $\overline{R}(\overline{R} \subset R)$ of absolute stability in the parameter space which may appear to be an overly strict region since (10) are only sufficient conditions for $\pi > 0, \forall \omega \geq 0$. Conditions (10), however, lead to a convenient interpretation of the stability regions.

INTERPRETATION PROCEDURE

After the inequalities (10) are specified, the problem of using them in practical problems is essentially one of interpretation. Since the practical problems may involve more than two parameters, an interpretation procedure for multi-parameter analysis is desired.

In general, to interpret the absolute stability region, let us imbed a parallelepiped Π into the convex region $\overline{R}$ determined by inequalities (10) which has sides perpendicular to the coordinate axes of the parameter space $(p_1, p_2, \ldots, p_\ell)$ and center at the known stable point $\overline{M}(\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_\ell)$. Let the volume v of Π be defined as

$$v = 2^\ell(p_1 - \overline{p}_1)(p_2 - \overline{p}_2) \ldots (p_\ell - \overline{p}_\ell) \tag{14}$$

Now, the function v should be maximized with respect to each inequality (10) separately considered as a constraint. Thus, a constraint

$$a_k(p_1, p_2, \ldots, p_\ell) = 0 \tag{15}$$

may be represented as

$$p_1 = p_1(p_2, p_3, \ldots, p_\ell) \tag{16}$$

Substituting (16) into (14) and extremizing, a necessary condition for $(p_2^o, p_2^o, \ldots, p_\ell^o)$ to occur at a maximum of v is that it be a solution to

$$\frac{\partial v}{\partial p_i} = 0, \quad (i = 2, 3, \ldots, \ell) \tag{17}$$

Standard sufficient conditions for this solution to be maximal are given in [8].

Let the solutions $(p_1^o, p_2^o, \ldots, p_\ell^o)_k$, $(k = 0, 1, \ldots, 2n)$ occur at maximum value of v subject to constraints (10), then the desired parallelepiped Π is given as

$$\Pi = \{(p_1, p_2, \ldots, p_\ell) \in R \,\Big|\, |p_i - \bar{p}_i| \leq \min_k |p_i - p_i^o|_k\}, \quad (i = 1, 2, \ldots, \ell) \tag{18}$$

Since each vertex point of Π is located in R containing the point $\bar{M}(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_\ell)$, it follows that the parallelepiped Π is completely imbedded in $\bar{R}$, i.e., $\Pi \subset \bar{R}$.

In case of the above specific example, let us choose the stable point $\bar{M}(0.2 \,;\, 0 \,;\, 0)$. The volume to be maximized is

$$v = 8(p_1 - 0.2)p_2p_3 \tag{19}$$

Maximization of v with respect to the constraint

$$49p_1 + 11p_2 - 6p_3 - 6 = 0 \tag{20}$$

yields: $p_1^o = 0.178$, $p_2^o = -0.123$, $p_3^o = 0.226$. According to these values of parameters, the parallelepiped Π is determined by

$$|p_1 - 0.2| \leq 0.022, \; |p_2| \leq 0.123, \; |p_3| \leq 0.226 \tag{21}$$

One can readily check that all the vertex points of Π satisfy the rest of the constraints of (13). Therefore, (21) is the solution of the interpretation problem under consideration.

It should be noted that some of the constraints in (10) may not contain all the parameters, as it is clear from inequalities (13). Then, some of the parameters in incomplete inequalities are arbitrary and to make the maximization of v meaningful, one should consider the arbitrary parameters as constants.

For example, the optimization of v in (19) with respect to the constraint $14p_1 - p_2 + 6 \geq 0$ of (13) should be performed with $p_3 = c$ $(c \neq 0)$. Then, the maximization of $v = 8c(p_1 - 0.2)p_2$ gives $p_1^o = -0.122$, $p_2 = 4.292$. In applying equation (18) to determine the parallelepiped Π, these values are discarded and Π is given by (21).

In case of time-varying parameters, by the arguments of reference [4] one can use the inequality (5) and prove the stability of either system (1) or (3). Then, as long as the parameters are varied inside the determined region $\bar{R}$ (or Π) the system is absolutely stable.

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