PLASMA RESEARCH

CASE INSTITUTE OF TECHNOLOGY

UNIVERSITY CIRCLE • CLEVELAND 6, OHIO

N68-22708

NSG-198

RADIATION FROM SOURCES IMMERSED IN A PARTLY IONIZED GAS

by

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Technical Report A-41

ABSTRACT

Using a three fluid model, wave equations are developed for the disturbances due to current or hydrodynamic force sources in a plasma. The formal solutions to the wave equations are obtained for an elementary current source. The power radiated from the source is given in terms of these solutions and, for the special case in which the electrons are much hotter than the other two constituents, the radiated power is determined in detail.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the advice and guidance of Professor B. Samuel Tanenbaum in the preparation of this thesis. He is also grateful to the Lewis Research Center of the National Aeronautics and Space Administration for their support of his graduate work.

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CHAPTER I

INTRODUCTION

Although a good deal of work has been done on the properties of plasma waves,[1-4] there has not been much effort directed towards the problem of radiation from sources in a warm plasma. Noted exceptions are the derivation of inhomogenous wave equations for a current source immersed in an isotropic plasma,[5] and a treatment of power radiated from a current source in an electron gas.[6]

In this thesis we first extend the results of reference 5 to include hydrodynamic force sources. We then solve the wave equations for an elementary current source and formulate a procedure for studying the radiated power. Lastly, we determine the radiated power for a range of cases for which solutions to the dispersion relation are available. M.K.S. units are used throughout.

CHAPTER II

THE PLASMA EQUATIONS

The starting point for our analysis is a set of three coupled hydrodynamic equations.[5] They consist respectively of the continuity equation, the momentum transport equation, and the adiabatic energy transport equation.

$$\frac{D}{Dt} N_a + N_a \nabla\cdot\underline{V}_a = 0 \tag{2.1}$$

$$\frac{D}{Dt}\underline{V}_a = \frac{q_a}{m_a}\left(\underline{E} + \mu_o \underline{V}_a x \underline{H}\right) - \frac{\nabla P_a}{m_a N_a}$$

$$-\sum_b \nu_{ab}\left(\underline{V}_a - \underline{V}_b\right) + \frac{\underline{F}_{as}}{m_a} \tag{2.2}$$

$$P_a N_a^{-\gamma} = \text{Const.} \tag{2.3}$$

The momentum equation has been modified to include an arbitrary hydrodynamic force source, $\underline{F}_{as}$. In the above, m_a, N_a, q_a, $\underline{V}_a$ and P_a are the mass, number density, charge, mean velocity, and pressure for particles of type a (a is the species index e, i, n for electrons, ions, and neutral particles respectively). The symbols $\underline{E}$ and $\underline{H}$ represent electric and magnetic field strength and μ_o is the permeability of free space. The symbol ν_{ab} denotes the effective collision frequency for momentum transfer for particles of type a with those of type b. We will use ν_a to denote the total collision frequency for particles of type a. The collision frequencies satisfy the relation[5]

$$m_a N_a \nu_{ab} = m_b N_b \nu_{ba} \tag{2.4}$$

The symbol $\frac{D}{Dt}$ is the hydrodynamic derivative $(\frac{\partial}{\partial t} + \underline{V} \cdot \nabla)$ and γ is the ratio of specific heats for constant pressure and volume.

In addition to the above we use the Maxwell curl equations:

$$\nabla \times \underline{E} = -\mu_o \frac{\partial \underline{H}}{\partial t} \tag{2.5}$$

$$\nabla \times \underline{H} = \epsilon_o \frac{\partial \underline{E}}{\partial t} + \underline{J} \tag{2.6}$$

In the above ϵ_o is the permittivity of free space and $\underline{J}$ is the source current plus the conduction current and, in a plasma medium, is given by

$$\underline{J} = e(N_i \underline{V}_i - N_e \underline{V}_e) + \underline{J}_s \tag{2.7}$$

We assume an initially quiescent plasma with no $\underline{E}$ or $\underline{H}$ field or ordered velocity and only zero order pressure and number density. If we then assume a small disturbance with time dependence $e^{-i\omega t}$ the other variables in the problem will take the form:

$$\underline{E} = \underline{E}(\underline{r})e^{-i\omega t} \tag{2.8a}$$

$$\underline{H} = \underline{H}(\underline{r})e^{-i\omega t} \tag{2.8b}$$

$$\underline{V}_a = \underline{V}_a(\underline{r})e^{-i\omega t} \tag{2.8c}$$

$$p_a = p_{ao} + p_a'(\underline{r})e^{-i\omega t} \tag{2.8d}$$

$$N_a = n_{ao} + n_a'(\underline{r})e^{-i\omega t} \tag{2.8e}$$

In the above n_{ao} is the ambient number density for particles of type a and

$$p_{ao} = n_{ao}kT_{ao} \tag{2.8f}$$

where p_{ao} is the ambient pressure for particles of type a. If we substitute the linearized variables into Equations (2.1) to (2.7) and combine (2.1) with (2.3) and (2.6) with (2.7) there results

$$\nabla \cdot \underline{V}_a = Q_a = \frac{i\omega p_a'}{\gamma\, p_{ao}} \tag{2.9}$$

$$-i\omega \underline{V}_a = \frac{q_a}{m_a}\underline{E} - \frac{\nabla p'_a}{m_a n_{ao}} - \sum_b \nu_{ab}(\underline{V}_a - \underline{V}_b) + \frac{\underline{F}_{as}}{m_a} \tag{2.10}$$

$$\nabla \times \underline{E} = i\omega\mu_o\underline{H} \tag{2.11}$$

$$\nabla \times \underline{H} = -i\omega\epsilon_o\underline{E} + en_o(\underline{V}_i - \underline{V}_e) + \underline{J}_s \tag{2.12}$$

Equation (2.9) may serve as the definition of Q_a.

In Equations (2.9) to (2.12) and henceforth we drop the $e^{-i\omega t}$.

CHAPTER III

THE TRANSVERSE WAVE EQUATION

It is possible to obtain a wave equation for the $\underline{H}$ field using Equations (2.10), (2.11), and (2.12). Combining Equation (2.11) with the curl of (2.12) we obtain

$$(\nabla^2+\frac{\omega^2}{c^2})\ \underline{H} + en_o\ \nabla x\ (\underline{V}_i-\underline{V}_e) = -\nabla x\ \underline{J}_s \tag{3.1}$$

We can eliminate explicit dependence on the velocities by taking the curl of Equation (2.10) and combining the result with (2.11). The result is

$$\omega^2\underline{S}_a+i\,\omega\sum_b \nu_{ab}(\underline{S}_a-\underline{S}_b) = -\frac{\omega^2\mu_o q_a\underline{H}}{m_a} + \frac{i\omega\nabla x\underline{F}_{as}}{m_a} \tag{3.2}$$

where $\underline{S}_a = \nabla x\underline{V}_a$. Equation (3.2) can now be expanded and solved for $(\underline{S}_i-\underline{S}_e)$ and the solution substituted into Equation (3.1).

There results

$$(\nabla^2+k_T^2)\underline{H} = -\nabla x\underline{J}_s + i\omega en_o\sum_a \frac{c_a}{m_a}\nabla x\ \underline{F}_{as} \tag{3.3}$$

In Equation (3.3) the c_a are given by

$$c_e = -\frac{\omega^2 + i\omega(\nu_{in} + \nu_n)}{D_T} \tag{3.4a}$$

$$c_i = \frac{\omega^2 + i\omega(\nu_{en} + \nu_n)}{D_T} \tag{3.4b}$$

$$c_n = \frac{i\omega\,(\nu_{in} - \nu_{en})}{D_T} \tag{3.4c}$$

and

$$k_T^2 = \frac{\omega^2}{c^2}\left[1 - \omega_e^2\,\frac{\omega^2 + i\omega\,(\nu_{in} + \nu_{ni} + \frac{m_e}{m_i}\nu_{en})}{D_T}\right] \tag{3.5a}$$

$$D_T = (\omega^2 + i\omega\nu_e)\left[\omega^2 + i\omega\,(\nu_{in} + \nu_n)\right] + \omega^2\nu_{en}\,(\nu_{ne} - \nu_{ie}) \tag{3.5b}$$

and

$$\omega_e{}^2 = \frac{n_o e^2}{m_e \epsilon_o} \tag{3.5c}$$

the electron plasma frequency.

To obtain Equations (3.4) and (3.5) it is necessary to assume only the relatively modest condition that the ion and neutral temperatures do not greatly exceed the electron temperature.

If, on the other hand, the temperature of the electrons does not greatly exceed that of the other two species Equation (3.5a) reduces to the familiar form

$$k_T^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_e^2}{\omega^2 + i\omega\nu_e}\right) \tag{3.6}$$

and in each of Equations (3.4) the last term in the denominator can be neglected. Equation (3.6) is valid for arbitrary plasma properties if the frequency satisfies the inequality:

$$\omega^2 \gg \nu_e\,(\nu_i + \nu_n) \tag{3.7}$$

If Equation (3.7) is satisfied, the c_a are given by

$$c_e = -\frac{1}{\omega^2 + i\omega\nu_e} \tag{3.8a}$$

$$c_i = -c_e \tag{3.8b}$$

$$c_n = \frac{i\omega\,(\nu_{in} - \nu_{en})}{\omega^2\,(\omega^2 + i\omega\nu_e)} \tag{3.8c}$$

Here and elsewhere we use "much greater than ($\gg$)" to denote an inequality at least of order $\sqrt{m_i/m_e}$.

CHAPTER IV

THE LONGITUDINAL WAVE EQUATION

The longitudinal waves can be described by the variables Q_a. We begin by taking the divergence of Equation (2.10) and substituting for $\underline{E}$ from Equation (2.12). The result written explicitely is

$$\nabla^2 \begin{bmatrix} Q_e \\ Q_i \\ Q_n \end{bmatrix} + \begin{bmatrix} A_{ee} & A_{ei} & A_{en} \\ A_{ie} & A_{ii} & A_{in} \\ A_{ne} & A_{ni} & A_{nn} \end{bmatrix} \begin{bmatrix} Q_e \\ Q_i \\ Q_n \end{bmatrix} =$$

$$\frac{e\nabla\cdot \underline{J}_s}{m_e \epsilon_o U_e^2} \begin{bmatrix} -1 \\ \mu_{ei}^2 \\ 0 \end{bmatrix} + i\omega \begin{bmatrix} (m_e U_e^2)^{-1}\ \nabla\cdot\ \underline{F}_{es} \\ (m_i U_i^2)^{-1}\ \nabla\cdot\ \underline{F}_{is} \\ (m_n U_n^2)^{-1}\ \nabla\cdot\ \underline{F}_{ns} \end{bmatrix} \qquad (4.1)$$

where

$$A_{aa} = \frac{\omega^2 - \omega_a^2 + i\omega\nu_a}{U_a^2} \qquad (4.2a)$$

$$A_{ab} = \frac{\omega_{ab}^2 - i\omega\nu_{ab}}{U_a^2} \qquad (4.2b)$$

$$\omega_{ab}^2 = \begin{cases} \omega_e^2 & \text{if } ab=ei \\ \omega_i^2 & \text{if } ab=ie \\ 0 & \text{if } a \text{ or } b = n \end{cases}$$

and

$$U_a^2 = \frac{\gamma p_{ao}}{m_a n_{ao}} \tag{4.2c}$$

the a species acoustic velocity.

We also have

$$\mu_{ab}^2 = \frac{n_{ao}T_{ao}}{n_{bo}T_{bo}} = \frac{p_{ao}}{p_{bo}} \tag{4.2d}$$

It can easily be shown that

$$\mu_{ab} A_{ab} = \mu_{ba} A_{ba} \tag{4.2e}$$

The set of Equations (4.1) can be decoupled by a matrix diagonalization procedure. In order to proceed further we assign the following symbols to the matrices in Equation (4.1):

$$[A] = \begin{bmatrix} A_{ee} & A_{ei} & A_{en} \\ A_{ie} & A_{ii} & A_{in} \\ A_{ne} & A_{ni} & A_{nn} \end{bmatrix} \tag{4.3a}$$

$$[Q] = \begin{bmatrix} Q_e \\ Q_i \\ Q_n \end{bmatrix} \tag{4.3b}$$

$$[S] = \frac{e\nabla\cdot\underline{J}_s}{m_e\epsilon_o U_e^2}\begin{bmatrix}-1\\ \mu_{ei}^2\\ 0\end{bmatrix} + i\omega\begin{bmatrix}(m_eU_e^2)^{-1}\ \nabla\cdot\underline{F}_{es}\\ (m_iU_i^2)^{-1}\ \nabla\cdot\underline{F}_{is}\\ (m_nU_n^2)^{-1}\ \nabla\cdot\underline{F}_{ns}\end{bmatrix} \qquad (4.3c)$$

For future use we also define

$$[P] = \begin{bmatrix}1 & 0 & 0\\ 0 & \mu_{ei} & 0\\ 0 & 0 & \mu_{en}\end{bmatrix} \qquad (4.4a)$$

$$[B] = [P]^{-1}\ [A]\ [P] \qquad (4.4b)$$

We also note at this point that B is symmetric i.e.

$$[B]_T = [B] \qquad (4.4c)$$

Equation (4.1) can be transformed to a set of scalar wave equations by the appropriate similarity transformation on the matrix $[A]$. Our first step is to use Equation (4.4b) to transform $[A]$ to the symmetric matrix $[B]$. An elementary theorem of matrix algebra states that a real symmetric matrix can be diagonalized by an orthogonal transformation, the column vectors of the transforming matrix being the normalized eigenvectors of the matrix to be diagonalized. It is easily shown that the above theorem is still correct with the qualification "real" deleted, provided we properly define orthogonal. The orthogonality property is

$$[X_j]_T [X_k] = \delta_{jk} \tag{4.5}$$

where δ_{jk} is the Kroniker delta and $[X_k]$ is the "k" th eigenvector. If we now form a modal matrix $[M]$ from the column vectors $[X_k]$ we have

$$[M]_T \, [B] \, [M] = [d] \tag{4.6}$$

where $[d]$ is a diagonal matrix composed of the eigenvalues of $[B]$ which, of course, are also the eigenvalues of $[A]$. The "j" th eigenvalue will be denoted by the symbol k_j^2.

It is thus appropriate to transform the set of variables $[Q]$ to a new set $[\phi]$ by the transformation

$$[Q] = [P] \, [M] \, [\phi] \tag{4.7}$$

If we substitute Equation (4.7) into (4.1) and premultiply by $[M]_T \, [P]^{-1}$ we have

$$\nabla^2 \, [\phi] + [d] \, [\phi] = [M]_T \, [P]^{-1} \, [S] \tag{4.8}$$

or in a more convenient notation

$$(\nabla^2 + k_j^2)\ \phi_j = S_j \tag{4.9}$$

where S_j is the "j" th element of the column matrix obtained by performing the matrix multiplication indicated on the right hand side of Equation (4.8). Here and elsewhere we do not sum on repeated indices unless indicated.

To compute the elements of $[M]$ we must first find the eigenvalues k_j^2 by solving the dispersion relation

$$\text{det.}\ \left[A - k^2 I\right] = 0 \tag{4.10}$$

This is a third order equation in k^2 and solutions are not available for completely arbitrary plasma properties. Approximate solutions are known,[8] however, for certain of the more interesting ranges of plasma parameters. Using those approximate eigenvalues we can find a modal matrix, valid in the same range, by the usual techniques.

CHAPTER V

PLASMA ENERGY RELATIONS

Before proceeding with solutions to the wave equations it is desirable to formulate the radiated power in terms of the variables $\underline{H}$ and ϕ_j. The total energy stored in the plasma is given by

$$W=\frac{1}{2}\int_V(\underline{H}\cdot\underline{B}+\underline{E}\cdot\underline{D})\,dV + \sum_a\int_V\int_{\underline{\xi}}\frac{1}{2}m_a\xi_a^2 f_a(\underline{\xi})\,d^3\xi\,dV \tag{5.1}$$

In Equation (5.1) the first integral is energy stored in the electric and magnetic fields. The second term is the total energy in fluid motion with f_a being the velocity distribution function for particles of type a and

$$\underline{\xi}_a = \underline{V}_a + \underline{c}_a \tag{5.2}$$

where $\underline{V}_a$ is the ordered velocity and $\underline{c}_a$ the random velocity of the particles of type a. Performing the integration over ξ we have

$$\int_\xi \xi_a^2 f_a d^3\xi = \int_\xi (V_a^2+c_a^2+2\underline{c}_a\cdot\underline{V}_a)\; f_a d^3\xi \tag{5.3}$$

Since $\underline{V}_a$ is a constant in the integration with respect to ξ and the average value of $\underline{c}_a$ is zero, the last term in Equation (5.3) vanishes. Also we have by definition:

$$\int_{\xi} f_a d^3\xi = N_a \tag{5.4}$$

and

$$m_a \int_{\xi} c_a^2 f_a d^3\xi = 3N_a kT_a = 3P_a \tag{5.5}$$

Combining these results with Equation (5.1) we have:

$$W = \frac{1}{2}\int_V \left[\underline{H}\cdot\underline{B} + \underline{E}\cdot\underline{D} + \sum_a \frac{1}{2}\rho_a V_a^2 + \frac{3}{2}P_a\right] dV \tag{5.6}$$

If we take a partial derivative with respect to time, substitute in Maxwell's curl equations and note that

$$\nabla\cdot\underline{E}\times\underline{H} = \underline{H}\cdot\nabla\times\underline{E} - \underline{E}\cdot\nabla\times\underline{H} \tag{5.7}$$

we have

$$\dot{W} = \int_V \left[-\nabla\cdot\underline{E}\times\underline{H} - \underline{J}\cdot\underline{E} + \sum_a \frac{V_a^2}{2}\dot{\rho}_a + \rho_a\underline{V}_a\cdot\dot{\underline{V}}_a + \frac{3}{2}\dot{P}_a\right] dV \tag{5.8}$$

If we now substitute Equations (2.1), (2.2), and (2.3) into Equation (5.8) we have:

$$\dot{W} = -\int_V \left[\nabla\cdot\underline{E}\times\underline{H} + \sum_a \nabla\cdot\left(\tfrac{1}{2}\rho_a V_a^2 + \tfrac{5}{2}p_a\right)\underline{V}_a \right] dV$$

$$- \int_V \left[\sum_{ab} \frac{\rho_a \nu_{ab}}{2} (\underline{V}_a - \underline{V}_b)\cdot(\underline{V}_a - \underline{V}_b) \right] dV \tag{5.9}$$

The last term in Equation (5.9) is negative definite and represents the energy fed from ordered motion into thermal motion via collisions. This energy is, of course, still in the medium. Had we used a more complete energy transport equation, including the effect of collisions, rather than the adiabatic version we would have an equal positive definite term showing the energy increase in thermal motion. In this same connection it should be noted that the term $-\underline{J}\cdot\underline{E}$ in the integrand of Equation (5.8) is not present in Equation (5.9). This term represents an energy transfer between the electric field and fluid motion and is cancelled by a like term resulting from the substitution of the momentum equation into Equation (5.8).

If we omit the heating term involving the collision frequencies and apply Gauss's Theorem, the rate at which energy crosses the boundary of a volume is seen to be:

$$\dot{W} = -\int_s \left[\underline{E} \times \underline{H} + \sum_a \left(\rho_a \frac{V_a^2}{2} + \frac{5}{2} P_a \right) \underline{V}_a \right] \cdot \underline{n} \, ds \qquad (5.10)$$

Equation (5.10) is then time averaged over one period of the applied frequency. It is, of course, necessary to retain terms to second order in computing power.

The electromagnetic term in the integrand of Equation (5.10) is easily time averaged to yield

$$\underline{S}_{em} = \frac{1}{2} \operatorname{Re} \underline{E} \times \underline{H}^* \qquad (5.11)$$

The second term in Equation (5.10) is third order in V_a so it is discarded. The third term gives two second order contributions, one involving the first order pressure times the first order velocity, and one involving the zero order pressure times the second order velocity. To avoid dealing with second order quantities we write

$$\frac{5}{2} P_a \underline{V}_a = \frac{5}{2} \left(\frac{P_a}{\rho_a} \right) \rho_a \underline{V}_a \qquad (5.12)$$

Expanding $\frac{P_a}{\rho_a}$ by use of the adiabatic energy transport equation we have, to first order

$$\frac{p_a}{\rho_a} = \frac{p_{ao}}{\rho_{ao}} \left[\frac{1 + p_a'/p_{ao}}{1 + \frac{3}{5}\, p_a'/p_{ao}} \right] = \frac{p_{ao}}{\rho_{ao}} + \frac{2}{5} \frac{p_a'}{\rho_{ao}} \tag{5.13}$$

hence

$$\frac{5}{2}\, p_a \underline{V}_a = \frac{5}{2}\, \rho_a \underline{V}_a \left[\frac{p_{ao}}{\rho_{ao}} + \frac{2}{5} \frac{p_a'}{\rho_{ao}} \right] \tag{5.14}$$

Transforming the surface integral of the first term back to a volume integral by use of Gauss's Theorem and substituting from the continuity equation we have

$$\int_s \frac{5}{2} \frac{p_{ao}}{\rho_{ao}}\, \rho_a \underline{V}_a \cdot \underline{n}\; ds = \frac{5}{2} \frac{p_{ao}}{\rho_{ao}} \int_V \nabla \cdot \rho_a \underline{V}_a dV$$

$$= - \frac{5}{2} \frac{p_{ao}}{\rho_{ao}} \int_V \frac{\partial \rho}{\partial t}\; dV \tag{5.15}$$

which averages out to zero since, on the average, the amount of fluid in a volume is unchanged. Time averaging the second term in Equation (5.14) in the usual way the fluid energy flux is seen to be

$$\underline{S}_{f1} = \frac{1}{2} \operatorname{Re} \sum_a p_a' \underline{V}_a^* \tag{5.16}$$

Equations (5.11) and (5.16) are combined to yield the total energy flux:

$$\underline{S} = \tfrac{1}{2} \,\mathrm{Re} \left[\underline{E} \times \underline{H}^* + \sum_a p_a' \underline{V}_a^* \right] \tag{5.17}$$

We can modify Equation (5.17) so that $\underline{S}$ depends only on $\underline{H}$ and Q_a by writing $\underline{E}$, p', and $\underline{V}$ in terms of these variables. The dependence on $\underline{E}$ and p' can be eliminated immediately by use of Equations (2.9) and (2.12), thus:

$$\underline{S} = \tfrac{1}{2}\mathrm{Re} \left\{ \frac{en_o}{i\omega\epsilon_o}(\underline{V}_i - \underline{V}_e) - \frac{1}{i\omega\epsilon_o} \nabla \times \underline{H} \right\} \times \underline{H}^* + \frac{\gamma}{i\omega} \sum_a p_{ao}\, Q_a\, \underline{V}_a^* \tag{5.18}$$

We have used Equation (2.12) in the source free form and thus intend Equation (5.18) to be valid everywhere in the plasma excluding the source.

To eliminate the $\underline{V}_a$ we consider the source free form of Equation (2.10) with $\underline{E}$ and p_a' replaced by the use of Equations (2.12) and (2.9) respectively.

$$\begin{bmatrix} A_{ee} & A_{ei} & A_{en} \\ A_{ie} & A_{ii} & A_{in} \\ A_{ne} & A_{ni} & A_{nn} \end{bmatrix} \begin{bmatrix} \underline{V}_{eT} \\ \underline{V}_{iT} \\ \underline{V}_{nT} \end{bmatrix} = \begin{bmatrix} 1 \\ -\mu_{ei}^{2} \\ 0 \end{bmatrix} \frac{e\nabla \times \underline{H}}{m_e \epsilon_o U_e^2} \tag{5.19a}$$

$$\begin{bmatrix} A_{ee} & A_{ei} & A_{en} \\ A_{ie} & A_{ii} & A_{in} \\ A_{ne} & A_{ni} & A_{nn} \end{bmatrix} \begin{bmatrix} \underline{V}_{eL} \\ \underline{V}_{iL} \\ \underline{V}_{nL} \end{bmatrix} = -\nabla \begin{bmatrix} Q_e \\ Q_i \\ Q_n \end{bmatrix} \tag{5.19b}$$

In Equations (5.19) we have split the velocity into a part that is the curl of a vector and is related to the transverse waves and a part that is the gradient of a scaler and is related to the longitudinal waves, i.e.

$$\underline{V}_a = \underline{V}_{aT} + \underline{V}_{aL} \tag{5.20}$$

By use of Equation (5.20) the energy density $\underline{S}$ decomposes naturally into four parts:

$$\underline{S} = \underline{S}_{TT} + \underline{S}_{TL} + \underline{S}_{LT} + \underline{S}_{LL} \tag{5.21}$$

where

$$\underline{S}_{TT} = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{n_o e}{i\omega\epsilon_o}\,(\underline{V}_{iT}-\underline{V}_{eT}) - \frac{1}{i\omega\epsilon_o}\,\nabla x\underline{H}\right] x\underline{H}^* \tag{5.22a}$$

$$\underline{S}_{TL} = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{n_o e}{i\omega\epsilon_o}\,(\underline{V}_{iL}-\underline{V}_{eL})\right] x\ \underline{H}^* \tag{5.22b}$$

$$\underline{S}_{LT} = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{\gamma}{i\omega}\sum_a p_{ao}\,Q_a\,\underline{V}_{aT}^*\right] \tag{5.22c}$$

$$\underline{S}_{LL} = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{\gamma}{i\omega}\sum_a p_{ao}\, Q_a\, \underline{V}_{aL}^{*}\right] \tag{5.22d}$$

We define $[G] = [A]^{-1}$ and G_{ab} = the "ab"th element of $[G]$. We then invert $[A]$, solve Equations (5.19) for the fluid velocities and substitute the results into Equations (5.22), obtaining:

$$\underline{S}_{TT} = \tfrac{1}{2}\,\mathrm{Re}\,\frac{i\omega\mu_o}{k_T^{2}}\,(\nabla\times\underline{H})\times\underline{H}^{*} \tag{5.23}$$

and

$$\underline{S}_{TL} + \underline{S}_{LT} = \tfrac{1}{2}\,\mathrm{Re}\,\frac{n_o e}{i\omega\epsilon_o}\sum_b (G_{1b}-G_{2b})\,\nabla\times Q_b\underline{H}^{*} \tag{5.24}$$

The reason for the split of $\underline{S}$ into four parts is now evident. The first part, $\underline{S}_{TT}$, involves only the $\underline{H}$ field and can thus be characterized as power in the transverse wave. The second and third parts, $\underline{S}_{TL}$ and $\underline{S}_{LT}$, involve products of transverse with longitudinal potentials and make no net contribution to radiated power. This can be demonstrated by recalling that $\underline{S}_{TL} + \underline{S}_{LT}$ enters power calculations as the integrand of a surface integral. Using Gauss's Theorem we can write the power carried by these waves as

$$P_{LT} + P_{TL} = \sum_b \tfrac{1}{2}\,\mathrm{Re}\,\frac{n_o e}{i\omega\epsilon_o} \int_V \nabla\cdot(G_{1b}-G_{2b})\,\nabla\times Q_b \underline{H}^*\, dV \qquad (5.25)$$

which vanishes since the integrand is the divergence of a curl.

The last term in Equation (5.21) is best handled in matrix notation. Equation (5.22d) can thus be written:

$$\underline{S}_{LL} = + \tfrac{1}{2}\,\mathrm{Re}\,\frac{\gamma p_{eo}}{i\omega}\,[Q]_T^*\,[P]^{-1}\,[P]^{-1}\,[G]\,\nabla\,[Q] \qquad (5.26)$$

If we substitute Equations (4.6) and (4.7) into Equation (5.26) we have:

$$S_{LL} = -\tfrac{1}{2}\,\mathrm{Re}\,\frac{i\gamma p_{eo}}{\omega}\,[\phi]_T^*\,[M]_T^*\,[M]\,[d]^{-1}\,\nabla\,[\phi] \qquad (5.27)$$

The above expression does not, in general, reduce to canonical form in ϕ_j. The reason is that $[M]$ is not orthogonal in the hermitian sense unless we neglect collisions thereby making all of the variables of the problem real. We do not evaluate the elements of $[M]$ or the above expression in the general case since a general solution to the dispersion relation is not available.

There are two special cases for which solutions to the dispersion relation are available. The first and most obvious is the collisionless case, for which the neutral wave is completely uncoupled from the other two, thus reducing the order of the dispersion relation. The other case for which good approximate solutions to the dispersion relation are available[8] is that in which the electrons are much hotter than the other two constituents. These cases will be treated separately in a later chapter.

CHAPTER VI

SOLUTIONS TO THE WAVE EQUATIONS FOR AN ELEMENTARY CURRENT SOURCE

The methods of solution of the transverse wave equation are quite analogous to the equivalent free space problems and the longitudinal wave equations necessitate little modification. The usual procedure is to solve the equations for an elementary source and use the result as a Green's Function for more complicated sources. We will obtain the solutions for an elementary current source described by

$$\underline{J}_s = A_o\,\delta(\underline{r})\,\underline{a}_z \tag{6.1}$$

where A_o is an amplitude constant with dimensions of amp-meters. The transverse wave equation for this source is

$$\begin{aligned}(\nabla^2 + k_T^2)\,\underline{H} &= -\,A_o\,\nabla \times \delta(\underline{r})\,\underline{a}_z \\ &= A_o\,\underline{a}_z \times \nabla\delta(\underline{r})\end{aligned} \tag{6.2}$$

The longitudinal wave equation for the elementary current source is

$$(\nabla^2+k_j^2)\ \phi_j = -\ (X_{1j}-\mu_{e1}X_{2j})\ \frac{eA_o}{m_e\,\epsilon_o U_e^2}\ \underline{a}_z\cdot\nabla\delta(\underline{r}) \qquad (6.3)$$

where X_{ij} is the "i"th element of the "j"th normalized eigenvector.

The solutions are obtained as follows:
We define the auxiliary functions Ψ and θ_j such that

$$\underline{H} = +\ A_o\ \underline{a}_z\ \times \nabla\Psi \qquad (6.4)$$

and

$$\phi_j = -\ (X_{1j}-\mu_{e1}\ X_{2j})\ \frac{eA_o}{m_e\,\epsilon_o U_e^2}\ \underline{a}_z\cdot\nabla\theta_j \qquad (6.5)$$

Since any linear vector operator commutes with the Laplacian we have

$$(\nabla^2+\ k_T{}^2)\,\Psi = -\delta(\underline{r}) \qquad (6.6a)$$

and

$$(\nabla^2\ +\ k_j{}^2)\ \theta_j = -\delta(\underline{r}) \qquad (6.6b)$$

The solutions are well known and are given by:

$$\Psi = \frac{1}{4\pi r}\ e^{\ ik_T r} \qquad (6.7a)$$

$$\theta_j = \frac{1}{4\pi r}\ e^{\ ik_j r} \qquad (6.7b)$$

Combining Equation (6.7a) with (6.4) and (6.7b) with (6.5) we obtain:

$$\underline{H} = A_o \sin\theta \left(ik_T - \frac{1}{r}\right) \frac{e^{ik_T r}}{4\pi r} \underline{a}_\phi \tag{6.8}$$

$$\phi_j = - \frac{eA_o \cos\theta \,(X_{1j} - \mu_{ei} X_{2j})}{m_e \epsilon_o U_e^2} \left(ik_j - \frac{1}{r}\right) \frac{e^{ik_j r}}{4\pi r} \tag{6.9}$$

The curl of Equation (6.8) and the gradient of Equation (6.9) are computed and substituted into Equations (5.23) and (5.27) respectively in order to calculate radiated power. This calculation, for the more complicated longitudinal waves, is deferred to the next chapter. For the transverse wave we retain in our power density only the term which dominates at a distance, from the source, large compared to a wavelength. The source indeed transfers energy to the plasma via the near field terms, but for waves which are not too strongly attenuated the energy dissipated in the near field is small. Thus, outside the source region, Equations (5.23) and (6.8) can be combined to yield:

$$\underline{S}_{TT} = \underline{a}_r \frac{\omega \mu_o A_o^2 \sin^2\theta \;\mathrm{Re}\, k_T}{32(\pi r)^2} \; e^{-2\mathrm{Im.}(k_T r)} \tag{6.10}$$

If we consider only the usual conditions for which $\omega_e >> \nu_e$ then the wave number is almost pure real above the plasma frequency and almost pure imaginary below it. Thus, at frequencies below the plasma frequency the transverse wave is strongly damped and no appreciable power is radiated beyond the near field. At frequencies above the plasma frequency the wave is lightly damped and

$$k_T \cong \frac{\sqrt{\omega^2-\omega_e^2}}{c} + \frac{i\nu_e\omega_e^2}{2\omega c\sqrt{\omega^2-\omega_e^2}} \qquad (6.11)$$

For $\omega^2-\omega_e^2 >> \frac{\nu_e\omega_e^2}{\omega}$ Equation (6.10) becomes:

$$\underline{S}_{TT}=\underline{a}_r \frac{A_o^2\sin^2\theta\,\omega\mu_o\sqrt{\omega^2-\omega_e^2}}{32c(\pi r)^2} \exp\left[-\frac{\nu_e\omega_e^2 r}{\omega c\sqrt{\omega^2-\omega_e^2}}\right] \qquad (6.12)$$

The total power carried away by this wave can be found by evaluating a closed surface integral. We shall choose a surface close enough to the source that the exponential damping can be neglected, yet beyond the range of the near field. The result is:

$$P_{TT} = \frac{A_o^2\omega\mu_o\sqrt{\omega^2-\omega_e^2}}{12\pi c} \qquad (6.13)$$

This, as expected, is in agreement with the result of Ref. 6* in which only electron motions are considered.

*There is a factor of 2 difference due to a trivial error introduced in Equation 35 of Ref. 6.

CHAPTER VII

RADIATION IN LONGITUDINAL WAVES

In order to obtain useful solutions to the longitudinal wave equations we must first have the roots of the dispersion relation (Equation (4.10)). As stated previously, approximate roots are available for two special cases. The first of these[2] which we shall call Case 1 applies to all situations in which collisions can be neglected completely. Solutions to the dispersion relation are also available[8] for a plasma in which the electrons are much hotter than the other two constituents. We shall divide this category into two cases: Case 2 if the degree of ionization is such that $p_{eo} >> p_{no}$, and Case 3 if the degree of ionization is so low that $p_{no} >> p_{eo}$ in spite of the high electron temperature. We will take these three cases in order, first calculating the modal matrices and then the power flow.

CALCULATION OF MODAL MATRICES

Case 1:

If we neglect collisions entirely the exact roots of the dispersion relation (Equation (4.10)) are given by:

$$k_{\pm}^2 = \frac{\omega^2-\omega_e^2}{2U_e^2} + \frac{\omega^2-\omega_i^2}{2U_i^2} \pm \sqrt{\left[\frac{\omega^2-\omega_e^2}{2U_e^2} + \frac{\omega^2-\omega_i^2}{2U_i^2}\right]^2 + \frac{\omega_e^2\omega_i^2}{U_e^2U_i^2}} \tag{7.1}$$

$$k_n^2 = \frac{\omega^2}{U_n^2} \tag{7.2}$$

Since there are no collisions, the neutral particle wave will not be excited by a current source and Equation (7.2) is of academic interest. Equation (7.1) reduces to a form which is more easily interpreted if we separately examine different portions of the spectrum.

If $\omega^2 >> (1 + \mu_{ie}^2)\omega_i^2$ we have:

$$k_1^2 \cong \frac{\omega^2-\omega_e^2}{U_e^2} \quad \text{(electron wave)} \tag{7.3a}$$

$$k_2^2 \cong \frac{\omega^2}{U_i^2} \quad \text{(ion wave)} \tag{7.3b}$$

Using these approximate roots the modal matrix can be calculated by the usual methods. In this frequency range it is given by:

$$[M] \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.4}$$

If, on the other hand, $\omega^2 \ll (1 + \mu_{ie}^2)\omega_i^2$ the two roots of Equation (7.1) reduce to:

$$k_1^2 \cong \frac{\omega^2}{U_p^2} \text{ ("ion" wave)} \tag{7.5a}$$

$$k_2^2 \cong - \frac{\omega_i^2}{U_\mu^2} \text{ (damped electron wave)} \tag{7.5b}$$

where

$$U_p^2 = \frac{\gamma k \; (T_e + T_i)}{n_o(m_e + m_i)} \tag{7.5c}$$

and

$$U_\mu^2 = \frac{\gamma k T_e T_i}{(T_e + T_i) m_i} \tag{7.5d}$$

The first of these is an oscillation involving both electrons and ions (although it is often called an "ion" wave) while the second root, also involving both electrons and ions, is an evanescent wave (for which there can be no power flow). In this frequency range the modal matrix is given by:

$$[M] \cong \begin{bmatrix} 1 & -\mu_{1e}\left(1+\frac{\omega^2}{\omega_1{}^2}\right) & 0 \\ \mu_{1e}\left(1+\frac{\omega^2}{\omega_1{}^2}\right) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.6}$$

In calculating the matrix elements of Equation (7.6) we have retained all terms to the order of $\frac{\omega^2}{\omega_1{}^2}$ (which is assumed to be very small). The reason for this is that using only the dominant terms in the appropriate eigenvector would give us a null result for the amplitude of the "ion" wave.

Case 2:

If the temperature of the electrons is much greater than that of the other two species, and the degree of ionization is high enough that $p_{eo} \gg p_{no}$, the high frequency ($\omega^2 \gg \omega_1{}^2$) roots are given by:

$$k_1^{\,2} \cong \frac{\omega^2-\omega_e^{\,2}+i\,\omega\nu_e}{U_e^{\,2}} \quad \text{(electron wave)} \tag{7.7a}$$

$$k_2^{\,2} \cong \frac{\omega^2}{U_i^{\,2}} \quad \text{(ion wave)} \tag{7.7b}$$

$$k_3^{\,2} \cong \frac{\omega^2}{U_n^{\,2}} \quad \text{(neutral wave)} \tag{7.7c}$$

The modal matrix we obtain for this set of conditions is given by:

$$[M] = \begin{bmatrix} 1 & \dfrac{\mu_{ie}(\omega_i^{\,2}-i\omega\nu_{ie})}{\omega^2} & \dfrac{-i\omega\mu_{ne}\nu_{ne}}{\omega^2} \\ -\mu_{ie}\dfrac{(\omega_i^{\,2}-i\omega\nu_{ie})}{\omega^2} & 1 & \dfrac{-i\omega\mu_{in}\nu_{in}U_-^{\,2}}{\omega^2 U_i^{\,2}} \\ \dfrac{i\omega\mu_{ne}\nu_{ne}}{\omega^2} & \dfrac{i\omega\mu_{in}\nu_{in}U_-^{\,2}}{\omega^2 U_i^{\,2}} & 1 \end{bmatrix} \tag{7.8}$$

where

$$U_-^2 = \frac{U_i^2 U_n^2}{U_i^2 - U_n^2} \tag{7.9}$$

The roots for the same plasma properties and the frequency range $\nu_n \ll \omega \ll \omega_i$ are given by:

$$k_1^2 \cong \frac{\omega^2}{U_p^2} \left(1 + \frac{i\nu'}{\omega}\right) \quad (\text{"ion" wave}) \tag{7.10a}$$

$$k_2^2 \cong \frac{-\omega_i^2 + i\omega\nu_i}{U_i^2} \quad (\text{damped plasma wave}) \tag{7.10b}$$

$$k_3^2 \cong \frac{\omega^2 + i\omega\nu_n}{U_n^2} \quad (\text{neutral wave}) \tag{7.10c}$$

where

$$\nu' = \nu_n + \nu_{in} + \frac{m_e}{m_i}\nu_{en} \tag{7.11}$$

The modal matrix for this frequency range is given by:

$$[M] \cong \begin{bmatrix} 1 & -\mu_{ie}(1+\frac{\omega^2}{\omega_i^2}) & \frac{-i\omega\mu_{ne}\nu_n}{\omega^2} \\ \mu_{ie}(1+\frac{\omega^2}{\omega_i^2}) & 1 & \frac{-i\omega\mu_{in}\nu_{in}}{\omega_i^2} \\ \frac{i\omega\mu_{ne}\nu_n}{\omega^2} & \frac{i\omega\mu_{in}\nu_{in}}{\omega_i^2} & 1 \end{bmatrix} \tag{7.12}$$

In computing the results for Equation (7.12) it was again necessary to calculate a higher approximation for X_{21} to avoid obtaining a null result for the weakly driven "ion" wave (with phase velocity U_p). The same correction was applied to X_{12} in the interest of consistency.

The low frequency ($\omega \ll \nu_n$) roots for this case are modified somewhat from the above. The "ion" wave becomes one which involves motion of all three species the root being given by:

$$k_1^2 \cong \frac{\omega^2}{U_T^2}\left[1 + \frac{i\omega\rho}{\rho_i\nu^r}\right] \text{ (total acoustic wave) } (7.13)$$

The damped plasma wave is still evanescent in this range and still has the same root (Equation

(7.10b)). The neutral wave also has the same root as above (Equation (7.10c)). The modal matrix for this range is given by:

$$[M] \cong \begin{bmatrix} 1 & -\mu_{ie} & \mu_{ne} \\ \mu_{ie}\left(1+\frac{\omega^2}{\omega_i^2}\right) & 1 & \frac{\mu_{ne}}{\mu_{ei}} - \frac{i\omega\mu_{in}\nu_{in}}{\omega_i^2} \\ -\mu_{ne} & \frac{\mu_{ne}}{\mu_{ei}} & 1 \end{bmatrix} \qquad (7.14)$$

In Equation (7.14) we have calculated a higher approximation for only these elements (X_{21} and X_{23}) which would otherwise give a null result for weakly driven wave amplitudes.

Case 3:

If the degree of ionization is so low that $p_{no} \gg p_{eo}$ in spite of $T_e \gg T_n$ the results of Case 2 must be modified at low frequencies. Specifically, for $\omega \ll \nu'$ the roots of the dispersion relation become:

$$k_1^2 \cong \frac{i\omega\nu''}{U_p^2} \quad \text{(damped "ion" wave)} \qquad (7.15a)$$

$$k_2^2 \cong \frac{-\omega_i^2 + i\omega\nu_i}{U_i^2} \quad \text{(damped plasma wave)} \qquad (7.15b)$$

$$k_3^2 \cong \frac{\omega^2}{U_T^2}\left(1+\frac{i\omega\Delta}{\nu'}\right) \quad \text{(total acoustic wave)} \qquad (7.15c)$$

where

$$\nu'' = \nu_{in} + \frac{m_e}{m_i}\nu_{en} \qquad (7.15d)$$

$$U_T^2 = \frac{\rho_i U_p^2 + \rho_n U_n^2}{\rho_i + \rho_n} \qquad (7.15e)$$

$$\Delta = \frac{\left(U_T^2 - U_n^2\right)^2}{U_T^4} \qquad (7.15f)$$

The modal matrix is given by:

$$[M] \cong \begin{bmatrix} 1 & -\mu_{ie} & \mu_{en} \\ \mu_{ie}\left(1+\frac{i\omega\nu_{in}}{\omega_i^2}\right) & 1 & \mu_{in}\left(1+\frac{\omega^2\nu_{in}}{\omega_i^2\mu_{ne}^2\nu_n}\right) \\ -\mu_{en} & \frac{i\omega\mu_{in}\nu_{in}}{\omega_i^2} & 1 \end{bmatrix} \qquad (7.16)$$

We have again calculated a higher approximation only for the elements X_{21} and X_{23}.

POWER RADIATED

The modal matrix is substituted into previous results in two places. Each column vector, along with the appropriate eigenvalue is substituted into Equation (6.9) to find the longitudinal wave potentials ϕ_j. In addition the modal matrix is substituted into Equation (5.27) in order to find the longitudinal wave power density.

It can be directly verified that, for each modal matrix calculated in this chapter, the dominant terms of the matrix product $[M]_T^*[M]$ are on the diagonal. That is, the diagonal terms are all approximately unity and the off diagonal terms are all much smaller. As a result, for all of the cases we are considering here, Equation (5.27) reduces to

$$\underline{S}_{LL} \cong \sum_j \frac{1}{2} \text{ Re } \frac{i\gamma p_{eo}}{\omega k_j^2} \phi_j^* \nabla \phi_j \tag{7.17}$$

The radiated power corresponding to each term in Equation (7.17) is found by substituting the solutions for ϕ_j into Equation (7.17) and taking a closed surface integral near enough, to the source, to neglect the exponential decay factor.

The general result is:

$$P_{LLj} = \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi\omega U_e^2} \left| X_{1j} - \mu_{ei} X_{2j} \right|^2 \operatorname{Re} k_j \qquad (7.18)$$

We note, at this point, that all of the solutions to the dispersion relation do not correspond to propagating waves. In each case the so called "damped plasma wave" (Equations (7.5b), (7.10b), and (7.15b)) is evanescent and cannot account for any power flow.

The appropriate eigenvector is substituted into Equation (7.18) in order to determine the power carried away by each wave. Results are computed separately for each frequency range and each case. The high frequency results for all these Cases are identical except for the absence of a neutral wave for Case 1. Hence, for all Cases and $\omega^2 \gg \omega_i^2(1+\mu_{ie}^2)$ we have:

$$P_{LL1} \simeq \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi\omega U_e^2} \operatorname{Re} \frac{\sqrt{\omega^2 - \omega_e^2}}{U_e} \qquad (7.19)$$

$$P_{LL2} \cong \frac{A_o^2 \omega_1^2 \mu_o c^2}{12\pi U_1^3} \tag{7.20}$$

Also for $\omega^2 \gg \omega_1^2(1+\mu_{1e}^2)$ and Cases 2 and 3 we have:

$$P_{LL3} \cong \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi U_e^2 U_n} \left[\frac{\mu_{1n}\nu_{1n}U_-^2}{\omega U_1^2}\right]^2 \tag{7.21}$$

The three waves, in this frequency range, are simply electron, ion, and neutral sound waves. As expected, Equation (7.19) agrees with the result of Ref. 6 in which only electron motion was considered.

In the lower frequency ranges we consider each case separately.

Case 1:

In a collisionless plasma there is only one wave excited by a current source in the frequency range $\omega^2 \ll \omega_1^2(1+\mu_{1e}^2)$. There can be no coupling to the neutral wave and the electron wave is evanescent. The power in the "ion" wave is given by:

$$p_{LL1} \cong \frac{A_o{}^2 \omega_e{}^2 \mu_o c^2 \omega^4}{12\pi \omega_i{}^2 U_p{}^3 \sqrt{1+\mu_{ie}{}^2}} \tag{7.22}$$

This result decreases very rapidly as ω becomes small compared to ω_i. As mentioned earlier, retaining only the dominant term in each element of the appropriate eigenvector would lead to a null result for this wave.

Case 2: $T_e >> T_i,\ T_n;\ p_{eo} >> p_{no}$

In the frequency range $\nu_n << \omega << \omega_i$ both the "ion" and neutral waves can propagate and both are excited by a current source. The power in each is given respectively by:

$$p_{LL1} \cong \frac{A_o{}^2 \mu_o c^2 \omega^4}{12\pi \omega_i{}^2 U_p{}^3} \tag{7.23}$$

$$p_{LL3} \cong \frac{A_o{}^2 \omega^2 \omega_i{}^2 \mu_o c^2}{12\pi U_p{}^2 U_n} \left(\frac{\mu_{ne} \nu_n}{\omega^2} - \frac{\mu_{en} \nu_{in}}{\omega_i{}^2} \right)^2 \tag{7.24}$$

In the low frequency range ($\omega << \nu_n$) the ions and electrons become coupled to the neutrals and the character of the "ion" wave changes to a total acoustic wave. The associated power flow is given by:

$$P_{LL1} \cong \frac{A_o^2 \omega^4 \mu_o c^2}{12\pi \omega_i^2 U_p^2 U_T} \tag{7.25}$$

In this same frequency range the power in the neutral wave is given by:

$$P_{LL3} \cong \sqrt{\frac{\omega \nu_n}{2}} \; \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi \omega U_e^2 U_n} \left(\frac{\omega^2 \mu_{en} \nu_n^2}{\omega_i^4} \right) \tag{7.26}$$

Case 3: $T_e \gg T_i,\ T_n;\quad p_{no} \gg p_{eo}$

In the frequency range $\nu' \ll \omega \ll \omega_i$ both "ion" and neutral waves can propagate and both are excited by a current source. The result for power in the "ion" wave is identical to the equivalent result for Case 2, i.e., Equation (7.23). The power in the neutral wave is also given by Equation (7.24). For $\omega \ll \nu'$ however, the results are quite different than those of Case 2. Here the "ion" wave maintains its identity all of the way down to $\omega = 0$ and the neutral wave turns into the total acoustic wave. The power in the "ion" wave is given by:

$$P_{LL1} \cong \sqrt{\frac{\omega \nu''}{2}} \; \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi \omega U_e^2 U_p} \; \frac{\omega^2 \nu_{in}^2}{\omega_i^4} \tag{7.27}$$

and the total acoustic wave power is:

$$P_{LL3} \simeq \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi U_e^2 U_T} \left[\frac{\mu_{en}^3 \omega^2 \nu_{in}}{\omega_i^2 \nu_n}\right]^2 \tag{7.28}$$

Qualitative results for power radiated are depicted graphically in Figs. 1, 2, and 3 for Cases 1, 2, and 3 respectively. All curves are normalized relative to the asymptotic value of the power in the electron wave, i.e.

$$P_o = \frac{A_o^2 \omega_e^2 \mu_o c^2}{12\pi U_e^3} \tag{7.29}$$

The segments of curves are connected by dashed lines in the frequency ranges where calculations were not made. The curves for each wave are labeled according to the high frequency identity of the wave.

In all cases the ion wave appears to be the most effective in removing power from the source at high frequencies. For frequencies not too far above ω_e, the electron wave also carries away much more power than the transverse wave.

At lower frequencies ($\omega \ll \omega_1$) the electron acoustic and transverse waves cease to propagate. The ion wave turns into the "ion" or electron-ion acoustic wave which, as noted earlier, is very weakly coupled to a current source. (Recall that it was necessary to use an improved approximation to the modal matrix in order to obtain a non-zero result for this wave.) As can be seen from the curves the amplitude of this wave falls off very rapidly with decreasing frequency.

In Case 2 the plasma acoustic wave becomes the total acoustic wave at very low frequency, whereas in Case 3 the neutral wave becomes the total acoustic wave at low frequency. In both cases the wave power has the frequency dependence ω^4 which is characteristic of these weakly driven waves.

Another interesting aspect of the curves is the null result (to this level of approximation) for the neutral wave at one particular frequency. Collisions with electrons and ions always oppose each other in establishing this wave and apparently, at this frequency, their effects cancel.

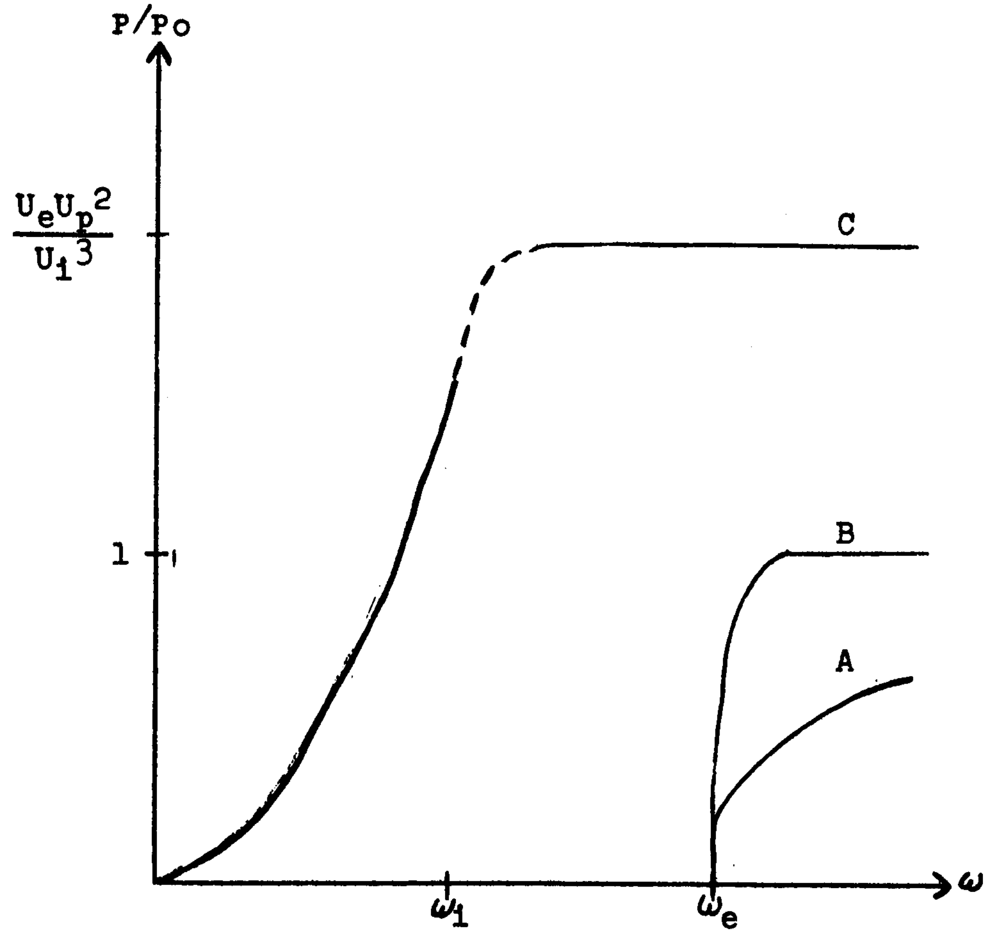

Fig. 1. Radiated Power for Case 1

A. Transverse wave
B. Electron wave
C. Ion wave

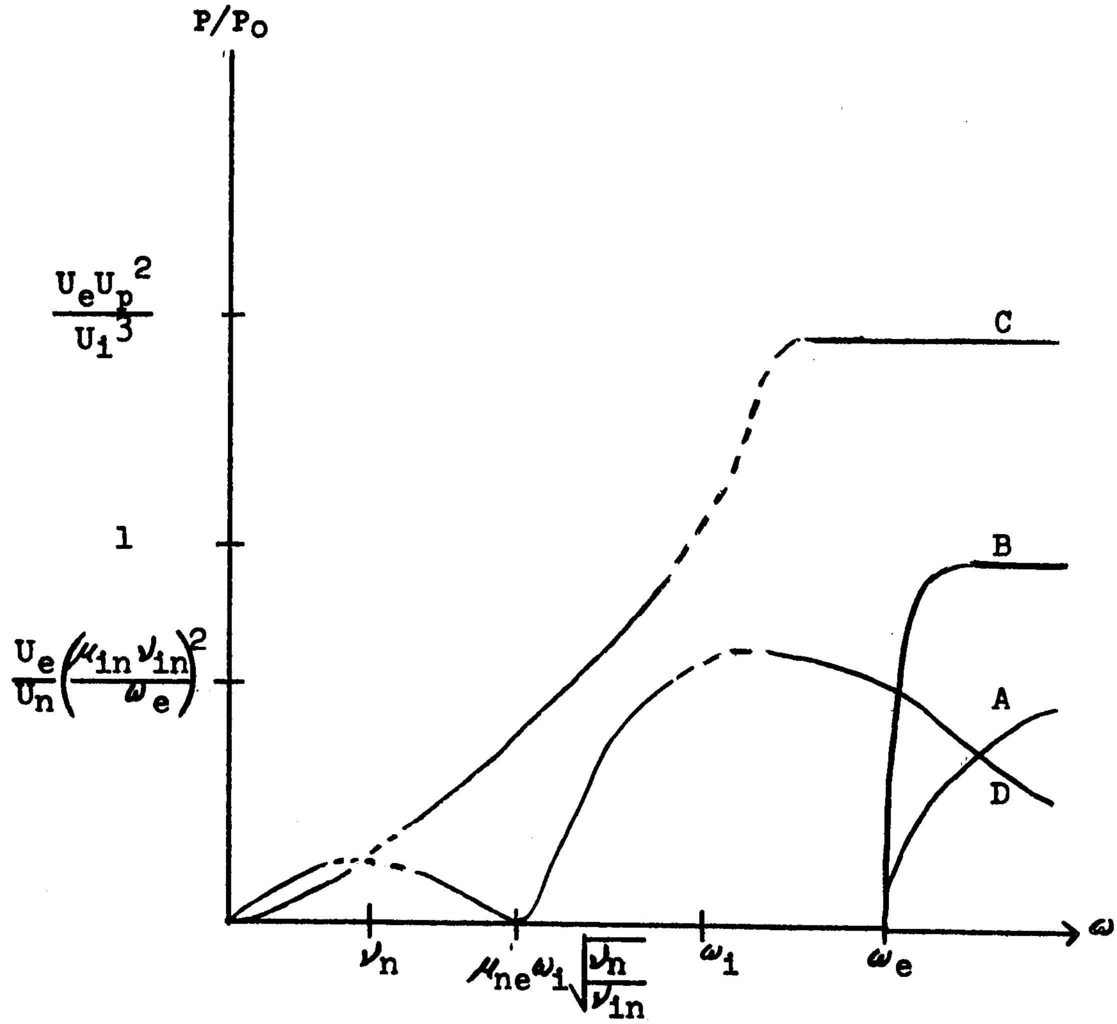

Fig. 2. Radiated Power for Case 2

A. Transverse wave
B. Electron wave
C. Ion wave
D. Neutral wave

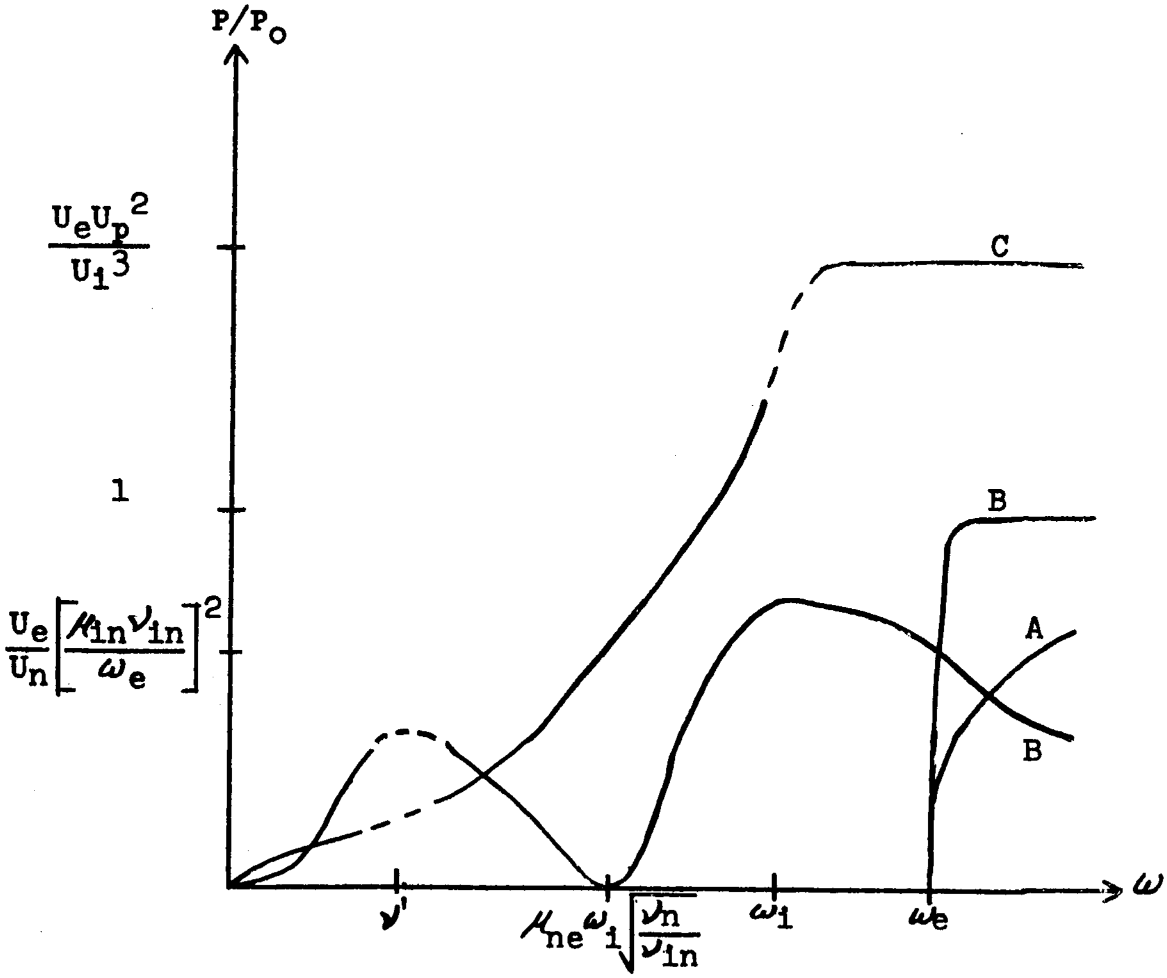

Fig. 3. Radiated Power for Case 3

A. Transverse wave
B. Electron wave
C. Ion wave
D. Neutral wave

CONCLUSION

In this thesis we have formulated a procedure for treating radiation from sources in a partly ionized gas. We have carried the method to its conclusion for the case of an elementary current source, calculating the power carried away from such a source by each wave and noting which waves can be strongly excited by a current source. The fact that certain waves cannot be strongly excited by a current source, irrespective of source geometry, is quite consistent with the accepted physical picture of these waves. All of the waves for which this fact was noted (the "ion" waves for $\omega \ll \omega_1$ and the total acoustic waves for $\omega \rightarrow 0$) involve collective, in phase, oscillations of either electrons and ions or electrons, ions, and neutrals. (Moreover, as the frequency decreases the phase coupling of electrons and ions gets better; hence the strong dependence on frequency.) Since a current source exerts oppositely directed forces on electrons and ions it should not be expected to generate such oscillations efficiently.

The relative magnitudes of power in our high frequency results for electron, ion, and transverse waves is probably unrealistic for most physical problems. The high proportion of power in the short wavelength ion waves is due to the assumed smallness of the source. The results of Ref. 6 for electron acoustic and transverse waves excited by the same type of source are identical to ours. It is further shown in Ref. 6, however, that as the source dimensions are increased beyond the wavelength of the electron wave, the proportion of power in this wave is decreased. As the size of the source approaches or exceeds the transverse wavelength the transverse wave dominates.

It is expected that increasing the source dimensions would effect our results in a like manner. Although no calculations have been made, it seems plausible that the ion wave would dominate only until the dimensions of the source approached the electron acoustic wavelength. The electron wave would then dominate until the source dimensions approached the transverse wavelength at which point the transverse wave would carry the most power.

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