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SIMILARITY STUDIES ON THE RADIATIVE

GAS DYNAMIC EQUATIONS

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by

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ABSTRACT

Spherically symmetric similarity solutions of the radiative gas dynamic equations are examined for problems of imploding and exploding shock waves. The system is reduced to similarity form by retaining the definitions of the radiant quantities as operators on the radiative intensity. Homology structure for the intensity is dictated by the governing equations. It is shown that stipulating a radiative transfer law constitutes a simple constraint on the system. Large classes of radiative transfer laws are compatible with a constant shock strength or with a limitingly strong shock. Because of radiative heating, similarity structure may be prescribed for the gas upstream as well as downstream of the shock wave. Constant shock strength is maintained by virtue of identical similarity homology in both regions.

The general Rankine-Hugoniot equations for an arbitrary radiative intensity are given. Initial conditions appropriate to self similar motion are given. It is shown that a sequential procedure for numerical solutions can be established. In the adiabatic problem, the solution for the velocity distribution is not contingent on the form of the radiative intensity. Computations are effected illustrating useful approximation schemes.

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INTRODUCTION

1.1 Perspective

Substantial interest in recent years has been focused on schemes for producing very high temperatures with the ultimate purpose of controlled fusion. As is well known, plasma heating may be accomplished by ohmic losses, magnetic compression and strong shock waves. Coexistent is the problem of containment. One contemporary solution employs high mirror ratio coils with open ended systems. There exists a genuine problem of confinement when attempting to attain initiating thermonuclear temperatures.

After the conclusion of World War II, in connection with the detonation of nuclear devices, shock waves from strong explosions received some attention. Symmetrical explosion and implosion studies were effected by Guderley,[1] Taylor[2] and Weizsacker.[3] The possibility of employing imploding waves presents itself as at least one method that should be investigated for producing high temperatures. Superficially, the method appears to obviate the confinement problem while heating occurs. In this spirit an investigation of a spherically symmetric implosion is not unwarranted. Extrapolation of similarity calculations for the neutral fluid dynamical equations tentatively indicate that the temperature which can be achieved in the region of convergence are limited only by a critical radius at which shock reflection might occur. This technique cannot rigorously substantiate the fluid property behavior at the convergence point.

Assuming the extrapolations are to some extent valid, the temperature ratio can be estimated by assuming a critical radius of the order of several mean free ion paths for a neutral mechanically driven fluid. The speculation follows that, if an imploding current sheet is preceeded by a region capable of supporting a magnetic field, the ion Larmor radius might replace the ion mean free path. The consequences are a diminished critical radius and higher temperature ratios. The feasibility of this approach is contingent on available energy loss mechanisms, a problem fundamental to all schemes proposed for controlled fusion. If an implosion proceeds into an ionized gas permeated by a magnetic field, minimally there will be energy loss due to bremstrahlung and cyclotron radiation.

In view of radiation shock smoothing, a critical radius of several ion mean free paths may totally lose significance. A more realistic characteristic dimension might be the radiation smoothed shock thickness. If this pessimism is substantiated the initial high temperature conjectures based on strong implosions, which do not take radiation into account, would be entirely spurious. Radiation trapping in the pre-shock core, should it become significant, would result in a net increase in the pressure and hence the reflection radius.

In order to discuss the influence of radiative effects on shock waves, the radiative gasdynamic equations may be used. It is expected that radiation effects of importance are incoherent and amenable to description via classical radiative transport theory. The nonlinearity of the fundamental system poses a formidable problem by virtue of a conspicuous lack of mature analytical mathematical techniques. Indeed, there exist but two broad areas of approach to the problem: the method of characteristics and that of self similarity. The radiative gas dynamic equations are hyperbolic, while the radiative transport equation is elliptic. Consequently the system, in total, cannot be analysed by the method of characteristics. This approach is still valuable if first a solution to the radiative transport problem or a constitutive hypothesis on the structure of the radiative intensity is made. It should be clear that the resulting equations except under specific assumptions will not form a reducible canonical system. With this formalism it would be possible to deduce the onset and development. The question of reflection remains uncertain. Nevertheless, this approach is mathematically accurate and it would be interesting to discover what minimal assumptions are required to make the problem tractable. The method of characteristics requires stipulating initial conditions which cannot be provided, as yet, satisfactorily by experiments.

The alternate approach of self similarity dispenses with the onset and development. Indeed, quite naturally, it requires the hypothesis that the motion be either totally or piecewise self similar. The formalism and consequences of a similarity approach lend themselves favorably to examination by experiment. It is frequently possible to deduce significant features of the dynamic problem without performing exorbitant computations. Similarity solutions are not generally applicable in arbitrarily small neighborhoods of the singular point of the transformation. Some heuristic extrapolation may be effected though conclusions based on such would be suspect. In past works there have been attempts to deal with one dimensional shock waves influenced by radiation. The approach revolves around stipulating a radiative transfer law, substituting it into the radiative gas dynamic equations and attempting to reduce the resultant equations to a self similar form. This has not been a completely successful approach. It has been necessary to introduce many approximations in the search for symmetric solutions. One of the primary objectives of this dissertation is to demonstrate a procedure enabling the reduction of the radiative gas dynamic system to self similar form for general radiative transfer laws. The complete solution of the feasibility of an implosion shock for controlled fusion is, of course, beyond the scope of a single dissertation. Consequently the attention is focused on developing the techniques of one possible approach, self similarity.

1.2 Literature Review

Guderley,[4] in a now classic paper, has examined strong spherical and cylindrical shock waves in the region of the convergence points. The starting point of his arguments are the ordinary fluid equations with an ideal gas law and isentropic energy equation. The shock waves are non-constant strength and limitingly strong (see page 60), propagating in a uniform region. The suggested justification of a similarity description in the regime of the convergence point revolves around an expansion of the shock position as a power series in time. The dominant term suggests the self similar form. When the equations are rendered in self similar form, the one parameter family of integral curves is examined to ascertain physically admissible solutions. Initial conditions issue from the similarity form of the limitingly strong shock jump relations. The divergent temperature and pressure ratios discussed in an experimental proposal by Winterberg[5] characterizes a particular solution from an infinity of solutions, which one reasonably would not expect to be valid in the presence of dissipation mechanisms such as radiation. Guderley makes no statements concerning the concept of critical radius. Only in a later paper by Sanger[6] is it suggested that, on the basis of chemical arguments, if a critical radius existed it would probably be of the order of several Debye lengths. Nevertheless, the optimistic point of view of Winterberg is based on neglecting the smoothing effect of radiation on the shock wave.

Weizsacker[7] concerns himself with obtaining numerical solutions to the similarity transformed ordinary differential equations. Using the hypothesis of a strong shock wave and fixing the shock trajectory and pre-shock initial conditions the jump equations establish starting post-shock conditions. Since the shock waves discussed are limitingly strong a sequence of solutions generated parametrically on (see page 22) are developed simultaneously with the appropriate parametric post-shock conditions.

Sponsered by the war effort in 1941, Taylor[8] has made calculations of a preliminary nature on the generation of very strong shock waves such as would be developed in high yield nuclear detonations. The computations were effected by the usual similarity approach. The validity of the solutions in retrospect was subject to some criticism. In addition to the constraint imposed by the limitingly strong shock condition, a particular γ is selected on the basis of an integral constraint imposed on the total energy. The applicability of such solutions pertains to areas in time such that the energy in the post-shock region greatly supersedes that in the pre-shock region. The energy equation contains no dissipative terms. Numerical computations are carried out on the transformed system and an approach for analytical approximation to this specific problem is suggested.

Pai and Speth[9] have derived a form of the Rankine-Hugoniot relations for a planar shock wave using asymptotic uniformity of the flow field. Their work is principally concerned with high temperature effects where a significant percentage of the gas is ionized and in which a magnetic field can exist. The radiation field is characterized by a Planckian distribution. The jump equation with radiative effects reflects this specialized assumption. An energy equation is suggested appropriate for the discussion of shock structure. Explicit cognizance of the work of Sen and Guess[10] is made in suggesting that a broadening of the shock thickness accompanies the inclusion of radiation.

The paper by J. Clarke[11] is concerned with calculations of a "radiation resisted" planar one dimensional shock wave and associated shock structure problem. The energy equation employed in this work structurally resembles that suggested by Speth and Pai in view of having omitted radiant energy and pressure by virtue of a dominant radiant flux. Refraining from arguing the relative importance of the characteristic diffusion lengths for radiation and heat conductivity, the discussion is concerned with the effect of the flux dissipation contribution in the energy equation. Both loss mechanisms are discussed by Sen and Guess. In all of these works pertaining to shock wave problems which intrinsically are strongly non equilibrium, the underlying assumption for the radiative field is equilibrium.

Marshak[12] for a one dimensional planar geometry has considered similarity solutions to the radiative gas dynamic equations. In this work again is the hypothesis of radiative equilibrium. The flux term in the energy equation is written to suggest a diffusion interpretation. The jump equations in essentially a blackbody gas are derived. No method is presented to dispense with equations in totality. The conjecture is made that what is common to a sequence of approximate similarity equations reflects the properties of more general non-self similar motion.

1.3 Definitions of Radiative Transport Quantities

The basic properties of the radiative transport phenomenon can be established from a scalar energy function. The method of formulation to some extent reflects an inability to cope with the many body problem. The techniques depend on global quantities which do not take explicit account of the microscopic structure of the radiation field. With a corpuscular perspective, the transport equation can be interpreted as deducible from a Boltzmann equation for photons. From a wave point of view it suggests a description of energy transport via a short time averaged Poynting vector. Consequently the classical formulation possesses the virtue of acquiescence to either the wave or particle interpretation.

The radiative intensity is a scalar point function of fixed coordinates and a vector direction defined as the rate of energy transport per unit frequency interval per area per solid angle per unit time:

(1) $$I_\nu \, d\nu \, d\omega \, (\vec{\Delta\sigma} \cdot \vec{L}) \, dt = d^6 E_\nu$$

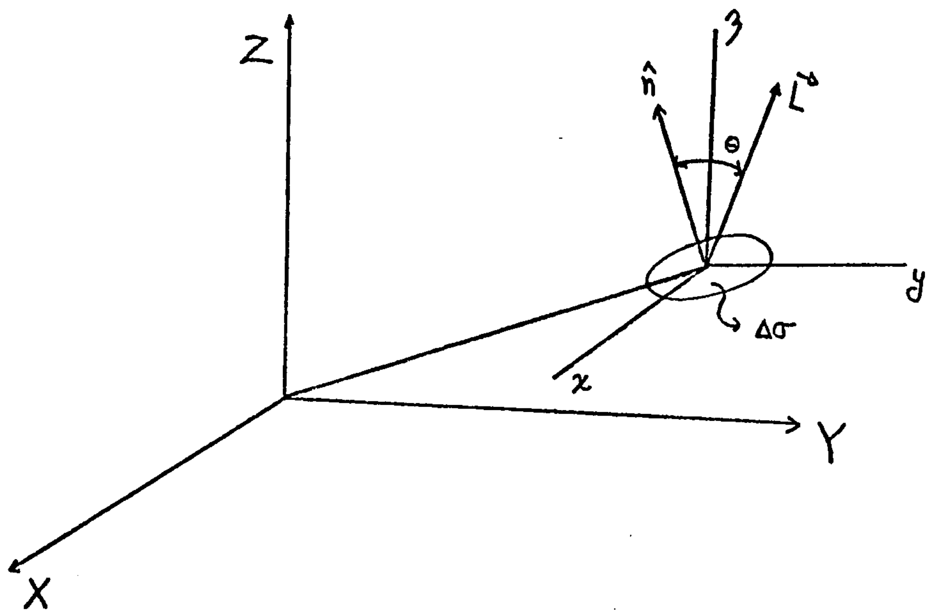


Figure 1. Diagram of Coordinates

The quantity I_ν is a local property of the medium and is an invariant under coordinate transformations. The total intensity is given:

(2) $$I = \int_0^\infty I_\nu \, d\nu$$

The transport of flux is defined:

$$(3) \qquad dF^R = \frac{I_\nu (\vec{\Delta\sigma} \cdot \vec{L})\, d\omega\, d\nu}{|\Delta\sigma|}$$

$$(4) \qquad dF^R = I_\nu \, l_i \, d\omega\, d\nu$$

Equation (4) is the differential flux vector. The quantities are the direction cosines associated with $\bar{L}$. The energy density is defined:

$$(5) \qquad dU^R = \frac{I_\nu}{c}\, d\nu\, d\omega$$

The momentum transfer associated with the radiation suggests constructing a radiation pressure tensor:

$$(6) \qquad dP_{ij} = l_j \frac{dF_i^R}{c} = I_\nu \, l_i \, l_j \, d\omega\, d\nu$$

The normal pressure stress on a surface is:

$$dP = I_\nu \cos^2\theta \, d\omega\, d\nu$$

Two additional quantities require definition: the source function and the absorption .

(7) $$d^7E_e = j_\nu \rho \, d\nu \, d\omega \, dt \left(\Delta\vec{\sigma} \cdot \frac{\vec{L}}{|L|}\right) c \, dt$$

(8) $$d^7E_a = A_\nu \rho \, d\nu \, d\omega \, dt \left(\Delta\vec{\sigma} \cdot \frac{\vec{L}}{|L|}\right) c \, dt$$

The structural qualities of the radiation field and the medium are embedded in the absorption and emission functions.. For a radiant gas the absorption may be written:

(9) $$A_\nu = \beta_\nu I_\nu$$

β_ν is defined to be the mass absorption coefficient. The emission is composed of two parts: scattering into a given direction from other directions $\mathcal{J}_\nu$, and true emission j^e_ν.

(10) $$j_\nu = \mathcal{J}_\nu + j^e_\nu$$

Similarly:

(11) $$\beta_\nu = \sigma_\nu + \kappa_\nu$$

where σ_ν is the loss from a given direction due to scattering in other directions and κ_ν is the true absorption. The radiative transfer equation is a rate equation for the energy transport.

(12) $$\frac{1}{c}\frac{D}{D\tau} I_\nu = \rho\, j_\nu - \rho\, \beta_\nu I_\nu$$

where:

(13) $$\frac{D}{D\tau} = \left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial \vec{s}} \right)$$

For time independent problems;

(14) $$\frac{\partial I_\nu}{\partial \vec{s}} = \rho\, j_\nu - \rho\, \beta_\nu I_\nu$$

if

(15) $$\frac{D}{D\tau} I_\nu \equiv 0$$

then

(16) $$\frac{j_\nu}{\beta_\nu} = I_\nu$$

With isotropy cancellation of scattering and no time dependence it follows:

(17) $$l_i \frac{\partial I_\nu}{\partial s} = -\rho \beta_\nu I_\nu l_i + j_\nu^e + J_\nu$$

(18) $$\int l_i l_j \frac{\partial I_\nu}{\partial r} d\nu d\omega = -\rho \int \kappa_\nu I_\nu l_i d\nu d\omega = -\rho \langle \kappa_\nu \rangle F^R$$

(19) $$\frac{\partial p^R}{\partial r} = -\rho \langle \kappa_\nu \rangle F^R \; ; \; \langle \kappa_\nu \rangle = \kappa$$

κ is a mean absorption.

A less heuristic, more accurate discussion of the fundamental concepts of radiative transport phenomenon may be found in the summary work of Goulard.[13]

1.4 The Birkhoff Search for Symmetric Solutions

A well known procedure for the reduction of a system of partial differential equations to ordinary differential equations is herein briefly paraphrased. The use of this procedure in the problems discussed is not necessary. The motivation for including it resides in its facility to cope with systems of equations not as well studied as those of fluid dynamics.

Let Σ be a set of differential equations with x_i and ξ_i the corresponding independent and dependent set of variables respectively. Define a transformation Γ as follows:

(20.1) $\bar{x}_r = a^{\alpha_r} x_r$

(20.2) $\bar{y}_j = a^{\gamma_j} y_j$

(20.3) $\Gamma(\Sigma) = \bar{\Sigma}$

The system Σ under Γ is required to be absolutely conformally invariant.

(21) $\Gamma(\Sigma) \equiv \Sigma$

A set of invariants are then constructed:

(22.1) $\eta^r = x_r / x_1^{\alpha_r/\alpha_1}$

(22.2) $f(\eta^r) = y_j / x_1^{\gamma_j/\alpha_1}$

Algebraic relations exist between the constants α_r and γ_j by virtue of requiring absolute conformal invariance. A rigorous and valuable contemporary discussion of self similarity and the Birkhoff search for symmetric solutions may be found in a book by Ames.[14]

1.5 Resume of Content

In Chapter II, the method of reduction and resulting fundamental equations are derived. The Rankine-Hugoniot equations are written for arbitrary radiative expressions. The significance of two sided similarity and proper initial conditions are given. Chapter III developes the implications of constraint conditions, limitingly small radiation and strong shock waves. Self similar motions are discussed. Chapter IV encompasses numerical and analytical approaches to the equations and Chapter V conjectures a variation of the fundamental symmetric search process. Concluding comments are pointed out in Chapter VI.

CHAPTER II

FUNDAMENTAL EQUATIONS

2.1 Initial Discussion

In general the distinction between radiant and non-radiant gas dynamics is the addition of a radiation pressure to the momentum equation and radiation energy and flux to the energy equation. Thus, the total energy is due to internal energy and radiant energy. The work done by pressure forces includes that done by radiation pressure. The flux enters as a dissipation which parallels the heat conduction in structure. The energy equation is a simple consequence of the first law of thermodynamics. That the entropy is not constant on stremalines is the effective manifestation of the dissipation mechanisms. Momentum follows from Newton's Law. The equations are as follows:[15]

(23) $$\frac{D\rho}{D\tau} + \rho\, u_{i,i} \qquad \text{Continuity}$$

(24) $$\rho \frac{Du_i}{D\tau} = \rho\, u_{i,i} + \rho\, u_j u_{i,j} = \rho X_i - P_{ij,i} - P^R_{ij,i} \qquad \text{Momentum}$$

(25) $$\rho \frac{D}{D\tau}\left(E + U^R/\rho\right) = -q_{i,i} - F^R_{i,i} - P_{ij}u_{i,j} - P^R_{ij}u_{i,j} \qquad \text{Energy}$$

(26) $$P = P(\rho, T)$$ Equation of State

(27) $$E = c_v T$$ Polytropic Assumption

(28) $$q_i = \lambda T_{,i}$$ Heat Conduction

The equation of state principally to be employed is the perfect gas law. This is employed for convenience in calculations and does not constitute a necessary requirement. A polytropic gas is by definition one in which the internal energy is proportional to the temperature. Hence the specific heat at constant volume is a constant. The law for the conduction of heat is not a primitive but empirical.

In addition to these, expression of the thermodynamic dependence of the radiative functions must be provided to complete the system. The radiant quantities are defined in terms of a single quantity, the radiative intensity:

(29) $$U^R = \int_0^\infty U_\nu^R \, d\nu \quad ; \quad U_\nu^R = \int_{4\pi} \frac{I_\nu}{c} \, d\omega$$

(30) $$F_i^R = \int_0^\infty F_{\nu_i}^R \, d\nu \quad ; \quad F_{\nu_i}^R = \int_{4\pi} I_\nu \ell_i \, d\omega$$

(31) $$P_{ij}^R = \int_0^\infty P_{\nu_{ij}} \, d\nu \quad ; \quad P_{\nu_{ij}}^R = \int_{4\pi} \frac{I_\nu}{c} \ell_i \ell_j \, d\omega$$

The ν integration is over frequency and ω over solid angle. The symbols ℓ_i and ℓ_j are direction cosines of $\vec{L}$ at the tip of an $\vec{r}$. U^R is the radiant energy, F^R the flux and P^R the pressure.

The thermodynamic form taken by I_ν reflects the internal state of the radiant media. The differential equation governing the rate process for radiative transport is the classical radiative transfer equation. The average microscopic state of the system is depicted through the lumped parameters describing the transport process. I_ν then represents a microscopic mean behavior.

$$(32) \qquad \frac{D}{Dt} I_\nu = \frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = -\rho \beta_\nu I_\nu + \rho j_\nu$$

$$(33) \qquad \frac{\partial}{\partial s} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu}$$

where $\qquad \mu = \cos\theta$

This equation does not couple into the radiative gas dynamic system since it in no way depends on the fluid velocity. Consequently from the radiative equation I_ν is ascertained as a function of space, time and thermodynamic variables. In this respect the view is adopted that this equation is a constitutive equation for the radiant medium. Throughout the course of this work the radiative transfer equation is never solved for particular structural assumption of emissivity and absorbtivity. It is one of the primary objectives of this dissertation to demonstrate a method by which general solutions for I_ν can be used to describe radiant transport properties consistent with a total reduction to self similarity of the radiative gas dynamic equations. For illustrative purposes structural forms of I_ν are assumed which are not meant to represent carefully constructed solutions to the radiative transfer equation, but serve to demonstrate the fact that by adopting the subsequent procedure these general forms are compatible. In all subsequent considerations these equations are analyzed in a spherically symmetric form. Shear stresses in the fluid and radiation pressure tensors are dropped from the equations. These terms can be included and the system reduced to self similar form provided appropriate similarity statements concerning the viscous coefficients are made. The heat conduction term with some contradiction is included to illustrate it requires one constraint. The coefficient of heat conduction is assumed constant. This term is never employed in the computations and serious considerations of it are not appropriate. The problem of reducing the multidimensional system to self similarity has not been investigated.

To effect total reduction to self similarity it is required to introduce the radiative quantities as operators on the radiative intensity. Designate:

(34) $$L^{0} = \int_{0}^{\infty}\int_{4\pi} (\qquad) \, d\nu \, d\omega$$

(35) $$L^{1} = \int_{0}^{\infty}\int_{4\pi} (\qquad) \cos\theta \, d\nu \, d\omega$$

(36) $$L^{2} = \int_{0}^{\infty}\int_{4\pi} (\qquad) \cos^{2}\theta \, d\nu \, d\omega$$

so that

(37) $$U^{R} = L^{0} I_{\nu}/c$$

(38) $$F^{R} = L^{1} I_{\nu}$$

(39) $$P^{R} = L^{2} I_{\nu}/c$$

It is obviously true that:

(40) $$\left[L^{(j)}, \frac{\partial}{\partial r} \right] = \left[L^{(j)}, \frac{\partial}{\partial t} \right] = 0$$

The space and time derivatives commute with the $L^{(j)}$ operators.

With these statements the fundamental equations to be considered are:

$$(41) \qquad \frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial}{\partial r}\left(P + L^{2} I_{\nu}/c\right) = 0 \qquad \text{Momentum.}$$

$$(42) \qquad \frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0 \qquad \text{Continuity.}$$

$$(43) \qquad \rho\frac{D}{Dt}\left(E + \frac{L^{0} I_{\nu}}{c\rho}\right) = -\left(P + L^{2}\frac{I_{\nu}}{c}\right)\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \frac{\partial}{\partial r}L^{1} I_{\nu} - 2\frac{L^{1} I_{\nu}}{r} - \lambda\frac{\partial^{2} T}{\partial r^{2}} - \frac{\lambda}{r}\frac{\partial T}{\partial r} \qquad \text{Energy}$$

In the formal language of Birkoff, the initial set is required to be conformally invariant under a one parameter group of transformations of the independent and dependent variables. With the elimination of one independent variable a set of invariants for the system is obtained together with the associated differential equations. It is not necessary to follow this formalism in equations so well studied as the fluid dynamical system. When one homology is stipulated the structure of the equations determines uniquely the homology of the remaining variables (see Appendix). However, when an attempt at a multidimensional reduction to self similarity is made, it is virtually mandatory to follow the Birkoff "search for symmetric solutions".

2.2 Reduction to Self Similarity

The independent similarity invariant is η defined:

$$\eta = r/\lambda \tau^{\delta} \tag{44}$$

λ is a scaling constant and δ is a fundamental constant governing the time evolution of the system.(16) In the non-self similar case $\delta = \delta(\tau)$. It follows:

$$\frac{\partial \eta}{\partial \tau} = -\frac{\delta}{\tau}\eta \;;\; \frac{\partial \eta}{\partial r} = \frac{\eta}{r} \tag{45}$$

The similarity forms to be associated with the fluid variables are:

$$\rho = r^{k} R(\eta) \tag{46}$$

$$T = r^{2}\tau^{-2} T(\eta) \tag{47}$$

$$u = r\tau^{-1} \mu(\eta) \tag{48}$$

where $R(\eta)$, $T(\eta)$, and $u(\eta)$ are the dependent similarity invariants associated with density, temperature and velocity. Now postulate the form of the radiant intensity:

$$(49) \qquad I_\nu = r^m \tau^p I_\nu(\eta)$$

The constants and are to be ascertained from the structure of the equations. To make provision for the possible variation of the index of refraction set:

$$(50) \qquad c = r\tau^{-1} \varsigma(\eta)$$

$$(51) \qquad P = r^{k+2}\tau^{-2} P(\eta)$$

This form is dictated by the momentum equation. The following algebraic manipulations are required to obtain the similarity forms for the differential system. For the energy equation:

$$(52) \qquad a r^k R \frac{D}{D\tau}\left(c_v r^2 \tau^{-2} T + \frac{L^0 \beta r^m \tau^p I_\nu}{a r^k R\, r\tau^{-1} \varsigma(\eta)} \right) + L^1 \frac{\partial}{\partial r} \beta r^m \tau^p I_\nu$$

$$= -2 L^1 \beta r^{m-1} \tau^p I_\nu - \left(a r^{k+2} \tau^{-2} P + \frac{L^2 r^m \tau^p I_\nu}{r\tau^{-1} \varsigma(\eta)} \right)\left(\frac{\partial}{\partial r} r\tau^{-1} u + \frac{2u}{\tau} \right)$$

The L. H. S. of the energy equation is expanded in the following manner: the first term is

$$c_v\left(-2r^2\tau^{-3}T - r^2\tau^{-3}T'\delta\eta\right) + \tag{53}$$

$$c_v\left(r\tau^{-1}U\right)\left(2r\tau^{-2}T + r\tau^{-2}T'\eta\right)$$

the second term

$$\frac{D}{D\tau}\frac{\beta}{a}\left(r^{m-(k+1)}\tau^{p+1}\frac{I_\nu}{R\varsigma}\right) = \left\{\left(\frac{r^{m-(k+1)}\tau^{p}I_\nu}{R\varsigma}\right)\left((p+1)+U(m-(k+1))\right)\right\} \tag{54}$$

$$+\left\{r^{m-(k+1)}\tau^{p}\eta\left[I_\nu' R^{-1}\varsigma^{-1} - I_\nu R' R^{-2} c^{-1} + I_\nu R^{-1} c^{2} c'\right]\left[U-\delta\right]\right\}$$

Finally

$$L.H.S. = a r^k R r^2 \tau^{-3} c_v\left[\eta T'(U-\delta) + 2T(U-1)\right] + \tag{55}$$

$$L^\circ \beta r^k R r^{m-(k+1)}\tau^p\Big[\left((p+1)+U(m-(k+1))\right)I_\nu R^{-1}c^{-1}$$

$$+\eta(U-\delta)\left(I_\nu' R^{-1}\varsigma^{-1} - I_\nu R^{-1} R^{-2}\varsigma^{-1} - I_\nu R^{-1}c^{-2}c'\right)\Big]$$

This result suggests:

$$(56) \qquad m-(k+1)=\lambda \;;\; p=-3$$

Therefore

$$(57) \qquad I_\nu = \beta r^{k+3} \tau^{-3} I_\nu(\eta)$$

The R. H. S. is established.

The first term is

$$(58) \qquad -L' \frac{\partial}{\partial r} \beta r^{k+3} \tau^{-3} I_\nu = -L'\beta \left\{ (k+3) r^{k+2} \tau^{-2} I_\nu + r^{k+2} \tau^{-3} I_\nu' \eta \right\}$$

The second term:

$$(59) \qquad r^{k+2} \tau^{-3} \left\{ a P + \beta \frac{L^2 I_\nu}{\varsigma} \right\} \left\{ \lambda U + \eta U' \right\}$$

The transformed energy equation is:

$$(60) \quad \left\{ \begin{aligned} & a R c_v \left[\eta T'(U-\delta) + \lambda T(U-1) \right] + \beta R \left[\lambda (U-1) \mathring{L} I_\nu R^{-1} \varsigma^{-1} \right. \\ & \left. + \eta (U-\delta) \left(\mathring{L} I_\nu' R^{-1} \varsigma^{-1} - I_\nu R' R^{-2} \varsigma^{-1} - I_\nu R^{-1} \varsigma^{-2} \varsigma' \right) \right] a \\ & = L' \beta \left[(k+5) I_\nu + \eta I_\nu' \right] - \left[a P + L^2 \frac{\beta I_\nu}{\varsigma} \right] \left[3U + \eta U' \right] \end{aligned} \right\}$$

Similarly for the Momentum equation:

$$(61) \quad \frac{\partial}{\partial \tau}(r\tau^{-1}U) + \frac{r}{\tau}U\frac{\partial}{\partial r}(r\tau^{-1}U) + \frac{1}{ar^kR}\left\{\frac{\partial}{\partial r}ar^{k+2}\tau^{-2}P + L^2\frac{\partial}{\partial r}I_\nu/c\right\}$$

Expanding this yields:

$$(62) \quad \left\{\eta U'(U-\delta) + U^2 - U + \frac{1}{R}\left[(k+2)P + \eta P'\right]\right.$$
$$\left. + \frac{L^2\beta}{aR}\left[(k+2)I_\nu\varsigma^{-1} + (I_\nu'\varsigma' - I_\nu c^{-2}c')\eta\right]\right\} = 0$$

For the continuity equation:

$$(63) \quad \frac{\partial}{\partial \tau}(ar^kR) + r\tau^{-1}U\frac{\partial}{\partial r}(ar^kR) + ar^kR\left[\frac{\partial}{\partial r}\left(r\tau^{-1}U + \frac{2U}{r}\right)\right] = 0$$

expanding

$$(64) \quad -\frac{\delta}{\tau}ar^kR'\eta + r\tau^{-1}U\left(akr^{k-1}R + ar^{k-1}R'\eta\right) = 0$$

rearranging

$$(65) \quad -\delta R'\eta + U(kR + \eta R') + R(3U + \eta U') = 0$$

The transformation of the heat conduction follows:

(66) $$q_r = -\lambda \frac{\partial T}{\partial r} \; ; \quad q_{i,i} = \lambda \frac{\partial^2 T}{\partial r^2} = -\lambda \frac{\partial^2}{\partial r^2} r^2 \tau^{-2} T$$

(67) $$\frac{\partial q_r}{\partial r} + \frac{2 q_r}{r} \Rightarrow -2\lambda t^{-2} \left[5\eta T' + 3T(\eta) + T'' \eta^2 \right]$$

Hence τ^{-2} must cancel $r^{K+2} \tau^{-3}$ or

(68) $$\tau^{-3} r^{K+2} \tau^2 = \eta^{\alpha}$$

and

(69) $$\delta = -1/_{K+2}$$

To summarize, the non-dimensional transformed equations used in the course of this work are:

Energy:

(70) $$\alpha R c_v \left[\eta T'(U-\delta) + 2T(U-1) \right] + \alpha\beta R \left[2(U-1) L^0 I_\nu R^{-1} C^{-1} + L^0 \eta (U-\delta) \left[I_\nu' R^{-1} C^{-1} \right.\right.$$
$$\left.\left. - I_\nu R' R^{-2} C^{-1} - I_\nu R^{-1} C^{-2} C' \right] \right] = -L'\beta \left[(k+5) I_\nu + \eta I_\nu' \right] - \left[\alpha P + \beta \frac{L^2}{C} I_\nu \right] \left[3U + \eta U' \right]$$

Momentum:

$$\eta U'[U-\delta] + U^2 - U + \frac{1}{R}\left[(k+2)P + \eta P'\right] + \frac{L^2\beta}{aR}\Big[(k+2)I_\nu C^{-1} \tag{71}$$

$$+ (I'_\nu C^{-1} - I_\nu C^{-2} C')\eta\Big] = 0$$

Continuity:

$$\eta R'[U-\delta] + (k+2)RU + \eta R U' = 0 \tag{72}$$

2.3 Shock Velocity

Initial observations suggest two possibilities for the shock path. Only one is correct, compatible with self similarity and the jump equations. Consider first $r = r(\tau)$ where $r(\tau) = \lambda\tau^\delta$ with $\eta = \bar{\eta}$ is some fixed value of η. In this case the shock path is described as a line of constant η and the shock velocity is:

$$\dot{r} = \delta\lambda\bar{\eta}\,\tau^{\delta-1} \tag{73}$$

In the second case set $r = r(\eta)$ then:

$$\frac{dr}{d\eta}\frac{d\eta}{d\tau} = \frac{dr}{d\tau} = -\delta\,\frac{r}{\tau}\,\frac{dr}{d\eta} \tag{74}$$

In order that there exist compatibility with an incoming shock wave the shock velocity is required to be negative. If $\delta > 0$ then either r or τ must be negative. The shock path is correctly described by equation (73). The reasoning follows from the diagrams.

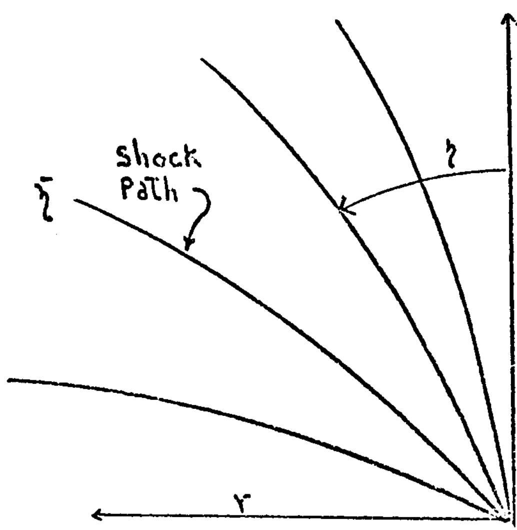


Figure 2 Correct Shock Path

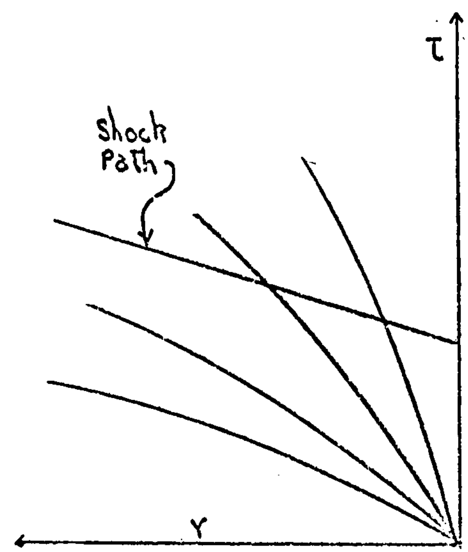


Figure 3 Incorrect Shock Path

In Figure (2) the shock lies on a path of constant $\eta = \bar{\eta}$. The region $\eta < \bar{\eta}$ in an implosion corresponds to the pre-shock region. In Figure (3) the shock does not lie on a line of constant η. Since any property which is a function of η is constant on lines of constant η then the path r = R(n) cannot divide a pre and post-shock wave region by virtue of the fact that it is intersected by lines of constant η.

2.4 Two Sided Similarity

Traditionally, in discussions of strong implosions or explosions, the pre-shock gas was stipulated to be uniform in concordance with physical structure. Shock waves in a radiant media are consistent with pre-shock uniformity when the constitutive behavior of the medium is such that the shock wave can be considered completely opaque. It will be shown in a later section that many functional forms of I_ν are compatible with the special constraints imposed on a constant finite strength shock wave. In problems dealing with a pre-shock nonuniform region it is reasonable to introduce similarity for both regions. This approach especially makes sense in the case of propagating nonopaque shock wave in a radiant gas. Energy transfer across the shock produces a nonuniform region. The jump equations impose no constraints for two sided similarity. All finite strength shocks are constant in strength. The three criteria for shock strength (see page 59) are compatible. To characterize a shock wave with two-sided similarity the homology of the dependent invariants are by mandate identical in both regions. The shock manifests itself as a discontinuity in the dependent invariants. In characterizing a shock in this way it is not necessary that all fluid variables be discontinuous. Thus the pressure and density may be discontinuous while the temperature is continuous compatible with the gas law and the strength measures of excess pressure and condensation. The excess pressure in the case of continuous temperature and for an ideal gas is precisely the condensation. The temperature can reasonably be discontinuous in the case of an opaque shock. That the usefulness of two-sided similarity discussions for non-opaque shock waves is not purely academic is easily recognized when it is remembered that in nuclear detonations a heat wave precedes the shock wave.

For definiteness, imagine that the pre and post-shock pressure distribution has been obtained by integrating into these regions from initial conditions given at the shock $\eta = \bar{\eta}$ compatible with the Rankine-Hugoniot equations in self similar form. Consider Figure (4).

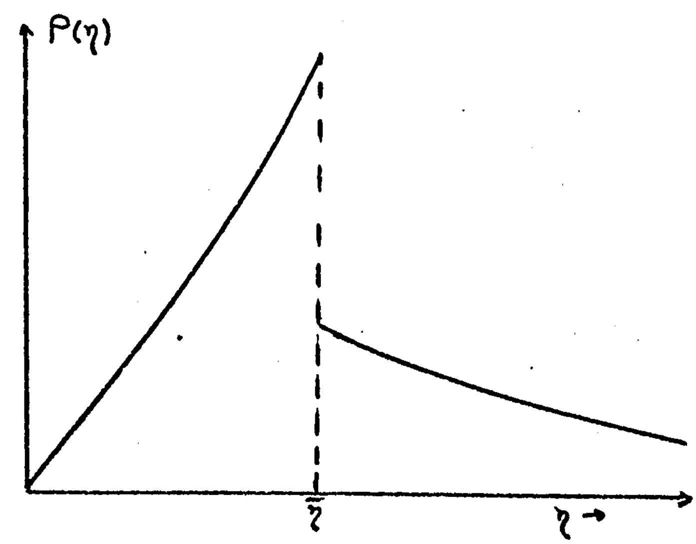


Figure 4 Discontinuous Dependent Invariant

Suppose:

(75) $$P(\eta) = \begin{cases} \alpha_1 \eta^2 & \eta \leq \bar{\eta} \\ \exp \alpha_2 \eta & \eta \geq \bar{\eta} \end{cases}$$

then

(76) $$\left[P(\eta)\right] = \left(\alpha_1 \eta^2 - \exp - \alpha_2 \eta\right)_{\eta = \bar{\eta}}$$

(77) $$\left[P\right] = r^{K+2} \tau^{-2} \left[P(\eta)\right]_{\eta = \bar{\eta}}$$

where r and τ are the time and position of the shock wave.

2.5 Rankine-Hugoniot Equations

The general jump equations with arbitrary radiative terms are obtainable by a process not extremely dissimilar from that used to derive the usual mechanical shock conditions. Introduce the coordinate system:

(78) $$\mathcal{R} = r - \int^{\tau} \mathcal{J} \, d\tau$$

(79) $$\left.\frac{\partial}{\partial r}\right)_{\tau} = \left.\frac{\partial}{\partial \mathcal{R}}\right)_{\tau} \; ; \; \left.\frac{\partial}{\partial \tau}\right)_{r} = \left.\frac{\partial}{\partial \tau}\right)_{\mathcal{R}} - \mathcal{J} \left.\frac{\partial}{\partial \mathcal{R}}\right)_{\tau}$$

$\mathcal{J}$ is the shock speed and $\mathcal{R}$ is the new spatial coordinate. In this frame of reference the differential conservation relations are:

Momentum equation

$$(80)\qquad \left.\frac{\partial}{\partial \tau}\right)_{\mathcal{R}} u - J \left.\frac{\partial u}{\partial \mathcal{R}}\right)_{\tau} + u \left.\frac{\partial u}{\partial \mathcal{R}}\right)_{\tau} + \frac{1}{\rho}\left(\frac{\partial P_T}{\partial \mathcal{R}}\right) = 0$$

Continuity equation

$$(81)\qquad \left.\frac{\partial \rho}{\partial \tau}\right)_{\mathcal{R}} - J \left.\frac{\partial \rho}{\partial \mathcal{R}}\right)_{\tau} + u \left.\frac{\partial \rho}{\partial \mathcal{R}}\right)_{\tau} + \rho \left.\frac{\partial u}{\partial \mathcal{R}}\right)_{\tau} + \frac{2u\rho}{\left[\mathcal{R} + \int J\, d\tau\right]} = 0$$

Energy equation

$$(82)\qquad \rho\left[\left.\frac{\partial}{\partial \tau}\right)_{\mathcal{R}} + (u - J)\left.\frac{\partial}{\partial \mathcal{R}}\right)_{\tau}\right] E_T = -\frac{\partial F^R}{\partial \mathcal{R}} - \frac{2F^R}{\left[\mathcal{R} + \int J\, d\tau\right]}$$

$$- P_T\left(\frac{\partial u}{\partial \mathcal{R}} + \frac{2u}{\left[\mathcal{R} + \int J\, d\tau\right]}\right) - \lambda \frac{\partial^2 T}{\partial \mathcal{R}^2} - \frac{\lambda 2}{\left[\mathcal{R} + \int J\, d\tau\right]} \frac{\partial T}{\partial r}$$

These differential relations are integrated over a control length of 2ϵ and the limit taken as $\epsilon \to 0$. All quantities that are not differentiated with respect $\mathcal{R}$ vanish in this process. This is a simple consequence of the following:

$$(83)\qquad I = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \Gamma \frac{\partial g}{\partial \tau}\, d\mathcal{R} = 0$$

(84) $$I = \left(\lim_{\epsilon \to 0} \epsilon\right)\left(\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \Gamma \frac{\partial g}{\partial \tau} \, d\eta\right) = 0$$

(85) $$I = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \Gamma \, d\eta = 0$$

Γ is arbitrary and the integrands have a finite number of finite discontinuities. Hence for the continuity equation:

(86) $$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{\partial}{\partial \eta} \rho (v - \mathcal{T}) \, d\eta = 0$$

Performing these oeprations similarly for the momentum and energy equations results in the system:

(87) $$\rho_1 (v_1 - \mathcal{T}) = \rho_0 (v_0 - \mathcal{T})$$ Continuity

(88) $$\rho_1 (v_1 - \mathcal{T}) \left[E_{T_1} - E_{T_0}\right] + F_1^R - F_0^R + P_{T_1} u_1 - P_{T_2} u_2 = 0$$ Energy

(89) $$\rho_1(u_1 - \mathcal{J})(u_1 - u_0) + P_{T_0} - P_{T_1} = 0$$

(90.1) $$P_T = (P^R + P)$$

(90.2) $$E_T = \left(c_v T + U^R/\rho + \frac{1}{2} u^2\right)$$

The energy equation used in deriving the latter relation was modified by multiplying the momentum equation by u and adding it to the original energy equation.

The similarity form of the jump equations is immediately derivable by substitution in the latter system. In the case of two-sided similarity these relations trivially impose no constraints. With $\mathcal{J} = \delta\, r/\tau$:

(91) $$\left\{ r^k R_1 \left[r/\tau (u_1 - \delta) \right] = r^k R_0 \left[r/\tau (u_0 - \delta) \right] \right\}$$

(92) $$\left\{ r^k R_1 \left[r/\tau (u_1 - \delta) \right] \left[r/\tau (u_1 - u_0) \right] + r^{k+2} \tau^{-2} \left[P_0 - P_1 \right] \right\} = 0$$

(93) $$\left\{ r^k R_1 \left[r/\tau (u_1 - \delta) \right] \left[E_{T_1} - E_{T_0} \right] r^2 \tau^{-2} + \left(F_1^R - F_0^R \right) r^{k+3} \tau^{-3} + \left(P_1 u_1 - P_0 u_0 \right) r^{k+3} \tau^{-3} \right\} = 0$$

The self similar forms of the jump equations are:

(94) $$\left\{ R_1[U_1-\delta] = R_0[U_0-\delta] \right\}$$

(95) $$\left\{ R_1[U_1-\delta]\left[C_v(T_1-T_0) + L^0\left(\frac{I_{\nu_1}}{R_1 C_1} - \frac{I_{\nu_0}}{R_0 C_0}\right) + L^1(I_{\nu_1} - I_{\nu_0})\right.\right.$$

$$\left. U_1\left[P_1 + \frac{L^2 I_{\nu_1}}{C_1}\right] - U_2\left[P_0 + \frac{L^2 I_{\nu_0}}{C_0}\right] + \frac{R_1}{2}(U_1-\delta)\left[U_1^2 - U_0^2\right]\right] \Big\} = 0$$

(96) $$\left\{ R_1(U_1-\delta)(U_1-U_0) + L^2\left(\frac{I_{\nu_1}}{C_1} - \frac{I_{\nu_0}}{C_0}\right) + (P_1 - P_0) \right\} = 0$$

2.6 <u>Initial Condition</u>

The differential similarity relations are of the form

(97) $$\frac{dy_j}{d\eta} + \alpha_{1j} y_j = \alpha_{2j}$$

where the y_j are the dependent variables and the α_{ij} are functions of y_j and η.

(98) $$\alpha_{ij} = \alpha_{ij}(Y_i, \eta)$$

The system of equations in similarity representation are ordinary nonlinear and coupled. In discussing initial conditions appropriate for integration no suggestion is made concerning sufficiency and necessity for uniqueness. The author is not aware of theorems appertaining to systems of this type. Hence the initial conditions are those required minimally to properly pose a numerical calculation to the computer and satisfy certain rudimentary physical requirements. The equations are all of first order. Consequently, to initiate a numerical procedure starting values of the functions must be be given at some point in the integration domain. In the case of zero or near zero thickness shock waves, the jump equations are point invariants of the conservation system. If the problem is not strongly time dependent, the jump equations can be viewed as linking asymptotic values of fluid variables in the pre and post-shock regions. The first interpretation is relevant to problems in this dissertation. In concordance with the physical requirement of satisfaction of the jump equations the following approach is suitable.

The classification of initial conditions is contingent on the continuity property of the dependent invariants. (Figure 6) This statement means that not all fluid variables need necessarily be discontinuous at the shock wave. Let $Q(\eta)$ be a property of the system defined on the interval (η_0, η_1) and to which corresponds a differential relation. If the property $Q(\eta)$ is continuous across the shock wave then $Q(\eta^*)$ for $(\eta_0 \leq \eta^* \leq \eta_i)$ is to be the given. It is conceivable to admit continuity

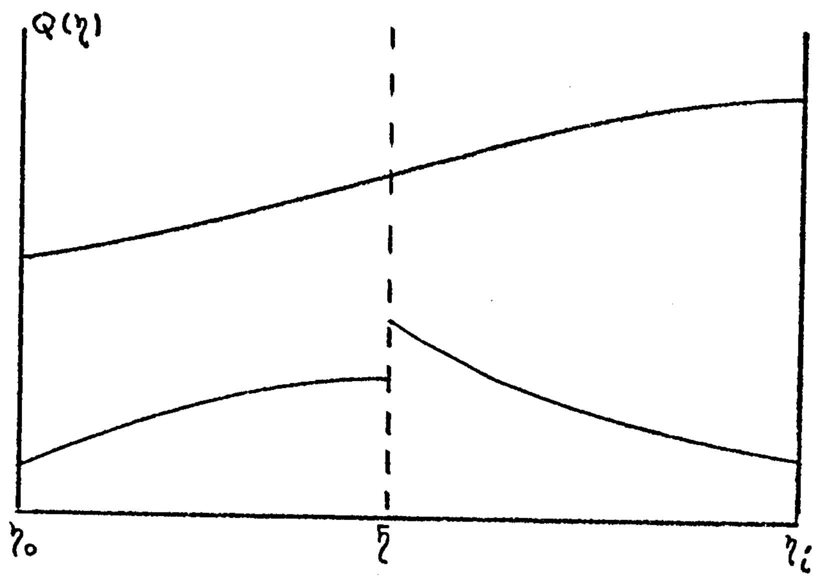


Figure 5 Admissible Invariant Structure

of all properties. If $Q(\eta)$ is discontinuous at the shock $\eta = \bar{\eta}$ then two alternatives arise. The jump and mean value of Q at $\eta = \bar{\eta}$ is given consistent with the saltus equations. Alternately $Q(\eta_1^*)$ or $Q(\eta_2^*)$ for $(\eta_0 \leq \eta_1^* \leq \bar{\eta})$ or $(\bar{\eta} \leq \eta_2^* \leq \eta_i)$ respectively are to be given together with the jump. The last criteria is probably most practical from an experimental point of view.

The shock velocity is presumed stipulated such that the jump system is determined. This information could be provided by experimental measurement or through appropriate values for conserved quantities when an integral constraint is employed to obtain a k-δ relation.

2.7 General Approach

In order to avoid difficulties arising from inconsistant guesses and to obtain valid solutions to the derived differential similarity system it is virtually manditory for a complete understanding of the results to follow the subsequent rules.

1. Construct an hypothesis on the type of problem to be considered, i.e. define the constraints which are physically appropriate.

2. Integrate numerically the shock Hugoniots for all initial doublet of sound speed and fluid speed.

3. From the differential system form the quantities $u^1 = u'(RTu)$ and $T^1 = T'(RTu)$ then form the quotient:

$$\frac{DT}{DU} = \frac{T'(R,T,U)}{U'(R,T,U)}$$

From the ideal gas law the unambiguous isentropic speed of sound is $\sqrt{\gamma T}$ Hence:

$$\frac{Dc}{Du} = \frac{c(R,U,c)}{u(R,U,c)}$$

4. Solve the above equation parametrically on appropriate initial conditions. The solution of this equation produces the vector field of integral curves corresponding to the problem. All subsonic, sonic, supersonic regimes can be read off this

vector field.

Minimally the shock Hugoniots should be found along with the singularities in the vector field.

CHAPTER III

SELF SIMILAR MOTION

3.1 Limitingly Small Radiation Effects

The question arises as to whether a smooth transition from problems involving radiation to those not involving radiation can be made. The admission of radiation effects, independent of the degree, fundamentally characterizes, through establishment of a (k,δ) relation, certain physical aspects of the problem. These aspects remain invariant throughout the spectrum of importance of the radiative phenomenon. For example, consider:

$$I_\nu = \sigma \rho^m T^n \tag{101}$$

for arbitrary m and n.

$$I_\nu = r^{mk-(k+3)+2n} \tau^{-3-2n} R^m(\eta) T^n(\eta) \sigma \tag{102}$$

and

$$\delta = -(3-2n)/(m-1)k+2n-3 \tag{103}$$

Consequently, if $k = 0$ then $\delta = 1$ for all values of m and n. Clearly ($\sigma \lambda^{2n-3} a^m$) contains two scaling parameters a and λ, hence the influence of the radiative terms in the equations can be varied. Nevertheless, with $\delta = 1$ the shock velocity is constant. The transition to Guderley's implosion non constant shock velocity problem can never be made continuously for the aforementioned class of radiative laws. It must be discontinuous and should not be considered as being reached by limitingly small value of σ, a or λ. Instead δ would behave discontinuously.

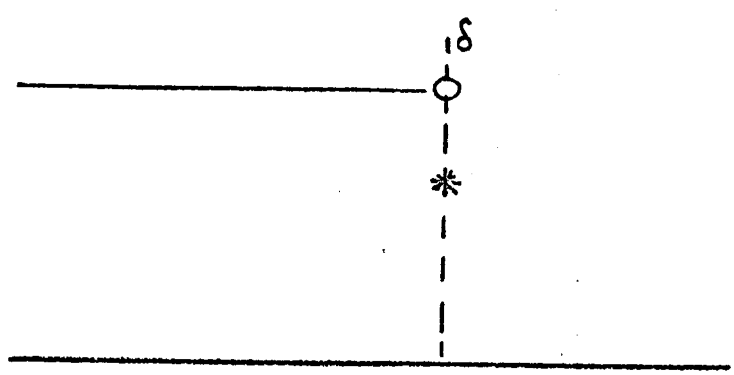


Figure 5 Discontinuous δ

It is only in this sense the transition to Guderley's problem is of a singular perturbation nature. With the class of radiative laws above what can be effected as a continuous transition is the problem discussed by Friedricks and Courant,[17] namely, an outwardly progressing spherical quasi simple wave preceded by a shock propagating into a quiescent region. That the coalescence of the field of integral curves occurs for small sound speeds and/or small effects of radiations (see Chapter 4) brings this point more sharply into focus. In other words the transition from one "type", in the sense of specified (k, δ), of a problem involving radiation to a different "type" not involving radiation must be thought of in the manner of a singular perturbation. Transitions between the same "type" problem can be made continuously. Hence, a warning is in order. If transitions of the first kind are expected, great care should be exercised in examining regions in which the self similar motion can be expected to be valid. Consider the following hypothetical example:

(104) $$\frac{D}{D\tau} I_\nu = -\kappa\rho I_\nu \; ; \; c = r/\tau \, c_0 \; ; \; \kappa\rho = \kappa_0\rho_0/r$$

The general solution to this equation is:

(105) $$I_\nu = I_{\nu_0} r^m \tau^p$$

for all values of m and p without restriction. Now:

(106) $$I_\nu = r^{k+3} \tau^{-3} I_\nu(\eta) = I_{\nu_0} r^m \tau^p$$

(107) $$I_\nu(\eta) = r^{m-(k+3)}\tau^{p+3} I_{\nu_0}$$

(108) $$\delta = -\frac{(p+3)}{m-(k+3)}$$

and for k = 0

(109) $$\delta = -\frac{p+3}{m-3}$$

Non constant shock velocities can also be included with k = 0 compatible with radiation laws. Hence for any particular "type", say the Guderley "type", one could take that value of δ and structure a radiation law compatible to it, and in the limit of no radiation the solution would go in a continuous manner to the non-radiative solution.

3.2 Constraints

It has been shown that the two parameter family of similarity transformations with two-sided similarity is consistent with a shock wave. The jump equations impose no constraints (when heat conduction is excluded). The inclusion of heat conduction can be viewed as a constraint on the system. The object of using constraints on the system is two fold. First it is desirable to obtain unique solutions to a specific problem. Secondly in modeling a specific practical problem by similarity techniques the constraints imply definite physical properties about the system. The reduction of a particular form of the intensity into an admissible similarity representation implies a constraint on the system and limits the class of homologies and shock velocities.

3.3 Equations of State

For an Isentropic law:

$$p\rho^{-\gamma} = \text{constant} \tag{110}$$

then

$$r^{k+2}\tau^{-2} P(\eta) r^{-\gamma k-\gamma} R(\eta) = f(\eta) \tag{111}$$

$$\eta^{\alpha} = r^{(k)(1-\gamma)+2}\tau^{-2} = r^{\alpha}\tau^{-\alpha\delta}\lambda^{-\alpha} \tag{112}$$

$$\alpha = k(1-\gamma)+2 \; ; \; \alpha\delta = 2 \tag{113}$$

$$\delta = 2/k(1-\gamma)+2 \quad \text{or} \quad \gamma = 1 - \frac{2(1-\delta)}{\delta k} \tag{114}$$

Hence, for constant strength shock waves preceded by a uniform state, k and δ are ascertained independent of the number of degrees of freedom of the gas. With two-sided similarity two additional constraints can be used compatibly to ascertain a and k provided the δ and k do not yield a non physical value for the ratio of the specific heats.

For an ideal gas:

$$P = \rho T \tag{115}$$

and

$$r^{k+2}\tau^{-2}P(\eta) = r^{k}R(\eta)\, r^{2}\tau^{-2}T(\eta) \tag{116}$$

Clearly no constraint is implied whatever. This result is expected by virtue of the way the homology for the variables was constructed from the fundamental equations.

For a real gas:

Generally the equation of state for a real gas is written as an expansion.

$$PV = A + \frac{B}{V} + \frac{C}{V^2} \cdots \tag{117}$$

For an ideal gas A = RT; for a Van der Waals gas[18] A = RT; B = RTb-a and C = RTb^2.

(118) $$P = \sum_n \rho^n A_n$$

(119) $$r^{k+2}\tau^{-2} P(\eta) = \sum_n r^{nk} R^n(\eta)\, A_n(r^2\tau^{-2}T(\eta))$$

That a real gas represented by a general virial expansion be compatible with similarity demands that k = 0, δ = 1. The coefficients can be arbitrary functions of the temperature. In any region in which a single term in the series dominates the latter conditions are relaxed. Constraints based on equations of state restrict similarity by constitutive assumption concerning the medium. The above discussion has shown that any gas law can be used with an outward progressing flow preceded by a shock and uniform pre-shock region. As will be shown subsequently many radiative transfer laws are compatible with these conditions.

3.4 Integral Constraints

For energy conservation consider:

(120) $$E = \int^{r} \left(\rho \frac{u^2}{2} + c_v T + P + U^R/\rho \right) r^2 dr$$

That the total integrated energy be a constant, where r is a variable upper limit, implies the following constraint:

$$(121) \qquad E = \int^{r} r^{K+4} \tau^{-2} \left(R(\eta) \frac{U^2}{2}(\eta) + c_v T(\eta) + P(\eta) + \frac{U^R(\eta)}{R(\eta)} \right) dr$$

Since

$$(122) \qquad \eta = r/\lambda \tau^{\delta}$$

$$(123) \qquad d\eta = \eta \, dr/r$$

For a fixed time τ and a given r there corresponds a specific η. Hence:

$$(124) \qquad E = \int^{\eta} r^{K+5} \tau^{-2} \frac{E(\eta)}{\eta} d\eta$$

Consequently:

$$(125) \qquad (\lambda \eta)^{\alpha} = r^{K+5} \tau^{-2} \rightarrow \delta = \frac{2}{K+5}$$

In the description of intense explosion Taylor(19) produced very good results by combining this constraint of total integral energy with the limitingly strong shock jump conditions. Thus for k = 0 and δ = 2/5 the shock speed is given:

(126) $$V_s = \frac{2}{5}(\lambda\bar{\eta})^{\frac{5}{2}} r^{-3/2}$$

(127) $$P = r^{-3}(\lambda\eta)^{-5} P(\eta)$$

Apparently this result was somewhat critisized since the hypothesis of strong shock would ultimately breakdown.[20]

For the momentum integral constraint:

(128) $$M = \int^{r} \rho u r^2 dr$$

That the total integrated momentum be a function only of η requires:

(129) $$M = \int^{r} r^{k+3} \tau^{-1} R(\eta) U(\eta) dr$$

(130) $$M = \int^{\eta} r^{k+4} \tau^{-1} \frac{M(\eta)}{\eta} d\eta$$

(131) $$\delta = 1/k+4$$

It is interesting to conjecture an alternative hypothesis to that of Taylor for a very strong explosion. Imagine that an appreciable amount of energy is dissipated, say in a nuclear explosion in the form of radiation. The intense heat wave in a nuclear blast precedes the shock. Hence the pre-shock gas is non uniform. This raises the possibility of employing two-sided similarity. Imagine, however, that the very strong shock condition with pre-shock uniformity is applicable, and the total impulse conserved. Then k = 0, $\delta = 1/4$.

(132) $$V_S = \frac{1}{4}(\lambda\xi)^4 r^{-3/4}$$

(133) $$p = r^{-3/2}(\lambda\eta)^{-4} P(\eta)$$

The pressure remains stronger farther out from the point of explosion.

For density set:

(134) $$\rho = \int^r r^K R(\eta)\, r^2 dr$$

(135) $$\rho = \int^\eta r^{K+3}\, \frac{R(\eta)}{\eta}\, d\eta$$

Consequently $k = -3$ for all values of δ. With this constraint the continuity equation is exactly integrable, i.e.,

$$[U-\delta]\eta R' + \eta R U' = 0 \quad ; \quad R = R_0 \frac{(U_0-\delta)}{(U-\delta)} \tag{136}$$

3.5 Inclusion of Radiation as a Constraint

If the two parameter family of transformation is uniquely determined by using constraints such as a uniform pre-shock region, integral constraints and gas laws, then the form of the radiative intensity, its functional dependence on thermodynamic parameters, is limited to those laws compatible with the constraints. There are always laws which can be constructed which are compatible with two-sided similarity. The shock conditions impose no constraints. Consequently, in lieu of this, the inclusion of more desirable radiative laws can be used to determine a k, δ relation. As a case in point, it is not possible to use the Rosseland Approximation together with a uniform pre-shock region and a finite constant strength shock wave.

A general form for I_ν that is compatible with a strong or limitingly strong shock wave can easily be given:

$$I_\nu = f(\nu, T, \hat{l}) \tag{137}$$

(138) $$I_\nu(\eta) = r^{-(k+3)}\tau^{3} f(\nu, r^2\tau^{-2}T(\eta), \hat{L})$$

Hence for k = 0 and δ = 1 .

(139) $$I_\nu(\eta) = \frac{f(\nu, (\lambda\eta)^2 T(\eta), \hat{L})}{(\lambda\eta)^3}$$

For a general power law on the temperature:

(140) $$I_\nu = \alpha_1(\nu)\,\alpha_2(\hat{L})\,T^{n}$$

(141) $$I_\nu = r^{2n}\tau^{-2n} r^{-(k+3)}\tau^{3}\, T^{n}(\eta)\,\alpha_1(\nu)\,\alpha_2(\hat{L})$$

The α_1 and α_2 are arbitrary functions of frequency and direction respectively.

(142) $$r^{2n-(k+3)}\tau^{3-2n} = (\lambda\eta)^{\alpha}$$

The implied constraint is:

(143) $$\delta = (2n-3)\big/(2n-3)-k$$

For n = 4

(144) $$\delta = \frac{5}{5-k}$$

It is frequently very useful, prior to formulating the constraint, to perform the frequency integration which is common to all three operators $L^{(j)}$, since the thermodynamic dependence may become less restrictive. Consider the following example:

(145) $$I_\nu = \left[\exp - \frac{h\nu}{kT}\right] \alpha_2(\hat{L})$$

This function is of the form first considered (Equation 139) and requires k = 0, δ = 1.

(146) $$I = \int_0^\infty I_\nu(\nu, T, \hat{L})\, d\nu = \alpha_2(\hat{L}) \int_0^\infty \exp - \frac{h\nu}{kT}\, d\nu$$

After integration

(147) $$I = -\alpha_2(\hat{L}) \frac{kT}{h}$$

The post integration form is significantly less restrictive and falls under the second category considered, i.e. n = 1 and $\delta = 1/k+4$.

Transformation for the Rosseland Flux :

In the Rosseland approximation the radiation energy and pressure are discarded while radiant flux dissipation is included in the energy equation. The flux in this theory has the form:

$$F^R = \frac{16}{3} \frac{\sigma}{\rho K} T^3 \frac{\partial T}{\partial r} \tag{148}$$

This expression is easily written in similarity form compatible with the energy equation as follows:

$$F^R(\eta)\, r^{k+3} \tau^{-3} = \frac{16}{3} \frac{\sigma r^6 \tau^{-6}}{r^k R(\eta)} \left(\frac{r}{\tau^2} T(\eta) + \eta T' \frac{r}{\tau^2} \right) \tag{149}$$

then

$$F^R(\eta) = \frac{16}{3} \frac{\sigma r^{7-k-(k+3)}}{R(\eta)} \tau^{-5} \left(T(\eta) + \eta T' \right) \tag{150}$$

and

$$\delta = 5/4-2k \tag{151}$$

(152) $$F^R(\eta) = \frac{16}{3}\frac{\sigma(\lambda\eta)^{4-2k}}{R(\eta)}\left(T(\eta)+\eta T'\right)$$

Imagine a nuclear explosion problem treated from the point of view of two-sided similarity using the Rosseland flux as a means of energy dissipation and assuming the conservation of impulse:

(153) $$\delta = 5/4\cdot 2k = 1/k+4$$

(154) $$k = -16/7 \;;\; \delta = 7/12$$

(155) $$V_S = 7/12\, r^{-5/7}(\lambda\eta)^{12/7}$$

(156) $$P = r^{-10/7}(\lambda\eta)^{24/7}P(\eta)$$

The alternate hypothesis herein mentioned are not intended necessarily to have physical content.

Several observations concerning the physical implications of integral constraints should be made. The basic requirement is that the integrand be a function of η. This means that the integrated quantity is conserved between the integration limits. More, however, is implied: namely, between any two η limits the quantity is conserved. Hence, between any two moving points in the fluid on lines of constant η the quantity is conserved. If the total quantity is defined to be fixed and finite throughout the η domain then the integral constraint can impose an asymptotic constraint on the value of the function. More over the stipulation of a defined quantity can fix simultaneous the shock velocity. These points are stressed by an example.

Consider the integral constraint on the energy:

$$(157) \qquad E_T = \int^{\eta} \eta^{K+4} E(\eta)\, d\eta$$

Imagine that two-sided similarity is to be used over an infinite η domain. Let the shock position be $\eta = \bar{\eta}$. $E_{total} = A$ $A < \infty$; where A is defined on physical grounds. Then

$$(158) \qquad \int_0^{\infty} \eta^{K+4} E(\eta)\, d\eta = A$$

If E(n) is continuous for $\eta > \bar{\eta}$ then as

$$(159) \qquad \eta \to \infty \ ; \ E(\eta) \to O\left(1/\eta^{K+6} \right)$$

and for $\eta < \bar{\eta}$

(160) $$\eta \to 0 \quad ; \; E(\eta) \to O\left(1/\eta^{k+4}\right)$$

Furthermore, since the integral is conserved between any two moving points:

(161) $$\int_0^{\bar{\eta}} \eta^{k+4} E(\eta)\, d\eta + \int_{\bar{\eta}}^{\infty} \eta^{k+4} E(\eta)\, d\eta = A$$

When E(n) is discontinuous at the shock wave equation (161) automatically determines the shock velocity as a function of the constant A. Of course, in an explosion preceded by a shock, the energy per unit volume decreases. The converse is concluded in the case of implosion when the integral energy constraint is used.

In closing this section it is noted the integral density constraint implies the existence of a contact surface. Indeed every line of constant is a contact surface for k = -3. This constraint can not be used simultaneously with a shock wave by virtue of contrary definitions.

3.6 Significance of Strong Shocks

The Rankine-Hugoniot equations in the absence of radiation are well known to be

(162) $$\rho_0 v_0 = \rho_1 v_1$$

(163) $\quad v_0 (\rho_0 u_0) + P_0 = (\rho_1 u_1) v_1 + P_1$ Momentum

(164) $\quad \rho_0 v_0 \left(\frac{1}{2} u_0^2 + e_0\right) - \left(\frac{1}{2} u_1^2 + e_1\right) \rho_1 v_1 = P_1 u_1 - P_0 u_0$ Energy

where $e = c_v T$

(165) $\quad v = u - U$ $\qquad U$ = shock velocity

and for an ideal gas when the internal energy is proportional to the temperature:

(166) $\quad P = A \rho^{\gamma}$

$A = A(s)$ is not necessarily constant.

(167) $\quad c^2 = \left.\frac{\partial P}{\partial \rho}\right)_s = \frac{\gamma P}{\rho} = \gamma T$

For the following discussion these relations will be used. Several definitions of shock strength are popularly accepted: the excess pressure ratio, condensation and the relative fluid velocity to sound speed. The subscripted s_j refers to these strengths.

(168) $$S_1 = \frac{P_1 - P_0}{P_0}$$

(169) $$S_2 = \frac{\rho_1 - \rho_0}{\rho_0}$$

(170) $$S_3 = \frac{U_1 - U_0}{c_0}$$

As before the similarity invariants employed are:

(171) $$\rho = r^{K} R(\eta) \quad ; \quad v = r\tau^{-1} U(\eta)$$

(172) $$T = r^2 \tau^{-2} T(\eta) \quad ; \quad c^2 = r^{K(\gamma-1)} R(\eta)^{(\gamma-1)}$$

When the pre-shock gas is uniform then ρ, U and T are constant and the aforementioned similarity applies to the post-shock region only.

For an ideal gas the jump relations:

(173) $$-\rho_0 U = \rho_1 (U_1 - U)$$

(174) $$\rho_0 T_0 = \rho_1 U_1 (U_1 - U) + \rho_1 T_1$$

(175) $$-\rho_0 U c_v T_0 - \left[\frac{1}{2} u_1^2 + c_v T \right] \rho_1 (u_1 - U) = \rho_1 T_1 u_1$$

3.7 Limiting Strong Shock Conditions

In the limit that T_0 is small using the initial continuity and momentum relations:

(176) $$\rho_0 V_0 (V_1 - V_0) = P_0 - P_1$$

(177) $$(\rho_0 - \rho_1) V_0 V_1 = P_0 - P_1$$

(178) $$V_0 V_1 = \frac{P_0 - P_1}{\rho_0 - \rho_1}$$

(179) $$U(u_1 - U) = P_1 \Big/ \frac{\rho_1}{\rho_0} - 1$$

In this case $S_1 = \infty$

$$S_2 = P_1 / U(u_1 - U)$$

For ρ_1, P_1, u_1, U stipulated then ρ_0 has a limiting non zero value. For this case the energy and momentum relations are:

$$U_1(U - u_1) = T_1 \tag{180}$$

$$\left[\tfrac{1}{2}u_1^2 + c_v T\right]\left[u_1 - U\right] = T_1 u_1 \tag{181}$$

It follows immediately from the above relation that for strong (not necessarily constant) shock waves no constraints are implied by momentum and energy on substitution of the similarity forms. The continuity equation still demands that $k = 0$. If in addition $P_o \rightarrow 0$, $\rho_o \rightarrow 0$ or ρ_o is sufficiently small then continuity implies no constraint.

It is not physically unreasonable to consider the latter possibility. Then $s_2 = s_1 = \infty$. Furthermore $c_o = \sqrt{\gamma T_o}$, then $s_3 = \infty$. All the definition of shock strength are consistent. Very strong explosions or implosions from a point could be approximated under these conditions.

If the shock wave is to be constant strength then U_1/c_o = const, constant directly behind the shock wave. The use of c_o is overly restrictive. Introduce U_1/c_1 = constant. Then

$$r\tau^{-1}U(\eta) \Big/ r^{k(\gamma-1)/2}\, R(\eta)^{\gamma-1/2} \tag{182}$$

is required to be constant directly behind the shock wave, therefore it must depend only on η.

(183) $$\frac{r\tau^{-1}}{r^{k(\gamma-1)/2}} = \left(\frac{r}{\lambda\tau^{\delta}}\right)^{\alpha}$$

(184) $$r^{1-\frac{k(\gamma-1)}{2}}\tau^{-1} = r^{\alpha}\tau^{-\alpha\delta}$$

(185) $$1/\delta = -k(\gamma-1)/2$$

This is the same result obtained in Friedrichs and Courant;[20] the k,δ relation for an isentropic gas and a constant shock strength. Since the shock trajectory is a line of constant and its strength is to be constant for all r and τ, then either r and τ must cancel from the jump equations or must be representable as functions of η. Suppose the homology in r and τ cannot be reduced to functions of η, then the jump equations depend on position and time. Hence the shock strength cannot be constant. This is evident in a straight forward way from the continuity equation. Therefore k = 0 renders the jump equations consistent and implies from the k-δ relation for an isentropic gas that $\delta = 1$. An ideal non isentropic gas is trivally consistent with k = 0, $\delta = 1$.

The most plausible hypothesis for explosions for example revolves around non constant shock strengths. The shock is initially strong and its strength decreases with distance. The jump equations using the similarity forms are:

(186) $$-\rho_0 r\tau^{-1} U = r^{k+1}\tau^{-1} R_1(\eta)\left[U_1(\eta) - U\right]$$

(187) $$\rho_0 T_0 = \frac{r^{k+2}}{\tau^2}\left[R_1(\eta)U_1(\eta)\left(U_1(\eta) - U\right) + R_1(\eta)T_1(\eta)\right]$$

(188) $$-\rho_0 U c_v T_0 \frac{r}{\tau} - \frac{r^{k+3}}{\tau^3}\left(\frac{1}{2}U_1^2 + c_v T_1(\eta)\right)R_1(U_1 - U)$$

$$= \frac{r^{k+3}}{\tau^3}\left(R_1(\eta)\,T_1(\eta)\,U_1(\eta)\right)$$

From these equations it is clear no possible way exists within the framework of pre-shock uniformity to consistently satisfy these relations for finite shock strengths without setting $k = 0$, $\delta = 1$. The only laternative is the use of the jump equations as initial conditions for integration, while the post shock initial conditions vary from one time interval to the next assuming the pre-shock conditions are constant. Hence a parametric family of solutions for independent invariants for the post-shock region could be developed on time or space. This may not be a totally unreasonable way of avoiding the partial differential equations and treating non constant finite strength shock waves.

The similarity expressions for the shock strength are:

(189.1) $$S_1 = \frac{r\tau^{-1} U_1(\eta)}{r\tau^{-1}\sqrt{\gamma T}}$$

or

$$S_1 = r\tau^{-1} U_1(\eta)/c_0$$

(189.2) $$S_2 = \left(\frac{r^k R(\eta)}{\rho_0} - 1\right)$$

(189.3) $$S_3 = \left(\frac{r^{k+2}\tau^{-2} P(\eta)}{P_0} - 1\right)$$

Suppose the jump equations are abandoned in favor of a modeled continuous solution across a transition region proceeded by a uniform region. From the similarity form of the shock strength relations it is still evidently not possible to discuss finite strengths shock except when k = 0 and δ = 1. With these statements in mind the initial motivation of employing two-sided similarity is clear. With two-sided similarity the jump equations impose no constraint. And

(190.1) $$S_1 = U_1(\eta)\Big/\sqrt{\gamma T(\eta)}$$

(190.2) $$S_2 = \left(\frac{R_1(\eta)}{R_0(\eta)} - 1 \right)$$

(190.3) $$S_3 = \left(\frac{P_1(\eta)}{P_0(\eta)} - 1 \right)$$

The shock strength can be made consistent for arbitrary k, δ pairs for finite strength shocks. The shocks, however, are still of constant strength. Indeed, it may be concluded that except for the limitingly strong shock problems self similarity fundamentally requires constant shock strength. A further remark, the physical implications of constant integral density is incompatible with a pre-shock uniformity since $k = -3$.

3.8 Self Similar Motions

Curves of constant η are traditionally of the form:

(191) $$r = \lambda \eta \tau^{\delta}$$

(192) $$\dot{r} = v_{shock} = \delta r \tau^{-1} = \delta \lambda \eta \tau^{\delta - 1} = r^{1-1/\delta} (\lambda \eta)^{1/\delta}$$

In modeling solutions to a particular problem it is not necessary to fix the shock wave to a path of constant η. A characteristic transition region may be introduced to facilitate description of physically realizable non zero thickness shock waves.

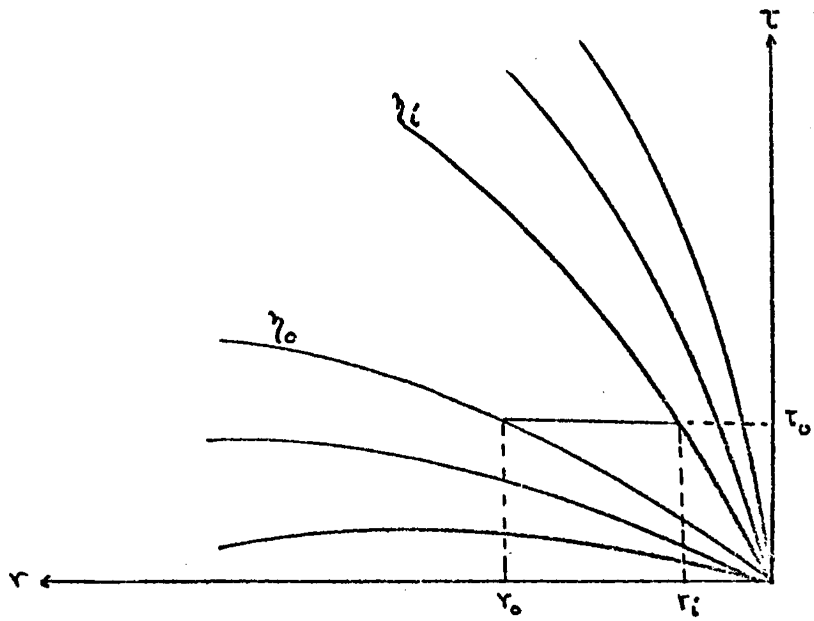


Figure 6 Transition Length

The assumptions inherent in a similarity formulation are visually apparent with the introduction of this length. While mathematically one locates the shock on a line of constant η, one recognizes a shock wave does not have infinitesmal thickness. The similarity approach dictates a behavior for this thickness. The equations derived in the previous chapter are not necessarily sufficient to describe a shock transition region because that the only loss mechanisms included are the radiative flux.

Let the space-time domain in Figure (6) threaded by curves of constant be divided in three angular regions:

(193.1) $(\eta_i \geq \eta \geq 0)$

(193.2) $(\eta_o \geq \eta \geq \eta_i)$

(193.3) $(\infty \geq \eta \geq \eta_o)$

These regions would correspond respectively to pre-shock, shock transition and post-shock regions. A simple formula for the shock thickness can now be written:

(194) $$\Delta_s = (r_o - r_i) = \lambda(\eta_o - \eta_i) t^{\delta}$$

For the case ($\delta > 0$); ($t \to 0$) and time increasing to infinity then $\Delta_s \to \infty$ corresponding to an explosion from a point. The jump conditions may be applied for pre and post-shock integrations initiating from η_o and η_i respectively. For the shock transition region values given at both end points over determines the integration proceedure. To discuss shock structure it may be required to produce continuous solutions to equations whose structure typifies the particular region and abandon the jump conditions. This question is open.

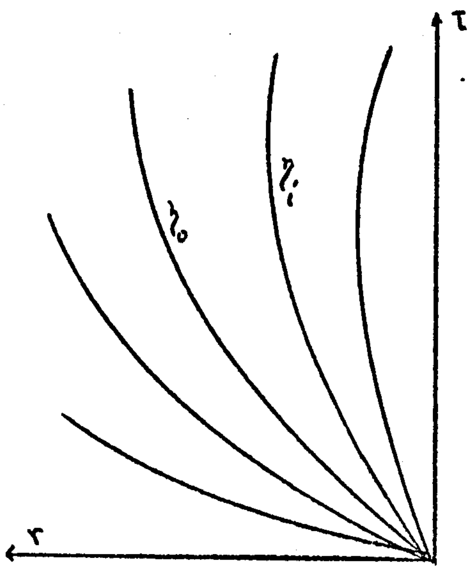


Figure 7 $\delta > 1$

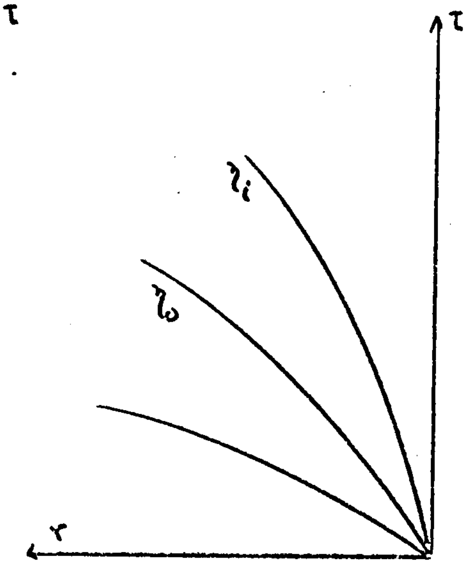


Figure 8 $1 > \delta > 0$

In Figure (7) where $\delta > 1$ and if $t(\infty \to 0)$ the lines of constant η bow toward the time axis. This motion is non physical and improper for describing implosions. The behavior is simply the inverse of the explosion problem. This fact is of course obvious since $\delta > 0$; $r > 0$ and $\tau > 0$ implies $V_{shock} > 0$. In Figure (8) for $1 > \delta \geq 0$. The lines of constant η bow toward the space axis and $\Delta_s \to \infty$ as $\tau \to \infty$ for $\tau > 0$ but more slowly. The division clearly occurs at $\delta = 1$.

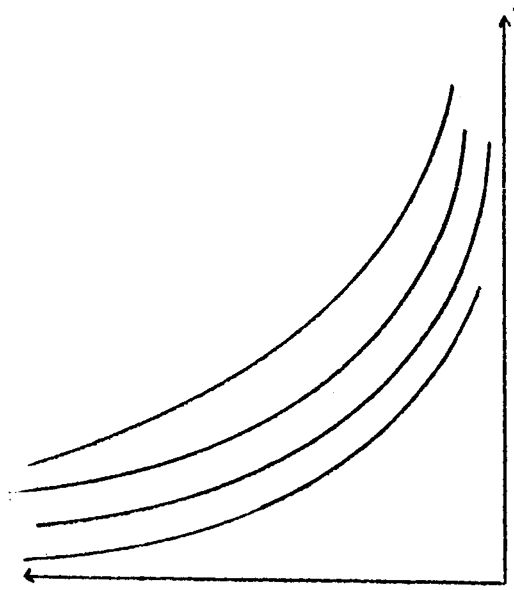


Figure 9 $\delta < 0$

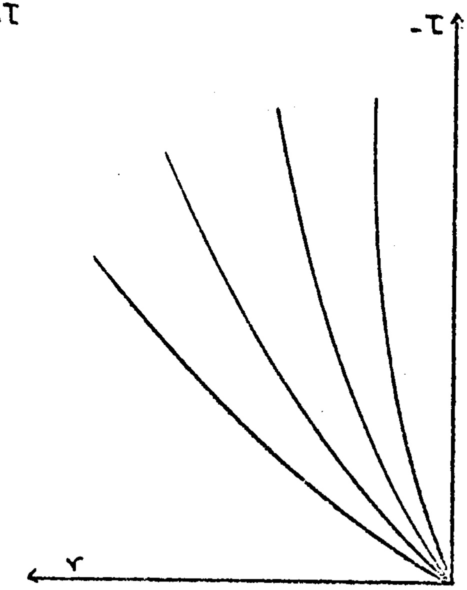


Figure 10 $\tau < 0$

Two alternatives for the description of implosion motions are properly described by Figures (9) and (10). In Figure (9) the motion is "hyperbolic" since δ is negative. For $\tau > 0$, $\delta < 0$ and $\tau \to \infty$ implies $\Delta_s \to 0$. The shock wave in this case takes an infinite amount of time to reach the origin. In Figure (10) time is negative therefore v_{shock} is negative $\tau(-\infty, 0)$ and $\delta > 0$ the shock converges to the origin in a finite time and $\Delta_s \to 0$.

3.9 Translation and Finite Surfaces

Traditionally curves of constant η are of the form $r = \lambda \eta \tau^{\delta}$. Discussions of similarity motions for explosions and implosions revolve around the singular point corresponding to $r = 0$, $\tau = 0$.

Lines of constant η converge or diverge from this point. The point at infinity is also a singular point. To reiterate, a contact surface can exist along the space or time axis in an infinite problem without imposing an integral constraint. It is instructive to envision the Euclidean plane with its lines of constant η mapped by stereographic projection onto a sphere located at the origin of coordinates. The two great circles of space and time divided this sphere into four sections.

3.10 The Effect of Translation

Consider the primary system of fluid equations:

(195) $$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

(196) $$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

(197) $$u = \frac{(x-x_o)}{(\tau-\tau_o)}\, U(\eta)$$

(198) $$\rho = (x-x_o)^{\kappa}\,\Omega(\eta)$$

(199) $$P = A\rho^{\gamma} = A(x-x_o)^{\gamma\kappa}\,\Omega^{\gamma k}(\eta)$$

(200) $$\eta = (x-x_o)/(\tau-\tau_o)^{\delta}$$

(201) $$\frac{\delta\eta}{\delta X} = \frac{\eta}{(X-X_0)} \; ; \; \frac{\delta\eta}{\delta\tau} = -\delta\frac{\eta}{(\tau-\tau_0)}$$

Transforming the Momentum: the LHS is;

(202) $$-\frac{(X-X_0)}{(\tau-\tau_0)^2}\left(U(\eta)+\delta\,U'(\eta)\right)+\frac{(X-X_0)\,U}{(\tau-\tau_0)}\left(\frac{U}{(\tau-\tau_0)}+\frac{\eta U'}{(\tau-\tau_0)}\right)$$

and

(203) $$\frac{(X-X_0)}{(\tau-\tau_0)^2}\left(\eta U'(\eta)(U-\delta)+U^2-U\right)=$$

$$\frac{A\kappa\gamma(X-X_0)^{\gamma(\kappa-1)}\left[\Omega^{\gamma\kappa}+\gamma\kappa\Omega^{\gamma(\kappa-1)}\Omega'(\eta)\right]}{(X-X_0)^{\kappa}\,\Omega^{\kappa}}$$

The isentropic equation of state implies:

(204) $$(\tau-\tau_0)^2(X-X_0)^{\gamma(\kappa-1)-(\kappa-1)} = \eta^{\alpha}$$

(205) $$\delta = -\,2/(\gamma-1)(\kappa-1)$$

Transforming the continuity equation:

$$(206)\quad -\frac{\delta (x-x_o)^K}{(\tau-\tau_o)}\eta\Omega' + \frac{(x-x_o)^K}{(\tau-\tau_o)}\left(\eta U' + U\right)\Omega + \frac{(x-x_o)^K}{(\tau-\tau_o)}(U\Omega') = 0$$

The above equations hold for arbitrary coordinates. The energy equations transforms similarly when isentropy is not assumed. Hence it has been shown that the equations are invariant under translations and maintain their self similar form. The similarity variable η is of the form:

$$(207)\qquad \eta = (x-x_o)/\tau^{\delta} \quad ; \; \tau_o = 0$$

In spherical coordinates the points $\eta = 0$ is a vertical line parallel to the time which describes the time revolution of a spherical surface of radius $r = r_o$. From all points of this finite surface the lines of constant η diverge (converge)

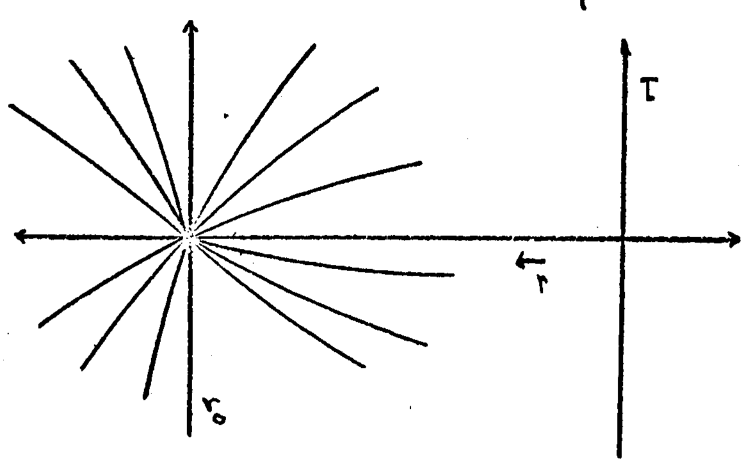


Figure 11 Finite Surface

In cylindrical coordinates with axial symmetry and z independence $\eta = 0$ corresponds to a cylindrical shell. Naturally, in plane coordinates with x and y independence it corresponds to a plane.

Looking into quadrant (1) where $\tau > 0$, $r > r_o$ with $r \rightarrow r + r_o$, explosions diverging from a finite surface can be investigated with a shock wave positioned on a line of constant η. In the region $\tau > 0$; $r < r_o$; $r \rightarrow (r_o - r)$ explosions from a spherical surface creating a wave propagating to the origin can be described. One notices that for this type of implosion:

$$\Delta_S = \lambda |\eta_i - \eta_o| \tau^{\delta} \qquad \text{for}; \ \delta > 0; \ \tau \rightarrow \infty \tag{208}$$

$$\Delta_S \rightarrow \infty \tag{209}$$

This represents a characteristic increase in the shock width as time evolves. The converse of the problems mentioned can be investigated by looking into the 3rd and 4th quadrants.

Some physical problems that can be associated with these motions might be mentioned. Using the radiative gas dynamic equations for $\delta > 0$ time $\tau \rightarrow \infty$, $\tau > 0$, it would not be implausible to describe a radiating exploding star in vacuum. A contact surface could be used to represent a matter-space interface implying k = -3. A gravitational force could be included.

Using $\tau < 0$, $\tau \to 0$ from $-\infty$, $\delta > 0$, $\lambda < 0$ or $\delta < 0$, $\tau > 0$, $\tau \to \infty$ $\lambda > 0$ an imploding shock wave in a radiant gas can be discussed. Further, a contact surface at infinity can be imagined to be the imploding force implying no constraint. In both cases $\eta = r/\lambda\tau^{\delta}$. For the finite spherical surface, focusing attention on the region $\tau > 0$ for $\delta > 0$; $r_o > r > 0$, a shock wave imploding to the origin can be discussed. In all cases these solutions are to be considered carefully for regions of physical applicability. They do not hold in all regions of space and time. Again, a shock and contact surface can only exist simultaneously if they are in coincidence, but this case is to be ruled out by contrary definitions (contact surface not at ∞).

If $\tau \to a\tau$, $r \to br$, where a and b are dimensionless numbers, this represents a uniform contraction or dilation and simply represents rotating the field of lines of constant η.

3.11 Superposition of Self Similar Motions for a Problem of an Imploding Shock

Almost definitely a given problem of physical interest will not be analysable, in totality, on the basis of self similarity. Hence to model a solution it may be necessary to model the motion in a piecewise self similar way from one region to the next. Consider Figure (12).

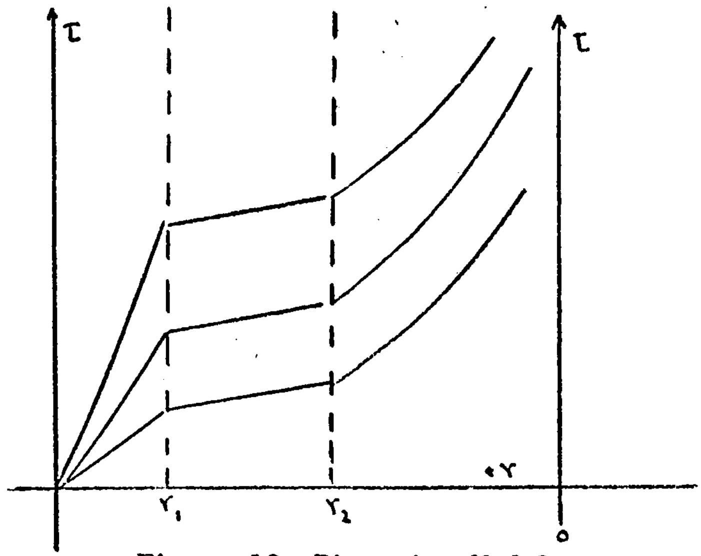


Figure 12 Piecewise Model

Let an explosion at $r = r_o$ on a finite surface occur generating a shock wave propagating inward and for $r_1 < r < r_o$; $\delta > 0$, $\tau \to \infty$ from zero. The solution can be joined at r_1 to a transition region where the shock velocity begins to diminish, i.e. $\delta > \delta^* > 0$. This region is joined to a third region where the shock takes an infinite amount of time to traverse the interval $(r_2 > r > 0)$. Here $\delta < 0$. At the points r_1 and r_2 curves of the same η for different δ can be joined. The solutions cannot be made valid generally at the juncture points r_1 and r_2. The homology which depends on k can be preserved but the invariants are generally discontinuous at these points. Failure of a solution in the neighborhood of a few points where solutions are joined is a small price to pay. In the region $(r_o > r > r)$, is of the form:

$$\eta = \left(\frac{r + r_o}{\lambda \tau^{\delta}} \right) \tag{210}$$

These remarks on piecewise self similarity can be viewed upon as concomitant with the hypothesis that δ in the general case of non self similarity is $\delta(\tau)$, and over the time intervals corresponding to the portions of space $(r_o > r > r_1)$, $(r_1 > r > r_2)$, $(r_2 > r > 0)$, $\delta(\tau)$ is approximately constant.

Shock Wave at the Origin

No adequate description within the framework of self similar motion exists to describe the reflection of the shock wave at the origin. If the criterion of critical reflection radius exists it cannot be formulated except to conjecture in an ad-hoc way that the self similar motion breaks down and is invalid beyond this point.

CHAPTER IV

ANALYTICAL AND NUMERICAL PROBLEMS

4.1 Introductory Remarks

The numerical problem herein discussed concerns an inward propagating shock wave which is completely opaque to radiation. From this problem the conclusion may be drawn that the effect of the radiative transfer is the cooling of the gas directly downstream of the shock wave. Consistent with a uniform pre-shock gas and the embedding of radiative transfer laws, the implosion velocity is constant. The problem is posed on the basis of the following statements.

a. The upstream (core) gas is quiescent with zero velocity, uniform density and uniform non-zero temperature.

b. A shock wave is defined to exist on a line of constant $\eta = \bar{\eta}$.

c. The shock wave is opaque to radiation. This means that in the heated radiant post-shock gas the net radiant flux diverges away from the shock.

d. The initial conditions for numerical integration are the initial post shock values i.e. $U(\eta)$, $T(\eta)$ and $R(\eta)$. Further, the shock velocity is stipulated.

In the model considered, the radiation energy and pressure are excluded. The radiation flux is maintained as a dissipative mechanism in the energy equation. This is reasonable because:

$$\frac{F^R}{p^R} = O(c) = \frac{F^R}{U^R} \tag{211}$$

The radiation quantities could be comparable in magnitude for dense materials or equivalently for a large index of refraction. In this case the radiant energy might be trapped over locally small volumes. The model problem presupposes that the post-shock gas is not optically dense.

Situations can arise in which all radiative terms should be included. An example of this might be a dense radiant star. In laboratory experiments the primary influence of radiative transfer effects, should they be of importance, is manifested by the flux dissipation in the energy equation.

Problems may be considered in which the heat flux (radiant plus conductive) has zero divergence. The radiation pressure and energy in this case augment the analogous fluid properties.

Equilibrium assumptions of some type are usually employed by experimentalist to establish a relation between the density, temperature and net radiation. The black body distribution is commonly used even by theoreticians. It is ludicrous to abjectly discard these theories as inaccurate. Deductions based on equilibrium hypothesis in a region quite strongly in non equilibrium (the shock wave region) should, however, be subject to scrutiny.

A procedure for analytical approximation is illustrated by application to the problem in which the divergence of the total heat flux is zero.

4.2 Embedding of a Radiative Transfer Law

In the numerical calculation the Rosseland flux hypothesis could be used. The proper similarity form for this term was derived in the section on constraints. In a strongly time dependent problem the validity of the assumptions required to formulate the Rosseland flux approximation break down. More convincing critical statements of this popularly accepted hypothesis cannot be made. The correct law to employ in a non-equilibrium region is unknown. The use of the Rosseland flux raises the order of the energy equation to two. In general the inclusion of the radiant energy and pressure terms with a Rosseland form for the flux, requires two k, δ relations. It has been shown that giving a similarity form to the radiative intensity, while retaining the expressions for the radiant quantities as integral operators on the intensity, requires but one constraint. This lends credence to the point of view adopted in Chapter II.

Unfortunately, appropriate choices of radiative transport functions appertaining to laboratory experiments is an exercise in academic conjecture. Therefore, a law is postulated in an ad-hoc manner. The numerical experiment is to be viewed as an illustration to which no profound physical significance should be attached. For just these reasons, laborious numerical analysis of the entire system of equations is without significance.

Postulate:

a. The material derivative of the radiant intensity is zero.

b. An anisotropic angular dependence of radiation.

c. Exclusion of scattering.

d. Black body frequency dependence.

e. Admissability of a mean absorption.

It is realized that these postulates are questionable.

The radiative transfer equation is:

(212) $$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = -\rho\beta_\nu I_\nu + j_\nu$$

(213) $$\beta_\nu = \sigma_\nu^s + \kappa_\nu$$

(214) $$j_\nu = j_\nu^e + j_\nu^s$$

The quantity σ_ν^s is the loss due to scattering and j_ν^s is an effective emission due to scattering in a given direction from other directions. With the five postulates:

(215) $$\rho \kappa_\nu I_\nu = \rho j_\nu^e = \rho \beta_\nu$$

A mean absorption may be defined:

(216) $$\langle \kappa_\nu \rangle = \frac{\int \kappa_\nu f \, d\nu}{\int f \, d\nu}$$

where f is an appropriate weighting function.

(217) $$I = \frac{1}{\kappa} \int B_\nu d\nu$$

(218) $$I \propto T^4$$

Then

(219) $$F^R \propto \int_0^\pi T^4 \alpha_1(\hat{L}) \cos\theta \, d\omega$$

Consequently

(220) $$F^R \propto T^4$$

The proportionality constant σ includes the Stephan-Boltzman constant, the mean absorption and integration constants. The only relevant part of this constant is its order of magnitude. Hence

(221) $$L^1 I_\nu = F^R = \sigma T^4$$

then

(222) $$\beta r^{k+3} \tau^{-3} L^1 I_\nu = \sigma r^8 \tau^{-8} T^4(\eta)$$

Using the homology for I_ν given in Chapter II compatible with the energy equation, one obtains:

(223) $$\beta L^2 I_\nu = r^{5-k} \tau^{-5} \sigma T^4(\eta)$$

That the R.H.S. be a function of η only implied:

(224) $$\delta = 5/5-k$$

and

(225) $$r^{5-k} \tau^{-5} = (\lambda \eta)^5$$

(226) $$\beta L' I_{\nu}(\eta) = \sigma(\lambda\eta)^5 T^4(\eta)$$

In spherical coordinates the divergence of the flux is:

(227) $$\nabla \cdot F^R = \frac{\partial F^R}{\partial r} + \frac{2F^R}{r}$$

In similarity form the R.H.S. becomes (see Chapter II):

(228) $$\beta(k+5) L' I_{\nu}(\eta) + \eta\beta L' I_{\nu}'(\eta)$$

Also

(229) $$\beta L' I_{\nu}'(\eta) = \sigma\lambda^5 \left[5\eta^4 T^4(\eta) + 4\eta^5 T^3 T' \right]$$

(230) $$\eta\beta L' I_{\nu}'(\eta) = L' I_{\nu}(\eta) \left[5 + 4\eta \frac{T'}{T} \right]$$

The flux dissipation becomes:

(231) $$(k+10)\beta L' I_{\nu}(\eta) + 4\beta L' I_{\nu}(\eta) \frac{\eta T'}{T}$$

The specific form of the function $\alpha_1(L)$ is not consequential.

The sign attached to the flux term prescribes the direction.

4.3 Numerical Procedure

The simplest procedure for numerical integration is seen from the following example.

(232) $$Y' = Y(x) \quad ; \; Y(x_0) = Y_0$$

then

(233) $$\Delta Y \cong Y' \Delta x \cong Y \Delta x$$

For fixed $\Delta x = h$, then:

(234) $$\Delta^n Y = hY(x_0 + nh)$$

and

(235) $$Y \cong Y_0 + \sum_n \Delta^n Y$$

Consider a system of coupled first order differential equations in which it is possible to solve explicitly for the derivative in terms of the functions. The equations can be put into the form:

(236) $$Y_j' = Y_j(Y_i, \eta)$$

A natural procedure presents itself for the integration. The equations in the problem to be treated are, of course, non-linear. Define a sequence $[k_n^j]$ where:

(237.1) $$k_{n+1}^j = Y_j^{(n)}\left(\eta_0 + \frac{n\Delta\eta}{2}, Y_{j_0} + \frac{k_n^j}{2}\right)\Delta\eta$$

The range of j equals the number of equations and n = 0, 1. For n = 2,3 use:

(237.2) $$\eta = \eta_0 + \frac{(\eta - 1)}{2}\Delta\eta$$

With this:

(238) $$\Delta Y^j \doteq 1/6\left(\sum_{n=1}^{4} k_n^j\right)$$

(239) $$y = y_0 + \Delta y^j$$

This scheme reduces to Simpson's rule when $y_j = y_1(\eta)$. Clearly the system under discussion is amenable to treatment by this method.[21] It is required to solve for $U'(n)$; $T'(n)$ and $R'(n)$ as functions of R, T, U.

4.4 The k-δ Values

With the condition that the pre-shock gas be uniform with a non zero temperature, the jump equations stipulate that $k = 0$, $\delta = 1$, which has already been shown. The law $F^R = T^n$ is compatible with these values for k and δ. Indeed, if it is desired to discuss a limitingly strong shock propagating into a uniform region then k must still be zero and with this radiation law $\delta = 1$. For the significance of this see the discussion on limitingly small radiation effects.

4.5 Immediate Deductions

Before any numerical computations are effected, the important physical consequence of the model can be ascertained. Since δ is constrained to be unity the shock velocity is a constant. Propagation is inward along a ray $r = \lambda\tau$. This is more than a consequence of pre-shock uniformity but also of the radiative law. If the similarity relation for $L'(I_\nu)$ is evaluated at $\eta = \bar{\eta}$, a relation is established between the shock velocity, radiant intensity and temperature in similarity form:

$$\beta L' I_\nu(\bar{\eta}) = \sigma (\lambda\bar{\eta})^5 T^4 = \sigma v_{shock}^5 T^4(\bar{\eta}) \tag{240.1}$$

Hence, if any two quantities can be observed experimentally then the third is determined. The radiation law hypothesized to apply for a given situation can be checked. If the model of an explosion from a finite spherical surface is used then for a finite "shock thickness":

(240.2) $$\Delta_s \to (\eta_i - \eta_o)\lambda\tau$$

Δ_s increases as τ increases or, in the case that τ goes from $-\infty$ to 0, Δ_s decreases. Hence the gross features of the model that can be checked by experiment are directly computable.

The object of any extended computation is to ascertain the structure of the variables away from the shock proper. Most usually this structure cannot be measured in present day experimental appartus.

4.6 Equations

(241) $$\eta R'[U-1] + 3RU + \eta U'R$$ Continuity

(242) $$[U-1][\eta U'+U] + [\lambda + \eta \frac{R'}{R}] T + \eta T'$$ Momentum

(243) $$c_v[U-1]\eta T' + 2c_v[U-1] T$$
$$= -T[3U+\eta U'] - \frac{L' \beta}{aR}(10 I_\nu + \eta I_\nu')$$ Energy

The ideal gas law has been substituted into the momentum equation. Rearranging the energy equation and substituting the similarity expression for I_ν yields:

(244) $$\eta T'\left[c_v(U-1)+\frac{4\bar{\sigma}\eta^5T^3}{R}\right]+T\left[2c_v(U-1)+3U+\eta U'\right] + \frac{10\bar{\sigma}\eta^5T^4}{R} = 0$$

where $\bar{\sigma} = \sigma/a$

In order to facilitate manipulation define the following:

(245) $$Q = [U-1]$$

(246) $$P = \left[c_vQ + \frac{4\bar{\sigma}\eta^5T^3}{R}\right]$$

Solving for T' in the energy equation and substitution in the momentum equation, and similarily solving for R' from the continuity and substitution yields:

(247) $$\left[\eta U'+U\right]Q + \left[2-\left(\frac{3U+\eta U'}{Q}\right)\right]T - \frac{10\bar{\sigma}\eta^5T^4}{RP} - \left(2\left[c_vQ\right]+\eta U'+3U\right)T/P = 0$$

and

(248) $$\eta U'\left[Q - \frac{T}{Q} - \frac{T}{P}\right] + \left[2 - \frac{3U}{Q} - \frac{2C_v Q}{P} - \frac{3U}{P}\right]T - \frac{10\bar{\sigma}\eta^5 T^4}{R} + UQ = 0$$

Forming the quotient and multiplying numerator and denominator by Q.

(249) $$U' = \frac{-\left[2PQ - 3UP - 2C_v Q^2 - 3UQ\right]T + \frac{10\bar{\sigma}\eta^5 T^4}{R}Q - UQ^2 P}{\eta\left[(Q^2 - T)P - QT\right]}$$

Now

$$2PQ = 2C_v Q^2 + \frac{8\bar{\sigma}\eta^5 T^4 Q}{R}$$

Substituting the above and dividing numerator and denominator by P

(250) $$U' = \frac{(3UT - UQ^2) + 3UTQ/P + \frac{2\bar{\sigma}\eta^5 T^4}{R}\frac{Q}{P}}{\eta\left[(Q^2 - T) - Q/P\, T\right]}$$

From the momentum equation, the derivative of the temperature is most easily expressed.

$$(251) \quad T' = -\left[\frac{[\eta U' + U][U-1] + \left[2 - \left(\frac{\eta U' + 3U}{(U-1)}\right)\right]T}{\eta}\right]$$

$$(252) \quad T' = \frac{[\eta U' + U]Q^2 - (\eta U' + U + 2)T}{\eta * Q}$$

The density is immediately determined.

$$(253) \quad R' = -\frac{R}{Q * \eta}\left[3U + \eta U'\right]$$

In the (k_n^j) sequence j goes from 1 to 3, where k_n^1 corresponds to the velocity increment k_n^2 to the density and k_n^3 to the temperature. Since U' is contained in the expression for R' and T', it is of course necessary to compute U' first then either R' or T'. In this computation the value 3/2 is used for the specific heat at constant volume. Assuming teh equipartition theorem, this corresponds to a gas of three degrees of freedom and yields a specific heat ratio of 5/3.

4.7 Singularities in the Field of Integral Curves

It is fruitful to examine the vector field of fluid velocity versus sound speed. This has been done for the case $k = 0$, $\delta = 1$

when no radiation is present.[22] In general it is a formidable undertaking to produce these curves. The singularities that are encountered are more easily divulged. Needless to say no deductions based on the curves with radiation absent need necessarily be extendable when arbitrary radiation laws are introduced. Each law determines a different field with different singularities. Since $P = \rho T$, then the sound speed is effectively $\sqrt{\gamma T}$. Taking the expression for the derivative of the temperature and dividing by the derivative of the velocity the following results:

$$(254)\quad \frac{dT}{dU} = \frac{\left[(\eta U' + U)Q^2 - (\eta U' + U + 2)T\right]\left[Q^2 - T - Q/\rho T\right]}{Q\left[(3UT - UQ^2) + 3UTQ/\rho + 2\bar{\sigma}\eta^5 T^4 Q/\rho\right]}$$

Resubstituting the expression for U' in the numerator:

$$(255)\quad \frac{dT}{dU} = \frac{\left[UQ^2 - (U+2)T\right][\alpha]}{Q[\beta]} + \frac{\eta U'\left[Q^2 - T\right][\alpha]}{Q[\beta]}$$

$$(256)\quad \frac{dT}{dU} = \left[UQ^2 - (U+2)T\right][\alpha] + \left[Q^2 - T\right][\beta]$$

call $\quad \gamma T = c^2$

$$(257)\quad dT = \frac{2c}{\gamma}\, dc$$

$$\text{(258)}\quad \frac{dc}{dU} = \frac{[UQ^2-(U+2)c^2/\gamma][\alpha]+[Q^2-c^2/\gamma][\beta]}{2cQ\left[(3Uc^2/\gamma-UQ^2)+\frac{3UQ}{P}\frac{c^2}{\gamma}+2\bar{\sigma}\eta^5T^4Q/P\right]}$$

The slope is infinite when the denominator is zero: c = 0; U = 0;

$$\text{(259)}\quad \left[(3U\frac{c^2}{\gamma}-UQ^2)+3\frac{UQ}{P}\frac{c^2}{\gamma}+2\bar{\sigma}\eta^5\frac{c^8}{\gamma^8}\frac{Q}{P}\right]=0$$

It is immediately noticed that with the introduction of radiation η enter's explicitly, a situation which hitherto did not occur. The additional contribution in the denominator due to the radiation is:

$$\text{(260)}\quad \left(\frac{2\bar{\sigma}\eta^5c^8Q}{\gamma^8P}\right)$$

When the radiation disappears the expression reduces to the old result.

4.8 Discussion of c_v

The ideal gas law has been written in the form

$$\text{(261)}\quad P=\rho T \quad ; \quad E=c_vT$$

and $E = c_v T$

The implications of this are immediate from the second law of thermodynamics:

(262.1) $$T\,ds = du + P\,dv$$

(262.2) $$ds = c_v \frac{dT}{T} + \rho\, d(1/\rho)$$

(262.3) $$\exp(s - s_0) = \frac{T}{\rho}^{c_v} \left(\rho_0 / T_0^{c_v} \right)$$

(262.4) $$P = A \exp(s - s_0)\, \rho^{\left(1 + \frac{1}{c_v}\right)}$$

Hence $\gamma = 1 + 1/c_v$

usually $\gamma = 1 + R/c_v$

Hence c_v as used in this formulation:

(263) $$c_v \rightarrow c_v/R = f/2$$

For three degrees of freedom $c_v = 3/2$ and $= 5/3$ as aforementioned.

4.9 Conditions for Numerical Integration

The shock wave trajectory is determined when is given together with . For an implosion wave, the shock velocity should be negative. This can be procued by any of the following conditions.

a. $\delta > 0$; $(\lambda < 0) \vee (\tau < 0) \wedge (\gamma > 0)$

b. $\delta < 0$; $(\lambda > 0) \wedge (\tau > 0) \wedge (\gamma > 0)$

c. $\delta > 0$; $(\lambda > 0) \wedge (\tau > 0) \vee (\gamma > 0)$

The case "a" is used in this problem.

A reference length and time are usually employed in the nondimensional formulation of the original system of equations. The quantities employed in this section can be thought of as nondimensionalized with respect to a reference length of one meter and a reference time of one second.

The dimensionless shock speed is fixed at -10^3. That is, $\lambda\bar{\zeta} = -10^3$. In view of the above paragraph, the conversion to physical units is effected by a multiplication by one meter per second. In the pre-shock gas the fluid variables are assumed uniform and are not stipulated. In order to vary the influence of the radiative flux term the post-shock initial condition for the temperature is varied. The post-shock conditions for the dependent similarity invariants associated with velocity and density are respectively $U(\eta) = .95$ and $R(\eta) = 1$. Consequently, the dimensional initial flow velocity would be -9.5×10^2 meters per second. In the shock frame of reference the post shock gas is moving away subsonically from the shock wave.

It is obviously not possible to stipulate all of these quantities consistent with the same pre-shock conditions for each value of the post-shock temperature. The jump equations impose constraints on the proper values. The shock jump conditions need to be solved numerically. In lieu of this, the initial conditions chosen are those which one might typically expect to find in a laboratory experiment.

To establish accurate mathematical and physical initial conditions for the solution of a given problem is an horrendous task. The shock Hugoniots would have to be solved numerically for triplets of pre-shock conditions and for positive and negative values of η. The jump equations are not necessarily invariant under a change of sign of η. Therefore, a set of pre-shock conditions suitable for a positive η regime are not necessarily suitable for a negative η regime. The field of integral curves should be determined together with all sonic, subsonic and supersonic regions and singularities. A particular problem may be represented by one curve in the vector field and appropriate initial conditions established by discovering where the integral curve intersects the surface generated by the jump conditions. A discussion of this approach for a non-radiative problem can be found in the paper of Guderley.[22] With radiation the difficulties are multiplied.

In the problem under consideration, the temperature across the shock wave is discontinuous. This discontinuity is compatible with an opaque shock wave.

4.10 Results of the Computation

The density and velocity fields for all practical purposes do not vary appreciably over the integration range for each computation. This result is not unexpected. Consequently, the curves for density and velocity are not reproduced.

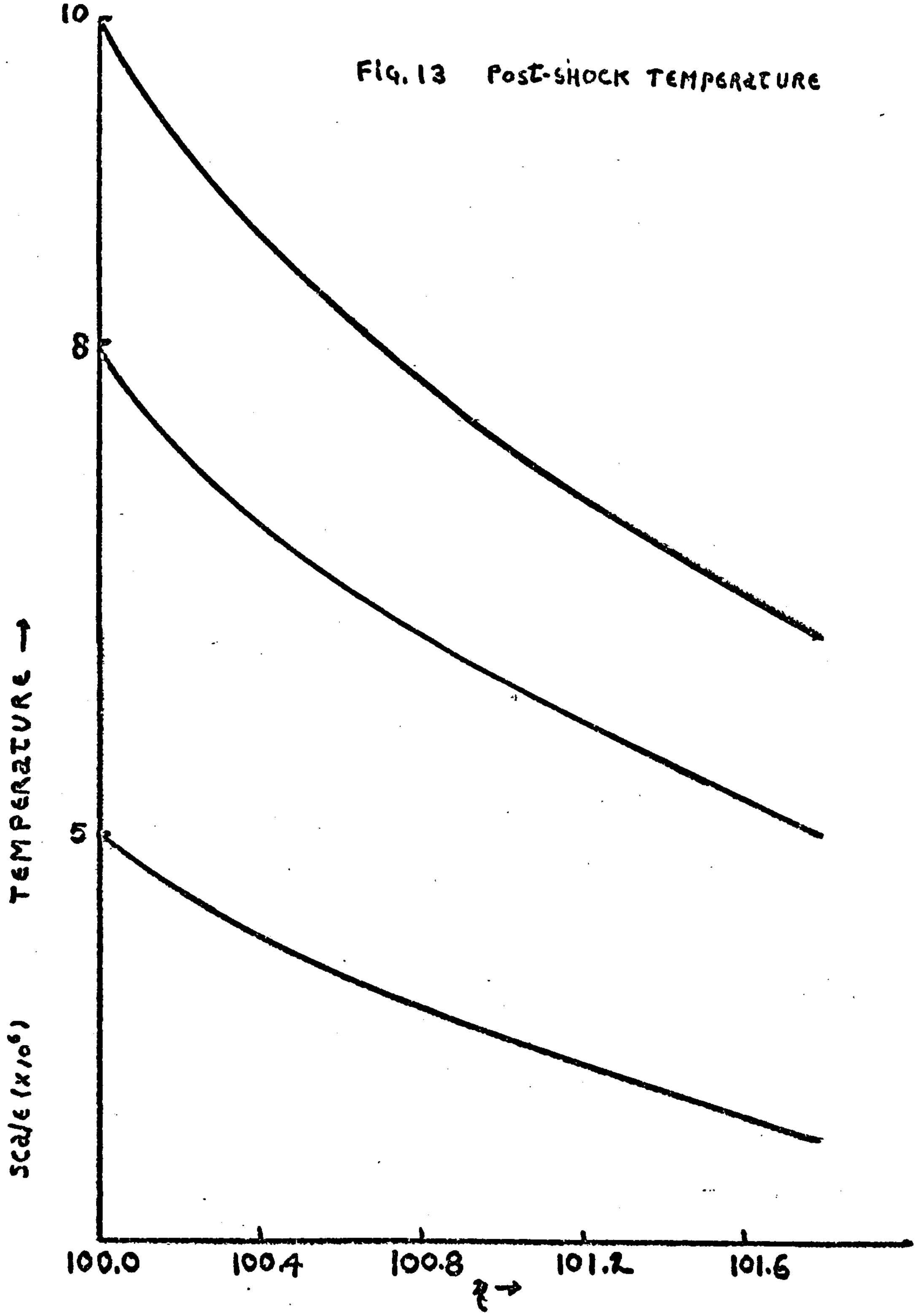
The effect of the radiative flux is to cool the gas directly downstream from the shock wave. The temperatures in this region are lower than what they normally would be without radiation. This statement is validated by considerations of section (4.9). The relative decrease of the temperature is, in each case the same. The asymptotic value approached is higher for higher values of the initial conditions. (See Figure 13).

The situation of constant shock velocity, a lower than normal downstream temperature and a constant shock strength are all compatible and represent a plausible model, at least in the gross features.

4.11 Analytical Approach

Frequently insight into the nature of the problem can be obtained by approximate solutions. Several calculations are effected in this section illustrating an available approach.

Fig. 13 Post-shock Temperature



A limiting case of the computer experiment is subsequently produced. The assumption is made that the contribution due to radiation is negligible. The quantity U' is then:

$$(264) \qquad U' \approx -\frac{U}{\eta}\,\frac{\left[Q^2 - 3T - 3QP/T\right]}{\left[Q^2 - T - Q/P\right]}$$

The statement $P \approx c_v Q$ can be made, but is not necessary, since by inspection:

$$(265) \qquad U' \approx -\frac{U}{\eta}\,\alpha$$

where α is a number between 1 and 3. From the continuity equation:

$$(266) \qquad \eta\,\frac{R'}{R} \approx -\frac{(3-\alpha)U_0}{(U_0 - \eta^{\alpha})}$$

Consequently:

$$(267) \qquad R = R_0 \exp \int -\frac{(3-\alpha)U_0}{\eta(U_0 - \eta^{\alpha})}\,d\eta$$

Hence generally:

(268) $$R = R_0 \exp - \frac{(3-\alpha)}{\alpha} \ln \left[\eta^{\alpha} / U_0 - \eta^{\alpha} \right]$$

(269) $$R = R_0 \left[\eta^{\alpha} / U_0 - \eta^{\alpha} \right]^{-\frac{(3-\alpha)}{\alpha}}$$

The temperature is established from the momentum equation, which has the form:

(270) $$f_0 T' + f_1 T + f_3 = 0$$

The solution to which is:

(271) $$T = \left[\exp - \int \frac{f_1}{f_0} d\eta \right] \left[\int - \frac{f_3}{f_0} \left[\exp \int \frac{f_1}{f_0} d\eta \right] d\eta \right]$$

(272) $$f_0 = \eta \;;\; f_1 = 2 + \frac{\eta R'}{R} \;;\; f_3 = (U-1)(\eta U' + U)$$

(273) $$T = \left[\exp - \int \left(\frac{2}{\eta} - \frac{(3-\alpha) U_0}{\eta (U_0 - \eta^{\alpha})} \right) d\eta \right] \times$$
$$\left[\int - \frac{(1-\alpha) U_0 \eta^{-\alpha} (U_0 \eta^{-\alpha} - 1)}{\eta} \exp \int \left(\frac{2}{\eta} - \frac{(3-\alpha) U_0}{\eta (U_0 - \eta^{\alpha})} \right) d\eta \right]$$

(274) $$T = \eta^{-2}\left(\frac{\eta^{\alpha}}{U_0-\eta^{\alpha}}\right)^{\frac{3-\alpha}{\alpha}}\left[-\int \frac{(1-\alpha)U_0\eta^{-\alpha}(U_0\eta^{-\alpha}-1)}{\eta}\cdot\eta^{2}\left(\frac{\eta^{\alpha}}{U_0-\eta^{\alpha}}\right)^{-\frac{(3-\alpha)}{\alpha}} d\eta\right]$$

(275) $$N = -\int \frac{(1-\alpha)U_0\,(\eta^{\alpha})^{-\left(\frac{2}{\alpha}+1\right)}}{(U_0-\eta^{\alpha})^{-3/\alpha}}\,d\eta$$

No general integral of N is available. The above form breaks down when α is one, setting $N = T_0$ produces the desired result.

For $\alpha = 1$, the forms follow:

<u>Similarity</u>	<u>Actual</u>
$U = U_0/\eta$	$U = U_0$
$R = R_0\left(\eta/U_0-\eta\right)^{-2}$	$R = R_0\left(\eta/U_0-\eta\right)^{-2}$
$T = \eta^{-2}\left(\frac{\eta}{U_0-\eta}\right)^{2}T_0$	$T = \left(\frac{\eta}{U_0-\eta}\right)^{2}T_0$

$\alpha = 1$ should imply for validity of the approximation that $3T/Q^2 \ll 1$: this is true for $\eta \gg 3$ since MAX $U_o = 1$. For $\alpha = 3$, $U = U_o\eta^{-3}$ and $R = R_o$.

$$N = \lambda \int \frac{U_o}{\eta^5}(U_o - \eta^3)\,d\eta \tag{276}$$

$$N = \left(-\frac{U_o^2}{2\eta^4} + \frac{2U_o}{\eta}\right) + \text{constant.} \tag{277}$$

$$T(\eta) = \frac{\lambda U_o}{\eta^3} - \frac{U_o^2}{\lambda\eta^6} \tag{278}$$

$$T = \left(\frac{2U_o}{\eta} - \frac{U_o^2}{\lambda\eta^4}\right) + \text{constant} \tag{279}$$

To digress, it is noted that the temperature diverges as $\eta \to 0$. Appropriate values for α are to be computed on the basis of initial conditions.

In the absence of radiation the energy invariable leads to an invariant when $k = 0$ and $\delta = 1$ in planar, cylindrical and spherical coordinates:

$$c_v[U-1]\eta T' + 2c_v T[U-1] = -T[\beta U + \gamma U'] \tag{280}$$

$\beta = 1, 2, 3$ corresponding to the coordinates mentioned above. But

$$- \frac{[\beta U + \eta U']}{[U-1]} = \frac{\eta R'}{R} \tag{281}$$

Hence dividing the energy equation by $[U - 1]\, Tc_v$ and integrating results in:

$$\frac{\eta^2 T}{R^{1/c_v}} = \text{constant} \tag{282}$$

This leads to the result $s = s_o$, a constant. Hence $k = 0$, $\delta = 1$ corresponds to an adiabatic motion as expected.

4.12 <u>The Iterative Scheme for Arbitrary Flux Laws</u>

$$Q = [U - \delta] \tag{283}$$

$$P = \left[c_v Q + \frac{\beta}{aR} \frac{\partial I_v}{\partial T} \right] \tag{284}$$

Writing the energy equation:

(285) $$c_v[U-\delta]\eta T' + 2c_v[U-1] = -T[3U+\eta U'] - (k+5)\beta I_\nu/aR - \beta/aR\,\eta\,\frac{DI_\nu}{D\eta}$$

(286) $$\eta\frac{DI_\nu}{D\eta} = \frac{\partial I_\nu}{\partial T}\eta T' + \frac{\partial I_\nu}{\partial R}\eta R' + \frac{\partial I_\nu}{\partial \eta}\eta$$

then

(287) $$\eta T' = -\frac{T}{P}\left[2c_v(U-1) + (3U+\eta U') + \frac{(k+5)\beta I_\nu}{aRT} + \frac{\beta}{TaR}\eta\frac{\partial I_\nu}{\partial \eta} + \frac{\partial I_\nu}{\partial R}\frac{\eta R'}{aRT}\right]$$

Substitution of this expression into the momentum equation produces the desired isolation of U' required for the previously suggested iteration scheme. Using the continuity to eliminate $\frac{R'}{R}$ in terms of U.

(288) $$\left\{Q\eta U' + U^2 - U + \left[(k+2) - \frac{[(k+3)U+\eta U']}{Q} - \frac{2c_v}{P}(U-1) - \frac{(3U+\eta U')}{P} - \frac{(k+5)}{aRT}\frac{\beta I_\nu}{P} - \frac{\beta\eta}{PaRT}\frac{\partial I_\nu}{\partial \eta} + \frac{\partial I_\nu}{\partial R}\frac{[(k+3)U+\eta U']}{PQaT}\right]\right\} = 0$$

(289) $$\eta U'\left[Q - \frac{T}{Q} - \frac{T}{P} + \frac{\partial I_\nu}{P\partial R\,Q a}\right] = \left[-U^2 + U - (k+2)T + \frac{(k+3)UT}{Q} + 2c_v(U-1)T/P + 3UT/P + (k+5)\beta I_\nu/aRP + \frac{\beta\eta}{aRP}\frac{\partial I_\nu}{\partial \eta} - \frac{\partial I_\nu}{\partial R}\frac{(k+3)U}{aP}\right]$$

(290) $$U' = \frac{(R.H.S)}{\eta * (L.H.S)}$$

(291) $$R' = -\frac{R}{\eta}\frac{\left[(3+k)U + (RHS)/(LHS)\right]}{Q}$$

and

(292) $$\eta T' = \frac{(RHS)}{(LHS)} - \frac{U^2}{\eta} + \frac{U}{\eta} - \left[(k+2) + \eta\frac{R'}{R}\right]T$$

Equations (290) through (292) are the general iterative scheme for arbitrary pairs of k and δ and for arbitrary flux laws.

Similarity representation for the entropy, From the second law:

(293) $$S = S_o + \ln\left(P\rho^{-\gamma}\right)$$

(293.1) $$S = S_o + \ln\left(r^{K+2}\tau^{-2} R(\eta)^{1-\gamma K} T(\eta) r^{-\gamma K}\right)$$

(293.2) $$S = S_o + \ln\left[r^{K(1-\gamma)+2}\tau^{-2}\right] T(\eta) R(\eta)^{1-\gamma K}$$

(293.3) $$\eta = r/\lambda\tau^{\delta}$$

(293.4 $$\tau^{-2} = \left(\frac{r}{\lambda\eta}\right)^{-2/\delta}$$

(293.5) $$S = S_0 + \mathrm{Ln}\left[r^{\kappa(1-\gamma)+2-2/\delta}\right]\left[\lambda\eta\right]^{2/\delta} T(\eta)\, R(\eta)^{(1-\gamma\kappa)}$$

4.13 Negligible Radiative Transfer-Heat Flux

The discussion in this section bears upon problems in which the divergence of the sum of the radiative flux the conductive heat flux is either negligible in comparison to other terms or zero. Explicitly:

(294) $$\nabla \cdot \left(q_c + F^R\right) = 0$$

The momentum and energy equations are respectively:

(295) $$\eta U'[U-\delta] + U^2 - U + \left[(k+2) + \frac{\eta R'}{R}\right]T + \eta T' + (k+2)\frac{L^2 I_\nu \beta}{aRC} + \eta\frac{L^2 \beta I_\nu'}{aRC} - \eta L^2 \frac{\beta I_\nu}{aRC}\frac{c'}{c} = 0$$

(296) $$[U-\delta]\left[\eta T' c_v + \eta\frac{\beta L^0 I_\nu'}{aRC} - \eta\frac{\beta L^0 I_\nu}{aRC}\frac{R'}{R} - \eta\frac{L^0 \beta I_\nu}{aRC}\frac{c'}{c}\right] + 2[U-1]\left[c_v T + \frac{\beta}{aRC}L^0 I_\nu\right] = -\left[T + \frac{\beta}{aRC}L^2 I_\nu\right]\left[3U + \eta U'\right]$$

In order to widen the class of readily manageable problems, an artiface is introduced. Define:

(297) $$c_v = \frac{L^0 I_\nu}{L^1 I_\nu}$$

Hence, whenever the ratio of the radiant energy to radiant pressure yields an acceptable value for the specific heat at constant volume of a gas, the statement may be made and will be proved that the velocity field is independent of the thermodynamic structure of the radiation field. With this artiface the specific heat is related to the angular distribution of the radiant intensity. The range of specific heats is determined from total isotropy of I_ν to total undirectionality. For these two antipodes of the spectrum of angular distribution the following results. When I_ν is totally isotropic:

(298) $$U^R = L^0 I_\nu = 4\pi I$$

(299) $$P^R = L^2 I_\nu = \frac{4\pi I}{3} = \frac{U^R}{3}$$

Hence $c_v = 3$. For I_ν strongly anistropic, The MAX P^R is U^R.

(300) $$\left| \int f(\theta) \cos^2\theta \sin\theta \, d\theta \right| \leq \left| \int f(\theta) \sin\theta \, d\theta \right|$$

Since MAX ($\cos^2\theta$) = 1 then $U^R = \rho^R$ and $c_v = 1$. Thus the artiface introduced as a mathematical convenience, usable on its own merit, is acceptable physically since realizable gases correspond to admissable quotients. To digress, this raises the question of the relation between the number of degrees of freedom of a radiant gas and the angular distribution of the radiant intensity. What would be the effect of a magnetic field? Define:

(301) $$z = \left[T + \frac{L^2 I_\nu}{aRc} \right]$$

Adding and subtracting $\frac{R'}{R} \frac{L^2 I_\nu}{aRc}$ from the momentum equation the energy and momentum equations in terms of z become:

(302) $$c_v \eta z'[U-\delta] + 2c_v z[U-1] = -z[3U+\eta U']$$

(303) $$\eta U'[U-\delta] + \left[(k+2) + \frac{\eta R'}{R}\right] z + \eta z' + U^2 - U = 0$$

The character of the above equations with z in place of T resembles the non radiative system. For the case $k = 0$, $\delta = 1$

the energy equation yields an invariant, dividing by c_v $z[U - 1]$:

(304) $$\frac{Z'}{Z} + \frac{2}{\eta} = \frac{R'}{c_v R}$$

(305) $$\frac{(Z\eta^2)}{R^{1/c_v}} = \text{constant}$$

(306) $$\frac{\eta^2 T(\eta)}{R^{1/c_v}} = \left(\text{Const.} - \frac{\eta^2 L^\circ I_\nu}{a R^\gamma}\right)$$

(307) $$S = S_0 + \text{Ln}\left(\text{Const} - \eta^2 \frac{L^\circ I_\nu}{a R^\gamma}\right)$$

Clearly from the energy equation:

(308) $$Z = Z_0 \exp \int \left[- \frac{[3U + \eta U']}{c_v \eta [U - \delta]} - \frac{2[U - 1]}{\eta [U - \delta]} \right] d\eta$$

Using the continuity equation, it follows that the momentum equation can be written solely in terms of the velocity field. This implies the velocity field is independent of the form of I_ν since the latter holds for arbitrary z.

4.14 Approximate Solution

Again, isolate U' from substitution of the energy and continuity expressions into the momentum equation.

(309) $$\left\{\eta U'[U-\delta]+U^2-U+\left[(k+2)-\frac{(k+3)U+\eta U'}{[U-\delta]}-\frac{[3U+\eta U']}{c_v[U-\delta]}\right]z - 2\frac{[U-1]}{[U-\delta]}z\right\}=0$$

(310) $$\eta U'\left[Q-(1+1/c_v)z/Q\right]=$$

$$-U^2+U-(k+2)z+(k+3+3/c_v)\frac{zU}{Q}+\frac{2[U-1]z}{Q}$$

(311) $$\eta U' = \frac{-U^2+U-(k+2)z+(k+3+3/c_v)\frac{zU}{Q}+\frac{2(U-1)z}{Q}}{Q-(1+1/c_v)z/Q}$$

Multiplying numerator and denominator by Q:

(312) $$\eta U' = \frac{-U(U-1)Q-(k+2)zQ+(k+3+3/c_v)zU+2(U-1)z}{Q^2-(1+1/c_v)z}$$

Rearranging the numerator:

(313) $$U' = \frac{1}{\eta}\,\frac{\left[-U\left(Q(U-1)-3\gamma z\right)+\left((k+2)\delta-2\right)z\right]}{Q^2-\gamma z}$$

Assuming $z/Q^2 \approx$ const.

(314) $$U' = \frac{1}{\eta}\left[-U\left[\left(\frac{U-1}{U-\delta}\right) - 3\gamma\alpha\right] + \left[(k+2)\delta - 2\right]\alpha\right] \Big/ (1-\gamma\alpha)$$

The physical significance of the assumption $z/Q^2 \approx$ constant is that in the shock frame of reference the work done by pressure forces is proportional to the translational energy of the gas. Consider the simplest case when $\delta = 1, \forall k$. Call: $a = \left[\frac{1-3\gamma\alpha}{1-\gamma\alpha}\right]$; $b = \frac{k}{(1-\gamma\alpha)}$.

(315) $$\eta U' = -aU + b$$

(316) $$U = b/a + \eta^{-a} U_0$$

(317) $$\frac{\eta R'}{R} = -\frac{\left[(k+3-a)(b/a + \eta^{-a} U_0) + b\right]}{\left[U_0 \eta^{-a} + b/a - \delta\right]}$$

(318) $$R = R_0 \exp - \int \frac{\left[(k+3-a)(U_0 + \eta^{a} b/a) + b\right]}{\left[U_0 + (\frac{b}{a} - \delta)\eta^{a}\right] \cdot \eta} d\eta$$

The momentum equation is of the form:

(319) $$f_0 z' + f_1 z + f_2 = 0$$

(320) $f_0 = \eta \ ; \ f_1 = \left(k+2 + \frac{\eta R'}{R} \right) \ ; \ f_2 = [U-\delta][\eta U' + U]$

The standard solution is:

(321) $$Z = \left[\exp - \int \frac{f_1}{f_0} d\eta \right] \left[\int \left[- \frac{f_2}{f_0} \exp \int \frac{f_1}{f_0} d\eta \right] d\eta \right]$$

These expressions for z, R, U are for arbitrary α and explicitly include parametric dependence on the homology number k.

To digress, in planar coordinates the similarity equations are altered by virtue of the divergence terms. The momentum and energy equations are:

(322) $$\eta U'(U-\delta) + U^2 - U + \left[(k+2) - \frac{[kU + \eta U']}{[U-\delta]} \right] Z + \eta Z' = 0$$

(323) $$\eta Z'[U-\delta] + 2[U-1] Z = - \frac{Z}{C_v} [U + \eta U']$$

The velocity derivative is:

(324) $$U' = \frac{[-U(U-1)Q + [1 + 1/c_v] Z U + [(k+2)\delta - 2] Z}{\eta [Q^2 - Z(1 + 1/c_v)]}$$

An equation which is almost identical to the previous. Set:

$$(k+2)\delta - 2 = 0$$

$$\delta = 1 \Rightarrow k = 0$$

then

(325.1) $$U' = -U/\eta$$

(325.2) $$U = U_0 \eta^{-1}$$

(326) $$R(\eta) = \text{const.}$$

(327) $$2z + \eta z' = 0 \; ; \; z = z_0 \eta^{-2} \quad \forall c_v$$

The temperature distribution is discovered when I_ν is expressly stipulated. This exactly solvable problem is the trivial equilibrium and uniform flow problem.

4.15 A Further Special Case

It is worthwhile to ask when the approximation schemes used in previous sections may be applied to the case where $\nabla \cdot (q_i + F_i^R) \neq 0$. If the heat conduction is written:

(328) $$q_i = \lambda_o \frac{\partial T}{\partial r}$$

if

(329) $$F_i^R = D^R \frac{\partial L^\circ I_\nu}{\partial r}$$

Clearly if $\lambda_o / D^R \equiv c_v$, the sum of the above expressions is nothing more than a constant multiplied into the gradient of the internal energy of the gas. The inclusion of this term raises the order of the energy equation to two. Furthermore, the similarity structure of the term parallels that of the heat conduction alone. To include it requires an additional constraint on the system unless the coefficients possess appropriate similarity structure. In addition, if the similarity form of $\nabla \cdot (q_i + F^R) = H(z, z'z'',)$ for an arbitrary function H, the conclusion that the velocity field is independent of the structure of the radiation still holds. The physical result that is concluded is that the real coupling between the material and radiation is the flux dissipation terms in the energy equation.

CHAPTER V

5.1 Remarks

The Birkhoff formalism was introduced in Chapter I. This approach was not needed in the main body of the work. The usual homology associated with the fluid variables can be ascertained when, for example, the homology associated with the density is stipulated. The proper homology for the radiative terms was discovered through requiring compatability with the energy equation. If it is desired to analyse equations not as well studied as the fluid equations, the Birkhoff formalism is an exceedingly useful guide in ascertaining admittable similarity forms.

It is natural to ask whether a less intuitive method for discovering proper self similar forms is feasable. This chapter constitutes an initial inquiry into an alternate procedure that could be used. The basic concepts introduced are not reduced to a rigorous mathematical formalism but are suggested as an interesting approach which to be properly useful would require further development adn substantiation.

5.2 The Boundary Value Problem in the Similarity Approach

The only really adequate formalism that can embrace properly posed boundary value problems for Hyperbolic systems is the method of characteristics. In some cases, however, problems of this type are amenable to solution compatible with similarity ideas. Usually, in inquiring into a similarity solution, boundary value problems are abandoned or conjured a' posteriori for purposes of compatability. A simple criterion exist by which it is possible to ascertain whether a particular boundary value problem can be reduced to self similarity. The following example illustrates this: Let Q(x,y) be a function to which there corresponds a differential relation. Let Q(x,y(x)) be given on the curve, y = y(x), refer to Figure 14. The similarity representation of Q is of the form:

$$x^m y^n Q(\eta) = Q(x,y) \tag{330}$$

at the boundary:

$$x^m y^n . Q(\eta)\Big|_{y=y(x)} = Q(x,y)\Big|_{y=y(x)} \tag{331}$$

$Q(\eta)$ is a constant on lines of constant η. Consequently, if lines of constant η thread the curve y = y(x), then the solution $Q(\eta)$ to the associated ordinary differential equation is established by initial conditions. If this were valid, the initial conditions would establish a solution for the evolution of the system independent of the structure of the equation. Obviously, this is not a correct approach. Therefore, the function $Q(\eta)$ associated with the similarity transformed differential relation must be a constant on the boundary arc. The appropriate constant constitutes an initial condition for integrating the transformed differential relation. The boundary curve must be a line of constant η. The admissable boundary curves are precisely:

(332) $$y(x) = \eta^* x^\delta$$

The appropriate boundary data must be of the form:

(333) $$Q(x,y)\Big|_{y=y(x)} = B x^p$$

where B and p are constants.

5.3 Basic Concept

The suggested technique depends on a combination of two classical concepts: transformation to general coordinates and the Birkhoff search for symmetric solutions. Prior to effecting a similarity treatment of a given system, the equations are written in a general coordinate system. The metrical coefficients associated with the general coordinates contribute to the system as assumed known functions. A reduction to self similarity is effected. In the process the metrical coefficients are given a similarity representation. The constraint that the system of equations be "absolutely conformally invariant" determines a class of general coordinates compatible with a similarity representation of the system. The inversion of the transformation for any member of this class produces a curve in the original system which can be represented by a constant value of the independent invariant in the transformed system, provided the transformation is not singular.

An extension to the Birkoff search can produce different curves compatible with similarity by properly designing the transformation Γ. Suppose there are two independent variables x_1 and x_2 and $\alpha_1 = 0$. Then define:

(334.1) $$\hat{x}_1 = x_1 + \ln a$$

(334.2) $$\hat{x}_2 = a^{\alpha_2} x_2$$

(335) $$\eta = x_2 / \exp x_1 \alpha_2$$

clearly $$\Gamma(\eta) = \eta$$

In retrospect, the latter procedure seems deceptively simple. The construction of general transformations requires considerable ingenuity. The value of the alternate method suggested is to reduce somewhat the need for intuitive guessing. The approach is pedestrian in comparison to the simple elegance depicted by Birkhoffs process. It could be more than competitive with the classical approach if the transformations could be inverted with some degree of generality. Indeed, if for a given system the transformation could be inverted, all similarity motion compatible with the system would then be determined. This unfortunately has not been accomplished. Only by trial and error is it possible to invert these at present.

5.4 Formulation

Introduce a system of differential relations by the symbol Σ in which there are p dependent variables $\{\xi_i\}$ and m independent variables $\{x_i\}$. Before using the Birkhoff formalism, introduce a general coordinate transformation on the system. Define:

(336) $$\bar{x}_i = \bar{x}_i(x_i)$$

(337) $$\frac{\partial}{\partial x_j} = \frac{\partial \bar{x}_i}{\partial x_j} \frac{\partial}{\partial \bar{x}_i}$$

and

$$(338) \qquad \Sigma_1 \rightarrow \bar{\Sigma}_1 \; ; \quad \zeta_i \rightarrow \bar{\zeta}_i$$

The transformation may be left unspecified until a later stage in the development of a particular problem. In the bar corrdinate system effect the transformation

$$(339) \qquad \hat{\bar{x}}_i = a^{\alpha_i} \bar{x}_i$$

$$(340) \qquad \hat{\bar{\zeta}}_i = a^{\gamma_i} \bar{\zeta}_i$$

$$(341) \qquad \Sigma \rightarrow \bar{\Sigma}'$$

$$(342) \qquad \frac{\delta \hat{\bar{x}}_i}{\delta x_j} = a^{\delta_{ij}} \frac{\delta \bar{x}_i}{\delta x_j}$$

α_i, γ_i and δ_{ij} are constants and the constant "a" is the generator of the transformation. The dependent invariants are:

$$(343) \qquad \tilde{\bar{\zeta}}_i = \bar{\zeta}_i \Big/ \bar{x}^{\gamma_i/\alpha_k} \qquad k \text{ fixed, } i = 1 \cdots p$$

The independent invariants are:

(345) $$\frac{\delta\tilde{\bar{x}}}{\delta x_j} = \frac{\delta \bar{x}_i}{\delta x_j} \Big/ \bar{x}_k^{(\delta_{ij}/\alpha_k)}$$

The requirement that the system of differential equations be absolutely conformally invariant is:

(346) $$\bar{\bar{\Sigma}} = \hat{\bar{\bar{\Sigma}}}$$

This produces ℓ constraints on the α_i, γ_i and δ_{ij} such that the number of unidentified constants in the formulation is:

(347) $$p + m + m^2 - \ell = \Delta$$

Now

(348) $$\frac{\delta\tilde{\bar{x}}_i}{\delta x_j} \equiv f_{ij}(\eta) .$$

where the f_{ij} are arbitrary functions.

(349) $$\frac{\delta \bar{x}_i}{\delta x_j} = \frac{\delta\tilde{\bar{x}}_i}{\delta x_j}\, \bar{x}_k^{(\delta_{ij}/\alpha_k)} = f_{ij}\, \bar{x}_k^{(\delta_{ij}/\alpha_k)}$$

The symbolic quadrature of equation (349) is:

$$x_j = x_j\left(\bar{x}_i,\ f_{ij}(\eta),\ \bar{x}^{\delta_{ij}/\alpha_k}\right) \tag{350}$$

Curves which can be represented by (350) are exactly transformable. There exists a transformation to a system in which they are representable by a constant value of the independent invariant.

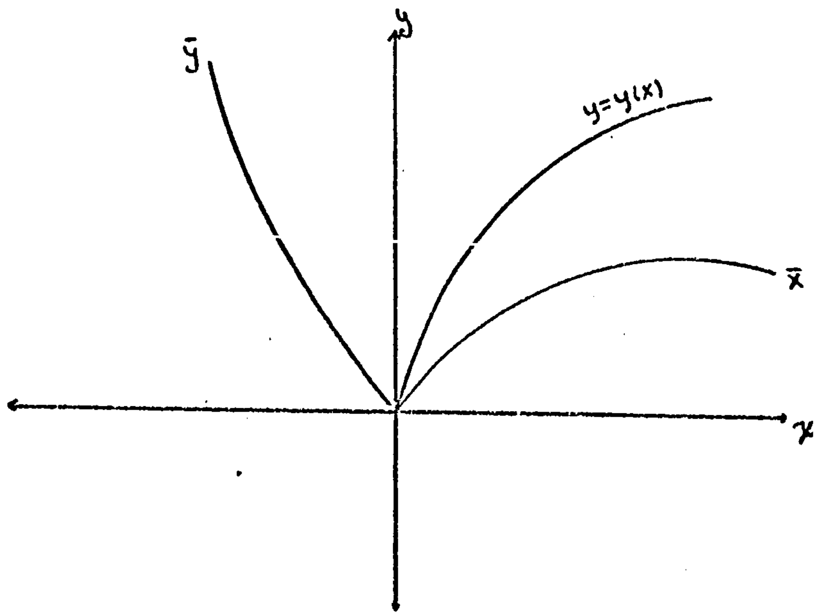


Figure 14 Sketch of Coordinates

Let y = y(x) (refer to Figure 14) be an initial data curve which is not of the form $y = x^{\delta}$ for some δ. In the x-y coordinate system it cannot be represented by some constant By first using the general coordinate transformation it is possible to ascertain a system $\bar{x} - \bar{y}$ in which the given curve can be expressed by a constant value of the independent invariant associated with the bar system.

From equation (349) the class of curves that are exactly transformable in two dimensions is:

(351.1) $$\frac{\delta \bar{x}}{\delta x} = f_{x\bar{x}}(\eta)\,\bar{x}^{\delta_{\bar{x}x}/\alpha_1}$$

(351.2) $$\frac{\delta \bar{x}}{\delta y} = f_{\bar{x}y}(\eta)\,\bar{x}^{\delta_{\bar{x}y}/\alpha_1}$$

(351.3) $$\frac{\delta \bar{y}}{\delta x} = f_{\bar{y}x}(\eta)\,\bar{x}^{\delta_{\bar{y}x}/\alpha_1}$$

(351.4) $$\frac{\delta \bar{y}}{\delta y} = f_{\bar{y}y}(\eta)\,\bar{x}^{\delta_{y\bar{y}}/\alpha_1}$$

For curves which satisfy the relations (351) no further constraints are imposed on the set of α_i, γ_i and δ_{ij}, other than those due to the differential system. .

5.5 Short Tabulation of Admissable Inversions

Case I:

(352) $\bar{x} = x^{\rho};\ \bar{y} = \exp(y - x^{m})$

then

(353) $$\frac{\partial \bar{y}}{\partial y} = \bar{Y} = \bar{x}^{\beta_3} f_{21}$$

(354) $$\frac{\partial \bar{y}}{\partial x} = -x^{m-1}\bar{y} = \bar{x}^{\beta_4} f_{22} = -\bar{x}^{(m-1)/\rho}\bar{Y}$$

with

(355) $$\beta_4 - \frac{(m-1)}{\rho} = \beta_3$$

then

(356) $$\eta = \bar{Y}/\bar{x}^{\alpha_2/\alpha_1} = \exp(y - x^{m})/x^{\rho\alpha_2/\alpha_1}$$

From this example the classical result (335) is easily produced by setting; $p = 1$, $m = 0$, $y = y + 1$. Further tabulations are computed without comment:

Case II:

(357.1) $$\bar{x} = x^{n}$$

(357.2) $$\bar{y} = y - ax^{m}$$

(357.3) $$\frac{\partial \bar{x}}{\partial x} = \bar{x}^{\beta_1} f_{11} = nx^{n-1} = n\bar{x}^{(n-1/n)}$$

(357.4) $$f_{11} = n \; ; \; \beta_1 = (1 - 1/n)$$

The β's abbreviate the homology of the metrical coefficients.

(357.5) $$\frac{\partial \bar{x}}{\partial y} = 0 \; ; \; f_{12} = 0$$

(357.6) $$\frac{\partial \bar{y}}{\partial y} = 1 \; ; \; f_{22} = 1 \; ; \; \beta_4 = 0$$

(357.7) $$\frac{\partial \bar{y}}{\partial x} = -max^{m-1} = -am\bar{x}^{(m-1)/n}$$

(357.8) $$\beta_3 = (m-1)/n$$

and

(357.9) $$\eta = \bar{Y}/\bar{x}^{\alpha_2/\alpha_1} = \left(\frac{y - a x^m}{x^{n\alpha_2/\alpha_1}} \right)$$

Case III:

Set:

(358) $$\bar{X} = \exp(x+iy)a$$

(358.1) $$\bar{y} = \exp(x-iy)b$$

(358.2) $$\frac{\partial \bar{x}}{\partial x} = a\bar{x} \ ; \ \frac{\partial \bar{x}}{\partial y} = ia\bar{x}$$

(358.3) $$\frac{\partial \bar{y}}{\partial y} = -ib\bar{y} \ ; \ \frac{\partial \bar{y}}{\partial x} = b\bar{y}$$

(358.4) $$\beta_1 = \beta_2 = 1 \ ; \ \beta_3 = \beta_4$$

(358.5) $$\eta = \exp\left[b(x-iy) - \frac{\alpha_2}{\alpha_1} a(x+iy)\right]$$

Case IV:

Set:

(359.1) $\bar{x} = \exp(x+iy) + \exp-(x+iy)$

(359.2) $\bar{y} = \exp(x+iy) - \exp-(x+iy)$

(359.3) $\frac{\partial \bar{x}}{\partial x} = \bar{y}$; $\frac{\partial \bar{y}}{\partial x} = \bar{x}$

(359.4) $\frac{\partial \bar{x}}{\partial y} = i\bar{y}$; $\frac{\partial \bar{y}}{\partial y} = i\bar{x}$

(359.5) $f_{12} = i/\eta$; $f_{22} = i$; $f_{11} = 1/\eta$; $f_{21} = i$

with $\beta_1 = \beta_2$

Set:

(360.1) $f_{x\bar{y}} = \eta_1$; $f_{y\bar{y}} = -\eta_1$ $\eta_1 =$ constant

(360.2) $f_{x\bar{x}} = f_{\bar{x}y} = \text{constant} = \eta_1$; $\beta_2 = 0$

(360.3) $d\bar{x} = \bar{x}^{\beta_1}\eta_1(dx + dy)$

(360.4) $$d\bar{y} = \eta_1 (dx - dy)$$

(360.5) $$\bar{x} = \left[\eta_1 (x+y+c_1)(1-\beta_1) \right]^{1/1-\beta_1}$$

(360.6) $$\bar{y} = \left[\eta_1 (x-y+c_2) \right]$$

Curves of constant η are:

(360.7) $$\eta = \frac{\left[\eta_1 (x-y+c_2) \right]}{\left[\eta_1 (x+y+c_1)(1-\beta_1) \right]^{\alpha_2/\alpha_1 (1-\beta_1)}}$$

5.6 Illustrative Computation

Let Σ_1' be the equation:

(361) $$U_y + UU_x + \nu U = 0$$

Under a transformation of coordinates this becomes:

(362) $$\left(\frac{\partial U}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y} + \frac{\partial U}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial y} \right) + U \left(\frac{\partial U}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial x} + \frac{\partial U}{\partial x} \frac{\partial \bar{x}}{\partial x} \right) + \nu U = 0$$

Consider the case (357.1) to (357.9)

(363) $$\bar{x} = x^{n} \; ; \; \bar{y} = y - a x^{m}$$

The above equation becomes:

(364) $$\frac{\partial U}{\partial \bar{y}} + U\left(\frac{\partial U}{\partial \bar{y}} (-a_m) \bar{x}^{\frac{m-1}{n}} + \frac{\partial U}{\partial \bar{x}} (n) \bar{x}^{\frac{n-1}{n}} \right) + \nu U = 0$$

Also

(365.1) $$\delta_{\bar{x}x} / \alpha_1 = (n-1)/n$$

(365.2) $$\delta_{\bar{y}x} / \alpha_1 = (m-1)/n$$

Then

(366.1) $$\hat{\bar{x}} = a^{\alpha_1} \bar{x}$$

(366.2) $$\hat{\bar{y}} = a^{\alpha_2} \bar{y}$$

(366.3) $$\hat{U} = a^{\gamma_1} U$$

(366.4) $$\hat{\nu} = a^{\gamma_2} \nu$$

The constraint conditions imposed by the differential equation are:

(376) $$(\gamma_1 - \alpha_2) = \left(\gamma_1 + \gamma_1 - \alpha_2 + \alpha_1 \frac{(m-1)}{n}\right) =$$

$$\left(\gamma_1 + \gamma_1 - \alpha_1 + \alpha_1 \frac{(n-1)}{n}\right) = (\gamma_1 + \gamma_2) = A$$

and

(368.1) $$\gamma_2 = -\alpha_2$$

(368.2) $$\alpha_1 \frac{(m-1)}{n} - \alpha_2 = \alpha_1 \left((n-1)/n - 1 \right)$$

(368.3) $$(\gamma_1 + \gamma_2) = \left(2\gamma_1 - \alpha_2 + \alpha_1 \frac{(m-1)}{n} \right)$$

and

(369.1) $$\gamma_1 = -\alpha_1 \frac{(m-1)}{n}$$

(369.2) $$\gamma_2 = -\alpha_1 \left(\frac{m-n}{n} - 1 \right)$$

(369.3) $$\alpha_2 = \alpha_1 \left(\frac{m-n}{n} + 1 \right)$$

and

(370) $$\alpha_1 = - \frac{An}{(2m-1)}$$

The class of curves and boundary conditions on these curves compatible with the differential equation and the choice of transformation are:

(371) $$\eta = \frac{(y - ax^m)}{x^m} = \left(\frac{y}{x^m} - a \right)$$

(372) $$U = x^{n\gamma_1/\alpha_1} U(\eta) = x^{1-m} U(\eta)$$

The solution is trivial, but sufficent to illustrate the approach. The solution $U(\eta)$ is still to be obtained.

5.7 Local Similarity

An approximation procedure for locally self similar solutions suggests itself. Designate:

(373) $$\Sigma = u_y + u_x + \nu u = 0$$

The requires curve is:

(374) $$y_s = y_s(x_s)$$

The boundary condition is:

(375) $$U(x,y)\Big|_{y_s = y_s(x_s)} = G(x)$$

Let a curve admissable with $G(x)$ be $y = y(x)$. Translate co-ordinates to a point x_s^*, y_s^* satisfying equation (374) by

(376) $$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} X_s & 0 \\ 0 & Y_s \end{pmatrix}$$

At the point x_s^*, y_s^* effect the transformation:

(377) $$\bar{x} = \bar{x}(\bar{\bar{x}}, \bar{\bar{y}})\ ;\ \bar{y} = \bar{y}(\bar{\bar{x}}, \bar{\bar{y}})$$

(378) $$\begin{pmatrix} \dfrac{\partial}{\partial \bar{y}} \\ \dfrac{\partial}{\partial \bar{x}} \end{pmatrix} = \bar{\bar{X}}^{(\delta_{ij}/\alpha_1)}\alpha_{ij}^{(s)} \begin{pmatrix} \dfrac{\partial}{\partial \bar{\bar{y}}} \\ \dfrac{\partial}{\partial \bar{\bar{x}}} \end{pmatrix}$$

The differential relation becomes:

(379)
$$\alpha_{11}^{(s)}\,\bar{\bar{x}}^{\,\delta_{11}/\alpha_1}\,\frac{\partial U}{\partial \bar{\bar{x}}} + \alpha_{12}^{(s)}\,\bar{\bar{x}}^{\,\delta_{12}/\alpha_1}\,\frac{\partial U}{\partial \bar{\bar{y}}}$$
$$+ \alpha_{21}^{(s)}\,\bar{\bar{x}}^{\,\delta_{21}/\alpha_1}\,\frac{\partial U}{\partial \bar{\bar{x}}} + \alpha_{22}^{(s)}\,\bar{\bar{x}}^{\,\delta_{22}/\alpha_1}\,\frac{\partial U}{\partial \bar{\bar{y}}} + \nu u = 0$$

Using the transformation:

(380.1)
$$\hat{\bar{\bar{y}}} = a^{\alpha_1}\bar{\bar{y}} \ ; \ \hat{\bar{\bar{x}}} = a^{\alpha_2}\bar{\bar{x}}$$

(380.2)
$$\hat{\bar{\bar{U}}} = a^{\gamma_1}\bar{\bar{U}} \ ; \ \hat{\bar{\bar{\nu}}} = a^{\gamma_2}\bar{\bar{\nu}}.$$

The similarity form of the differential equation becomes:

(381)
$$J_1(\alpha_{ij}^{(s)};\eta)\,\frac{d\tilde{U}}{d\eta} + J_2(\alpha_{ij}^{(s)},\eta,\nu(\eta))\,\tilde{U} = 0$$

(382)
$$\tilde{U}(\eta) = \exp - \int \frac{J_2}{J_1}\,d\eta$$

The accuracy and usefulness of this approach depends upon how close the curve $y_s = y_s(x)$ can be approximated by $y = y(x)$

in a neighborhood of the point x_s^*, y_s^*.

CHAPTER VI

CONCLUDING COMMENTS

6.1 Conclusions

It has been shown that a total reduction to self similarity can be achieved with the radiative gas dynamic equations with spherical symmetry. This reduction was effected in a one coordinate system by preserving the definitions of the radiant quantities as integral operators on the radiative intensity and asking for the proper homology for I_ν compatible with the system of equations. The findings of this paper have thrown serious doubt on the validity of several published results. Specifically, in the paper by E. I. Zababakhin and V. A. Simonenko[23] the statement is made that "the heat wave and shock wave arrive successively at the center. Each of these near the center is described by its own self similar motion, a general self similar motion for the entire process does not exist." In another paper Marshak states explicitly[24] that "In order to obtain a solution of any kind for eq. (9) to (11), two fundamental assumptions must be made:

Assumption I - The radiation energy and pressure are negligible compared with the material energy and pressure. The radiation flux must of course not be neglected in the flux term.

Assumption II - The material plane."

Clearly in the light of the present work the latter statements are incorrect.

Indeed, the constraint conditions normally imposed by the requirement of preshock uniformity can be relaxed to nonuniformity by virtue of radiative energy transport in this region. It has been found also that conditions for unitingly strong shock waves or even that the more severe restrictions imposed by preshock uniformity can be used provided the correct form of the radiative intensities are employed.

The constraint conditions imposed by gas laws, invariant integrals and specific forms of the radiative intensity have been developed. It has been shown that approximations to radiative transfer phenomenon before reduction to self similarity renders a complete reduction more difficult. The Rosseland approximation is a case in point. The jump equations are derived in generality for arbitrary radiative intensities. The proper shock paths for self similar motions for implosions from points and finite symmetrical surfaces have been given. The significance and implications of strong shock waves have been discussed. The impossibility of discussing simultaneously contact surfaces and shock waves in the framework of similarity is proven. The appropriate physical interpretation of limitingly small radiation effects is derived.

In conjunction with radiative effects two-sided similarity is introduced and its utility explored. The significance of the jump conditions for discontinuous dependent invariants is discussed and appropriate initial conditions for integration are given.

Throughout the course of this work it has been shown that the essential physical characteristics are manifest from the following:

1. The system constraints ascertain the homology of the dependent variables and the fundamental constant ∂ which determines the space time evolution of the system.

2. The introduction of the similarity form of a shock transition tickness suggest an origin of coordinates and permits testability of similarity.

3. General relations are deriveable relating the intensity of radiation to the magnitude of the shock velocity and other thermodynamic functions.

4. On the basis of comments (2) and (3) one also is provided with a connection between the radiation intensity at the shock wave, the shock transition thickness and shock strength.

In general deductions based on the statements (1) through (4) do not require extended computation. They represent what the writer believes are conclusions which are most directly experimentally tested.

In addition, it has been shown that under appropriate physical conditions, the velocity field is independent of the form of the radiative intensity. An exact solution to the similarity equations for a uniform flow problem exists in planar coordinates. In spherical coordinates another exact solution which is non-physical and corresponds to a negative specific heat.

A numerical computation is performed for an inward propagating opaque shock wave. In this manner a numerical procedure is established which is applicable to arbitrary flux laws. The legitimate physical conclusions that may be drawn on the basis of the computations and the statements given above are made in sections (4.5) and (4.10). A further physical statements would be highly speculative.

It is the writer's contention that in general the application of similarity techniques is limited. The procedures in this dissertation depend critically on the system possessing a high element of symmetry and a space time evolution governed by self similarity.

6.2 Suggestions for Future Work

The useful application of the techniques developed in this work are found through the relationships that can be established with a minimum of computation. These testable observables are; the shock trajectory, the behavior of the shock "thickness", the similarity conditions of the shock strength and the expressions relating the radiation to the shock velocity and thermodynamic variables. The implication of formulas obtained for particular combinations of constraints ultimately should be examined experimentally. When the relevance of a particular model has been satisfactorally established experimentally one might legitimately ask for a structure calculation of the fluid variables. A scheme for such calculation has been effected and executed on the sample problem of an opaque imploding shock in Chapter 2 .

From a theoretical point of view the problem of a covergent shock wave in a radiant gas should be approached by the method of characteristics.

APPENDIX

Homology Determination:

A.1 $$\rho = r^k \Omega(\eta) \; ; \; \eta = \tau r^{-\lambda}$$

A.2 $$P = \rho R T$$

Consider

A.3 $$\rho_\tau + u \rho_r + \rho \left(u_r + \frac{2u}{r} \right) = 0$$

A.4 $$\rho_\tau = r^k \Omega' \frac{\partial \eta}{\partial \tau} = r^{k-\lambda} \Omega'$$

Let

A.5 $$u = r^a f(\eta)$$

Then

A.6 $$u_r = a r^{a-1} f + r^a f' \frac{\partial \eta}{\partial r}$$

A.7 $$\frac{\partial \eta}{\partial r} = -\lambda t\, r^{-(\lambda+1)} = -\lambda t\, r^{-\lambda} r^{-1} = -\lambda \eta r^{-1}$$

A.8 $$u_r = a r^{a-1} f - \lambda r^{a-1} \eta f'$$

A.9 $$\rho_r = k r^{k-1} \Omega - \lambda \eta r^{k-1} \Omega'$$

A.10 $$r^{k-\lambda} \Omega' + r^a f \left[k r^{k-1} \Omega - \lambda \eta r^{k-1} \Omega' \right]$$

$$+ r^k \Omega \left[a r^{a-1} f - \lambda r^{a-1} \eta f' + 2 r^{a-1} f \right] = 0$$

A.11 $$r^{k-\lambda} \Omega' + r^{a+k-1} f \left[k\Omega - \lambda \eta \Omega' \right] + r^{k+a-1} \Omega \left[af - \lambda \eta f' + 2f \right]$$
$$= 0$$

require: $r^{-\lambda - a + 1} = \text{const.} \Rightarrow -\lambda - a + 1 = 0$

or $a = 1 - \lambda$

Therefore

A.12 $$u = r^{1-\lambda} f \; ; \; u = \frac{r}{t} \eta f(\eta)$$

Momentum:

A.13 $$U_t + UU_r + P_r/\rho = 0$$

A.14 $$U_t = r^{1-\lambda} f' r^{-\lambda} = r^{1-2\lambda} f'$$

A.15 $$U_r = r^{a-1}\left[af - \lambda\eta f\right] = r^{-\lambda}\left[af - \lambda\eta f'\right]$$

Set

A.16 $$P = r^{b} g(\eta)$$

then

A.17 $$P_r = b r^{b-1} g - r^{b-1}\lambda\eta g' = r^{b-1}\left[bg - \lambda\eta g'\right]$$

A.18 $$r^{1-2\lambda} f' + r^{1-\lambda} f r^{-\lambda}\left[af - \lambda\eta f\right] + \frac{r^{b-1} r^{-k}}{\Omega}\left[bg - \lambda\eta g'\right]$$

A.19.1 $$1 - 2\lambda = b - 1 - k$$

A.19.2 $$b = 2 - 2\lambda + k \; ; \; P = r^{2+k/2\lambda} g(\eta)$$

but

A.20 $$r^{-2\lambda} = \eta^2/\tau^2$$

A.21 $$p = \frac{r^{k+2}}{\tau^2}\eta^2 g(\eta) \equiv \frac{r^{k+2}}{\lambda^2\tau^2} P(\eta)$$

REFERENCES

1. G. Guderley, Luftfahrt-Forsch. 19, 302 (1942).

2. G. I. Taylor, The Formation of a Blast Wave by a Very Intense Explosion, Ministry of Home Security R. C. 210, (II, 5-153), (1941).

3. C. F. v. Weizacker, Z. Naturforschg. 9a 269 (1954).

4. G. Guderley, Luftfahrt-Forsch. 19, 302 (1942).

5. O. Winterberg, On a Method to Reach Extremely High Temperatures by Convergent Shock Waves, Case Institute of Technology, Plasma Research Report, Technical Report No. A-7 (1962).

6. Sanger, Luftfahrt-Forsch. 19, 313 (1942).

7. C. F. v. Weizsacker, Z. Naturforschg. 9a, 269 (1954).

8. G. I. Taylor, The Formation of a Blast Wave by a Very Intense Explosion, Ministry of Home Security R. C. 210, (II, 5-153), (1941).

9. S. I. Pai and A. I. Speth, Phys. Fluids, 4 (1961).

10. A. W. Guess and H. K. Sen, Conference on Extremely High Temperature (John Wiley and Sons, Inc. New York, p. 111, 1958).

11. J. Clarke, Phys. Fluids 4, 1239 (1961).

12. R. E. Marshak, Phys. Fluids 1, 24 (1957).

13. R. Goulard, The High Temperature Aspects of Hypersonic Flow, ed. Nelson, (Macmillan Company, New York, 1964).

14. W. F. Ames, Non-linear Partial Differential Equations for Engieers, (Academic Press, New York, 1959).

15. R. Goulard, The High Temperature Aspects of Hypersonic Flow, ed. Nelson, (Macmillan Company, New York, 1964).

16. Sedov, Similarity and Dimensional Methods in Mechanics, (Academic Press, New York, 1959).

17. K. Friedrichs and R. Courant, Supersonic Flow and Shock Waves, (Interscience Publishers, New York, 1948).

18. F. Sears, Thermodynamics, (Addison Wesley, Massachusetts,(1953).

19. C. F. v. Weizsacker, Z. Naturforschg. 9a, 269 (1954).

20. K. Friedrichs and R. Courant, Supersonic Flow and Shock Waves, (Interscience Publishers, New York, 1948).

21. I. Scarborough, Numerical Mathematical Analyses, (John Hopkins Press, Baltimore, 1958).

22. K. Friedrichs and R. Courant, Supersonic Flow and Shock Waves, (Interscience Publishers, New York, 1948).

23. E. J. Zababakhin and V. A. Simonenko, P. M. M., vol. 29, No. 2, 1965.

24. R. E. Marshak, Phys. Fluids 1 , 24 (1957).