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Finite State Stochastic Games: Existence Theorems and Computational Procedures

by

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and

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Abstract

Let $\{X_n\}$ be a Markov process with finite state space and transition probabilities $p_{ij}(u_i,v_i)$ depending on u_i and v_i. State 0 is the capture state (where the game ends; $p_{oi} \equiv \delta_{oi}$), $u = \{u_i\}$ and $v = \{v_i\}$ are the pursuer and evader strategies, resp., and are to be chosen so that capture is advanced or delayed and the cost $C_i^{u,v} = E[\sum_0^\infty k(u(X_n),v(X_n),X_n) | X_0 = i]$ is minimaxed (or maximined), where $k(\alpha,\beta,0) \equiv 0$. The existence of a saddle point and optimal strategy pair or ϵ-optimal strategy pair are considered. Recursive schemes for computing the optimal or ϵ-optimal pairs are given. The lattice (and complete lattice) structure for the set of vectors $\bigcup_{u,v} C^{u,v}$, $\bigcup_u \sup_v C^{u,v}$, $\bigcup_v \inf_u C^{u,v}$ are given under the relevant partial orderings.

I. Introduction

Recently, there has been some interest in the engineering literature in stochastic games of pursuit and evasion [7]-[9], [11]. The approach of some of this work, such as that of [7] for a stochastic differential game, while obviously of importance for many reasons, has the drawback that it is limited essentially to 'linear-quadratic' problems whose analytic solutions of the complicated derived non-linear partial differential equations can be obtained. A general theory concerning the existence of a saddle point, ϵ approximations to a saddle point, or numerical procedures, does not yet exist. It is suggested, then, that more attention be given to the simpler, yet inadequately treated, discrete state and time game problem. This paper is devoted to a number of questions for this problem. Let $X_0, X_1, \ldots$ be a Markov process on the state space $[0,1,\ldots,N]$ of $N+1$ points. Let the probability transition function $p_{ij}(u_i, v_i)$ be indexed by two variables u_i and v_i taking values in sets U_i and V_i, resp. The u_i and v_i may also be random variables (independent of u_j and v_j for $j \neq i$). Stochastic games will be considered, where the vector $u = (u_1, \ldots, u_N)$ is identified as the strategy of the <u>pursuer</u> and $v = (v_1, \ldots, v_N)$ as the strategy of the <u>evader</u>. The game terminates when the (absorbing -- or capture) state 0 is reached and we assume that $p_{oi}(\alpha, \beta) \equiv 0$ for $i \neq 0$. The cost, for any fixed u and v (assuming, for the moment, that it is well defined) is

$$C_i^{u,v} \equiv E[\sum_0^\infty k(u(X_n), v(X_n), X_n) | u \text{ and } v \text{ used and } X_0 = i]$$

where $k(\alpha,\beta,0) \equiv 0$ is assumed. The object is for the evader to choose v so that $C^{u,v} = \{C_i^{u,v}, i = 1,\dots,N\}$ is maximized in some sense, and for the pursuer to choose u so that $C^{u,v}$ is simultaneously minimized in some sense. Write $P(u,v)$ for the matrix $\{p_{ij}(v_i,v_i);\ i = 1,\dots,N;\ j = 1,\dots,N\}$. Then one minus the sum of the entries in the i^{th} row of $P^n(u,v)$ is the probability of reaching state 0 in n steps, when starting in state i.

In stochastic games, the following questions arise. Are the imposed conditions adequate to insure that the game will terminate in a finite time w.p.1? Given any two strategies u^1 and u^2, is there a strategy u^3 so that† $\sup_v C^{u^3,v} \leq \min[\sup_v C^{u^1,v}, \sup_v C^{u^2,v}]$? For any two strategies v^1 and v^2, is there a v^3 so that $\inf_u C^{u,v^3} \geq \max[\inf_u C^{u,v^1}, \inf_u C^{u,v^2}]$? Are the infima or suprema attained by admissible strategies? Does the expression $\sup_v \inf_u C^{u,v}$ make sense, and does it equal $\inf_u \sup_v C^{u,v}$? Is the value $\sup_v \inf_u C^{u,v} = \inf_u \sup_v C^{u,v}$ attained at some $u = \bar{u}$ and $v = \bar{v}$; i.e., is there a saddle point (or solution) for the game? For any $\epsilon > 0$, is there a pair $u(\epsilon)$ and $v(\epsilon)$ so that, for $i = 1,\dots,N$, $\inf_u \sup_v C_i^{u,v} - \epsilon \leq C_i^{u(\epsilon),v(\epsilon)} \leq \sup_v \inf_u C_i^{u,v} + \epsilon$? Can $\bar{u}, \bar{v}$, $u(\epsilon)$ and $v(\epsilon)$ and the corresponding costs be computed recursively?†† If so, how?

The theorems answer these questions under given conditions. Two types of conditions are considered. The first is that $P^N(u,v)$

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† $\max(A,B)$ or $A \leq B$ refers to the maximum or inequality, component by component. The $\sup_v A$ (or $\inf_u A$) is also the vector of the suprema (or infima) of the components.

†† The discrete time and state results are highly relevant to the continuous state and time problem, since finite difference schemes for solving the continuous time problem often involve a reduction (deliberate or unintentional) to an equivalent discrete time and state problem. See [1] and [2] for a fuller elaboration of this point for the control problem.

is a contraction map (from N-space to N-space) uniformly in (the pure strategies) u and v; i.e. the row sums of $P^N(u,v)$ are less than $1-\epsilon$, where $\epsilon > 0$ is independent of u and v. The second is that there is <u>some</u> pure strategy $\tilde{u}$ so that $P^N(\tilde{u},v)$ is a contraction uniformly in v and $\inf_{i,\alpha,\beta} k(\alpha,\beta,i) \geqq \delta > 0$. The first condition is not unusual in examples. The condition asserts that, no matter what strategies are used, the probability that the game will end in N steps is no less than ϵ; the condition is implied if, for each u and v and initial state i, there is some chain of different states leading to state 0, and the probability of the chain is $\geqq \epsilon' > 0$, where ϵ' is independet of i,u and v. If state 0 is reachable in some number of states, then it is reachable in N states (see [1]-[4]). If $P^n(u,v)$ is a contraction for some $n \geqq 1$, then it is a contraction for $n = N$. Generally, $P^N(u,v)$ would be a contraction, but $P(u,v)$ would not. Under somewhat stronger conditions (k is constant and $P(u,v)$ itself is a contraction uniformly in u and v) some of the results were obtained by Zachrissen [5].

The second condition is also quite natural. In fact, the condition on $P^N(\tilde{u},v)$ for some pure strategy $\tilde{u}$, is a necessary condition for the existence of a pure strategy saddle point for the game. For, otherwise, for any u, there is some v so that the game may not terminate for some initial state i. (Similarly, a necessary condition for the existence of a random saddle point, is that the $P^N(\tilde{u},v)$ be a contraction, uniformly in v for some random strategy $\tilde{u}$.) The second condition (concerning $\tilde{u}$) is weaker and somewhat more natural than the first condition, since, in any game, there may be useless or self-defeating strategies open to the pursuer -- but he would not select them. Yet, within the context of proofs (known to the authors) under the first condition, it is often difficult to eliminate such useless strategies (for which the first condition would not hold).

The proofs of some of the statements, under the first condition, can be had by a straight forward extension of the arguments in [5]. The second condition is more subtle, and requires a more elaborate proof; hence, for economy, the methods for the proof under the second condition are also used for the proof under the first condition. The proof is motivated by that, for the control situation, given in [4].

Most of the results are extendable to suitable formulations of the discrete time and continuous space case. The results may also be applied to the numerical solution of the non-linear difference equations arising when the process is a diffusion, much as is done in [1] or [2] for the control situation.

II. Discrete Markov Games

The development proceeds roughly in the following way. Lemma 1 gives some criteria under which products of certain matrices are contractions, and will be used often in the sequel. Theorem 1 is concerned with conditions which actually assure that a saddle point exists and with numerical procedures for its computation. The proof is written so that the proofs of subsequent results will be facilitated without undue repetition of arguments. The proof, although not hard, is somewhat long.

Corollaries 1 and 2 are concerned with both cases when a saddle point actually exists, and when it can only be approximated within ϵ, in a given sense. Approximation results are proved and the lattice structure is given (e.g., for any u^1, u^2, there is a u^3 so that $C^{u^3,v} \leq \min [C^{u^1,v}, C^{u^2,v}]$, where the minimum is component by component, or $\sup_v C^{u^3,v} \leq \min [\sup_v C^{u^1,v}, \sup_v C^{u^2,v}]$, etc.) Such results assure the existence of a suitable 'ϵ-approximate' saddle point. Theorem 2 is concerned with the numerical procedure for the computation of the ϵ-approximate saddle point.

Lemma 1. A. Let $P^N(u,v)$ be a contraction operator, uniformly in v. Let $Y^{n+1} \leq P(u,v)Y^n + K(u,v)$. Let $M \geq Y^0 \geq -M > -\infty$, and let the components of $K(u,v)$ be uniformly bounded. Then the Y^n are bounded uniformly in n,v and Y^0. Let† $\prod_m^{N+m} P(u,v^i)$ be a contraction uniformly in m, and $Y^{n+1} \leq P(u,v^n)Y^n + K(u,v^n)$. Then, under the other conditions on Y^0 and $K(u,v)$, the Y^n are uniformly bounded.

B. Let $\infty > \hat{Y} \geq Y^{n+1} \geq P(u,v)Y^n + K(u,v)$ and suppose that the components of the $K(u,v)$ are bounded from below by a positive number δ, uniformly in u and v. Then $P^N(u,v)$ is a contraction, uniformly in all u,v for which the inequality holds.

C. Assume (B) except that $\infty > \hat{Y} \geq Y^{n+1} \geq P(u^n,v^n)Y^n + K(u^n,v^n)$. Then†

$$\sum_{n=0}^{\infty} \prod_{i=n}^{\infty} P(u^i,v^i) < M_1 < \infty.$$

Proof. From (A), $Y^n \leq P^n(u,v)Y^0 + \sum_0^{n-1} P^i(u,v)K(u,v)$, and the

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† $\prod_k^{\ell} P(u^i,v^i) = P(u^{\ell},v^{\ell}) \cdots P(u^k,v^k)$.

first statement of (A) follows since $P^N(u,v)$ is a contraction, uniformly in v. The second statement follows from the second hypothesis and $Y^{n+1} \leqq \prod_0^n P(u,v^i)Y^o + \sum_{j=0}^n \prod_{i=j+1}^n P(u,v^i)K(u,v^j)$. (B) follows from the evaluation $\hat{Y} \geqq Y^n \geqq P^n(u,v)Y + \sum_0^{n-1} P^i(u,v)K(u,v)$, and the strict positivity of the components of $K(u,v)$.

Similarly (C) follows from the evaluation

$$\hat{Y} \geqq Y^{n+1} \geqq \prod_0^n P(u^i,v^i)Y^o + \sum_{j=0}^n \prod_{i=j+1}^n P(u^i,v^i)K(u^j,v^j)$$

(where the product is taken in the obvious order), and the strict positivity of the components of $K(u,v)$. Q.E.D.

The proof of Theorem 1 is written so that the proofs of the corollaries and of Theorem 2 are either contained in it or follow easily from it. An admissible u and v are any vectors so that $u_i \in U_i$ and $v_i \in V_i$, where U_i and V_i are part of the problem statement.

<u>Theorem 1. Let</u> $\sup_{i,\alpha,\beta} |k(\alpha,\beta,i)| < \infty$. <u>Assume either</u> (A-1): $P^N(u,v)$ <u>is a contraction uniformly in the admissible pure strategies</u> u <u>and</u> v <u>or</u> (A-2): $\inf_{\alpha,\beta,i} k(\alpha,\beta,i) \geqq \delta > 0$, <u>and there is some pure strategy</u> $\tilde{u}$ <u>so that</u> $P^N(\tilde{u},v)$ <u>is a contraction uniformly in the admissible</u> v. <u>Assume</u> (A-3): $\sup_\beta \inf_\alpha [\sum_{j=1}^N p_{ij}(\alpha,\beta)q_j + k(\alpha,\beta,i)] = \inf_\alpha \sup_\beta [\sum_{j=1}^N p_{ij}(\alpha,\beta)q_j + k(\alpha,\beta,i)]$ <u>for any set of real numbers</u> $q_1,\ldots,q_N$, <u>and</u> (A-4): $p_{ij}(u_i,v_i)$ <u>and</u> $k(u_i,v_i,i)$ <u>are continuous for each</u> i,j <u>where</u> u_i <u>and</u> v_i <u>range over the compact sets</u> U_i <u>and</u> V_i, <u>resp. Then there is an optimal pure strategy pair (saddle point)</u> $(\bar{u},\bar{v})$: $C^{\bar{u},v} \leqq C^{\bar{u},\bar{v}} \leqq C^{u,\bar{v}}$.

<u>The cost</u> $C = C^{\bar{u},\bar{v}}$ <u>is the unique solution of</u>†, ††, †††

(1) $$C = \inf_u \sup_v [P(u,v)C + K(u,v)] = \sup_v \inf_u [P(u,v)C + K(u,v)]$$

<u>and the inf sup is attained at</u> $(\bar{u},\bar{v})$. <u>The sequence</u>

(2) $$C^n = \inf_u \sup_v [P(u,v)C^{n-1} + K(u,v)] = \sup_v \inf_u [P(u,v)C^{n-1} + K(u,v)]$$

<u>converges to</u> C <u>for any finite</u> C^o.

<u>Proof.</u> 1^o. Fix u. Let the row sums of $P^N(u,v)$ be less than $1-\epsilon$ for some $\epsilon > 0$ and each v. There is some $\epsilon^1 > 0$ so that, uniformly in i and v, the probability of going from state i to state 0 within N steps, and without stopping at the same state more than once (except for state 0) is greater than ϵ^1. This implies that the row sums of $\prod_1^N P(u,v^i)$ are less than $1-\epsilon^1$ for any sequence $\{v^i\}$. Thus (A-1) and the second part of (A-2) are equivalent to this stronger condition.

The following extension will be helpful in the sequel, when considering ϵ-approximate saddle points. Let $P^N(u,v)$ be uniformly contracting in u and v, where u varies over a family F_1, and v over a family F_2. Let u^{nm} be in F_1 and v^{nm} in F_2, and let the limit $\lim_n P(u^{nm},v^{nm}) \equiv P_m$ exist. Then P_m^N is contracting and so is $\prod_1^N P_m = P_N \cdots P_1$, and the row sums are less than $1-\epsilon^1$ for some $\epsilon^1 > 0$.

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† u ranges over $\prod_1^N U_i$ and v over $\prod_1^N V_i$.

†† The inf sup is taken component by component; the assertion is that there is some $\bar{u},\bar{v}$ so that $C_i = \inf_{u_i} \sup_{v_i} [\sum_1^N p_{ij}(u_i,v_i)C_j + k(u_i,v_i,i)]$ and the inf sup is attained at $\bar{u}_i,\bar{v}_i$.

††† $K(u,v) = \{k(u_i,v_i,i),\ i = 1,\ldots,N\}$.

2°. Fix the strategies u and v. Under (A-1), there is a cost $C^{u,v}$ which is the unique solution to

$$C^{u,v} = P(u,v)C^{u,v} + K(u,v). \tag{3}$$

In fact $|C_i^{u,v}| \leq N \sup_{i,\alpha,\beta} |k(\alpha,\beta,i)|/\epsilon^1$. Under (A-2), $C^{u,v}$ solves (3), and the solution is unique if $u = \tilde{u}$ (since $P^N(\tilde{u},v)$ is a contraction).

3°. Fix the strategy u, and consider the problem of choosing v to maximize $C^{u,v}$. Under (A-1) and (A-4), there is an optimal control $v(u)$ and†

$$C^u \equiv \sup_v C^{u,v} = \sup_v[P(u,v)C^u + K(u,v)] \tag{4a}$$

Similarly, for fixed v, the strategy minimizing $C^{u,v}$ is given by

$$C^v \equiv \inf_u C^{u,v} = \inf_u[P(u,v)C^v + K(u,v)], \tag{4b}$$

and the solutions to (4a) and (4b) are unique.

Now assume (A-2) and (A-4), and fix the strategy v. First, it will be shown that the set of vectors $S = \bigcup_u C^{u,v}$ is a complete lattice††, under the 'min' partial ordering. Let u^1 and u^2 be strategies and denote $C^{u^i,v} \equiv C^{i,v}$. Order the states so that $C_i^{1,v} \leq$

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†The existence and uniqueness proofs are consequences of the more general arguments of the sequel. The details are very close to the details for the condition (A-2) below. The result of 3° is essentially that of [4], for the control problem, with a few minor changes and corrections, and is also in [1] and [2].

††Fix v. The set S is partially ordered under the ordering $C^{u,v} \leq C^{\hat{u},v}$ if $C_i^{u,v} \leq C_i^{\hat{u},v}$ for $i = 1,\ldots,N$. The statement asserts that S is a lattice; i.e., for any u^1,u^2, there is a u^3 so that $C^{u^3,v} \leq \min[C^{u^1,v},C^{u^2,v}]$, (where the 'min' is component by component), and that S is complete under the 'min' ordering; i.e., let $C^{u^n,v} \downarrow A$, then there is some $\hat{u}$ so that $A = C^{\hat{u},v}$.

For the set $S^1 = \bigcup_v C^{u,v}$, the 'max' ordering will be used.

$C_i^{2,v}$, $i = 1,\dots,\ell$ and $C_i^{2,v} < C_i^{1,v}$, $i = \ell+1,\dots,N$. Denote $u^3 = [u_1^1,\dots,u_\ell^1, u_{\ell+1}^2,\dots,u_N^2]$. It will first be shown that $C^{3,v} \leqq \min[C^{1,v},C^{2,v}]$. No generality is lost by supposing $C^{1,v} < \infty$ and $C^{2,v} < \infty$. Denote $Y^o = [C_1^{1,v},\dots,C_\ell^{1,v}, C_{\ell+1}^{2,v},\dots,C_N^{2,v}] = \min[C^{1,v},C^{2,v}]$. Define Y^n and the transformation $T(u^3,v)$ by

$$(5) \qquad Y^n = P(u^3,v)Y^{n-1} + K(u^3,v) \equiv T(u^3,v)Y^{n-1}.$$

Now $T(u^3,v)$ is a monotone transformation ($X \geqq Y$ implies $T(u^3,v)X \geqq T(u^3,v)Y$) and $Y^1 \leqq Y^o$. Thus $T^n(u^3,v)Y^o = Y^n$ is a non-increasing sequence tending to some $Y \geqq 0$. Y satisfies $Y = P(u^3,v)Y + K(u^3,v)$.

By (B) of Lemma 1, $P^N(u^3,v)$ is a contraction (let $\hat{Y} = Y^o$, the $u = u^3$, and replace $\geqq$ by $=$). This implies that the solution to the equation for Y is unique and $Y = C^{3,v} \leqq \min[C^{1,v},C^{2,v}] = Y^o$.

Since S is a lattice, $0 \leqq \inf_u C^{u,v}$ is well defined and there is a sequence u^n so that $C^{n,v} \downarrow C^v \equiv \inf_u C^{u,v}$ (for all components simultaneously) and

$$(6) \qquad C^v \equiv \lim_n C^{n,v} = \lim_n \inf_u [P(u,v)C^{n,v} + K(u,v)].$$

Now the expression $\inf_u [P(u,v)C + K(u,v)]$ is continuous in C (whether or not (A-4) holds). Thus, since $C^{n,v} \downarrow C^v$, (6) implies that†

$$(7) \qquad C^v = \inf_u [P(u,v)C^v + K(u,v)],$$

and under (A-4), there is some $u(v)$ at which the $\inf_u$ is attained.

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†Note that Lemma 1(B) implies that the solution to (7) is unique.

A similar argument (still supposing (A-2)) shows that for any v^1 and v^2, there is a v^3 so that $\infty \geqq C^{u,v^3} \geqq \max[C^{u,v^1}, C^{u,v^2}]$ where some components may be infinite $(+\infty)$. There is a sequence v^n so that $0 \leqq C^{u,v^n} \uparrow \sup_v C^{u,v} \equiv C^u \leqq \infty$. C^u satisfies (4a), and, by (A-4), the supremum is attained by some $v(u) = v$. If $u = \tilde{u}$, then $C^{\tilde{u}}$ is the unique solution to (4a).

4^o. Next, it is shown that the expressions $\sup_v C^v \equiv \overline{C}$ and $\inf_u C^u \equiv \underline{C}$ make sense[†] and that, $\overline{C} = C^{\overline{v}}$ and $\underline{C} = C^{\underline{u}}$ for some $\overline{v}$ and $\underline{u}$.

First assume (A-1), and denote C^{v^i} by C^i. Fix v^1 and v^2 and order the states so that $C_i^1 \geqq C_i^2$, $i = 1,\ldots,\ell$ and $C_i^2 > C_i^1$, $i = \ell+1,\ldots,N$. Write $v^3 = [v_1^1,\ldots,v_\ell^1, v_{\ell+1}^2,\ldots,v_N^2]$. It will first be shown that $C^3 \geqq \max[C^1,C^2] = (C_1^1,\ldots,C_\ell^1, C_{\ell+1}^2,\ldots,C_N^2) \equiv Y^o$. Define Y^n and the transformation $T(v^3)$ by

$$Y^n = \inf_u[P(u,v^3)Y^{n-1} + K(u,v^3)] \tag{8a}$$

$$Y^n = T(v^3)Y^{n-1}. \tag{8b}$$

Now $X \geqq Y$ implies that $T(v^3)X \geqq T(v^3)Y$. Using this and the definition of Y^o gives $Y^1 \geqq Y^o$ and $Y^2 \equiv T(v^3)Y^1 \geqq T(v^3)Y^o = Y^1$ and, in general, $Y^n \geqq Y^{n-1}$. (A-1) (see second paragraph of 1^o) implies that the Y^n have a finite upper limit, which is denoted by Y, whether or not (A-4) holds. Then

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[†] I.e., $\bigcup_u C^u$ and $\bigcup_v C^v$ are complete lattices under the appropriate partial orderings.

(9) $$Y = \lim_n \inf_u [P(u,v^3)Y^n + K(u,v^3)] = \inf_u [P(u,v^3)Y + K(u,v^3)]$$

(A-1) implies that $C^{v^3} = Y$ is the unique solution of (9). Thus $C^{v^3} \geq \max[C^{v^1}, C^{v^2}]$. By (A-4) the infimum in (9) is attained at some $u = u(v^3)$. A similar calculation yields that for any u^1 and u^2, there is a u^3 so that $C^{u^3} \leq \min[C^{u^1}, C^{u^2}]$.

Next (still assuming (A-1)), let v^n be a sequence such that (monotically non-decreasing) $C^{v^n} \uparrow \sup_v C^v \equiv \overline{C}$. Such a sequence exists by the argument of the previous paragraph. By (A-1) (see 2°) $\overline{C}$ is finite. Also

$$C^{v^n} = \inf_u [P(u,v^n)C^{v^n} + K(u,v^n)]$$

By (A-4), there is a subsequence of the v^n which converges (in each component) to, say, $\overline{v}$, and

(10a) $$\overline{C} = \inf_u [P(u,\overline{v})\overline{C} + K(u,\overline{v})].$$

By (A-1), $C^{\overline{v}} = \overline{C}$ is the unique solution to (10a). Under (A-4), the infimum is attained at some $\overline{u}$. An analogous argument shows that there is a sequence u^n so that $C^{u^n} \downarrow \underline{C} \equiv C^{\underline{u}}$ for some $\underline{u}$ and that $\underline{C}$ is the unique solution to

(10b) $$\underline{C} = \sup_v [P(\underline{u},v)\underline{C} + K(\underline{u},v)].$$

The supremum is attained at some $\underline{v}$, under (A-4).

Now, <u>assume</u> (A-2). The proof is almost the same as under (A-1), and only the differences will be given. By the definition (8a),

$$Y^n \leq P(\tilde{u},v^3)Y^{n-1} + K(\tilde{u},v^3) \tag{11}$$

where $\tilde{u}$ is defined in the Theorem statement. But $P^N(\tilde{u},v^3)$ is a contraction. Thus, the Y^n are uniformly bounded by Lemma 1 (A), and by the argument given under (A-1) we conclude that $C^{v^3} \geq \max[C^{v^1},C^{v^2}]$. The second part of (A-2) now implies that $\overline{C}$ is finite since $\overline{C} \leq \sup_v C^{\tilde{u},v} < \infty$, and the rest of the previous argument carries over . The details relating to $\underline{C}$ are similar to those for the condition (A-1).

Uniqueness of the solution to (10a) is proved as follows. Let the finite vectors C^i, $i = 1,2$, solve (10a) with corresponding u^i, $i = 1,2$. Then, by (10a), $C^i = P(u^i,\overline{v})C^i + K(u^i,\overline{v})$. By Lemma 1 (B), the $P^N(u^i,\overline{v})$ are contractions (let $Y^n \equiv C^i$). Using (10a), $C^i \leq P(u^j,\overline{v})C^i + K(u^j,\overline{v})$, where $j \neq i$, $j,i = 1,2$, and $C^i = P(u^i,\overline{v})C^i + K(u^i,\overline{v})$. Thus, we obtain

$$P(u^1,\overline{v})(C^1-C^2) \leq C^1-C^2 \leq P(u^2,\overline{v})(C^1-C^2)$$
$$P^N(u^1,\overline{v})(C^1-C^2) \leq C^1-C^2 \leq P^N(u^2,\overline{v})(C^1-C^2)$$

which implies $C^1 = C^2$. The details for the uniqueness of (10b) are also straightforward and are omitted.

5°. Assume (A-1) or (A-2). It will now be shown that†

$$\overline{C} = \sup_v \inf_u [P(u,v)\overline{C} + K(u,v)] \tag{12a}$$

$$\underline{C} = \inf_u \sup_v [P(u,v)\underline{C} + K(u,v)]. \tag{12b}$$

† The $\sup_v \inf_u$ and $\inf_u \sup_v$ are, of course, to be understood to apply to each component of the vector separately.

By the definition of 'sup',

$$(13) \qquad \sup_v\{\inf_u[P(u,v)\overline{C} + K(u,v)]\} \geq \inf_u[P(u,\overline{v})\overline{C} + K(u,\overline{v})] = \overline{C}.$$

Suppose that the inequality (13) is strict for some component of the vectors, say the q^{th} (i.e., there is some v^1 so that the q^{th} component of the bracketed term on the left of (13) is strictly greater than the q^{th} component of the right of (13)). Define $Y^o = \inf_u[P(u,v^1)\overline{C} + K(u,v^1)]$. Using the transformation $T(v)$ of (8b), define $Y^n = T(v^1)Y^{n-1}$. Since $Y^o = T(v^1)\overline{C} \geq \overline{C}$ (with a strict inequality in the q^{th} component), the arguments of 4^o yield that $Y^n \uparrow Y = C^{v^1} \geq \overline{C}$, with a strict inequality in the q^{th} component. The contradiction yields (12a). (12b) is obtained similarly.

6^o. By (A-3), the $\sup_v \inf_u$ and $\inf_u \sup_v$ in (12) may be interchanged. Thus, for either (A-1) or (A-2), and any u,v,

$$(14a) \qquad P(\overline{u},v)\overline{C} + K(\overline{u},v) \leq \overline{C} = P(\overline{u},\overline{v})\overline{C} + K(\overline{u},\overline{v}) \leq P(u,\overline{v})\overline{C} + K(u,\overline{v})$$

$$(14b) \qquad P(\underline{u},v)\underline{C} + K(\underline{u},v) \leq \underline{C} = P(\underline{u},\underline{v})\underline{C} + K(\underline{u},\underline{v}) \leq P(u,\underline{v})\underline{C} + K(u,\underline{v})$$

Assume (A-2). Then using the left side of (14) and Lemma† 1(C), $\prod_o^n P(\overline{u},v^i) \to 0$ and $\prod_o^n P(\underline{u},v^i) \to 0$ as $n \to \infty$ for any sequence $\{v^i\}$. In fact $\sum_o^\infty \prod_n^\infty P(\overline{u},v^i) \leq M < \infty$, for any sequence $\{v^i\}$. (The assertion is clearly true for (A-1)).

Now assume either (A-1) or (A-2). On the left of (14a), let $v = \underline{v}$ and let $u = \overline{u}$ on the right of (14b),

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† $\prod_k^\ell P(u^i,v^i) \equiv P(u^\ell,v^\ell)\ldots P(u^k,v^k)$.

$$\underline{C}-\overline{C} \leq P(\overline{u},\underline{v})(\underline{C}-\overline{C})$$

or

$$\underline{C}-\overline{C} \leq P^n(\overline{u},\underline{v})(\underline{C}-\overline{C}).$$

The right side tends to zero as $n \to \infty$. Thus $\underline{C} \leq \overline{C}$. The reverse inequality follows from the sequence

$$\sup_v C^{u,v} \geq C^{u,v}$$

$$\underline{C} = \inf_u \sup_v C^{u,v} \geq \inf_u C^{u,v}$$

$$\underline{C} \geq \sup_v \inf_u C^{u,v} = \overline{C}.$$

Thus $\underline{C} = \overline{C}$ and the game has a saddle point with value $\underline{C} = \overline{C}$; either pair $(\overline{u},\overline{v})$ or $(\underline{u},\underline{v})$ is a saddle point.

7^o. Let C^o be an arbitrary finite vector. Define the sequence C^n by

$$C^{n+1} = \inf_u \sup_v [P(u,v)C^n + K(u,v)] = \sup_v \inf_u [P(u,v)C^n + K(u,v)]$$

By (A-4), the $\inf_u \sup_v$ is realized at some u^n, v^n. Then for any u,v

(15a) $$P(u^n,v)C^n + K(u^n,v) \leq C^{n+1} \leq P(u,v^n)C^n + K(u,v^n)$$

(15b) $$C^{n+1} \leq P(\tilde{u},v^n)C^n + K(\tilde{u},v^n)$$

Either (15b),(A-1) and 1^o or (15b) , (A-2), 1^o and Lemma 1 (A), imply that the C^n are uniformly bounded. Then the left side of (15a) and (A-1) or (A-2) and Lemma 1(C) imply that $\prod_o^n P(u^i,\overline{v}) \to 0$

Similarly, either (A-1) and 1^o or (A-2), (14a), Lemma 1(B) and 1^o imply that $\prod_0^n P(\bar{u},v^i) \to 0$ as $n \to \infty$. Next, in (15a), let $u = \bar{u}$, $v = \bar{v}$ and let $v = v^n$ and $u = u^n$ in (14a). Then (14a) and (15a) yield

$$(16) \qquad P(\bar{u},v^n)(\bar{C}-C^n) \leq \bar{C} - C^{n+1} \leq P(u^n,\bar{v})(\bar{C}-C^n)$$

or

$$(17) \qquad \prod_0^n P(\bar{u},v^i)(\bar{C}-C^o) \leq \bar{C}-C^{n+1} \leq \prod_0^n P(u^i,\bar{v})(\bar{C}-C^o).$$

Since both sides of (17) go to zero as $n \to \infty$, $C^n \to \bar{C}$ as $n \to \infty$. Q.E.D.

Let $\mathcal{E}_\epsilon$ denote a vector, each of whose components has the value ϵ. Recall that $C^u = \sup_v C^{u,v}$ and $C^v = \inf_u C^{u,v}$.

<u>Corollary 1.</u> <u>Assume (A-1) or (A-2) and that the components of $K(u,v)$ are uniformly bounded in u and v. Then, for any u^1 and u^2, there is some u^3 so that $C^{u^3,v} \leq \min[C^{u^1,v},C^{u^2,v}]$. For each $\epsilon > 0$, there is a $u(\epsilon)$ so that</u>

$$\mathcal{E}_\epsilon + \inf_u C^{u,v} \geq C^{u(\epsilon),v}.$$

<u>For each v^1 and v^2, there is a v^3 so that $C^{u,v^3} \geq \max[C^{u,v^1},C^{u,v^2}]$. For each $\epsilon > 0$, there is a $v(\epsilon)$ so that</u>

$$\sup_v C_i^{u,v} - \epsilon \leq C_i^{u,v(\epsilon)} \quad i = 1,\ldots,N$$

<u>if $C_i^u < \infty$, and $C_i^{u,v(\epsilon)} \geq 1/\epsilon$ if $C_i^u = \infty$.</u>

Furthermore

$$C^v = \inf_u [P(u,v)C^v + K(u,v)] \tag{18a}$$

$$C^u = \sup_v [P(u,v)C^u + K(u,v)] \tag{18b}$$

The solution to (18a) is unique (if finite). If $u = \tilde{u}$, the solution to (18b) is unique (if finite).

Proof. Most of the proof is a direct consequence of the arguments of 3^o, 4^o. Also most of the statements are essentially contained in [4]. Only the uniqueness for (18a) will be proved, under (A-2). The uniqueness under (A-1) is obvious. Assume (A-2), and fix $\epsilon > 0$ so that $\inf_{i,\alpha,\beta} k(\alpha,\beta,i) - \epsilon \equiv \delta > 0$. There is a $u(\epsilon)$ so that

$$C^v \geq P(u(\epsilon),v)C^v + K(u(\epsilon),v) - \mathcal{E}_\epsilon \tag{19}$$

Thus

$$C^v \geq P^n(u(\epsilon),v)C^v + \sum_0^{n-1} P^i(u(\epsilon),v)[K(u(\epsilon),v) - \mathcal{E}_\epsilon] \geq 0 \tag{20}$$

(20), together with $\delta > 0$, and Lemma 1(B) implies that $P^N(u(\epsilon),v)$ is a contraction, uniformly in ϵ; i.e., $\|\sum_0^\infty P^i(u(\epsilon),v)\| \leq M < \infty$ for some real number M and all $0 < \epsilon < \delta$. Let C^v and $\tilde{C}$ be solutions to (18a). Then, combining

$$\tilde{C} \leq P(u(\epsilon),v)\tilde{C} + K(u(\epsilon),v)$$

with (19) yields

$$\tilde{C}-C^{v} \leq P(u(\epsilon),v)(\tilde{C}-C^{v}) + \varepsilon_\epsilon$$
$$\leq P^{n}(u(\epsilon),v)(\tilde{C}-C^{v}) + \sum_{0}^{n-1} P^{i}(u(\epsilon),v)\varepsilon_\epsilon$$
$$\leq M\varepsilon_\epsilon .$$

Since the argument is symmetric in $\tilde{C}$ and C^{v}, $-M\varepsilon_\epsilon \leq \tilde{C}-C^{v} \leq M\varepsilon_\epsilon$. Since ϵ is arbitrarily small, the uniqueness follows. Q.E.D.

<u>Corollary 2</u>. <u>Assume</u> (A-1) <u>or</u> (A-2) <u>and that the components of</u> $K(u,v)$ <u>are uniformly bounded in</u> u <u>and</u> v. <u>Then, for each</u> u^1 <u>and</u> u^2, <u>there is a</u> u^3 <u>so that</u>

$$\sup_{v} C^{u^3,v} \leq \min[\sup_{v} C^{u^1,v}, \sup_{v} C^{u^2,v}].$$

<u>For each</u> v^1 <u>and</u> v^2, <u>there is a</u> v^3 <u>so that</u>

$$\inf_{u} C^{u,v^3} \geq \max[\inf_{u} C^{u,v^1}, \inf_{u} C^{u,v^2}].$$

<u>For each</u> $\epsilon > 0$, <u>there are</u> $\underline{u}(\epsilon)$, $\underline{v}(\epsilon)$, $\overline{u}(\epsilon)$, $\overline{v}(\epsilon)$ <u>so that</u>

(21a) $$C^{\underline{u}(\epsilon),\underline{v}(\epsilon)} + \varepsilon_\epsilon \geq \sup_{v} C^{\underline{u}(\epsilon),v} \leq \inf_{u}\sup_{v} C^{u,v} + \varepsilon_\epsilon$$

(21b) $$C^{\overline{u}(\epsilon),\overline{v}(\epsilon)} - \varepsilon_\epsilon \leq \inf_{u} C^{u,\overline{v}(\epsilon)} \geq \sup_{v}\inf_{u} C^{u,v} - \varepsilon_\epsilon .$$

<u>Proof</u>. The first paragraph of Corollary 2 follows from the arguments of part 3^{o}, 4^{o} of the proof of the Theorem. The right sides of (21a) and (21b) are obvious since (see 4^{o}) $C^{u^n} = \sup_{v} C^{u^n,v} \downarrow \inf_{u}\sup_{v} C^{u,v}$ and $C^{v^n} = \inf_{u} C^{u,v^n} \uparrow \sup_{v}\inf_{u} C^{u,v}$ (merely choose ϵ and let $\underline{u}(\epsilon)$ =

$= u^n$, $\bar{v}(\epsilon) = v^n$, where n is large enough so that $C_i^{u^n} - \inf_u C_i^u < \epsilon$ and $\sup_v C_i^v - C_i^{v^n} < \epsilon$ for $i = 1,\ldots,N$). Finally, the left sides of (21a) and (21b) follow from Corollary 1. Q.E.D.

<u>Theorem 2</u>. <u>Assume</u> (A-1) <u>and</u> (A-3) <u>or</u> (A-2) <u>and</u> (A-3) <u>and</u> <u>let the components of</u> $K(u,v)$ <u>be uniformly bounded in</u> u <u>and</u> v. <u>Let</u> C^0 <u>be a finite vector</u>. <u>Define</u> C^n <u>by</u>

$$C^{n+1} = \inf_u \sup_v [P(u,v)C^n + K(u,v)]. \tag{22}$$

<u>Then</u>

$$C^n \to \inf_u \sup_v C^{u,v} = \sup_v \inf_u C^{u,v}. \tag{23}$$

<u>Fix</u> $\epsilon > 0$, <u>so that, in the case of</u> (A-2), ϵ <u>is smaller than some lower bound to the components of</u> $K(u,v)$. <u>Let</u> $u^n(\epsilon)$ <u>and</u> $v^n(\epsilon)$ <u>satisfy</u>

$$P(u^n(\epsilon),v)C^n + K(u^n(\epsilon),v) - \mathcal{E}_\epsilon \leq C^{n+1} \leq P(u,v^n(\epsilon))C^n + K(u,v^n(\epsilon)) + \mathcal{E}_\epsilon. \tag{24}$$

<u>Then, there is a finite constant</u> M_1 <u>so that, for sufficiently large</u> n,

$$-M_1\mathcal{E}_\epsilon \leq C^{u^n(\epsilon),v^n(\epsilon)} - \inf_u \sup_v C^{u,v} \leq M_1\mathcal{E}_\epsilon. \tag{25}$$

[(23) and (25) imply the existence of an ϵ-optimal strategy pair.]

<u>Proof</u>. First the analogs of (12) and (14) will be obtained. Fix $\epsilon > 0$ and, in case of (A-2), let $\inf_{i,\alpha,\beta} k(\alpha,\beta,i) - \epsilon \equiv \delta > 0$. Let $\mathcal{E}_\epsilon(q)$ be a vector whose q^{th} component is <u>strictly greater than</u> ϵ and

the other components are zero. Denote $C^{\overline{u}(\epsilon);\overline{v}(\epsilon)}$ by $\overline{C}^{\epsilon}$ and $C^{\underline{u}(\epsilon),\underline{v}(\epsilon)}$ by $\underline{C}^{\epsilon}$. Suppose that (c.f. (13)), there is some v^1 and some q so that

$$\inf_u [P(u,v^1)\overline{C}^{\epsilon} + K(u,v^1)] \geq \inf_u [P(u,\overline{v}(\epsilon))\overline{C}^{\epsilon} + K(u,\overline{v}(\epsilon))] + \mathcal{E}_{\epsilon}(q).$$

Define Y^o and $T(v^1)$ as below (13). Then $Y^o \leq Y^n \uparrow Y$, where Y satisfies (18a). Since the solution to (18a) is unique (Corollary 1) and, for $v = v^1$, equals C^{v^1}, the evaluation $\overline{C}^{\epsilon} + \mathcal{E}_{\epsilon}(q) \leq Y^o \leq \lim Y^n = C^{v^1}$ contradicts the definition of $\overline{C}^{\epsilon}$. Thus, it is concluded that there is no such v^1 and

$$\text{(26a)} \qquad \overline{C}^{\epsilon} \geq \sup_v \inf_u [P(u,v)\overline{C}^{\epsilon} + K(u,v)] - \mathcal{E}_{\epsilon}$$

Similarly

$$\text{(26b)} \qquad \underline{C}^{\epsilon} \leq \inf_u \sup_v [P(u,v)\underline{C}^{\epsilon} + K(u,v)] + \mathcal{E}_{\epsilon}$$

is obtained. By (A-3), the $\inf_u \sup_v$ and $\sup_v \inf_u$ may be interchanged and (27) (the analog of (14)) obtained.

$$\text{(27)} \qquad -\mathcal{E}_{\epsilon} + P(u(\epsilon),v)C^{\epsilon} + K(u(\epsilon),v) \leq C^{\epsilon} = P(u(\epsilon),v(\epsilon))C^{\epsilon} + K(u(\epsilon),v(\epsilon))$$
$$\leq P(u,v(\epsilon))C^{\epsilon} + K(u,v(\epsilon)) + \mathcal{E}_{\epsilon}$$

where C^{ϵ}, $u(\epsilon)$ and $v(\epsilon)$ are either $\overline{C}^{\epsilon}$, $\overline{u}(\epsilon)$, $\overline{v}(\epsilon)$, or $\underline{C}^{\epsilon}$, $\underline{u}(\epsilon)$, $\underline{v}(\epsilon)$, resp. Let $\Delta_{\epsilon} = \underline{C}^{\epsilon} - \overline{C}^{\epsilon}$. Then (27) yields (compare (14) and the development following it)

$$P(\underline{u}(\epsilon),\overline{v}(\epsilon))\Delta_\epsilon - 2\varepsilon_\epsilon \leqq \Delta_\epsilon \leqq P(\overline{u}(\epsilon),\underline{v}(\epsilon))\Delta_\epsilon + 2\varepsilon_\epsilon$$

which implies that (under (A-1) or (A-2))

$$(28) \qquad -M\varepsilon_\epsilon \leqq \Delta_\epsilon \leqq M\varepsilon_\epsilon$$

where M is some real number (see proof of Corollary 1). The left hand side of (27) implies by (A-1) or (A-2) and Lemma 1(c) that, for any sequence $\{v^i\}$, $\sum_{j=0}^{\infty} \prod_{i=j+1}^{\infty} P(u(\epsilon),v^i)K(u(\epsilon),v^j)$ is bounded uniformly in ϵ for $u(\epsilon) = \overline{u}(\epsilon)$ or $\underline{u}(\epsilon)$).

Then (28) and the arbitrariness of ϵ imply that $\overline{C} = \underline{C}$.

The proof of (25) is along the lines of the proof in the Theorem, with the obvious alterations, and only the outline will be given. Each C^n is finite since the components of $K(u,v)$ are finite. Then the existence of $u^n(\epsilon)$ and $v^n(\epsilon)$ follows by (A-3). In (27), let

$$C^\epsilon = \overline{C}^\epsilon, u(\epsilon) = \overline{u}(\epsilon),\ v(\epsilon) = \overline{v}(\epsilon),\ u = u^n(\epsilon),\ v = v^n(\epsilon).$$

Then

$$(29) \qquad -K_1\epsilon + P(\overline{u}(\epsilon),v^n(\epsilon))(\overline{C}^\epsilon - C^n) \leqq \overline{C}^\epsilon - C^{n+1} \leqq P(u^n(\epsilon),\overline{v}(\epsilon))(\overline{C}^\epsilon - C^n) + K_1\epsilon$$

for some $\infty > K_1 > 0$. (29) implies that the elements of $\overline{C}^\epsilon - C^{n+1}$ differ by no more than $\pm M_1\epsilon$, for some real number $M_1 < \infty$. Q.E.D.

References

1. H. J. Kushner, A. Kleinman, "Numerical Methods for the Solution of the Degenerate Non-linear Elliptic Equations Arising in Optimal Stochastic Control Theory," to appear in 1968 J.A.C.C.

2. H. J. Kushner, "Methods for the Numerical Solution of Degenerate Linear and Nonlinear Elliptic Boundary Value Problems," Report 67-7, Center for Dynamical Systems, Brown University, Providence, R.I., Nov. 1967.

3. W. Feller, Probability Theory and its Applications, Vol. 1., Wiley, New York, 1950.

4. J. H. Eaton, L. A. Zadeh, "Optimal Pursuit Strategies in Discrete-State Probabilistic Systems," J. Basic Engr. (ASME), March, 1962, pp. 23-29.

5. L. E. Zachrisson, "Markov Games," in Annals of Math. Studies, No. 52, Princeton Univ. Press, Princeton, N. J., (1964) , pp. 211-253.

6. L. S. Shapley, "Stochastic Games", Proc. Nat. Acad. Sci., Vol. 39, 1953, pp. 1095-1100.

7. Y. C. Ho, A. E. Bryson Jr., S. Baron, "Differential Games and Optimal Pursuit-Evasion Strategies", IEEE Trans. Automatic Control, Vol. AC-10, No. 4, October, 1965, pp. 385-389.

8. Y. C. Ho, "Optimal Terminal Maneuver and Evasion Strategy", J. SIAM Control, Vol. 4, No. 3, 1966, pp. 421-428.

9. J. L. Speyer, "A Stochastic Differential Game with Controllable Statistical Parameters", IEEE Trans. Systems Science and Cyb, Vol. SSC-3, No. 1, June, 1967, pp. 17-20.

10. R. Isaacs, Differential Games, Wiley, New York, 1965.

11. R. Chattopadhyay, "A New Approach to the Solution of Linear Pursuit-Evasion Problems", 1967 JACC Preprints, pp. 797-803.