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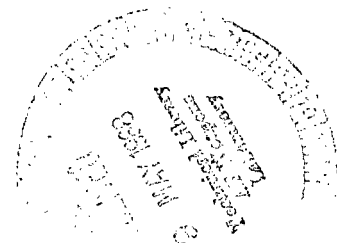# METHOD OF CALCULATING THE NORMAL MODES AND FREQUENCIES OF A BRANCHED TIMOSHENKO BEAM

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

A method is presented in this report for calculating the normal modes and frequencies of a branched Timoshenko beam. The method is essentially a modified Stodola method and requires an iteration procedure to determine the normal modes and frequencies of the system. In this method, an arbitrary deflection, consistent with the boundary condition, is assumed. Also, because of the presence of the spring-mass system, the frequency is not a constant factor in the governing equations and must also be assumed. Knowing the deflection and frequency, the corresponding shears, moments, slopes, and new deflection can be determined by integrating the governing differential equations of the system. The new deflection is then adjusted to satisfy the boundary conditions, and a new frequency is subsequently calculated. The process is then repeated, and the frequency is used as the criterion for convergence. Although the method can be applied for any type of boundary conditions, particular attention was given to those boundary conditions of interest in launch-vehicle dynamics, namely, the free-free beam and the cantilevered-free beam.

In general, the iteration routine will always converge to the lowest or fundamental mode of vibration. To determine the higher modes, the lower mode components are removed from the assumed shear and moment distribution by utilizing the second orthogonality condition of the branched beam. The iteration routine will then converge to the lowest mode whose components are not removed.

INTRODUCTION

Experimental and theoretical studies have shown that a launch vehicle airframe can be idealized as a nonuniform beam for purposes of studying its lateral bending dynamics. When major components of the vehicle are cantilevered from the airframe, the vehicle

can be represented as a beam with cantilevered branch beams. With such a representation, the bending dynamics of the components and their effect on gross vehicle dynamics can be studied. For example, the engine shroud or the engine thrust structure could be idealized as a branched beam at the aft portion of the vehicle, while the payload or the nose fairing could be represented as a branched beam at the forward portion. For multi-staged vehicles, the engine and engine-support structure frequently are suspended in the interstage-adapter well, and this component can be represented as an intermediate branch.

Because of the low aspect ratio (i. e., L/D) of current vehicles, the bending dynamics of airframes are accurately described by beam theory, only if the effects of shear flexibility and rotational inertia of the beam elements are included. This beam theory, known as the Timoshenko beam theory, is given in reference 1.

There are several methods for determining the normal modes and frequencies of a nonuniform Timoshenko beam. For example, reference 2 presents a modified Myklestad method which takes into account both shear flexibility and rotary inertia of the beam elements, while reference 3 gives a matrix iteration technique which accounts for rotational inertia. Shear flexibility could also be handled in the second method by including the shear deformation in the matrix of influence coefficients. Reference 4 gives a modified Stodola method which also includes the effects of shear flexibility and rotational inertia of beam elements. The effects of flexible branches could be handled by extending either of these methods or by using the component mode method presented in reference 5. This report extends the method of reference 4 to account for the effect of the branches on the normal modes and frequencies of the system.

SYMBOLS

A_s	effective shear area
a_i	distance from tip of ith branch beam to center of gravity of ith rigid mass
b_i	length of ith branch beam
E	Young's modulus of elasticity
G	modulus of elasticity in shear
I	bending moment of inertia
I_{cgi}	mass moment of inertia of ith rigid mass about center of gravity
J_i	mass moment of inertia of ith rigid mass about tip of ith branch beam
K_i	torsion spring rate attached to i^{th} branch beam

l	length of primary beam
$M_{bi}(\xi_i)$	moment distribution along i^{th} branch beam
M_{ei}	moment transferred to end of ith branch beam due to motion of spring-mass system
$m(x,t), M(x)$	moment distribution along primary beam
m_{ci}	mass of rigid body attached to i^{th} branch beam
$m_i(t), M_i$	moment transferred to primary beam due to motion of ith beam
p_i	natural frequency of ith spring-mass system
$Q_{bi}(\xi_i)$	shear distribution along i^{th} branch beam
Q_{ei}	shear transferred to end of ith branch beam due to motion of spring-mass system
$q(x,t), Q(x)$	shear distribution along primary beam
$q_i(t), Q_i$	vertical shear transferred to primary beam due to motion of ith branch beam
x	longitudinal coordinate of primary beam
x_i	longitudinal distance from the end of primary beam to attachment point of ith branch beam
$Y(x,t), y(x)$	vertical deflection function of primary beam
$\overline{Y}(x,t), \overline{y}(x)$	bending deflection of primary beam assumed fixed at $x = 0$
$Y_0(t), y_0$	vertical deflection of primary beam at $x = 0$
$Y_0'(t), y_0'$	bending slope of primary beam at $x = 0$
α_i	total angle of rotation of spring-mass system relative to horizontal
β_i	bending slope at tip of ith branch beam
γ_i	bending slope of primary beam at attachment of ith branch beam
Δ_i	vertical deflection at tip of ith branch beam
$\overline{\delta}$	vertical deflection of primary beam at attachment station of ith branch beam
$\eta_i(\xi_i)$	vertical deflection function of ith branch beam
$\overline{\eta}_i(\xi_i)$	bending deflection of ith branch beam assumed fixed at attachment station
θ_i	angle of rotation of spring-mass system relative to static equilibrium position

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μ_j	generalized mass corresponding to j^{th} normal mode
ξ_i	longitudinal coordinate of ith branch beam
ρA	mass per unit length
$\overline{\Phi}(x,t), \varphi(x)$	bending slope of primary beam
$\Phi(x)$	shear deflection of primary beam
$\psi(\xi_i)$	bending slope of ith branch beam
ω	circular frequency

Subscripts:

| bi, ei, i | integers referring to branch beams |
| j, k, p, r, v | integers referring to normal modes |

THEORETICAL ANALYSIS

Consider a launch vehicle idealized as a nonuniform beam with a cantilevered branch beam attached at station x_i as shown in figure 1. To make the analysis more general by showing how a spring-mass component can be handled, assume that a rigid body mass is attached to the end of the branch beam by a pin connection with a torsion spring whose spring rate is K_i. The governing equations will first be developed for a primary beam with one branch, and then the equations will be generalized to include (in principle) any number of branches.
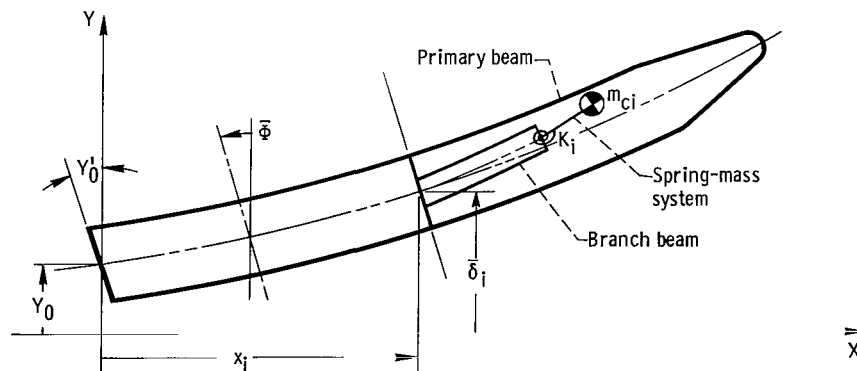

Figure 1. - Idealized launch vehicle coordinate system.

Primary Beam Equations

An element taken along the length of the primary beam is shown in figure 2. The concentrated moment and shear transferred to the primary beam from the branch beam are represented on the differential element by $m_i\delta(x - x_i)dx$ and $q_i\delta(x - x_i)dx$, respectively. The expression $\delta(x - x_i)$ is the Dirac delta function. Some of the properties of the Dirac delta function that will be used in the subsequent analysis are as follows:

$$\int_0^X \delta(x - x_i)dx = h(x - x_i) = \begin{cases} 0 & x < x_i \\ \\ 1 & x > x_i \end{cases} \tag{1}$$

$$\int_0^X \int_0^X \delta(x - x_i)dx\ dx = \int_0^X h(x - x_i)dx = \begin{cases} 0 & x < x_i \\ \\ x - x_i & x > x_i \end{cases} \tag{2}$$

where $h(x - x_i)$ is the unit step function. The equations of motion of the primary beam are determined by summing forces and moments acting on the element and neglecting differentials of higher order. Thus, for forces,

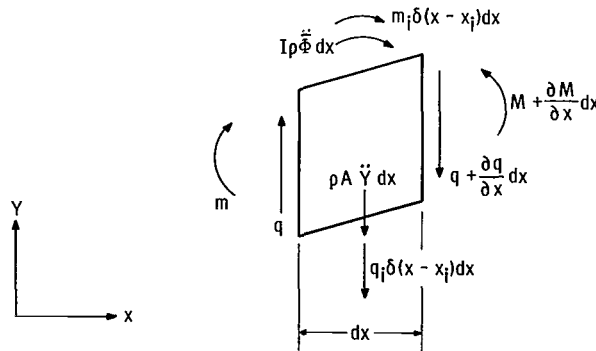$$\frac{\partial q(x, t)}{\partial x} + q_i(x, t)\delta(x - x_i) = -\rho A \ddot{Y}(x, t) \tag{3}$$

Figure 2. - Forces and moments acting on typical primary beam element.

and for moments,

$$\frac{\partial m(x,t)}{\partial x} - q(x,t) - m_i \delta(x - x_i) = I_\rho \ddot{\overline{\Phi}}(x,t) \qquad (4)$$

Note that the rotatary inertia term in equations (1) to (4) depends only on the bending slope $\overline{\Phi}$, since the shear forces produce only distortion of the cross section and no rotation. From the geometry of the system (fig. 1), the bending slope is given by

$$\overline{\Phi}(x,t) = \frac{\partial \overline{Y}(x,t)}{\partial x} + Y_0'(t) \qquad (5)$$

where $Y_0'(t)$ is the bending slope at $x = 0$ and $\partial \overline{Y}/\partial x$ is the bending slope of the beam cantilevered at $x = 0$. The total slope will be the bending slope minus the shear slope. The shear slope is subtracted from the bending slope in this case because a positive shear force tends to decrease the total slope as shown in figure 2; therefore,

$$\frac{\partial Y(x,t)}{\partial x} = \frac{\partial \overline{Y}(x,t)}{\partial x} + Y_0'(t) - \beta(x,t) \qquad (6)$$

From elementary beam theory, the angle of shear β at the neutral axis is given by

$$\beta(x,t) = \frac{q(x,t)}{A_s G} \qquad (7)$$

where A_s is the effective shear area. Substituting equation (6) into equation (7) gives

$$\frac{\partial Y(x,t)}{\partial x} = \frac{\partial \overline{Y}(x,t)}{\partial x} + Y_0'(t) - \frac{q(x,t)}{A_s G} \qquad (8)$$

The moment-curvature relations for the Bernoulli-Euler beam and the Timoshenko beam are the same; therefore,

$$m(x,t) = EI \frac{\partial \overline{\Phi}(x,t)}{\partial x} = EI \frac{\partial^2 \overline{Y}(x,t)}{\partial x^2} \qquad (9)$$

To eliminate the time dependency from equations (2) to (5), (8), and (9), only harmonic motions are considered; that is, it is assumed that the shear, moment, slope, and

deflection functions are of the form

$$m(x, t) = M(x)e^{i\omega t}, \quad q_i(t) = Q_i e^{i\omega t}$$

$$q(x, t) = Q(x)e^{i\omega t}, \quad Y(x, t) = y(x)e^{i\omega t}$$

$$m_i(t) = M_i e^{i\omega t}, \quad \overline{\Phi}(x, t) = \varphi(x)e^{i\omega t}$$

(10)

From equations (3) to (6) and (8) to (10), the following ordinary differential equations are obtained:

$$\frac{dQ(x)}{dx} = \omega^2 \rho A y(x) - Q_i \delta(x - x_i)$$

$$\frac{dM(x)}{dx} = Q(x) + M_i \delta(x - x_i) - \omega^2 I\rho\varphi(x)$$

$$\varphi(x) = \frac{d\overline{y}(x)}{dx} + y'_0$$

$$\frac{dy(x)}{dx} = \frac{d\overline{y}(x)}{dx} + y'_0 - \frac{Q(x)}{A_s G}$$

$$M(x) = EI \frac{d^2\overline{y}(x)}{dx^2}$$

(11a)

Equations (11a) represent the required differential equations of the primary beam. In the Stodola method, a deflection function y(x) is assumed and equations (11a) are integrated to obtain a new improved deflection. The constants of integration are adjusted such that the boundary conditions are satisfied. The process is then repeated until it converges. The values of M_i and Q_i in these equations depend on the dynamics of the branch beam.

Boundary Conditions For Primary Beam

The boundary conditions for two cases will be considered: (1) a free-free beam and

(2) a free-cantilevered beam. Other types of boundary conditions can be handled in an analogous manner.

  Free-free beam. - For a beam free at $x = 0$ and $x = l$, the shears and moments at the boundaries must vanish. Thus,

$$Q(0) = Q(l) = 0 \tag{12}$$

$$M(0) = M(l) = 0 \tag{13}$$

The first of equations (11a) and (12) yield the relation

$$\omega^2 \int_0^l \rho A y(x) dx - Q_i = 0 \tag{14}$$

The second of equations (11a) and (13) yield the relation

$$\int_0^l Q(x) dx + M_i - \omega^2 \int_0^l I\rho\varphi(x) dx = 0 \tag{15}$$

The first of equations (11a) yields

$$Q(x) = \omega^2 \int_0^x \rho A y(x) dx - Q_i h(x - x_i) \tag{16}$$

Substituting equation (16) into (15) results in

$$M_i + \omega^2 \int_0^l \int_0^x \rho A y(x) dx \, dx - Q_i \int_0^l h(x - x_i) dx - \omega^2 \int_0^l I\rho\varphi(x) dx(x) = 0 \tag{17}$$

It is easily shown (ref. 6) that

$$\int_0^l \int_0^x \rho A y(x) dx \, dx = \int_0^l \rho A(l - x) y(x) dx \tag{18}$$

Substituting equation (18) into (19) and rearranging yield

$$l\left[\omega^2 \int_0^l \rho A y(x) dx - Q_i\right] - \omega^2 \int_0^l I\rho\varphi(x) dx - \omega^2 \int_0^l \rho A x y(x) dx + Q_i x_i + M_i = 0 \tag{19}$$

The first term in brackets in equation (19) is simply the shear at $x = l$, which is zero for this case. Thus, equation (19) becomes

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$$\omega^2 \int_0^l \rho A x y(x) dx + \omega^2 \int_0^l I \rho \varphi(x) dx - Q_i x_i - M_i = 0 \tag{20}$$

Equations (14) and (20) represent the equations which must be satisfied to ensure a vanishing moment and shear at the boundaries of the primary beam.

Free-cantilever beam. - For a beam free at $x = 0$ and fixed at $x = l$, the boundary conditions are

$$\left. \begin{array}{cc} Q(0) = 0 & M(0) = 0 \\[2em] y'(l) = -\dfrac{Q(l)}{A_s G} & y(l) = 0 \end{array} \right\} \tag{21}$$

Branch Beam Equations

The governing differential equations of the branch beam can be developed in a manner analogous to those of the primary beam. The coordinate system of the branch is as shown in figure 1, and an element along the length is as shown in figure 2, except that the concentrated shears and moments q_i and m_i will not be present. The governing equations of the branch, corresponding to equations (11a) of the primary beam, will therefore be given by

$$\left. \begin{array}{c} \dfrac{dQ_{bi}(\xi_i)}{d\xi_i} = \omega^2 (\rho A)_{bi} \eta_i(\xi_i) \\[2em] \dfrac{dM_{bi}(\xi_i)}{d\xi_i} = Q_{bi}(\xi_i) - \omega^2 (I\rho)_{bi} \psi_i(\xi_i) \\[2em] \psi_i = \dfrac{d\overline{\eta}_i(\xi_i)}{d\xi_i} + \gamma_i \\[2em] \dfrac{d\eta_i(\xi_i)}{d\xi_i} = \dfrac{d\overline{\eta}_i(\xi_i)}{d\xi_i} + \gamma_i - \dfrac{Q_{bi}(\xi_i)}{(A_s G)_{bi}} \\[2em] M_{bi} = (EI)_{bi} \dfrac{d^2 \overline{\eta}_i(\xi_i)}{d\xi_i^2} \end{array} \right\} \tag{22}$$

where the subscripts bi refer to the ith branch (for one branch $i = 1$) and γ_i is the bending slope at the attachment station x_i.

Branch Beam Boundary Conditions

To obtain expressions for the boundary conditions of the branch beam, the free-body diagram of the branch beam shown in figure 3 is considered. In keeping with beam theory, the slopes β_i and γ_i are assumed to be small (i. e., $\tan \beta_i = \beta_i$ and $\tan \gamma_i = \gamma_i$).

For small deflections, Newton's second law gives the following equations for the spring-mass system in harmonic motion:

$$\left. \begin{array}{l} Q_{ei} = -m_{ci}\omega^2[\Delta_i + a_i(\theta_i + \beta_i)] \\[2em] M_{ei} = \omega^2 I_{cgi}(\theta_i + \beta_i) - Q_{ei}a_i \end{array} \right\} \qquad (23)$$

or

$$M_{ei} = K_i\theta_i = \omega^2[J_i(\theta_i + \beta_i) + m_{ci}a_i\Delta_i] \qquad (24)$$

where J_i, the mass moment of inertia of the spring-mass system about the tip of the branch beam, may be expressed as
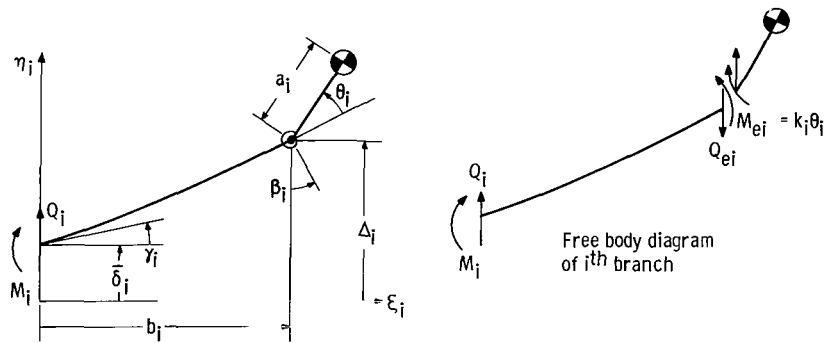
$$J_i = I_{cgi} + m_{ci}a_i^2$$

Figure 3. - Coordinate system for i^{th} branch beam.

Solving equation (24) for θ_i yields

$$\theta_i = \frac{\dfrac{\omega^2}{p_i^2}}{1 - \dfrac{\omega^2}{p_i^2}} \left(\beta_i + \frac{m_{ci} a_i \Delta_i}{J_i} \right) \tag{25}$$

where p_i is the natural frequency of the simple spring-mass system defined by

$$p_i = \sqrt{\frac{K_i}{J_i}}$$

Substituting equation (25) into equations (23) and (24) yields the following expressions for shear and moment at the right side of the branch beam.

$$Q_{ei} = \frac{-\omega^2 m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}} \left[\left(1 - \frac{I_{cgi}\omega^2}{K_i} \right) \Delta_i + a_i \beta_i \right] \tag{26}$$

$$M_{ei} = K_i \theta_i = \frac{\omega^2}{1 - \dfrac{\omega^2}{p_i^2}} \left(J_i \beta_i + m_{ci} a_i \Delta_i \right) \tag{27}$$

The shear and moment at the left side of the branch beam can be obtained by integrating the first and second of equations (22). Thus,

$$Q_{bi}(b_i) = \omega^2 \int_0^{b_i} (\rho A)_{bi} \eta_i(\xi_i) d\xi_i + Q_{bi}(0) \tag{28}$$

and

$$M_{bi}(b_i) = \int_0^{b_i} Q_{bi}(\xi_i) d\xi_i - \omega^2 \int_0^{b_i} (I\rho)_{bi} \psi_i(\xi_i) d\xi_i + M_{bi}(0) \tag{29}$$

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The first integral in equation (29) can be written in the following form:

$$\int_0^{b_i} Q_{bi}(\xi_i)d\xi_i = \omega^2 \int_0^{b_i}\left[\int_0^{\xi_i} (\rho A)_{bi}\eta_i(s)ds\right]d\xi_i + b_i Q_{bi}(0)$$

$$= b_i Q_{bi}(b_i) - \omega^2 \int_0^{b_i} (\rho A)_{bi}\xi_i\eta_i(\xi_i)d\xi_i \qquad (30)$$

Using this relation in equation (29) yields

$$M_{bi}(b_i) = M_{bi}(0) + b_i Q_{bi}(b_i) - \omega^2\left[\int_0^{b_i} (\rho A)_{bi}\xi_i\eta_i(\xi_i)d\xi_i + \int_0^{b_i} (I\rho)_{bi}\psi_i(\xi_i)d\xi_i\right] \qquad (31)$$

For the adopted sign convention,

$$\left.\begin{array}{ll} M_{bi}(0) = -M_i & M_{bi}(b_i) = M_{ei} \\[2mm] Q_{bi}(0) = Q_i & Q_{bi}(b_i) = Q_{ei} \end{array}\right\} \qquad (32)$$

Then, from equations (26) to (29) and (32),

$$Q_i = -\omega^2 \int_0^{b_i} (\rho A)_{bi}\eta_i(\xi_i)d\xi_i + Q_{ei} \qquad (33)$$

$$M_i = -M_{ei} + b_i Q_{ei} - \omega^2\left[\int_0^{b_i} (\rho A)_{bi}\xi_i\eta_i(\xi_i)d\xi_i + \int_0^{b_i} (I\rho)_{bi}\psi_i(\xi_i)d\xi_i\right] \qquad (34)$$

Eliminating M_{ei} and Q_{ei} from these last two equations yields

$$Q_i = -\omega^2\left\{\int_0^{b_i} (\rho A)_{bi}\eta_i(\xi_i)d\xi_i + \left(\frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\right)\left[\left(1 - \frac{I_{cgi}\omega^2}{K_i}\right)\Delta_i + a_i\beta_i\right]\right\} \qquad (35)$$

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$$M_i = -\omega^2 \left\{ \int_0^{b_i} (\rho A)_{bi} \xi_i \eta_i(\xi_i) d\xi_i + \int_0^{b_i} (I\rho)_{bi} \psi_i(\xi_i) d\xi_i \right.$$

$$\left. + \left(\frac{m_{ci}}{1 - \frac{\omega^2}{p_i^2}} \right) \left[a_i + \left(1 - \frac{I_{cgi}\omega^2}{K_i} \right) b_i \right] \Delta_i + \left(\frac{1}{1 - \frac{\omega^2}{p^2}} \right) (J_i + m_{ci} a_i b_i) \beta_i \right\} \qquad (36)$$

These two equations represent the shear and moment transferred to the primary beam as a result of the dynamics of the branch-beam and spring-mass system.

Orthogonality Conditions

To determine the orthogonality conditions for the branched Timoshenko beam, eliminate m and q from equations (3) and (4) by using equations (8) and (9) to obtain

$$\left. \begin{aligned} \frac{\partial}{\partial x}\left(EI \frac{\partial \overline{\Phi}}{\partial x} \right) + A_s G \left(\frac{\partial Y}{\partial x} - \overline{\Phi} \right) - m_i \delta(x - x_i) = I\rho \ddot{\overline{\Phi}} \\ \frac{\partial}{\partial x}\left[A_s G \left(\frac{\partial Y}{\partial x} - \overline{\Phi} \right) \right] - q_i \delta(x - x_i) = \rho A \ddot{Y} \end{aligned} \right\} \qquad (37)$$

For a system vibrating in the jth normal mode,

$$\left. \begin{aligned} Y(x, t) = y_j(x) e^{i\omega jt}, \quad m_i(t) = M_{ij} e^{i\omega jt} \\ \overline{\Phi}(x, t) = \varphi_j(x) e^{i\omega t}, \quad q_i(t) = Q_{ij} e^{i\omega jt} \end{aligned} \right\} \qquad (38)$$

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From equations (37) and (38),

$$\left.\begin{array}{l} (EI\varphi_j')' + [A_sG(y_j' - \varphi_j)] - M_{ij}\delta(x - x_i) = -\omega_j^2 I\rho\varphi_j \\[4mm] [A_sG(y_j' - \varphi_j)]' - Q_{ij}\delta(x - x_i) = -\omega_j^2\rho A y_j \end{array}\right\} \tag{39}$$

where

$$y_j = y_j(x), \quad \varphi_j = \varphi_j(x)$$

and the primes denote d/dx. Let y_k and φ_k represent the deflection and slope of the beam vibrating in the kth normal mode. Then, multiplying the first of equations (39) by $\varphi_k \, dx$ and the second by $y_k \, dx$, integrating over the length, and adding the resulting equations yields

$$-\omega_j^2 \int_0^l (\rho A y_j y_k + I\rho\varphi_j\varphi_k)dx + M_{ij}\gamma_{ik} + Q_i\bar{\delta}_{ik} = \int_0^l (EI\varphi_j)' \, \varphi_k \, dx$$

$$+ \int_0^l A_sG(y_j' - \varphi_j)' \, y_k \, dx + \int_0^l [A_sG(y_j' - \varphi_j)\varphi_k \, dx] \tag{40}$$

Integrating the first two integrals on the right side of equation (40) by parts yields

$$-\omega_j^2 \int_0^l (\rho A y_j y_k + I\rho\varphi_j\varphi_k)dx + M_{ij}\gamma_{ik} + Q_{ij}\bar{\delta}_{ik} = \left[EI\varphi_j'\varphi_k\right]_0^l - \int_0^l EI\varphi_j'\varphi_k' \, dx$$

$$+ [A_sG(y_j' - \varphi_j)y_k]_0^l - \int_0^l A_sG(y_j' - \varphi_j)y_k' \, dx + \int_0^l A_sG(y_j' - \varphi_j)\varphi_k \, dx \tag{41}$$

Now, for conventional boundary conditions (i. e., free, pinned, or fixed), the terms in brackets in equation (41) vanish. Thus,

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$$-\omega_j^2 \int_0^l (\rho A y_j y_k + I\rho \varphi_j \varphi_k)dx + M_{ij}\gamma_{ik} + Q_{ij}\bar{\delta}_{ik} = -\int_0^l EI\varphi_j'\varphi_k' \, dx$$

$$-\int_0^l A_s G(y_j' - \varphi_j)(y_k' - \varphi_k)dx \qquad (42)$$

Following the same procedure for the branch beam, the equation corresponding to equation (41) will be

$$-\omega_j^2 \int_0^{b_i} [(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}]d\xi_i = [(EI)_{bi}\psi_{ij}'\psi_{ik}]_0^{b_i} + [(A_s G)_{bi}(\eta_{ij}' - \psi_{ij})\eta_{ik}]_0^{b_i}$$

$$-\int_0^{b_i} (EI)_{bi}\psi_{ij}'\psi_{ik}' \, d\xi_i - \int_0^{b_i} (A_s G)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})d\xi_i \qquad (43)$$

Evaluating the terms in brackets

$$\left.\begin{aligned}
[(EI)_{bi}\psi_{ij}'\psi_{ik}]_0^{b_i} &= M_{eij}\beta_{ik} - (-M_{ij}\gamma_{ik}) \\
\\
-[(A_s G)_{bi}(\psi_{ij} - \eta_{ij}')\eta_{ik}]_0^{b_i} &= -(Q_{eii}\Delta_{ik} - Q_{ij}\bar{\delta}_{ik})
\end{aligned}\right\} \qquad (44)$$

then, substituting equations (44) into equation (43) and adding the resulting equation to equation (42) gives

$$\omega_j^2\left\{\int_0^l (\rho A y_j y_k + I\rho \varphi_j \varphi_k)dx + \int_0^{b_i} [(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}]d\xi_i\right\}$$

$$= \int_0^l [EI\varphi_j'\varphi_k' + A_s G(y_j' - \varphi_j)(y_k' - \varphi_k)]dx$$

$$+ \int_0^{b_i} [(EI)_{bi}\psi_{ij}'\psi_{ik}' + (A_s G)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})]d\xi_i - M_{eij}\beta_{ik} + Q_{eij}\Delta_{ik} \qquad (45)$$

Also, from equations (23) and (24)

$$M_{eij}\beta_{ik} - Q_{eij}\Delta_{ik} = -\omega_j^2\theta_{ik}(J_i\alpha_{ij} + m_{ci}a_i\Delta_{ij})$$

$$+ \omega_j^2[I_{cgi}\alpha_{ij}\alpha_{ik} + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})(\Delta_{ik} + a_i\alpha_{ik})] \qquad (46)$$

From equation (24),

$$J_i\alpha_{ij} + m_{ci}a_i\Delta_{ij} = \frac{K_i\theta_{ij}}{\omega_j^2} \qquad (47)$$

Substituting equation (47) into equation (46) and substituting the resulting equation into equation (45) gives

$$\omega_j^2\left\{\int_0^l (\rho A y_j y_k + I\rho\varphi_j\varphi_k)dx + \int_0^{b_i}[(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}]d\xi_i + I_{cgi}\alpha_{ij}\alpha_{ik}\right.$$

$$\left. + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})(\Delta_{ik} + a_i\alpha_{ik})\right\} = \left\{\int_0^l [EI\varphi_j'\varphi_k' + A_sG(y_j' - \varphi_j)(y_k' - \varphi_k)]dx\right.$$

$$\left. + K_i\theta_{ij}\theta_{ik} + \int_0^{b_i}[(EI)_{bi}\psi_{ij}'\psi_{ik}' + (A_sG)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})]d\xi_i\right\} \qquad (48)$$

All the terms which appear in equation (48) are symmetric with respect to j and k except ω_j^2. Therefore, if the system is considered to be vibrating in the kth mode and the same procedure is followed as for the jth mode, an equation identical to equation (48) is obtained except ω_j^2 will be replaced by ω_k^2; that is,

$$\omega_k^2\left\{\int_0^l (\rho A y_k y_j + I\rho\varphi_j\varphi_k)dx + \int_0^{b_i}[(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}]d\xi_i + I_{cgi}\alpha_{ij}\alpha_{ik}\right.$$

$$\left. + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})(\Delta_{ik} + a_i\alpha_{ik})\right\} = \left\{\int_0^l [EI\varphi_j'\varphi_k' + A_sG(y_j' - \varphi_j)(y_k' - \varphi_k)]dx\right.$$

$$\left. + K_i\theta_{ij}\theta_{ik} + \int_0^{b_i}[(EI)_{bi}\psi_{ij}'\psi_{ik}' + (A_sG)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})]d\xi_i\right\} \qquad (49)$$

Subtracting equation (49) from equation (48) and assuming $\omega_j^2 \neq \omega_k^2$ gives

$$\int_0^{b_i} (\rho A y_k y_j + I\rho \varphi_j \varphi_k)dx + \int_0^{b_i} [(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}]d\xi_i$$

$$+ I_{cgi}\alpha_{ij}\alpha_{ik} + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})(\Delta_{ik} + a_i\alpha_{ik}) = 0 \qquad (50)$$

Equation (50) represents the first orthogonality condition of the branched beam. To obtain the second orthogonality condition, multiply equation (48) and (49) by ω_k^2 and ω_j^2, respectively, and subtract. Then, for $\omega_j^2 \neq \omega_k^2$,

$$\int_0^l [EI\varphi_j'\varphi_k' + A_sG(y_j' - \varphi_j)(y_k' - \varphi_k)]dx + K_i\theta_{ij}\theta_{ik}$$

$$+ \int_0^{b_i} [(EI)_{bi}\psi_{ij}'\psi_{ik}' + (A_sG)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})]d\xi_i = 0 \qquad (51)$$

An alternate form for the second orthogonality condition is obtained by using the relations

$$M = EI\varphi', \quad Q = A_sG(\varphi - y')$$

$$M_{bi} = (EI)_{bi}\psi_i', \quad Q_{bi} = (A_sG)_{bi}(\psi_i - \eta_i')$$

in equation (51). Thus,

$$\int_0^l \frac{M_j M_k}{EI} dx + \int_0^l \frac{Q_j Q_k}{A_sG} dx + \int_0^{b_i} \frac{M_{bij}M_{bik}}{(EI)_{bi}} d\xi_i$$

$$+ \int_0^{b_i} \frac{Q_{bij}Q_{bik}}{(A_sG)_{bi}} d\xi_i + \frac{M_{eik}M_{eij}}{K_i} = 0 \qquad (52)$$

Generalized mass. - Equations (50) to (52) are valid for $\omega_j^2 \neq \omega_k^2$. To obtain the general expression for these orthogonality conditions, let

$$\mu_j = \int_0^l \left(\rho A y_j^2 + I\rho \varphi_j^2\right) dx + \int_0^{b_i} \left[(\rho A)_{bi}\eta_{ij}^2 + (I\rho)_{bi}\psi_{ij}^2\right] d\xi_i + I_{cgi}\alpha_{ij}^2 + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})^2$$

(53a)

where μ_j is referred to as the generalized mass of the system vibrating in the jth normal mode. Then, in terms of the generalized mass, the first and second orthogonality conditions become

$$\int_0^l (\rho A y_j y_k + I\rho \varphi_j \varphi_k) dx + \int_0^{b_i} [(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}] d\xi_i$$

$$+ I_{cgi}\alpha_{ij}\alpha_{ik} + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})(\Delta_{ik} + a_i\alpha_{ik}) = \mu_j \delta_{jk} \qquad (54a)$$

and

$$\int_0^l [(EI\varphi_j'\varphi_k' + A_s G(y_j' - \varphi_j)(y_k' - \varphi_k)] dx + K_i \theta_{ij}\theta_{ik}$$

$$+ \int_0^{b_i} [(EI)_{bi}\psi_{ij}\psi_{ik} + (A_s G)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})] d\xi_i = \omega_j^2 \mu_j \delta_{jk} \qquad (55a)$$

where δ_{jk} is the Kronecker delta function defined by

$$\delta_{jk} = \begin{cases} 1 & j = k \\ \\ 0 & j \neq k \end{cases}$$

Equations (54a) and (55a) satisfy equation (48) for $j = k$ and equations (50) and (51) for $j \neq k$.

Natural frequency. - An expression for the natural frequency of the system can be obtained by setting $j = k$ in equation (48). Thus,

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$$\omega_j^2 = \cfrac{\displaystyle\int_0^l \left(\frac{M_j^2}{EI} + \frac{Q_j^2}{A_sG}\right)dx + \int_0^{b_i}\left[\frac{M_{bij}^2}{(EI)_{bi}} + \frac{Q_{bij}^2}{(A_sG)_{bi}}\right]d\xi_i + \frac{M_{eij}^2}{K_i}}{\displaystyle\int_0^l \left(\rho A y_j^2 + I\rho\varphi_j^2\right)dx + \int_0^{b_i}\left[(\rho A)_{bi}\eta_{ij}^2 + (I\rho)_{bi}\psi_{ij}^2\right]d\xi_i + I_{cgi}\alpha_{ij}^2 + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})^2}$$

<div style="text-align:right">(56a)</div>

Primary Beam with Several Branches

The equations developed on the preceding pages are for a primary beam with one branch. The results can be immediately generalized to include any number of branches by replacing $Q_i\delta(x - x_i)$ and $M_i\delta(x - x_i)$ with $\sum_i Q_i\delta(x - x_i)$ and $\sum_i M_i\delta(x - x_i)$, respectively, in the equilibrium equations. The development will then proceed in the same manner, and the resulting equations will be similar to those previously developed, except that the branch beam parameters will be summed over all the branches. For example, assume that there are n branches. Then, a summary of the pertinent equations will be as follows.

Governing differential equations of primary beam:

$$\left.\begin{aligned}
\frac{dQ(x)}{dx} &= \omega^2\rho A y(x) - \sum_{i=1}^{n} Q_i\delta(x - x_i) \\[2em]
\frac{dM(x)}{dx} &= Q(x) + \sum_{i=1}^{n} M_i\delta(x - x_i) - \omega^2 I\rho\varphi(x) \\[2em]
\varphi(x) &= \frac{d\bar{y}(x)}{dx} + y_0' \\[2em]
\frac{dy(x)}{dx} &= \frac{d\bar{y}(x)}{dx} + y_0' - \frac{Q(x)}{A_sG} \\[2em]
M(x) &= EI\,\frac{d^2\bar{y}(x)}{dx^2}
\end{aligned}\right\}$$

<div style="text-align:right">(11b)</div>

The last three equations are independent of the number of branches and are included herein only for later reference.

Free-free boundary conditions:

$$\left.\begin{array}{l} \omega^2 \int_0^l \rho A y(x) dx - \sum_{i=1}^{n} Q_i = 0 \\[20pt] \omega^2 \int_0^l \rho A x y(x) dx + \omega^2 \int_0^l I\rho\varphi(x) dx - \sum_{i=1}^{n} Q_i x_i - \sum_{i=1}^{n} M_i = 0 \end{array}\right\} \qquad (53b)$$

Generalize mass:

$$\mu_j = \int_0^l \left(\rho A y_j^2 + I\rho\varphi_j^2\right) dx + \sum_{i=1}^{n} \int_0^{b_i} \left[(\rho A)_{bi}\eta_{ij}^2 + (I\rho)_{bi}\psi_{ij}^2\right] d\xi_i$$

$$+ \sum_{i=1}^{n} \left[I_{cgi}\alpha_{ij}^2 + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})^2\right] \qquad (53b)$$

Orthogonality conditions:

$$\int_0^l (\rho A y_j y_k + I\rho\varphi_j\varphi_k) dx + \sum_{i=1}^{n} \int_0^{b_i} [(\rho A)_{bi}\eta_{ij}\eta_{ik} + (I\rho)_{bi}\psi_{ij}\psi_{ik}] d\xi_i$$

$$+ \sum_{i=1}^{n} [I_{cgi}\alpha_{ij}\alpha_{ik} + m_{ci}(\Delta_{ij} + a_i\alpha_{ij})(\Delta_{ik} + a_i\Delta_{ik})] = \mu_j\delta_{jk} \qquad (54b)$$

$$\int_0^l [EI\varphi_j'\varphi_k' + A_s G(y_j' - \varphi_j)(y_k' - \varphi_k)] dx + \sum_{i=1}^{n} K_i\theta_{ij}\theta_{ik}$$

$$+ \sum_{i=1}^{n} \int_0^{b_i} [(EI)_{bi}\psi_{ij}'\psi_{ik}' + (A_s G)_{bi}(\eta_{ij}' - \psi_{ij})(\eta_{ik}' - \psi_{ik})] d\xi_i = \omega_j^2\mu_j\delta_{ik} \qquad (55b)$$

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Frequency equation:

$$\omega_j^2 = \frac{\displaystyle\int_0^\ell \left(\frac{M_j^2}{EI} + \frac{Q_j^2}{A_sG}\right)dx + \sum_{i=1}^n \left\{\int_0^{b_i}\left[\frac{M_{bij}^2}{(EI)_{bi}} + \frac{Q_{bij}^2}{(A_sG)_{bi}}\right]d\xi_i + \frac{M_{eij}^2}{K_i}\right\}}{\displaystyle\int_0^\ell \left(\rho A y_j^2 + I\rho\varphi_j^2\right)dx + \sum_{i=1}^n\left\{\int_0^{b_i}\left[(\rho A)_{bi}\eta_{ij}^2 + (I\rho)_{bi}\psi_{ij}^2\right]d\xi_i + I_{cgi}\alpha_{ij} + m_{ci}(\Delta_i + a_i\alpha_{ij})^2\right\}}$$

(56b)

MODIFIED STODOLA METHOD

In the previous section, the governing differential equations of a branched Timoshenko beam have been developed. Also, the boundary conditions for the free-free and the free-cantilevered primary beam have been expressed in integral equation form. In this section, a modified Stodola method will be applied to solve the differential equations of motion subject to the free-free and free-cantilevered type boundary conditions.

The Stodola method, as applied to beam vibration, starts by assuming a deflected shape of the beam which satisfies the boundary condition. Based on this assumed deflection, the corresponding shears, moments, slopes, and new deflections can be determined by integrating the governing differential equations. Then, by using the boundary condition equations, the new deflection is adjusted to satisfy the boundary conditions. At this point, the natural frequency of vibration can be determined. Using the new improved deflection, the process is repeated until the natural frequency converges. If the branched beam has an attached spring mass, as in the case considered in this report, it is also necessary to assume a frequency of vibration at the start of each iteration. The reason for this is that the frequency does not appear as a constant factor in the governing equations as developed. This point will be clarified as the Stodola method is applied in detail.

For the free-free case, the iteration routine is started by assuming that the deflected shape of the primary beam is a polynomial of the form

$$y(x) = Y_p^{(o)}(x) = \prod_{v=1}^{v=p}\left[x - \left(\frac{v}{p+2}\right)\ell\right]$$

(57)

where the superscript (o) refers to the first iteration and the subscript p referes to the mode being determined. For a cantilevered beam, clamped at $x = l$, the initial assumed primary beam deflection can be taken as

$$y(x) = Y_p^{(o)}(x) = (x - l) \prod_{v=1}^{p} \left(x - \frac{vl}{p}\right) \tag{58}$$

A deflection must also be assumed for each branch beam. For the ith branch, the deflected shape can be taken as

$$\eta_{bi}^{(o)}(\xi_i) = \bar{\delta}_i^{(o)} + \gamma_i^{(o)} \xi_i + \xi_i^2 \tag{59}$$

where $\bar{\delta}_i^{(o)}$ is the deflection of the primary beam at the attachment station and

$$\gamma_i^{(o)} = Y_p'^{(o)}(x_i)$$

is the slope at the attachment station.

Before starting the iteration routine, it will be convenient to normalize the deflection to unity at the tip of the primary beam. The branch beams should be multiplied by the same normalizing factor as the primary beam. Also for convenience, ω^2, which appears as a factor, is set equal to one, since the mode shapes can only be determined to within a constant factor. The ω^2 terms which do not appear as a factor are set equal to one in the first iteration and then for subsequent iterations, the value determined by the previous iteration is used.

First, consider the iteration routine of the ith branch beam. For the assumed shape, the shear and moment at the tip of the branch beam is given by equations (26) and (27) as

$$Q_{ei} = \frac{-m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}} \left[\left(1 - \frac{I_{cgi}}{K_i}\omega^2\right)\Delta_i + a_i\beta_i\right] \tag{60}$$

$$M_{ei} = \frac{1}{1 - \dfrac{\omega^2}{p_i^2}} (J_i\beta_i + m_{ci}a_i \Delta_i) \tag{61}$$

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Then, the shear and moment at the attachment station will be given by equations (33) and (34), respectively.

$$Q_i = -\int_0^{b_i} (\rho A)_{bi} \eta_i(\xi_i) d\xi_i + Q_{ei} \tag{62}$$

$$M_i = -\left[\int_0^{b_i} (\rho A)_{bi} \xi_i \eta_i(\xi_i) d\xi_i + \int_0^{b_i} (I\rho)_{bi} \psi_i(\xi_i) d\xi_i\right] + Q_{ei} b_i - M_e \tag{63}$$

Since ψ_i or β_i is not known in the first iteration, it is set equal to zero, and for subsequent iterations the values from the previous iteration are used.

The shear and moment distribution along the branch beam can now be calculated by integrating the first and second of equations (22) and noting that, for the adopted sign convention,

$$Q_{bi}(0) = Q_i, \quad M_{bi}(0) = -M_i \tag{64}$$

Therefore,

$$Q_{bi}(\xi_i) = \int_0^{\xi_i} (\rho A)_{bi} \eta_i(\xi_i) d\xi_i + Q_i \tag{65}$$

$$M_{bi}(\xi_i) = \int_0^{\xi_i} Q_{bi}(\xi_i) d\xi_i - \int_0^{\xi_i} (I\rho)_{bi} \psi_i d\xi_i - M_i \tag{66}$$

When the shear and moment distribution is known, a new improved deflection of the branch beam can be determined by utilizing the third, fourth, and fifth of equations (22); thus,

$$\eta_i'(\xi_i) = \int_0^{\xi_i} \frac{M_{bi}(\xi_i)}{(EI)_{bi}} d\xi_i + \gamma_i - \frac{Q_{bi}(\xi_i)}{(A_s G)_{bi}} \tag{67}$$

and

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$$\eta_i(\xi_i) = \int_0^{\xi_i} \int_0^{\xi_i} \frac{M_{bi}(\xi_i)}{(EI)_{bi}} \, d\xi_i \, d\xi_i - \int_0^{\xi_i} \frac{Q_{bi}(\xi_i)}{(A_sG)_{bi}} \, d\xi_i + \gamma_i\xi_i + \overline{\delta}_i \qquad (68)$$

To determine the values of $\overline{\delta}_i$ and γ_i in equations (67) and (68), the deflection and slope of the primary beam at the attachment point of the ith branch must be determined. This can be done by an integration procedure similar to that used for the branched beam.

For the primary beam, the shear and moment distribution can be determined by integrating the first and second of equations (11b). Thus,

$$Q(x) = \int_0^x \rho A y(x) dx - \sum_{i=1}^n Q_i h(x - x_i) \qquad (69)$$

and

$$M(x) = \int_0^x \int_0^x \rho A y(x) dx \, dx - \int_0^x I\rho\varphi(x) dx - \sum_{i=1}^n Q_i \int_0^x h(x - x_i) dx + \sum_{i=1}^n M_i h(x - x_i)$$

$$(70)$$

where $h(x - x_i)$ and $\int_0^x h(x - x_i) dx$ are given by equations (1) and (2), respectively. Then, from the third, fourth, and fifth of equations (11b),

$$y(x) = \int_0^x \int_0^x \frac{M(x)}{EI} \, dx \, dx - \int_0^x \frac{Q(x)}{A_sG} \, dx + y_0'x + y_0 \qquad (72)$$

The constants of integration y_0 and y_0 must now be adjusted to satisfy the boundary conditions. First, for compactness, the following quantities are defined:

$$\overline{y}(x) = \int_0^x \int_0^x \frac{M(x)}{EI} \, dx \, dx \qquad \Phi(x) = \int_0^x \frac{Q(x)}{A_sG} \, dx$$

$$\bar{\eta}_i(\xi_i) = \int_0^{\xi_i} \int_0^{\xi_i} \frac{M_{bi}(\xi_i)}{(EI)_{bi}} \, d\xi_i \, d\xi_i \qquad\qquad \zeta(\xi_i) = \int_0^{\xi_i} \frac{Q_{bi}(\xi_i)}{(A_s G)_{bi}} \, d\xi_i$$

Using these substitutions, the deflection and bending slope of the beam can be written as

$$\left.\begin{array}{c}
y(x) = \bar{y}(x) - \Phi(x) + y_0' x + y_0 \\[2mm]
\varphi(x) = \bar{y}'(x) + y_0' \\[2mm]
\eta_i(\xi_i) = \bar{\eta}_i(\xi_i) - \zeta_i(\xi_i) + [\bar{y}'(x_i) + y_0']\xi_i + [\bar{y}(x_i) - \Phi(x_i) + y_0' x_i + y_0] \\[2mm]
\psi_i(\xi_i) = \eta_i'(\xi_i) + [\bar{y}'(x_i) + y_0']
\end{array}\right\} \qquad (73)$$

For the free-free beam, the boundary constants y_0' and y_0 must be adjusted to satisfy equations (14) and (20). The expressions for Q_i and M_i, which should be used in these equations, are given by equations (35). Therefore, the following equations must be satisfied

$$\left.\begin{array}{l}
\displaystyle\int_0^l \rho A y(x) dx + \sum_{i=1}^n \int_0^l (\rho A)_{bi} \eta_i(\xi_i) d\xi_i + \sum_{i=1}^n \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[\left(1 - \frac{I_{cgi}\omega^2}{K_i}\right)\Delta_i + a_i\beta_i\right] = 0 \\[8mm]
\displaystyle\int_0^l \rho A x y(x) dx + \int_0^l I\rho\varphi(x) dx + \sum_{i=1}^n \int_0^{b_i} (\rho A)_{bi}(x_i + \xi_i)\eta_i(\xi_i) d\xi_i + \sum_{i=1}^n \int_0^{b_i} (I\rho)_{bi}\psi_i(\xi_i) d\xi_i \\[8mm]
\displaystyle + \sum_{i=1}^n \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[\left(1 - \frac{I_{cgi}\omega^2}{K_i}\right)(x_i + b_i) + a_i\right]\Delta_i + \sum_{i=1}^n \frac{1}{1 - \dfrac{\omega^2}{p_i^2}}[m_{ci}a_i(x_i + b_i) + J_i]\beta_i = 0
\end{array}\right\} \qquad (74)$$

If equations (73) are substituted into equations (74), two equations in two unknown constants are obtained, namely, y_0' and y_0. Note that in equations (74) the expressions for Δ_i and β_i are given by

$$\Delta_i = \eta_i(b_i) = \overline{\eta}_i(b_i) - \zeta_i(b_i) + [\overline{y}'(x_i) + y_0']b_i + \overline{y}(x_i) - \Phi(x_i) + y_0'x_i + y_0 \left.\right\}$$

$$\beta_i = \psi_i(b_i) = \overline{\eta}(b_i) + \overline{y}'(x_i) + y_0' \qquad\qquad\qquad\qquad\qquad (75)$$

Substituting equations (73) and (75) into (74) yields

$$\int_0^l \rho A[\overline{y}(x) - \Phi(x) + y_0'x + y_0]dx + \sum_{i=1}^{n} \frac{m_{ci}a_i}{1 - \dfrac{\omega^2}{p_i^2}} [\overline{\eta}_i'(b_i) + \overline{y}'(x_i) + y_0']$$

$$+ \sum_{i=1}^{n} \int_0^{b_i} (\rho A)_{bi}\left\{\overline{\eta}_i(\xi_i) - \zeta_i(\xi_i) + [\overline{y}'(x_i) + y_0']\xi_i + \overline{y}(x_i) - \Phi(x_i) + y_0'x_i + y_0\right\}d\xi_i$$

$$+ \sum_{i=1}^{n} \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left(1 - \frac{I_{cgi}\omega^2}{J_i p_i^2}\right)\left\{\overline{\eta}_i(b_i) - \zeta_i(b_i)\right.$$

$$\left. + [\overline{y}'(x_i) + y_0']b_i + \overline{y}(x_i) - \Phi_i(x_i) + y_0'x_i + y_0\right\} = 0 \qquad\qquad (76)$$

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$$\int_0^l \rho A x[\bar{y}(x) - \Phi(x) + y_0'x + y_0]dx + \int_0^l I\rho[\bar{y}(x) + y_0']dx + \sum_{i=1}^n \int_0^{b_i} (I\rho)_{bi}[\bar{\eta}_i'(\xi_i) + \bar{y}'(x_i) + y_0']d\xi_i + \sum_{i=1}^n \int_0^{b_i} \rho A(x_i + \xi_i)\{\bar{\eta}_i(\xi_i) - \zeta_i(\xi_i)$$

$$+ [\bar{y}'(x_i) + y_0']\xi_i + \bar{y}(x_i) - \Phi(x_i) + y_0'x_i + y_0\}d\xi_i + \sum_{i=1}^n \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[\left(1 - \frac{I_{cgi}\omega^2}{J_i p_i^2}\right)(x_i + b_i) + a_i\right]\{\bar{\eta}_i(b_i) - \zeta_i(b_i) + [\bar{y}'(x_i) + y_0']b_i + \bar{y}(x_i) - \Phi(x_i) + y_0'x_i + y_0\}$$

$$+ \sum_{i=1}^n \frac{1}{1 - \dfrac{\omega^2}{p_i^2}}[m_{ci}a_i(x_i + b_i) + J_i][\bar{\eta}_i'(b_i) + \bar{y}'(x_i) + y_0'] = 0 \tag{77}$$

Collecting terms in y_0' and y_0 and rearranging yields equations (78a) and (79a).

Shear boundary condition:

$$\int_0^l \rho A[\bar{y}(x) - \Phi(x)]dx + \sum_{i=1}^n \int_0^{b_i} (\rho A)_{bi}[\bar{\eta}_i(\xi_i) - \zeta_i(\xi_i)]d\xi_i + \sum_{i=1}^n \frac{m_{ci}a_i}{1 - \dfrac{\omega^2}{p_i^2}}\bar{\eta}'(b_i) + \sum_{i=1}^n \bar{y}'(x)\left\{\int_0^{b_i} (\rho A)_{bi}\xi_i\, d\xi_i + \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[(b_i + a_i) - \frac{I_{cgi}b_i\omega^2}{K_i}\right]\right\}$$

$$+ \sum_{i=1}^n \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left(1 - \frac{I_{cgi}\omega^2}{K_i}\right)[\bar{\eta}_i(b_i) - \zeta_i(b_i) + \bar{y}(x_i) - \Phi(x_i)] + \sum_{i=1}^n [\bar{y}(x_i) - \Phi(x_i)]\left[\int_0^{b_i} (\rho A)_{bi}\, d\xi_i\right]$$

$$+ y_0'\left\{\int_0^l \rho A x\, dx + \sum_{i=1}^n \int_0^{b_i} (\rho A)_{bi}(x_i + \xi_i)d\xi_i + \sum_{i=1}^n \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[x_i + b_i + a_i - \frac{I_{cgi}}{K_i}(b_i + x_i)\omega^2\right]\right\}$$

$$+ y_0\left[\int_0^l \rho A\, dx + \sum_{i=1}^n \int_0^{b_i} (\rho A)_{bi}\, d\xi_i + \sum_{i=1}^n \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left(1 - \frac{I_{cgi}\omega^2}{K_i}\right)\right] = 0 \tag{78a}$$

Moment boundary condition:

$$\int_0^l \rho A x[\overline{y}(x) - \Phi(x)]dx + \int_0^l I\rho \overline{y}'(x)dx + \sum_{i=1}^{n} \int_0^{b_i} (\rho A)_{bi}(x_i + \xi_i)[\overline{\eta}_i(\xi_i) - \zeta_i(\xi_i)]d\xi_i + \sum_{i=1}^{n} \int_0^{b_i} (I\rho)_{bi}\overline{\eta}_i'(\xi_i)d\xi_i$$

$$+ \sum_{i=1}^{n} [\overline{y}(x_i) - \Phi(x_i)]\int_0^{b_i}(\rho A)_{bi}(x_i + \xi_i)d\xi_i + \sum_{i=1}^{n} \frac{1}{1 - \dfrac{\omega^2}{p_i^2}}[m_{ci}a_i(x_i + b_i + a_i) + I_{cgi}\overline{\eta}_i'(b_i)$$

$$+ \sum_{i=1}^{n} \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[(x_i + b_i + a_i) - \frac{I_{cgi}}{K_i}(x_i + b_i)\omega^2\right][\overline{\eta}_i(b_i) - \zeta_i(b_i) + \overline{y}(x_i) - \Phi(x_i)] + \sum_{i=1}^{n} \overline{y}'(x_i)\left\langle \int_0^{b_i}(\rho A)_{bi}(x_i + \xi_i)\xi_i\,d\xi_i \right.$$

$$+ \int_0^{b_i}(I\rho)_{bi}\,d\xi_i + \left(\frac{1}{1 - \dfrac{\omega^2}{p_i^2}}\right)\left\{m_{ci}(x_i + b_i + a_i)(b_i + a_i) + I_{cgi}\left[1 - \frac{m_{ci}}{K_i}(x_i + b_i)b_i\omega^2\right]\right\}\right) + y_0'\left\langle \int_0^l \rho A x^2\,dx + \int_0^l I\rho\,dx \right.$$

$$+ \sum_{i=1}^{n} \int_0^{b_i}(I\rho)_{bi}\,d\xi_i + \sum_{i=1}^{n} \int_0^{b_i}(\rho A)_{bi}(x_i + \xi_i)^2\,d\xi_i + \sum_{i=1}^{n} \frac{1}{1 - \dfrac{\omega^2}{p_i^2}}\left\{m_{ci}(x_i + a_i + b_i)^2 + I_{cgi}\left[1 - \frac{m_{ci}}{K_i}(x_i + b_i)^2\omega^2\right]\right\}\right\rangle$$

$$+ y_0\left\{\int_0^l \rho A x\,dx + \sum_{i=1}^{n} \int_0^{b_i}(\rho A)_{bi}(x_i + \xi_i)d\xi_i + \sum_{i=1}^{n} \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}}\left[x_i + a_i + b_i - \frac{I_{cgi}}{K_i}(x_i + b_i)\omega^2\right]\right\} = 0 \qquad (79a)$$

To put equations (78a) and (79a) into a more convenient form, suitable for calculations, we define the constants given in the appendix. The subscripts r which appear with the constants K in this appendix signify that K depends on the particular mode being sought. The other constants are independent of the mode and depend only on the mass and geometric properties of the system. Using the appendix, equations (78a) and (79a) can be written as

$$A_{1r}y_0' + A_{2r}y_0 = C_{1r} \tag{78b}$$

$$A_{3r}y_0' + A_{1r}y_0 = C_{2r} \tag{79b}$$

where

$$A_{1r} = N_0 + \sum_{i=1}^{n} \left[N_{bi} + \frac{1}{1 - \dfrac{\omega^2}{p_i^2}} \left(N_{ci} - \frac{I_{cgi}\omega^2}{K_i} \right) \right]$$

$$A_{2r} = m_0 + \sum_{i=1}^{n} \left[m_{bi} + \frac{m_{ci}}{1 - \dfrac{\omega^2}{p_i^2}} \left(1 - \frac{I_{cgi}\omega^2}{K_i} \right) \right]$$

$$A_{3r} = M_0 + P_0 + \sum_{i=1}^{n} \left\{ P_{bi} + M_{bi} + \frac{1}{1 - \dfrac{\omega^2}{p_i^2}} \left[M_{ci} + I_{cgi}\left(1 - \frac{M_{di}\omega^2}{K_i} \right) \right] \right\}$$

$$C_{1r} = K_{1r} + \sum_{i=1}^{n} (K_{1ri} - K_{4ri} + K_{5ri} + K_{9ri} - K_{10ri})$$

$$C_{2r} = K_{2r} - K_{3r} + \sum_{i=1}^{n} (K_{2ri} - K_{3ri} - K_{6ri} + K_{7ri} + K_{8ri} - K_{11ri})$$

Solving equations (78b) and (79b) yields, for y_0' and y_0,

$$y_0' = \frac{\begin{vmatrix} C_{1r} & A_{2r} \\ C_{2r} & A_{1r} \end{vmatrix}}{\begin{vmatrix} A_{1r} & A_{2r} \\ A_{3r} & A_{1r} \end{vmatrix}} \qquad y_0 = \frac{\begin{vmatrix} A_{1r} & C_{1r} \\ A_{3r} & C_{2r} \end{vmatrix}}{\begin{vmatrix} A_{1r} & A_{2r} \\ A_{3r} & A_{1r} \end{vmatrix}} \qquad (80)$$

Equations (80) give the boundary constants for the free-free case. For the free-cantilevered case, we obtain from equations (21) the following expressions for y_0' and y_0:

$$y_0' = -y'(l) \qquad y_0 = \Phi(l) - y(l) + l \cdot y'(l) \qquad (81)$$

After the integration constants have been determined from either equations (80) or (81), a new deflection can then be obtained from equations (73). The frequency of vibration can now be calculated from equation (56b). Before starting the next iteration, it will be necessary, in general, to recalculate y_0 and y_0' for the free-free case based on the new improved frequency. This is required because the assumed deflection must satisfy the boundary conditions. Then, after determining a new improved deflection, the deflection is normalized, and the process is repeated until two successive frequencies are equal. A flow chart of the iteration routine is given in figure 4.
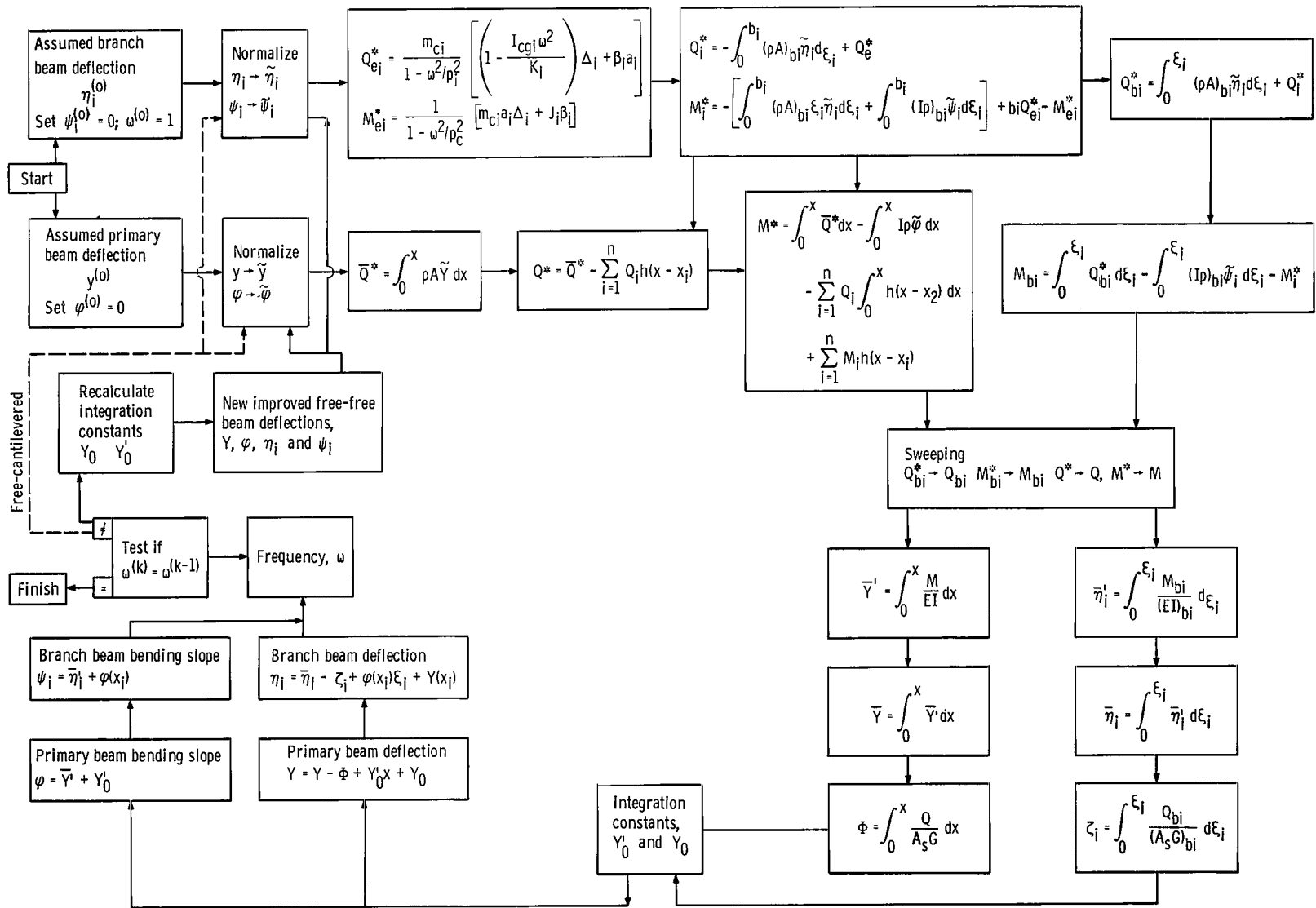
Figure 4. – Iteration routine for branched beam.

High Modes of Vibration

The iteration routine described in the previous section will always converge to the first or fundamental mode of vibration. To determine the higher modes of vibration, it is necessary to remove all lower mode components from the assumed shape. Let $y_r^*(x)$ and $\eta_{r,i}^*(\xi)$ represent the assumed shape of the rth mode. Assume that the mode shapes of all the $v = r - 1$ modes are known and that it is required to determine the rth mode. Since any arbitrary shape can be represented by a series involving the normal modes of the system,

$$
\left.
\begin{aligned}
y_r^*(x) &= C_1 y_1(x) + C_2 y_2(x) + \ldots + C_v y_v(x) + \ldots \\[2ex]
\eta_{ri}^*(\xi) &= C_1 \eta_{1i}(\xi) + C_2 \eta_{2i}(\xi) + \ldots + C_v \eta_{vi}(\xi) + \ldots
\end{aligned}
\right\}
\tag{82}
$$

where the C_v's are unknown constants that must be determined. The shear and moment distribution, corresponding to the assumed deflection, can be expressed by the series

$$
\left.
\begin{aligned}
Q_r^*(x) &= C_1 Q_1(x) + C_2 Q_2(x) + \ldots + C_v Q_v(x) + \ldots \\[2ex]
M_r^*(x) &= C_1 M_1(x) + C_2 M_2(x) + \ldots + C_v M_v(x) + \ldots \\[2ex]
Q_{ri}(\xi) &= C_1 Q_{1i}(\xi) + C_2 Q_{2i}(\xi) + \ldots + C_v Q_{vi}(\xi) + \ldots \\[2ex]
M_{ri}(\xi) &= C_1 M_{1i}(\xi) + C_2 M_{2i}(\xi) + \ldots + C_v M_{vi}(\xi) + \ldots
\end{aligned}
\right\}
\tag{83}
$$

For $\xi = b_i$, the last equation becomes

$$
M_{eri}^* = C_1 M_{e1i} + C_2 M_{e2i} + \ldots + C_v M_{evi} + \ldots
\tag{84}
$$

Next, multiply the first of equations (83) by $[Q_v(x)/A_s G]dx$, the second by $[M_v(x)/EI]dx$, the third by $\left[Q_{vi}(\xi)/(A_s G)_{bi}\right]d\xi$, and the fourth by $[M_{vi}(\xi)/(EI)_{bi}]d\xi_i$, and integrate over their respective lengths. Then multiply equation (84) by M_{evi}/K_i. These operations yield the following set of equations:

$$\int_0^l \frac{Q_r^* Q_v \, dx}{A_s G} = C_1 \int_0^l \frac{Q_1 Q_v \, dx}{A_s G} + \ldots + C_v \int_0^l \frac{Q_v^2 \, dx}{A_s G} + \ldots$$

$$\int_0^l \frac{M_r^* M_v \, dx}{EI} = C_1 \int_0^l \frac{M_1 M_v \, dx}{EI} + \ldots + C_v \int_0^l \frac{M_v^2 \, dx}{EI} + \ldots$$

$$\int_0^{b_i} \frac{Q_{ri}^* Q_{vi}}{(A_s G)_{bi}} \, d\xi = C_1 \int_0^{b_i} \frac{Q_{1i} Q_{vi}}{(A_s G)_{bi}} \, d\xi + \ldots + C_v \int_0^{b_i} \frac{Q_{vi}^2}{(A_s G)_{bi}} \, d\xi + \ldots$$

$$\int_0^{b_i} \frac{M_{ri}^* M_{vi}}{(EI)_{bi}} \, d\xi = C_1 \int_0^{b_i} \frac{M_{1i} M_{vi}}{(EI)_{bi}} \, d\xi + \ldots + C_v \int_0^{b_i} \frac{M_{vi}^2}{(EI)_{bi}} + \ldots$$

$$\frac{M_{eri}^* M_{evi}}{K_i} = C_1 \frac{M_{eli} M_{evi}}{K_i} + \ldots \cdot \frac{C_v M_{evi}^2}{K_i} + \ldots$$

$$\left. \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \qquad (85)$$

Adding equations (85) and using the second orthogonality condition which is given by equation (55b) yield

$$C_v = \frac{\displaystyle\int_0^l \left(\frac{M_r^* M_v}{EI} + \frac{Q_r^* Q_v}{A_s G} \right) dx + \int_0^{b_i} \left[\frac{M_{ri}^* M_{vi}}{(EI)_{bi}} + \frac{Q_{ri}^* Q_{vi}}{(A_s G)_{bi}} \right] d\xi + \frac{M_{eri}^* M_{evi}}{K_i}}{\omega_v^2 \mu_v} \qquad (86)$$

where ω_v is the natural frequency of the vth mode and μ_v is the generalized mass of the vth mode given by equation (53b).

For n branches, equation (86) becomes

$$C_v = \frac{\int_0^l \left(\frac{M_r^* M_v}{EI} + \frac{Q_r Q_v}{A_s G} \right) dx + \sum_{i=1}^{n} \left\{ \int_0^{b_i} \left[\frac{M_{ri}^* M_{vi}}{(EI)_{bi}} + \frac{Q_{ri}^* Q_{vi}}{(A_s G)_{bi}} \right] d\xi_i + \frac{M_{eri}^* M_{evi}}{K_i} \right\}}{\omega_v^2 \mu_v} \tag{87}$$

Knowing a_v, the purified shear and moment distribution becomes

$$\left. \begin{aligned} Q_r &= Q_r^* - \sum_{v=1}^{q-1} C_v Q_v \\[1em] M_r &= M_r^* - \sum_{v=1}^{q-1} a_v M_v \\[1em] Q_{ri} &= Q_{ri}^* - \sum_{v=1}^{q-1} C_v Q_{vi} \\[1em] M_{ri} &= M_{ri}^* - \sum_{v=1}^{q-1} C_v M_{vi} \end{aligned} \right\} \tag{88}$$

Equations (88) represent the shear and moment distribution which must be used in the iteration routine to determine the higher modes of vibration (i. e. , 2, 3, ...).

CONCLUDING REMARKS

A computer program, based on the analysis presented in this report, has been written for finding the normal modes and frequencies of a branched Timoshenko beam. Several test cases have been run and the results obtained show that the iteration routine converges in a reasonable length of time. A flow chart showing the iteration routine is given in figure 4. An examination of this flow chart shows that a number of integrals must be evaluated during each iteration. Any of the standard methods of numerical integration can be used to evaluate these integrals, and the best method will depend, in general, on

the mass and stiffness distribution along the beam. The summation method, which is thoroughly covered in reference 7, is particularly suitable for most practical cases.

Lewis Research Center,
    National Aeronautics and Space Administration,
        Cleveland, Ohio, October 23, 1967,
           180-06-06-03-22.

APPENDIX - DEFINITION OF CONSTANTS USED IN EQUATIONS (80)

$$m_0 = \int_0^l \rho A \ dx$$

$$m_{bi} = \int_0^{b_i} (\rho A)_{bi} \ d\xi_i$$

$$N_{di} = m_{ci}(x_i + b_i)$$

$$N_0 = \int_0^l \rho A x \ dx$$

$$N_{bi} = \int_0^{b_i} (\rho A)_{bi}(x_i + \xi_i) d\xi_i$$

$$N_{ci} = m_{ci}(x_i + b_i + a_i)$$

$$M_0 = \int_0^l \rho A x^2 \ dx$$

$$M_{bi} = \int_0^{b_i} (\rho A)_{bi}(x_i + \xi_i)^2 \ d\xi_i$$

$$M_{ci} = m_{ci}(x_i + b_i + a_i)^2$$

$$P_0 = \int_0^l I\rho \ dx$$

$$P_{bi} = \int_0^l (I\rho)_{bi} \ d\xi_i$$

$$M_{di} = m_{ci}(x_i + b_i)^2$$

$$K_{1r} = \int_0^l \rho A[\Phi(x) - \bar{y}(x)]dx$$

$$K_{1ri} = \int_0^{b_i} (\rho A)_{bi}[\zeta_i(\xi_i) - \bar{\eta}_i(\xi_i)]d\xi_i$$

$$K_{2r} = \int_0^l \rho A x[\Phi(x) - \bar{y}(x)]dx$$

$$K_{2ri} = \int_0^{b_i} (\rho A)_{bi}(x_i + \xi_i)[\zeta_i(\xi_i) - \eta_i(\xi_i)]d\xi_i$$

$$K_{3r} = \int_0^l I\rho\bar{y}'(x)dx$$

$$K_{3ri} = \int_0^{b_i} (I\rho)_{bi}\bar{\eta}_i'(\xi_i)d\xi_i$$

$$K_{4ri} = \bar{y}'(x_i)\left\{N_{bi} - m_{ci}x_i + \left(\frac{1}{1 - \frac{\omega^2}{p_i^2}}\right)\left[N_{ci} - m_{ci}x_i - \omega^2\frac{I_{cgi}}{K_i}(N_{di} - m_{ci}x_i)\right]\right\}$$

$$K_{5ri} = \frac{m_{ci}}{1 - \frac{\omega^2}{p_i^2}}\left(1 - \frac{I_{cgi}\omega^2}{K_i}\right)[\zeta_i(b_i) - \bar{\eta}_i(b_i) + \Phi(x_i) - \bar{y}(x_i)]$$

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$$K_{6ri} = \overline{y}'(x_i) \left\{ M_{bi} - N_{bi}x_i + P_{bi} + \left(\frac{1}{1 - \dfrac{\omega^2}{p_i^2}} \right) \left[M_{ci} - N_{ci}x_i + I_{cgi} - \frac{I_{cgi}\omega^2}{K_i}(M_{di} - N_{di}x_i) \right] \right\}$$

$$K_{7ri} = [\Phi(x_i) - \overline{y}(x_i)]N_{bi}$$

$$K_{8ri} = \frac{1}{1 - \dfrac{\omega^2}{p_i^2}} \left(N_{ci} - \frac{I_{cgi}\omega^2}{K_i} N_{di} \right) [\zeta_i(\xi_i) - \overline{\eta}_i(\xi_i) + \Phi(x_i) - \overline{y}(x_i)]$$

$$K_{9ri} = m_{bi}[\Phi(x) - \overline{y}(x)]$$

$$K_{10ri} = \left(\frac{m_{ci}a_i}{1 - \dfrac{\omega^2}{p_i^2}} \right) \overline{\eta}_i'(b_i)$$

$$K_{11ri} = \frac{1}{1 - \dfrac{\omega^2}{p_i^2}} (N_{ci}a_i + I_{cgi})\overline{\eta}'(b_i)$$

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