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PERIODIC, SMALL-AMPLITUDE SOLUTIONS TO THE SPATIALLY UNIFORM PLASMA CONTINUITY EQUATIONS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

The coupled set of first-order nonlinear differential equations

$$\dot{\chi} = C_0 + C_1\,\chi + C_2\,y + C_3\,\chi y + C_4\,\chi^2 + C_5\,y^2$$

$$\dot{y} = A_0 + A_1\,\chi + A_2\,y + A_3\,\chi y + A_4\,\chi^2 + A_5\,y^2$$

are solved by an approximate method which gives $y(t)$ in closed form for the particular case in which the variables χ and y vary periodically in time, the coefficients C_i and A_i are real, and the peak-to-peak variation of χ is of small amplitude relative to the mean value of χ. The peak-to-peak amplitude of y, however, is not necessarily small compared with the mean value of y. These equations are a generalized form of Volterra's problem of two conflicting populations, in which χ is the population density of the prey, and y is the population density of the predator, or vice versa. Similar equations may also be derived from the spatially uniform neutral and charged-particle continuity equations in the field of plasma physics. These equations then describe relatively low-frequency oscillations in the number density of electrons and neutral atoms in a partially ionized plasma.

The functional dependence of $y(t)$ is given in closed form by Jacobian elliptic functions when the assumption of small amplitude is made (i.e., $\Delta\chi/\chi_1 \ll 1$), and when the parameters, C_i, A_i, and the mean value of χ, χ_1, are such that periodic motion occurs. The amplitude, period, and waveform of the fluctuations in y are given in terms of the parameters, C_i, A_i, (which may be positive or negative), and χ_1. When periodic fluctuations are observed, it is possible to predict the amplitude, period, and waveform if C_i, A_i, and χ_1 are known (as is usually the case in plasma physics applications). It is also possible to infer relations among C_i and A_i if χ_1 and the period, amplitude, and waveform are known (as is usually the case in the predator-prey problem). When $C_5 = A_5 = 0$, the amplitude, period, and functional form of these Jacobian elliptic functions are derived for all cases in which periodic motion occurs. In the general case in which $C_5 \neq 0$ and/or $A_5 \neq 0$, it is shown that the closed-form solutions may also be expressed in terms of Jacobian elliptic functions, and the problem is left in a general form that may be solved for particular cases of interest. This procedure does not provide the waveform or amplitude of $\chi(t)$, the quantity whose peak-to-peak amplitude is assumed to be small.

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INTRODUCTION

The equations to be studied herein are

$$\dot{\chi} = C_0 + C_1 \chi + C_2 y + C_3 \chi y + C_4 \chi^2 + C_5 y^2 \tag{1}$$

$$\dot{y} = A_0 + A_1 \chi + A_2 y + A_3 \chi y + A_4 \chi^2 + A_5 y^2 \tag{2}$$

(All symbols are defined in appendix A.) These equations are a generalized form of Volterra's classical problem of two conflicting populations (refs. 1 to 9). A less general form of these equations, comprising fewer terms on the right side, was originally studied by Lotka, in an attempt to formulate a mathematical theory of the behavior of two conflicting species in a state of nature (refs. 1 and 2), and of certain oscillatory chemical reactions (ref. 3). This work was somewhat later refined by Volterra (refs. 4 and 5), whose work is summarized by Davis (ref. 6). When a mathematical model of the predator-prey problem is formulated, and the terms of the differential equations are multiplied by the proper coefficients, the solutions of the equations are periodic (ref. 4), which is consistent with the observed periodic fluctuations in the population of predators and prey (ref. 5).

The mathematical theory of coupled, first order nonlinear equations such as equations (1) and (2) has been discussed extensively by Lotka, Volterra, Davis, and by M. Frommer (ref. 7). The nonlinear nature of these equations has precluded the obtaining of general solutions and has limited investigations of this problem to obtaining particular solutions to simplified versions of equations (1) and (2), to studying the stability and existence of periodic solutions to the equations, or to obtaining a numerical solution to the equations for particular values of A_i and C_i.

Most of the literature on this problem is concerned with determining the conditions under which the population of one or both species will remain bounded or under which periodic fluctuations of the populations will occur (refs. 6, 8, and 9). Little emphasis has been placed on determining the peak-to-peak amplitude, period, and waveform of the solutions to equations (1) and (2) because of the extreme difficulty of the exact, nonlinear problem. Analytical solutions have been obtained for the linearized case in which the peak-to-peak amplitude of both x and y are small by comparison with their mean values (refs. 3 and 4). The waveforms are sinusoidal in this case.

In the present analysis, analytical solutions to equations (1) and (2) were obtained under the assumption that the fluctuations of only one of the two variables are small by comparison with its mean value. The class of solutions of interest in this analysis are those in which steady, periodic oscillation exists. Solutions which decay to a constant

value and those wherein either population increases without bound in violation of the small-amplitude approximation were not examined in any detail.

APPLICATION OF EQUATIONS

Physical Significance of Terms in Differential Equations for

Classical Problem of Predator and Prey

Consider the following pair of coupled, nonlinear first-order differential equations, in which χ is the population density of the prey and y is the population density of predators:

$$\dot\chi = C_0 + C_1\,\chi + C_2\,y + C_3\,\chi y + C_4\,\chi^2 + C_5\,y^2 \tag{1}$$

$$\dot y = A_0 + A_1\,\chi + A_2\,y + A_3\,\chi y + A_4\,\chi^2 + A_5\,y^2 \tag{2}$$

Equations (1) and (2) are a generalized form of Volterra's problem, in which the terms on the right side of each equation have the following physical significance: The terms A_0 and C_0 represent, respectively, the rate at which the predator or prey migrates across the boundary of the region of interest, when this rate is constant. The terms containing A_2 and C_1 represent the e-folding growth rate at which the predator or prey would increase in the absence of the other species, in the absence of a constant rate of migration, in the absence of competition among the members of each species, but in the presence of an unlimited food supply for both species. The terms containing A_3 and C_3 represent the rate at which the predator and prey interact, and this rate measures the rate at which the population of prey is depleted by the total number of predators. It is further assumed for mathematical convenience that the population of predators is augmented by the amount $A_3\chi y$ immediately upon the act of predation, which diminished the population of prey by an amount equal to $C_3\chi y$.

The terms containing C_4 and A_5 represent the effects of interaction among the members of each species. The terms containing C_2 and A_1 represent the beneficial or harmful effects on one species of the mere presence of the other, effects that are different from those of predation and might, for example, consist of a form of symbiosis. Finally, the terms containing A_4 and C_5 represent the effects on the population of one species of competition among the members of the other species. This might come about, for example, if the prey fed in part on the carcasses of the predator.

One of the major problems in setting up a mathematical model of the predator-prey interaction is that there is often reason to suppose that the coefficients C_i and A_i are themselves time dependent, varying with a seasonal periodicity (refs. 2 and 4). This variability greatly complicates the problem and makes the obtaining of closed-form solutions much more difficult. It is assumed in this analysis that the coefficients C_i and A_i are constant in time. It is also possible that more complex interactions between and among the two populations may result in a less simple interpretation of the individual coefficients, even when the total process is accurately represented by equations of the given form.

Physical Significance of Terms in Differential Equations

in Plasma Physics Applications

Restricted forms of equations (1) and (2) have appeared in the literature of plasma physics (ref. 10), in which one equation was the energy equation and the other a continuity equation. Reference 10, however, contains no closed-form solution to these equations. Apparently it has not been realized that if χ is identified with the neutral density and y with the ion density, the plasma continuity equations themselves can be written in the form of equations (1) and (2), and the periodic solutions identified with a previously unrecognized plasma instability (ref. 11).

Equations (1) and (2) can be derived in a plasma physics context by writing the continuity equations for each of the three components (ions, electrons, and neutrals) of a partially ionized gas and assuming that the spatial variations are small over the region of interest. One of the three equations may be eliminated by assuming that the Debye distance is small compared with the apparatus dimensions, so that the ion and electron densities are equal at each point. There will remain (in the absence of three-body processes) two equations similar in form to equations (1) and (2). The variable χ might represent the number density of neutral particles and y the number density of electrons, or vice-versa.

If the former is true, the terms in equations (1) and (2) containing C_0 and A_0 represent the rate at which neutrals and charged particles are injected or extracted from the volume of interest. The parameters C_1 and A_2 represent the e-folding rate at which the density of neutrals and ions would change in the absence of other processes as a result, for example, of effusion across the boundary of the containment volume. The term containing C_3 accounts for the decrease of neutral density due to ion-neutral or electron-neutral ionizing collisions. The term containing A_3 gives the rate of increase of the ion density due to ion-neutral or electron-neutral ionization processes; this term also accounts for the scattering by neutral atoms or molecules of ions and/or electrons

into the escape cone of magnetic mirror-type confinement geometries. The terms containing C_4 and A_4 account for the change in ion or neutral number density due to neutral-neutral ionization processes, one neutral of which may be in an excited state. The term containing C_5 gives the rate of increase of the neutral density due to ion-electron recombination processes. The term containing the parameter A_5 accounts for the decrease in ion density due to ion-electron recombination processes, and also for the scattering into the loss cone by electron-electron, electron-ion, or ion-ion Coulomb interactions. The terms containing C_2 and A_1 represent the rate at which the number density of one species changes because of the mere presence of the other. These latter two terms therefore do not appear to have physical significance in a plasma physics context, but they are included in the mathematical analysis for the sake of generality. The coefficients C_i and A_i will usually be constant in plasma physics applications if the mean particle energy is independent of time.

The mathematical analysis developed herein is sufficiently general so that the peak-to-peak amplitudes of $\chi(t)$ and $y(t)$ may or may not have the same magnitude. However, if $\chi(t)$ refers to the density of neutral atoms or molecules in a partially ionized plasma and $y(t)$ to the ion density, it is plausible that in many plasma physics applications the peak-to-peak amplitude of their fluctuations are about equal. This equality arises from the fact that ionizing collisions result in a one-to-one correspondence between neutrals lost and ions gained. The assumption that $\Delta\chi/\chi_1 \ll 1$, therefore, also implies that $\Delta y \ll \chi_1$. The peak-to-peak fluctuation in ion density must then be small by comparison with the neutral density. This state of affairs is assured if the percentage of ionization is low at all times.

The interpretation of χ as the ion density and y as the neutral density is also possible. In this case it would still be true that $\Delta\chi \approx \Delta y$, and the assumption that $\Delta\chi/\chi_1 \ll 1$ would imply that $\Delta y \ll \chi_1$ or that the change in neutral density is small by comparison with the total charged-particle density. The mathematical theory developed herein can therefore be applied to the case of a spatially uniform plasma, the percentage of ionization of which remains either low or high at all times.

A detailed and extensive investigation of particular plasma physics applications of equations (1) and (2) is beyond the scope of this analysis. It is intended herein only to lay the mathematical groundwork required to apply equations (1) and (2) to plasma physics.

ANALYSIS

Objectives and Assumptions

For an oscillatory system, a closed-form solution can show the relative importance of the various terms, as well as predict the effect of changes in the system on the char-

acter of the oscillation. It is an easy matter to put equations (1) and (2) on a computer and obtain numerical solutions for a particular set of C_i and A_i. However, it is most desirable to have a closed-form solution for $\chi(t)$ and $y(t)$ in terms of the coefficients C_i and A_i of the differential equations, in order to compare experiment with the predictions of the mathematical model. With such a solution the trend of the period and amplitude of the oscillation can be predicted as a function of experimentally determined quantities, which are contained in the constant coefficients C_i and A_i.

The nonlinear nature of equations (1) and (2) defeated an attempt to find an unrestricted closed-form solution to this set of equations. The most obvious approach to obtaining a closed-form solution is to linearize these equations through a "double-perturbation" analysis, in which both χ and y are assumed to have constant mean values, χ_1 and y_{00}. The time-varying portions of χ and y ($\chi_2(t)$ and $y_{01}(t)$) are assumed to be much smaller than their respective mean values, so that $\chi_1 \gg \chi_2(t)$ and $y_{00} \gg y_{01}(t)$. This method was used by Lotka (ref. 3) and by Volterra (ref. 5) in their analyses of a less general form of equations (1) and (2), and yields exponential and sinusoidal solutions.

This double-perturbation approach, which is illustrated in appendix C, has at least two disadvantages. The first is that, in many interesting physical situations, the peak-to-peak amplitude of either χ or y may not be small compared with the respective mean value. The second disadvantage of the double-perturbation approach is that the amplitudes $\chi_2(t)$ and $y_{01}(t)$ must be regarded as given small quantities, which cannot be obtained as functions of the coefficients C_i and A_i. While the double-perturbation approach is perhaps more familiar and mathematically simple, the two disadvantages cited are deemed sufficient reason to study the more general problem discussed in the next paragraph.

The small-amplitude approach used in this analysis starts with the assumption that the peak-to-peak amplitude of only one of the two variables is small by comparison with its mean value. Removal of this constraint from the second variable is a desirable generalization; it permits the amplitude of the second variable to be calculated as a function of the coefficients C_i and A_i. This greater generality is achieved at some cost in mathematical complexity, since the resulting perturbation equations must be solved in terms of Jacobian elliptic functions rather than the real or imaginary exponential functions that suffice for the double-perturbation approach. In addition, the amplitude and waveform of the first variable (whose peak-to-peak amplitude is assumed to be small) cannot, in general, be obtained in closed form. The period of both variables is, of course, the same. To recapitulate, the double-perturbation approach provides the waveform of both variables but the amplitude of neither; while the small-amplitude approach considered herein provides both the amplitude and the waveform of the variable with arbitrary peak-to-peak amplitude, but neither the amplitude nor the waveform of the variable whose peak-to-peak amplitude is assumed to be small.

An approximate closed-form solution is obtained herein by assuming that the peak-to-peak amplitude of the fluctuations in χ is small compared with the mean value of χ

$$\delta \equiv \frac{\Delta\chi}{\chi_1} \ll 1 \tag{3}$$

as is illustrated in figure 1.

(a) y_0 maximum.

(b) y_0 minimum.

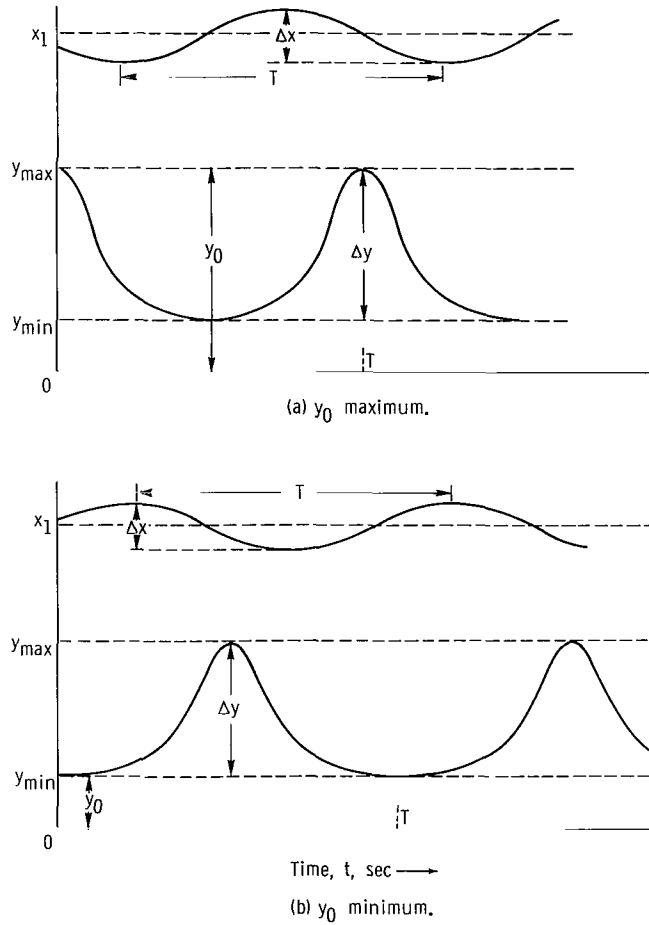Figure 1. - Schematic illustrations of periodic solutions.

The period of the waveform is defined as the time elapsed between two successive maxima of the functions χ or y. In the following analysis, it is assumed in all cases that the function y is periodic in time and that the boundary conditions at time $t = 0$ are given by equation (4).

$$y(t = 0) = y_0 \qquad \dot{y}(t = 0) = 0 \tag{4}$$

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It is further assumed that y_0 is positive definite and may be either the maximum value (fig. 1(a)) or the minimum value (fig. 1(b)) of $y(t)$.

No assumption was made about the relative magnitude of Δy and y, so that the assumption $\Delta\chi/\chi_1 \ll 1$ could also cover the case in which the relative amplitude Δy is either comparable to or small by comparison with the maximum value of y. In addition, the mathematical development is sufficiently general so that χ_1 may be either larger or smaller than the mean value of y.

It appears that these assumptions, while restricting the range of application of the solutions, still cover an interesting and significant range of phenomena and should permit a contact to be made between theory and experiment in physical situations in which equations (1) and (2) apply. There are, however, certain limitations in the applicability of equations (1) and (2). One important limitation lies in the assumption that the coefficients C_i and A_i are not functions of time. In the predator-prey problem, not only do many processes have a seasonal dependence, but there is also a time lag between the appearance of a new member of the population and its participation in reproduction. Another limitation lies in the assumption that all the relevant processes can be included in a limited number of independent terms of no higher than quadratic degree. Because of this limitation, three-body recombination processes must be excluded from consideration in plasma physics applications.

Development of Small-Amplitude Approximation

The development was kept as general as possible, and the small-amplitude approximation introduced at the last possible stage in the analysis. This procedure is both necessary and desirable, since information regarding the behavior of the variables is lost by introducing an approximation and this information does not appear in subsequent steps of the analysis. Premature introduction of the small-amplitude approximation therefore may result in obtaining a trivial or specialized solution.

The basic equations of concern herein have already been given.

$$\dot{\chi} = C_0 + C_1\,\chi + C_2\,y + C_3\,\chi y + C_4\,\chi^2 + C_5\,y^2 \tag{1}$$

$$\dot{y} = A_0 + A_1\,\chi + A_2\,y + A_3\,\chi y + A_4\,\chi^2 + A_5\,y^2 \tag{2}$$

Immediate introduction of the small-amplitude approximation, by setting $\chi(t) = \chi_1 + \Delta\chi(t)$ and then obtaining a solution for $y(t)$ by ignoring terms containing $\Delta\chi(t)$, would be premature since it would merely decouple equation (2) from the variation of $\chi(t)$ and

lead to a family of trivial solutions for $y(t)$. Of interest is the more general case, in which neither $\dot{\chi}$ nor $\dot{y}$ is zero, in which the waveform of $y(t)$ depends on $\chi(t)$ and on the coefficients C_i (which in part determine the magnitude and time behavior of $\Delta\chi(t)$), as well as on the coefficients A_i in equation (2). If equation (2) is differentiated with respect to time,

$$\ddot{y} = A_1 \dot{\chi} + A_2 \dot{y} + A_3 \dot{\chi}y + A_3 \chi\dot{y} + 2A_4 \chi\dot{\chi} + 2A_5 y\dot{y} \tag{5}$$

The first-order time derivatives in equation (5) may be replaced by substituting equations (1) and (2) for them, to obtain

$$
\begin{aligned}
\ddot{y} = &\Big[\big(A_1 C_0 + A_2 A_0\big) + \big(A_1 C_1 + A_1 A_2 + A_0 A_3 + 2A_4 C_0\big)\chi \\
&\quad + \big(A_1 C_4 + A_2 A_4 + A_1 A_3 + 2A_4 C_1\big)\chi^2 + \big(A_3 A_4 + 2A_4 C_4\big)\chi^3\Big] \\
&+ \Big[\big(A_1 C_2 + A_2^2 + A_3 C_0 + 2A_5 A_0\big) + \big(A_1 C_3 + 2A_2 A_3 + A_3 C_1 \\
&\quad + 2A_4 C_2 + 2A_5 A_1\big)\chi + \big(A_3 C_4 + A_3^2 + 2A_4 C_3 + 2A_4 A_5\big)\chi^2\Big]y \\
&+ \Big[\big(A_1 C_5 + 3A_2 A_5 + A_3 C_2\big) + \big(A_3 C_3 + 3A_3 A_5 + 2A_4 C_5\big)\chi\Big]y^2 \\
&+ \Big[A_3 C_5 + 2A_5^2\Big]y^3
\end{aligned} \tag{6}
$$

in which the coefficients of the powers of $y(t)$ are grouped in brackets and the coefficients of various powers of $\chi(t)$ in parentheses.

Equation (6) is still completely general, but it is even more difficult of exact solution than the original equations (1) and (2). The only two time-varying quantities appearing in equation (6) are $y(t)$ and $\chi(t)$. When the variable $\chi(t)$ is written as $\chi(t) = \chi_1 + \Delta\chi(t)$, the terms within each bracket can be separated into two parts, one containing constants and the other depending on $\Delta\chi(t)$. When the sum of the constant terms in each bracket is much larger than the sum of the time-varying terms containing $\Delta\chi(t)$ (bearing in mind that $\Delta\chi(t) \ll \chi_1$), these small terms may be considered negligible and omitted. This requirement is not automatically satisfied by the small-amplitude assumption for all possible cases but usually follows for particular numerical values of A_i, C_i, and χ_1. It is also possible that some of the expressions in parentheses or even those in brackets are negligibly small by comparison with other expressions of the same type. If so, their inclusion or exclusion is immaterial.

The variation of $y(t)$ is expected to be dominated by the large, constant terms in equation (6) and only slightly perturbed by the small time-varying part containing $\Delta\chi(t)$. It should therefore be possible to obtain an approximation to $y(t)$ by neglecting the small time-varying terms containing $\Delta\chi(t)$. Equation (6) thus may be approximated by an equation in the single variable $y(t)$ by introducing in it the small-amplitude approximation given by equation (3).

If it is assumed, therefore, that the peak-to-peak variation of $\chi(t)$ is sufficiently small that $\chi(t)$ in equation (6) may be approximated by its mean value χ_1,

$$\chi = \chi_1 + \Delta\chi \approx \chi_1 \tag{7}$$

Equation (6) may then be written in terms of the constants A_i, C_i, and χ_1 as

$$
\begin{aligned}
\ddot{y} \approx & \left[\left(A_1 C_0 + A_2 A_0 \right) + \left(A_1 C_1 + A_1 A_2 + A_0 A_3 + 2A_4 C_0 \right)\chi_1 \right. \\
& \left. + \left(A_1 C_4 + A_2 A_4 + A_1 A_3 + 2A_4 C_1 \right)\chi_1^2 + \left(A_3 A_4 + 2A_4 C_4 \right)\chi_1^3 \right] \\
& + \left[\left(A_1 C_2 + A_2^2 + A_3 C_0 + 2A_5 A_0 \right) + \left(A_1 C_3 + 2A_2 A_3 + A_3 C_1 \right. \right. \\
& \left. + 2A_4 C_2 + 2A_5 A_1 \right)\chi_1 + \left(A_3 C_4 + A_3^2 + 2A_4 C_3 + 2A_4 A_5 \right)\chi_1^2 \Big] y \\
& + \left[\left(A_1 C_5 + 3A_2 A_5 + A_3 C_2 \right) + \left(A_3 C_3 + 3A_3 A_5 + 2A_4 C_5 \right)\chi_1 \right] y^2 \\
& + \left[A_3 C_5 + 2A_5^2 \right] y^3 \tag{8}
\end{aligned}
$$

Since the terms in brackets are constant for a given problem, equation (8) can be written in the simplified form

$$\ddot{y} = l_0 + l_1 y + l_2 y^2 + l_3 y^3 \tag{9}$$

where the constants l_i are defined by

$$
\begin{aligned}
l_0 \equiv & \left[\left(A_1 C_0 + A_2 A_0 \right) + \chi_1 \left(A_1 C_1 + A_1 A_2 + A_0 A_3 + 2A_4 C_0 \right) \right. \\
& \left. + \chi_1^2 \left(A_1 C_4 + A_2 A_4 + A_1 A_3 + 2A_4 C_1 \right) + \chi_1^3 \left(A_3 A_4 + 2A_4 C_4 \right) \right] \tag{10}
\end{aligned}
$$

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$$l_1 \equiv \left[\left(A_1C_2 + A_2^2 + A_3C_0 + 2A_5A_0\right) + \chi_1\left(A_1C_3 + 2A_2A_3\right.\right.$$

$$\left.\left. + A_3C_1 + 2A_4C_2 + 2A_5A_1\right) + \chi_1^2\left(A_3C_4 + A_3^2 + 2A_4C_3 + 2A_4A_5\right)\right] \tag{11}$$

$$l_2 \equiv \left[\left(A_1C_5 + 3A_2A_5 + A_3C_2\right) + \chi_1\left(A_3C_3 + 3A_3A_5 + 2A_4C_5\right)\right] \tag{12}$$

$$l_3 \equiv \left[\left(A_3C_5 + 2A_5^2\right)\right] \tag{13}$$

The small-amplitude approximation has therefore reduced the general problem of finding the closed-form solutions of $\chi(t)$ and $y(t)$ to the assumption that $\chi(t)$ is approximately equal to χ_1; $y(t)$ is given by the solution to equation (9). The merit of the procedure used in the preceding calculations is that equation (9) happens to be one of the few non-linear differential equations whose solutions are available in closed form. Equation (9) is recognizable as the differential equation whose periodic solutions are given in terms of Jacobian elliptic functions (ref. 6, p. 129).

The analytical procedure following equation (5) may be regarded in an equivalent but conceptually different light. In equation (5) $\chi(t)$ can be approximated by its mean value χ_1. Equations (1) and (2), with $\chi(t)$ approximated by χ_1, can then be regarded as approximations to $\dot{\chi}$ and $\dot{y}$, which may also be substituted in equation (5). These approximations to $\chi(t)$ ($\dot{\chi}$ and $\dot{y}$ in eq. (5)) result in an approximate equation for $\ddot{y}$ that is algebraically identical to equation (9).

Equation (9) may be integrated by multiplying both sides by $\dot{y}(t)$ and eliminating dt to obtain (ref. 12)

$$d\left(\dot{y}^2\right) = 2\left(l_0 + l_1 y + l_2 y^2 + l_3 y^3\right)dy \tag{14}$$

When this expression is integrated and the assumed boundary conditions given in equation (4) ($y = y_0$ and $\dot{y} = 0$ at $t = 0$) are applied, equation (14) becomes

$$\dot{y}^2 = 2l_0\left(y - y_0\right) + l_1\left(y^2 - y_0^2\right) + \frac{2}{3}l_2\left(y^3 - y_0^3\right) + \frac{1}{2}l_3\left(y^4 - y_0^4\right) \tag{15}$$

This equation is of a relatively well-known type (ref. 6, p. 129), whose solutions are either exponential or sinusoidal when $l_2 = l_3 = 0$. The solutions are given by Jacobian elliptic functions when $l_2 \neq 0$ and/or $l_3 \neq 0$, provided that the constant coefficients l_i are real and are nondegenerate such that periodic motion occurs. Even the less interesting non-periodic solutions have implications for the long-term behavior of the system when the l_i are such that it is not oscillating periodically. Therefore, all solutions to

equation (15) for real l_i were investigated.

In the following analysis, it is convenient to shift the origin of the variable y so that

$$Z = y - y_0 \tag{16}$$

$$\dot{Z} = \dot{y} \tag{17}$$

and

$$\left.\begin{array}{l} Z(t = 0) = 0 \\[2ex] \dot{Z}(t = 0) = 0 \end{array}\right\} \tag{18}$$

It should be noted that Z can be positive or negative, depending on whether y_0 is the minimum or maximum value of y. The analysis is concerned with finding the total peak-to-peak amplitude of the fluctuation of y,

$$Z_{max} \equiv y_{max} - y_{min} \tag{19}$$

the functional form of $Z(t)$, and the period T of the oscillation, where

$$Z(t = 0) = Z(t = T) \tag{20}$$

It is assumed that $y(t)$ is real and positive but may be zero at one point on its waveform.

Solution of the Case $l_3 \neq 0$

The most general solution to equation (15) occurs when $l_3 \neq 0$. Referring to equations (13), (1), and (2) reveals that this term is nonzero only if the y^2 term is present in either or both equations (1) and (2).

If $\chi(t)$ is regarded as the population density of the prey and $y(t)$ as the population density of the predator, then l_3 is nonzero only when competition among the predators significantly affects their own population or directly affects that of the prey. In plasma physics applications, if $\chi(t)$ is the density of neutral particles and $y(t)$ is the charge density, l_3 is nonzero only if direct ion-electron recombination is a significant process or if significant numbers of collisions of two charged particles result in the loss of one of them, as by knocking one into an escape cone in a magnetic mirror confinement geometry.

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Removing a factor of $y - y_0$ from equation (15) leaves

$$\dot{y}^2 = \left(y - y_0\right)\left[2l_0 + l_1\left(y + y_0\right) + \frac{2}{3}l_2\left(y^2 + yy_0 + y_0^2\right) + \frac{1}{2}l_3\left(y^3 + y^2 y_0 + yy_0^2 + y_0^3\right)\right] \quad (21)$$

Using equations (16) and (17) to convert equation (21) into an equation in the variable Z gives

$$\dot{Z}^2 = Z\left[2\left(l_0 + l_1 y_0 + l_2 y_0^2 + l_3 y_0^3 + l_3 y_0^3\right) + \left(l_1 + 2l_2 y_0 + 3l_3 y_0^2\right)Z\right.$$
$$\left. + \left(\frac{2}{3}l_2 + 2l_3 y_0\right)Z^2 + \frac{1}{2}l_3 Z^3\right] \quad (22)$$

Defining the auxiliary constants

$$j \equiv \frac{4\left(\frac{1}{3}l_2 + l_3 y_0\right)}{l_3} \quad (23)$$

$$f \equiv \frac{2\left(l_1 + 2l_2 y_0 + 3l_3 y_0^2\right)}{l_3} \quad (24)$$

and

$$h \equiv \frac{4\left(l_0 + l_1 y_0 + l_2 y_0^2 + l_3 y_0^3\right)}{l_3} \quad (25)$$

makes it possible to rewrite equation (22) in the much simpler form

$$\dot{Z}^2 = \frac{1}{2}l_3 Z\left(Z^3 + jZ^2 + fZ + h\right) \quad (26)$$

When the boundary conditions of equation (18) are used, this equation may be rewritten in integral form as

$$t\left(\frac{l_3}{2}\right)^{1/2} = \int_0^Z \frac{dZ}{\left[Z\left(Z^3 + jZ^2 + fZ + h\right)\right]^{1/2}} \quad (27)$$

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This integral equation may be written in terms of the four roots of the denominator as

$$t\left(\frac{l_3}{2}\right)^{1/2} = \int_0^Z \frac{dZ}{\left[(Z - Z_1)(Z - Z_2)(Z - Z_3)(Z - Z_4)\right]^{1/2}} \tag{28}$$

where $Z_1 = 0$ in all cases, and the roots, Z_2, Z_3, and Z_4, are functions of the constant parameters, j, f, and h, given by the three roots of the cubic equation

$$Z^3 + jZ^2 + fZ + h = 0 \tag{29}$$

The integral of equation (28) is immediately recognizable as being of a general type that gives rise to Jacobian elliptic functions (see, e.g., section 260 of ref. 13). The particular elliptic function involved and the expressions for the maximum amplitude of the fluctuations in y, the period of oscillation, and the elliptic modulus k all depend on the sign of l_3 and on the relation of the range of integration to the values of the four roots, Z_1, Z_2, Z_3, and Z_4. An exhibition of all possible solutions to equation (28) would be extremely tedious and is not attempted. In a specific instance in which $l_3 \neq 0$, however, it should not be difficult to determine the roots of equation (29) once the sign and magnitude of j, f, and h are known. It is then a relatively easy matter to integrate equation (28) (for which ref. 13 is most helpful) and to determine whether or not finite, periodic solutions exist and their specific form.

Solutions of the Case $l_3 = 0$

The next most complex case is that in which $l_2 \neq 0$ but $l_3 = 0$. The values of l_0 and l_1 are initially assumed to be arbitrary. In this case, equation (15) may be written

$$\dot{y}^2 = 2l_0(y - y_0) + l_1(y^2 - y_0^2) + \frac{2}{3}l_2(y^3 - y_0^3) \tag{30}$$

Factoring out $y - y_0$ yields

$$\dot{y}^2 = (y - y_0)\left[2l_0 + l_1(y + y_0) + \frac{2}{3}l_2(y^2 + yy_0 + y_0^2)\right] \tag{31}$$

Substituting equations (16) and (17) into equation (31) gives

$$\dot{z}^2 = z\left[\frac{2}{3}l_2 z^2 + (l_1 + 2y_0 l_2)z + 2(l_0 + y_0 l_1 + y_0^2 l_2)\right] \tag{32}$$

Defining two additional constants

$$b \equiv \frac{3\left(l_1 + 2y_0 l_2\right)}{2l_2} \tag{33}$$

$$c \equiv \frac{3\left(l_0 + y_0 l_1 + y_0^2 l_2\right)}{l_2} \tag{34}$$

enables equation (32) to be written in the simplified form

$$\dot{Z}^2 = \frac{2}{3} l_2 Z\left(Z^2 + bZ + c\right) \tag{35}$$

Equation (35) may also be written in terms of the roots of the right side as

$$\dot{Z}^2 = \frac{2}{3} l_2 \left(Z - Z_1\right)\left(Z - Z_2\right)\left(Z - Z_3\right) \tag{36}$$

where $Z_3 \geq Z_2$ and

$$Z_1 = 0 \tag{37}$$

$$Z_2 = -\frac{1}{2}\left[b + \left(b^2 - 4c\right)^{1/2}\right] \tag{38}$$

and

$$Z_3 = -\frac{1}{2}\left[b - \left(b^2 - 4c\right)^{1/2}\right] \tag{39}$$

In order that the amplitude of Z have real values, the signs of l_2, Z, and the roots in equation (36) must always combine to give a positive definite value to the right side of equation (35). The solution to the differential equation of equation (36) under the boundary conditions of equation (18) is given by the integral

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$$t\left(\frac{2l_2}{3}\right)^{1/2} = \int_0^Z \frac{dZ}{\left[(Z - Z_1)(Z - Z_2)(Z - Z_3)\right]^{1/2}} = gu_1 \qquad (40)$$

The integral, or the right side of equation (40), is immediately recognizable as an elliptic integral of a type tabulated by Byrd and Friedman (ref. 13). Unfortunately, the functional form of the solution depends critically on the relation of the range of integration to the roots, Z_1, Z_2, and Z_3. A variety of solutions to equation (40) are presented in appendix B, and their ranges of validity and the nature of their solutions are indicated on table I. For the sake of concreteness, a bounded, periodic solution to equation (40) (case N of appendix B) is presented in the following paragraph.

When both of the roots given by equations (38) and (39) are positive and the range of Z is confined to $Z_2 \leq Z \leq 0$, the solution to the basic integral of equation (40) is given by formula 233.00 of reference 13. The argument of the elliptic function is given by

$$u_1(k, t) = \frac{t}{g}\left(\frac{2l_2}{3}\right)^{1/2} \qquad (41)$$

where

$$g = \frac{2^{3/2}}{\left[-b + \left(b^2 - 4c\right)^{1/2}\right]^{1/2}} = \frac{2}{Z_3^{1/2}} \qquad (42)$$

The amplitude of Z has the functional dependence

$$Z = Z_2 \, SN^2 u_1(k) = -\frac{1}{2}\left[b + \left(b^2 - 4c\right)^{1/2}\right]SN^2 u_1(k) = y - y_0 \qquad (43)$$

The total peak-to-peak amplitude of Z is then

$$Z_{max} = Z_2 = -\frac{1}{2}\left[b + \left(b^2 - 4c\right)^{1/2}\right] \qquad (44)$$

and the modulus of the elliptic function is

TABLE I. - SUMMARY OF CASES PRESENTED IN APPENDIX B FOR WHICH $l_3 = 0$

Case	Range of variables							Nature of solution $y(t)$
	l_0	l_1	l_2	b	c	$b^2 - 4c$	$Z \equiv y - y_0$	
A	Arbitrary[a]	Arbitrary	>0	Arbitrary	>0	<0	≥ 0	Periodic, positively infinite maxima
B		<0		Arbitrary	>0	<0	≤ 0	Periodic, negatively infinite minima
C		Arbitrary		0	0	0	>0	Monotone decreasing, unbounded
D		>0		>0	$b^2/4$		≥ 0	Periodic, positively infinite maxima
E		>0		<0			≥ 0	Monotone increasing, bounded
F		<0		>0			≤ 0	Monotone decreasing, bounded
G		<0		<0			≤ 0	Periodic, negatively infinite minima
H		>0		>0	0	>0	$\geq -b$	Monotone, singular
I		<0		>0			Arbitrary	Monotone, singular
J		>0		<0			Arbitrary	Monotone, singular
K		<0		<0			$\leq -b$	Monotone, singular
L		>0		Arbitrary	Arbitrary		$Z \geq 0 > Z_3 > Z_2$	Periodic, positively infinite maxima
M		<0					$0 \geq Z > Z_3 > Z_2$	Periodic, bounded
N		>0					$Z_3 > Z_2 \geq Z \geq 0$	Periodic, bounded
O		<0					$Z_3 > Z_2 > 0 \geq Z$	Periodic, negatively infinite minima
P		<0					$Z_3 \geq Z \geq 0 > Z_2$	Periodic, bounded
Q		>0					$Z_3 > 0 \geq Z \geq Z_2$	Periodic, bounded
R			0	----------	----------	--------	Arbitrary	Periodic solutions possible
S		0	0	----------	----------	--------	Arbitrary	Periodic solutions not possible

[a]The parameter so designated must, however, be restricted within a range appropriate to the limitations imposed on the other parameters for that case.

$$k^2 = \frac{z_2}{z_3} \approx \frac{b + \left(b^2 - 4c\right)^{1/2}}{b - \left(b^2 - 4c\right)^{1/2}} \qquad (45)$$

The solution given by equation (43) has a peak-to-peak period of $u_1(T) = 2K(k)$. The period of oscillation is then given by

$$T = \frac{4K(k)}{\left[-b + \left(b^2 - 4c\right)^{1/2}\right]^{1/2}} \left(\frac{3}{\ell_2}\right)^{1/2} \qquad (46)$$

Equation (43) does represent a solution of physical interest, since it is both periodic and finite.

APPLICATION AND LIMITATIONS OF THE METHOD

A requirement for application of the method presented in the preceding section is that $\Delta\chi(t) \ll \chi_1$. It is not easy to determine whether this inequality holds when a particular example of equations (1) and (2) is under consideration. If the equations correctly represent an actual physical situation and it is known that one of the two variables is oscillating by this mechanism with a small amplitude with respect to its mean value, this information may be used to justify application of the analytical procedure developed in this paper. This "physical" justification of the use of the small-amplitude approximation is appropriate, for example, to the equations describing the oscillations in a lightly ionized gas (ref. 11).

Circumstances can arise, however, in which equations (1) and (2) will be given with a particular set of numerical coefficients and the nature of their solutions will not be known a priori. When this is the case, either the equations can be solved numerically on a computer to verify that $\Delta\chi(t) \ll \chi_1$, or the "phase-plane" method of determining the limits of χ and y can be applied. The former procedure is exact but renders a closed-form solution to the equations unnecessary for the particular numerical values of the coefficients A_i and C_i so calculated. The phase-plane method presumes a knowledge of the literature on the class of equations represented by equations (1) and (2). This method is discussed in general terms in references 2 and 6. Application of this method to specialized cases of equations (1) and (2) (when certain of the A_i and/or C_i are zero) is discussed in the journal literature, of which references 1, 3, 5, 7, 8, and 11 are examples.

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The phase-plane method has been developed in the literature to determine the nature of the solutions to coupled sets of equations such as (1) and (2). This method predicts whether the solutions are periodic, steady-state, growing, or damped. This method determines the trajectory that the variables χ and y trace out on the χ - y (phase) plane. The equation for this phase-plane trajectory is derived by dividing equation (2) by equation (1), which eliminates time as an explicit variable and yields

$$\frac{dy}{d\chi} = \frac{A_0 + A_1 \chi + A_2 y + A_3 \chi y + A_4 \chi^2 + A_5 y^2}{C_0 + C_1 \chi + C_2 y + C_3 \chi y + C_4 \chi^2 + C_5 y^2} \tag{47}$$

The requirement that the $\chi(t)$ and $y(t)$ be periodic requires the existence of at least one closed trajectory on the phase plane. If more than one exist, the possibility of stable oscillations of differing amplitudes is implied. The peak-to-peak amplitude of these curves in the χ-direction is determined by the initial conditions and the zeros of the denominator; the peak-to-peak amplitude of y is determined by the initial conditions and the zeros of the numerator.

The requirement that $\Delta\chi(t) \ll \chi_1$ and that the solutions be periodic implies that the phase trajectories be narrow closed curves and that their width be small compared with the mean value of χ, χ_1, at the center of the curve. This situation on the phase plane is illustrated schematically in figure 2. The closed-form solutions given in references 3 and 5 are appropriate only for small closed curves about χ_1 and y_{01}, where the peak-to-peak amplitudes of χ and y are both small by comparison with their mean values. A further discussion of the phase-plane technique appears in references 2 and 6.

The peak-to-peak amplitudes of $\chi(t)$ and $y(t)$ are determined jointly by the initial conditions and the values of the coefficients A_i and C_i. For a given set of initial conditions on χ and y, the requirement that $\Delta\chi(t) \ll \chi_1$ places restrictions on the values
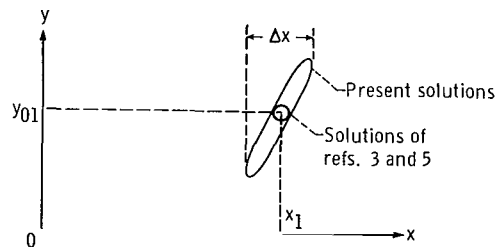

Figure 2. - Schematic illustration of phase-plane trajectory.

of A_i and C_i for which the present procedure will yield valid closed-form solutions. The restrictions on A_i and C_i implied by $\Delta\chi(t) \ll \chi_1$ exclude many cases of interest. The advantage of the procedure developed herein is that the requirement of references 3 and 5, that both $\Delta\chi(t) \ll \chi_1$ $\underline{\text{and}}$ $\Delta y(t) \ll y_{01}$, is clearly more restrictive of the coefficients A_i and C_i than the present case, where only one amplitude is assumed to be small.

If the coefficients of equations (1) and (2) are such that $\Delta\chi(t) \ll \chi_1$, the method of solution given herein may nonetheless fail if the time-dependent terms in the brackets of equation (6) are not small by comparison with the constant terms within these brackets. The latter terms are given by the l_i in equations (10) to (13). Applying the procedure developed herein requires checking not only to see whether the coefficients are such that $\Delta\chi(t) \ll \chi_1$, but also to see whether the following inequalities are satisified:

$$l_0 \gg \Delta\chi(t)\left(A_1C_1 + A_1A_2 + A_0A_3 + 2A_4C_0\right) + 2\chi_1 \Delta\chi(t)\left(A_1C_4 + A_2A_4\right)$$

$$+ A_1A_3 + 2A_4C_1\big) + 3\chi_1^2 \Delta\chi(t)\left(A_3A_4 + 2A_4C_4\right) \tag{48}$$

$$l_1 \gg \Delta\chi(t)\left(A_1C_3 + 2A_2A_3 + A_3C_1 + 2A_4C_2 + 2A_5A_1\right)$$

$$+ 2\chi_1 \Delta\chi(t)\left(A_3C_4 + A_3^2 + 2A_4C_3 + 2A_4A_5\right) \tag{49}$$

and

$$l_2 \gg \Delta\chi(t)\left(A_3C_3 + 3A_3A_5 + 2A_4C_5\right) \tag{50}$$

where l_0, l_1, and l_2 are given by equations (10), (11), and (12), respectively. The better these inequalities are satisfied, the more nearly will equation (6) approximate the Jacobian elliptic equation with constant coefficients. The solutions of equation (9) will then become better and better approximations to the small-amplitude solutions of equations (1) and (2).

This procedure has a further difficulty in common with all procedures involving a small-amplitude approximation. There is no assurance that oscillatory (or monotone) solutions to the small-amplitude approximation given herein will, for the same A_i, C_i, and χ_1, correspond to oscillatory (or monotone) solutions to the exact equations (1) and (2). This state of affairs comes about because the time-dependent terms in the l_i, however small they may be, may possibly introduce damping into solutions whose small-amplitude approximation is periodic, or make periodic a solution whose small-amplitude approximation is monotone.

CONCLUSIONS

A method has been developed which provides approximate solutions for the coupled set of first-order, nonlinear equations

$$\dot{\chi} = C_0 + C_1 \chi + C_2 y + C_3 \chi y + C_4 \chi^2 + C_5 y^2 \tag{1}$$

$$\dot{y} = A_0 + A_1 \chi + A_2 y + A_3 \chi y + A_4 \chi^2 + A_5 y^2 \tag{2}$$

where χ and y are variables and A_i and C_i are constants. The method provides solutions for $y(t)$ and requires that the fluctuations in the other variable $\chi(t)$ be small compared with its mean value. Additional requirements for the applicability of the method are set forth. This method is of particular value in obtaining oscillatory solutions for $y(t)$, for which it yields amplitude as well as period and waveform. In general, the solutions for $y(t)$ are given in terms of Jacobian elliptic functions. These functions may be approximately sinusoidal under certain conditions. The waveform and amplitude of $\chi(t)$ are not given by this procedure.

Classes of solutions are described for various combinations of the coefficients C_i and A_i. The most general case, with C_5 and A_5 not equal to zero, is left in the form of an integral. With these two coefficients set equal to zero, nineteen different combinations of coefficients were treated. Among these combinations, five permitted finite-amplitude oscillatory solutions.

The analysis presented herein extends the solutions available in the literature (refs. 3 and 5) to the case in which the peak-to-peak amplitude of $y(t)$ may be comparable to its mean value. In appendix C it is shown that the results obtained in this analysis reduce to those given in references 3 and 5 in the limit of small peak-to-peak fluctuations in $y(t)$. The present analysis therefore contains the previously obtained closed-form solution to equations (1) and (2) as a special case.

Lewis Research Center,
    National Aeronautics and Space Administration,
        Cleveland, Ohio, November 14, 1967,
           129-02-03-05-22.

APPENDIX A

SYMBOLS

A_i constant parameters appearing in eq. (2)

b   constant defined by eq. (33)

C_i constant parameters appearing in eq. (1)

CN   Jacobian elliptic cosine function (see ref. 13)

c   constant defined by eq. (34)

f   constant defined by eq. (24)

g   constant parameter

h   constant defined by eq. (25)

i $\sqrt{-1}$

j   constant defined by eq. (23)

K(k)   complete elliptic integral of first kind

k   modulus of Jacobian elliptic functions

k'   complementary elliptic modulus, equal to $1 - k^2$

l_0 constant defined by eq. (10)

l_1 constant defined by eq. (11)

l_2 constant defined by eq. (12)

l_3 constant defined by eq. (13)

SN   Jacobian elliptic sine function (see ref. 13)

T   peak-to-peak period of oscillation, sec

TN   Jacobian elliptic tangent (see ref. 13)

t   time, sec

u_1 argument of elliptic function, incomplete elliptic integral of first kind

y   variable appearing in eq. (2)

y_{max} maximum value of y

y_0 minimum value of y

y_{00} mean value of y

y_{01} amplitude of fluctuation in y

Z   relative amplitude of y, defined by eq. (16)

Z_{max} peak-to-peak amplitude of y, defined by eq. (19)

δ small perturbation amplitude, defined by eq. (3)

ν frequency, Hz

χ variable appearing in eq. (1)

$\Delta\chi$ peak-to-peak amplitude of χ

χ_1 mean value of χ

χ_2 amplitude of fluctuation in χ

ω frequency, rad/sec

APPENDIX B

SOLUTIONS FOR $y(t)$ WHEN $l_3 = 0$

The algebraic manipulations required to obtain closed-form solutions to equation (9) can be somewhat tedious and may involve mathematical functions unfamiliar to the reader. It was therefore deemed desirable to exhibit the solutions for the case $l_3 = 0$ and, thus make it easier for the reader to obtain $y(t)$ from a knowledge of the coefficients A_i and C_i, or vice-versa. The solutions for the cases listed in table I (p. 17) are therefore presented here.

Case $l_2 \neq 0$

Case A, $l_2 > 0$, $(b^2 - 4c) < 0$, $Z \geq 0$:

When l_2 is greater than zero and the discriminant $b^2 - 4c$ is less than zero, the roots Z_3 and Z_2 are complex conjugates, and the lower limit in the range of integration is equal to the only real root of the denominator, $Z_1 = 0$. Under these circumstances, the integral in equation (40) is given by formula 239:00 of reference 13, which is

$$u_1(k, t) = \frac{t}{g}\left(\frac{2l_2}{3}\right)^{1/2}$$
(41)
(B1)

where

$$g = c^{-1/4}$$
(B2)

the amplitude Z has the functional form

$$Z = c^{1/2}\left(\frac{1 - CNu_1}{1 + CNu_1}\right) = y - y_0$$
(B3)

And the elliptic modulus k^2 has the value

$$k^2 = \frac{1}{2}\left(1 - \frac{b}{2c^{1/2}}\right)$$
(B4)

The requirement that k^2 be real and positive places the following requirements on b and c

$$c > 0 \qquad (B5)$$

$$-1 \le \frac{b}{2c^{1/2}} \le 1 \qquad (B6)$$

An examination of equation (B3) reveals that Z has the period $u_1(T) = 4K(k)$, where K is the complete elliptic integral of the first kind. Therefore,

$$T = \frac{4K(k)}{c^{1/4}} \left(\frac{3}{2\ell_2}\right)^{1/2} \text{ sec} \qquad (B7)$$

However, during one period the amplitude will become infinite when the elliptic argument u_1 is some multiple of $2K(k)$. Thus, while the solutions given by equation (B3) are periodic, they involve infinite amplitudes, which are unrealistic for the solution to a physical problem and beyond the scope of the small-amplitude approximation.

Case B, $\ell_2 < 0$, $(b^2 - 4c) < 0$, $Z \le 0$:

This case is identical to the previous one, except that the argument of CN is imaginary and Z is negative.

$$u_2(k', t) = iu_1(k', t) = i\frac{t}{g} \left(\frac{2\ell_2}{3}\right)^{1/2} \qquad (B8)$$

In reference 13, expression 125.02 is a formula for the CN function of imaginary argument, where

$$CN \, i(u_1(k)) = \left[CN \, u_1(k')\right]^{-1} \qquad (B9)$$

and

$$k'^2 = 1 - k^2 = \frac{1}{2} \left(1 + \frac{b}{2c^{1/2}}\right) \qquad (B10)$$

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If u_1 is now understood to be a function of k', equation (B7) for the amplitude of Z may be written as

$$Z = c^{1/2} \left(\frac{CNu_1 - 1}{CNu_1 + 1} \right) = y - y_0 \tag{B11}$$

Thus, if $l_2 < 0$, the amplitude Z is infinite when $u_1 = 2K(k')$. Equation (B11) may also be obtained from formula 243.00 in reference 13. Therefore, case B also does not lead to physically realistic solutions within the scope of the small-amplitude approximation.

Case C, l_2 arbitrary, $b^2 - 4c = 0$, $b = c = 0$:

In this case, the integral of equation (40) assumes the much simpler form

$$t \left(\frac{2l_2}{3} \right)^{1/2} = \int_0^Z \frac{dZ}{Z^{3/2}} \tag{B12}$$

This integral is obviously monotone and divergent, regardless of the value of l_2. Case C therefore does not result in bounded periodic solutions.

Case D, $l_2 > 0$, $b^2 - 4c = 0$, $b < 0$, $Z \geq 0$:

In this case $Z_3 = Z_2 = -b/2$ and the basic integral (eq. (40)) is given by

$$t \left(\frac{2l_2}{3} \right)^{1/2} = \int_0^Z \frac{dZ}{Z^{1/2} \left(Z + \frac{b}{2} \right)} = \frac{2^{3/2}}{\sqrt{|b|}} \tan^{-1} \left(\frac{2Z}{b} \right)^{1/2} \tag{B13}$$

The amplitude of the function Z is therefore

$$Z = \frac{b}{2} \tan^2 \left[t \left(\frac{l_2 b}{12} \right)^{1/2} \right] = y - y_0 \tag{B14}$$

The function Z will have a peak-to-peak period of π radians, which results in a period of

$$T = \pi \left(\frac{12}{\ell_2 b} \right)^{1/2} \text{ sec} \tag{B15}$$

The amplitude of Z becomes infinite, however, and so equation (B14) cannot represent physically realistic solutions within the scope of the small amplitude approximation.

Case E, $\ell_2 > 0$, $\left(b^2 - 4c \right) = 0$, $b < 0$, $Z \geq 0$:

When the constant b is negative, the integral in equation (B13) is given by

$$Z = \frac{|b|}{2} \tanh^2 \left[t \left(\frac{|\ell_2 b|}{12} \right)^{1/2} \right] = y - y_0 \tag{B16}$$

This solution is bounded, monotone, and a physically realistic solution to the problem.

Case F, $\ell_2 < 0$, $\left(b^2 - 4c \right) = 0$, $b > 0$, $Z \leq 0$:

When the identity $\tan i\chi = i \tanh \chi$ is used, the solution for this case is obtained from equation (B14) as follows:

$$Z = -\frac{b}{2} \tanh^2 \left[t \left(\frac{|\ell_2 b|}{12} \right)^{1/2} \right] = y - y_0 \tag{B17}$$

This solution is not of interest in the present analysis since it is nonperiodic.

Case G, $\ell_2 < 0$, $\left(b^2 - 4c \right) = 0$, $b < 0$, $Z \leq 0$:

The functional form for this case can be obtained by using the identity $\tanh i\chi = i \tan \chi$ in equation (B16) to yield

$$Z = -\frac{|b|}{2} \tan^2 \left[t \left(\frac{|\ell_2 b|}{12} \right)^{1/2} \right] \tag{B18}$$

This solution has the period given by equation (B15), but it is not physically realistic since it has infinite amplitudes and is therefore beyond the scope of the small-amplitude approximation.

Case H, $l_2 > 0$, $(b^2 - 4c) > 0$, $c = 0$, $b > 0$:

From equations (35) and (40) it is apparent that in this case the basic integral is given by

$$t\left(\frac{2l_2}{3}\right)^{1/2} = \int_0^Z \frac{dZ}{Z(Z+b)^{1/2}} = \frac{1}{b^{1/2}} \ln \left| \frac{(Z+b)^{1/2} - b^{1/2}}{(Z+b)^{1/2} + b^{1/2}} \right|_0^Z \qquad \text{(B19)}$$

Because a singularity exists at the lower limit of integration, this case cannot represent a small-amplitude solution to the differential equation.

Case I, $l_2 < 0$, $(b^2 - 4c) > 0$, $c = 0$, $b > 0$:

The negative value of l_2 results in the left side of equation (B19) being multiplied by $\sqrt{-1}$. This process does not remove the singularity at $Z = 0$; therefore, this case also cannot represent a small-amplitude solution.

Case J, $l_2 > 0$, $(b^2 - 4c) > 0$, $c = 0$, $b < 0$:

and

Case K, $l_2 < 0$, $(b^2 - 4c) > 0$, $c = 0$, $b < 0$:

In both these cases, the basic integral, given by equation (B19), is divergent at its lower limit. Therefore, the basic differential equation does not have a small-amplitude solution.

Case L, $l_2 > 0$, $(b^2 - 4c) > 0$, $Z \geq 0 > Z_3 > Z_2$:

In this case, it is assumed that the parameter l_2 is positive, that the discriminant $b^2 - 4c$ is positive, and that the constant parameters are such that the roots Z_3 and Z_2, given by equations (38) and (39), are both negative. In this case, the value of the basic integral (eq. (40)) is given by formula 237.00 of reference 13.

The solution in this case is

$$u_1(k, t) = \frac{t}{g}\left(\frac{2l_2}{3}\right)^{1/2} \qquad \text{(B1)}$$

where

$$g = \frac{2^{3/2}}{\left[b + \left(b^2 - 4c\right)^{1/2}\right]^{1/2}} \tag{B20}$$

The amplitude of Z has the functional form

$$Z = \frac{b - \left(b^2 - 4c\right)^{1/2}}{2} TN^2 u_1(k) = y - y_0 \tag{B21}$$

and the modulus of the elliptic function is equal to

$$k^2 = \frac{2\left(b^2 - 4c\right)^{1/2}}{b + \left(b^2 - 4c\right)^{1/2}} \tag{B22}$$

The requirement that $0 \leq k^2 \leq 1$ is automatically satisfied when both roots Z_2 and Z_3 are negative. Equation (B21) has a peak-to-peak period of $u_1(T) = 2K(k)$, which results in the following period of oscillation:

$$T = \frac{4K(k)}{\left[b + \left(b^2 - 4c\right)^{1/2}\right]^{1/2}} \sqrt{\frac{3}{|\ell_2|}} \text{ sec} \tag{B23}$$

It is clear from equation (B21) that Z becomes infinite at $u_1 = K$, $3K$, etc. Therefore, this solution cannot represent a small-amplitude solution.

Case M, $\ell_2 < 0$, $(b^2 - 4c) > 0$, $0 > Z \geq Z_3 > Z_2$:

This case is closely related to the previous one, in which equation (B1) implies that

$$u_2(k, t) = iu_1(k, t) \tag{B24}$$

Using formula 125.02 of reference 13 yields

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$$TN\, iu_1(k) = i\, SN\, u_1(k')$$ (B25)

where the complementary modulus is given by

$$k'^2 = \frac{b - \left(b^2 - 4c\right)^{1/2}}{b + \left(b^2 - 4c\right)^{1/2}}$$ (B26)

and equation (B21) becomes for this case

$$Z = -\frac{b - \left(b^2 - 4c\right)^{1/2}}{2}\, SN^2 u_1(k') = y - y_0$$ (B27)

The period is given by equation (B23) with k replaced by k'. The amplitude Z is finite, negative, and periodic; therefore, equation (B27) is a physically interesting solution to the problem. Equation (B27) could also have been obtained from expression 236.00 of reference 13.

Case N, $l_2 > 0$, $(b^2 - 4c) > 0$, $Z_3 > Z_2 \geq Z > 0$:

When both of the roots given by equations (38) and (39) are positive and the range of Z is confined to $Z_2 \leq Z \leq 0$, the solution to the basic integral of equation (40) is given by formula 233.00 of reference 13.

$$u_1(k, t) = \frac{t}{g}\left(\frac{2l_2}{3}\right)^{1/2}$$ (B1)

where

$$g = \frac{2^{3/2}}{\left[-b + \left(b^2 - 4c\right)^{1/2}\right]^{1/2}}$$ (B28)

The amplitude of Z has the functional dependence

$$Z = Z_2 SN^2 u_1(k) = -\frac{1}{2}\left[b + \left(b^2 - 4c\right)^{1/2}\right] SN^2 u_1(k) = y - y_0$$ (B29)

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The total peak-to-peak amplitude of Z is then

$$Z_{max} = Z_2 = -\frac{1}{2}\left[b + \left(b^2 - 4c\right)^{1/2}\right]$$

(B30)

and the modulus of the elliptic function is

$$k^2 = \frac{Z_2}{Z_3} = \frac{b + \left(b^2 - 4c\right)^{1/2}}{b - \left(b^2 - 4c\right)^{1/2}}$$

(B31)

The solution given by equation (B29) has a peak-to-peak period of $u_1(T) = 2K(k)$. The period of oscillation is then given by

$$T = \frac{4K(k)}{\left[-b + \left(b^2 - 4c\right)^{1/2}\right]^{1/2}}\left(\frac{3}{l_2}\right)^{1/2}$$

(B32)

Equation (B29) does represent a solution of physical interest, since it is both periodic and finite.

Case O, $l_2 < 0$, $(b^2 - 4c) > 0$, $Z_3 > Z_2 > 0 > Z$:

This case is identical to the previous one, except that the argument u_1 is imaginary and $Z < 0$. Using formula 125.02 in reference 13 yields

$$SN\, iu_1(k) = i\, TN\, u_1(k')$$

(B33)

Equation (B29) may then be written

$$Z = -Z_2\, TN^2 u_1(k') = \frac{1}{2}\left[b + \left(b^2 - 4c\right)^{1/2}\right]TN^2 u_1(k') = y - y_0$$

(B34)

where the complementary modulus is

$$k'^2 = 1 - k^2 = \frac{-2\left(b^2 - 4c\right)^{1/2}}{b - \left(b^2 - 4c\right)^{1/2}}$$

(B35)

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and the period is given by equation (B32) with k replaced by k'. Equation (B34) cannot represent a small-amplitude solution to the problem since the amplitude becomes infinite. Equation (B34) may also be obtained from expression 232.00 of reference 13.

Case P, $l_2 < 0$, $(b^2 - 4c) > 0$, $Z_3 \geq Z > 0 > Z_2$:

In the case in which there is only one positive root and the range of integration is restricted between $Z_3 \geq Z \geq 0$, the solution to the basic integral of equation (40) is given by formula 235.00 of reference 13. For this case,

$$u_1(k, t) = \frac{t}{g} \sqrt{\frac{2|l_2|}{3}} \tag{B36}$$

where

$$g = \frac{2}{\left(b^2 - 4c\right)^{1/4}} \tag{B37}$$

The amplitude of Z has the functional dependence

$$Z = \frac{Z_2 Z_3 \, SN^2 u_1(k)}{Z_3 \, SN^2 u_1(k) - Z_3 + Z_2} = -\frac{-c \, SN^2 u_1(k)}{\left(b^2 - 4c\right)^{1/2} - \frac{1}{2}\left[-b + \left(b^2 - 4c\right)^{1/2}\right] SN^2 u_1(k)} \tag{B38}$$

where the modulus k is equal to

$$k^2 = \frac{Z_3}{Z_3 - Z_2} = \frac{1}{2}\left[1 - \frac{b}{\left(b^2 - 4c\right)^{1/2}}\right] \tag{B39}$$

The requirement that $0 \leq k^2 \leq 1.0$ will be met, since $c < 0$ in this case. The total peak-to-peak amplitude of the fluctuations in Z is given by

$$Z_{max} = Z_3 = \frac{1}{2}\left[-b + \left(b^2 - 4c\right)^{1/2}\right] \tag{B40}$$

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Equation (B38) has a peak-to-peak period of $u_1(T) + 2K(k)$, so that the period of the oscillation is given by

$$T = 2K(k)g \sqrt{\frac{3}{2|l_2|}} = \frac{4K(k)}{\left(b^2 - 4c\right)^{1/4}} \left(\frac{3}{2l_2}\right)^{1/2} \text{ sec} \qquad (B41)$$

The solution of equation (B38) is positive, periodic, and finite.

Case Q, $l_2 > 0$, $(b^2 - 4c) > 0$, $Z_3 > 0 > Z \geq Z_2$:

The solution to this case is identical to the previous one, except that the argument $u_1(k)$ is imaginary. With the use of equation (B33), equation (B38) may be written as

$$Z = \frac{-Z_2 Z_3 TN^2 u_1(k')}{Z_2 - Z_3 - Z_3 TN^2 u_1(k')} \qquad (B42)$$

This solution is periodic and finite and is therefore a desired solution to the problem. The period of Z is given by equation (B41), with k replaced by k'. The complementary modulus k' is equal to $1 - k^2$, where k^2 is given by equation (B39) and Z_{max} is equal to Z_2. Equation (B42) can also be obtained from expression 234.00 of reference 13.

Case R, $l_2 = l_3 = 0$, $l_1 \neq 0$:

The next situation, of lesser mathematical complexity, arises when $l_1 \neq 0$, $l_2 = l_3 = 0$, and l_0 may have any value. For this case, equation (9) may be written

$$\ddot{y} = l_0 + l_1 y \qquad (B43)$$

In terms of the variable Z defined in equation (16), this equation may be written

$$\ddot{Z} = l_0 + l_1 y_0 + l_1 Z \qquad (B44)$$

If ω^2 is defined as

$$\omega^2 \equiv -l_1 \qquad (B45)$$

equation (B44) may be written as

$$\ddot{Z} + \omega^2 Z = l_0 - \omega^2 y_0 \tag{B46}$$

The solution to this equation is

$$Z = \left(y_0 + \frac{l_0}{l_1}\right)(\cos \omega t - 1) \tag{B47}$$

which satisfies the boundary conditions of equation (18). The requirement that the solutions be periodic implies that

and therefore,
$$\left.\begin{array}{c} l_1 < 0 \\ \\ \\ \omega^2 > 0 \end{array}\right\} \tag{B48}$$

And the requirement that the amplitude of $y(t)$ be either zero or positive places the following restriction on the permissible value of l_0 with respect to l_1 and y_0:

$$l_0 > \frac{1}{2} y_0 |l_1| = \frac{1}{2} y_0 \omega^2 \tag{B49}$$

The constant l_0 must therefore be positive and larger than indicated in equation (B49) if periodic motion which satisfies the boundary conditions and has a positive amplitude is to occur.

The period of the fluctuation in y is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(-l_1\right)^{1/2}} \tag{B50}$$

and the total amplitude of the fluctuation in y is

$$\Delta y = Z_{max} = 2\left|y_0 - \frac{l_0}{\omega^2}\right| = 2\left|y_0 + \frac{l_0}{l_1}\right| \tag{B51}$$

Case S, $l_1 = l_2 = l_3 = 0$:

The most elementary special case occurs when only l_0 is nonzero. In this case, equation (9) may be written

$$\ddot{y} = \ddot{Z} = l_0 \qquad \text{(B52)}$$

Integrating twice and applying the boundary conditions of equation (18) gives

$$Z = \frac{l_0}{2} t^2 \qquad \text{(B53)}$$

There are no periodic solutions to equation (B52), and the only solution is monotone in time.

APPENDIX C

COMPARISON WITH PRIOR LITERATURE

One of the solutions to equations (1) and (2) which appears in appendix B can be compared with the solution of a much more restricted case that has appeared in the literature. Lotka (ref. 3) and Volterra (ref. 5) both considered periodic, small-perturbation solutions to the system of equations

$$\dot{\chi} = C_1 \chi + C_3 \chi y \tag{C1}$$

$$\dot{y} = A_2 y + A_3 \chi y \tag{C2}$$

where these equations have been written in the notation used herein. These equations differ from equations (1) and (2) in that

$$C_0 = C_2 = C_4 = C_5 = 0 \tag{C3}$$

and

$$A_0 = A_1 = A_4 = A_5 = 0 \tag{C4}$$

In the present analysis, it is assumed that the fluctuations in χ are small by comparison with the mean value of χ, but that the peak-to-peak amplitude of y can be either smaller or larger than the mean value of y. Lotka and Volterra assumed that the perturbations of both χ and y were small with respect to their mean values. These authors expanded χ and y in the form

$$\chi = \chi_1 + \chi_2(t) \tag{C5a}$$

$$\chi_1 \gg \chi_2(t) \tag{C5b}$$

$$y = y_{00} + y_{01}(t) \tag{C6a}$$

$$y_{00} \gg y_{01}(t) \tag{C6b}$$

where χ_1 and y_{00}, the time-independent mean values of χ and y, are defined as

$$\chi_1 \equiv -\frac{A_2}{A_3} \qquad (C7)$$

$$y_{00} \equiv -\frac{C_1}{C_3} \qquad (C8)$$

Substituting equations (C5a), (C6a), (C7), and (C8) into equations (C1) and (C2) yields

$$\dot{\chi}_2 = C_3 y_{01}(\chi_1 + \chi_2) \qquad (C9)$$

$$\dot{y}_{01} = A_3 \chi_2 (y_{00} + y_{01}) \qquad (C10)$$

Differentiating equation (C10) with respect to time and eliminating the first order time derivatives yield the second-order equation

$$\ddot{y}_{01} = A_3 C_3 y_{01}(\chi_1 + \chi_2)(y_{00} + y_{01}) + A_3^2 \chi_2^2 (y_{00} + y_{01}) \qquad (C11)$$

At this point the approximations given by equations (C5b) and (C6b) must be introduced. When these approximations are introduced and the products of small quantities are neglected, equation (C11) becomes

$$\ddot{y}_{01} = A_3 C_3 \chi_1 y_{00} y_{01} \qquad (C12)$$

By making use of equations (C7) and (C8), the frequency may be written

$$\omega \equiv \left(-A_3 C_3 \chi_1 y_{00}\right)^{1/2} = \left(-A_2 C_1\right)^{1/2} \qquad (C13)$$

A solution of equation (C12) which is consistent with the initial conditions of equation (4) is

$$y_{01}(t) = \delta \cos \omega t \qquad (C14)$$

where δ is (in this appendix) the time-independent amplitude of the perturbation in y.

The waveform of the oscillation in y is then obtained by substituting equation (C14) into equation (C6a), which gives

$$y(t) = y_{00} + \delta \cos \omega t \tag{C15}$$

where

$$y_0 = y_{00} + \delta \tag{C16}$$

is considered to be the maximum value of y and to occur at $t = 0$. The frequency of the oscillations is, from equation (C13),

$$\nu = \frac{\left(-A_3 C_3 \chi_1 y_{00}\right)^{1/2}}{2\pi} = \frac{\left(-A_2 C_1\right)^{1/2}}{2\pi} \text{ cps (Hz)} \tag{C17}$$

and the period is

$$T = \frac{2\pi}{\left(-A_3 C_3 \chi_1 y_{00}\right)^{1/2}} = \frac{2\pi}{\left(-A_2 C_1\right)^{1/2}} \text{ sec} \tag{C18}$$

which is the result obtained by Lotka (ref. 3) and Volterra (ref. 5).

This prior result will be compared with the results of the present analysis, in the limiting situation given by equations (C5b) and (C6b). Substituting equations (C3) and (C4) into equations (10) to (13) yields

$$\ell_0 = \ell_3 = 0 \tag{C19}$$

$$\ell_1 = \left(A_2 + A_3 \chi_1\right)^2 + A_3 C_1 \chi_1 \tag{C20}$$

$$\ell_2 = A_3 C_3 \chi_1 \tag{C21}$$

If $\chi(0) = \chi_1$ is chosen as an initial condition, the initial conditions on $y(t)$ given by equation (4) then imply

$$A_2 + A_3 \chi_1 = 0 \tag{C22}$$

so that equation (C20) becomes

$$\ell_1 = A_3 C_1 \chi_1 \tag{C23}$$

Since $\ell_3 = 0$, the results presented in appendix B are appropriate. In actual physical applications, χ and y will represent the magnitude of real physical quantities and will be positive. As can be seen by manipulating equations (C1) and (C2), either A_3 or C_3 must be negative in order for periodic solutions to exist. The parameter ℓ_2 in equation (C21) will therefore be negative. As in the analysis leading to equation (C15), it is assumed here that y_0 is the maximum value of y, occurring at $t = 0$. This assumption will result in the ancillary variable Z, defined by equation (16), being negative. The three conditions, $\ell_3 = 0$, $\ell_2 < 0$, and $Z \leq 0$, therefore imply that case M of appendix B is appropriate for comparison with the results of Lotka and Volterra. If a positive parameter

$$\eta \equiv - \frac{C_1}{C_3 y_0} \tag{C24}$$

is defined, the parameters b and c given by equations (33) and (34) are equal to

$$b = \frac{3}{2} y_0 (2 - \eta) \tag{C25}$$

$$c = 3 y_0^2 (1 - \eta) \tag{C26}$$

Equations (38) and (39) for the roots are

$$Z_2 = -\frac{3}{4} y_0 \left\{ (2 - \eta) + \left[\left(\eta - \frac{2}{3} \right)(\eta + 2) \right]^{1/2} \right\} \tag{C27}$$

$$Z_3 = -\frac{3}{4} y_0 \left\{ (2 - \eta) - \left[\left(\eta - \frac{2}{3} \right)(\eta + 2) \right]^{1/2} \right\} \tag{C28}$$

From equation (B27), it can be seen that the waveform predicted by the present analysis is

$$y(t) = y_0 + Z_3 SN^2 u_1 (k') \tag{C29}$$

An inspection of equation (C28) reveals that a necessary and sufficient condition for the small-amplitude limit $Z_3 \ll y_0$ is $\eta \to 1.0$. As η approaches 1 as a limit, equations (C24) to (C28) become

$$C_3 y_0 \approx -C_1 \tag{C30}$$

$$b \approx \frac{3}{2} y_0 \tag{C31}$$

$$c \approx 0 \tag{C32}$$

$$Z_2 \approx -\frac{3}{2} y_0 \tag{C33}$$

$$Z_3 \approx 0 \tag{C34}$$

In this limit, the period of oscillation given by equation (B23) is equal to

$$T \approx \frac{4K(k')}{\left(-y_0 l_2\right)^{1/2}} = \frac{4K(k')}{\left(-y_0 \chi_1 A_3 C_3\right)^{1/2}} \text{ sec} \tag{C35}$$

From equation (B26), in the limit $\eta \to 1.0$,

$$k'^2 = \frac{Z_3}{Z_2} \approx 0 \tag{C36}$$

Thus, when $\eta \to 1.0$,

$$K(k') \approx \frac{\pi}{2} \tag{C37}$$

Substituting equations (C37), (C30), and (C22) into equation (C35) shows the period of oscillation to be

$$T \approx \frac{2\pi}{\left(-y_0 \chi_1 A_3 C_3\right)^{1/2}} = \frac{2\pi}{\left(-C_1 A_2\right)^{1/2}} \text{ sec} \tag{C38}$$

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which, if $y_0 \approx y_{00}$, is in agreement with the period derived by Lotka and Volterra in equation (C18).

As the elliptic modulus k'^2 approaches zero, the Jacobian elliptic sine becomes equal to the conventional trigonometric sine (ref. 13).

$$SN u_1(k') \to \sin u_1(0) \tag{C39}$$

The argument $u_1(k' \to 0)$ may be obtained from equations (B19) and (B20).

$$u_1(0) \approx \frac{1}{2} t \left(-y_0 \chi_1 A_3 C_3\right)^{1/2} = \frac{\omega t}{2} \tag{C40}$$

When this argument is substituted into equation (C29) the small-amplitude limit is

$$y(t) = y_0 + Z_3 \sin^2 \frac{\omega t}{2} = y_0 + \frac{Z_3}{2} (1 - \cos \omega t) \tag{C41}$$

Identifying Z_3 with the small amplitude δ,

$$Z_3 \equiv -2\delta \tag{C42}$$

and using equation (C16) to rewrite equation (C41) yield the limiting small-amplitude expression

$$y(t) \approx y_{00} + \delta \cos \omega t \tag{C43}$$

Thus, the frequency and waveform predicted by this analysis in the small-amplitude variation of both $\chi(t)$ and $y(t)$ agree with the small-perturbation results of Lotka and Volterra, which are given by equation (C15).

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