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THE APPROXIMATION OF STELLAR ENERGY DISTRIBUTIONS AND MAGNITUDES FROM MULTI-COLOR PHOTOMETRY

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Paul B. Davenport

SUMMARY

A procedure is developed for approximating the energy distribution of a star from present day multi-color photometry. This approximation is then used to predict stellar magnitude relative to a light sensitive device with known spectral response. Using magnitudes in the U, B, V system it is found that the energy distributions of stars can be approximated by this procedure fairly accurately over the range of 3000-8000 angstroms. The stellar magnitude relative to a sensor whose dominant response lies in this region can then be determined with an accuracy of about 0.1 magnitude.

THE APPROXIMATION OF STELLAR ENERGY DISTRIBUTIONS AND MAGNITUDES FROM MULTI-COLOR PHOTOMETRY

INTRODUCTION

With the increasing applications of star trackers in air and space navigation it is imperative that procedures be developed which accurately predict the magnitude of a star relative to a given tracker. It certainly is not practical, nor in general feasible, to observe every star of interest each time a tracker with a different response becomes available. There have been many internally published documents from various organizations giving the photoelectric magnitudes of the brighter stars. All of these lists, however, have been generated in practically the same manner i.e. black body radiation was assumed (the temperature obtained from various sources as a function of the Draper spectral type) to generate an index which is added to the Harvard visual magnitudes. Unfortuntely, this assumption as well as the basic data used is not accurate when compared with modern observations. These methods have also assumed that all sensors with the same S-number have the same spectral response which can be obtained from a handbook containing the nominal response. The fact is, however, that even when two devices from the same manufactor are designated the same their measured response may be quite different causing deviations of up to a tenth of a magnitude. (This value was derived from our own calculations based upon the measured responses of dozens of photomultipliers of the same type. See also [1]).

Here, we present a procedure for determining stellar magnitudes which uses the latest available data, namely multi-color photometry such as the U, B, V system [2], [3], and [4]. The approach here is to obtain information about the energy distribution of a star from the observed color magnitudes which in turn can be used to determine the stellar magnitude relative to a sensor whose response lies in the region covered by the responses defining the color magnitudes. Since a large number of stars have now been observed in the U, B, V system, and the dominant part of the response curves of many sensors lies in that part of the spectrum covered by the U, B, and V filters the techniques developed here have a very general use.

THE MAGNITUDE EQUATION

The stellar magnitude (m_s) of an astronomical radiation source reduced to outside the earth's atmosphere and incorporating the interstellar absorption is given by

$$m_s = m_{s0} - 2.5 \log \frac{I_s}{I_{s0}} = - 2.5 \log I_s + k_s \ , \qquad (1)$$

where

$$I_s = \int_0^\infty J(\lambda)\, \sigma_s(\lambda)\, d\lambda \ . \qquad (2)$$

$J(\lambda)$ is the spectral distribution of light intensity from the source, σ_s is the spectral response function of the optical train observing the radiation, and the constant I_{s0} is the value of I_s for a standard source of radiation with magnitude m_{s0} . If a black-body distribution is assumed then

$$J(\lambda) = \frac{c_1 \lambda^{-5}}{e^{c_2/T\lambda} - 1} \qquad (3)$$

where T is the temperature of the source.

To obtain information about the spectral distribution of light intensity from a multi-color magnitude system such as Johnson's & Morgan's UBV system we use the constant energy concept widely practiced in photometric work. Since many observers today use the effective wave number concept we adopt this procedure here [5].

Let $x = 1/\lambda$, then the effective intensity integral equation (2) becomes

$$I_s = \int_0^\infty x^{-2}\, J(x)\, \sigma_s(x)\, dx$$

or

$$I_s = \int_0^\infty F(x)\, \sigma_s(x)\, dx$$

where

$$F(x) = x^{-2}\, J(x) \ .$$

For a black-body distribution

$$F(x) = \frac{c_1 x^3}{e^{c_2/TX} - 1} \quad . \tag{4}$$

Letting

$$x_0 = \frac{\int_0^\infty x \sigma_s(x)\, dx}{\int_0^\infty \sigma_s(x)\, dx}$$

and expanding F(x) in a Taylor series about x_0 the effective intensity integral becomes

$$I_s = \left[F(x_0) \int_0^\infty \sigma_s(x)\, dx + \frac{1}{2} \frac{d^2F}{dx^2}\bigg|_{x=x_0} \int_0^\infty (x - x_0)^2 \sigma_s(x)\, dx + \ldots \right]$$

$$= \int_0^\infty \sigma_s(x)\, dx \left[F(x_0) + \Delta F \right]$$

where

$$\Delta F = \sum_{n=2}^{\infty} \frac{1}{n!} \frac{d^nF}{dx^n}\bigg|_{x=x_0} \mu_n(x_0)$$

and

$$\mu_n(x_0) = \frac{\int_0^\infty (x - x_0)^n \sigma_s(x)\, dx}{\int_0^\infty \sigma_s(x)\, dx}$$

The stellar magnitude is then given by

$$m_s = -2.5 \log \left[F(x_0) + \Delta F \right] + k'_s \,,$$

$$= -2.5 \left[\log F(x_0) + \log (1 + \epsilon) \right] + k'_s \,,$$

where

$$\epsilon = \frac{\Delta F}{F(X_0)} \quad . \tag{5}$$

Since

$$\log (1 + \epsilon) = (\log e) \left(\epsilon - \frac{1}{2} \epsilon^2 + \ldots \right) \quad ,$$

provided $|\epsilon| < 1$

$$m_s = - 2.5 \left[\log F(x_0) + (\log e) \left(\epsilon - \frac{1}{2} \epsilon^2 + \ldots \right) \right] + \kappa'_s \quad .$$

Thus, if the band pass of the receiver is narrow enough or the second and higher derivatives of the energy distribution F(x) are sufficiently small the magnitude is a measure of the monochromatic flux at x_0 i.e.

$$m_s = - 2.5 \log F(x_0) + k'_s \quad .$$

For the U, B, and V bandpasses we assume that this is the case. Hence,

$$U = - 2.5 \log F(x_1) + k_1$$

$$B = - 2.5 \log F(x_2) + k_2 \tag{6}$$

$$V = - 2.5 \log F(x_3) + k_3$$

where

$$X_i = \frac{\int_0^\infty X \sigma_i(X)\, dx}{\int_0^\infty \sigma_i(X)\, dx}, \quad i = 1, 2, 3 \tag{7}$$

and σ_i (X) is the corresponding response function. Thus the U, B, V system provides a means of approximating the energy distribution at three points within the response of many photomultipliers and other sensors.

Since the band width of most sensors is at least three times as wide as each filter (U, B, or V) we assume that the energy distribution over the range of interest is quadratic.

Thus, the S magnitude, m_s, is given by

$$m_s = -2.5 \left[\log F(X_s) + (\log e)\, \epsilon \right] + k_s \tag{8}$$

where ϵ is defined by equation (5) with the third and higher derivatives assumed to be zero, x_s is given by equation (7) and the response function used in the equations is that of the measured response of the sensor of interest.

From the three values of $y(x) = \log F(x)$ obtained by the U, B, and V magnitudes (equation 6) a quadratic approximation to $y(x)$ can be determined by constructing a quadratic function passing through the three points i.e.

$$y(X) = A_1X^2 + A_2X + A_3 \quad , \tag{9}$$

where the coefficients A_i are determined from the matrix equation

$$\begin{pmatrix} X_1^2 & X_1 & 1 \\ X_2^2 & X_2 & 1 \\ X_3^2 & X_3 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} y(X_1) \\ y(X_2) \\ y(X_3) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} .$$

The explicit solution is given by

$$A_1 = \frac{1}{D} \left[(X_1 - X_3)(y_3 - y_2) + (X_3 - X_2)(y_3 - y_1) \right] ,$$

$$A_2 = \frac{1}{D} \left[(X_3^2 - X_1^2)(y_3 - y_2) + (X_2^2 - X_3^2)(y_3 - y_1) \right] , \tag{10}$$

$$A_3 = \frac{1}{D} \left[X_2X_3(X_2 - X_3)y_1 + X_1X_3(X_3 - X_1)y_2 + X_1X_2(X_1 - X_2)y_3 \right] ,$$

where

$$D = (X_3 - X_2)(X_3 - X_1)(X_1 - X_2) .$$

To determine the S magnitude as given by equation (8) we need $y(X_s) = \log F(X_s)$ and ϵ. $y(X_s)$ is obtained directly from equation (9) i.e.

$$y(X_s) = \log F(X_s) = A_1 X_s^2 + A_2 X_s + A_3 \quad . \tag{11}$$

Now

$$\epsilon = \frac{\Delta F}{F(X_s)} = \frac{\left.\frac{1}{2}\frac{d^2F}{dX^2}\right|_{x=x_s}}{F(X_s)} \; \mu_2(X_s)$$

where

$$\mu_2(X_s) = \frac{\int_0^\infty (X - X_s)^2 \sigma_s(X)\, dx}{\int_0^\infty \sigma_s(X)\, dx} \; .$$

Since

$$F(X) = 10^{y(x)} \quad ,$$

$$\frac{dF}{dX} = \ln 10 \; 10^{y(X)} \frac{dy}{dX} \quad ,$$

$$\frac{d^2F}{dX^2} = \ln 10 \left[10^{y(x)} \frac{d^2y}{dX^2} + \ln 10 \; 10^{y(x)} \left(\frac{dy}{dX}\right)^2 \right] ,$$

then

$$\frac{\frac{d^2F}{dX^2}}{F(X)} = \ln 10 \left[\frac{d^2y}{dX^2} + \ln 10 \left(\frac{dy}{dX}\right)^2 \right] .$$

From equation (9)

$$\left.\frac{dy}{dX}\right|_{x=x_s} = 2A_1 X_s + A_2 \quad ,$$

$$\frac{d^2y}{dX^2} = 2A_1 \quad ,$$

thus

$$\epsilon = \frac{1}{2} \ln 10 \left[2A_1 + \ln 10 \, (2A_1 X_s + A_2)^2 \right] \mu_2(X_s) \quad . \tag{12}$$

In summary, the constants X_1, X_2, X_3, and X_s are determined from the known response of the U, B, V, and S bandpasses respectively $\left(\mu_2(X_2)\right.$ is obtained from the S response and $\left.x_s\right)$. y_1, y_2, and y_3 are obtained from equation (6), then A_1, A_2, and A_3 from equation (10) log $F(X_s)$ from equation (11), ϵ from equation (12), and finally the S magnitude (m_s) from equation (8).

By convention the values of U, B, and V are not given directly, but as a single magnitude (V) and two color indices (B-V and U-V or U-B). Since the data is often obtained in terms of V, B-V, and U-V [4] we will write our solution in terms of these variables. Thus

$$A_1 = \frac{1}{2.5D}\left[(X_1 - X_3)(B - V - C_B) + (X_3 - X_2)(U - V - C_u)\right]$$

$$A_2 = \frac{1}{2.5D}\left[(X_3^2 - X_1^2)(B - V - C_B) + (X_2^2 - X_3^2)(U - V - C_u)\right]$$

$$A_3 = \frac{1}{2.5}\left\{(k_s - V) - \frac{1}{D}\left[X_2 X_3(X_2 - X_3)(U - V - C_u)\right.\right.$$

$$\left.\left. + X_1 X_3(X_3 - X_1)(B - V - C_B)\right]\right\}$$

where

$$C_u = k_1 - k_3\ , \quad C_B = k_2 - k_3\ .$$

Therefore,

$$m_s = V + b_1(U - V) + b_2(B - V) + b_0 + \Delta m\ , \tag{13}$$

where

$$b_1 = \frac{X_s^2 - X_s(X_2 + X_3) + X_2 X_3 + \mu_2(X_s)}{(X_1 - X_2)(X_1 - X_3)}\ ,$$

$$b_2 = \frac{X_s^2 - X_s(X_1 + X_3) + X_1 X_3 + \mu_2(X_s)}{(X_3 - X_2)(X_1 - X_2)}\ ,$$

$$\Delta m = -\frac{1}{5}\,\ell n\, 10\ \mu_2(X_s)\left[C_1(U - V) + C_2(B - V) + C_3\right]^2$$

$$C_1 = \frac{X_2 - 2X_s + X_3}{(X_1 - X_2)(X_1 - X_3)} \quad , \quad C_2 = \frac{X_1 - 2X_s + X_3}{(X_1 - X_2)(X_3 - X_2)} ,$$

$$C_3 = - (C_u C_1 + C_B C_2) \quad .$$

The constant b_0 is arbitrary and serves as a zero point for the magnitude system. The only constant dependent upon stellar energy distributions is C_3 which does not require absolute measures but only relative ones i.e. ultraviolet and blue relative to visual. Besides, the constant C_3 is involved only in the second order term. Since the effective wave numbers are also relative measurements it appears that the constants in equation (13) can be determined quite accurately. The primary source of error, therefore, will probably be due to the assumptions that the flux is linear over each bandpass and quadratic over the entire interval of interest.

CONSTANT ENERGY EFFECTIVE WAVE NUMBERS

Using the published response curves of the U, B, and V filters [2] and the reflectivity of two aluminum reflections [6] we obtain

$$x_1 = 2.889, \qquad x_2 = 2.296, \qquad x_3 = 1.826,$$

which is consistent with Code's results [5].

If the energy distributions were linear over each bandpass then the plots of the distributions versus wave number for different stars (with the same magnitude) would intersect at a point. This point of intersection would give the effective wave number and the value of the flux at the effective wave number. Actually the plots will not intersect at a point since the flux is not linear, however, a point of minimum deviation will exist which approximates the effective wave number.

Using Willstrop's data for $V = 0.0$ [7] we find that the intersection occurs near $X = 1.83$ and $F(1.83) = 1.14 \times 10^{-5}$ (erg/cm^2/sec) μ . The effective wave number obtained on this manner agrees with that obtained directly from the V sensitivity-curve. The value of $F(1.83)$ is consistent with Code's value of 1.13×10^{-5} (erg/cm^2/sec) μ using Minnaert's derived fluxes and Stebbins & Kron's apparent visual magnitude of the sun. It is also consistent with Code's direct measurement of 1.20 at 1.80. We shall adopt the value of 1.14×10^{-5} (erg/cm^2/sec) μ at 1.83 ($1/\mu$) for V = 0.0. Thus,

$$V \cong -2.5 \log F(1.83) + 0.14$$

From Willstrop's data for B = 0.0 the effective wave number of the B band-pass is 2.295 and the average value of F(2.295) is 1.24×10^{-5} (erg/cm^2/sec) μ. Using the adopted expression for V and Code's expression for the monochromatic magnitudes we obtain

$$\log F(X) = 0.4 \left[m(1.83) - m(X) - V + 0.142 \right]$$

where m(x) is Code's monochromatic magnitudes. From Code's data using the observed values of B-V and selecting V such that B = 0.0 we find that the effective wave number for the B band pass is 2.295 and the average value of F(2.295) is 1.26×10^{-5} (erg/cm^2/sec) μ for B = 0.0. Oke's data [8] is for the stars whose spectral types are very similiar hence the plotting method using this data does not yield a satisfactory approximation to the effective wave numbers i.e. the curves are nearly parallel. The average value of the flux at the effective wave numbers, however, should be consistent with that of Willstrop's and Code's. Indeed this is the case, using Oke's monochromatic magnitudes in the same manner as Code's we obtain an average value of 1.29×10^{-5} (erg/cm^2/sec) μ at 2.29 ($1/\mu$) for B = 0.0. Thus, the effective wave number as obtained from Johnson's, Willstrop's, and Code's data is in agreement and the value for the flux at the effective wave number from Willstrop's, Code's, and Oke's data is also consistent. Hence,

$$B \cong -2.5 \log F(2.295) + 0.251 \ .$$

From Code's data, the observed values of U-V, and selecting V such that V = 0.0 we obtain an effective wave number for the U band pass of 2.905 with an average value of 0.50×10^{-5} (erg/cm^2/sec) μ for the flux. Oke's data gives an average value of 0.41 (erg/cm^2/sec) μ at 2.90. Thus, we adopted the value of 0.45 (erg/cm^2/sec) μ at 2.89 ($1/\mu$) for U = 0.0. Hence,

$$U \cong -2.5 \log F(2.89) - 0.867$$

A summary of the results of the various investigators and the adopted values are given below.

Effective Wave Number

U	B	V	Remarks
2.89	2.29	1.83	Sensitivity – curves by Johnson & Morgan – calculated by Code
2.889	2.296	1.826	Sensitivity – curves by Johnson & Morgan – calculated by author
	2.295	1.83	Willstrop's data
2.905	2.295		Code's data
2.89	2.295	1.83	Adopted

Flux at Effective Wave Number			
U = 0 X = 2.90	B = 0 X = 2.295	V = 0 X = 1.83	
		1.13	Code using Minnaert's flux and Stebbins & Kron V of sun
	1.24	1.14	Willstrop's data
0.50	1.26		Code's data
0.41	1.29		Oke's data
0.45	1.26	1.14	Adopted

AN EXAMPLE

From the nominal S4 response [9] we obtain as the effective number of the S4 response the value 2.49 and $\mu_2(2.49) = 0.155$. With these values plus the values given above we obtain

$$b_1 = 0.439, \qquad b_2 = 0.413,$$

$$c_1 = -1.321, \qquad c_2 = 0.890,$$

$$c_3 = -1.43$$

Hence,

$$m_{s4} = V + 0.439(U - V) + 0.413(B - V) + 0.145 \tag{14}$$

$$-0.071 \left[- 1.32(U - V) + 0.89(B - V) - 1.43 \right]^2$$

where the zero point has been selected such that $m_{s4} - V = 0$ when $U - V = B - V = 0$

To test the accuracy of this formula we generated the flux of seven stars (ϵ Ori, β Ori, α Lyr, β Ari, σ Boo, λ Ser & α Tau) from Code's data. The seven stars were selected so as to cover a wide range of U - V and B - V values. The values of V used to generate these data were chosen so that B = 0 merely to keep the values of the flux within the same numerical range.

The values beyond X = 2.94 were obtained by linear extrapolation using the values at X = 2.74 and X = 2.94. Values at even increments were obtained by linear interpolation. Using these energy distributions and the response curves the deflection y, the blue-yellow color index c_b, and the ultraviolet-yellow color index c_μ were computed using numerical integration. Values of V, B - V, and U - V were then computed using a zero point so as to minimize the mean difference between the observed and computed values.

The relationships were

$$V = - 2.5 \log y + 0.22 ,$$

$$U - V = 2.5 \log \frac{y}{\mu} - 0.05 ,$$

$$B - V = 2.5 \log \frac{y}{b} + 0.79 .$$

Except for the U - V of β Ori and σ Boo the agreement between the observed and computed values are quite good and the extrapolation in the ultraviolet can easily account for the error of 0.1 mag. in U - V for these two stars.

The energy distributions from Code were then assumed to be correct and the computed V, U - V, and B - V (from these energy distributions plus the U, B, and V response functions) to be correct also. The S4 magnitudes were then computed by integration and by equation (14), (actually the value of C and the zero point were adjusted to agree with Code's data rather than the adopted means). The maximum error is only 0.06 magnitudes which includes uncertainties in the sensitivity-curves as well as the assumed energy distributions. Since the constants used in equation (14) are based on data which is consistent from several different sources it appears that the S4 magnitude can be predicted from the observed V, U - V, B - V magnitude & colors to within 0.1 magnitude (assuming that the S4 spectral response is accurately known).

Actually, the energy distributions can be approximated much better by using the relationships

$$F(X) = 10^{y(x)} ,$$

$$y(x) = A_1 X^2 + A_2 X + A_3 ,$$

rather than retaining only quadratic terms in the expansion of $10^{y(x)}$. This additional accuracy in the energy distrituions, however, affects the S4 magnitudes very little. This is due to the fact that the third and higher moments $(\mu_3, \mu_4 \ldots ,)$ about the effective wave number for this particular curve are very small. For a response curve with a wider band pass of a large third moment about the effective wave number the better approximation may be necessary.

CONCLUSION

The method presented here was specifically formulated with the intent of being applied to a S4 response and particularly to obtain the guide star list for the Orbiting Astronomical Observatory launched in April of 1966. As our example shows, the method is quite accurate for this type of response. Since that time we have found that as suspected the accuracy falls off for sensors which respond to a wider range of light. However, with more recent data and minor modifications to our method here it is reasonable to expect that similar procedures will be applicable for any sensor whose response lies in the 3000-8000 Angstrom range with an accuracy of better than 0.1 magnitudes. The results of these late improvements will be presented at a later date.

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