ON A NONLINEAR THEORY OF ELASTIC SHELLS#

by

H. Cohen* and C. N. DeSilva**

SUMMARY. The present paper develops a general nonlinear theory of elastic shells for which the strain measures depend on first and second order deformation gradients. The reduction to theories of previous work is indicated.

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*University of Manitoba, Winnipeg, Manitoba

**Wayne State University, Detroit, Michigan

1. Introduction

In a recent paper [1], the authors presented a nonlinear theory of elastic shells by treating the deformation of an elastic surface. Subsequently Naghdi [2] suggested that the theory of Ref [1] could be improved by defining a surface stress tensor which is not symmetric. In developing shell theories from the equations of three-dimensional continua, it is natural to define a nonsymmetric surface stress tensor. In this paper, the authors extend the results of Ref [1] by considering a less restricted class of deformations while simultaneously defining a non-symmetric surface stress tensor.

Two principles are postulated: a principle of virtual work and a principle of material objectivity. (An appropriate formulation of the principle of virtual work is the key to introducing a non-symmetric stress tensor.) These principles are applied to obtain a theory of elastic surfaces within the framework of the assumed form of the strain energy function. The stress tensor and the couple stress tensor, both non-symmetric surface tensors, depend on deformation tensors $\underset{\sim}{L}$, $\underset{\sim}{K}$ and $\underset{\sim}{C}$ which are defined. The quantities which in Ref [1] were energetically undetermined are now explicitly given in terms of the strain energy function by a constitutive equation. These quantities, which are interpreted as double stresses, are characteristic of introducing second order gradients into a theory.

Finally, it is shown how the theory of Ref [1] may be recovered as a special case of the present theory.

2. Kinematics

Let X^K be rectangular Cartesian coordinates defined in an E_3. We define a surface S or an R_2 imbedded in this E_3 by

$$X^K = X^K (U^\Delta) \tag{2.1}$$

where U^Δ are general curvilinear coordinates in S. Here and in all cases Latin and Greek indices take on the values 1, 2, 3 and 1, 2 respectively. The surface given by Eq. (2.1) is described by the fundamental surface tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with components given by:

$$A_{\Delta\Sigma} = X^K_{;\Delta} X^K_{;\Sigma} \quad ; \qquad B_{\Delta\Sigma} = N^K X^K_{;\Delta\Sigma}$$

where $\underset{\sim}{N}$ with components N^K is the unit vector normal to S and where the semi-colon denotes the total covariant derivative (Ericksen [3]). We recall that the tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ satisfy the usual equations of surface theory associated with the names of Gauss, Weingarten and Mainardi-Codazzi.

We now assume that S deforms into a surface s given by

$$x^i = x^i(u^\delta) \tag{2.2}$$

Majuscule and miniscule Latin and Greek letters and indices will be associated with S and s respectively. We assume $\underset{\sim}{X}$ in S maps into $\underset{\sim}{x}$ in s. If we specify the mapping

$$u^\delta = u^\delta (U^\Delta) \tag{2.3}$$

then the deformation of S into s is given by Eqs (2.1), (2.2) and (2.3).

We define, for a differential description of the strain, the deformation tensors:

$$L_{\Delta\Sigma} = x^i_{;\Delta} x^i_{;\Sigma} = a_{\delta\sigma} u^\delta_{;\Delta} u^\sigma_{;\Sigma} \tag{2.4}$$

$$K_{\Delta\Sigma} = n^i x^i_{;\Delta\Sigma} = b_{\delta\sigma} u^\delta_{;\Delta} u^\sigma_{;\Sigma} \tag{2.5}$$

$$C_{\Delta\Sigma\Gamma} = x^i_{;\Delta} x^i_{;\Sigma\Gamma} = a_{\delta\sigma} u^\delta_{;\Delta} u^\sigma_{;\Sigma\Gamma} \ . \tag{2.6}$$

$\underset{\sim}{L}$ and $\underset{\sim}{K}$ are the Kirchhoff-Love tensors which enable us to calculate length and angle changes and normal curvature changes respectively. $\underset{\sim}{C}$, like $\underset{\sim}{L}$, is associated with changes in the intrinsic aspect of surface geometry.

3. Principles of Virtual Work and Objectivity

We define the virtual work associated with an arbitrary virtual displacement by

$$\mathcal{A} = \oint_c \left[s_k \ \delta x^k + m_k^{\sigma} \ \delta x^k{}_{,\sigma} \right] dc \ + \int_{\sigma} \gamma \left[f_k \ \delta x^k \ + \bar{\ell}_k^{\delta} \ \delta x^k{}_{,\delta} \right] d\sigma \qquad (3.1)$$

where c is a circuit enclosing a surface σ in s, γ is the surface mass density and the comma denotes ordinary covariant differentiation. The quantities $\underset{\sim}{s}$ and $\underset{\sim}{f}$ will later be associated with the stress and body force vectors while $\underset{\sim}{m}$ and $\underset{\sim}{\bar{\ell}}$ will define the double stress and the double body vectors respectively.

We assume that the deformation is such that there exists a strain energy function ϵ which is of the form

$$\epsilon = \epsilon \left(x^i_{;\Delta} \ ; \quad x^i_{;\Delta\Sigma} \right) \qquad (3.2)$$

and that the total energy of deformation W is given by

$$W = \int_{\sigma} \gamma \epsilon \ d\sigma \qquad (3.3)$$

We now postulate

(1) that the mechanical behavior of the surface is governed by a principle of virtual work which requires as a necessary condition for equilibrium:

$$\mathcal{A} = \delta W \qquad (3.4a)$$

$$\delta(\gamma \, d\sigma) = 0$$

for arbitrary variations δx^k.

An equivalent formulation of the principle of virtual work is obtained by introducing Lagrange multipliers t_k^δ (Courant and Hilbert [4]) and requiring as a necessary condition for equilibrium:

$$\delta W + \int_\sigma t_k^\delta \; \delta\left(\frac{\partial x^k}{\partial u^\delta} - x^k{}_{,\delta}\right) d\sigma = \mathscr{A} \tag{3.4c}$$

$$\delta(\gamma \, d\sigma) = 0 \tag{3.4b}$$

for arbitrary variations δx^k, $\delta x^k{}_{,\alpha}$. This latter formulation allows x^k and $x^k{}_{,\alpha}$ to be regarded as independent variables.

(ii) the energy function ϵ is objective, i.e.

$$\delta\epsilon = 0 \tag{3.5a}$$

when

$$\delta x^i = a^i{}_p \, x^p \tag{3.5b}$$

where $a^i{}_p$ is an arbitrary constant skew-symmetric tensor.

4. Basic Equations of an Elastic Surface

If we set

$$\tau_k^{\ \delta} = \gamma\left(\frac{\partial \epsilon}{\partial x^k_{;\Delta}} u^{\delta}_{;\Delta} + \frac{\partial \epsilon}{\partial x^k_{;\Delta\Sigma}} u^{\delta}_{;\Delta\Sigma}\right) \tag{4.1}$$

$$\bar{\mu}_k^{\ \delta\sigma} = \gamma \frac{\partial \epsilon}{\partial x^k_{;\Delta\Sigma}} u^{\delta}_{;\Delta} u^{\sigma}_{;\Sigma} \tag{4.2}$$

apply the principle of virtual work given by (3.4 b, c), and use Eqs (3.1), (3.2) and (3.3) as well as Green's theorem, we obtain that, for arbitrary and independent δx^i and $\delta x^i_{,\delta}$, the following must hold:

on c:

$$s_k = t_k^{\ \delta} \nu_\delta \ ; \quad m_k^{\ \delta} = \bar{\mu}_k^{\ \delta\sigma} \nu_\sigma \tag{4.3}$$

where $\underset{\sim}{\nu}$ is the unit normal surface vector to c.

on Γ:

$$t_{k,\delta}^{\ \delta} + \gamma f_k = 0 \tag{4.4a}$$

$$\bar{\mu}_{k\ ,\sigma}^{\ \delta\sigma} + t_k^{\ \delta} - \tau_k^{\ \delta} + \gamma \bar{\ell}_k^{\ \delta} = 0 \tag{4.4b}$$

Eqs (4.3) are the boundary conditions, Eqs (4.4) the equilibrium conditions.

We now apply the postulate of material indifference. We note that Eqs (3.5)a, b may be replaced, or using Eqs (3.2), (4.1) and (4.2), by the following equation:

$$\frac{\partial \epsilon}{\partial x^{[i}_{;\Delta}} x^{j]}_{;\Delta} + \frac{\partial \epsilon}{\partial x^{[i}_{;\Delta\Sigma}} x^{j]}_{;\Delta\Sigma} = 0 \tag{4.5}$$

We next consider Eq (4.5) as a set of 3 independent first order partial differential equations in the 15 independent variables $x^i_{;\Delta}$, $x^i_{;\Delta\Sigma}$. The set of 12 functionally independent solutions of Eq (4.5) is given by

$$\mathscr{S} = (L_{\Delta\Sigma},\ K_{\Delta\Sigma},\ C_{\Delta\Sigma\Gamma},) \tag{4.6}$$

where $\underset{\sim}{L}$, $\underset{\sim}{K}$ and $\underset{\sim}{C}$ are given by Eqs (2.4)-(2.6). Hence we have

$$\epsilon = \epsilon\,(\underset{\sim}{L},\ \underset{\sim}{K},\ \underset{\sim}{C}) \tag{4.7}$$

With the use of Eqs (2.4), (2.5), (2.6), (4.5), (4.7), the constitutive equations (4.1), (4.2) may be written:

$$\tau^{(\alpha\beta)} = 2\gamma\left[\frac{\partial \epsilon}{\partial L_{\Delta\Sigma}} u^{\alpha}_{;\Delta} u^{\beta}_{;\Sigma} + \frac{\partial \epsilon}{\partial C_{\Gamma\Delta\Sigma}} u^{(\alpha}_{;\Gamma} u^{\beta)}_{;\Delta\Sigma}\right] \tag{4.8}$$

$$\tau^{[\alpha\beta]} = 0$$

$$\tau^{\delta} = \gamma\, b_{\gamma\sigma} \frac{\partial \epsilon}{\partial C_{\Delta\Gamma\Sigma}} u^{\gamma}_{;\Gamma} u^{\sigma}_{;\Sigma} u^{\delta}_{;\Delta} = b_{\gamma\sigma} \bar{\lambda}^{\delta\gamma\sigma} \tag{4.9}$$

$$\bar{\lambda}^{\delta\sigma\gamma} = \gamma \frac{\partial \epsilon}{\partial C_{\Delta\Sigma\Gamma}} u^{\delta}_{;\Delta} u^{\sigma}_{;\Sigma} u^{\gamma}_{;\Gamma} \tag{4.10}$$

$$\bar{\mu}^{(\delta\sigma)} \doteq \gamma \frac{\partial \epsilon}{\partial K_{\Delta\Sigma}} u^{\delta}_{;\Delta} u^{\sigma}_{;\Sigma} \tag{4.11}$$

when we decompose $\underset{\sim}{\tau}$ and $\underset{\sim}{\bar{\mu}}$ into their surface representation i.e. we set

$$\tau^{k\delta} = \tau^{\alpha\delta} x^{k}_{,\alpha} + \tau^{\alpha} n^{k} \tag{4.12}$$

$$\bar{\mu}^{k\delta\sigma} = \bar{\lambda}^{\alpha\delta\sigma} x^{k}_{,\alpha} + \bar{\mu}^{\delta\sigma} n^{k} \tag{4.13}$$

Finally we decompose the equationsof equilibrium into their surface representation. Eq (4.14a) in such a representation yields the so-called stress equations of equilibrium:

$$t^{\alpha\delta}_{,\delta} - b^{\alpha}_{\delta} t^{\delta} + \gamma f^{\alpha} = 0 \tag{4.14a}$$

$$t^{\delta}_{,\delta} + b_{\alpha\beta} t^{\alpha\beta} + \gamma f = 0 \tag{4.14b}$$

Similarly Eq (4.14b) may be written with the additional use of Eq (4.9):

$$\bar{\lambda}^{\alpha\delta\sigma}_{,\sigma} + t^{\alpha\delta} - \tau^{\alpha\delta} - b^{\alpha}_{\sigma} \bar{\mu}^{\delta\sigma} + \gamma \bar{\ell}^{\alpha\delta} = 0 \tag{4.15}$$

$$\bar{\mu}^{\delta\sigma}_{,\sigma} + 2b_{\alpha\sigma} \bar{\lambda}^{[\alpha\delta]\sigma} + t^{\delta} + \gamma \bar{\ell}^{\delta} = 0 \tag{4.16}$$

If we take in order the skew-symmetric and the symmetric part of Eq (4.15) we obtain:

$$t^{[\alpha\delta]} + \bar{\lambda}^{[\alpha\delta]\sigma}_{,\sigma} - b^{[\alpha}_{\sigma} \bar{\mu}^{\delta]\sigma} + \gamma \bar{\ell}^{[\alpha\delta]} = 0 \tag{4.17}$$

$$t^{(\alpha\delta)} + \bar{\lambda}^{(\alpha\delta)\sigma}_{,\sigma} - b^{(\alpha}_{\sigma} \bar{\mu}^{\delta)\sigma} - \tau^{(\alpha\delta)} + \gamma \bar{\ell}^{(\alpha\delta)} = 0 \tag{4.18}$$

Equation (4.16) expresses the usual two equations of moment equilibrium. Equation (4.17) may be regarded as the form in this theory corresponding to the third equation of moment equilibrium. Equation (4.18) expresses the symmetric part of the surface stress tensor $t^{\alpha\beta}$ in terms of quantities previously defined or specified. The boundary conditions (4.3) may be written:

on c

$$s^{\alpha} = t^{\alpha\delta}\nu_{\delta} \quad , \quad s = t^{\alpha}\nu_{\alpha} \tag{4.19a}$$

$$m^{\alpha\delta} = \bar{\lambda}^{\alpha\delta\sigma}\nu_{\sigma} \quad , \quad m^{\alpha} = \bar{\mu}^{\alpha\sigma}\nu_{\sigma} \tag{4.19b}$$

5. Concluding Remarks

We have shown that it is possible to introduce a surface stress tensor $t^{\alpha\beta}$ which is not symmetric in general. Its skew symmetric part $t^{[\alpha\beta]}$ can be determined from the "third" equation of moment equilibrium. The surface couple stress tensor $\bar{\mu}^{\alpha\beta}$ is also not symmetric. Its symmetric part $\bar{\mu}^{(\alpha\beta)}$ is determined from the constitutive equation (4.11). We note also that Eqs (4.14), (4.16), (4.17) and (4.18) correspond in order to Eqs (5.19), (5.21)a, b and (5.22) of Ref [1]. We note that the theory is determinate.

If we write

$$x^k_{;\Delta\Sigma} = x^k_{/\Delta\Sigma} + x^k_{,\delta} \, u^\delta_{;\Delta\Sigma}$$

$$x^k_{/\Delta\Sigma} = x^k_{,\delta\sigma} \, u^\delta_{;\Delta} \, u^\sigma_{;\Sigma}$$

and assume a homogeneous deformation defined by $u^\delta_{;\Delta\Sigma} = 0$, we can recover the theory of Ref [1]. In this case we set $t^{\alpha\beta} = \tau^{\alpha\beta}$, $t^\alpha = \bar{t}^\alpha$ and although from a geometrical constraint $\underset{\sim}{C} = 0$, we keep $\underset{\sim}{\bar{\lambda}}$ as an energetically undetermined constraint.

For a theory which neglects second order gradients $u^\delta_{;\Delta\Sigma}$, we set $\underset{\sim}{C} = \underset{\sim}{\bar{\lambda}} = 0$. Then τ^α vanishes and the constitutive equation for $\tau^{(\alpha\beta)}$ involves only the derivative of $\underset{\sim}{\xi}$ with respect to $\underset{\sim}{L}$. The stress equilibrium equations (4.14) are unchanged, but in (4.16) - (4.18), we omit the terms involving $\underset{\sim}{\bar{\lambda}}$. The surface tensors $\underset{\sim}{t}$ and $\underset{\sim}{\bar{\mu}}$ are still non-symmetric but this restricted theory is still determinate. The reduction to a linear Kirchoff-Love theory is clear.

Finally we point out that the general theory of this paper can be brought into correspondence with the theory of simple force dipoles given by Balaban, Green and Naghdi [5] as recorded in equations (6.1) and (8.16) - (8.23) of this Reference. $N^{i\alpha}$ correspond to $t^{i\alpha}$, $G^{i\alpha}$ to $\tau^{i\alpha}$, $N^{3\alpha\beta}$ to $\bar{\mu}^{\alpha\beta}$, $G^{\beta\alpha\delta}$ $(= G^{\beta(\alpha\delta)} = N^{\beta\alpha\delta} = N^{\beta(\alpha\delta)})$ to $\bar{\lambda}^{\beta\alpha\delta}$. We point out also that the theory of this paper is the two-dimensional analogue of the results of Toupin [6].

References

1. H. Cohen and C. N. DeSilva, Nonlinear Theory of Elastic Surfaces, J. Math. Phys. 7, 246, 1966.

2. P. M. Naghdi, Personal Communication.

3. J. L. Ericksen, "Tensor Fields" in Handbuch der Physik, Vol. III/I (Springer Verlag, Berlin), 1960.

4. R. Courant and D. Hilbert, Methoden der Mathematische Physik (Springer Verlag, Berlin), 1931.

5. M. M. Balaban, A. E. Green and P. M. Naghdi, College of Engineering Report AM-66-8, ONR Contract Nonr-222(-69), Project NR-064-436. University of California (Berkeley) 1966.

6. R. A. Toupin, Theories of Elasticity with Couple Stress, Arch. Ratl. Mech. Anal. 17, 85, 1964.