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APPLICATION OF GROUP AND TRANSFORMATION THEORY TO THE SOLUTION OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Prepared under Contract No. NAS 8-20286 by
R. H. Martin, Jr., C. N. Driskell, Jr.,
and L. J. Gallaher

GEORGIA INSTITUTE OF TECHNOLOGY

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama

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ABSTRACT

This report is concerned with the application of transformation groups to the solution of systems of ordinary differential equations and, in particular, partial differential equations. These groups are Lie groups in the usual sense, but it is the transformation properties rather than the group structure that is used.

The principal tool used here, referred to as Lie's theorem, gives a method for finding an integrating factor for a system of ordinary differential equations when the appropriate invariance group or groups can be found. Lie's theorem is extended to partial differential equations by considering a partial differential equation as a continuously infinite system of coupled ordinary differential equations. For a system of ordinary differential equations the integrating factor is a matrix. For a partial differential equation the integrating factor is a continuously infinite matrix.

The proof of Lie's theorem and its use for partial differential equations depends on constructing an adequate theory of continuously infinite matrices; this is done here through the use of distributions or generalized functions.

TABLE OF CONTENTS

I. INTRODUCTION

This report is concerned with the application of transformation groups to the solution of systems of ordinary differential equations and in particular partial differential equations. These groups are Lie groups in the usual sense, but it is the transformation properties rather than the group structure that is used.

The principal theorem, referred to as Lie's theorem, gives a method for finding an integrating factor for a system of ordinary differential equations when the appropriate invariance group or groups can be found. Lie's theorem can be extended to partial differential equations by considering a partial differential equation as a continuously infinite system of coupled ordinary differential equations. For a system of ordinary differential equations the integrating factor is matrix. For a partial differential equation the integrating factor is then a continuously infinite matrix.

The proof of Lie's theorem and its use for partial differential equations depends on having an adequate theory of continuously infinite matrices; this is done here through the use of distributions or generalized functions.

Chapter II treats systems of ordinary differential equations. Lie's theorem is derived in such a way that it can be readily applied to the discrete approximation of a partial differential equation. Examples given are the discrete approximation to the heat flow and wave equations considered as initial value problems, and it is shown that the limiting form of the solutions obtained are those given by other more familiar techniques.

In Chapter III Lie's theorem is derived for and applied directly to partial differential equations without the necessity of using a discrete approximation. The heat flow equation is again used as an example.

While the examples given are linear equations, there is nothing in the method that restricts it to linear problems. Lie's theorem can in principle be applied to non-linear partial differential equations, but in practice it has been difficult to find a non-linear example.

Chapters II and III are oriented towards the engineer, physicist or chemist whose prime interest is application and the practical solution of problems. The proofs or derivations here would not be considered satisfactory by standards of the mathematics of this century.

In the Appendix, however, an effort was made to achieve rigor in the proofs given. It is here that the foundations of a theory of continuously infinite matrices based on distribution theory and generalized functions[29-33] is given.

It will be noted that the notation of Chapters II and III and of the Appendix are not always consistent with each other. Where differences occur it is usually due to an effort to maintain a notation consistent with that of the reference from which the material was obtained.

While we have not been able to find in the literature Lie's theorem for partial differential equations (or even for systems of ordinary differential equations), it seems unlikely that the work here is completely new.

Notation

The notation used in connection with matrices in Chapter II is as follows: A doubly indexed quantity will be called a matrix. If A_{ij} are the elements of a matrix, then the matrix is referred to as A. Singly indexed quantities will be called vectors so that B_i are the elements of the vector called B. The transpose of A and B will be denoted A^T and B^T respectively. The inverse of A is A^I and its elements written as A^I_{ij}.

The summation convention will be used so that any repeated index is understood to be summed unless stated otherwise. For example, if A and C are matrices, the product AC will be written as $A_{im}C_{mj}$. The index m is understood to be summed. The product A^TC is written as $A_{mi}C_{mj}$, etc. These sums run over the entire range for which the index is defined.

To shorten notation when partial derivatives are used, the comma notation will be used. That is $\frac{\partial\phi}{\partial t}$ (x,t) and $\frac{\partial\phi}{\partial x}$ (x,t) will be written as $\phi,_t$ and $\phi,_x$ respectively. If a quantity is a function of a set of indexed variables, for example

$$\Psi(y_1,y_2,y_3, \ldots) = \Psi(y),$$

then

its partial derivatives $\frac{\partial\Psi}{\partial y_k}$ will be written as $\Psi,_k$. For example

$$\frac{d}{dt} Q(y_1(t),\ y_2(t),\ \ldots,\ t) = Q,_t + Q,_i\ \dot{y}_i$$

where

$$\dot{y}_i = \frac{dy_i(t)}{dt} .$$

In the chapter dealing with the application of continuously infinite matrices to the solution of partial differential equations, the notation can become quite complex and under some circumstances ambiguous. Some of the conventions and notations used there will be described as follows.

The partial derivative will have its usual meaning. That is, if $\phi \equiv \phi(x,t,z)$ for example, then

$$\frac{\partial\phi}{\partial x} \equiv \frac{d\phi}{dx}\Bigg|_{\substack{t\,=\,\text{constant}\\ z\,=\,\text{constant}}} \quad \text{and} \quad \frac{\partial\phi}{\partial t} \equiv \frac{d\phi}{dt}\Bigg|_{\substack{x\,=\,\text{constant}\\ z\,=\,\text{constant}.}}$$

If z happens to be a function of x and t, that is $z \equiv z(x,t)$ then

$$\phi \equiv \phi(x,t,z(x,t)) \equiv \Psi(x,t)$$ and

$$\frac{\partial\Psi}{\partial x} \equiv \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial z}\,\frac{\partial z}{\partial x}\,(x,t),\ \frac{\partial\Psi}{\partial t} \equiv \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial z}\,\frac{\partial z}{\partial t}\,(x,t).$$

The comma notation will also be used for the partial derivatives, i.e.

$$\phi,_t \equiv \frac{\partial\phi}{\partial t},\ \phi,_x \equiv \frac{\partial\phi}{\partial x},$$ and

$$f,_j \equiv \frac{\partial}{\partial y_j}\,f(y_1,y_2,y_3\ .\ .\ .\ ).$$

Where this will cause no confusion, the prime and dot notation will also be used for partial or total derivatives with respect to x and t respectively. That is

$$\dot{y} \equiv \frac{\partial y}{\partial t}\,(x,t),\ \ddot{y} \equiv \frac{\partial^2 y}{\partial t^2},\quad y' \equiv \frac{\partial y}{\partial x}\,(x,t),\ y'' \equiv \frac{\partial^2 y}{\partial x^2},\ \text{etc.}$$

A functional notation will also be used. Parentheses will be used to indicate parameters of functions or distributions and square brackets indicate functional parameters. Thus $\phi\ (x)[y] \equiv \phi\ (x,z)$ with

$$z = \int_{-\infty}^{+\infty} f(\bar{x},y(\bar{x}),y',y''\ .\ .\ .\ )d\bar{x},$$

where f is some function of the indicated parameters. That is, ϕ can be regarded as a function (or distribution) in the variable x and as a functional in the quantity y.

The variational derivative* will indicate a derivative with respect to a functional parameter. That is

$$\frac{\delta\phi(x)[y]}{\delta y(s)} \equiv \frac{\partial\phi}{\partial z}(x,z)\Bigg|_{z=\int_{-\infty}^{+\infty} f(\bar{x},y(\bar{x}),y',\ldots)d\bar{x}} \cdot \frac{\delta}{\delta y(s)}\int_{-\infty}^{+\infty} f(\bar{x},y(\bar{x}),y',y'',\ldots)d\bar{x}$$

and

$$\frac{\delta}{\delta y(s)}\int_{-\infty}^{+\infty} f(\bar{x},y(\bar{x}),y',\ldots)d\bar{x} \equiv \int_{-\infty}^{+\infty}\frac{\partial}{\partial z_0} f(\bar{x},z_0,y',\ldots)\Bigg|_{z_0=y(\bar{x})} \cdot \frac{\delta y(\bar{x})}{\delta y(s)}\,d\bar{x}$$

$$+\int_{-\infty}^{+\infty}\frac{\partial}{\partial z_1} f(\bar{x},y(\bar{x}),z_1\ldots)\Bigg|_{z_1=y'(\bar{x})} \cdot \frac{\delta}{\delta y(s)}y'(\bar{x})d\bar{x}$$

$$+ \text{etc.}$$

and

$$\frac{\delta y(\bar{x})}{\delta y(s)} \equiv \delta(\bar{x}-s),\ \frac{\delta}{\delta y(s)}y'(\bar{x}) \equiv \frac{d\delta(z)}{dz}\Bigg|_{z=\bar{x}-s},$$

*Frechét derivative

$$\frac{\delta}{\delta y(s)} y''(\bar{x}) \equiv \frac{d^2}{dz^2}\delta(z)\Bigg|_{z=\bar{x}-s}, \qquad \text{etc.}$$

so that

$$\frac{\delta\phi(x)[y]}{\delta y(s)} = \frac{\partial\phi}{\partial z}(x,z)\Bigg|_{z=\int_{-\infty}^{+\infty} f d\bar{x}} \cdot \sum_{0\leq k}\left(\frac{-d}{ds}\right)^k \frac{\partial f}{\partial z_k}(s,z_0,z_1,z_2,\ \ldots)\Bigg|_{z_m=\frac{d^m y(s)}{ds^m}}$$

Here $\delta(x)$ is the Dirac delta distribution or generalized function. The parentheses designate either function or distribution parameters; no distinction between function or distribution parameters will be made. However, it is understood that in any integration associated with a matrix multiplication, one of the occurrences of the variable is a distribution and the other a fairly good function (in the sense of Lighthill [29]).

Distributions, the general theory of continuously infinite matrices, definitions, and theorems associated with these topics are given in the appendices. It should be noted that a functional parameter can also be a function (or distribution) in some variable. In that case, account must be taken in expressing the total and partial derivatives. Thus if

$$\Phi \equiv \Phi(t)[y] \equiv \Phi(t,z)\Bigg|_{z=\int_{-\infty}^{+\infty} f(x,y(x,t),y'(x,t),y''(x,t)\ \ldots)dx}$$

then

$$\frac{\partial\Phi}{\partial t} \equiv \frac{\partial}{\partial t}\Phi(t,z)\Bigg|_{z=\int_{-\infty}^{+\infty} f\,dx}$$

and

$$\frac{d\Phi}{dt} \equiv \frac{\partial\Phi}{\partial t} + \frac{\partial}{\partial z}\Phi(t,z)\Bigg|_{z=\int_{-\infty}^{+\infty} f\,dx} \cdot \frac{d}{dt}\int_{-\infty}^{+\infty} f(x,y(x,t),y'(x,t),\ .\ .\ .\ )dx$$

$$= \frac{\partial\Phi}{\partial t} + \int_{-\infty}^{+\infty} \frac{\delta\ \Phi(t)[y]}{\delta y\ (x,t)}\ \frac{\partial y}{\partial t}\ (x,t)dx$$

II. LIE'S THEOREM FOR A FINITE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Lie developed the theory of one parameter continuous transformation groups for the purpose of studying ordinary differential equations [1]. This technique has become a standard tool for the solution of first order ordinary differential equations and is derived and discussed in most text books on the subject [2-4]. For some reason, not too apparent, the extension of Lie's theorem to systems of first order differential equations seems to have been neglected. This extension is made in the first report on this contract [26] where Lie's theorem is proved for systems of equations and examples given. The proofs will be repeated here in this report in a slightly altered form, one that is more easily extended to partial differential equations and more examples given. However, we will not repeat here many of the definitions and elementary concepts of group and transformation theory discussed in the first report but will refer the reader to this report [26] or to the standard text books on these subjects.

A. Lie's Theorem for Systems of Ordinary Differential Equations

Consider the system of M (total) differential equations in M + 1 variables

$$P_{jk}(y_1, \ldots y_M, t)\, dy_k + Q_j(y_1, y_2, \ldots y_M, t)dt = 0, \qquad \text{(II-1)}$$

$j = 1, 2, \ldots M$. The summation convention for repeated indexes is used here. Let $\phi_j\ (y_1, y_2, \ldots y_M, t) = c_i$ (constants) $j = 1, 2, \ldots M$, be a family of solutions to II-1. That is

$$\phi_{j,k} \equiv \frac{\partial \phi_j}{\partial y_k} = \lambda_{ji} P_{ik} \qquad \text{(II-2a)}$$

$$\phi_{j,t} = \frac{\partial \phi_j}{\partial t} = \lambda_{ji} Q_i \qquad \text{(II-2b)}$$

where the λ_{ji} may be functions of the y and t but are independent of the index k. λ is called an integrating factor and is an M by M matrix. Thus if an integration factor exists that satisfies II-2, each ϕ_j must satisfy the partial differential equation

$$\phi_{j,k} P^I_{ki} Q_i - \phi_{j,t} = 0 \ , \qquad \text{(II-3)}$$

where P^I_{ki} is the k,i element of the inverse of the matrix P provided P^I exists.

Assume that the $\phi_j = c_j$ are invariant as a family under the groups U_n, n = 1, 2, . . . M,

$$U_n = \alpha_{nk}(y_1, y_2, \ldots y_M, t) \frac{\partial}{\partial y_k} + \beta_n(y_1, y_2, \ldots y_M, t) \frac{\partial}{\partial t}$$

that is

$$U_n \phi_j = g_{nj}(\phi) \qquad \text{(II-4)}$$

for n = 1, 2, . . . M, and j = 1, 2, . . . M, where the g_{nj} are some functions of the ϕ's. Introduce Φ_s defined as

$$\Phi_s = \int g^I_{is} (\phi) \, d\phi_i \qquad \text{(II-5)}$$

so that $\Phi_s = C_s$ is identical with the family, $\phi_j = c_j$. The notation here is that g^I_{is} is the i, s component of the inverse of the matrix $g(\phi)$, assuming that this inverse exists. The right hand side of II-5 is meant to indicate a line integral in ϕ space, i. e.

$$\int g^I_{is}\, d\phi_i \equiv \int_{R_1}^{\phi_1} g^I_{1s}(w,R_2,R_3,\ . \ . \ .\ )dw$$

$$+ \int_{R_2}^{\phi_2} g^I_{2s}\ (\phi_1,w,R_3,R_4,\ . \ . \ .\ )dw + \int_{R_3}^{\phi_3} g^I_{3s}\ (\phi_1,\phi_2,w,R_4,\ . \ . \ .\ )dw$$

$$+ \text{ etc.},$$

so that

$$\frac{\partial \Phi_s}{\partial \phi_i} = g^I_{is} \ .$$

Here the R_i are arbitrary constants.

Then

$$U_m \Phi_s = U_m \phi_i \frac{\partial \Phi_s}{\partial \phi_i} = g_{mi}\ g^I_{is} = \delta_{ms}\ , \qquad \text{(II-6)}$$

where $\delta_{ms} = 1$ if m = s or 0 if m $\neq$ s. It is also seen that the Φ_s obey the same partial differential equation II-4 as do the ϕ_i, that is

$$\Phi_{s,k} P^I_{ki} Q_i - \Phi_{s,t} = 0\ . \qquad \text{(II-7)}$$

Equation II-6 and II-7 can be combined to solve for $\Phi_{s,k}$ and $\Phi_{s,t}$ in terms P, Q, α and β giving

$$\Phi_{s,k} \equiv \frac{\partial \Phi_s}{\partial y_k} = (P\alpha^T + Q\beta^T)^I{}_{si} P_{ik} \tag{II-8a}$$

and

$$\Phi_{s,t} \equiv \frac{\partial \Phi_s}{\partial t} = (P\alpha^T + Q\beta^T)^I{}_{si} Q_i \tag{II-8b}$$

Here the notation $(P\alpha^T + Q\beta^T)^I{}_{si}$ refers to the s, i component of the inverse of the sum of the matrix products P with the transpose of α and Q with the transpose of β, provided this inverse exists. (Note that $Q\beta^T$ is a square matrix).

From equation II-7 it is seen that under the assumptions made, an integration factor or matrix exists of the form $(P\alpha^T + Q\beta^T)^I$, and matrix multiplication with equation II-1 gives a perfect differential in the sense that

$$d\Phi_s = \frac{\partial \Phi_s}{\partial y_k} dy_k + \frac{\partial \Phi_s}{\partial t} dt = (P\alpha^T + Q\beta^T)^I{}_{sj} P_{jk} dy_k + (P\alpha^T + Q\beta^T)^I{}_{sj} Q_j dt.$$

The function Φ can be found by a line integral in the y, t space along some convenient path, represented by

$$\Phi_s = \int d\Phi_s = \int (P\alpha^T + Q\beta^T)^I{}_{sj} P_{jk} dy_k + \int (P\alpha^T + Q\beta^T)^I{}_{sj} Q_j dt$$

$$= K_s \qquad s = 1, 2, \ldots M \tag{II-9 }$$

where the K_s are constants. The equations $\Phi_s(y_1, y_2, \ldots, t) = K_s$ represent then the general solution to the set of equations II-1.

There are two points to be noted in connection with this result. The first is that instead of the matrix equation $P \frac{dy}{dt} + Q = 0$, it would be just as general to have used the equation $\frac{dy}{dt} + \bar{Q} = 0$, where $\bar{Q} = P^I Q$ since a necessary and sufficient condition for the existence of a solution to the first is that P^I exist.

The second point to note is that there is no need to consider transformations of the variable t. That is the transformation

$$t \leftarrow t + \epsilon\beta_n$$

is exactly the same transformation (as far as the equation $dy/dt + Q(y,t) = 0$ is concerned) as the transformation

$$y_k \leftarrow y_k (t - \epsilon\beta_n)$$ since

$$y_k(t - \epsilon\beta_n) \approx y_k(t) - \epsilon\frac{dy}{dt} \beta_n = y_k + \epsilon Q_k \beta_n \ .$$

Thus the transformations

$$U_n = \alpha_{nk} \frac{\partial}{\partial y_k} + \beta_n \frac{\partial}{\partial t}$$

are identical to the transformations

$$U_n = (\alpha_{nk} + Q_k \beta_n)\frac{\partial}{\partial y_k}$$

with respect to the equation $dy/dt + Q = 0$.

The remainder of this chapter will be concerned with the equation $dy/dt + Q = 0$ and transformation $y \leftarrow y + \epsilon\alpha$ only.

A formal statement of the theorem used in this report for the solution of differential equations, which we will refer to as Lie's Theorem, then, is as follows:

"If the differential equation $\frac{dy(t)}{dt} + Q(y(t), t) = 0$, where y and Q are vectors, and t a scalar, is invariant with respect to the set of transformations specified by

$$U_n \equiv \alpha_{ns}(y,t)\frac{\partial}{\partial y_s}$$

where α is a square matrix and $\frac{\partial}{\partial y}$ a vector operator, then provided α^{TI} exist, the general solution to the differential equation is

$$\int \alpha^{TI}(dy + Qdt) = K$$

where the integral is understood as a line integral in y, t space along any convenient path, and K is an arbitrary vector constant."

The paragraphs in this chapter leading up to a statement of this theorem can, in fact, be considered a proof of the theorem, but an alternate form of the proof will now be given.

The differential equation to be integrated is

$$\frac{dy}{dt} + Q(y,t) = 0 \ . \qquad \text{(II-10)}$$

If this equation is to be invariant with respect to the transformation specified by

$$U_n \equiv \alpha_{nk} \frac{\partial}{\partial y_k} \qquad \text{(II-11)}$$

for all n it must be invariant with respect to the infinitesimal transformations

$$\begin{aligned} y_k &\leftarrow y_k + \epsilon\alpha_{nk}(y,t) \\ Q_k(y,t) &\leftarrow Q_k(y_j + \epsilon\alpha_{nj},t) \end{aligned} \qquad \text{(II-12)}$$

to first order in ϵ for all k and n. Here ϵ is an infinitesimal parameter. Making this transformation gives

$$\frac{dy_k}{dt} + Q_k + \epsilon\left\{\frac{d}{dt}\,\alpha_{nk} + \alpha_{nm}\,Q_{k,m}\right\} + \epsilon^2(\,.\;.\;.\,) = 0 \qquad \text{(II-13)}$$

This equation is invariant up to first order in ϵ if an only if

$$\frac{d}{dt}\,\alpha_{nk}(y(t),t) = -\alpha_{nm}\,Q_{k,m} \qquad \text{(II-14)}$$

Letting $\alpha^T_{kn} = \alpha_{nk}$ and α^{TI} be the inverse of α^T, II-14 gives

$$\alpha^T_{kn}\,\frac{d}{dt}\,\alpha^{TI}_{ns} = Q_{k,s} \qquad \text{(II-15)}$$

The left side of this equation is the right Volterra derivative* of α^{TI}.

It will now be shown that every solution to II-15 is an integrating factor of II-10. Let λ_{km} be an integrating factor of II-10 for each k; that is

*MacDuffie, C. C., The Theory of Matrices, Chelsea Publishing Co., New York, (1946) page 103.

$$d\phi_k = \lambda_{km}\, dy_m + \lambda_{km}\, Q_m\, dt$$

where $d\phi_k$ is a perfect differential for each k.

Then

$$\lambda_{km} = \phi_{k,m} \qquad \text{and}$$

$$\lambda_{km}\, Q_m = \phi_{k,t} \; .$$

Thus ϕ_k is a solution of the partial differential equation

$$\phi_{k,t} - Q_m\, \phi_{k,m} = 0 \; . \tag{II-16}$$

Differentiating (II-16) with respect to y_s gives the partial differential equation for λ,

$$\lambda_{ks,t} - Q_m\, \lambda_{ks,m} = \lambda_{km}\, Q_{m,s}$$

or

$$\lambda^{I}_{kn} \left[\frac{\partial}{\partial t} - Q_m \frac{\partial}{\partial y_m} \right] \lambda_{ns} = Q_{k,s} \; .$$

The operator in square brackets is the total derivative with respect to t, so that λ satisfies the equation

$$\lambda^{I}_{kn} \frac{d}{dt} \lambda_{ns} = Q_{k,s} \; . \tag{II-17}$$

Now if C is an invertible constant matrix, then

$$\lambda^{I}_{kn}\, C^{I}_{np} \frac{d}{dt} (C_{pq}\, \lambda_{qs}) = Q_{k,s} \qquad \text{and}$$

$$C_{pq} \lambda_{qs} = (C_{pq} \phi_q)_{,s} ,$$

so that $C\lambda$ is both an integrating factor and a solution to II-17 for every invertible constant C.

Suppose two invertible matrices R(t) and S(t) have equal Volterra derivatives, that is

$$R^I \frac{d}{dt} R = S^I \frac{d}{dt} S.$$

Then

$$R^I \frac{d}{dt} R + \left(\frac{d}{dt} S^I\right) S = 0,$$

provided $\frac{d}{dt} S^I$ exists. Multiplying on the left by R and on the right by S^I gives

$$\frac{dR}{dt} S^I + \frac{RdS^I}{dt} = 0 \qquad \text{or}$$

$$\frac{d}{dt}\left(RS^I\right) = 0.$$

Then

$$RS^I = C$$

where C is some invertible constant matrix. That is, if two matrices have the same Volterra derivative, they are proportional to each other through some invertible constant matrix.

In this way it is shown that every solution of II-17 (or II-15a) is proportional to every other solution through some invertible constant matrix. Thus if some solution is an integrating factor, every solution is an integrating factor. The matrix $\alpha^{TI}(y(t),t)$ is then an integrating factor of the matrix equation

$$\frac{dy(t)}{dt} + Q(y,t) = 0 .$$

Then since

$$d\phi_m = \frac{\partial\phi_m}{\partial y_k} dy_k + \frac{\partial\phi_m}{\partial t} dt = \alpha^{TI}_{mj}(dy_j + Q_j\, dt) , \qquad \text{(II-18)}$$

the line integral in y, t space

$$\phi_m = \int \alpha^{TI}_{mj}(dy_j + Q_j\, dt) = K_m \qquad \text{(II-19)}$$

is a solution to the system of differential equations II-10 for each constant vector K. That is, the K_m are the constants of integration.

This completes the proof of Lie's theorem for systems of differential equations, the basic theorem on which the methods and results of this report are based.

We note at this point that the Lie's theorem is proved here for a finite system of equations. The extension to countably infinite systems depends on an adequate theory of countably infinite matrices. The proof would be unchanged for a system of countably infinite matrices that form an algebra, that is, a system which is closed under addition and multiplication.* The extension to

*C. C. MacDuffie (op. cit.) page 106.

continuously infinite matrices which forms the basis of the application to partial differential equations will be discussed in a later chapter.

B. Examples

1. The One-Dimensional Heat Flow Equation

As an example of the use of Lie's theorem to solve systems of ordinary differential equations, the method will be applied to the discrete form of the one-dimensional heat flow equation.

Consider the partial differential equation

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial^2}{\partial x^2} y(x,t) = 0 \ , \qquad \text{(II-20)}$$

with initial conditions

$$y(x,0) = y^{\circ}(x)$$

and the periodic boundary conditions

$$y(x + 2L, t) = y(x,t) \ .$$

Using the lowest order difference approximation for the derivative with respect to x gives the system of equations

$$\frac{d}{dt} y_n(t) - \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = 0 \qquad n = 0, \pm 1, \pm 2 \ldots N \qquad \text{(II-21)}$$

Here $y_n(t) = y(nh,t)$ where h is the discretization interval ($Nh = L$). There are only 2N independent equations since $y_n(t) = y_{n+2N}(t)$ by virtue of the periodic boundary conditions. The initial conditions are

$$y_n(0) = y_n^\circ .$$

Considered as a system of coupled differential equations, II-21 is of the form

$$\frac{d}{dt} y_n + Q_n(y,t) = 0$$

where

$$Q_n(y,t) = -(y_{n+1} - 2y_n + y_{n-1})/h^2 .$$

Equation II-21 is invariant with respect to the transformation

$$y_n(t) \leftarrow y_n(t) + \epsilon y_{j+n}(t)$$

for n = 0, ±1, . . N and j = 0, ±1, ±2, . . N. The operators characterizing the set of transformation are

$$U_j = y_{j+s} \frac{\partial}{\partial y_s} \qquad j = 0, \pm 1, \ldots N .$$

This is of the form

$$U_j = \alpha_{js} \frac{\partial}{\partial y_s} ,$$ with

$$\alpha_{js} = y_{j+s} = \alpha_{sj}^T .$$

α is a square 2N by 2N matrix.

The matrix α, whose elements are y_{j+s}, is called an anticirculant matrix. Much is known about anticirculants. In particular, the inverse of an anticirculant is an anticirculant, if it exists. It is straightforward to show that if q_{m+k} are the elements of the inverse of the matrix whose elements are y_{m+j} $(=\alpha^T_{jm} = \alpha_{mj})$, so that

$$\alpha^{TI}_{mk}\,\alpha^{T}_{jm} = q_{m+k}\,y_{m+j} = \delta_{kj}\ ,$$

then

$$q_k = \frac{1}{2N}\,\frac{e^{\pi iks/N}}{e^{\pi ijs/N}y_j}$$

(Note here that the summation convention is used on all repeated indices, the sums running from -N+1 to N, and q_k is periodic with period 2N.) This inverse can be obtained in a variety of ways. It can be obtained from the theory of finite Fourier expansions. Also, by showing that powers of the Nth roots of unity form a unitary matrix that diagonalizes every N by N anticirculant, this inverse can be obtained from the reciprocal of the eigenvalues of y_{m+j}.

With this inverse, then, the integration factor is

$$\alpha^{TI}_{mk} = q_{m+k} = \frac{1}{2N}\,\frac{e^{\pi i(k+m)s/N}}{e^{\pi ijs/N}y_j}\ .$$

There exist, then, perfect differentials, $d\Phi_m$

$$d\Phi_m = \Phi_{m,k}\,dy_k + \Phi_{m,t}\,dt$$

such that

$$\frac{\partial\Phi_m}{\partial y_k} = \alpha_{mk}^{TI} \ , \quad \frac{\partial\Phi_m}{\partial t} = \alpha_{mk}^{TI} Q_k \quad ,$$

$$\Phi_{m,t} = -\alpha_{mk}^{TI}(y_{k+1} - 2y_k + y_{k-1})/h^2 = -(\delta_{m1} - 2\delta_{m0} + \delta_{m-1})/h^2 \ .$$

The functions $\Phi_m(y_{-N+1}, y_{-N+2}, \ldots y_0, \ldots y_N, t)$ can then be calculated by integrating $d\Phi_m$ along some convenient path in the 2N + 1 dimensional space of the y's and t. A convenient path of integration is as follows:

(1) $y_k = \delta_{k0}$ along t from t = 0 to t

(2) $t = t$, $y_k = 0 (k \neq 0)$ along y_o from $y_o = 1$ to $y_o(t)$

(3) $t = t$, $y_k = 0(k \neq 0,1)$ $y_o = y_o(t)$, along y_1 from $y_1 = 0$ to $y_1(t)$

(4) $t = t$, $y_k = 0(k \neq 0,1,-1)$ $y_o = y_o(t)$, $y_1 = y_1(t)$, along y_{-1} from $y_{-1} = 0$ to $y_{-1}(t)$

etc. Written out with the summation signs, this is

$$-\int_0^t dt(\delta_{m1} - 2\delta_{mo} + \delta_{m-1})/h^2$$

$$+\frac{1}{2N} \sum_{-N<s\leq N} \left\{ \int_1^{y_o} dw \frac{e^{\pi i(0+m)s/N}}{\sum_{0<j<1} e^{\pi ijs/N} y_j + e^{\pi i0s/N} w} \right.$$

$$+ \int_0^{y_1} dw \frac{e^{\pi i(1+m)s/N}}{\sum_{-1<j<1} e^{\pi ijs/N} y_j + e^{\pi i1s/N} w}$$

$$+ \int_0^{y_{-1}} dw \quad \frac{e^{\pi i(-1+m)s/N}}{\sum_{-1<j<2} e^{\pi ijs/N} y_j + e^{\pi i(-1)s/N} w}$$

$+ \ . \ . \ .$

$$+ \int_0^{y_N} dw \quad \frac{e^{\pi i(N+m)s/N}}{\sum_{-N<j<N} e^{\pi ijs/N} y_j + e^{\pi iNs/N} w} \Bigg\} \ .$$

This particular path of integration gives

$$\Phi_m(y,t) = \frac{e}{2N}^{\pi ims/N} \ln(e^{\pi ijs/N} y_j) - \frac{t}{h^2} (\delta_{m1} - 2\delta_{m0} + \delta_{m-1}) \ . \quad \text{(II-22)}$$

That this is in fact $\Phi_m(y,t)$ can be readily verified by calculating that

$$\Phi_{m,k} = \alpha^{TI}_{km} \ , \qquad \text{and}$$

$$\Phi_{m,t} = -(\delta_{m1} - 2\delta_{m0} + \delta_{m-1})/h^2 \ .$$

A completely general solution to equation II-21 is given by $\Phi_m = K_m$ where the K_m are arbitrary constants.

While this solution does not look particularly useful, it can be solved explicitly for the y_j by introducing new integration constants M_s where $\ln M_s = e^{-\pi ims/N} K_m$.

Then taking (discrete) Fourier transforms of both sides of

$$\frac{e}{2N}^{\pi ims/N} \ln\left(e^{\pi ijs/N} y_j \right) - \frac{t}{h^2} (\delta_{m1} - 2\delta_{m0} + \delta_{m-1}) = K_m$$

gives

$$\ln(e^{\pi ijs/N} y_j) + 4t \sin^2(\pi s/2N)/h^2 = \ln M_s \ , \qquad \text{or}$$

$$y_j(t) = \frac{e}{2N}^{(-\pi ijs/N - 4t \sin^2(\pi s/2N)/h^2)} M_s \ .$$

The constants M_s can then be related to initial conditions by noting that at $t = 0$

$$y_j(0) = \frac{1}{2N} e^{-\pi ijs/N} M_s \qquad \text{or}$$

$$M_s = e^{\pi ims/N} y^{\circ}_m \ .$$

In terms of the initial conditions then, the solution to II-21 is

$$y_n(t) = \frac{e}{2N}^{-i\pi(n-m)s/N} e^{-4t \sin^2(\pi s/2N)/h^2} y^{\circ}_m \qquad \text{(II-23)}$$

(Note the sum over both the repeated indices m and s.)

While this solution to the one-dimensional heat flow equation may not look familiar, by passing to the limits $h \to 0$, $N \to \infty$ with $Nh = L$ it can be seen that this is the usual solution for the initial value problem.

Introducing the notation

$$x_n \equiv nh,$$

$$x'_m \equiv mh$$

$$\Delta x' \equiv h,$$

and writing out the summation signs explicitly,

$$y(x,t) = \lim_{\substack{N\to\infty \\ h\to 0 \\ (Nh=L)}} \sum_{-N\leq m\leq N} \Delta x' \sum_{-N\leq s\leq N} \frac{e^{-2\pi i(x_n - x'_m)s/2L}}{2Nh} e^{-4t \sin^2(\pi s/2N)/h^2} y^\circ_m$$

$$= \int_{-L}^{+L} dx' \frac{1}{2L} \sum_{-\infty\leq s\leq\infty} e^{-2\pi i(x - x')s/2L} e^{-4t(\pi s/2L)^2} y(x',0) .$$

If the limit $L \to \infty$ is now taken one obtains the solution to the heat flow equation valid over the entire real axis for the initial value problem. Intro- the notation

$$p_s \equiv s/2L , \Delta p \equiv 1/2L$$

gives

$$\lim_{\substack{L\to\infty \\ (\Delta p\to 0)}} \int_{-L}^{+L} dx' \sum_{\infty\leq s\leq -\infty} \Delta p\ e^{-2\pi i(x-x')p_s - 4t(\pi p_s)^2} y(x',0)$$

$$= \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dp\ e^{-2\pi i(x-x')p - (2\pi p)^2 t} y(x',0).$$

This is the usual Fourier transform solution for the infinite interval. The integration over p can be carried out and gives

$$y(x,t) = \int_{-\infty}^{+\infty} dx' \frac{e^{-(x-x')^2/4\sqrt{t}}}{2\sqrt{\pi t}} y(x',0). \qquad \text{(II-24)}$$

This is the standard solution to the initial value problem for the infinite interval.

1.1 Discussion

The above technique for using Lie's theorem to solve the heat flow equation is quite complicated and gives well-known solutions that are much more easily obtained in other ways. It is used here only to illustrate this method. The equation II-20 is linear, but Lie's theorem can be applied to the non-linear problems. The technique of discretization of the partial differential equation, followed by the application of Lie's theorem to a finite system of ordinary differential equations, followed in turn by taking the limit back to the continuous system, can be applied to other partial differential equations but is an extremely awkward way of proceeding. A more desirable method would be to obtain a form of Lie's theorem applicable directly to partial differential equations without introducing the discrete approximation. This subject will be taken up in later chapters of this report.

2. The Wave Equation

The second example given here will be the application to a second order partial differential equation, the wave equation

$$\frac{\partial^2}{\partial t^2} y(x,t) - \frac{\partial^2}{\partial x^2} y(x,t) = 0 \quad , \qquad \text{(II-25)}$$

with initial values $y(x,0) = y^\circ(x)$ and $\frac{\partial}{\partial t} y(x,t)\big|_{t=0} = \dot{y}^\circ(x)$.

Since Lie's theorem is applicable to first order differential equations, it will be necessary to reduce II-25 to a pair of first order differential equations. This can be done by introducing two new variables $\dot{y}$ and y' defined as $\dot{y} \equiv \frac{\partial y}{\partial t}(x,t)$, $y' \equiv \frac{\partial y}{\partial x}(x,t)$. Then the single second order partial differential equation II-25 can be written as a pair of first order coupled differential equations

$$\frac{\partial}{\partial t}\dot{y} - \frac{\partial}{\partial x}y' = 0 \qquad \text{(II-26a)}$$

$$\frac{\partial}{\partial t}y' - \frac{\partial}{\partial x}\dot{y} = 0 \; . \qquad \text{(II-26b)}$$

This pair of coupled equations can be reduced to a pair of uncoupled equations defining

$$u \equiv \dot{y} + y'$$

$$v \equiv \dot{y} - y' \quad ,$$

which satisfy the equations

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \qquad \text{(II-27a)}$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \; . \qquad \text{(II-27b)}$$

The solution of II-25 is related to the solution of II-27 by

$$y(x,t) = y(x,0) + \frac{1}{2}\int_0^t \{u(x,\bar{t}) + v(x,\bar{t})\}d\bar{t} \; .$$

Equations II-27 represent a pair of uncoupled first order partial differential equations equivalent to II-25. These can be solved separately and the equation for u only will be solved here since that for v can be obtained by reversing the sign of t in the solution for u.

The discrete version of II-27a is

$$\frac{d}{dt} u_n - \frac{u_{n+1} - u_{n-1}}{2h} = 0 \ . \qquad \text{(II-28)}$$

If the periodic boundary conditions $u_n = y_{n+2N}$ are used, this equation is invariant with respect to the set of infinitesimal transformations

$$u_n \leftarrow u_n + \epsilon u_{n+k} \qquad k = 0, \pm1, \pm2 \ . \ . \ . \ N \ ,$$

characterized by the operator

$$U_k = u_{k+n} \frac{\partial}{\partial y_n} \ .$$

Equation II-28 is in the form

$$\frac{du_n}{dt} + Q_n = 0 \ ,$$

where

$$Q_n = -(u_{n+1} - u_{n-1})/2h \ ,$$

and U is in the form

$$U_k = \alpha_{kn} \frac{\partial}{\partial u_n} \ ,$$

where

$$\alpha_{kn} = u_{k+n} \ .$$

Since $\alpha = \alpha^T$ and $\alpha^I = \alpha^{TI}$, the T superscript will be dropped from α.

The matrix α, whose elements are u_{k+n}, is called an anticirculant matrix. The inverse of an anticirculant is an anticirculant, if it exists. It is straightforward to show that if q_{j+k} are the elements of the inverse of the matrix whose elements are u_{k+n} so that

$$q_{j+k}\, u_{k+n} = \delta_{jn} \ , \qquad \text{then}$$

$$q_k = \frac{1}{2N} \frac{e^{\pi iks/N}}{e^{\pi ijs/N} u_j} \ , \qquad \text{and}$$

$$\alpha^I{}_{km} = q_{k+m} = \frac{1}{2N} \frac{e^{\pi i(k+m)s/N}}{e^{\pi ijs/N} u_j} \ .$$

(Note the summation over both repeated indexes s and j, and that q is periodic with period 2N.) There exists then a function

$$\Phi_m(u_{-N+1}, \ . \ . \ u_{-1}, u_0, u_1, \ . \ . \ . \ u_N, t) \qquad \text{such that}$$

$$\frac{\partial \Phi_m}{\partial u_k} = q_{k+m} \qquad \text{and}$$

$$\frac{\partial \Phi_m}{\partial t} = -q_{m+n}(u_{n+1} - u_{n-1})/2h = -(\delta_{m1} - \delta_{m-1})/2h \ .$$

Φ_m can be obtained from a line integral in the 2N + 1 dimensional space of u and t. A convenient path of integration is the same used above for the heat flow problem, (page 21). This can be written

$$\sum_{0\le k\le N} \int_{\delta_{ko}}^{u_k} dw \left\{ \frac{1}{2N} \sum_{-N<s\le N} \frac{e^{\pi i(k+m)s/N}}{\sum_{-k<j<k} e^{\pi ijs/N} u_j + e^{\pi iks/N} w} \right\}$$

$$+ \sum_{-N<k\le -1} \int_0^{u_k} dw \left\{ \frac{1}{2N} \sum_{-N<s\le N} \frac{e^{-\pi i(k+m)s/N}}{\sum_{k<j<-k+1} e^{\pi ijs/N} u_j + e^{\pi iks/N} w} \right\}$$

$$+ \int_0^t dt(\delta_{m1} - \delta_{m-1})/2h \; .$$

Integrating along this path gives

$$\Phi_m(u,t) = \frac{e^{\pi ims/N}}{2N} \quad \ln(e^{\pi ijs/N} \; u_j) - t(\delta_{m1} - \delta_{m-1})/2h \; .$$

It is readily verified that this Φ_m gives the correct $\partial\Phi_m/\partial u_k$ and $\partial\Phi_m/\partial t$. The relations

$$\Phi_m(u,t) = K_m$$

give the general solutions to equations II-28, where the K_m are the 2N arbitrary constants. Introducing new constants M_s such that

$$\ln M_s = e^{-\pi ims/N} \; K_m \; ,$$

one can solve for u_j as

$$u_j(t) = \frac{e}{2N}^{-(\pi ijs/N + it\sin(\pi s/N)/h)} M_s \; .$$

The M_s are related to the initial values of $u(u(x,t) = \frac{\partial}{\partial t} y(x,t) + \frac{\partial}{\partial x} y(x,t))$ by $M_s = e^{\pi irs/N} \; u^\circ_r$, where $u^\circ_j = u_j(0)$ is the value of u at t = 0. The M_s are the discrete Fourier transforms of the initial u_j. In terms of the u_j° then

$$u_j(t) = \frac{e^{-\pi i(j-k)s/N}}{2N} e^{-it\sin(\pi s/N)/h} u_k^\circ \ .$$

Taking the limit as $h \to 0$, $N \to \infty$, with $hN = L$ gives the continuous solution to II-28 for periodic boundary conditions with period 2L. Introducing the notation

$$x_j \equiv jh$$

$$x_k' \equiv kh$$

$$\Delta x' \equiv h \quad ,$$

and writing in the summations explicitly, we have

$$u(x,t) = \lim_{\substack{h \to 0 \\ N \to \infty \\ (Nh=L)}} \sum_{-N<s\leq N} \sum_{-N<k\leq N} \Delta x' \frac{e^{-\pi i(x_j - x_k')s/hN}}{2hN} e^{-it\sin(\pi hs/hN)/h} u_k^\circ$$

$$= \sum_{-\infty<s\leq\infty} \frac{1}{2L} \int_{-L}^{+L} dx' \, e^{\pi i(x' - x+t)s/L} u^\circ(x') \ .$$

Taking the limit as $L \to \infty$ with

$$p \equiv s/2L, \ \Delta p \equiv 1/2L \ ,$$ gives

$$u(x,t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx' \, e^{2\pi i(x' - x+t)p} u^\circ(x')$$

$$= \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx' \, e^{2\pi i x' p} \, u^{\circ}(x' + x - t)$$

$$= u^{\circ}(x-t) \; .$$

The solution for $v(s,t)$ is similar to the solution for u, except with the sign of t reversed, and can be worked out to give $v(x,t) = v^{\circ}(x+t)$. Since

$$u(x,t) = \dot{y}(x,t) + y'(x,t)$$

$$v(x,t) = \dot{y}(x,t) - y'(x,t)$$

and

$$y(x,t) = y(x,0) + \int_0^t \tfrac{1}{2} \{u(x,T) + v(x,T)\} dT \; ,$$

we have

$$y(x,t) = y(x,0) + \int_0^t \tfrac{1}{2} \{\dot{y}(x+T,0) + \dot{y}(x-T,0) + y'(x+T,0) - y'(x-T,0)\} dT$$

$$= \tfrac{1}{2} \{y(x+t,0) + y(x-t,0) + \int_{x-t}^{x+t} \dot{y}(T,0) dT\} \; .$$

This is the usual form of d' Alembert's solution to the wave equation.

Again this is a long and involved way of finding a well known solution that is much more easily obtained by other methods. The purpose here is to illustrate the method and principles involved in applying Lie's theorem.

In treating second order differential equations by Lie's method it is necessary to first reduce the problem to a pair of first order equations. If these two equations can then be uncoupled as was done above, the mechanics of the solution become much simpler, but the method still applies even if the equations remain coupled.

III. LIE'S THEOREM FOR PARTIAL DIFFERENTIAL EQUATIONS

This chapter deals with the extension of Lie's theorem directly to partial differential equations without the need for the discretization introduced in Chapter II. The extension of theorems for finite matrices to continuously infinite matrices is given in the appendix.

A. Lie's Theorem

A statement of Lie's theorem for use with partial differential equations is as follows:

"If the partial differential equation

$$\frac{\partial y}{\partial t}(x,t) + Q(x,t)[y] = 0$$

is invariant with respect to the transformations

$$y(x,t) \leftarrow y(x,t) + \epsilon\alpha(\bar{x},x,t)[y]$$

for all relevant x, $\bar{x}$ and t, and provided an $\alpha^I(\bar{x},x,t)[y]$ exists such that

$$\int dx\, \alpha^I(\bar{x},x,t)\, \alpha(x,\bar{\bar{x}},t) = \delta(\bar{x} - \bar{\bar{x}}), \qquad \text{then}$$

$$\alpha^{TI}(x,\bar{x},t) \equiv \alpha^I(\bar{x},x,t)$$

is an integrating factor of the partial differential equation. That is, there exists a $\phi(x,t)[y]$ such that

$$\frac{\delta}{\delta y(\bar{x})}\phi(x,t)[y] = \alpha^{TI}(x,\bar{x},t) \qquad \text{and}$$

$$\frac{\partial\phi}{\partial t}(x,t)[y] = \int d\bar{x}\, \alpha^{TI}(x,\bar{x},t)\, Q(\bar{x},t)[y]."$$

The proof can be constructed along the lines of the discrete version given in the previous chapter. Such a proof depends on the construction of a satisfactory theory of continuously infinite matrices. This theory is outlined in the appendixes and the reader will be referred to there for the necessary definitions and theorems as needed.

Proof: The differential equation to be integrated,

$$\frac{\partial}{\partial t} y(x,t) + Q(x,t)[y] = 0 \, , \tag{III-1}$$

is to be invariant with respect to the infinitesimal transformation

$$y(x,t) \leftarrow y(x,t) + \epsilon\alpha(\bar{x},x,t)[y] \tag{III-2a}$$

$$Q(x,t)[y] \leftarrow Q(x,t)[y + \epsilon\alpha] \tag{III-2b}$$

to first order in ϵ for all relevant x and $\bar{x}$. Making this transformation gives

$$\frac{\partial}{\partial t} y(x,t) + Q(x,t)[y] + \epsilon\left\{\frac{d}{dt}\,\alpha(\bar{x},x,t)[y]\right.$$

$$\left. + \int d\bar{\bar{x}}\, \frac{\delta}{\delta y(\bar{\bar{x}})}\, Q(x,t)[y]\; \alpha(\bar{x},\bar{\bar{x}},t)[y]\right\} + \epsilon^2(\,.\;.\;.\,) + \; = 0 \, .$$

For the coefficient of ϵ to be zero,

$$\frac{d}{dt}\,\alpha(\bar{x},x,t)[y] = -\int d\bar{\bar{x}}\, \frac{\delta}{\delta y(\bar{\bar{x}})}\, Q(x,t)[y]\; \alpha(\bar{x},\bar{\bar{x}},t)[y] \, .$$

Considering $\alpha(\bar{x},x)$ as a matrix in the parameters $\bar{x}$ and x, and provided that the inverse of its transpose exists as specified in the statement of the theorem, then

$$\int d\bar{x}\, \frac{d}{dt}\, \alpha^{T}(x,\bar{x},t)\ \alpha^{TI}(\bar{x},\bar{\bar{x}},t) = -\frac{\delta}{\delta y(\bar{\bar{x}})}\, Q(x,t) \ .$$

Here $\alpha^{T}(x,\bar{x}) = \alpha(\bar{x},x)$ and α^{TI} is the inverse of α^{T}. From the properties of the matrices

$$\int d\bar{x}\, \frac{d}{dt}\, \alpha^{T}(x,\bar{x},t)\ \alpha^{TI}(\bar{x},\bar{\bar{x}},t) = -\int dx\ \alpha^{T}(x,\bar{x},t)\, \frac{d}{dt}\, \alpha^{TI}(\bar{x},\bar{\bar{x}},t) \ ,$$

and

$$\int d\bar{x}\ \alpha^{T}(x,\bar{x},t)\, \frac{d}{dt}\, \alpha^{TI}(\bar{x},\bar{\bar{x}},t) = \frac{\delta}{\delta y(\bar{\bar{x}})}\, Q(x,t) \ . \qquad \text{(III-3)}$$

(The functional dependence on y is understood here, and will not be written where this would cause no confusion.)

The left side of equation III-3 is the right Volterra derivative of the matrix α^{TI}. It is straightforward* to show that if two matrices have the same Volterra derivative, they are proportional to each other through a non-singular (matrix) constant. It is also clear that if a matrix is an integration factor, a constant matrix, multiplied by the integration factor, is also an integration factor. Thus it only needs to be proven that the integration factor, λ, also satisfies the equation

$$\int d\bar{x}\ \lambda^{I}(x,\bar{x},t)\, \frac{d}{dt}\, \lambda(\bar{x},\bar{\bar{x}},t) = \frac{\delta}{\delta y(\bar{\bar{x}})}\, Q(x,t) \ . \qquad \text{(III-4)}$$

To show this we note that an integration factor for equation III-1 is defined so that

$$\lambda(x,\bar{x},t) = \frac{\delta}{\delta y(\bar{x})}\phi(x,t)[y] \ , \qquad \text{(III-5a)}$$

*See Appendix, Section F, page 86.

and

$$\int d\bar{x}\ \lambda(x,\bar{x},t)\ Q(\bar{x},t) = \frac{\partial}{\partial t}\ \phi(x,t)[y] \ , \qquad \text{(III-5b)}$$

and $\lambda(x,\bar{x},t)[y]$ is nonsingular. Taking a partial derivative with respect to t of the first of these two equations and substituting $\partial\phi/\partial t$ from the second gives

$$\frac{\partial}{\partial t}\ \lambda(x,\bar{x},t) = \frac{\delta}{\delta y(\bar{x})} \int d\bar{\bar{x}}\ \lambda(x,\bar{\bar{x}},t)\ Q(\bar{\bar{x}},t)$$

$$= \int d\bar{\bar{x}}\ \frac{\delta\lambda(x,\bar{\bar{x}},t)}{\delta y(\bar{x})}\ Q(\bar{\bar{x}},t) + \int d\bar{\bar{x}}\ \lambda(x,\bar{\bar{x}},t)\ \frac{\delta}{\delta y(\bar{x})}\ Q(\bar{\bar{x}},t) \ ,$$

and

$$\frac{d}{dt}\ \lambda(x,\bar{x},t)[y] \equiv \frac{\partial}{\partial t}\ \lambda(x,\bar{x},t) + \int d\bar{\bar{x}}\ \frac{\delta}{\delta y(\bar{\bar{x}})}\lambda(x,\bar{x},t)\ \frac{\partial}{\partial t}\ y(\bar{\bar{x}},t)$$

$$= \int d\bar{\bar{x}}\ \lambda(x,\bar{\bar{x}},t)\ \frac{\delta}{\delta y(\bar{x})}\ Q(\bar{\bar{x}},t) \ . \qquad \text{(III-6)}$$

Multiplying by the inverse of λ on both sides gives

$$\int dx\ \lambda^{I}(\bar{\bar{x}},x,t)\ \frac{d}{dt}\ \lambda(x,\bar{x},t) = \frac{\delta}{\delta y(\bar{x})}\ Q(\bar{\bar{x}},t) \ ,$$

showing that the integration factor and α^{TI} have the same Volterra derivative. They are then proportional to each other through a nonsingular constant* matrix and thus α^{TI} is also an integration factor, completing the proof.

In comparing this version of Lie's Theorem with the discrete version, we note no mention is made here of obtaining a solution to the differential

*See Appendix, Section F, page 86.

equation by performing the line integral in y, t space. The theorem for the continuous case only gives an integrating factor and not ϕ directly. While a line integral in a discrete (even infinite) vector space is a straightforward concept, a line integral in a continuously infinite-dimensional vector space is not so readily achieved. In practice to perform a line integral in a continuously infinite vector space, one would discretize the problem, apply the line integral to the finite (or countable) dimensional vector space and then perform a limiting process.

In the absence of a solution by a line integral it appears that the theorem is not very powerful. In fact, Lie's theorem only allows one to change the partial differential equation into an equivalent variational equation. That is, the partial differential equation

$$\frac{\partial}{\partial t} y(x,t) + Q(x,t)[y] = 0$$

and the variational equation

$$\frac{\partial}{\partial t} \Phi(x,t)[y] - \int d\bar{x}\; Q(\bar{x}) \frac{\delta}{\delta y(\bar{x})} \Phi(x,t)[y] = 0$$

are equivalent to each other. It may or may not be more convenient to solve the variational equation by a "pseudo line integral" than to attack the original equation. The following sections examine the heat flow equation and others in view of the continuous form of the Lie Theorem.

B. Examples

1. The one-dimensional heat flow equation

Here the heat-flow equation will be analyzed again, this time using the continuous form of the Lie theorem stated in the previous section of this chapter.

The partial differential equation to be solved is

$$\frac{\partial}{\partial t} y(x,t) - \frac{\partial^2}{\partial x^2} y(x,t) = 0 \;, \qquad \text{(III-10)}$$

with initial conditions

$$y(x,0) = y^{\circ}(x) \;,$$

defined everywhere on the real x axis. (This is equivalent to setting

$$Q(x,t)[y] = -\int d\bar{x} \frac{\partial^2}{\partial x^2} \delta(x - \bar{x})\; y(\bar{x},t) \;) \;.$$

Equation III-10 is invariant with respect to the transformation

$$y(x,t) \leftarrow y(x,t) + \epsilon y(\bar{x} + x,\; t)$$

for all $\bar{x}$ and x. That is, α for the transformation is given by

$$\alpha(\bar{x},x,t) = y(\bar{x} + x,t) \;.$$

α is symmetric ($\alpha^T = \alpha$) and is an anticirculant continuous matrix. Its inverse is also an anticirculant. It is straightforward to show that if

$$q(x,t) = \int_{-\infty}^{+\infty} dp \frac{e^{2\pi i x p}}{\int_{-\infty}^{+\infty} d\bar{x}\; e^{2\pi i \bar{x} p}\; y(\bar{x},t)} \;, \qquad \text{(III-11)}$$

then

$$\int_{-\infty}^{+\infty} d\bar{x}\; q(\bar{x} + x)\; y(\bar{x} + \bar{\bar{x}}) = \delta(\bar{\bar{x}} - x) \;.$$

Thus we have

$$\frac{\delta\ \phi(x,t)}{\delta y\ (\bar{x},t)} = \alpha^{TI}(x,\bar{x},t) = q(x+\bar{x}) = \int_{-\infty}^{+\infty} dp \frac{e^{2\pi i(\bar{x}+x)p}}{\int_{-\infty}^{+\infty} d\bar{\bar{x}}\ e^{2\pi i\bar{\bar{x}}p}\ y(\bar{\bar{x}},t)} \tag{III-12a}$$

and

$$\frac{\partial\ \phi(x,t)}{\partial t} = -\int_{-\infty}^{+\infty} d\bar{\bar{x}} \int_{-\infty}^{+\infty} d\bar{x}\ \alpha^{TI}(x,\bar{\bar{x}},t) \frac{\partial^2}{\partial \bar{\bar{x}}^2} \delta(\bar{\bar{x}}-\bar{x})\ y(\bar{x},t)$$

$$= -\frac{\partial^2}{\partial x^2} \delta(x) \ . \tag{III-12b}$$

The general solution to the partial differential equation then is

$$\phi(x,t)[y] = K(x)$$

where K is an arbitrary "constant" vector. (K(x) is a constant in the sense that $\frac{\partial}{\partial t} K(x) = 0$ and $\frac{\delta}{\delta y(\bar{x})} K(x) = 0$.)

Finding $\phi(x,t)[y]$ from $\delta\phi/\delta y$ and $\partial\phi/\partial t$ is something of a problem. In this particular case it is possible to look at the discrete version of this problem and figure out what ϕ ought to be in the continuous case. The discrete case can be solved by taking a line integral in a finite-dimensional space. In the continuous case one must essentially guess the integral and verify by substitution in III-12. While this may appear crude, it is, nevertheless, the way all quadrature is done, the problem here being more complex in that it is the entire line integral that must be guessed, rather than the individual components of a line integral.

By inspection of III-12 it is not difficult to see that

$$\phi(x,t)[y] = \int_{-\infty}^{+\infty} dp\ e^{2\pi i x p} \ln\left(\int_{-\infty}^{+\infty} d\bar{\bar{x}}\ e^{2\pi i \bar{\bar{x}} p}\ y(\bar{\bar{x}},t)\right) - t \frac{\partial^2}{\partial x^2} \delta(x) \quad \text{(III-13)}$$

gives the correct $\delta\phi/\delta y$ and $\partial\phi/\partial t$. The general solution is $\phi(x,t) = K(x)$ where K is independent of y and t, but can depend on x. To relate this solution to the initial value problem where

$$y(x,0) = y^{\circ}(x) ,$$

let

$$\ln(M(p)) = \int_{-\infty}^{+\infty} dx\ e^{-2\pi i p x}\ K(x)$$

and take the Fourier transform of both sides of

$$\int_{-\infty}^{+\infty} dp\ e^{2\pi i x p} \ln\left(\int_{-\infty}^{+\infty} d\bar{\bar{x}}\ e^{2\pi i \bar{\bar{x}} p}\ y(\bar{\bar{x}},t)\right) - t \frac{\partial^2}{\partial x^2} \delta(x) = K(x) \quad \text{(III-14)}$$

giving

$$\ln\left(\int_{-\infty}^{+\infty} d\bar{\bar{x}}\ e^{2\pi i \bar{\bar{x}} p}\ y(\bar{\bar{x}},t)\right) - t(2\pi i p)^2 = \ln M(p) .$$

Solving for y:

$$y(x,t) = \int_{-\infty}^{+\infty} dp\ e^{-2\pi i x p\ -\ (2\pi p)^2 t}\ M(p) .$$

M(p) then, is the Fourier transform of $y^{\circ}(x)$,

$$M(p) = \int_{-\infty}^{+\infty} d\bar{x}\ e^{2\pi i\bar{x}p}\ y^{\circ}(\bar{x}) \ ,$$

so that y(x,t) in terms of $y^{\circ}(x)$ is

$$y(x,t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} d\bar{x}\ e^{-2\pi i(x - \bar{x})p - (2\pi p)^2 t} y^{\circ}(\bar{x}) \ . \qquad \text{(III-15)}$$

Doing the p integration first gives the usual form of the solution to the initial value problem:

$$y(x,t) = \int_{-\infty}^{+\infty} d\bar{x}\ \frac{e^{-(x - \bar{x})^2/4\sqrt{t}}}{2\sqrt{\pi t}}\ y^{\circ}(\bar{x}) \ . \qquad \text{(III-16)}$$

It will be noted here in comparing with the procedure for solving the discrete form of the heat flow equation, that there is a one-to-one correspondence between the steps in each. The discrete solution can be used as a model or guide in following the continuous case or vice versa. The continuous case is possibly easier to follow because of the absences of the discretization and limiting processes. One notes that all the discrete Fourier transforms are replaced by the corresponding continuous transformation and these are only introduced to relate the general solution ϕ = K to the initial value type solution.

The one point in the continuous case that is possibly more complex than the discrete case is relating the gradients of ϕ (with respect to y and t) to ϕ, itself. In the discrete case this can be done by a line integral in a finite-dimensional space; the analog in the continuous case would be a line integral in a continuously infinite-dimensional space -- a rather difficult concept. In any event, Lie's theorem reduces the problem of integrating a differential equation to finding a quadrature or set of quadratures, provided the appropriate invariance group can be found.

The wave equation example given in the preceding chapter can be worked out in a manner similar to the heat flow equation without recourse to discretizing. The two treatments are so similar that this will not be done here.

2. A class of linear problems

From the heat-flow and wave equations it can be seen that there is a general class of first order linear initial value problems that can be solved by use of the same transformation. Consider the partial differential equation of the form

$$\left\{\frac{\partial}{\partial t} + f\left(t,\frac{\partial}{\partial x}\right)\right\} y(x,t) = 0 \qquad \text{(III-20)}$$

where f(t,z) is integrable in t and a fairly good function* of z. This equation is invariant with respect to the infinitesimal transformation

$$y(x,t) \leftarrow y(x,t) + \epsilon y(\bar{x} + x,t) \ ;$$

thus

$$\alpha(\bar{x},x,t) = y(\bar{x} + x,t) \ , \qquad \text{and}$$

*In the sense of Lighthill [30]

$$\alpha^{TI}(\bar{x},\bar{\bar{x}},t) = \int_{-\infty}^{+\infty} dp \frac{e^{2\pi ip(\bar{x} + \bar{\bar{x}})}}{\int_{-\infty}^{+\infty} ds\ e^{2\pi ips}\ y(s,t)} \quad . \tag{III-21}$$

The functional ϕ is given by

$$\phi(x,t)[y] = \int_{-\infty}^{+\infty} dp\ e^{2\pi ixp}\ \ln\left(\int_{-\infty}^{+\infty} d\bar{x}\ e^{2\pi ip\bar{x}}\ y(\bar{x},t)\right)$$

$$+ \int_0^t d\bar{t}\ f(\bar{t},\frac{\partial}{\partial x})\ \delta(x) \quad , \tag{III-22}$$

and the general solution to III-20 is

$$\phi(x,t)[y] = K(x) \quad .$$

In terms of the initial conditions, y(x,t) is given by

$$y(x,t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} d\bar{x}\ e^{2\pi i(\bar{x} - x)p - \int_0^t d\bar{t}\ f(\bar{t},2\pi ip)}\ y(\bar{x},0) \quad . \tag{III-23}$$

3. Partial differential equations in more than two independent variables

Lie's theorem is also applicable to partial differential equations that are first order in t and have several independent variables x_1, x_2, x_3 . . . . In this case, the statement of the theorem is modified so as to replace x by the vector $\vec{x} = \{x_1,x_2,x_3\ .\ .\ .\}$, and dx by the volume element in x space, $d\vec{x} = dx_1,\ dx_2,\ dx_3\ .\ .\ .\ .$

The family of partial differential equations mentioned above in Section 2 of this chapter can then be generalized to

$$\left\{\frac{\partial}{\partial t} + f(t,\frac{\vec{\partial}}{\partial x})\right\} y(\vec{x},t) = 0 \qquad \text{(III-24)}$$

where $\vec{\partial}/\partial x$ is the gradient operation with respect to the components of $\vec{x}$. This equation is invariant with respect to the transformation

$$y(\vec{x},t) \leftarrow y(\vec{x},t) + \epsilon y(\vec{x} + \vec{\bar{x}},t) \; .$$

The integrating factor is

$$\alpha^{TI}(\vec{\bar{x}},\vec{\bar{\bar{x}}},t) = \int_{-\infty}^{+\infty} d\vec{p}\, \frac{e^{2\pi i\vec{p}\,\cdot\,(\vec{\bar{x}} + \vec{\bar{\bar{x}}})}}{\int_{-\infty}^{+\infty} d\vec{s}\; e^{2\pi i\vec{p}\,\cdot\,\vec{s}}\; y(\vec{s},t)} \; . \qquad \text{(III-25)}$$

The functional ϕ is given by

$$\phi(\vec{x},t)[y] = \int_{-\infty}^{+\infty} d\vec{p}\; e^{2\pi i\vec{x}\,\cdot\,\vec{p}}\; \ln\left(\int_{-\infty}^{+\infty} d\vec{\bar{x}}\; e^{2\pi i\vec{p}\,\cdot\,\vec{\bar{\bar{x}}}}\; y(\vec{\bar{\bar{x}}},t)\right) + \int_0^t d\bar{t}\; f(\bar{t},\frac{\vec{\partial}}{\partial x})\; \delta(\vec{x}) \; ,$$

and the solution to III-24 is

$$\phi(\vec{x},t)[y] = K(\vec{x}) \; .$$

In terms of the initial conditions, $y(\vec{x},t)$ is given by

$$y(\vec{x},t) = \int_{-\infty}^{+\infty} d\vec{p} \int_{-\infty}^{+\infty} d\vec{\bar{x}}\; e^{2\pi i(\vec{\bar{x}} - \vec{x}) \cdot \vec{p} - \int_0^t d\bar{t}\; f(\bar{t},2\pi i\vec{p})}\; y(\vec{\bar{x}},0) \quad .(\text{III-26})$$

The notation used here in that $d\vec{x}$, $d\vec{p}$ represent volume elements in $\vec{x}$ and $\vec{p}$ space respectively, $\vec{x} \cdot \vec{p}$ is the scalar product, i.e. $\vec{x} \cdot \vec{p} \equiv x_1p_1 + x_2p_2 + \ldots$, and $\delta(\vec{x})$ is the multi-dimensional delta (generalized) function

$$\delta(\vec{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\;.\;.\;.\;.$$

All of the results derived with a scalar x can be carried over to the case where x is a vector.

IV. RESULTS AND CONCLUSIONS

A. Results

The main result of this investigation is that it is possible to apply Lie's theorem to the integration of partial differential equations. This can be done in two ways.

The first method is to discretize the partial differential equation so that it is approximated by a system of coupled ordinary differential equations and then apply the form of Lie's theorem for a system of ordinary differential equations. A limiting process can then be used to get from the solution of the discrete approximation back to the continuous case. This is an awkward procedure but has certain advantages. The main advantage is that in the discrete case, Lie's theorem gives a prescription both for the construction of an integrating factor and for integrating the resulting equation by way of a line integral in a finite dimensional space.

The second method uses a form of Lie's theorem applicable directly to the partial differential equation without introducing a discrete approximation. But here one obtains a prescription for constructing the integrating factor only. The actual integration becomes a line integral in a continuously infinite dimensional space. Such line integrals are not as obvious as in the discrete case.

Lie's theorem for systems of coupled ordinary differential equations is as follows:

"If the differential equation $\frac{dy(t)}{dt} + Q(y(t),t) = 0$, where y and Q are vectors, and t a scalar, is invariant with respect to the set of transformations specified by

$$U_n = \alpha_{ns}(y,t)\frac{\partial}{\partial y_s} ,$$

where α is a square matrix and $\frac{\partial}{\partial y}$ a vector operator, then provided α^{TI} exists, the general solution to the differential equation is

$$\int \alpha^{TI}(dy + Qdt) = K$$

where the integral is understood as a line integral in y, t space along any convenient path, and K is an arbitrary vector constant." (Summation over repeated indexes is understood.)

Lie's theorem for partial differential equations is as follows:

"If the partial differential equation

$$\frac{\partial}{\partial t} y(x,t) + Q(x,t)[y] = 0$$

is invariant with respect to the transformations

$$y(x,t) \leftarrow y(x,t) + \epsilon\alpha(\bar{x},x,t)[y]$$

for all relevant x, $\bar{x}$, and t, and provided an $\alpha^I(\bar{x},x,t)[y]$ exists such that

$$\int dx\; \alpha^I(\bar{x},x,t)\; \alpha(x,\bar{\bar{x}},t) = \delta(\bar{x} - \bar{\bar{x}}) ,$$

then $\alpha^{TI}(x,\bar{x},t) \equiv \alpha^I(\bar{x},x,t)$ is an integrating factor of the partial differential equation. That is, there exists a $\phi(x,t)[y]$ such that

$$\frac{\delta \phi}{\delta y(\bar{x})}(x,t)[y] = \alpha^{TI}(x,\bar{x},t) \qquad \text{and}$$

$$\frac{\partial \phi}{\partial t}(x,t)[y] = \int d\bar{x}\ \alpha^{TI}(x,\bar{x},t)\ Q(\bar{x},t)[y] \ ."$$

Proof of the first version is straightforward. Proof of the second involves a theory of continuously infinite matrices. A development of continuous matrices based on distribution theory or generalized functions is given in the Appendix.

The examples given here (heat flow equation and wave equation) are all of linear partial differential equations. There is nothing in Lie's theorem that restricts it to linear problems but no example of non-linear equations, solvable using Lie's theorem, have been found.

B. Conclusions

The method of solution of partial differential equations by use of Lie's theorem has both advantages and disadvantages. Among the advantages are the following:

1. Where it can be applied, Lie's theorem gives a completely general solution to the differential equation. It is general enough so that, in principle, any boundary conditions can be accommodated.

2. The method as given here applies to single first order partial differential equations, but can be extended both to higher order equations and systems of partial differential equations.

3. While a knowledge of group theory would be useful, the method does not depend on the general theory of Lie groups or the structures of the Lie algebras for its use.

The disadvantages are as follows:

1. It is not clear what class of partial differential equations can be solved by Lie's theorem and which cannot. There is probably a large class of differential equations that cannot be solved in this manner.

2. It is not usually easy to find an appropriate transformation necessary to apply Lie's theorem to a particular differential equation. There is no straightforward prescription for finding such a transformation. There are undoubtedly many equations for which the required kind of transformation does not exist.

3. Even when a suitable transformation group can be found it is not always easy to find the inverse matrix that is the integrating factor.

4. If the inverse matrix is found, it may still be difficult to actually do the necessary line integral.

5. Finally, if the line integral can be done, the solution may be in an awkward form (possibly as an integral relation) that is not easy to use or for applying initial conditions.

C. Recommendations for Further Study

Several improvements and extensions of Lie's theorem, and the application of group theory to partial differential equations can be suggested.

1. Lie's theorem as stated here applies to a single first order partial differential equation of the form

$$\frac{\partial}{\partial t} y(x,t) + \mathbf{Q}(x,t)[y] = 0 \ . \qquad \text{(IV-1)}$$

While it is straightforward to extend this method to higher order partial differential equations or systems of equations, it is difficult to use these extensions. Work remains to be done on examples of the higher order and systems of partial differential equations.

2. It should also be possible to find an extension of Lie's theorem in such a way as to allow its application to systems of equations of the form

$$f_i\left(\frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^3 y}{\partial t^3}, \ldots \frac{\partial^n y}{\partial t^n}, x, t\right)[y] = 0, \; i = 1, 2 \ldots$$

This would eliminate the necessity of bringing equations in to the form IV-1 above before solving.

3. Lie's method is applicable to non-linear partial differential equations but so far no such examples have been found. It ought to be possible, for example, to set up transformations and then find rather general forms of partial differential equations that are invariant with respect to these transformations. In this way, tables of equations and transformationscould be made, and used (much as tables of integrals are used) for finding integration factors. Here, one would expect that classical group theory and the structure of Lie groups would be useful in classifying and correlating the equations and integrating factors.

4. In the case of a single ordinary (or total) differential equation, if two distinct integrating factors can be found, their ratio (set equal to a constant) represents a solution. There should be similar theorems for systems of ordinary differential equations and for partial differential equations but these are not known.

Respectfully submitted,

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V. APPENDIX

DISTRIBUTIONS AND CONTINUOUSLY INFINITE MATRICES

A. Mathematical Background

Notation: N will denote the set (0,1,2, . . .) of natural numbers, Z the set (0,±1,±2, . . .) of integers, R the field of real numbers, and C the field of complex numbers. If K is any of the sets above, then K^* will denote the same set without the zero element.

Dfn: A non-empty set E is said to be a vector space over the field C of complex numbers if there is a binary operation + from E x E into E and a binary operation · from C x E into E such that if x, y, $z \epsilon E$, a, $b \epsilon C$,

(1) $x + y = y + x$.

(2) $(x + y) + z = x + (y + z)$.

(3) There exists an element $0 \epsilon E$ such that $x + 0 = 0 + x = x$.

(4) There exists an element $-x \epsilon E$ such that $x + -x = -x + x = 0$.

(5) $a \cdot (x + y) = a \cdot x + a \cdot y$.

(6) $(a + b) \cdot x = a \cdot x + b \cdot x$.

(7) $(ab) \cdot x = a \cdot (bx)$.

(8) $1 \cdot x = x$.

Dfn: A non-empty set E is said to be a topological space if there exists a family T of subsets of E such that

(1) $\emptyset$, $E \epsilon T$

(2) If $(0_\lambda)_{\lambda \epsilon L}$ is a family of sets in T, then $\bigcup_{\lambda \epsilon L} 0_\lambda \epsilon T$

(3) If $0_1, 0_2, . . ., 0_n \epsilon T$, then $\bigcap_{i=1}^{n} 0_i \epsilon T$.

The elements of T are called open sets.

Dfn: A map f from a topological space E into a topological space F is said to be continuous at a point $x \in E$, if for any open set W of F containing f(x), there is an open set V of E containing x such that $f(V) \subset W$. f is said to be continuous if it is continuous of every point of E.

Dfn: If E is a topological space and $x \in E$, then a set $N \subset E$ is said to be a neighborhood of x if there exists an open set O in E such that $x \in O \subset N$. A family of neighborhood $(N_\lambda)_{\lambda \in L}$ of $x \in E$ is said to be a fundamental system of neighborhoods if for any neighborhood N of x, there is a $\lambda_0 \in L$ such that $N_{\lambda_0} \subset N$.

Proposition: If E is a topological space, $x \in E$, $(N_\lambda)_{\lambda \in L}$ a fundamental system of neighborhoods of x, f a map from E into of topological space F, $(M_\mu)_{\mu \in M}$ a fundamental system of neighborhoods of f(x) in F, then f is continuous at x if, and only if, for each $\mu_0 \in M$, there exists a $\lambda_0 \in L$ such that $f(N_{\lambda_0}) \subset N_{\mu_0}$.

Proof: trivial.

Dfn: Let E, F, G be topological spaces. Then a function $f : E \times F \to G$ is said to be continuous at $(x_0, y_0) \in E \times F$ if for any neighborhood W of $f(x_0, y_0)$ in G there exist neighborhoods V_1 of X_0 in E and V_2 of y_0 in F such that $f(V_1 \times V_2) \subset W$.

Dfn: A non-empty set E which is both a vector space over C and a topological space is said to be a topological vector space over C if the maps

(1) $(x,y) \to x + y$ from $E \times E \to E$

(2) $(a,x) \to ax$ from $C \times E \to E$

are continuous. (where C is endowed with the normal topology).

Dfn: If E is a vector space over C then a subset A of E is said to be convex if for any x, yϵA, a, bϵC, a, b $\geq$ 0, a + b = 1, then ax + byϵA.

Dfn: A topological vector space E is said to be a locally convex space if each point in E has a fundamental system of convex neighborhoods.

Note: It is easy to check that if $(N_\lambda)_{\lambda \epsilon L}$ is a fundamental system of neighborhoods of zero in a topological vector space E then $(x + N_\lambda)_{\lambda \epsilon L}$ is a fundamental system of neighborhoods of any xϵE. In particular, E is locally convex if, and only if, zero has a fundamental system of convex neighborhoods.

Dfn: Let $R_+ = \{x \epsilon R: x \geq 0\}$. Then a function q: E $\rightarrow$ R_+ where E is a vector space is called a semi-norm or E on the following holds:

(1) $q(ax) = |a| q(x)$ for all aϵC, xϵE

(2) $q(x + y) \leq q(x) + q(y)$ for all x, yϵE

THEOREM: Let E be a vector space over C and $(q_\iota)_{\iota \epsilon I}$ a family of semi-norms of E. Then there exists a unique topology on E associated with the family $(q_\iota)_{\iota \epsilon I}$ which makes E into a locally convex space. A fundamental system of neighborhoods of zero is given by

$$N_{m,\epsilon} = \left\{ x \epsilon E: \quad q_{\iota_k}(x) \leq \epsilon, \ 0 \leq k \leq m \right\}$$

where $\epsilon > 0$, mϵN, ι_k, $0 \leq k \leq m$ a finite subset of I.

Proof: See Horvath [33] pp. 88-89.

Dfn: Let E and F be vector spaces. A map f: E $\rightarrow$ F is said to be linear if for all x, yϵE, aϵC,

(1) $f(x + y) = f(x) + f(y)$

(2) $f(ax) = af(x)$.

Proposition: A linear map f from a topological vector space E into a topological vector space F is continuous if and only if it is continuous at the origin.

Proof: See Horvath [33] p. 97.

Dfn: If E and F are non-empty sets, a map $f: E \to F$ is said to be injective (one-to-one) if for any $x, y \in E$ such that $f(x) = f(y)$, then $x = y$. f is said to be surjective (onto) if for any $y \in F$, there exists an $x \in E$ such that $f(x) = y$. If f is both injective and surjective, it is called bijective.

Proposition: If $f: E \to F$ is a bijective map and for each $y \in F$, we define $g(y) = x$ if and only if $f(x) = y$, then $g: F \to E$ is a bijective map. g is called the inverse of f and denoted f^{-1}.

Proof: trivial.

Dfn: If E and F are topological spaces, then a continuous bijective map $f: E \to F$ is called a homeomorphism if f^{-1} is continuous.

Dfn: If $f: E \to F$, $g: F \to G$ are functions then we denote the function $x \to g(f(x))$ from E into G by $g \circ f$.

Note: if $f: E \to F$ is bijective, then $f^{-1} \circ f = I_E$ and $f \circ f^{-1} = I_F$.

B. Rapidly Decreasing Functions and Temperate Distributions

Dfn: A function $f: R \to C$ is said to vanish at infinity if given $\epsilon > 0$, there exists a $M \geq 0$ such that

$$|f(\gamma)| < \epsilon \quad \text{for all} \quad |\gamma| > M .$$

Dfn: T will denote the set of infinitely differentiable functions $\varphi: R \to C$ such that for each $R \in Z$, $p \in N$, the function $\gamma \to (1 + \gamma^2)^R \varphi^{(p)}(\gamma)$ vanishes at infinity.

Dfn: O_M will denote the set of infinitely differentiable functions α: $R \to C$ such that for each $p\epsilon N$, there exists a $k\epsilon Z$ such that the function $\gamma \to (1 + \gamma^2)^k \alpha^{(p)}(\gamma)$ vanishes at infinity.

Dfn: O_C will denote the set of infinitely differentiable functions β: $R \to C$ for which there exists a $k\epsilon Z$ such that the functions $\gamma \to (1 + \gamma^2)^k$ $\beta^{(p)}(\gamma)$ vanishes at infinity for all $p\epsilon N$.

The elements of T are called rapidly decreasing functions. It is easy to see that if $\varphi\epsilon T$, that $\varphi^{(p)}\epsilon T$ for all $p\epsilon N$. Also, $T \subset L^1(R)$.

If for each $k\epsilon Z$, $p\epsilon N$, we define $q_{k,p}$: $T \to R_+$ such that

$$q_{k,p}(\varphi) = \max_{\gamma\epsilon R} \left\{(1 + \gamma^2)^k \ |\varphi^{(p)}(\gamma)|\right\}$$

then $q_{k,p}$ is a semi-norm on T. Thus, the family $(q_{k,p})_{(k,p)\epsilon Z \times N}$ defines a unique locally convex topology on the vector T which makes T into a topological vector space. A fundamental system of neighborhoods of $0\epsilon T$ is given by

$$N_{k,m,\epsilon} = \left\{\varphi\epsilon T: \quad (1 + \gamma^2)^k|\varphi^{(p)}(\gamma)| < \epsilon, p \leq m\right\}$$

where k, $m\epsilon N$, $\epsilon > 0$. (See Horvath [33] pp. 90-91).

For each $k\epsilon Z$, let T_k denote the class of all infinitely differentiable functions f: $R \to C$ such that the functions $\gamma \to (1 + \gamma^2)^k$ $f^{(p)}(\gamma)$ vanishes at infinity for each $p\epsilon N$. For each $k\epsilon Z$, we let the family $(q_{k,p})_{p\epsilon N}$ of semi-norm defined above determine the topology on T_k which makes T_k into a topological vector space.

It is easy to check that $O_C = \bigcup_{k\epsilon Z} T_k$. Furthermore, a fundamental systems of neighborhoods of T_k is given by

$$N_{k,m,\epsilon} = \left\{ f\epsilon\ T_k: \quad (1+\gamma^2)^k \ |f^{(p)}(\gamma)| < \epsilon, p \le m \right\} .$$

If i_k: $T_k \to O_C$ is the canonical injection (i.e. $i_k(\varphi) = \varphi$ for all $\varphi \epsilon T_k$), then we equip O_C with the finest locally convex topology for which the family $(i_k)_{k\epsilon Z}$ of maps are continuous. (See Horvath [33] p. 157).

Proposition: If G is a locally convex space and g: $O_C \to G$ is a linear map, then g is continuous if, and only if, the maps $g \circ i_k$: $T_k \to G$ are continuous (i.e. g is continuous if, and only if, $g|T_k$ is continuous for all $k\epsilon Z$).

Proof: See Horvath [33] p. 159.

Dfn: If V is any vector space over a field K, a linear function f: $V \to K$ is called a linear form (functional) on V.

Dfn: The set of continuous linear forms on T will be called temperate distributions. This set will be denoted by T'.

If $T\epsilon T'$, $\varphi\epsilon T$, then we will denote $T(\varphi)$ by either $<T, \varphi>$ or $\int_{-\infty}^{+\infty} T(\gamma)\varphi(\gamma)d\gamma$.

Dfn: The set of continuous linear forms on O_C will be called rapidly decreasing distributions. This set will be denoted by O_C'.

If $S\epsilon O_C'$, $\beta\epsilon O_C$, we will denote $S(\beta)$ by either $<S, \beta>$ or $\int_{-\infty}^{+\infty} S(\gamma)\beta(\gamma)d\gamma$.

Examples:

(1) If $f\epsilon L^1(R)$, and $<T, \varphi> = \int_{-\infty}^{+\infty} f(\gamma)\varphi(\gamma)d\gamma$ for all $\varphi\epsilon T$, it can be shown that $T\epsilon T'$. T is usually denoted T_f or $\mathring{f}$.

(2) If $<T, \varphi> = \varphi(0)$ for all $\varphi\epsilon T$ then $T\epsilon T'$ and T is usually denoted by δ and called the Dirac delta measure. (i.e. $\int_{-\infty}^{+\infty} \delta(\gamma)\varphi(\gamma)d\gamma = \varphi(0)$).

(3) If $\varphi\epsilon T$, and $<S_\varphi, \beta> = \int_{-\infty}^{+\infty} \varphi(\gamma)\beta(\gamma)d\gamma$ then $S_\varphi \ \epsilon \ O_c'$. For let $k\epsilon Z$

and $\varepsilon > 0$ be given. Then if $M_k = \max_{\gamma\varepsilon R} \left\{(1 + \gamma^2)^{-k+1} \, |\varphi(\gamma)|\right\} < \infty$ and

$$N_k = \left\{\beta\varepsilon T_k : \; (1 + \gamma^2)^k \, |\beta(\gamma)| < \delta\right\}, \text{ where } \delta = \frac{\varepsilon}{M_k \int_{-\infty}^{+\infty} \frac{d\gamma}{1+\gamma^2} + 1}$$

then N_k is a neighborhood of $0\varepsilon T_k$, and if $\beta\varepsilon N_k$,

$$|<S_\varphi, \beta>| = \left| \int_{-\infty}^{+\infty} \varphi(\gamma)\beta(\gamma)d\gamma \right| \le \int_{-\infty}^{+\infty} |\varphi(\gamma)|\,|\beta(\gamma)|d\gamma$$

$$\le \int_{-\infty}^{+\infty} M_k (1 + \gamma^2)^{k-1} \cdot \delta \cdot (1 + \gamma^2)^{-k} \, d\gamma$$

$$= M_k \cdot \delta \cdot \int_{-\infty}^{+\infty} (1 + \gamma^2)^{-1} \, d\gamma$$

$$= M_k \cdot \int_{-\infty}^{+\infty} \frac{d\gamma}{(1 + \gamma^2)} \cdot \frac{\varepsilon}{M_k \int_{-\infty}^{+\infty} \frac{d\gamma}{1+\gamma^2} + 1} < \varepsilon$$

Thus, $S_\varphi | T_k$ is continuous at the origin and hence continuous. Hence, by the previous proposition, S_φ is continuous on O_c. Therefore, $S_\varphi \varepsilon O_c'$.

(4) If S is such that $<S, \beta> = \beta(0)$ for all $\beta\varepsilon O_c$, then $S\varepsilon O_c'$.

THEOREM: If for each $\varphi\varepsilon T$, we define $\hat{\varphi}(\eta) = F[\varphi](\eta) = \int_{-\infty}^{+\infty} e^{-2\pi i \eta\gamma} \varphi(\gamma)d\gamma$, the map $F: \varphi \to \hat{\varphi}$ is a linear homeomorphism from $T \to T$. The inverse of F is given by the map $\bar{F}^1: \varphi \to \tau$ where $\tau(\eta) = \int_{-\infty}^{+\infty} e^{2\pi i\eta\gamma} \varphi(\gamma)d\gamma$ for each $\eta\varepsilon R$.

Furthermore, if for each $T\varepsilon T'$ we define $F[T]$ such that $<F[T], \varphi> = <T, F[\varphi]>$ and $\bar{F}^1[T]$ such that $<\bar{F}^1[T],\varphi> = <T,\bar{F}^1[\varphi]>$ then $F[T]$ and $\bar{F}^1[T]$ are in T'.

Proof: See Horvath [33] pp. 408-411.

The maps F and $\bar{F}^1$ are called the Fourier and inverse Fourier transformation respectively.

THEOREM: if α: $R \to C$ is a function, then $\alpha\varphi\epsilon T$ for all $\varphi\epsilon T$ if and only if $\alpha\epsilon O_M$. The map M_α: $\varphi \to \alpha\varphi$ is a continuous linear map from T into T if and only if $\alpha\epsilon O_M$.

Furthermore, if for each $T\epsilon T'$, $\alpha\epsilon O_M$, we define αT such that $\langle\alpha T, \varphi\rangle = \langle T, \alpha\varphi\rangle$ for all $\varphi\epsilon T$, then $\alpha T\epsilon T'$.

Proof: See Horvath [33] pp. 417-419.

Note: It is easy to see that if $\alpha\epsilon O_M$, then M_α is a homeomorphism from T onto T if and only if $\frac{1}{\alpha}\ \epsilon O_M$, and if M_α is a homeomorphism, $M_\alpha^{-1} = M_{\frac{1}{\alpha}}$.

THEOREM: If $p\epsilon N$, and we define $D^p[\varphi] = \varphi^{(p)}$ then D^p is a continuous linear map from T into T. Furthermore, if $T\epsilon T'$, and we define $D^p[T]$ such that $\langle D^p[T], \varphi\rangle = (-1)^p \langle T, D^{(p)}[\varphi]\rangle$ for all $\varphi\epsilon T$, then $D^p[T]\epsilon T'$.

Proof: See Horvath [33] pp. 411-412.

Dfn: Let f: $R \to C$ be a function. Then for each $h\epsilon R$, $a\epsilon R^*$, we define $\tau_h f$ to be the function $\gamma \to f(\gamma - h)$ and $\mu_a f$ to be the function $\gamma \to f(a\gamma)$.

Proposition: the function T_h: $\varphi \to \tau_h\varphi$ and R_a: $\varphi \to \mu_a\varphi$ where $a\epsilon R^*$, $h\epsilon R$ are linear homeomorphisms from T onto T.

Proof: It is clear that these maps are linear and bijective. Furthermore, since $T_h^{-1} = T_{-h}$ and $R_a^{-1} = R_{1/a}$, we need only show they are continuous. Let

$$N_{k,m,\epsilon} = \left\{\tau\epsilon T: \quad (1 + \gamma^2)^k \ |\tau^{(p)}(\gamma)| < \epsilon,\ p \le m\right\}$$

where $k\epsilon Z$, $m\epsilon N$, $p\epsilon N$, $p \le m$, and $\epsilon > 0$. First we show T_h is continuous: Let

$$M_{k,m+1,S} = \left\{\tau\epsilon T \cdot \ (1 + \gamma^2)^k \ |\tau^{(p)}(\gamma)| < S,\ p \le m + 1\right\}$$

where $S = 1/2\ \frac{\epsilon}{1 + |h|}$. Then if $\varphi\epsilon\ M_{k,m+1,S}$ and $p \le m$, then

$$\max_{\gamma\epsilon R}\left\{(1+\gamma^2)^k\ |\varphi^{(p)}(\gamma-h)-\varphi^{(p)}(\gamma)|\right\}=\max_{\gamma\epsilon R}\left\{|h|(1+\gamma^2)^k\ |\varphi^{(p+1)}(\gamma)|\right\}<\epsilon/2\ .$$

In particular,

$$(1+\gamma^2)^k\ |\varphi^{(p)}(\gamma-h)|<\epsilon/2+(1+\gamma^2)^k\ |\varphi^{(p)}(\gamma)|<\epsilon\ .$$

Hence, if $\varphi\epsilon\ M_{k,m+1,S}$, we have $T_h[\varphi]\epsilon\ N_{k,m,\epsilon}$ so T_h is continuous for all $h\epsilon R$.

To prove R_a is continuous, we take $N_{k,m,\epsilon}$ as above and choose

$$M_{k,m,S}=\left\{\tau\epsilon T:\ \ (1+\gamma^2)^k\ |\tau^{(p)}(\gamma)|<S,\ p\le m\right\}$$

where $S=\epsilon/[n^{2k}(1+|a|)^m]$ and $n\epsilon N^*$ is large enough so that $|a|>\frac{1}{n}$. Then if $\varphi\epsilon\ M_{k,m,S},\ p\le m$,

$$\begin{aligned}(1+\gamma^2)^k\ |(R_a[\varphi]^p(\gamma)| &= (1+\gamma^2)^k\ |a^p\varphi^{(p)}(a\gamma)|\\ &< (1+\gamma^2)^k\ |a|^p(1+(a\gamma)^2)^{-k}\cdot S\\ &< (1+\gamma^2)^k(n^2+\gamma^2)^{-k}\ n^{2k}\ |a|^p\ S\\ &< \epsilon\ .\end{aligned}$$

Therefore $R_a[\varphi]\epsilon\ N_{k,m,\epsilon}$ whenever $\varphi\epsilon\ M_{k,m,s}$ so R_a is continuous. Q.E.D.

Dfn: We also let τ_h, $h\epsilon R$ be the map $x\to x-h$ and μ_a, $a\epsilon R^*$ be the map $x\to ax$ from R onto R.

From the above definition, we see that $\tau_h\varphi=\varphi\circ\tau_h$ and $\mu_a\varphi=\varphi\circ\mu_a$. Thus, if μ and ν are any of the maps μ_a or τ_a, we define $\mu\nu\varphi$ to be the map $\varphi\circ\nu\circ\mu$. Thus, we see that $\mu_a\tau_h\varphi$ is the map $\gamma\to\varphi(a\gamma-h)$ and $\tau_h\mu_a\varphi$ is the map $\gamma \to \varphi(a\gamma - ah)$. Also, if $h_1, h_2 \epsilon R$, $\tau_{h_1} \cdot \tau_{h_2} \varphi = \tau_{h_1 + h_2}\varphi$ and if $a, b\epsilon R^*$, $\mu_a\mu_b\varphi = \mu_b\mu_a\varphi = \mu_{ab}\varphi$.

Note: If $a, b\epsilon R^*$, $h_1, h_2\epsilon R$, then $R_a \circ R_b = R_b \circ R_a = R_{ab}$ and $T_{h_1} \circ T_{h_2} = T_{h_2} \circ T_{h_1} = T_{h_1 + h_2}$. Also, $R_1 = T_o = I$.

THEOREM: if $T\epsilon T'$, $h\epsilon R$, and we define $\tau_h T$ such that $\langle\tau_h T,\varphi\rangle = \langle T,\tau_{-h}\varphi\rangle$ for $\varphi\epsilon T$, then $\tau_h T\epsilon T'$. Also if $a\epsilon R^*$ and we define $\mu_a T$ such that $\langle\mu_a T,\varphi\rangle = \frac{1}{|a|} \langle T,\mu_{1/a}\varphi\rangle$ for all $\varphi\epsilon T$, then $\mu_a T\epsilon T'$.

Proof: Immediate since $\tau_h T = T \circ T_{-h}$ and $\mu_a T = T \circ R_a$, all of which are continuous and linear.

Remark: If $f\epsilon L^1(R)$ and $T = T_f$, then from the definitions and theorems above it is easy to check that

$$F[T_f] = T_{\hat{f}},\ D^p[T_f] = T_{f^{(p)}},\ \alpha T_f = T_{\alpha f},\ \tau_h T_f = T_{\tau_h f},\ \text{and}\ \mu_a T_f = T_{\mu_a f}\ .$$

Proposition: If $T\epsilon T'$, $\varphi\epsilon T$ and we define the convolution $T^*\varphi$ by

$$T^*\varphi(\eta) = \langle T,\tau_\eta\mu_{-1}\varphi\rangle$$

for all $\eta\epsilon R$, then $T^*\varphi\epsilon\ O_c$ and the function $\varphi \to T^*\varphi$ from T into O_c is continuous for each $T\epsilon T'$.

Proof: See Horvath [33] p. 420.

Note: If $f\epsilon L^1(R)$, $T = T_f$, then

$$T_f{}^*\varphi(\eta) = \langle T_f,\tau_\eta\mu_{-1}\varphi\rangle = \int_{-\infty}^{+\infty} f(\gamma)(\tau_\eta\mu_{-1}\varphi)(\gamma)\varphi\gamma$$

$$= \int_{-\infty}^{+\infty} f(\gamma)\ \varphi(\eta - \gamma)\varphi\gamma = \int_{-\infty}^{+\infty} f(\eta - \gamma)\ \varphi(\gamma)\varphi\gamma$$

$$= f^*\varphi(\eta)\ .$$

Dfn: If $S\epsilon O_c'$, $T\epsilon T'$, $\varphi\epsilon T$, we define the convolution T^*S such that

$$<T^*S, \varphi> = <S, (\mu_{-1}T)^* \varphi>$$

Note: This is well defined for, by the previous proposition $(\mu_{-1}T)^*\varphi\epsilon O_c'$. Furthermore, $T^*S\epsilon T'$ since $\varphi \to (\mu_{-1}T)^*\varphi$ and S are continuous. Also, this definition agrees with the previous one if $S = S_\tau$, $\tau\epsilon T$.

THEOREM: The Fourier transform maps O_c' isomorphically onto O_M, its inverse is the inverse Fourier transform which maps O_M isomorphically onto O_c' and if $S\epsilon O_c'$, $T\epsilon T'$, then

(1) $F[T*S] = F[T] \cdot F[S]$.

Furthermore, F also maps O_M isomorphically onto O_c', F^{-1} is its isomorphic inverse, and for $\alpha\epsilon O_M$, $T\epsilon T'$,

(2) $F[\alpha T] = F[\alpha]*F[T]$.

Formulas (1) and (2) also hold with F replaced by $\bar{F}^1$.

Proposition: If S_1, S_2, S_3 $\epsilon O_c'$, then

(1) $S_1{}^*S_2 \ \epsilon O_c'$.

(2) $(S_1{}^*S_2)^*S_3 = S_1{}^*(S_2{}^*S_3)$

(3) $S_1{}^*S_2 = S_2{}^*S_1$

Proof:

(1) If S_1, S_2 $\epsilon O_c'$ then $F[S_1]$, $F[S_2]$ ϵO_M and $F[S_1{}^*S_2] = F[S_1] \cdot F[S_2] \ \epsilon O_M$

$\bar{F}^1[F[S_1{}^*S_2]] = S_1{}^*S_2 \ \epsilon O_c'$.

(2) $F[(S_1{}^*S_2)\ {}^*S_3] = F[(S_1{}^*S_2)] \cdot F[S_3] = F[S_1] \cdot F[S_2] \cdot F[S_3]$

$$= F[S_1]\ F[(S_2{}^*S_3)] = F[S_1{}^*(S_2{}^*S_3)]$$

(2) If α: $R \to C$ is a function the map $M_\alpha[\varphi] = \alpha\varphi$ for all $\varphi \epsilon T$ is in $L(T)$ if, and only if, $\alpha \epsilon O_M$. Furthermore, M_α is invertible if and only if $1/\alpha \epsilon O_M$ and in this case, $\bar{M}_\alpha^1 = M_{1/\alpha}$.

(3) The previously defined functions D^p, T_h, $h \epsilon R$, R_a, $a \epsilon R^*$ are in $L(T)$.

Notation: For convenience we will denote the elements of T' by $\mathring{f}$, $\mathring{g}$, $\mathring{h}$, etc. and for $\varphi \epsilon T$ we will usually write $\int_{-\infty}^{+\infty} \mathring{f}(\gamma)\varphi(\gamma)d\gamma$ for $<\mathring{f},\varphi>$.

THEOREM: If $\mathring{f} \epsilon T'$ and for each $\varphi \epsilon T$ we define $C_{\mathring{f}}[\varphi] = f^*\varphi$, a necessary and sufficient condition for $C_{\mathring{f}}$ to be in $L(T)$ in that $f \epsilon O_c'$ (or equivalently, $F[\mathring{f}] \epsilon O_M$).

Proof: Suppose $C_{\mathring{f}} \epsilon L(T)$. Then $F \circ C_{\mathring{f}} \epsilon L(T)$ and if $\varphi \epsilon T$, $F \circ C_{\mathring{f}}[\varphi] = F[\mathring{f}^*\varphi] = F[\mathring{f}] \cdot F[\varphi] \epsilon T$. But every element in T can be expressed in the form $F[\varphi]$ so by a previous theorem, $F[\mathring{f}] \epsilon O_M$ (i.e. $\alpha\tau \epsilon T$ for all $\tau \epsilon T$ if, and only if, $\alpha \epsilon O_M$).

Conversely, if $\mathring{f} \epsilon O_c'$, then $F[\mathring{f}] \epsilon O_M$ so $M_{F[\mathring{f}]} \epsilon L(T)$. Furthermore, for each $\varphi \epsilon T$,

$$\bar{F}^1 \circ M_{F[\mathring{f}]} \circ F[\varphi] = \bar{F}^1[F[\mathring{f}] \cdot F[\varphi]] = \bar{F}^1[F[\mathring{f}^*\varphi]] = \mathring{f}^*\varphi = C_{\mathring{f}}[\varphi] .$$

Thus

$$C_{\mathring{f}} = \bar{F}^1 \circ M_{F[\mathring{f}]} \circ F \epsilon L(T). \quad \text{Q.E.D.}$$

Dfn: If $\mathring{f} \epsilon O_c'$, the transformation $C_{\mathring{f}}[\varphi] = \mathring{f}^*\varphi$ is in $L(T)$ and is called the convolution transformation of $\mathring{f}$. The set of convolution transformations in $L(T)$ will be denoted $L_c(T)$.

Proposition: If $C_{\mathring{f}} \epsilon L_c(T)$ and F the Fourier transformation then

$$F \circ C_{\mathring{f}} = M_{F[\mathring{f}]} \circ F.$$

By taking $\bar{F}^1$ of both sides we get (2).

(3) $F[S_1 * S_2] = F[S_1]\ F[S_2] = F[S_2]\ F[S_1] = F[S_2 * S_1]$ Q.E.D.

C. Continuous Linear Transformations on the Rapidly Decreasing Functions

Dfn: L(T) will denote the class of continuous linear transformation from T into T.

Dfn: If E is a non-empty set, we say E is an associative algebra over C if E is a vector space over C and there is a binary operation $\cdot$ from E x E into E such that for x, y, zϵE, aϵC,

(1) $(x + y) \cdot z = x \cdot z + y \cdot z$

(2) $x(y + z) = x \cdot y + x \cdot z$

(3) $a(x \cdot y) = (ax) \cdot y = x \cdot (ay)$

(4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Furthermore, if there is an element eϵE such that $e \cdot x = x \cdot e = x$ for all xϵE then we say E is an associative algebra over C with identity.

Dfn: If U, VϵL(T), zϵC, and $\varphi\epsilon$T, then we define the following:

(1) $(U + V)[\varphi] = U[\varphi] + V[\varphi]$

(2) $(Z \cdot U)[\varphi] = Z \cdot U[\varphi]$

(3) $(U \circ V)[\varphi] = U[V[\varphi]]$.

This definition makes L(T) into an associative algebra over C with identity. This identity is the function I such that $I[\varphi] = \varphi$ for all $\varphi\epsilon$T.

Dfn: An element UϵL(T) is said to be invertible if there exists an element $\bar{U}^1\epsilon L(T)$ such that $U \circ \bar{U}^1 = \bar{U}^1 \circ U = I$. $\bar{U}^1$ is called the inverse of U.

Examples:

(1) The Fourier transformation F is in L(T). Also $\bar{F}^1$ is in L(T) and since $F \circ \bar{F}^1[\varphi] = F[\bar{F}^1[\varphi]] = \varphi = I[\varphi]$ and $\bar{F}^1 \circ F[\varphi] = I[\varphi]$ for all $\varphi\epsilon$T, $\bar{F}^1$ is the inverse of F.

Note: $M_{F[\overset{\circ}{f}]} \epsilon L(T)$ since $F[\overset{\circ}{f}] \epsilon O_M$.

Proof: If $\varphi \epsilon T$ we have $F \circ C_{\overset{\circ}{f}}[\varphi] = F[\overset{\circ}{f} * \varphi] = F[\overset{\circ}{f}] \cdot F[\varphi]$, and $M_{F[\overset{\circ}{f}]} \circ F[\varphi]$ $= M_{F[\overset{\circ}{f}]} [F[\varphi]] = F[f] \cdot F[\varphi]$. Q.E.D.

Corollary: If $C_{\overset{\circ}{f}} \epsilon L_c(T)$ and F, $\bar{F}^1$ are the Fourier and inverse Fourier transformations respectively, then

$$F \circ C_{\overset{\circ}{f}} \circ \bar{F}^1 = M_{F[\overset{\circ}{f}]} .$$

Proof: By the above proposition,

$$F \circ C_{\overset{\circ}{f}} \circ \bar{F}^1 = (F \circ C_{\overset{\circ}{f}}) \circ \bar{F}^1 = (M_{F[\overset{\circ}{f}]} \circ F) \circ \bar{F}^1 = M_{F[\overset{\circ}{f}]} \circ (F \circ \bar{F}^1) = M_{F[\overset{\circ}{f}]} . \quad \text{Q.E.D.}$$

Proposition: If $C_{\overset{\circ}{f}}$ is a convolution transformation and $\frac{1}{F[\overset{\circ}{f}]} \epsilon O_M$, then $C_{\overset{\circ}{f}}$ is invertible and

$$\bar{C}^1_{\overset{\circ}{f}} = \bar{F}^1 \circ M_{1/F[\overset{\circ}{f}]} \circ F.$$

Furthermore, if $\overset{\circ}{g} = \bar{F}^1[\frac{1}{F[\overset{\circ}{f}]}]$, then $\overset{\circ}{g} \epsilon O'_c$ and $\bar{C}^1_{\overset{\circ}{f}} = C_{\overset{\circ}{g}}$.

Proof: Using the previous corollary and since

$$\bar{M}^1_{F[\overset{\circ}{f}]} = M_{1/F[\overset{\circ}{f}]} ,$$

we have

$$C_{\overset{\circ}{f}} \circ (\bar{F}^1 \circ M_{1/F[\overset{\circ}{f}]} \circ F) = (\bar{F}^1 \circ M_{F[\overset{\circ}{f}]} \circ F)\ (\bar{F}^1 \circ M_{1/F[\overset{\circ}{f}]} \circ F)$$

$$= \bar{F}^1 \circ M_{F[\overset{\circ}{f}]} \circ M_{1/F[\overset{\circ}{f}]} \circ F$$

$$= \bar{F}^1 \circ I \circ F = I .$$

Similarly $(\bar{F}^{1} \circ M_{1/F[\mathring{f}]} \circ F) \circ C_{\mathring{f}} = I$. Thus, the first part of the proposition is true. Since

$$\frac{1}{F[\mathring{f}]} \epsilon O_M \ , \qquad \text{then}$$

$$g = \bar{F}^{1}[\frac{1}{F[\mathring{f}]}] \epsilon O_c'$$

so $\bar{C}^{1}_{\underset{g}{\circ}} \epsilon L(T)$. Furthermore, if $\varphi \epsilon T$, then

$$\bar{C}^{1}_{\underset{f}{\circ}}[\varphi] = (\bar{F}^{1} \circ M_{1/F[f]} \circ F)[\varphi] = \bar{F}^{1}[\frac{1}{F[\mathring{f}]} \cdot F[\varphi]]$$

$$= \bar{F}^{1}[\frac{1}{F[\mathring{f}]}] * \bar{F}^{1}[F[\varphi]] = \mathring{g} * \varphi = C_{\mathring{g}}[\varphi] \ . \quad \text{Q.E.D.}$$

Examples:

(1) If $f \epsilon L^1(R)$, $F[\mathring{f}] \epsilon O_M$, then $C_{\mathring{f}} \epsilon L_c(T)$ and for each $\varphi \epsilon T$,

$$C_{\mathring{f}}[\varphi](\eta) = \mathring{f} * \varphi(\eta) = \int_{-\infty}^{+\infty} f(\eta - \gamma)\varphi(\gamma)\varphi\gamma \ .$$

(2) Let $\rho \epsilon N$ and $\delta^{(\rho)}$ be the ρ^{th} derivative of the Dirac measure. Then $\delta^{(\rho)} \epsilon O_c'$ and for $\varphi \epsilon T$,

$$C_{\delta^{(\rho)}}[\varphi](\eta) = \delta^{(\rho)} * \varphi(\eta) = < \delta^{(\rho)}, \tau_{\eta\mu_{-1}}\varphi >$$

$$= (-1)^{\rho} < \delta, (\tau_{\eta\mu_{-1}}\varphi)^{(\rho)} >$$

$$= (-1)^{\rho} < \delta, (-1)^{\rho}(\tau_{\eta\mu_{-1}}\varphi^{(\rho)}) >$$

$$= \tau_{\eta\mu^{-1}}\varphi^{(\rho)}(0) = \varphi^{(\rho)}(\eta)$$

Hence, $D^\rho \epsilon L_c(T)$ and $D^\rho = C_\delta(\rho)$. In particular, $I = C_\delta$.

(3) Let $f(\gamma) = \frac{\pi}{a} e^{-2\pi a|\gamma|}$, $a > 0$. Then $f \epsilon L^1(R)$ and

$$F[f](\eta) = \int_{-\infty}^{+\infty} e^{-2\pi i\eta\gamma}(\frac{\pi}{a} e^{-2\pi a|\gamma|})d\gamma = \frac{1}{a^2+\eta^2} \epsilon O_M .$$

Thus, $C_f \epsilon L_c(T)$. Furthermore, $\frac{1}{F[f]}(\eta) = a^2 + \eta^2 \epsilon O_M$ so C_f is invertible and $C_f = C_{\mathring{g}}$ where $\mathring{g} = \bar{F}^1[a^2 + \eta^2]$. But

$$\bar{F}^1[a^2 + \eta^2] = \bar{F}^1[a^2] + \bar{F}^1[\eta^2] = a^2\delta - \frac{1}{4\pi^2}\delta''$$

so

$$\mathring{g} = a^2\delta - \frac{1}{4\pi^2}\delta'' .$$

Proposition: If $L_c(T)$ denotes the class of all convolution transformations in $L(T)$, then $L_c(T)$ is a commutative subalgebra of $L(T)$ which contains the composition identity and the composition inverses if they exist.

Proof: Let $C_{\mathring{f}}, C_{\mathring{g}} \epsilon L_c(T)$ and $Z \epsilon C$. It is clear that $C_{\mathring{f}} + C_{\mathring{g}}, ZC_{\mathring{f}} \epsilon L_c(T)$ since $C_{\mathring{f}} + C_{\mathring{g}} = C_{\mathring{f}+\mathring{g}}$ and $ZC_{\mathring{f}} = C_{Z\mathring{f}}$. Since $\mathring{f}, \mathring{g} \epsilon O_c'$, we know $\mathring{f} * \mathring{g} \epsilon O_c'$ and if $\varphi \epsilon T$,

$$C_{\mathring{f}} \circ C_{\mathring{g}}[\varphi] = C_{\mathring{f}}[\mathring{g} * \varphi] = \mathring{f} * (\mathring{g} * \varphi)$$

But $\varphi \epsilon O_c'$ and by a previous proposition $\mathring{f} * (\mathring{g} * \varphi) = (\mathring{f} * \mathring{g}) * \varphi$ so

$$C_{\mathring{f}} C_{\mathring{g}} = C_{\mathring{f} * \mathring{g}} .$$

Furthermore, since $f * g = g * f$ we have

$$C_{\overset{\circ}{f}} \circ C_{\overset{\circ}{g}} = C_{\overset{\circ}{f} * \overset{\circ}{g}} = C_{\overset{\circ}{g} * \overset{\circ}{f}} = C_{\overset{\circ}{g}} * C_{\overset{\circ}{f}} .$$

Since we have already shown $C_\delta \epsilon L_c(T)$ is the identity and if $C_{\overset{\circ}{f}}$ is invertible, then $\bar{C}^1_{\overset{\circ}{f}} = C_{\overset{\circ}{g}}$ where $\overset{\circ}{g} = \bar{F}^1[\frac{1}{F[\overset{\circ}{f}]} \epsilon O'_c$ then $\bar{C}^1_{\overset{\circ}{f}} \epsilon L_c(T)$ and the proof is complete.

Dfn: For each $a \epsilon R^*$, let R_a be the transformation $R_a[\varphi] = \mu_a \varphi$ for all $\varphi \epsilon T$ and let $L_c a(T)$ denote the set of all transformations which can be written in the form

$$A_{a,\overset{\circ}{f}} = C_{\overset{\circ}{f}} \circ R_a$$

where

$$C_{\overset{\circ}{f}} \epsilon L_c(T) .$$

Note: $L_{c'}(T) = L_c(T)$.

Proposition: For each $a \epsilon R^*$, $L_c a(T)$ is a sub vector space of $L(T)$ and the map L_a: $L_c(T) \rightarrow L_c a(T)$ such that $L_a(C_{\overset{\circ}{f}}) = A_{a,\overset{\circ}{f}}$ is a linear isomorphism from the sub vector space $L_c(T)$ into $L_c a(T)$.

Proof: It is trivial that $L_c a(T)$ is a sub vector space of $L(T)$ since $A_{a,\overset{\circ}{f}} + A_{a,\overset{\circ}{g}} = (C_{\overset{\circ}{f}} + C_{\overset{\circ}{g}}) \circ R_a = C_{\overset{\circ}{f} + \overset{\circ}{g}} \circ R_a$ and $Z \cdot A_{a\overset{\circ}{f}} = C_{Z\overset{\circ}{f}} \circ R_a$.

Furthermore, since R_a is invertible, $C_{\overset{\circ}{f}} R_a = 0$ if, and only if, $C_{\overset{\circ}{f}} = 0$ so L_a is injective. It is surjective from the definition of $L_c a(T)$ and the linearity follows since $C_{\overset{\circ}{f}} + C_{\overset{\circ}{g}} = C_{\overset{\circ}{f} + \overset{\circ}{g}}$ and $ZC_{\overset{\circ}{f}} = C_{Z\overset{\circ}{f}}$. Q.E.D.

Proposition: If $a \epsilon R^*$, $A_{a,\overset{\circ}{f}} \epsilon L_c a(T)$, and $\overset{\circ}{g} \frac{1}{|a|} \mu_{1/a} \overset{\circ}{f}$, then $g \epsilon O'_c$ and $A_{a,\overset{\circ}{f}} = R_a \circ C_{\overset{\circ}{g}}$.

Proof: It is clear that $g \epsilon 0'_c$. Let $\varphi \epsilon T$, $\eta \epsilon R$. Then $A_{a,\mathring{f}} = C_{\mathring{f}} \circ R_a$ and

$$C_{\mathring{f}} \circ R[\varphi](\eta) = C_{\mathring{f}}[\mu_a \varphi](\eta) = \mathring{f} * (\mu_a \varphi)(\eta)$$

$$= < \mathring{f}, \tau_\eta \mu_{-1} \mu_a \varphi > = < \mathring{f}, \tau_\eta \mu_{-a} \varphi > .$$

Also

$$R_a \circ C_{\mathring{g}}[\varphi](\eta) = R_a[\mathring{g} * \varphi(a\eta)] = \mathring{g} * \varphi(a\eta)$$

$$= < \mathring{g}, \tau_{a\eta} \mu_{-1} \varphi > = \frac{1}{|a|} < \mu_{1/a} \mathring{f}, \tau_{a\eta} \mu_{-1} \varphi >$$

$$= < \mathring{f}, \mu_a \tau_{a\eta} \mu_{-1} \varphi > .$$

But if $\gamma \epsilon R$, we have

$$(\tau \eta \mu_{-a}) \varphi(\gamma) = \varphi(\mu_{-a} \tau_\eta(\gamma)) = \varphi(\mu_{-a}(\gamma - \eta)) = \varphi(a\eta - a\gamma)$$

and

$$(\mu_a \tau_{a\eta} \mu_{-1}) \varphi(\gamma) = \varphi(\mu_{-1} \tau_{a\eta} \mu_a(\gamma)) = \varphi(\mu_{-1} \tau_{a\eta}(a\gamma))$$

$$= \varphi(\mu_{-1}(a\gamma - a\eta)) = \varphi(a\eta - a\gamma) .$$

Hence, $\tau_\eta \mu_{-a} \varphi = \mu_a \tau_{a\eta} \mu_{-1} \varphi$ and it follows that $C_{\mathring{f}} \circ R_a = R_a \circ C_{\mathring{g}}$. Q.E.D.

Corollary: If $a \epsilon R^*$, $U = R_a \circ C_{\mathring{g}}$ where $\mathring{g} \epsilon 0'_c$, then $U \epsilon L_C a(T)$ where $U = C_{\mathring{f}} \circ R_a$ and $\mathring{f} = |a| \mu_a \mathring{g}$.

Proof: From the above proposition $C_{\mathring{f}} \circ R_a = R_a \circ C_{\mathring{h}}$ where $\mathring{h} = \frac{1}{|a|} \mu_{1/a} \mathring{f}$ $= \frac{1}{|a|} \mu_{1/a} |a| \mu_a \mathring{g} = \mathring{g}$. Q.E.D.

Corollary: If a, $b \in R^*$, $A_{a,\mathring{f}} \in L_C a(T)$ and $A_{b,\mathring{g}} \in L_C b(T)$ where $\mathring{f}, \mathring{g} \in O_C'$, then

$$A_{a,\mathring{f}} \circ A_{b,\mathring{g}} = A_{ab,\mathring{h}} \in L_C ab(T)$$

where $\mathring{h} = \mathring{f} * (|a|\mu_a \mathring{g})$.

Proof: By the above corollary we have

$$A_{a,\mathring{f}} \circ A_{b,\mathring{g}} = (C_{\mathring{f}} \circ R_a) \circ (C_{\mathring{g}} \circ R_b) = C_{\mathring{f}} \circ (R_a \circ C_{\mathring{g}}) \circ R_b$$

$$= C_{\mathring{f}} \circ (C_{|a|\mu_a \mathring{g}}) \circ R_a \circ R_b = C_{\mathring{f} * (|a|\mu_a \mathring{g}} \circ R_{ab} \text{ . Q.E.D.}$$

Note: $A_{-1,\mathring{f}} \circ A_{-1,\mathring{g}} = C_{\mathring{f} * \mu_{-1}\mathring{g}} \in L_C(T)$.

D. Continuously Infinite Matrices

Theorem: Let $U \in L(T)$ and for each $\eta \in R$, define $< \mathring{f}_\eta, \varphi > = U[\varphi](\eta)$ for all $\varphi \in T$. Then $\mathring{f}_\eta \in T'$ for all $\eta \in R$.

Proof: $\mathring{f}_\eta$ is clearly a linear form on T so let $N_\epsilon = \{Z \in C: |Z| < \epsilon\}$ where $\epsilon > 0$ be a neighborhood of 0 in C. Now let $M_\epsilon = \{\tau \in T: |\tau(\gamma)| < \epsilon, \gamma \in R\}$, then M_ϵ is a neighborhood of 0 in T so, from the continuity of U, there exists a neighborhood $\bar{M}$ of 0 in T such that $U[\varphi] \in M$ for all $\varphi \in \bar{M}$. Hence, if $\varphi \in \bar{M}$, the $| < \mathring{f}_\eta, \varphi > | = |U[\varphi](\eta)| < \epsilon$ so $< \mathring{f}_\eta, \varphi > \in N_\epsilon$ and f_η is continuous. Thus, $f_\eta \in T'$ for all $\eta \in R$.

Dfn: If $U \epsilon L(T)$ and the $\mathring{f}_\eta$, $\eta \epsilon R$ are as above then we say the family $(\mathring{f}_\eta)_{\eta \epsilon R}$ of elements in T' determines U.

Since, for each $\eta \epsilon R$, $\varphi \epsilon T$, we have $U[\varphi](\eta) = < \mathring{f}_\eta, \varphi > = \int_{-\infty}^{+\infty} \mathring{f}_\eta(\gamma)\varphi(\gamma)d\gamma$ we will denote the family $(\mathring{f}_\eta)_{\eta \epsilon R}$ by $[\mathring{f}(\eta,\gamma)]$ and call $[\mathring{f}(\eta,\gamma)]$ the matrix of U and write $U[\varphi](\eta) = \int_{-\infty}^{+\infty} \mathring{f}(\eta,\gamma) \ \varphi(\gamma)d\gamma$ or $U \sim [\mathring{f}(\eta,\gamma)]$.

Note: η is considered as a parameter while γ is the "variable of integration" of the temperate distribution $\mathring{f}_\eta$.

Examples:

(1) If F is the Fourier transform then $F \sim [e^{-2\pi i \eta \gamma}]$.

(2) If I is the Identity transformation then $I \sim [\tau_\eta \delta(\gamma)]$ since, if $\varphi \epsilon T$, then $< \tau_\eta \delta, \varphi > = < \delta, \tau_{-\eta} \varphi > = \int_{-\infty}^{+\infty} \delta(\gamma) \ \varphi(\gamma + \eta)d\gamma = \varphi(\eta) = I[\varphi](\eta)$.

Dfn: Two matrices $[\mathring{f}(\eta,\gamma)]$ and $[\mathring{g}(\eta,\gamma)]$ are said to be equivalent if they determine the same linear transformation U in L(T).

The two matrices $[\tau_\eta \ \delta(\gamma)]$ and $[\tau_\eta \ \mu_{-1}\delta]$ are equivalent since they both determine I. We consider two matrices as being equal if they determine the same linear transformation in L(T).

Dfn: Let M denote the class of matrices which is associated with some $U \epsilon L(T)$. Then M can be made into an associative algebra over C where if $Z \epsilon C$, $[\mathring{f}(\eta,\gamma)] \sim U$, $[\mathring{g}(\eta,\gamma)] \sim V$, U, $V \epsilon L(T)$, we define

(1) $[\mathring{f}(\eta,\gamma)] + [\mathring{g}(\eta,\gamma)] = [\mathring{h}_1(\eta,\gamma)]$ where $[\mathring{h}_1(\eta,\gamma)] \sim U + V$

(2) $z[\mathring{f}(\eta,\gamma)] = [\mathring{h}_2(\eta,\gamma)]$ where $[\mathring{h}_2(\eta,\gamma)] \sim zU$

(3) $[\mathring{f}(\eta,\gamma)] \cdot [\mathring{g}(\eta,\gamma)] = [\mathring{h}_3(\eta,\gamma)]$ where $[\mathring{h}_3(\eta,\gamma)] \sim U \circ V$.

Since $\int_{-\infty}^{+\infty} (\mathring{f}(\eta,\gamma) + \mathring{g}(\eta,\gamma)) \ \varphi(\gamma)d\gamma = < \mathring{f}_\eta + \mathring{g}_\eta, \varphi > = < \mathring{f}_\eta, \varphi > + < \mathring{g}_\eta, \varphi >$ $= \int_{-\infty}^{+\infty} \mathring{f}(\eta,\gamma) \ \varphi(\gamma)d\gamma + \int_{-\infty}^{+\infty} \mathring{g}(\eta,\gamma) \ \varphi(\gamma)d\gamma$ we have $\mathring{h}_1(\eta,\gamma) = \mathring{f}(\eta,\gamma) + \mathring{g}(\eta,\gamma)$.

Similarly,

$$\int_{-\infty}^{+\infty} z \cdot \mathring{f}(\eta,\gamma)\ \varphi(\gamma)d\gamma = z \cdot \int_{-\infty}^{+\infty} \mathring{f}(\eta,\gamma)\ \varphi(\gamma)d\gamma$$

so we have $\mathring{h}_2(\eta,\gamma) = z \cdot f(\eta,\gamma)$. Furthermore, from the definition of composition we have

$$\int_{-\infty}^{+\infty} \mathring{f}(\rho,\eta)\left(\int_{-\infty}^{+\infty} \mathring{g}(\eta,\gamma)\ \varphi(\gamma)d\gamma\right)d\eta = \int_{-\infty}^{+\infty} \mathring{h}_3(\rho,\gamma)\ \varphi(\gamma)d\gamma\ ,$$

so we usually write

$$\int_{-\infty}^{+\infty} \mathring{f}(\rho,\eta)\ \mathring{g}(\eta,\gamma)d\eta = \mathring{h}_3(\rho,\gamma)\ .$$

Notation: If $f \epsilon T'$, and $[\tau_\eta \mathring{f}(\gamma)]$ is a matrix, we will denote this matrix by $[\mathring{f}(\gamma - \eta)]$. Also we will denote the matrix $[\tau_{-\eta}\mathring{f}(\gamma)]$ by $[\mathring{f}(\eta + \gamma)]$.

Examples:

(1) Since $[\delta(\gamma - \eta) \sim I$ and $I + I = 2I$, $I \circ U = U \circ I = U$ for all $U \epsilon L(T)$, we have

$$[\delta(\gamma - \eta)] + [\delta(\gamma - \eta)] = [2\delta(\gamma - \eta)]\ ,$$

and

$$[\delta(\gamma - \eta)]\ [\mathring{f}(\eta,\gamma)] = [\mathring{f}(\eta,\gamma)] \cdot [\delta(\gamma - \eta))] = [f(\eta,\gamma)],$$

$$(\text{i.e.} \int_{-\infty}^{+\infty} \delta(\eta - \rho)\ \mathring{f}(\eta,\gamma)d\eta = \int_{-\infty}^{+\infty} \mathring{f}(\rho,\eta)\ \delta(\gamma - \eta)d\eta = \mathring{f}(\rho,\gamma))\ .$$

Since $[\delta(\eta - \gamma)] = [\delta(\gamma - \eta)]$ the same results are obtained if $[\delta(\gamma - \eta)]$ is replaced by $[\delta(\eta - \gamma)]$.

(2) Since $F \sim [e^{-2\pi i \eta\gamma}]$, $\bar{F}^1 \sim [e^{2\pi i\eta\gamma}]$ and $F \circ \bar{F}^1 = \bar{F}^1 \circ F = I$, we have

$$[e^{-2\pi i\eta\gamma}] = [e^{2\pi i\eta\gamma}]^{-1}$$

and

$$[e^{-2\pi i\eta\gamma}]\ [e^{2\pi i\eta\gamma}] = [e^{2\pi i\eta\gamma}]\ [e^{-2i\eta\gamma}] = [\delta(\eta - \gamma)]$$

or

$$\int_{-\infty}^{+\infty} e^{-2\pi i\rho\eta}\ e^{2\pi i\eta\gamma}\ d\eta = \int_{-\infty}^{+\infty} e^{2\pi i\rho\eta}\ e^{-2\pi i\eta\gamma}\ d\eta = \delta(\rho - \gamma)\ .$$

(3) $R_{-1} \sim [\tau_{-\eta}\delta(\gamma)] = [\delta(\gamma + \eta)]$ since if $\varphi\epsilon T$, $\eta\epsilon R$,

$$< \tau_{-\eta}\delta, \varphi > = < \delta, \tau_{\eta}\varphi > = \varphi(-\eta) = R_{-1}[\varphi](\eta)\ .$$

(4) $D^{\rho} \sim [(-1)^{\rho}\quad \delta^{(\rho)}(\gamma - \eta)]$ since if $\varphi\epsilon T$, $\eta\epsilon R$,

$$< (-1)^{\rho}\ \tau_{\eta}\delta^{(\rho)},\ \varphi > = (-1)^{\rho} < \delta,\ (-1)^{\rho}\ \tau_{-\eta}\varphi^{(\rho)} > = \varphi^{(\rho)}(\eta) = D^{\rho}[\varphi](\eta)\ .$$

(5) If $\alpha\epsilon O_M$ then $\mu_a \sim [\alpha(\eta)\ \delta(\gamma - \eta)] = [\alpha(\gamma)\ \delta(\gamma - \eta)] =$ since if $\varphi\epsilon T$, $\eta\epsilon R$,

$$< \alpha(\eta)\tau_{\eta}\delta,\ \varphi > = \alpha(\eta) < \delta,\ \tau_{-\eta}\varphi > = \alpha(\eta)\ \varphi(\eta) = \mu_a[\varphi](\eta)$$

and

$$< \alpha(\tau_{\eta}\delta),\ \varphi > = < \tau_{\eta}\delta,\ \alpha\varphi > = < \delta,\ \tau_{-\eta}(\alpha\varphi) >$$

$$= \alpha(\eta)\ \varphi(\eta) = \mu_a[\varphi](\eta)\ .$$

Remark: Since $M_\alpha \circ D^\rho[\varphi](\eta) = \alpha(\eta)\ \varphi^{(\rho)}(\eta)$ then $M_\alpha \circ D^\rho \sim [\alpha(\eta)(-1)^\rho \delta^{(\rho)}(\gamma - \eta)]$ for if $\varphi \epsilon T$, $\eta \epsilon R$,

$$< \alpha(\eta)(-1)^\rho\ \tau_\eta \delta^{(\rho)}, \varphi > = (-1)^\rho\ \alpha(\eta) < \delta, (-1)^\rho\ \tau_{-\eta}\varphi^{(\rho)} > = \alpha(\eta)\ \varphi^{(\rho)}(\eta)\ .$$

Thus,

$$[\alpha(\eta)\ \delta(\gamma - \eta)][(-1)^{(\rho)}\delta^{(\rho)}(\gamma - \eta)] = [\alpha(\eta)(-1)^\rho\delta^{(\rho)}(\gamma - \eta)]$$

or

$$\int_{-\infty}^{+\infty} \alpha(\rho)\ \delta(\eta - \rho)(-1)^{(\rho)}\delta^{(\rho)}(\gamma - \eta)d\eta = \alpha(\rho)(-1)^\rho\delta^{(\rho)}(\gamma - \rho)\ .$$

(6) If $C_{\mathring{f}} \epsilon L_C(T)$ then $C_{\mathring{f}} \sim [\mu_{-1}\tau_{-\eta}f(\gamma)]$ since if $\varphi \epsilon T$, $\eta \epsilon R$,

$$< \mu_{-1}\tau_{-\eta}\mathring{f}, \varphi > = < \mathring{f}, \tau_\eta\mu_{-1}\varphi > = \mathring{f} * \varphi(\eta) = C_{\mathring{f}}[\varphi](\eta)\ .$$

If f is a function then $\mu_{-1}\tau_{-\eta}f(\gamma) = f(\eta - \gamma)$ so we denote the matrix of $C_{\mathring{f}}$ by $[\mathring{f}(\eta - \gamma)]$. Matrices determining convolution transformations will be called circulant.

If $C_{\mathring{f}}$, $C_{\mathring{g}} \epsilon L_C(T)$, $C_{\mathring{f}} \sim [\mathring{f}(\eta - \gamma)]$, $C_{\mathring{g}} \sim [\mathring{g}(\eta - \gamma)]$, then $C_{\mathring{f}} \circ C_{\mathring{g}} = C_{\mathring{f} * \mathring{g}}$ so that

$$[\mathring{f}(\eta - \gamma)][\mathring{g}(\eta - \gamma)] = [\mathring{f} * \mathring{g}(\eta - \gamma)]$$

or

$$\int_{-\infty}^{+\infty} \mathring{f}(\rho - \eta)\ \mathring{g}(\eta - \gamma)d\eta = \mathring{f} * \mathring{g}(\rho - \gamma)\ .$$

Furthermore, if $C_{\mathring{f}}$ is invertible, then $C_{\mathring{f}}^{-1} = C_{\mathring{h}}$ where $\mathring{h} = F^{-1}[\frac{1}{F[\mathring{f}]}]$ so that

(8) Let $L_{C_a}(T)$ denote the elements of $L(T)$ which can be written in the form $B_{a,f} = R_a \circ C_{\mathring{f}}$ where $C_{\mathring{f}} \epsilon L_C(T)$. Then $B_{a,\mathring{f}} \sim [\mu_{-1}\tau_{-a\eta}\mathring{f}]$ since if $\varphi \epsilon T$, $\eta \epsilon R$, we have

$$< \mu_{-1}\tau_{-a\eta}\mathring{f},\varphi > = < \mathring{f},\tau_{a\eta}\mu_{-1}\varphi > = R_a[< \mathring{f},\tau_{\eta}\mu_{-1}\varphi >] = R_a \circ C_{\mathring{f}}[\varphi](\eta)$$

If f is a function, then $\mu_{-1}\tau_{a\eta}f(\gamma) = f(a\eta - \gamma)$, so we denote the matrix of $B_{a,\mathring{f}} \epsilon L_{C_a}(T)$ by $[\mathring{f}(a\eta - \gamma)]$.

(9) Since $A_{a,\mathring{f}} \circ A_{b,\mathring{g}} = A_{ab,\mathring{f}} * (|a|\mu_a\mathring{g})$ we have

$$[\frac{1}{|a|}\mathring{f}(\eta - \frac{1}{a}\gamma)][\frac{1}{|b|}\mathring{g}(\eta - \frac{1}{b}\gamma)] = [\frac{1}{|ab|}\mathring{f} * (|a|\mu_a\mathring{g})(\eta - \frac{1}{ab}\gamma)]$$

or

$$\int_{-\infty}^{+\infty}\frac{1}{|a|}\mathring{f}(\rho - \frac{1}{a}\eta)\frac{1}{|b|}\mathring{g}(\eta - \frac{1}{b}\gamma)d\eta = \frac{1}{|ab|}\mathring{f} * (|a|\mu_a\mathring{g})(\rho - \frac{1}{ab}\gamma)$$

$$= \frac{1}{|b|}\mathring{f} * (\mu_b\mathring{g})(\rho - \frac{1}{ab}\gamma)$$

Note: If f, $g\epsilon L^1(R)$, then

$$\int_{-\infty}^{+\infty}\frac{1}{|a|}f(\rho - \frac{1}{a}\eta)\frac{1}{|b|}g(\eta - \frac{1}{b}\eta)d\eta = \frac{1}{|a||b|}\int_{-\infty}^{+\infty}f(\rho - \frac{1}{a}\eta)\ g(\eta - \frac{1}{b})d\eta\ .$$

Letting $\bar{\eta} = \frac{1}{a}\eta$, we have

$$\int_{-\infty}^{+\infty}\frac{1}{|a|}f(\rho - \frac{1}{a}\eta)\frac{1}{|b|}g(\eta - \frac{1}{b}\gamma)d\eta = \frac{1}{|a||b|}\int_{-\infty}^{+\infty}f(\rho - \bar{\eta})\ g(a\bar{\eta} - \frac{1}{b}\gamma)|a|d\bar{\eta}$$

$$= \frac{1}{|b|}\int_{-\infty}^{+\infty}f(\rho - \bar{\eta})(\mu_a g)(\bar{\eta} - \frac{1}{ab}\gamma)d\bar{\eta}$$

$$[\mathring{f}(\eta - \gamma)][\bar{F}^1[\frac{1}{F[\mathring{f}]}](\eta - \gamma)] = [\delta(\eta - \gamma)]$$

or

$$\int_{-\infty}^{+\infty} \mathring{f}(\rho - \eta)\ \bar{F}^1[\frac{1}{F[\mathring{f}]}](\eta - \gamma)d\eta = \delta(\eta - \gamma) \ .$$

By writing $\bar{F}^1$ and F in its integral notation, we have

$$\int_{-\infty}^{+\infty} f(\rho - \eta)[\int_{-\infty}^{+\infty} e^{2\pi i s(\eta - \gamma)}(\int_{-\infty}^{+\infty} e^{-2\pi i s\xi}\ \mathring{f}(\xi)d\xi)^{-1}ds]d\eta = \delta(\rho - \gamma) \ .$$

(7) Let $A_{a,\mathring{f}} = C_{\mathring{f}} \circ R_a \in L_C(T)$. Then $A_{a,\mathring{f}} \sim [\frac{1}{|a|}\mu_{1/a}\mu_{-1}\tau_{-\eta}\mathring{f}(\gamma)]$ since if $\varphi \in T, \eta \in R$,

$$< \frac{1}{|a|}\mu_{1/a}\mu_{-1}\tau_{-\eta}\mathring{f},\varphi > = \frac{1}{|a|} < \mu_{-1/a}\tau_{-\eta}\mathring{f},\varphi >$$

$$= < \mathring{f},\tau_{\eta}\mu_{-a}\varphi > = < \mathring{f},\tau_{\eta}\mu_{-1}\mu_a\varphi >$$

$$= < \mathring{f},\tau_{\eta}\mu_{-1}(R_a[\varphi]) > = \mathring{f} * R_a[\varphi](\eta)$$

$$= C_{\mathring{f}} \circ R_a[\varphi](\eta) \ .$$

If f is a function we have

$$\frac{1}{|a|}\mu_{1/a}\mu_{-1}\tau_{-\eta}f(\gamma) = \frac{1}{|a|} f(\tau_{-\eta} \cdot \mu_{-1} \cdot \mu_{1/a}(\gamma)) = \frac{1}{|a|} f(\eta - \frac{1}{a}\gamma)$$

so we usually denote $[\frac{1}{|a|}\mu_{1/a}\mu_{-1}\tau_{-\eta}\mathring{f}(\gamma)]$ by $[\frac{1}{|a|} f(\eta - \frac{1}{a}\gamma)]$. In particular,

$$A_{-1,\mathring{f}} \sim [\mathring{f}(\eta + \gamma)].$$

$$= \frac{1}{|b|}\int_{-\infty}^{+\infty} f([\rho - \frac{1}{ab}\gamma] - \bar{\eta})\ \mu_a g(\bar{\eta})d\bar{\eta} = \frac{1}{|b|}\ f * (\mu_a g)(\rho - \frac{1}{ab}\gamma) .$$

Hence, we see the integrals agree if $\mathring{f} = f$, $\mathring{g} = g \epsilon L^1(R)$.

(10) Let a, $b\epsilon R^*$, $B_{a,\mathring{f}}$, $B_{b,\mathring{g}} \epsilon L_{C_a}(T)$ where f, $g\epsilon O'_C$. Then by using a previous proposition we have

$$B_{a,\mathring{f}} \circ B_{b,\mathring{g}} = (R_{\mathring{a}} \circ C_{\mathring{f}})(R_{\mathring{b}} \circ C_{\mathring{g}}) = R_a \circ (C_{\mathring{f}}R_b)C_{\mathring{g}}$$

$$= R_{ab} \circ (C(\frac{1}{|b|}\ \mu_{1/b}\mathring{f}) * \mathring{g}) = B_{ab}(\frac{1}{|b|}\ \mu_{1/b}\mathring{f}) * \mathring{g}\epsilon L_{C_{ab}}(T).$$

Thus,

$$[\mathring{f}(a\eta - \gamma)][\mathring{g}(b\eta - \gamma)] = [(\frac{1}{|b|}\ \mu_{1/b}\mathring{f}) * \mathring{g}(ab\eta - \gamma)] ,$$

or

$$\int_{-\infty}^{+\infty} \mathring{f}(a\rho - \eta)\ \mathring{g}(b\eta - \gamma)d\eta = \frac{1}{|b|}\ (\mu_{1/b}\mathring{f}) * \mathring{g}(ab\eta - \gamma) .$$

Note: If f, $g\epsilon L^1(R)$ then

$$\int_{-\infty}^{+\infty} f(a\rho - \eta)\ g(b\eta - \gamma)d\eta = \int_{-\infty}^{+\infty} f(a\rho - \frac{1}{b}\bar{\eta})\ g(\bar{\eta} - \gamma)\ \frac{1}{|b|}\ d\bar{\eta} \text{ where } \bar{\eta} = b\eta .$$

$$= \frac{1}{|b|}\int_{-\infty}^{+\infty} f(\ \frac{1}{b}(ab\rho - \bar{\eta}))\ g(\bar{\eta} - \gamma)d\bar{\eta}$$

$$= \frac{1}{|b|}\int_{-\infty}^{+\infty} (\mu_{1/b}f)(ab\rho - \bar{\eta})\ g(\bar{\eta} - \gamma)d\bar{\eta}$$

$$= \frac{1}{|b|}\int_{-\infty}^{+\infty} (\mu_{1/b}f)(ab\rho - \gamma - \bar{\eta})\ g(\bar{\eta})d\bar{\eta}$$

Hence, this definition of the integral agrees if $\mathring{f}, \mathring{g} \epsilon L^1(R)$.

(11) Let $\alpha \epsilon O_M$, $\mathring{f} \epsilon O_C'$, and $a \epsilon R^*$. Then if $\varphi \epsilon T$, $\eta \epsilon R$,

$$M_\alpha \circ B_{a,f}[\varphi](\eta) = M_\alpha[R_a \circ C_{\mathring{f}}[\varphi]](\eta) = \alpha(\eta)\mathring{f} * \varphi(a\eta) = U[\varphi](\eta) .$$

Hence, since $M_\alpha \sim [\alpha(\eta)\ \delta(\eta - \gamma)]$, $B_{a,\mathring{f}} \sim [\mathring{f}(a\eta - \gamma)]$ we have

$$[\alpha(\eta)\ \delta(\eta - \gamma)][\mathring{f}(a\eta - \gamma)] = [\alpha(\eta)\mathring{f}(a\eta - \gamma)]$$

or

$$\int_{-\infty}^{+\infty} \alpha(\rho)\ \delta(\rho - \eta)\ \mathring{f}(a\eta - \gamma)d\eta = \alpha(\rho)\ \mathring{f}(a\rho - \gamma) .$$

The transpose of a continuously infinite matrix:

Dfn: For each $U \epsilon L(T)$ we define the transpose of U - denoted ${}^tU$ - to be the map from T' into T' such that for each $\mathring{f} \epsilon T'$, ${}^tU[\mathring{f}] = \mathring{f} \circ U$ (U is continuous so $\mathring{f} \circ U \epsilon T'$).

Note: For each $\varphi \epsilon T$, $< {}^tU[\mathring{f}], \varphi > = < \mathring{f}, U[\varphi] >$.

Note: If $U \epsilon L(T)$ then ${}^tU$ is linear for if $\mathring{f}, \mathring{g} \epsilon T'$, $z \epsilon C$, and $\varphi \epsilon T$,

$$< {}^tU[\mathring{f} + \mathring{g}], \varphi > = < \mathring{f} + \mathring{g}, U[\varphi] > = < \mathring{f}, U[\varphi] > + \mathring{g}, U[\varphi] >$$

$$= < {}^tU[\mathring{f}], \varphi > + < {}^tU[\mathring{g}], \varphi >$$

and

$$< {}^tU[z\mathring{f}], \varphi > = < zf, U[\varphi] > = z < \mathring{f}, U[\varphi] >$$

$$= z \cdot < {}^tU[\mathring{f}], \varphi > = < z \cdot {}^tU[\mathring{f}[, \varphi > .$$

Note: If F is the Fourier transformation in L(T) then for each $\mathring{f}\epsilon T'$, it is easy to see that $F[\mathring{f}] = {}^tF[\mathring{f}]$.

Proposition: If U, $V\epsilon L(T)$, then ${}^t(U + V) = {}^tU + {}^tV$, and ${}^t(U \circ V) = {}^tV \circ {}^tU$.

Proof: Let $\mathring{f}\epsilon T'$, $\varphi\epsilon T$, then

$$< {}^t(U + V)[f], \varphi > = < f,(U + V)[\varphi] > = < f, U[\varphi] > + < f, V[\varphi] >$$

$$= < {}^tU[f] + {}^tV[f], \varphi > = < ({}^tU + {}^tV)[f], \varphi > ,$$

and

$$< {}^t(U \circ V)[\mathring{f}], \varphi > = < \mathring{f}, U \circ V[\varphi] > = < ({}^tU[\mathring{f}], V[\varphi] >$$

$$= < {}^tV[{}^tU[\mathring{f}]], \varphi > = < {}^tV \circ {}^tU[\mathring{f}], \varphi > .$$

Proposition: If $I\epsilon L(T)$ is the identity transformation, the ${}^tI$ is the identity transformation on T'. If $U\epsilon L(T)$ and $\bar{U}^1\epsilon L(T)$, then ${}^tU$ is invertible and $({}^tU)^{-1} = {}^t(\bar{U}^1)$.

Proof: If $\varphi\epsilon T$, $\mathring{f}\epsilon T'$, the

$$< {}^tI[\mathring{f}], \varphi > = < \mathring{f}, I[\varphi] > = < \mathring{f}, \varphi > .$$

Hence ${}^tI[\mathring{f}] = \mathring{f}$ for all $\mathring{f}\epsilon T'$.

Using the previous proposition and letting I' be the identity map on T', we have

$${}^tU \circ {}^t(\bar{U}^1) = {}^t(\bar{U}^1 \circ U) = {}^tI = I'$$

and similarly ${}^t(\bar{U}^1) \circ {}^tU = I'$. Hence ${}^t(\bar{U}^1) = ({}^tU)^{-1}$ Q.E.D.

E. The Finite Matrix as a Special Case of the Continuously Infinite

Dfn: Let Θ denote the set of $\theta \epsilon T$ such that $\theta(0) = 1$ and $\theta(n) = 0$, $n = \pm 1, \pm 2, \ldots$.

Dfn: For each $\theta \epsilon \Theta$, $i, j \epsilon Z$, define

$$V_{\theta_i}^j [\varphi](\eta) = \varphi(j)\theta(\eta - i)$$

for all $\varphi \epsilon T$, $\eta \epsilon R$.

Proposition: $V_{\theta_i}^j \epsilon L(T)$ for all $\theta \epsilon \Theta$, $i, j \epsilon Z$.

Proof: Let $i = 0$ and suppose $N_\epsilon = \{\Psi \epsilon T: \ (1 + \gamma^2)^k |\Psi^{(p)}(\gamma)| < \epsilon,$ $p \leq m\}$ is a neighborhood of 0 in T. Take $M_\epsilon = \{\Psi \epsilon T: \ |\Psi(\gamma)| < \delta\}$ where

$$\delta = (\max_{\substack{p \leq m \\ \gamma \epsilon R}} \{(1 + \gamma^2)^k |\theta^{(p)}(\gamma)|\})^{-1} \cdot \epsilon ,$$

then if $\varphi \epsilon M_\epsilon$, it is clear that $V_{\theta_o}^j [\varphi] = \varphi(j)\theta \epsilon N_\epsilon$ so that $V_{\theta_o}^j$ is continuous at the origin. But $V_{\theta_o}^j$ is linear so it is continuous. $V_{\theta_i}^j$ is continuous for any $i \epsilon Z$ since $V_{\theta_i}^j = T_i \circ V_{\theta_o}^j$ where T_i: $\varphi \rightarrow \tau_i \varphi$ which is continuous. Q.E.D.

Dfn: For each $n \epsilon N^*$, $\theta \epsilon \Theta$, let $L_\theta^n(T)$ denote the subset of $L(T)$ which can be written in the form:

$$U \epsilon L_\theta^n(T) \ = > U = \sum_{j=1}^{n} (\sum_{j=1}^{n} a_{ij} \ V_{\theta_i}^j)$$

where $a_{ij} \epsilon C$ and $V_{\theta_i}^j$ is as defined above.

Proposition: If $i, j, k, l \epsilon N$, $\theta \epsilon \Theta$, then

$$V_{\theta_i}^j \circ V_{\theta_k}^l = \begin{cases} 0 & \text{if } j \neq k \\ V_{\theta_i}^l & \text{if } j = k \end{cases} .$$

Proof: If $\varphi \epsilon T$, $\rho \epsilon R$, we have

$$V^j_{\theta_i} \circ V^1_{\theta_k}[\varphi(\gamma)](\rho) = V^j_{\theta_i}[\varphi(1)\theta(\eta - k)](\rho) = \varphi(1)\theta(j - k)\theta(\rho - i)$$

$$= \begin{cases} 0 \text{ if } j \neq k \text{ (since } \theta(j - k) = 0) \\ \varphi(1)\theta(\rho - i) \text{ if } j = k \text{ (since } \theta(0) = 1) \end{cases}$$

$$= \begin{cases} 0 \text{ if } j \neq k \\ V^1_{\theta_i}[\varphi](\rho) \text{ if } j = k \end{cases} \qquad \text{Q.E.D.}$$

Proposition: For each $n \epsilon N$, $\theta \epsilon \Theta$, $L^n_\theta(T)$ form a subalgebra of $L(T)$. Furthermore, if

$$U_1 = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij} V^j_{\theta_i}), \quad U_2 = \sum_{i=1}^{n}(\sum_{j=1}^{n} b_{ij} V^j_{\theta_i}),$$

and $Z \epsilon C$, then

(1) $U_1 + U_2 = \Sigma\Sigma(a_{ij} + b_{ij})V^j_{\theta_i}$

(2) $ZU_1 = \Sigma\Sigma\, Za_{ij} V^j_{\theta_i}$

(3) $U_1 \circ U_2 = \Sigma\Sigma\, c_{ij} V^j_{\theta_i}$ where $C_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Proof: (1) and (2) are trivial. Also

$$U_1 \circ U_2 = (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} V^j_{\theta_i}) \circ (\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} V^j_{\theta_i})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{ij}b_{kl} V^j_{\theta_i} \circ V^l_{\theta_u} .$$

But $V_{\theta_i}^{j} \circ V_{\theta_n}^{1} = 0$ if $j \neq k$ and $= V_{\theta_i}^{1}$ if $j = k$ so we have

$$U_1 \circ U_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kl} V_{\theta_i}^{1} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{kl}\right) V_{\theta_i}^{1} \qquad \text{Q.E.D.}$$

The following corollary follows immediately from the above proposition.

Corollary: If $M_n(C)$ denotes the n x n square matrices with entries in C, the maps f_n,θ: $L_\theta^n \to M_n(C)$ such that for each $U \epsilon L_\theta^n(T)$,

$$f_{n,\theta}(U) = (a_{ij})_{i,j = 1,n}$$

where $U = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} V_{\theta_i}^{j}\right)$ are algebra isomorphisms.

Hence, we can consider the finite matrices as a subset of the continuously infinite matrices associated with L(T). In a similar fashion we can show that the n x m matrices can be considered as a subset of the continuously infinite matrices.

Note: $V_{\theta_i}^{j} \sim [\theta(\eta - i)(\tau_j \delta(\gamma))] = [\theta(\eta - i)\delta(\gamma - j)]$ for if $\varphi \epsilon T$, $\eta \epsilon R$, we have

$$< \theta(\eta - i)\tau_j \delta, \varphi > = \theta(\eta - i) < \delta, \tau_{-j}\varphi > = \theta(\eta - i)\varphi(j) = V_{\theta_i}^{j}[\varphi](\eta) \ .$$

Thus, for each $n \epsilon N$, $\theta \epsilon \Theta$, matrices of the form $\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}[\theta(\eta - i)\delta(\gamma - j)]\right)$ are isomorphic to $M_n(C)$.

F. Matrices with Function Entries

1. Finite Matrices

Dfn: Let $n \epsilon N^*$ and for each $1 \leq i, j \leq n$ let σ_{ij}: $R \to R$ be a function. If for each $t \epsilon R$, we define $S(t)$ to be the matrix $(\sigma_{ij}(t))$, we say that S is a matrix whose entries are the function σ_{ij} and write $S = (\sigma_{ij})$.

Dfn: If $S = (\sigma_{ij})$, $T = (\tau_{ij})$ are matrices then we define $S + T$ to be the matrix (υ_{ij}) where $\upsilon_{ij}(t) = \sigma_{ij}(t) + \tau_{ij}(t)$ and $S \cdot T$ to be the matrix (μ_{ij}) where $\mu_{ij}(t) = \sum_{k=1}^{n} \sigma_{ik}(t)\tau_{kj}(t)$ for all $t \epsilon R$.

Note: If δ_{ij}: $R \to R$ is much that for each $t \epsilon R$, $\delta_{ij}(t) = 0$ if $i \neq j$ and $\delta_{ij}(t) = 1$ if $i = j$ then the matrix $I = (\delta_{ij})$ is the identity matrix, for if $S \cdot I = (\mu_{ij})$ then

$$\mu_{ij}(t) = \sum_{k=1}^{n} \sigma_{ij}(t)\delta_{kj}(t) = \sigma_{ij}(t)\delta_{jj}(t) = \sigma_{ij}(t)$$

for all $t \epsilon R$, so $S \cdot I = S$. Similarly, $I \cdot S = S$.

Dfn: If $S = (\sigma_{ij})$ in a matrix, then for each $t \epsilon R$, we define $\nabla(t) = |(\sigma_{ij}(t))|$ where $|(\sigma_{ij}(t))|$ is the determinant of the matrix $(\sigma_{ij}(t))$. We call the function ∇ the determinant of S.

Dfn: If $S = (\sigma_{ij})$ is a matrix, then we say S is invertible if there exists a matrix $S^{-1} = (\sigma_{ij})$ such that $S \cdot S^{-1} = S^{-1} \cdot S = I$.

Note: If $S = (\sigma_{ij})$ is invertible and $S^{-1} = (\bar{\sigma}_{ij})$ is the inverse of S, then for each $t \epsilon R$, $S(t) \cdot S^{-1}(t) = S^{-1}(t) \cdot S(t) = I(t)$ so we see that S is invertible if, and only if, $S(t)$ is invertible for all $t \epsilon R$. Hence, we know S is invertible if, and only if, $\nabla(t) \neq 0$ for any $t \epsilon R$.

Dfn: If f: $R \to R$ is a function, $S = (\sigma_{ij})$ a matrix, then we define $f \cdot S$ to be the matrix (μ_{ij}) where $\mu_{ij}(t) = f(t) \cdot \sigma_{ij}(t)$ for all $t \epsilon R$.

Dfn: If $S = (\sigma_{ij})$ is a matrix, then for each $t \epsilon R$, we define adj(S)(t) to be the adjoint of the matrix $S(t) = (\sigma_{ij}(t))$. We call the matrix adj(S) the adjoint of the matrix S. Furthermore, for each $t \epsilon R$ we define $S^T(t)$ to be the transpose of the matrix $(\sigma_{ij}(t))$. We call the matrix S^T the transpose of S.

Note: It is easy to see that $S^T = (\hat{\sigma}_{ij})$ where $\hat{\sigma}_{ij} = \sigma_{ji}$.

Dfn: If $S = (\sigma_{ij})$ is a matrix, we say S is differentiable if σ_{ij} is differentiable for all $1 \le i, j \le n$ and defined dS to be the matrix (σ'_{ij}).

We say S is continuously differentiable if σ'_{ij} is continuous for all $1 \le i, j \le n$.

Proposition: If $S = (\sigma_{ij})$ is differentiable (respectively continuously differentiable) then the determinant ∇ is differentiable (respectively continuously differentiable).

Proof: Immediate since ∇ can be written as sums, differences, and products of the σ_{ij}.

Proposition: If $S = (\sigma_{ij})$ is a differentiable (respectively continuously differentiable) matrix then S^T and adj(S) are differentiable (respectively continuously differentiable).

Proof: S^T is clearly differentiable (respectively continuously differentiable) and since entry is the adj(S) in a sum, difference, and product of the entries of S, it is clearly differentiable (respectively continuously differentiable).

Proposition: If $S = (\sigma_{ij})$ is invertible, then $S^{-1} = \frac{1}{\nabla}(\text{adj} \cdot (S))^T$.

Proof: Immediate, since $S^{-1}(t) = \frac{1}{\nabla}(t)(\text{adj } (S(t)))^T$ for all $t \in R$.

Proposition: If $f\colon R \to R$ is differentiable (respectively continuously differentiable) and $S = (\sigma_{ij})$ is a differentiable(respectively continuously differentiable) matrix then $f \cdot S$ is a differentiable (respectively continuously differentiable) matrix.

Proof: trivial.

Proposition: If $S = (\sigma_{ij})$ is differentiable (respectively continuously differentiable) and S^{-1} exists, then S^{-1} is differentiable (respectively continuously differentiable).

Proof: We know ∇ is differentiable (respectively continuously differentiable) and since S^{-1} exists, $\nabla(t) \neq 0$ for every $t \epsilon R$ so $\frac{1}{\nabla}$ is also. By the three previous propositions we have $S^{-1} = \frac{1}{\nabla}(\mathrm{adj}(S))^T$ is differentiable (respectively continuously differentiable). Q.E.D.

Dfn: If $S = (\sigma_{ij})$ is a differentiable matrix and S is invertible, then we define the right Volterra derivative of S to be the matrix $S^{-1}dS$ and denote it by DS.

Note: From the previous proposition we have dS^{-1} exists and hence, $DS^{-1} = SdS^{-1}$.

Proposition: If $S = (\sigma_{ij})$ and $T = (\tau_{ij})$ are differentiable matrices, then $S \cdot T$ is differentiable and $d(S \cdot T) = (dS) \cdot T + S \cdot (dT)$.

Proof: $S \cdot T = (\mu_{ij})$ where

$$\mu_{ij} = \sum_{k=1}^{n} \sigma_{ik}\tau_{kj} \ .$$

Hence μ_{ij} is differentiable so $S \cdot T$ is differentiable and

$$\mu'_{ij} = (\sum_k \sigma_{ik}\tau_{kj})' = \sum_k (\sigma'_{ik}\tau_{kj} + \sigma_{ik}\tau'_{kj}) = \sum_k \sigma'_{ik}\tau_{kj} + \sum_k \sigma'_{ik}\tau_{kj} \ .$$

But clearly if $\mu_{ij} = \sum_k \sigma'_{ik}\tau_{kj}$ and $\rho_{ij} = \sum_k \sigma_{in}\tau'_{kj}$ then $(dS) \cdot T = (\mu_{ij})$ and $S \cdot (\alpha T) = (\rho_{ij})$. Hence $d(S \cdot T) = (dS) \cdot T + S \cdot (dT)$. Q.E.D.

Corollary: If $C = (C_{ij})$ is a matrix of constant functions, then C is differentiable and $dC = 0$. Furthermore, if S is differentiable, $d(C \cdot S) = C \cdot dS$.

Proof: Trivial.

Note: If S is a differentiable matrix where $dS = 0$, then it is easy to see that S is a constant matrix.

THEOREM: Let $S = (\sigma_{ij})$ and $T = (\tau_{ij})$ be two differentiable and invertible matrices. Then a necessary and sufficient condition for $DS = DT$ is that there exists an invertible constant matrix C such that $S = CT$.

Proof: Suppose $S = CT$ where C is constant and invertible. Then $DS = D(CT) = (CT)^{-1}d(CT) = (T^{-1}C^{-1})CdT = T^{-1}dT = DT$. Hence the condition is sufficient.

Conversely, suppose that $DS = DT$. Then $S^{-1}dS - T^{-1}dT = 0$ and $S^{-1}dS + d(T^{-1})T = 0$, since $d(T^{-1})$ exists for all $T \epsilon R$. Multiplying on the left by S and on the right by T^{-1} gives $(dS)T^{-1} + Sd(T^{-1}) = 0$ or $d(ST^{-1}) = 0$. $ST^{-1} = C$ where C is a constant invertible matrix, since both S and T^{-1} are invertible.

$$\therefore S = CT \text{ or } T = C^{-1}S \text{ .}$$

2. Infinite Matrices

Dfn: Let $(\varphi_t)_{t\epsilon R}$ be a family of elements in T. Then we say $\lim_{t\to t_o} \varphi_t = \psi$ if for any neighborhood N of zero in T, there exists a $\delta > 0$ such that

$$|t - t_o| < \delta$$

implies $\varphi_t - \psi \epsilon N$. If $\psi = \varphi_{t_o}$ then we say the family $(\varphi_t)_{t\epsilon R}$ is continuous at t_o. If $(\varphi_t)_{t\epsilon R}$ is continuous for all t_o in R, we simply say the family is continuous.

Dfn: Let $(U_t)_{t\epsilon R}$ be a family of linear maps in L(T). We say the family is continuous in t if for each $t_o \epsilon R$, $\varphi \epsilon T$, $\lim_{t\to t_o} U_t[\varphi] = U_{t_o}[\varphi]$. We say this family is strongly continuous if for each $t_o \epsilon R$, and any neighborhood N of zero in T, there exists a $\delta > 0$ and a neighborhood M of zero in T such that for all $t \epsilon R$, $|t - t_o| < \delta$ and all $\varphi \epsilon M$, we have $U_t[\varphi] - U_{t_o}[\varphi] \epsilon N$.

If $[\mathring{f}_t(\eta,\gamma)] \sim U_t$ for all $t\epsilon R$ then we say $[\mathring{f}_t(\eta,\gamma)]$ is continuous (strongly continuous) if $(U_t)_{t\epsilon R}$ is continuous (strongly continuous).

Dfn: We say the family (U_t) is differentiable if there is a family (V_t) in L(T) such that for each $t_o \epsilon R$, $\varphi \epsilon T$,

$$\lim_{t\to t_o} \frac{(U_t[\varphi] - U_{t_o}[\varphi])}{t - t_o} = V_{t_o}[\varphi] .$$

(V_t) is called the derivative of (U_t) and denoted (U'_t).

If $[\mathring{f}_t(\eta,\gamma)] \sim U_t$ for all $t\epsilon R$ then we say this family of matrices is differentiable if (U_t) is differentiable and denote its derivative by $\frac{\partial}{\partial_t} [\mathring{f}_t(\eta,\gamma)]$.

Note: It is easy to see that if $\frac{\partial}{\partial t} [\mathring{f}_t(\eta,\gamma)]$ exists then the function

$$t \to \int_{-\infty}^{+\infty} \mathring{f}_t(\eta,\gamma)\varphi(\gamma)\varphi\gamma$$

from R into C is differentiable for all $\eta\epsilon R$, $\varphi\epsilon T$ and its derivative is the function

$$t \to \int_{-\infty}^{+\infty} \frac{\partial}{\partial t} \mathring{f}_t(\eta,\gamma)\varphi(\gamma)d\gamma .$$

Proposition: $\frac{\partial}{\partial t} [\mathring{f}_t(\eta,\gamma)] = 0$ if, and only if, $[\mathring{f}_t(\eta,\gamma)]$ is constant in t.

Proof: It is clearly true that if $[\mathring{f}_t(\eta,\gamma)]$ is constant in t then

$$\frac{\partial}{\partial t} [\mathring{f}_t(\eta,\gamma)] = 0.$$

Conversely, if $\frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)] = 0$ then by the note above the function

$$t \to \int_{-\infty}^{+\infty} \mathring{f}_t(\eta,\gamma)\varphi(\gamma)d\gamma$$

is constant and hence $\mathring{f}_t(\eta,\gamma) = \mathring{f}_s(\eta,\gamma)$ for all s, $t \in R \Rightarrow [\mathring{f}_t(\eta,\gamma)]$ is constant. Q.E.D.

THEOREM: If $[\mathring{f}_t(\eta,\gamma)]$ and $[\mathring{g}_t(\eta,\gamma)]$ are differentiable and $[\mathring{f}_t(\eta,\gamma)]$ is strongly continuous, then $[\mathring{f}_t(\eta,\gamma)] \cdot [\mathring{g}_t(\eta,\gamma)]$ is differentiable and

$$\frac{\partial}{\partial t}\left([\mathring{f}_t(\eta,\gamma)] \cdot [\mathring{g}_t(\eta,\gamma)]\right) = \frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)][\mathring{g}_t(\eta,\gamma)] + [\mathring{f}_t(\eta,\gamma)]\frac{\partial}{\partial t}[\mathring{g}_t(\eta,\gamma)]$$

Proof: Let $t_o \in R$ and $\varphi \in T$. Then if $(U_t) \sim [\mathring{f}_t(\eta,\gamma)]$ and $(V_t) \sim [\mathring{g}_t(\eta,\gamma)]$ then

$$\frac{U_t \circ V_t[\varphi] - U_{t_o} \circ V_{t_o}[\varphi]}{t - t_o} = \frac{U_t\left[V_t[\varphi]\right] - U_{t_o}\left[V_{t_o}[\varphi]\right]}{t - t_o}$$

$$= \frac{U_t\left[V_t[\varphi] - V_{t_o}[\varphi] + V_{t_o}[\varphi]\right] - U_{t_o}\left[V_{t_0}[\varphi]\right]}{t - t_o} = \frac{U_t\left[V_t[\varphi] - V_{t_0}[\varphi]\right]}{t - t_o} +$$

$$\frac{U_t\left[V_{t_o}[\varphi]\right] - U_{t_o}\left[V_{t_o}[\varphi]\right]}{t - t_o}$$

But U_t is strongly continuous and since

$$\frac{V_t[\varphi] - V_{t_o}[\varphi]}{t - t_o} \to V'_{t_o}[\varphi] \;, \quad \frac{U_t\left[V_{t_o}[\varphi]\right] - U_{t_o}\left[V_{t_o}[\varphi]\right]}{t - t_o} \to U'_{t_o}\left[V_{t_o}[\varphi]\right] \text{ as } t \to t_o \,,$$

we have $(U_{t_o} V_t)' = U_{t_o} \circ V_t' + U_{t_o}' \circ V_t$ and hence the same result corresponding to the matrices. Q.E.D.

Note: It is easy to see that

$$\frac{\partial}{\partial t}\Big([f_t(\eta,\gamma)] + [g_t(\eta,\gamma)]\Big) = \frac{\partial}{\partial t}\Big[f_t(\eta,\gamma)\Big] + \frac{\partial}{\partial t}\Big[g_t(\eta,\gamma)\Big]$$

Examples:

(1) If $h(t)$ is differentiable for all $t \epsilon R$ and $[\mathring{f}(\eta,\gamma)]$ is a matrix, the family $[h(t) \cdot \mathring{f}(\eta,\gamma)]$ is differentiable and

$$\frac{\partial}{\partial t}\Big[h(t) \cdot \mathring{f}(\eta,\gamma)\Big] = \Big[h'(t) \cdot \mathring{f}(\eta,\gamma)\Big]$$

(2) For each $t \epsilon R$, let $\mathring{y}(\gamma,t)$ be a temperate distribution. Then if $\frac{\partial}{\partial t}\mathring{y}(\gamma,t)$ exists and $[\mathring{y}(\gamma - \eta,t)]$, $[\frac{\partial}{\partial t}\mathring{y}(\gamma - \eta,t)]$ are matrices then $\frac{\partial}{\partial t}[\mathring{y}(\gamma - \eta,t)] = [\frac{\partial}{\partial t}\mathring{y}(\gamma - \eta,t)]$ for if $\varphi \epsilon T$, $t,t_o \epsilon R$,

$$\frac{\int_{-\infty}^{+\infty}\mathring{y}(\eta - \gamma,t)\ \varphi(\gamma)d\gamma - \int_{-\infty}^{+\infty}\mathring{y}(\eta - \gamma,\ t_o)\ \varphi(\gamma)d\gamma}{t - t_o} - \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\ y(\eta - \gamma,t_o)\ \varphi(r)d\gamma$$

$$\int_{-\infty}^{+\infty}\left(\frac{\mathring{y}(\eta - \gamma,t) - \mathring{y}(\eta - \gamma,t_o)}{t - t_o} - \frac{\partial}{\partial t}\ y(\eta - \gamma,t_o)\right)\varphi(\gamma)d\gamma \to 0 \text{ as } t \to t_o\ .$$

Dfn: Let $[\mathring{f}_t(\eta,\gamma)]$ be a differentiable family of invertible matrices. Then we define

$$D_t\Big[\mathring{f}_t(\eta,\gamma)\Big] = \Big[\mathring{f}_t(\eta,\gamma)\Big]^{-1}\frac{\partial}{\partial t}\Big[\mathring{f}_t(\eta,\gamma)\Big]\ .$$

THEOREM: If $[\mathring{f}_t(\eta,\gamma)]$, $[\mathring{g}_t(\eta,\gamma)]$ are differentiable families of invertible matrices where both $[\mathring{f}_t(\eta,\gamma)]^{-1}$ and $[\mathring{g}_t(\eta,\gamma)]^{-1}$ are differentiable, then exists an invertible matrix $[\mathring{h}(\eta,\gamma)]$ such that

$$[\mathring{f}_t(\eta,\gamma)] = [\mathring{h}(\eta,\gamma)] \cdot [\mathring{g}_t(\eta,\gamma)]$$

if and only if

$$D_t[\mathring{f}_t(\eta,\gamma)] = D_t[\mathring{g}_t(\eta,\gamma)]$$

Proof: If $[\mathring{f}_t(\eta,\gamma)] = [\mathring{h}(\eta,\gamma)] \cdot [\mathring{g}_t(\eta,\gamma)]$, then

$$D_t[\mathring{f}(\eta,\gamma)] = [\mathring{f}_t(\eta,\gamma)]^{-1} \frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)] = \Big([\mathring{h}(\eta,\gamma)][\mathring{g}_t(\eta,\gamma)]\Big)^{-1} \frac{\partial}{\partial t}\Big([\mathring{h}(\eta,\gamma)][\mathring{g}_t(\eta,\gamma)]\Big)$$

$$= [\mathring{g}_t(\eta,\gamma)]^{-1} [\mathring{h}(\eta,\gamma)]^{-1} [\mathring{h}(\eta,\gamma)] \frac{\partial}{\partial t} [\mathring{g}_t(\eta,\gamma)]$$

$$= [\mathring{g}_t(\eta,\gamma)]^{-1} \frac{\partial}{\partial t}[\mathring{g}_t(\eta,\gamma)]$$

$$= D_t[\mathring{g}_t(\eta,\gamma)] \ .$$

Conversely, suppose $D_t[\mathring{f}_t(\eta,\gamma)] = D_t[\mathring{g}_t(\eta,\gamma)]$. Then since

$$[\mathring{f}_t(\eta,\gamma)][\mathring{f}_t(\eta,\gamma)]^{-1} = [\delta(\eta - \gamma)] \ ,$$

$$\frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)] \cdot [\mathring{f}_t(\eta,\gamma)]^{-1} + [\mathring{f}_t(\eta,\gamma)] \frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)]^{-1} = 0$$

and we have

$$\frac{\partial}{\partial t}\Big([\mathring{f}_t(\eta,\gamma)]^{-1}\Big) = -[\mathring{f}_t(\eta,\gamma)]^{-1} \Big(\frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)]\Big) [\mathring{f}_t(\eta,\gamma)]^{-1} \ .$$

Thus, since

$$[\mathring{f}_t(\eta,\gamma)]^{-1}\frac{\partial}{\partial t}[\mathring{f}_t(\eta,\gamma)] = [\mathring{g}_t(\eta,\gamma)]^{-1}\frac{\partial}{\partial t}[\mathring{g}_t(\eta,\gamma)]$$

we have by multiplying by $[\mathring{f}_t(\eta,\gamma)]^{-1}$ that

$$-\frac{\partial}{\partial t}\left([\mathring{f}_t(\eta,\gamma)]^{-1}\right) - \left[\mathring{g}_t(\eta,\gamma)\right]^{-1}\left(\frac{\partial}{\partial t}\left[\mathring{g}_t(\eta,\gamma)\right]\right)\left[\mathring{f}_t(\eta,\gamma)\right]^{-1} = 0$$

or

$$\left[\mathring{g}_t(\eta,\gamma)\right]\frac{\partial}{\partial t}\left([\mathring{f}_t(\eta,\gamma)]^{-1}\right) + \left(\frac{\partial}{\partial t}[\mathring{g}(\eta,\gamma)]\right)\left[\mathring{f}_t\,(\eta,\gamma)\right]^{-1} = 0$$

or

$$\frac{\partial}{\partial t}\left([\mathring{g}_t(\eta,\gamma)][\mathring{f}_t(\eta,\gamma)]^{-1}\right) = 0\ .$$

Hence

$$[\mathring{g}_t(\eta,\gamma)][\mathring{f}_t(\eta,\gamma)]^{-1} = [\mathring{h}(\eta,\gamma)]$$

for all $t\in R$. Q.E.D.

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