ON THE OBSERVABILITY OF THE VELOCITY OF LIGHT

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N 68-23585

INTRODUCTION

The well-known general space vehicle trajectory estimation problem consists of processing a body of tracking and other observational data to estimate not only the parameters of the trajectory itself, but also a set of other uncertain parameters which are significant in the problem. Such other parameters include the tracking station locations, measurement biases, and assorted astrodynamic constants.

It is also conceivable to include in this list the velocity of light, since its uncertainty has a significant effect on certain types of measurements. Thus, in principle, it would seem reasonable to add the velocity of light to the list of state variables, and thereby obtain a better estimate not only of the trajectory parameters, but also of the velocity of light itself.

The purpose of this paper is to show that the hope of improving the knowledge of the velocity of light in this way is an illusion, that the velocity of light is an unobservable parameter in trajectory estimation, and that its inclusion in the problem is not only inappropriate but also can lead to erroneous results. Although many readers of this paper may already know and understand this, the analysis presented here is felt to be useful for others who have had difficulty understanding the fact, and also may give some insights into other observability problems.

OBSERVABILITY IN SEQUENTIAL ESTIMATION

Observability generally means the ability of a measurement or observation scheme to "see" and separate out in some sense all the various states of a multivariable system. Thus, ordinarily, observability is regarded as a property of the measurement scheme itself. However, there is more to observability than this in the case of estimation based on noisy observations. This is particularly clear for sequential estimation, as illustrated by the well-known equations for sequential minimum-variance processing of data in a linear system:

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$$\hat{x}' = \hat{x} + \frac{PH^T}{HPH^T + q}[y - H\hat{x}] \tag{1}$$

$$P' = P - \frac{(PH^T)(HP)}{HPH^T + q} \tag{2}$$

Here, $\hat{x}$ and $\hat{x}'$ are the estimates of the state vector x, respectively, before and after incorporating the information contained in the scalar observation, $y = Hx + e$, where e is a random observation error with zero mean and variance q; P is the covariance matrix of the error in estimate, $P = E(x - \hat{x})(x - \hat{x})^T$.

Observability has to do with the information content of the observation relative to the state x. Note that if an element of the PH^T vector is zero, then the estimate of the corresponding element of the state remains unchanged, regardless of the size of the residual, $(y - H\hat{x})$. Also, the corresponding row and column of $\Delta P = (PH^T)(HP)/(HPH^T + q)$ will be zero, so that no change occurs in these elements of the covariance matrix. For the purposes of this paper, a state element is said to be unobservable if the observations do not change the estimate of that element. Thus, so far as the observation $y = Hx + e$ is concerned, the ith element of the vector PH^T is zero.

Observability thus depends not only upon the H row vector which characterizes the observation, but also upon P. It is, therefore, crucial that the correct P matrix be used in considering questions of observability in sequential estimation. In particular, one must be concerned with the initial P matrix for a given problem, and be sure that it properly summarizes the a priori knowledge of the state x as it is supposed to. This, it will be noted, is in contrast to statements which sometimes have been made to the effect that the initial P matrix is not too material, or can be specified somewhat arbitrarily.

SPECIFICATIONS OF A PRIORI COVARIANCE MATRIX

For a typical trajectory estimation problem, the state elements are vehicle position and velocity (six elements), tracking station location (three elements per station), earth radius, earth-moon distance, earth mass, moon mass, velocity of light, etc. The initial covariance matrix summarizing the a priori distribution of error in the estimates of all these variables can be constructed from independent measurements of each of the variables. A typical example is the set of observations used initially to determine the tracking station locations. Assuming that surveying techniques were used, which employ (among other things) distance measuring equipment (DME), it can be argued that the uncertainty in the velocity of light enters in the following way. Assume that DME uses a two-way phase measurement,

$$\varphi = 4\pi f\left(\frac{R}{c}\right) \tag{3}$$

where φ is the phase shift of an electromagnetic wave of frequency f traversing the distance $2R$, and c is the velocity of propagation. If the state variables are taken to be R and c, then the linearized form of equation (3) is

$$\Delta\varphi \cong \frac{4\pi f}{c}\Delta R - \frac{4\pi f R}{c}\frac{\Delta c}{c} \tag{4}$$

where ΔR and $\Delta c/c$ may be defined as the state variables of the linearized system. In terms of the measurement of ΔR, equation (4) may be written as

$$\Delta R_M = \frac{c\Delta\varphi}{4\pi f} \cong \Delta R - R\frac{\Delta c}{c} \tag{5}$$

This linearized observation equation implies an H matrix (with respect to the variables ΔR, $\Delta c/c$), which is

$$H = [1 \quad -R] \tag{6}$$

The analysis of all prior observations on the state space is, of course, more complicated than the simple case described above, inasmuch as all observations must be expressed in terms of a common reference system. However, it can be shown that as long as the totality of the prior observations "span" the state space (that is, P_o is nonsingular), they can be summarized by a set of equivalent measurements having H matrices of the same form as (6), namely, they are direct with respect to all the state variables except $\Delta c/c$, and indirect with respect to the latter. For the three components of tracking station location, the complete H matrix (considering, for the moment, only those variables which directly affect the surveying measurements) is

$$H_{sta} = \begin{bmatrix} 1 & 0 & 0 & -x_{sta} \\ 0 & 1 & 0 & -y_{sta} \\ 0 & 0 & 1 & -z_{sta} \end{bmatrix} \tag{7}$$

For the various astrodynamic constants considered a similar analysis can be applied, the details of which are not shown here, which results in similar equivalent H matrices. For the radius of the earth and the earth-moon distance, for instance, we obtain

$$\left.\begin{aligned} H_{R_E} &= [1 \quad -R_E] \\ H_{R_{EM}} &= [1 \quad -R_{EM}] \end{aligned}\right\} \tag{8}$$

and for the masses of the earth and of the moon,

$$\left.\begin{aligned} H_{\mu_E} &= [1 \quad -3\mu_E] \\ H_{\mu_M} &= [1 \quad -3\mu_M] \end{aligned}\right\} \tag{9}$$

(In equations (8) and (9) zero elements in H have been omitted for simplicity.) Equations (9) indicate that the sensitivity of the equivalent measurements of the masses to the velocity of light uncertainty are minus three times the respective μ's. The H matrix for the direct a priori velocity of light measurements is simply

$$H_c = [0 \quad 0 \quad \ldots \quad 0 \quad 1] \tag{10}$$

since it is not dependent on any of the other state variables.

In regard to the a priori observations of the spacecraft trajectory parameters, a simple analysis is not possible, so we will use here a heuristic argument. We will assume that the trajectory parameters used are the components of the instantaneous position and velocity vectors, and that the time of injection is t_o. The a priori (i.e., before t_o) observations include all measurements made at the launch site, and tracking and other measurements during the launch. The totality of these observations can be summarized in terms of equivalent direct measurements of x_V, y_V, z_V, $\dot{x}_V$, $\dot{y}_V$, and $\dot{z}_V$, with sensitivity to the velocity of light uncertainty $\Delta c/c$, just as in the case with the previously considered variables. Analysis to determine the $\Delta c/c$ sensitivity coefficients is complicated, but it is reasonable to expect that they would turn out to be precisely the coefficients which would result if real direct measurements with DME could be, and were, made. As has been shown, these coefficients are the negatives of the respective state variables. Thus, we assume that the equivalent H matrix for this part of the state is

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -x_V \\ 0 & 1 & 0 & 0 & 0 & 0 & -y_V \\ 0 & 0 & 1 & 0 & 0 & 0 & -z_V \\ 0 & 0 & 0 & 1 & 0 & 0 & -\dot{x}_V \\ 0 & 0 & 0 & 0 & 1 & 0 & -\dot{y}_V \\ 0 & 0 & 0 & 0 & 0 & 1 & -\dot{z}_V \end{bmatrix} \tag{11}$$

again excluding all zero columns.

Collecting all the equivalent measurement matrices, equations (7) to (11), we obtain a total a priori H matrix of the form

$$H_o = \left[\begin{array}{c|c} I & a \\ \hline 0 & 1 \end{array}\right] \tag{12}$$

where the last column is the set of sensitivity coefficients for the $\Delta c/c$ uncertainty. Associated with the set of equivalent measurements there is an equivalent measurement error covariance matrix, Q_o, which, partitioned in the same manner as H_o, can be represented as

$$Q_o = \left[\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & q_c \end{array}\right] \tag{13}$$

Here, the zeroes mean that it is assumed that the random errors in the direct measurement of c are uncorrelated with the random errors in the other measurements.

With the a priori H_o and Q_o matrices, it is a simple matter to form the a priori estimation covariance matrix, if a least-squares reduction of the a priori data is assumed:

$$\begin{aligned} P_o &= H_o^{-1} Q_o H_o^{-T} \\ &= \left[\begin{array}{c|c} Q_1 + q_c a a^T & -q_c a \\ \hline -q_c a^T & q_c \end{array}\right] \end{aligned} \tag{14}$$

VELOCITY OF LIGHT OBSERVABILITY IN TRAJECTORY ESTIMATION

Having a representation of the a priori P_o matrix, we now consider the processing of a scalar observation characterized by an H matrix

$$H = [H_2 \mid b] \tag{15}$$

where b is the (scalar) sensitivity of the observation to the $\Delta c/c$ uncertainty. It is seen that the $P_o H^T$ vector is given by

$$P_o H^T = \left[\begin{array}{c} (Q_1 + q_c a a^T) H_2^T - q_c a b \\ \hline -q_c (a^T H_2^T - b) \end{array}\right] \tag{16}$$

where the lower element is the one associated with the $\Delta c/c$ variable. As previously stated, $\Delta c/c$ is unobservable if and only if the last element of $P_o H^T$ is zero, which requires that

$$b = H_2 a \tag{17}$$

The question in considering the observability of $\Delta c/c$ in trajectory estimation is whether or not the observations made of the vehicle have the property (17). But before taking up this matter, it is necessary to consider the fact that in sequential trajectory estimation where the epoch is current time, the dynamics of the problem cause P_o to change. In P_o, as represented by equation (14), it is noted that the last column (except one element) is proportional to the a vector. Now, since the linearized system (state) equation is of the form

$$\dot{x} = Fx + Gu \tag{18}$$

the P matrix, in the absence of observations, varies with time according to the differential equation:

$$\dot{P} = FP + PF^T + GRG^T \tag{19}$$

where $R = E[uu^T]$. From equation (19) it is easy to show that in the present problem the last column of P, which we may call p_c (i.e., $P = [P_1 \mid p_c]$), obeys the differential equation

$$\dot{p}_c = Fp_c \tag{20}$$

which is the same as the system equation (18) except that there is no random forcing function equivalent to u. Thus, where p_c is initially

$$p_c(t_o) = q_c \begin{bmatrix} -a \\ \hline 1 \end{bmatrix} \tag{21}$$

at a later time it would become

$$p_c(t) = \Phi(t,t_o)p_c(t_o) \tag{22}$$

where Φ is the transition matrix of the system represented by equation (18). In partitioned form,

$$\Phi(t,t_o) = \left[\begin{array}{c|c} \Phi_1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{23}$$

and

$$p_c(t) = q_c \begin{bmatrix} -\Phi_1 a \\ \hline 1 \end{bmatrix} \tag{24}$$

Now, the element of the PH^T vector which affects the estimate of c is the scalar product of the p_c and H^T vectors:

$$\begin{aligned} (PH^T)_{\Delta c/c} &= Hp_c \\ &= q_c[-H_2\Phi_1 a + b] \end{aligned} \tag{25}$$

Considering that Φ_1 is the matrix of partial derivations of the state vector at t with respect to the state vector at t_o,

$$\Phi_1 = \left[\frac{\partial x(t)}{\partial x(t_o)}\right] \tag{26}$$

and H_2 and b are, respectively, the partials of the measurement quantity M with respect to the state variables and $\Delta c/c$, at time t,

$$\left.\begin{aligned} H_2 &= \left[\frac{\partial M(t)}{\partial x(t)}\right]^T \\ b &= \frac{\partial M(t)}{\partial \Delta c/c} \end{aligned}\right\} \tag{27}$$

equation (25) can be written as

$$(PH^T)_{\Delta c/c} = q_c \left\{ - \left[\frac{\partial M(t)}{\partial x(t)}\right]^T \left[\frac{\partial x(t)}{\partial x(t_o)}\right] a + \frac{\partial M(t)}{\partial \Delta c/c} \right\} \tag{28}$$

which, by factoring out the scalar $\partial M(t)/\partial M(t_o)$, becomes

$$(PH^T)_{\Delta c/c} = q_c \frac{\partial M(t)}{\partial M(t_o)} \left\{ -\left[\frac{\partial M(t_o)}{\partial x(t_o)}\right]^T a + \frac{\partial M(t_o)}{\partial \Delta c/c} \right\} \tag{29}$$

It is seen, therefore, that if the unobservability condition - namely, the quantity in brackets in (29) is zero - at any time (after all, t_o is arbitrary), then it will also be satisfied at any later time. That is to say, the unobservability condition is time invariant, and only the situation at t_o need be considered to establish whether or not c is observable.

Now let us consider the properties of certain types of observations. First, for range measurements from an earth-based tracking station, where range is given by

$$R = [(x_v - x_{sta})^2 + (y_v - y_{sta})^2 + (z_v - z_{sta})^2]^{1/2} \tag{30}$$

it is easy to show that the nonzero elements of the H matrix (the partials of range measurement with respect to all relevant state variables) are as given in Table I.

TABLE I

Variable	x_v	y_v	z_v	x_{sta}	y_{sta}	z_{sta}	$\Delta c/c$
p_c/q_c	x_v	y_v	z_v	x_{sta}	y_{sta}	z_{sta}	1
H_R	$\frac{x_v - x_{sta}}{R}$	$\frac{y_v - y_{sta}}{R}$	$\frac{z_v - z_{sta}}{R}$	$-\frac{x_v - x_{sta}}{R}$	$-\frac{y_v - y_{sta}}{R}$	$-\frac{z_v - z_{sta}}{R}$	$-R$

In the table the elements of the p_c vector, normalized by q_c, are also given. The last element of H_R is $-R$, which is b for the range measurement. It is evident here that the unobservability condition is satisfied, i.e., $H_2 a = b$. Note that this is the same as saying that the dot product of the H_R and p_c vectors is zero.

Next, consider the case of range-rate measurements. Range rate is

$$\dot{R} = [(\dot{x}_v - \dot{x}_{sta})^2 + (\dot{y}_v - \dot{y}_{sta})^2 + (\dot{z}_v - \dot{z}_{sta})^2]^{1/2} \qquad (31)$$

The partials which make up $H_{\dot{R}}$ are more complex than those of H_R. Although the variables $\dot{x}_{sta}$, $\dot{y}_{sta}$, $\dot{z}_{sta}$ need not be introduced as part of the state due to their functional dependence on x_{sta}, y_{sta}, z_{sta}, it is convenient to do so here. The partials are given in Table II, together with the corresponding elements of p_c/q_c.

TABLE II

Variables	$\bar{R}_v^T$	$\dot{\bar{R}}_v^T$	$\bar{R}_{sta}^T$	$\dot{\bar{R}}_{sta}^T$	$\Delta c/c$
p_c/q_c	$\bar{R}_v^T$	$\dot{\bar{R}}_v^T$	$\bar{R}_{sta}^T$	$\dot{\bar{R}}_{sta}^T$	1
$H_{\dot{R}}$	$\left(\frac{\dot{\bar{R}}}{R} - \frac{\dot{R}\bar{R}}{R^2}\right)^T$	$\left(\frac{\dot{\bar{R}}}{\dot{R}}\right)^T$	$-\left(\frac{\dot{\bar{R}}}{R} - \frac{\dot{R}\bar{R}}{R^2}\right)^T$	$-\left(\frac{\dot{\bar{R}}}{\dot{\bar{R}}}\right)^T$	$-\dot{R}$

In Table II, vector notation has been used for brevity. With a little algebra it can be seen that $H_{\dot{R}} p_c = 0$, although this is not so obvious as in the case of H_R. Thus, $\Delta c/c$ is unobservable with $\dot{R}$ measurements.

The results indicated by the analysis given here have been verified by means of a computer simulation of range/range-rate tracking of a space vehicle, which shows that range/range-rate tracking does not give an improvement in the knowledge of c provided that the correct initial P matrix is used. Results have also been obtained which show that if the initial correlations between $\Delta c/c$ and the other state variables are selected to be zero, then an order of magnitude (or better) improvement is indicated in the knowledge of c in certain trajectory estimation situations. This indicates clearly that the specification of proper initial correlations is crucial.

Now consider some angle-type measurements. One case is the measurement of the antenna pointing angle for an earth-based tracking station. This is equivalent in a planar problem to the measurement of the angle between the range vector $\bar{R}$ and the earth-vehicle vector $\bar{R}_v$:

$$\delta = \cos^{-1} \frac{\bar{R} \cdot \bar{R}_v}{R R_v} \qquad (32)$$

For a simple planar problem, the corresponding H matrix is given in Table III.

TABLE III

Variable	x_v	y_v	x_{sta}	y_{sta}	$\Delta c/c$
p_c/q_c	x_v	y_v	x_{sta}	y_{sta}	1
H_δ	$\frac{(y_v - y_{sta})^2}{R^3 \sin \delta}$	$-\frac{(x_v - x_{sta})(y_v - y_{sta})}{R^3 \sin \delta}$	$-\frac{(y_v - y_{sta})^2}{R^3 \sin \delta}$	$\frac{(x_v - x_{sta})(y_v - y_{sta})}{R^3 \sin \delta}$	0

As is seen, the measurement of δ is not affected by the $\Delta c/c$ uncertainty, that is, the b element of H_δ is zero. Nevertheless, we see that $H_\delta p_c = 0$, and δ observations, therefore, give no information on c.

On-board-type angle measurements can also be considered. Two types are possible: (1) the direction of the line of sight from vehicle to earth (or other planet), as obtained from sextant or theodolite observations, and (2) the subtended angle of the earth (or planet). Considering the following three measurements,

$$\left.\begin{aligned} \alpha &= \text{declination} \\ \beta &= \text{right ascension} \\ \gamma &= \text{half-subtense} \end{aligned}\right\} \text{ of earth from vehicle}$$

the corresponding H matrix is given in Table IV.

TABLE IV

Variable	x_V	y_V	z_V	R_E	$\Delta c/c$
p_c/q_c	x_V	y_V	z_V	R_E	1
H_α	$\frac{x_V z_V}{R_V^2(x_V^2 + y_V^2)^{1/2}}$	$\frac{y_V z_V}{R_V^2(x_V^2 + y_V^2)^{1/2}}$	$-\frac{(x_V^2 + y_V^2)^{1/2}}{R_V^2}$	0	0
H_β	$-\frac{y_V}{x_V^2 + y_V^2}$	$\frac{x_V}{(x_V^2 + y_V^2)}$	0	0	0
H_γ	$-\frac{R_E x_V}{R_V^2}$	$-\frac{R_E y_V}{R_V^2}$	$-\frac{R_E z_V}{R_V^2}$	1	0

Again, it is seen that such observations contain no information relative to the velocity of light.

Although the observations described here are the most common types considered in trajectory estimation, this is by no means an exhaustive list. Furthermore, no easy generalization is apparent. Hence, it is not possible at this point to state unequivocally that there are no observation types in trajectory estimation which would improve the knowledge of the velocity of light. The pattern is convincing nonetheless. What appears to be the case is that only direct measurements of the velocity of light, that is, repetitions of the laboratory experiments, can be expected to improve the estimate of c.

ELIMINATION OF $\Delta c/c$ FROM THE ESTIMATION PROBLEM

From the foregoing it appears that there is no point in complicating the trajectory estimation problem by including the velocity of light as a parameter to be estimated. One way of eliminating this parameter follows directly from the preceding analysis. It is clear that repeated observations of the trajectory improve the knowledge only of a subspace of the state space, and are equivalent to reducing the Q_1 observation error covariance matrix in equation (14). The limiting P matrix thus can be obtained by letting Q_1 approach zero in equation (14):

$$P_{min} = q_c \left[\begin{array}{c|c} aa^T & -a \\ \hline -a^T & 1 \end{array}\right] \tag{33}$$

Obviously, observations having H matrices of the form $H = [H_2 \mid H_2 a]$ have no effect on P_{min}, that is, $P_{min}H^T = 0$. Thus, $\Delta c/c$ can be eliminated from the estimation problem by subtracting P_{min} from the initial P_0 matrix. This leaves only zeroes in the last row and column of the reduced P matrix, and since these remain zero throughout the problem they need not be carried. At any time the P_{min} matrix can be added onto P, of course, if a representation is desired of the total uncertainty in the estimate of the state.

Another way of eliminating the velocity of light from consideration is by using a different definition of the problem at the outset. It should be recognized that all of the really accurate distance measurements utilize electromagnetic radiation, and, in effect, measure not distance as such but rather the time it takes light to traverse the distance. Thus, the scale of things in the universe must always be uncertain to the same degree as is the velocity of light. It should be noted, however, that this uncertainty really is a matter of the definition of the unit of length. Perhaps we should have noted in the beginning that the modern definition of the meter is so many wavelengths of the orange-red line of Krypton 86, not the platinum bar in a vault in France. In terms of the modern standard, then, no observations, a priori or otherwise, are affected by the uncertainty in the velocity of light, and the latter simply does not enter the picture. What this means is that in terms of the modern standard meter the scale of things is known more accurately than in terms of the old meter. The difference is the uncertainty in the calibration of the two meters relative to each other, which is the same as the uncertainty in the velocity of light.

CONCLUSIONS

It appears that the velocity of light is not observable in trajectory estimation problems - or for that matter, in any other problem in which the velocity of light enters only indirectly. It should, therefore, be eliminated from consideration in such problems.

It has also been shown that when the velocity of light uncertainty is considered it is crucial that the correct a priori correlations be used in the estimation covariance matrix. Otherwise the results would show an erroneous "improvement" in the knowledge of the velocity of light. This fact suggests the probability that proper a priori correlations are important for other variables as well in order not to obtain erroneous, overly optimistic results in estimation problems.