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THE EQUATIONS OF MOTION FOR OPTIMIZED PROPELLED FLIGHT EXPRESSED IN DELAUNAY AND POINCARÉ VARIABLES AND MODIFICATIONS OF THESE VARIABLES

by William E. Miner

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SUMMARY

This document presents methods for developing the ordinary differential equations (o.d.e.) of motion in canonical form equivalent to the forms of Delaunay and Poincaré. It also presents modifications to these forms so that three variables, which are constants of motion, result while the forms remain canonical.

The equations of motion are for a vehicle propelled by constant thrust magnitude with a constant mass flow rate. The vehicle is moving in a central force field. The trajectories are optimum in the sense of classical calculus of variations in a neighborhood definable by the boundary conditions of the specific problem. Specific problems are not discussed in this document.

The value of the document lies in two major areas:

1. The possible economics in numerical calculations which may result from using these ordinary differential equations, and

2. The application of the general perturbation theory of classical celestial mechanics to approximate solutions of these ordinary differential equations.

This document has been written to record the results of the investigation and was not meant to be a tutorial treatment of the subject. For such treatment, references 1 through 4 are recommended by the author.

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1. Bliss, G.A. : Lectures on the Calculus of Variations. University of Chicago Press, Chicago, Ill., 1961.
2. Goldstein, H. : Classical Mechanics. Addison-Wesley Publishing Co., Inc., Cambridge, Mass., March 1956.
3. Ford, L.R. : Differential Equations. McGraw-Hill Book Co., Inc., N.Y., 1933.
4. Smart, W.M. : Celestial Mechanics. Longmans, Green, and Co., Ltd., London, 1953.

I. INTRODUCTION

This report addresses itself to the development of the ordinary differential equations (o.d.e.) governing the motion of a vehicle propelled by constant thrust magnitude with constant mass flow rate in a central force field. The trajectories are optimum in the sense of classical calculus of variations. The variables used to define the trajectory are canonical and based on the solution of the trajectory when the thrust magnitude is zero.

The Hamiltonian for the full problem is given by

$$H = \lambda_1\left(\frac{v^2}{r^3} + \frac{w^2}{r^3\cos^2\theta} - \frac{k}{r^2}\right) - \lambda_2\frac{w^2}{r^2}\sec^2\theta\tan\theta + \rho_1 u$$

$$+ \rho_2\frac{v}{r^2} + \rho_3\frac{w}{r^2}\sec^2\theta - \lambda_7\sigma + \frac{F}{m}\Delta(\chi) \qquad (1)$$

where

$$\sigma = -\dot{m}$$

$$u = \dot{r}$$

$$v/r^2 = \dot{\theta}$$

$$(w/r^2)\sec^2\theta = \dot{\phi}$$

$$\Delta(\chi) = \sqrt{\lambda_1^2 + r^2\lambda_2^2 + r^2\cos^2\theta\,\lambda_3^2}$$

and λ_1, λ_2, λ_3, ρ_1, ρ_2, ρ_3, and λ_7 are the Lagrange multipliers.

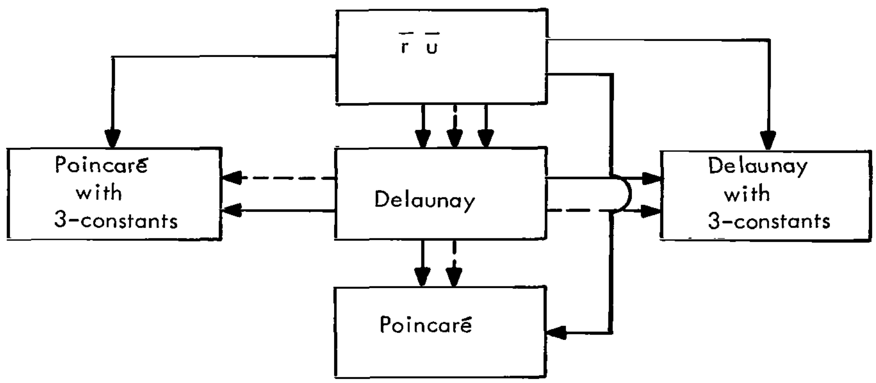
We consider here

$$H = H_o + H_1 \qquad (2)$$

where $H_1 = \frac{F}{m}\Delta(\chi)$, and we present various solutions of H_o (base problem).

This report proposes only to present methods of deriving the various transformations and differential equations. Only enough detail is given so that the results may be reproduced. This paper is concerned only with methods. The basic theory may be found in the references.

The approach was based on solving the problem for the Delaunay variables, and then transforming the Delaunay variables to other variables. This was accomplished with ease. However, the inverse sets of transformation created problems in defining the new variables in the initial coordinates of $(r, \theta, \phi, u, v, w, \bar{\lambda}, \bar{\rho}, \lambda_7)$. These were algebraic problems and were solvable. However, this does not assure a correct answer. Therefore, a more direct approach was sought. Noting that these transformations form a group, one is assured of a direct transformation from the initial coordinate systems to the final desired coordinate system. The following diagram illustrates the problem:



In the diagram, the dashed lines represent the first approach and the solid ones the approach used in this document.

The problem given by Eq. (1) has four constants: the Hamiltonian and three others. If these last three constants are made to be coordinates in the transformed sense, then there are three less integrals to evaluate in numerical integration or in perturbation procedures. Therefore, in two of the sets of variables these are made coordinates.

The author desired one transfer function in which the constants were to be modified for all four cases. This was not accomplished. However, two transfer functions did do the job except that for the modified Poincaré variables added transformations are needed. The Delaunay and Poincaré variables are on state only. One transfer function, based primarily on state, serves for this type of transformation. The cases with the three constants were dependent for some variables on both state and Lagrange multipliers. One transfer function, with additional transformations for the modified Poincaré variables, serves for this type of transformation.

The approach used shortens the path to solutions of the problem but does not eliminate much detailed algebra and differentiation. Therefore, support for this type of work was obtained by use of FORMAC.* It was decided that FORMAC could be used favorably for the following steps in the development:

1. Obtaining the partial derivatives of the generating function, with respect to the coordinates Q and substituting them into the Hamilton-Jacobi partial differential equation to confirm the generating function;
2. Obtaining the first partial derivatives of the generating function with respect to the transformed momenta [K (α)] for the transformed coordinate L or β;
3. Obtaining the second partial derivatives of the generating function with respect to the coordinates and momenta to display information for forming the ordinary differential equations of the full problem.

The above work was supplemented by hand calculations at points so that the programs could remain general and manageable.

*There were many detailed discussions and close liaison between the author of this document and Mr. D. Valenzuela of IBM. His report on the FORMAC work is in preparation. Further information may be obtained from Mr. Valenzuela, IBM, 1730 Cambridge Street, Cambridge, Massachusetts.

The results of this work have the following potential uses:

1. Development of approximate closed-form solutions for trajectory problems by using the methods of celestial mechanics perturbation theory;

2. Study of the characteristics of types of trajectories using the methods of celestial mechanics;

3. Development of computer programs for numerical integration of the ordinary differential equations.

Because of the canonical form this formulation should be easily expanded to include variations of the gravitational field. Also potentially large steps may be made in numerical integration because of the form of the variables.

The information in sections II and III has been treated separately because of the designation of the variables as momenta or coordinates. This designation is different for the Delaunay and Poincaré variables and the modified variables. The organization of the text follows:

1. The Hamiltonian for the base problem is repeated.

2. A transformation to designate convenient momenta and coordinates is made for this set of problems.

3. The Hamilton-Jacobi equation for this set of problems is presented.

4. An indicated procedure for solution is presented and the general generating function for the transformation is presented.

5. The coordinates are then presented in transformed form.

6. A discussion of the ordinary differential equations is given.

7. The perturbation function is presented.

8. The logic for selecting the $K_i(\alpha_j)$ for the particular coordinates is given (either Delaunay or Poincaré) and the generating function for the transformation presented.

9. Steps 5, 6, and 7 are then repeated.

The procedures for obtaining the information presented are not unique. The choice of the procedures used was based on the following:

1. The use of FORMAC to minimize errors;

2. The ease in presentation; and

3. The overall logic of the approach.

The reader is cautioned that the standard astronomical notations of α and β are used in both Poincaré and Delaunay computations. Thus:

	D	P
K	α	α
L	β	β

It will be obvious to the reader from the text what α and β represent. The same holds true for the modified Delaunay and Poincaré variables.

GLOSSARY

$\frac{d(\ )}{dt} = (\dot{\ })$

r = radius vector to vehicle

θ = angle out of reference plane (θ_{max} = inclination)

ϕ = angle in reference plane

u = momentum conjugate to r ($\dot{r}$)

v = momentum conjugate to θ ($r^2 \dot{\theta}$)

w = momentum conjugate to ϕ ($r^2 \cos^2 \theta \dot{\phi}$)

t = time

σ = negative of mass flow rate ($-\dot{m}$)

k = gravitational constant

m = mass

λ_1 = Lagrange multiplier connected to u

λ_2 = Lagrange multiplier connected to v

λ_3 = Lagrange multiplier connected to w

λ_7 = Lagrange multiplier connected to m

ρ_1 = Lagrange multiplier connected to r

ρ_2 = Lagrange multiplier connected to θ

ρ_3 = Lagrange multiplier connected to ϕ

The above terms are basic to this document. Other terms are included, but since the thesis of this work is based on a series of variables, no attempt will be made to define each and every one because of the confusion it would probably create.

II. DELAUNAY AND POINCARÉ VARIABLES

This section presents a method of deriving a generating function in terms of parameters (K) which are to be regarded as functions of the new momenta of the problem. The base two-body Hamiltonian for the trajectory problem may be written as:

$$H_o = \lambda_1 \left(\frac{v^2 + w^2 \sec^2\theta}{r^3} - \frac{k}{r^2} \right) - \lambda_2 \frac{w^2}{r^2} \sec^2 \theta \tan \theta$$

$$+ \rho_1 u + \rho_2 \frac{v}{r^2} + \rho_3 \frac{w}{r^2} \sec^2 \theta - \lambda_7 \sigma \tag{3}$$

where

$$\sigma = -\dot{m} = \text{constant}$$

$$u = \dot{r}$$

$$v/r^2 = \dot{\theta}$$

$$\frac{w}{r^2} (\sec^2 \theta) = \dot{\phi}$$

λ_1, λ_2, ρ_1, ρ_2, ρ_3, and λ_7 are the Lagrange multipliers.

The first step will be to transform this equation by the following transfer function:

$$S = P_1\lambda_1 + P_2\lambda_2 + P_3\lambda_3 + P_4 r + P_5\rho_2 + P_6\rho_3 + P_7 m \tag{4}$$

The results of this tranformation give the relationships:

$$\begin{aligned}
Q_1 &= \lambda_1, & -u &= P_1 = \frac{\partial S}{\partial Q_1}, \\
Q_2 &= \lambda_2 & -v &= P_2 = \frac{\partial S}{\partial Q_2} \\
Q_3 &= \lambda_3 & -w &= P_3 = \frac{\partial S}{\partial Q_3} \\
Q_4 &= r & \rho_1 &= P_4 = \frac{\partial S}{\partial Q_4} \\
Q_5 &= \rho_2 & -\theta &= P_5 = \frac{\partial S}{\partial Q_5} \\
Q_6 &= \rho_3 & -\phi &= P_6 = \frac{\partial S}{\partial Q_6} \\
Q_7 &= m & \lambda_7 &= P_7 = \frac{\partial S}{\partial Q_7}
\end{aligned} \tag{5}$$

where the partial derivitives are listed for ease in writing the Hamilton-Jacobi equation which follows:

$$0 = \frac{\partial S}{\partial t} + Q_1 \left[\frac{\left(\frac{\partial S}{\partial Q_2}\right)^2 + \left(\frac{\partial S}{\partial Q_3}\right)^2 \sec^2 \frac{\partial S}{\partial Q_5}}{Q_4^3} - \frac{k}{Q_4^2} \right]$$

$$+ Q_2 \frac{\left(\frac{\partial S}{\partial Q_3}\right)^2}{Q_4^2} \sec^2 \frac{\partial S}{\partial Q_5} \tan \frac{\partial S}{\partial Q_5} - \frac{\partial S}{\partial Q_4} \frac{\partial S}{\partial Q_1} - Q_5 \frac{\partial S}{\partial Q_2} \frac{1}{Q_4^2}$$

$$- Q_6 \frac{\partial S}{\partial Q_3} \frac{\sec^2 \frac{\partial S}{\partial Q_5}}{Q_4^2} - \frac{\partial S}{\partial Q_7} \sigma \tag{6}$$

We note that Q_7, Q_3, and t do not appear in the equations, so that, by separation of variables, we have:

$$\frac{\partial S}{\partial t} = K_1, \ \sigma \frac{\partial S}{\partial Q_7} = +K_2, \ \frac{\partial S}{\partial Q_3} = -K_3 \tag{7}$$

This new partial differential equation has the following differential equations of the characteristic strip:

$$\frac{\dot{\partial S}}{\partial Q_2} = -K_3^2 \frac{\sec^2 \frac{\partial S}{\partial Q_5} \tan \frac{\partial S}{\partial Q_5}}{Q_4^2} \tag{8}$$

$$\frac{\dot{\partial S}}{\partial Q_5} = \frac{\partial S}{\partial Q_2} \frac{1}{Q_4^2} \tag{9}$$

$$\frac{\dot{\partial S}}{\partial Q_1} = - \left[\frac{\left(\frac{\partial S}{\partial Q_2}\right)^2 + K_3^2 \sec^2 \frac{\partial S}{\partial Q_5}}{Q_4^3} - \frac{k}{Q_4^2} \right] \tag{10}$$

$$\dot{Q}_4 = - \frac{\partial S}{\partial Q_1} \tag{11}$$

$$\frac{\dot{\partial S}}{\partial Q_6} = - K_3 \frac{\sec^2 \frac{\partial S}{\partial Q_5}}{Q_4^2} \tag{12}$$

where the dot indicates the derivative with respect to a parameter τ

$$\left(\frac{\partial \dot{S}}{\partial Q_i} = \frac{d}{d\tau} \frac{\partial S}{\partial Q_i} \right)$$

These equations are simply the two-body equations which we integrate as follows: From Eqs. (8) and (9), we obtain an integral. We substitute this integral into Eq. (10). Next, we combine Eqs. (10) and (11) to obtain a second integral. The next integral is obtained from Eqs. (9) and (12). The last integral is obtained by use of the solution of Eqs. (8) and (9) in combination with Eqs. (9) and (11).

The solutions are:

$$\left(\frac{\partial S}{\partial Q_2} \right)^2 = K_4^2 - K_3^2 \sec^2 \frac{\partial S}{\partial Q_5} = v^2 \tag{13}$$

$$\left(\frac{\partial S}{\partial Q_1} \right)^2 = -K_5 + \frac{2k}{Q_4} - \frac{K_4^2}{Q_4^2} = u^2 \tag{14}$$

$$\frac{\partial S}{\partial Q_6} = \sin^{-1} \frac{K_3 \tan \frac{\partial S}{\partial Q_5}}{\sqrt{K_4^2 - K_3^2}} - K_6 \tag{15}$$

$$K_7 = \cos^{-1} \frac{K_4^2 - k Q_4}{Q_4 \sqrt{k^2 - K_4^2 K_5}} + \sin^{-1} \frac{K_4 \sin \frac{\partial S}{\partial Q_5}}{\sqrt{K_4^2 - K_3^2}} \tag{16}$$

$$\frac{\partial S}{\partial Q_5} = \sin^{-1} \sin i \sin (K_7 - f) \tag{17}$$

where

$$\cos f = \frac{K_4^2 - kQ_4}{Q_4 \sqrt{k^2 - K_4^2 K_5}}$$

$$\sin i = \frac{\sqrt{K_4^2 - K_3^2}}{K_4}$$

The terms u, v, f, and i used in the above equations are to be considered as symbols and not variables.

Eqs. (13) through (17) substituted into the Hamilton-Jacobi equation (6) yield an expression for $\frac{\partial S}{\partial Q_4}$.

On integration one obtains:

$$
S^{(Q_4)} = (K_1 - K_2)\left[\frac{uQ_4}{K_5} - \frac{k}{K_5^{3/2}}\cos^{-1}\frac{k - K_5 Q_4}{\sqrt{k^2 - K_4^2 K_5}}\right] - Q_1 u - Q_2 v
$$

$$
+ Q_5 \sin^{-1}\frac{\sin(K_7 - f)}{\sin i} + Q_6\left[-K_6 + \sin^{-1}\frac{\cos i \sin\left(K_7 - f\right)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}}\right] \tag{18}
$$

This equation was combined with those derived before to obtain a generating function

$$
S = K_1 t + \frac{K_2}{\sigma} Q_7 + (K_1 - K_2)\left[\frac{u\,Q_4}{K_5} - \frac{k}{K_5^{3/2}}\cos^{-1}\frac{k - K_5 Q_4}{\sqrt{k^2 - K_5 K_4^2}}\right] \tag{19}
$$

$$
-K_3 Q_3 - Q_1\sqrt{-K_5 + \frac{2k}{Q_4} - \frac{K_4^2}{Q_4^2}} + Q_5 \sin^{-1}[\sin i \sin(K_7 - f)]
$$

$$
-Q_2\frac{K_4 \sin i \cos(K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} + Q_6\left[-K_6 + \sin^{-1}\frac{\cos i \sin(K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}}\right]
$$

where S is considered as $S(\bar{K}, \bar{Q}, t) = S$ and

$$
K_1 = -H_o,\ K_2 = \sigma\,\lambda_7,\ K_3 = w,\ K_4^2 = v^2 + w^2 \sec^2\theta
$$

$$
K_5 = -u^2 + \frac{2k}{r} - \frac{K_4^2}{r^2},\ K_6 = \phi - \sin^{-1}\frac{K_3 \tan\theta}{\sqrt{K_4^2 - K_3^2}} \tag{20}
$$

$$
K_7 = \cos^{-1}\frac{K_4^2 - kr}{r\sqrt{k^2 - K_5 K_4^2}} - \sin^{-1}\frac{K_4 \sin\theta}{\sqrt{K_4^2 - K_3^2}}
$$

The first partials of S with respect to K_i produce the conjugate variables L_i ($\frac{\partial S}{\partial K_i} = L_i$, $i = 1, 2 \ldots 7$). These are presented below and information to reproduce them will be given in a document to be published by IBM (see footnote on page 3).

$$L_1 = t + \frac{Q_4 u}{K_5} - \frac{k}{K_5^{3/2}} \cos^{-1} \frac{k - K_5 Q_4}{\sqrt{k^2 - K_4^2 K_5}}$$

$$L_2 = \frac{Q_7}{\sigma} + (t - L_1)$$

$$L_3 = Q_2 \frac{\cos i \cos (K_7 - f)}{\sin i \left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} - Q_5 \frac{\cos i \sin (K_7 - f)}{\sin i \sqrt{1 - \sin^2 i \sin^2 (K_7 - f)} \quad K_4}$$

$$+ Q_6 \frac{\sin (K_7 - f) \cos (K_7 - f)}{K_4 \left[1 - \sin^2 i \sin^2 (K_7 - f)\right]} - Q_3$$

$$L_7 = Q_2 \frac{K_4 \sin i \cos^2 i \sin (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} + Q_5 \frac{\sin i \cos (K_1 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$+ Q_6 \frac{\cos i}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]} \tag{21}$$

$$L_6 = -Q_6$$

$$L_5 = Q_1 \frac{1}{2u} + \frac{K_4 L_7 (K_4^2 - k Q_4)}{2u (k^2 - K_4^2 K_5) Q_4} + (K_1 - K_2) \left\{ -\frac{Q_4}{uK_5} + \frac{3}{2K_5} (t - L_1) \right.$$

$$\left. + \frac{K_4^2 (K_4^2 - k Q_4)}{2 u Q_4 K_5 (k^2 - K_4^2 K_5)} \right\}$$

$$L_5 = \frac{Q_1}{2 K_5} \left\{ - \frac{3 (K_4^2 - k Q_4)(t - L_1)}{Q_4^3} + \frac{K_4^4 u}{Q_4^2 (k^2 - K_4^2 K_5)} - u \right\}$$

$$+ \frac{L_7 K_4}{2 K_5 Q_4} \left\{ - \frac{3 (t - L_1)}{Q_4} + \frac{u (K_4^2 + k Q_4)}{\left(k^2 - K_4^2 K_5\right)} \right\} - \frac{\rho_1}{2 K_5} \left\{ 3u (t - L_1) + \frac{Q_4 u^2}{K_5} \right.$$

$$\left. - \frac{(k + K_5 Q_4)}{K_5} + \frac{k^2 (K_4^2 - k Q_4)}{K_5 Q_4 (k^2 - K_4^2 K_5)} \right\}$$

$$L_4 = (K_1 - K_2)\frac{K_4(K_4^2 - kQ_4)}{uQ_4(k^2 - K_4^2K_5)} - \frac{K_3}{K_4}(L_3 + Q_3) + \frac{L_7(2k^2 - K_5kQ_4 - K_4^2K_5)}{Q_4u(k^2 - K_4^2K_5)}$$

$$-Q_2\frac{\sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}} + \frac{K_4Q_1}{uQ_4^2}$$

$$L_4 = \frac{Q_1K_4^3 u}{Q_4^2 (k^2 - K_4^2 K_5)} - \cos i\,(L_3 + Q_3) - \rho_1 \frac{(K_4^2 - kQ_4) K_4}{Q_4(k^2 - K_4^2K_5)}$$

$$+ L_7\frac{(K_4^2 + kQ_4)u}{Q_4(k^2 - K_4^2K_5)} - Q_2 \frac{\sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

The ordinary differential equations will be developed here by taking the time differentials of the transformations and substituting in the original ordinary differential equations of r, θ, ϕ, u, v, w, m, $\bar{\lambda}$, $\bar{\rho}$, and λ_7. One notes that all terms will contain F as a multiple. These ordinary differential equations are presented for reference below and are used in the later discussions:

$$\begin{aligned}
\dot{u} &= \frac{F}{m}\frac{\lambda_1}{\Delta(\chi)} + \left(\frac{v^2}{r^3} + \frac{w^2\sec^2\theta}{r^3} - \frac{k}{r^2}\right) \\
\dot{v} &= \frac{F}{m}\frac{r^2\lambda_2}{\Delta(\chi)} - \frac{w^2}{r^2}\sec^2\theta\tan\theta \\
\dot{w} &= \frac{F r^2 \lambda_3 \cos^2\theta}{m\Delta(\chi)} \\
\dot{r} &= u \\
\dot{\theta} &= \frac{v}{r^2} \\
\dot{\phi} &= \frac{w}{r^2\cos^2\theta} \\
\dot{\lambda}_1 &= -\rho_1 \\
\dot{\lambda}_2 &= -\lambda_1\frac{2v}{r^3} - \rho_2\frac{1}{r_2} \\
\dot{\lambda}_3 &= -\lambda_1\frac{2w}{r^3}\sec^2\theta + \lambda_2\frac{2w}{r^2}\sec^2\theta\tan\theta - \rho_3\frac{\sec^2\theta}{r^2}
\end{aligned} \tag{22}$$

$$\dot{\rho}_1 = \lambda_1 \left(\frac{3\,(v^2 + w^2 \sec^2\theta)}{r^4} - \frac{2k}{r^3} \right) - \lambda_2 \frac{2w^2}{r^3} \sec^2\theta \tan\theta$$

$$+ \rho_2 \frac{2v}{r^3} + \rho_3 \frac{2w}{r^3} \sec^2\theta - \frac{F}{m} \frac{r\,(\lambda_2^2 + \lambda_3^2 \cos^2\theta)}{\Delta(X)}$$

$$\dot{\rho}_2 = -\lambda_1 \frac{w^2}{r^3}\, 2\sec^2\theta \tan\theta + \lambda_2 \frac{w^2}{r^2}\,(2\sec^2\theta \tan^2\theta + \sec^4\theta)$$

$$- \rho_3 \frac{w}{r^2}\, 2\sec^2\theta \tan\theta + \frac{F}{m} \frac{r^2 \lambda_3^2 \sin\theta\cos\theta}{\Delta(X)}$$

$$\dot{\rho}_3 = 0$$

$$\dot{m} = -\sigma$$

$$\dot{\lambda}_7 = \frac{F}{m^2}\,\Delta(X)$$

The ordinary differential equations for $\dot{K}_i(K_i)$ are easily derived by combining Eqs. (20) and (22) in the following order: K_2, K_3, K_4, K_5, K_6, K_7, and K_1. One then obtains the following expressions:

$$\dot{K}_2 = \frac{F}{m^2}\,\sigma\,\Delta(X)$$

$$\dot{K}_3 = \frac{F\,Q_4^2\,Q_3 \cos^2\theta}{m\,\Delta(X)}$$

$$\dot{K}_4 = \frac{F}{m} \frac{Q_4^2(v\,Q_2 + K_3\,Q_3)}{\Delta(X)\,K_4} \tag{23}$$

$$\dot{K}_5 = -2\,\frac{F}{m} \frac{Q_1 u + Q_2 v + K_3 Q_3}{\Delta(X)}$$

$$\dot{K}_6 = \frac{F}{m} \frac{\tan\theta\; Q_4^2}{\Delta(X)} \left(\frac{K_3 Q_2 - v\cos^2\theta\; Q_3}{\left(K_4^2 - K_3^2\right)} \right)$$

$$\dot{K}_7 = \frac{F}{m\,\Delta(X)} \left\{ \frac{Q_4^2\,K_3 \tan\theta\,(Q_2 K_3 - v\cos^2\theta\; Q_3)}{K_4\,(K_4^2 - K_3^2)} \right.$$

$$\left. + \frac{K_4^2\,(K_4^2 - kQ_4)\,Q_1 - (Q_2 v + Q_3 K_3)\,Q_4^2\,u\,(K_4^2 + k\,Q_4)}{Q_4(k^2 - K_4^2 K_5)\,K_4} \right\}$$

where

$$v = \frac{K_4 \sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$u^2 = -K_5 + \frac{2k}{Q_4} - \frac{K_4^2}{Q_4^2}$$

$$\cos \theta = \sqrt{1 - \sin^2 i \sin^2 (K_7 - f)} \tag{24}$$

$$\sin \theta = - \sin i \sin (K_7 - f)$$

$$\dot{K}_1 = \frac{F}{Q_7^2} \sigma \Delta (x) + \frac{F}{m \Delta (x)} \Big[\rho_1 Q_1 + Q_5 Q_2 - Q_3 L_6 + Q_1 Q_2 \frac{2v}{Q_4}$$

$$+ Q_1 Q_3 \frac{2K_3}{Q_4} - Q_2 Q_3 \, 2K_3 \tan \theta - u Q_4 (Q_2^2 + Q_3^2 \cos^2 \theta) + \cos \theta \sin \theta \, v Q_3^2 \Big]$$

where ρ_1 and $\Delta (x)$ will be defined later. All other variables are K's, Q's, and t.

To obtain the $\dot{L}$'s, one observes that

$$L_i = \frac{\partial S(\overline{K}, \overline{Q}, t)}{\partial K_i}$$

Therefore, the general equation for the time derivatives is given by

$$\dot{L}_i = \frac{\partial^2 S}{\partial K_i \, \partial \overline{K}} \dot{\overline{K}} + \frac{\partial^2 S}{\partial K_i \partial \overline{Q}} \dot{\overline{Q}} + \frac{\partial^2 S}{\partial K_i \, \partial t} \tag{25}$$

where

$$\frac{\partial^2 S}{\partial K_i \partial \overline{q}} \dot{\overline{q}} = \sum_{j=1}^{7} \frac{\partial^2 S}{\partial K_i \partial q_j} \dot{q}_j$$

The

$$\frac{\partial^2 S}{\partial K_i \partial K_j}, \quad \frac{\partial^2 S}{\partial K_i \partial Q_j} \quad \text{and} \quad \frac{\partial^2 S}{\partial K_i \partial t}$$

will be given in a document (previously cited*) which will be published in the near future. Remembering that the terms must be a multiple of F, we may write:

$$\overline{\dot{Q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{F}{m} \dfrac{r^2 \lambda_3^2 \sin\theta \cos\theta}{\Delta(x)} \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

ρ_1 may be defined from the Hamiltonian, so that

$$\rho_1 = \frac{\partial S}{\partial Q_4} = -\frac{1}{u}\left[K_1 - K_2 + Q_1\left(\frac{K_4^2}{Q_4^3} - \frac{k}{Q_4^2}\right) - Q_2 \frac{w^2}{Q_4^2} \sec^2\theta \tan\theta + Q_5 \frac{v}{Q_4^2} + Q_6 K_3 \frac{\sec^2\theta}{Q_4^2}\right] \tag{27}$$

where

$$Q_6 = -L_6$$

$$Q_7 = L_2\sigma - \sigma(t - L_1) \tag{28}$$

Q_1, Q_2, Q_3, and Q_5 are obtained from Eqs. (21):

$$Q_1 = 2uL_5 - \frac{K_4 L_7 (K_4^2 - kQ_4)}{(k^2 - K_4^2 K_5)} - (K_1 - K_2)\left\{-\frac{Q_4}{K_5} - \frac{2u^2 Q_4}{K_5^2} + \frac{3k}{2K_5^{5/2}} \cos^{-1}\frac{k - K_5 Q_4}{\sqrt{k^2 - K_5 K_4^2}} + \frac{k\left(k K_4^2 + K_5 K_4^2 Q_4 - 2k^2 Q_4\right)}{K_5^2 (k^2 - K_4^2 K_5)}\right\} \tag{29}$$

$$\begin{bmatrix} Q_2 \\ Q_5 \\ Q_3 \end{bmatrix} = A^{-1} \begin{bmatrix} G_7 \\ G_3 \\ G_4 \end{bmatrix} = A^{-1}\,\overline{g} \tag{30}$$

where A, A^{-1}, and $\overline{g}$ are given below.

*See footnote on page 3.

$$A = \begin{bmatrix} \dfrac{K_4 \sin i \cos^2 i \sin(K_7 - f)}{[1 - \sin^2 i \sin^2(K_7 - f)]^{3/2}} & \dfrac{+\sin i \cos(K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} & 0 \\ \dfrac{\cos i \cos(K_7 - f)}{\sin i [1 - \sin^2 i \sin^2(K_7 - f)]^{3/2}} & \dfrac{-\cos i \sin(K_7 - f)}{\sqrt{K_4^2 - K_3^2}\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} & -1 \\ \dfrac{-\sin i \cos(K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} & 0 & -\cos i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \dfrac{\cos^2 i \sin(K_7 - f)}{\sqrt{K_4^2 - K_3^2}\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} & \dfrac{\cos i \sin i \cos(K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} & \dfrac{-\sin i \cos(K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} \\ \dfrac{\cos(K_7 - f)[1 - \sin^4 i \sin^2(K_7 - f)]}{\sin i [1 - \sin^2 i \sin^2(K_7 - f)]^{3/2}} & \dfrac{-K_4 \sin i \cos^3 i \sin(K_7 - f)}{[1 - \sin^2 i \sin^2(K_7 - f)]^{3/2}} & \dfrac{K_4 \sin i \cos^2 i \sin(K_7 - f)}{[1 - \sin^2 i \sin^2(K_7 - f)]^{3/2}} \\ \dfrac{-\cos i \sin(K_7 - f)\cos(K_7 - f)}{K_4[1 - \sin^2 i \sin^2(K_7 - f)]} & \dfrac{-\sin^2 i \cos^2(K_7 - f)}{[1 - \sin^2 i \sin^2(K_7 - f)]} & \dfrac{-\cos i}{[1 - \sin^2 i \sin^2(K_7 - f)]} \end{bmatrix} \quad (31)$$

$$\bar{g} = \begin{bmatrix} L_7 + L_6 \dfrac{\cos i}{1 - \sin^2 i \sin^2(K_7 - f)} \\ L_3 + L_6 \dfrac{\cos(K_7 - f)\sin(K_7 - f)}{K_4[1 - \sin^2 i \sin^2(K_7 - f)]} \\ L_4 - \dfrac{Q_1 u K_4^3}{Q_4^2(k^2 - K_4^2 K_5)} + L_3 \cos i + \rho_1 \dfrac{K_4(K_4^2 - kQ_4)}{Q_4(k^2 - K_4^2 K_5)} - L_7 \dfrac{u(K_4^2 + kQ_4)}{Q_4(k^2 - K_4^2 K_5)} \end{bmatrix}$$

Q_4 is obtained by iterating a solution of

$$t - L_1 = -\frac{Q_4 u}{K_5} + \frac{k}{K_5^{3/2}} \cos^{-1} \frac{k - K_5 Q_4}{\sqrt{k^2 - K_4^2 K_5}}$$

with t given.

$\Delta(\chi)$ is defined as

$$\Delta(\chi) = \sqrt{\lambda_1^2 + \lambda_2^2 \; r^2 + \lambda_3^2 \; r^2 \cos^2 \theta} \tag{32}$$

where $\lambda_1 = Q_1$ is given by Eq. (29) and λ_2 and λ_3 are Q_2 and Q_3 given by Eqs. (30) and (31).

The new Hamiltonian is given by

$$H = -K_1 + \frac{F\Delta(\chi)}{\sigma\left[L_2 - (t-L_1)\right]} \tag{33}$$

The Delaunay variables may be given by

$$\alpha_5 = \frac{2k}{r} - u^2 - \frac{\alpha_6^2}{r^2}$$

$$\alpha_6^2 = v^2 + \alpha_7^2 \sec^2 \theta \tag{34}$$

$$\alpha_7 = w$$

$$\alpha_3 = \phi - \sin^{-1} \frac{\alpha_7 \tan \theta}{\sqrt{\alpha_6^2 - \alpha_7^2}}$$

$$\alpha_4 = \cos^{-1} \frac{\alpha_6^2 - kr}{r\sqrt{k^2 - \alpha_5\alpha_6^2}} - \sin^{-1} \frac{\alpha_6 \sin \theta}{\sqrt{\alpha_6^2 - \alpha_7^2}}$$

$$\frac{\alpha_5^{3/2}}{k} t + B_1 = -\frac{ruk}{\alpha_5^{1/2}} + \cos^{-1} \frac{k - \alpha_5 r}{\sqrt{k^2 - \alpha_5\alpha_6^2}}$$

If we compare these equations with Eqs. (20) and the first equation of Eqs. (21), we see that

$$K_5 = \alpha_5 \qquad K_7 = \alpha_4 \qquad K_4 = \alpha_6$$

$$K_6 = \alpha_3 \qquad K_3 = \alpha_7 \qquad K_1 = -\alpha_1 \frac{\alpha_5^{3/2}}{k} \tag{35}$$

We let

$$K_2 = \alpha_2 \frac{\alpha_5^{3/2}}{k} ,$$

and then our generating function becomes

$$S = -\alpha_1 \frac{\alpha_5^{3/2}}{k} t + \alpha_2 \frac{\alpha_5^{3/2}}{\sigma\ k} Q_7 + (\alpha_1 + \alpha_2) \left[- \frac{u\ Q_4\ \alpha_5^{1/2}}{k} + \cos^{-1} \frac{k - \alpha_5 Q_4}{\sqrt{k^2 - \alpha_5 \alpha_6^2}} \right] - \alpha_7 Q_3 - Q_1 \sqrt{-\alpha_5 + \frac{2k}{Q_4^2} - \frac{\alpha_6^2}{Q_4^2}}$$
$$+ Q_5 \sin^{-1} \left[\sin i \sin (\alpha_4 - f) \right] - Q_2 \frac{\alpha_6 \sin i \cos (\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2 (\alpha_4 - f)}}$$
$$+ Q_6 \left[-\alpha_3 + \sin^{-1} \frac{\cos i \sin (\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2 (\alpha_4 - f)}} \right] \tag{36}$$

where

$$\sin i = \frac{\sqrt{\alpha_6^2 - \alpha_7^2}}{\alpha_6} , \quad \cos f = \frac{\alpha_6^2 - k\ Q_4}{Q_4 \sqrt{k^2 - \alpha_5 \alpha_6^2}} , \quad u^2 = -\alpha_5 + \frac{2k}{Q_4} - \frac{\alpha_6^2}{Q_4^2} . \tag{37}$$

We treat S as before, so that

$$\beta_1 = - \frac{\alpha_5^{3/2}}{k} t - \frac{u Q_4 \alpha_5^{1/2}}{k} + \cos^{-1} \frac{k - \alpha_5 Q_4}{\sqrt{k^2 - \alpha_5 \alpha_6^2}}$$

$$\beta_2 = \frac{\alpha_5^{3/2}}{k} \frac{Q_7}{\sigma} + \frac{\alpha_5^{3/2}}{k} t + \beta_1$$

$$\beta_3 = - Q_6$$

$$\beta_7 = Q_2 \frac{\cos i \cos (\alpha_4 - f)}{\sin i \left[1 - \sin^2 i \sin^2 (\alpha_4 - f)\right]^{3/2}} - Q_5 \frac{\sin (\alpha_4 - f) \cos i}{\alpha_6 \sqrt{1 - \sin^2 i \sin^2 (\alpha_4 - f)} \sin i}$$

$$+ Q_6 \frac{\sin(\alpha_4 - f) \cos(\alpha_4 - f)}{\alpha_6 \left[1 - \sin^2 i \sin^2 (\alpha_4 - f)\right]} - Q_3 \tag{38}$$

$$\beta_4 = Q_2 \frac{\alpha_6 \sin i \cos^2 i \sin (\alpha_4 - f)}{\left[1 - \sin^2 i \sin^2 (\alpha_4 - f)\right]^{3/2}} + Q_5 \frac{\sin i \cos (\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2 (\alpha_4 - f)}}$$

$$+ Q_6 \frac{\cos i}{\left[1 - \sin^2 i \sin^2 (\alpha_4 - f)\right]}$$

$$\beta_5 = \frac{\beta_4 \alpha_6 (\alpha_6^2 - kQ_4)}{2 u Q_4 (k^2 - \alpha_5 \alpha_6^2)} + Q_1 \frac{1}{2 u} + \frac{3 \alpha_5^{1/2}}{2} \left(\alpha_2 \frac{Q_7}{\sigma k} - \frac{\alpha_1}{k} t \right)$$

$$+ (\alpha_1 + \alpha_2) \left[- \frac{u Q_4}{2 \alpha_5^{1/2} k} + \frac{Q_4^2 \alpha_5^{1/2}}{2 u Q_4 k} - \frac{(\alpha_6^2 k + \alpha_5 \alpha_6^2 Q_4 - 2 k^2 Q_4)}{2 \alpha_5^{1/2} Q_4 u (k^2 - \alpha_5 \alpha_6^2)} \right]$$

$$\beta_5 = \frac{3 \alpha_5^{1/2}}{2} \left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 \frac{t}{k} \right) + \beta_4 \frac{\alpha_6 \alpha_5 u (\alpha_6^2 + k Q_4)}{2 \alpha_5^2 Q_4 (k^2 - \alpha_5 \alpha_6^2)}$$

$$+ \rho_1 \left[\frac{Q_4}{\alpha_5} - \frac{\alpha_6^2 (\alpha_6^2 - kQ_4)}{2 \alpha_5 Q_4 (k^2 - \alpha_5 \alpha_6^2)} \right] + Q_1 \left[- \frac{u}{2\alpha_5} + \frac{\alpha_6^4 u}{2 Q_4^2 \alpha_5 (k^2 - \alpha_5 \alpha_6^2)} \right]$$

$$\beta_6 = - \frac{(\alpha_1 + \alpha_2) \alpha_5^{3/2} \alpha_6 (\alpha_6^2 - kQ_4)}{u Q_4 k (k^2 - \alpha_5 \alpha_6^2)} + Q_1 \frac{\alpha_6}{u Q_4^2} - Q_2 \frac{\sin i \cos(\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2 (\alpha_4 - f)}}$$

$$- (\beta_7 + Q_3) \cos i + \beta_4 \frac{2 k^2 - \alpha_5 \alpha_6^2 - \alpha_5 k Q_4}{u Q_4 (k^2 - \alpha_5 \alpha_6^2)}$$

$$\beta_6 = Q_1 \frac{\alpha_6^3 u}{Q_4^2 (k^2 - \alpha_5\alpha_6^2)} - \frac{\alpha_6 (\alpha_6^2 - kQ_4)}{Q_4 (k^2 - \alpha_5\alpha_6^2)} \rho_1 - Q_2 \frac{\sin i \cos (\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2 (\alpha_4 - f)}}$$

$$- (\beta_7 + Q_3) \cos i + \beta_4 \frac{u(\alpha_6^2 + kQ_4)}{Q_4(k^2 - \alpha_5\alpha_6^2)}$$

$$\alpha_1 = -\alpha_2 + \frac{k}{\alpha_5^{3/2}} \left[Q_1 \frac{\alpha_6^2 - kQ_4}{Q_4^3} + \rho_1 u + \frac{\alpha_5}{Q_4^2} \beta_4 \right]$$

We repeat the operations for determining the ordinary differential equations. Eqs. (22) are still applicable to this problem and are used:

$$\dot{\alpha}_7 = \frac{FQ_4^2 Q_3 \cos^2 \theta}{m\Delta (x)}$$

$$\dot{\alpha}_6 = \frac{FQ_4^2 (vQ_2 + \alpha_7 Q_3)}{m\Delta (x) \alpha_6}$$

$$\dot{\alpha}_5 = -2 \frac{F(Q_1 u + Q_2 v + \alpha_7 Q_3)}{m\Delta (x)}$$

$$\dot{\alpha}_4 = \frac{F Q_4^2 \tan \theta (\alpha_7 Q_2 - vQ_3 \cos^2 \theta) \alpha_7}{m\Delta (x) (\alpha_6^2 - \alpha_7^2)\alpha_6}$$

$$+ \frac{\left[\alpha_6^2 (\alpha_6^2 - kQ_4) Q_1 - (\alpha_6^2 + kQ_4) Q_4^2 u (vQ_2 + \alpha_7 Q_3) \right] F}{\alpha_6 Q_4 (k^2 - \alpha_6^2 \alpha_5) m\Delta (x)}$$

$$\dot{\alpha}_3 = \frac{F \tan \theta Q_4^2 (\alpha_7 Q_2 - vQ_3 \cos^2 \theta)}{m\Delta (x) (\alpha_6^2 - \alpha_7^2)} \tag{39}$$

$$\dot{\alpha}_2 = \frac{F \sigma \Delta (x) k}{Q_7^2 \alpha_5^{3/2}} - \frac{3\alpha_2}{2\alpha_5} \dot{\alpha}_5$$

$$\dot{\alpha}_1 = \frac{F}{Q_7^2}\,\sigma\,\Delta(X) + \frac{F}{m\Delta(X)}\Big[\rho_1 Q_1 + Q_5 Q_2 - Q_3\beta_3$$

$$+ Q_1 Q_2 \frac{2v}{Q_4} + Q_1 Q_3 \frac{2\alpha_7}{Q_4} - Q_2 Q_3 2\alpha_7 \tan\theta - uQ_4 (Q_2^2 + Q_3^2 \cos^2\theta)$$

$$+ \cos\theta \sin\theta\, v Q_3^2\Big] - \frac{3\alpha_1}{2\alpha_5}\,\dot{\alpha}_5$$

where

$$v = \frac{\alpha_6 \sin i \cos(\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}}$$

$$u^2 = -\alpha_5 + \frac{2k}{Q_4} - \frac{\alpha_6^2}{Q_4^2}$$

$$\cos\theta = \sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}$$

$$\sin\theta = -\sin i \sin(\alpha_4 - f)$$

The terms ρ_1 and $\Delta(X)$ will be defined later. All other variables are Q's, α's, and t. We use the same procedures as before to determine β's; that is:

$$\beta_i = \frac{\partial S(\bar{\alpha}, \bar{Q}, t)}{\partial \alpha_i},$$

so that

$$\dot{\beta}_i = \frac{\partial^2 S}{\partial \alpha_i \partial \bar{\alpha}}\,\dot{\bar{\alpha}} + \frac{\partial^2 S}{\partial \alpha_i \partial \bar{Q}}\,\dot{\bar{Q}} + \frac{\partial^2 S}{\partial \alpha_i \partial t}\,. \tag{40}$$

Again $\frac{\partial^2 S}{\partial \alpha_i\, \partial \alpha_j}$, $\frac{\partial^2 S}{\partial \alpha_i \partial Q_j}$, and $\frac{\partial^2 S}{\partial \alpha_i \partial t}$ will be given in the report cited in the footnote on page 3. Equation (26) is used to define $\dot{\bar{Q}}$ and ρ_1 is given by

$$\rho_1 = -\frac{1}{u}\left\{-(\alpha_1 + \alpha_2)\frac{\alpha_5^{3/2}}{k} + Q_1\left(\frac{\alpha_6^2}{Q_4^3} - \frac{k}{Q_4^2}\right) - Q_2 \frac{\alpha_7^2}{Q_4^2}\sec^2\theta \tan\phi + Q_5 \frac{v}{Q_4^2} + \beta_3 \alpha_7 \frac{\sec^2\theta}{Q_4^2}\right\}$$

$$Q_7 = \left[\frac{k(\beta_2 - \beta_1)}{\alpha_5^{3/2}} - t\right]\sigma \tag{41}$$

$$Q_1 = 2u\beta_5 - \frac{3u\alpha_5^{1/2}}{k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right)$$

$$- \frac{(\alpha_1 + \alpha_2)\,\alpha_5^{1/2}}{kQ_4}\left[2r^2 - \frac{\alpha_6^2(\alpha_6^2 - kQ_4)}{k^2 - \alpha_5\alpha_6^2}\right] + \frac{\alpha_6\beta_4(\alpha_6^2 - kQ_4)}{Q_4(k^2 - \alpha_6^2\alpha_5)}$$

We use the same formulations as before for computing Q_4 and $\Delta(x)$ where

$$A = \begin{bmatrix} \dfrac{\alpha_6 \sin i \cos^2 i \sin(\alpha_4 - f)}{\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]^{3/2}} & \dfrac{\sin i \cos(\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}} & 0 \\ \dfrac{\cos i \cos(\alpha_4 - f)}{\sin i\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]^{3/2}} & \dfrac{-\cos i \sin(\alpha_4 - f)}{\alpha_6 \sin i\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}} & -1 \\ \dfrac{-\sin i \cos(\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}} & 0 & -\cos i \end{bmatrix} \tag{42}$$

$$A^{-1} = \begin{bmatrix} \dfrac{\cos^2 i \sin(\alpha_4 - f)}{\sqrt{\alpha_6^2 - \alpha_7^2}\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}} & \dfrac{\cos^2 i \sin i \cos(\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}} & \dfrac{-\sin i \cos(\alpha_4 - f)}{\sqrt{1 - \sin^2 i \sin^2(\alpha_4 - f)}} \\ \dfrac{\cos(\alpha_4 - f)\left[1 - \sin^4 i \sin^2(\alpha_4 - f)\right]}{\sin i\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]^{3/2}} & \dfrac{-\alpha_6 \sin i \cos^3 i \sin(\alpha_4 - f)}{\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]^{3/2}} & \dfrac{\alpha_6 \sin i \cos^2 i \sin(\alpha_4 - f)}{\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]^{3/2}} \\ \dfrac{-\cos i \sin(\alpha_4 - f)\cos(\alpha_4 - f)}{\alpha_6\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]} & \dfrac{-\sin^2 i \cos^2(\alpha_4 - f)}{\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]} & \dfrac{-\cos i}{\left[1 - \sin^2 i \sin^2(\alpha_4 - f)\right]} \end{bmatrix}$$

$$
\bar{g} = \begin{bmatrix}
\beta_4 + \beta_3 \dfrac{\cos i}{\left[1 - \sin^2 i \sin^2 (\alpha_4 - f)\right]} \\[2ex]
\beta_7 + \beta_3 \dfrac{\sin (\alpha_4 - f) \cos (\alpha_4 - f)}{\left[1 - \sin^2 i \sin^2 (\alpha_4 - f)\right]} \\[2ex]
\beta_6 - \dfrac{Q_1 \alpha_6^3 u}{Q_4^2 (k^2 - \alpha_5 \alpha_6^2)} + \beta_7 \cos i + \rho_1 \dfrac{\alpha_6 (\alpha_6^2 - kQ_4)}{Q_4 (k^2 - \alpha_5 \alpha_6^3)} - \beta_4 \dfrac{u (\alpha_6^2 + kQ_4)}{Q_4 (k^2 - \alpha_5 \alpha_6^2)}
\end{bmatrix}
$$

For the Poincaré variables, we wish to eliminate the singularities of $K_4^2 = K_3^2$ (zero inclination) and $k^2 = K_5 K_4^2$ (zero eccentricity). If we view Eqs. (13) through (17) and Eq. (19), we see that this cannot be done with the functions used. A look at Eqs. (15) and (16) suggests adding or subtracting K_6 and K_7 to try to eliminate the $K_4^2 = K_3^2$ singularity. When this is done, one has

$$
K_6 - K_7 = \phi - \cos^{-1} \frac{K_4^2 - kr}{r\sqrt{k^2 - K_4^2 K_5}} + \sin^{-1} \frac{v \sin \theta}{K_4 + K_3} \tag{43}
$$

where $K_4 > 0$ and $K_3 > 0$ by convention.

We next note that if we choose $K_1 = -\alpha_1 \frac{\alpha_5^{3/2}}{k}$, we remove the factor $\frac{k}{\alpha_5^{3/2}}$ from the $\cos^{-1} \frac{k - K_5 Q_4}{\sqrt{k^2 - K_5 K_4^2}}$ term and have possibilities of combining it with the $K_4 - K_3$ term. Also, note that the β_1 term that results is a position-velocity term (no Lagrange multiplier -- part of the unadjoined two-body problem). Assume again an addition or subtraction of the β_1 term with the $K_4 - K_3$ term. This results in the elimination of the $k^2 = K_5 K_4^2$ singularity. When this is done, one has

$$
\frac{\partial S}{\partial \alpha_1} = \phi + \sin^{-1} \frac{v \sin \theta}{K_4 + K_3} - t \frac{K_5^{3/2}}{k} - \frac{ru\, K_5^{1/2}}{k} + \cos^{-1} \left[1 - \frac{ru^2}{k + K_5^{1/2} K_4}\right]. \tag{44}
$$

We have now established one new variable. If we solve Eq. (15) for $\tan \frac{\partial S}{\partial Q_5}$ $(-\tan\theta)$, we may write:

$$\sin K_6 = \frac{v \sin\phi - w \tan\theta \cos\phi}{\sqrt{K_4^2 - K_3^2}}$$

$$\cos K_6 = \frac{v \cos\phi + w \tan\theta \sin\phi}{\sqrt{K_4^2 - K_3^2}}$$

When multiplying both sides by $\sqrt{K_4 - w}$, we obtain:

$$\sqrt{K_4 - w}\ \sin K_6 = \frac{v \sin\phi - w \tan\theta \cos\phi}{\sqrt{K_4 + W}} = \alpha_3$$

$$\sqrt{K_4 - w}\ \cos K_6 = \frac{v \cos\phi + w \tan\theta \sin\phi}{\sqrt{K_4 + w}} = \alpha_4 \tag{45}$$

If we operate on Eq. (16) in a like manner, we obtain:

$$\alpha_6 = \sqrt{k - K_4 K_5^{1/2}}\ \sin(K_6 - K_7)$$

$$= \frac{(K_4 \cos\theta + K_3 \sec\theta)\left[\sin\phi\,(K_4^2 - kr) - \cos\phi\, K_4 ru\right] + v \sin\theta\,(\cos\phi\,(K_4^2 - kr) + \sin\phi\, K_4 ur)}{r\,(K_4 + K_3)\sqrt{k + K_4 K_5^{1/2}}} \tag{46}$$

$$\alpha_7 = \sqrt{k - K_4 K_5^{1/2}}\ \cos(K_6 - K_7)$$

$$= \frac{\left[(K_4 \cos\theta + K_3 \sec\theta)(\cos\phi\,(K_4^2 - kr) + \sin\phi\, K_4 ru) - v \sin\theta\,(\sin\phi\,(K_4^2 - kr) - \cos\phi\, K_4 ur)\right]}{r(K_4 + K_3)\sqrt{k + K_4 K_5^{1/2}}}$$

Let $\alpha_5 = K_5$, and we have given α_5, α_3, α_4, α_6, and α_7 from K_5, K_4, K_3, K_6, K_7 or rather $(u, v, w, r\ \theta, \phi)$ where the α's have no zero in the denominator at zero inclination or eccentricity.

Our transformations become:

$$\alpha_5 = -u^2 + \frac{2k}{r} - \frac{v^2 + w^2 \sec^2 \theta}{r^2} \tag{47}$$

$$\alpha_3 = \frac{v \sin \phi - w \tan \theta \cos \phi}{\sqrt{\sqrt{v^2 + w^2 \sec^2 \theta} + w}}$$

$$\alpha_4 = \frac{v \cos \phi + w \tan \theta \sin \phi}{\sqrt{\sqrt{v^2 + w^2 \sec^2 \theta} + w}}$$

$$K_4 = \sqrt{v^2 + w^2 \sec^2 \theta}$$

$$\alpha_6 = \frac{(K_4 \cos \theta + w \sec \theta) (\sin \phi (K_4^2 - kr) - \cos \phi K_4 ur)}{r (K_4 + w) \sqrt{k + K_4 \alpha_5^{1/2}}}$$

$$+ \frac{v \sin \theta \left[(\cos \phi (K_4^2 - kr) + \sin \phi K_4 ur)\right]}{r (K_4 + w) \sqrt{k + K_4 \alpha_5^{1/2}}}$$

$$\alpha_7 = \frac{(K_4 \cos \theta + w \sec \theta) (\cos \phi (K_4^2 - kr) + \sin \phi K_4 ru)}{r (K_4 + w) \sqrt{k + K_4 \alpha_5^{1/2}}}$$

$$- \frac{v \sin \theta \left[\sin \phi (K_4^2 - kr) - \cos \phi K_4 ur\right]}{r (K_4 + w) \sqrt{k + K_4 \alpha_5^{1/2}}}$$

Next we write the K_i's as functions of α_i $(i = 3, 4, 5, 6, 7)$

$$K_3 = \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2$$

$$K_4 = \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}}$$

$$K_5 = \alpha_5 \tag{48}$$

$$K_6 = \tan^{-1} \frac{\alpha_3}{\alpha_4}$$

$$K_7 = \tan^{-1} \frac{\alpha_3 \alpha_7 - \alpha_4 \alpha_6}{\alpha_4 \alpha_7 + \alpha_3 \alpha_6}$$

We next note that, if $S(\overline{K}, \overline{Q}, t)$ is a solution of the Hamilton-Jacobi equation, then so is $S(\overline{K}, \overline{Q}, t) + S^*(\overline{K})$. Let

$$K_1 = -\alpha_1 \frac{\alpha_5^{3/2}}{k}$$

$$K_2 = \alpha_2 \frac{\alpha_5^{3/2}}{k}$$

in S and note $S = S_{(old)} + S^*$

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 = -\frac{\alpha_5^{3/2}}{k} t - \frac{uQ_4 \alpha_5^{1/2}}{k} + \cos^{-1} \frac{k - \alpha_5 Q_4}{\sqrt{(\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}} + \frac{\partial S^*}{\partial \alpha_1}$$

$$\frac{\partial S^*}{\partial \alpha_1} = K_6 - K_7 = \tan^{-1} \frac{\alpha_3}{\alpha_4} - \tan^{-1} \frac{\alpha_3 \alpha_7 - \alpha_4 \alpha_6}{\alpha_4 \alpha_7 + \alpha_3 \alpha_6}$$

$$= \tan^{-1} \frac{\alpha_6}{\alpha_7},$$

so that

$$S^* = (\alpha_1 + \alpha_2) \tan^{-1} \frac{\alpha_6}{\alpha_7}$$

The resulting new S is now written as $S_{(new)} = S_{(old)} + S^*$.

$$S = \frac{\alpha_5^{3/2}}{k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) + (\alpha_1 + \alpha_2)\left(-\frac{uQ_4\alpha_5^{1/2}}{k} + \tan^{-1}\frac{\alpha_6}{\alpha_7}\right.$$

$$\left. + \cos^{-1}\frac{k - \alpha_5 Q_4}{\sqrt{(\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}}\right) - Q_3\left(\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)$$

$$- Q_1\sqrt{-\alpha_5 + \frac{2k}{Q_4} - \frac{\left[k - \alpha_6^2 - \alpha_7^2\right]^2}{\alpha_5 Q_4^2}} - Q_6 \tan^{-1}\frac{\alpha_3}{\alpha_4} \tag{49}$$

$$- Q_2 \frac{K_4 \sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}} + Q_5 \sin^{-1}[\sin i \sin (K_7 - f)]$$

$$+ Q_6 \sin^{-1}\frac{\cos i \sin (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

where

$$\sin i = \frac{\alpha_5^{1/2}\sqrt{\left(\alpha_3^2 + \alpha_4^2\right)\left(2\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}{(k - \alpha_6^2 - \alpha_7^2)}$$

$$\cos f = \frac{\left[k - \alpha_6^2 - \alpha_7^2\right]^2 - \alpha_5 k Q_4}{\alpha_5 Q_4 \sqrt{(\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}} \tag{50}$$

$$K_4 = \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}}$$

$$\tan K_7 = \frac{\alpha_3 \alpha_7 - \alpha_4 \alpha_6}{\alpha_4 \alpha_7 + \alpha_3 \alpha_6}$$

$$u^2 = -\alpha_5 + \frac{2k}{Q_4} - \frac{\left[k - \alpha_6^2 - \alpha_7^2\right]^2}{\alpha_5 Q_4^2}$$

$$\beta_1 = \phi + \sin^{-1} \frac{v \sin \theta}{2 \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2} - t \frac{\alpha_5^{3/2}}{k} - \frac{Q_4 u \alpha_5^{1/2}}{k}$$

$$+ \cos^{-1} \left[1 - \frac{Q_4 u^2}{2k - \alpha_6^2 - \alpha_7^2}\right]$$

$$\beta_2 = \frac{\alpha_5^{3/2}}{k} \left[\frac{Q_7}{\sigma} + t\right] + \beta_1 \tag{51}$$

$$\begin{bmatrix} \beta_3 \\ \\ \beta_4 \end{bmatrix} = 2 \left[Q_3 + \frac{\frac{\partial Z}{\partial i}}{\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2 \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}} \right] \begin{bmatrix} \alpha_3 \\ \\ \alpha_4 \end{bmatrix}$$

$$+ \left[\frac{Q_6}{\alpha_4^2 + \alpha_3^2} - \frac{\frac{\partial Z}{\partial (K_7 - f)}}{\alpha_3^2 + \alpha_4^2} \right] \begin{bmatrix} -\alpha_4 \\ \\ \alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} \beta_6 \\ \\ \beta_7 \end{bmatrix} = 2 \left[\frac{-Q_1 (k - \alpha_6^2 - \alpha_7^2)^3 u}{Q_4^2 \alpha_5^2 (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)} + \frac{\rho_1 k (k - \alpha_6^2 - \alpha_7^2)\left[(k - \alpha_6^2 - \alpha_7^2) - k\alpha_5 Q_4\right]}{\alpha_5^2 Q_4 \, k(2k - \alpha_6^2 - \alpha_7^2)(\alpha_6^2 + \alpha_7^2)} \right.$$

$$\left. + \frac{Q_3}{\alpha_5^{1/2}} - \frac{\partial Z}{\partial K_4} \frac{1}{\alpha_5^{1/2}} + \frac{\partial Z}{\partial i} \frac{1}{k - \alpha_6^2 - \alpha_7^2} \sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2 \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}} \right.$$

$$
- \frac{\partial Z}{\partial (K_7 - f)} \frac{u\left[(k - \alpha_6^2 - \alpha_7^2)^2 + kQ_4\,\alpha_5\right]}{\alpha_5^{3/2} Q_4 (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}\Bigg]\begin{bmatrix}\alpha_6\\ \alpha_7\end{bmatrix} + \Bigg\{ \frac{\rho_1 ku}{\alpha_5^{3/2}(\alpha_6^2 + \alpha_7^2)}
$$

$$
+ \frac{kQ_1\left[(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5\, kQ_4\right]}{\alpha_5^{5/2}(\alpha_6^2 + \alpha_7^2)\,Q_4^3} + \frac{\partial Z}{\partial (K_7 - f)} \frac{\left[k\,(k - \alpha_6^2 - \alpha_7^2) - \alpha_5^2\, Q_4^2\right]}{\alpha_5^2\, Q_4^2\, (\alpha_6^2 + \alpha_7^2)}\Bigg\}\begin{bmatrix}\alpha_7\\ -\alpha_6\end{bmatrix}
$$

$$
\begin{bmatrix}\beta_6\\ \\ \beta_7\end{bmatrix} = 2\Bigg[\frac{(\alpha_1 + \alpha_2)\,\alpha_5^{1/2}\,(k - \alpha_6^2 - \alpha_7^2)\left[\frac{(k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5} - kQ_4\right]}{uQ_4 k\,(\alpha_6^2 + \alpha_7^2)\,(2k - \alpha_6^2 - \alpha_7^2)}
$$

$$
+ \frac{Q_3}{\alpha_5^{1/2}} - \frac{Q_1\,(k - \alpha_6^2 - \alpha_7^2)}{uQ_4^2\,\alpha_5} - \frac{1}{\alpha_5^{1/2}}\frac{\partial Z}{\partial K_4} + \frac{\frac{\partial Z}{\partial i}}{k - \alpha_6^2 - \alpha_7^2}\sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}
$$

$$
- \frac{\partial Z}{\partial (K_7 - f)} \frac{k(k - \alpha_5\, Q_4) + (\alpha_6^2 + \alpha_7^2)\,(2k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{1/2}\, uQ_4\,(\alpha_6^2 + \alpha_7^2)\,(2k - \alpha_6^2 - \alpha_7^2)}\Bigg]\begin{bmatrix}\alpha_6\\ \\ \alpha_7\end{bmatrix}
$$

$$
+ \frac{1}{\alpha_6^2 + \alpha_7^2}\left(\alpha_1 + \alpha_2 - \frac{\partial Z}{\partial (K_7 - f)}\right)\begin{bmatrix}\alpha 7\\ \\ -\alpha_6\end{bmatrix}
$$

$$
\beta_5 = \frac{3\alpha_5^{1/2}}{2k}\left(\alpha_2\,\frac{Q_7}{\sigma} - \alpha_1 t\right) - Q_1\,\frac{u}{2\alpha_5} + \frac{\beta_3\,\alpha_3 + \beta_4\,\alpha_4}{4\alpha_5}
$$

$$
+ \rho_1\,\frac{Q_4}{\alpha_5} + \frac{Q_3}{2\alpha_5}\left[\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right] + Q_2\,\frac{\sin i\,\cos\,(K_7 - f)\,(k - \alpha_6^2 - \alpha_7^2)}{2\alpha_5^{3/2}\sqrt{1 - \sin^2 i\,\sin^2\,(K_7 - f)}}
$$

$$\beta_5 = \frac{3\alpha_5^{1/2}}{2k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) + \frac{\alpha_5^{1/2} Q_4 (\alpha_1 + \alpha_2)}{ku} + \frac{Q_3 (k - \alpha_6^2 - \alpha_7^2)}{2\alpha_5^{3/2}} + \frac{\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_7^2)^2}{2u\,\alpha_5^2 Q_4^2} Q_1$$

$$+ \frac{\partial Z}{\partial i} \frac{1}{2\alpha_5} \sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2 \dfrac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}} - \frac{\partial Z}{\partial K_4} \frac{k - \alpha_6^2 - \alpha_7^2}{2\alpha_5^{3/2}} - \frac{\partial Z}{\partial (K_7 - f)} \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{3/2} u Q_4}$$

where

$$\frac{\partial Z}{\partial i} = -Q_2 \frac{\left(k - \alpha_6^2 - \alpha_7^2\right)\cos(\phi - K_6)\left[\cos^2 i + \sin^2 i \sin^2(\phi - K_6)\right]}{\alpha_5^{1/2} \cos i}$$

$$- Q_5 \sin(\phi - K_6) + Q_6 \tan i \cos(\phi - K_6) \sin(\phi - K_6) \tag{52}$$

$$\frac{\partial Z}{\partial (K_7 - f)} = -Q_2 \frac{(k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{1/2}} \tan i \sin(\phi - K_6) \left[\cos^2 i + \sin^2 i \sin^2(\phi - K_6)\right]$$

$$+ Q_5 \sin i \cos(\phi - K_6) + Q_6 \frac{\cos^2 i + \sin^2 i \sin^2(\phi - K_6)}{\cos i}$$

$$\frac{\partial Z}{\partial K_4} = -Q_2 \sin i \cos(\phi - K_6).$$

One notes that, when $\alpha_3 = \alpha_4 = 0$ and $\alpha_6 = \alpha_7 = 0$, there are singularities in the above equations.

If one defines the following:

$$A_1 = \frac{\alpha_3 \beta_3 + \alpha_4 \beta_4}{2} = (\alpha_3^2 + \alpha_4^2) Q_3 + \frac{\partial Z}{\partial i} \sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2 \dfrac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}$$

$$A_2 = \alpha_4 \beta_3 - \alpha_3 \beta_4 = -Q_6 + \frac{\partial Z}{\partial (K_7 - f)} \tag{53}$$

$$A_3 = \alpha_7 \beta_6 - \alpha_6 \beta_7 = \alpha_1 + \alpha_2 - \frac{\partial Z}{\partial (K_7 - f)}$$

$$A_3 = \frac{\rho_1 ku}{\alpha_5^{3/2}} + \frac{Q_1 k \left[(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5 k Q_4\right]}{\alpha_5^{5/2} Q_4^3} + \frac{\partial Z}{\partial (K_7 - f)} \frac{k(k - \alpha_6^2 - \alpha_7^2) - \alpha_5^2 Q_4^2}{\alpha_5^2 Q_4^2}$$

$$A_4 = \frac{\alpha_6\beta_6 + \alpha_7\beta_7}{2} = \frac{(\alpha_1 + \alpha_2)\,\alpha_5^{1/2}\,(k - \alpha_6^2 - \alpha_7^2)\left[\frac{(k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5} - kQ_4\right]}{uQ_4\,k\,(2k - \alpha_6^2 - \alpha_7^2)}$$

$$+ \frac{(\alpha_6^2 + \alpha_7^2)}{\alpha_5^{1/2}}\left(Q_3 - \frac{Q_1\,(k - \alpha_6^2 - \alpha_7^2)}{uQ_4^2\,\alpha_5^{1/2}} - \frac{\partial Z}{\partial K_4}\right)$$

$$+ \frac{\partial Z}{\partial i}\,\frac{\alpha_6^2 + \alpha_7^2}{k - \alpha_6^2 - \alpha_7^2}\sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}$$

$$- \frac{\partial Z}{\partial (K_7 - f)}\;\frac{k(k - \alpha_5 Q_4) + (\alpha_6^2 + \alpha_7^2)\,(2k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{1/2} uQ_4\,(2k - \alpha_6^2 - \alpha_7^2)}$$

$$= - Q_1\,\frac{(k - \alpha_6^2 - \alpha_7^2)^3\,u}{\alpha_5^2\,Q_4^2\,(2k - \alpha_6^2 - \alpha_7^2)} + \rho_1\,\frac{(k - \alpha_6^2 - \alpha_7^2)\left[(k - \alpha_6^2 - \alpha_7^2)^2 - k\alpha_5 Q_4\right]}{\alpha_5^2\,Q_4(2k - \alpha_6^2 - \alpha_7^2)}$$

$$+ \frac{(\alpha_6^2 + \alpha_7^2)}{\alpha_5^{1/2}}\left(Q_3 - \frac{\partial Z}{\partial K_4}\right) + \frac{\partial Z}{\partial i}\,\frac{\alpha_6^2 + \alpha_7^2}{k - \alpha_6^2 - \alpha_7^2}\sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}$$

$$- \frac{\partial Z}{\partial (K_7 - f)}\;\frac{\left[\left(k - \alpha_6^2 - \alpha_7^2\right)^2 + k\alpha_5\,Q_4\right]u}{\alpha_5^{3/2}\,Q_4(2k - \alpha_6^2 - \alpha_7^2)}$$

and uses these along with the equations for β_1, β_2, and β_5, then the system is non-singular at $\alpha_6 = \alpha_7 = 0$ and $\alpha_3 = \alpha_4 = 0$.

$$
\begin{bmatrix} \dot{\alpha}_3 \\ \\ \dot{\alpha}_4 \end{bmatrix} = \frac{FQ_4^2}{m\Delta(\chi)} \left\{ \frac{\begin{bmatrix} \sin\phi & -\cos\phi \\ \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} Q_2 \\ Q_3 \cos\theta \sin\theta \end{bmatrix}}{\sqrt{2\dfrac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}} \right.
$$

$$
\left. - \frac{\alpha_5^{1/2}\left[Q_2 v + Q_3 (K_3 + K_4 \cos^2\theta)\right]}{2\,(k - \alpha_6^2 - \alpha_7^2)\left(2\dfrac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)} \begin{bmatrix} \alpha_3 \\ \\ \alpha_4 \end{bmatrix} \right\}
$$

$$
\begin{bmatrix} \dot{\alpha}_6 \\ \\ \dot{\alpha}_7 \end{bmatrix} = \frac{F}{m\Delta(\chi)\, K_4\, (K_4 + w)} \left\{ \frac{\begin{bmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}{\sqrt{K_4 + w}} \left\{ (K_4 \cos\theta + w \sec\theta) \begin{bmatrix} (vQ_2 + wQ_3)\, Q_4^2 u + Q_1 K_4^2 \\ \\ 2Q_4 K_4\, (vQ_2 + wQ_3) \end{bmatrix} \right. \right.
$$

$$
+ Q_4 \cos\theta\, (vQ_2 + (w + K_4)Q_3) \begin{bmatrix} K_4\, uQ_4 \\ \\ K_4^2 - kQ_4 \end{bmatrix} + \sin\theta \left(K_4 Q_4 Q_2 \begin{bmatrix} -K_4^2 + kQ_4 \\ \\ K_4\, uQ_4 \end{bmatrix} \right. \tag{54}
$$

$$
\left. \left. + v \begin{bmatrix} -2Q_4 K_4\, (vQ_2 + wQ_3) \\ \\ (vQ_2 + wQ_3)\, Q_4^2 u + Q_1 K_4^2 \end{bmatrix} \right) \right\} - \left(Q_4^2 \left[vQ_2 + Q_3\, (w + K_4 \cos^2\theta) \right] \right.
$$

$$
\left. \left. + \frac{\left[(Q_2 v + Q_3 w)\, (Q_4^2 \alpha_5 - K_4^2) - Q_1\, uK_4^2 \right] (K_4 + w)}{2\alpha_5^{1/2}\, (k + K_4\, \alpha_5^{1/2})} \right) \begin{bmatrix} \alpha_6 \\ \\ \alpha_7 \end{bmatrix} \right\}
$$

$$
\dot{\alpha}_5 = -2\, \frac{F(Q_1 u + Q_2 v + Q_3 w)}{m\Delta(\chi)}
$$

$$
\dot{\alpha}_2 = \frac{kF\,\sigma\,\Delta(\chi)}{\alpha_5^{3/2}\, m^2} - \frac{3\alpha_2}{2\alpha_5}\, \dot{\alpha}_5
$$

$$\dot{\alpha}_1 = -\dot{\alpha}_2 - \frac{3(\alpha_1+\alpha_2)}{2\alpha_5}\dot{\alpha}_5 - \frac{Fk}{\alpha_5^{3/2}\, m\Delta(x)}\Big[\rho_1 Q_1 + Q_5 Q_2 + Q_3 Q_6$$

$$+ Q_1 Q_2 \frac{2v}{Q_4} + Q_1 Q_3 \frac{2w}{Q_4} - 2Q_2Q_3 w \tan\theta - u Q_4 (Q_2^2 + Q_3^2 \cos^2\theta)$$

$$+ \cos\theta \sin\theta v Q_3^2\Big]$$

$$I_t = \frac{FQ_4^2 \cos(K_7 - f)}{m\Delta(x)K_4}\left[\frac{\cos i\, Q_2}{\sqrt{1 - \sin^2 i \sin^2(K_7 - f)}} - Q_3 \sin i \cos(K_7 - f)\right]$$

$$\left[K_7 - f\right]_t = -\frac{\cos i \sin(K_7 - f)}{\sin i \cos(K_7 - f)} I_t = \frac{\sin(\phi - K_6)}{\sin i \cos(\phi - K_6)} I_t \tag{55}$$

The following equations may be used to evaluate the Q's needed in the problem and give information for $\Delta(x)$.

Adding A_2 and A_3 of Eqs. (53), one obtains:

$$Q_6 = \alpha_1 + \alpha_2 - A_2 - A_3 = \alpha_1 + \alpha_2 + \alpha_3\beta_4 + \alpha_6\beta_7 - \alpha_7\beta_6 - \alpha_4\beta_3$$

From equations for A_1 and A_2 of Eqs. (53), one obtains:

$$g_1 = \sin i \cos(\phi - K_6) A_1 + \sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}\ \sin(\phi - K_6)\left[A_2 - Q_6\,\frac{(1 - \cos i)}{\cos i}\right]$$

$$= (\alpha_3^2 + \alpha_4^2) \sin i \cos(\phi - K_6) Q_3 - \frac{\left[\cos^2 i + \sin^2 i \sin^2(\phi - K_6)\right](\alpha_3^2 + \alpha_4^2)}{\cos i} Q_2$$

Next we eliminate $\frac{\partial Z}{\partial(K_7 - f)}$ and $\frac{\partial Z}{\partial i}$ from A_4 and β_5 by Eqs. (53) for A_1 and A_2 and obtain:

$$g_2 = A_4 - \frac{(\alpha_1 + \alpha_2)\alpha_5^{1/2}(k - \alpha_6^2 - \alpha_7^2)\left[\frac{(k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5} - k\,Q_4\right]}{uQ_4\, k\,(2k - \alpha_6^2 - \alpha_7^2)}$$

$$+ A_3^* \frac{k\,(k - \alpha_5 Q_4) + (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{1/2}\, uQ_4\,(2k - \alpha_6^2 - \alpha_7^2)} - A_1 \frac{(\alpha_6^2 + \alpha_7^2)}{k - \alpha_6^2 - \alpha_7^2}$$

$$= (\alpha_6^2 + \alpha_7^2)\left\{ - \frac{k - \alpha_6^2 - \alpha_7^2}{uQ_4^2\,\alpha_5} Q_1 + \frac{\sin i \cos(\phi - K_6)}{\alpha_5^{1/2}} Q_2 + Q_3 \frac{\cos i}{\alpha_5^{1/2}} \right\}$$

$$g_3 = 2\alpha_5\beta_5 - \frac{3\alpha_5^{3/2}}{k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) - \frac{2\alpha_5^{3/2}\, Q_4(\alpha_1 + \alpha_2)}{ku}$$

$$+ \frac{2(k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{1/2}\, uQ_4} A_3^* - A_1 = \frac{\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_7^2)^2}{u\alpha_5\, Q_4^2} Q_1$$

$$+ Q_2 \frac{(k - \alpha_6^2 - \alpha_7^2)\sin i \cos(\phi - K_6)}{\alpha_5^{1/2}} + Q_3 \left(\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2 \right)$$

where

$$A_3^* = \alpha_1 + \alpha_2 - A_3 .$$

Next we write:

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$ where $\begin{bmatrix} B \end{bmatrix}$ is a 3 x 3 matrix.

or

$$[B]^{-1}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

and $\Delta^2(x) = Q_1^2 + Q_4^2 Q_2^2 + Q_4^2 \cos^2\theta\, Q_3^2$.

$$B = \begin{bmatrix} 0 & \dfrac{-\left[\cos^2 i + \sin^2 i \sin^2(\phi - K_6)\right](\alpha_3^2 + \alpha_4^2)}{\cos i} & (\alpha_3^2 + \alpha_4^2)\sin i \cos(\phi - K_6) \\ \dfrac{-(\alpha_6^2 + \alpha_7^2)\,(k - \alpha_6^2 - \alpha_7^2)}{uQ_4^2\alpha_5} & \dfrac{(\alpha_6^2 + \alpha_7^2)}{\alpha_5^{1/2}}\sin i \cos(\phi - K_6) & \dfrac{\alpha_6^2 + \alpha_7^2}{\alpha_5^{1/2}}\cos i \\ \dfrac{\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_7^2)^2}{uQ_4^2\alpha_5} & \dfrac{(k - \alpha_6^2 - \alpha_7^2)\sin i \cos(\phi - K_6)}{\alpha_5^{1/2}} & \dfrac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0 & \dfrac{-uK_4}{(\alpha_6^2 + \alpha_7^2)\,\alpha_5^{1/2}} & \dfrac{u}{\alpha_5} \\ \dfrac{-\cos i}{\alpha_3^2 + \alpha_4^2} & \dfrac{\sin i \cos(\phi - K_6)\,(\alpha_5 Q_4^2 - K_4^2)}{Q_4^2\alpha_5^{1/2}(\alpha_6^2 + \alpha_7^2)} & \dfrac{K_4 \sin i \cos(\phi - K_6)}{Q_4^2\alpha_5} \\ \dfrac{\sin i \cos(\phi - K_6)}{\alpha_3^2 + \alpha_4^2} & \dfrac{\left[\cos^2 i + \sin^2 i \sin^2(\phi - K_6)\right](\alpha_5 Q_4^2 - K_4^2)}{\cos i\; \alpha_5^{1/2} Q_4^2(\alpha_6^2 + \alpha_7^2)} & \dfrac{K_4\left[\cos^2 i + \sin^2 i \sin^2(\phi - K_6)\right]}{\cos i\, Q_4^2\alpha_5} \end{bmatrix}$$

The equations for ρ_1, and Q_5 are given here for reference:

$$\rho_1 = \frac{\alpha_5}{Q_4}\beta_5 + \frac{3\alpha_5^{3/2}}{2kQ_4}\left(\alpha_1 t - \alpha_2 \frac{Q_7}{\sigma}\right) + Q_1 \frac{u}{2Q_4} - \frac{\beta_3\alpha_3 + \beta_4\alpha_4}{4Q_4} - \frac{Q_3}{2Q_4}\left[\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right]$$

$$- \frac{Q_2 \sin i\, \alpha_5^{1/2}\left(\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)(k - \alpha_6^2 - \alpha_7^2)\,\alpha_5^{1/2}}{2\sqrt{(\alpha_3^2 + \alpha_4^2)\left[(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5\left(2\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)^2\right]}\, Q_4\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$Q_5 = Q_3 \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} \sin i \sin(\phi - K_6) + \frac{(k - \alpha_6^2 - \alpha_7^2)(\beta_3 \cos\phi - \beta_4 \sin\phi)}{\alpha_5^{1/2}\sqrt{2\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}$$

$$+ \sqrt{\frac{\alpha_3^2 + \alpha_4^2}{2\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}} \left[A_1 \sin(\phi - K_6) + Q_6 \cos(\phi - K_6)\right]$$

We may now proceed to determine $\Delta(x)$ and H as in the two former cases shown. Here we solve for ϕ instead of Q_4 (r) as before.

III. MODIFIED DELAUNAY AND POINCARÉ VARIABLES

This section presents a method of introducing the three known constants of motion for the full problem as coordinates of the base problem. We will start with the Hamiltonian of section II:

$$H = \lambda_1\left(\frac{v^2}{r^3} + \frac{w^2}{r^3 \cos^2\theta} - \frac{k}{r^2}\right) - \lambda_2 \frac{w^2}{r^2} \sec^2\theta \tan\theta + \rho_1 u$$

$$+ \rho_2 \frac{v}{r^2} + \rho_3 \frac{w}{r^2} \sec^2\theta - \lambda_7 \sigma + \frac{F}{m} \Delta(x) \tag{56}$$

where

$$\Delta(x) = \sqrt{\lambda_1^2 + r^2 \lambda_2^2 + r^2 \cos^2\theta \, \lambda_3^2}$$

The first step is to transform the Hamiltonian and consider only the base problem. The transformation is:

$$S = P_1 \lambda_1 + P_7 m + P_2 \lambda_2 + P_3 w + P_4 r + P_5 \theta + P_6 \phi. \tag{57}$$

The results of this transformation give the following relationships:

$$\begin{aligned}
\lambda_1 &= Q_1 & -u &= P_1 = \partial S/\partial Q_1 \\
\lambda_2 &= Q_2 & -v &= P_2 = \partial S/\partial Q_2 \\
r &= Q_4 & \rho_1 &= P_4 = \partial S/\partial Q_4 \\
m &= Q_7 & \lambda_7 &= P_7 = \partial S/\partial Q_7 \\
w &= Q_3 & \lambda_3 &= P_3 = \partial S/\partial Q_3 \\
\theta &= Q_5 & \rho_2 &= P_5 = \partial S/\partial Q_5 \\
\phi &= Q_6 & \rho_3 &= P_6 = \partial S/\partial Q_6
\end{aligned} \tag{58}$$

The Hamilton-Jacobi equation becomes:

$$0 = \frac{\partial S}{\partial t} - \sigma \frac{\partial S}{\partial Q_7} + Q_1 \left[\frac{\left(\frac{\partial S}{\partial Q_2}\right)^2 + Q_3^2 \sec^2 Q_5}{Q_4^3} - \frac{k}{Q_4^2} \right]$$

$$- \frac{\partial S}{\partial Q_4} \frac{\partial S}{\partial Q_1} - Q_2 \frac{Q_3^2 \sec^2 Q_5 \tan Q_5}{Q_4^2} + \frac{\partial S}{\partial Q_6} \frac{Q_3 \sec^2 Q_5}{Q_4^2} - \frac{\partial S}{\partial Q_5} \frac{\frac{\partial S}{\partial Q_2}}{Q_4^2} . \tag{59}$$

We note that Q_6, t, and Q_7 do not appear, so that by separation of variables we have

$$\frac{\partial S}{\partial t} = K_1, \quad \sigma \frac{\partial S}{\partial Q_7} = K_2, \quad \frac{\partial S}{\partial Q_6} = K_3 . \tag{60}$$

We have proceeded to this point as in section II. Here, we will introduce the known constants. These constants come from the following relationships in cartesian coordinates:

$$\overline{\lambda} \times \ddot{\overline{x}} + \overline{x} \times \ddot{\overline{\lambda}} = 0$$

This integrates to form:

$$\overline{\lambda} \times \dot{\overline{x}} + \overline{x} \times \dot{\overline{\lambda}} = \overline{M} \tag{61}$$

where $\overline{M}$ is a constant vector of three components. The resulting equations in our coordinates are:

$$\begin{aligned} M_1 &= \sin\phi \; A + \cos\phi \; B \\ M_2 &= \cos\phi \; A - \sin\phi \; B \\ M_3 &= \rho_3 \end{aligned} \tag{62}$$

where

$$A = \lambda_3 v - \lambda_2 w \sec^2 \theta + \rho_3 \tan\theta$$

$$B = \rho_2 - \lambda_3 w \tan\theta .$$

It can be shown by Poisson brackets that using M_1, M_2, and M_3 as new momenta is impossible. However, we may use the following:

$$\begin{aligned} K_7^2 &= M_1^2 + M_2^2 = A^2 + B^2 \\ M_3 &= \rho_3 \end{aligned} \tag{63}$$

One notes that $K_7 = K_7(\lambda_2, v, \lambda_3, w, \rho_2, \phi, \rho_3)$. We, therefore, proceed with the following differential equations of the characteristic strip:

$$\frac{\dot{\partial} S}{\partial Q_2} = \frac{Q_3^2 \sec^2 Q_5 \tan Q_5}{Q_4^2} \tag{64}$$

$$\dot{Q}_5 = -\frac{\partial S/\partial Q_2}{r^2} \tag{65}$$

$$\frac{\dot{\partial} S}{\partial Q_1} = -\left(\frac{\left(\frac{\partial S}{\partial Q_2}\right)^2 + Q_3^2 \sec^2 Q_5}{r^3} - \frac{k}{r^2}\right) \tag{66}$$

$$\frac{\dot{\partial} S}{\partial Q_4} = \frac{\partial S}{\partial Q_1} \; . \tag{67}$$

We integrate in a manner similar to that used in section II to obtain:

$$\left(\frac{\partial S}{\partial Q_2}\right)^2 = K_4^2 - Q_3^2 \sec^2 Q_5 \tag{68}$$

$$\left(\frac{\partial S}{\partial Q_1}\right)^2 = -K_5 + \frac{2k}{Q_4} - \frac{K_4^2}{Q_4^2} \tag{69}$$

We may now rewrite the Hamilton-Jacobi equation with the above constants in the form:

$$(K_1 - K_2)r^2 + Q_1 \left(\frac{K_4^2}{r} - k\right) - \frac{\partial S}{\partial Q_4} \frac{\partial S}{\partial Q_1} r^2 = K_6 K_4$$

$$= Q_2 Q_3^2 \sec^2 Q_5 \tan Q_5 - K_3 Q_3 \sec^2 Q_5 + \frac{\partial S}{\partial Q_5} \frac{\partial S}{\partial Q_2} \tag{70}$$

From this separation of variables, we may write:

$$K_6 K_4 = (K_1 - K_2)\, r^2 + Q_1 \left(\frac{K_4^2}{r} - k\right) + \frac{\partial S}{\partial Q_4} ur^2 \tag{71}$$

$$K_4 K_6 = Q_2 Q_3^2 \sec^2 Q_5 \tan Q_5 - K_3 Q_3 \sec^2 Q_5 + \frac{\partial S}{\partial Q_5} \frac{\partial S}{\partial Q_2} \tag{72}$$

From Eq. (63) we have:

$$K_7^2 = \left(-\frac{\partial S}{\partial Q_3}\frac{\partial S}{\partial Q_2} - Q_2 Q_3 \sec^2 Q_5 + K_3 \tan Q_5\right)^2$$

$$+ \left(\frac{\partial S}{\partial Q_5} - \frac{\partial S}{\partial Q_3} Q_3 \tan Q_5\right)^2 . \tag{73}$$

If one expands this expression and solves for $\frac{\partial S}{\partial Q_3}$, one obtains:

$$\frac{\partial S}{\partial Q_3} = \frac{-Q_2 \frac{\partial S}{\partial Q_2} Q_3 \sec^2 Q_5 + K_3 \frac{\partial S}{\partial Q_2} \tan Q_5 + \frac{\partial S}{\partial Q_5} Q_3 \tan Q_5}{(K_4^2 - Q_3^2)} \tag{74}$$

$$\pm \frac{\sqrt{M_7^2 (K_4^2 - Q_3^2) - (K_4 K_6 + K_3 Q_3)^2}}{(K_4^2 - Q_3^2)}$$

Formally we may write:

$$S^{(Q_3)} = \int^{Q_3} \frac{-Q_2 Q_3 \frac{\partial S}{\partial Q_2} \sec^2 Q_5 + K_3 \frac{\partial S}{\partial Q_2} \tan Q_5 + \frac{\partial S}{\partial Q_5} Q_3 \tan Q_5}{(K_4^2 - Q_3^2)} \, dQ_3$$

$$\pm \int^{Q_3} \frac{\sqrt{M_7^2 (K_4^2 - Q_3^2) - (K_4 K_6 + K_3 Q_3)^2}}{(K_4^2 - Q_3^2)} \, dQ_3 \tag{75}$$

One notes that the last expression on the right does not affect the Hamilton-Jacobi equation (59), so that we may augment our transfer function S by this quantity.

One notes that $K_7^2 + K_3^2$ appears in the S function and does not affect the Hamiltonian. We let

$$K_3 = K_3^*, \quad M_7 = K_7$$

and

$$K_7^2 + K_3^2 = K_7^{*2}$$

and drop the asterisk(*). Our S function then becomes:

$$S = K_1 t + \frac{K_2}{\sigma} Q_7 + (K_1 - K_2)\left[\frac{uQ_4}{K_5} - \frac{k}{K_5^{3/2}} \cos^{-1} \frac{k - K_5 Q_4}{\sqrt{k^2 - K_4^2 K_5}}\right]$$

$$- Q_1 u - Q_2 v + K_3 \left[Q_6 - \sin^{-1} \frac{Q_3 \tan Q_5}{\sqrt{K_4^2 - Q_3^2}} \right]$$

$$+ K_6 \left[\cos^{-1} \frac{K_4^2 - kQ_4}{Q_4 \sqrt{k^2 - K_4^2 K_5}} - \sin^{-1} \frac{K_4 \sin Q_5}{\sqrt{K_4^2 - Q_3^2}} \right]$$

$$+ K_7 \sin^{-1} \frac{K_7^2 Q_3 + K_4 K_3 K_6}{K_4 \sqrt{K_7^2 - K_3^2} \sqrt{K_7^2 - K_6^2}} + K_3 \cos^{-1} \frac{K_4 K_6 + K_3 Q_3}{\sqrt{K_4^2 - Q_3^2} \sqrt{K_7^2 - K_3^2}} \tag{76}$$

$$+ K_6 \cos^{-1} \frac{K_3 K_4 + K_6 Q_3}{\sqrt{K_4^2 - Q_3^2} \sqrt{K_7^2 - K_6^2}}$$

Our K's are now defined as

$$
\begin{aligned}
K_1 &= -H_o \\
K_2 &= \sigma \lambda_7 \\
K_3 &= \rho_3 \\
K_4^2 &= v^2 + Q_3^2 \sec^2 Q_5 \\
K_5 &= \frac{2k}{Q_4} - \frac{K_4^2}{Q_4^2} - u^2 \\
K_6 &= \frac{1}{K_4} (\lambda_2 Q_3^2 \sec^2 Q_5 \tan Q_5 - \rho_3 Q_3 \sec^2 Q_5 - \rho_2 v) \\
K_7^2 &= (\lambda_3 v - \lambda_2 Q_3 \sec^2 Q_5 + K_3 \tan Q_5)^2 + (\rho_2 - \lambda_3 Q_3 \tan Q_5)^2 + K_3^2 \quad .
\end{aligned}
\tag{77}
$$

We may now record the time derivatives of the K's:

$$
\begin{aligned}
\dot{K}_2 &= \frac{F\sigma}{m^2}\,\Delta(x) \\
\dot{K}_3 &= 0 \\
\dot{K}_4 &= \frac{F(vQ_2 + Q_3\lambda_3)\,r^2}{m\Delta(x)K_4} \\
\dot{K}_5 &= -2\,\frac{F}{m\Delta(x)}\,(v\,Q_2 + \lambda_3 Q_3 + Q_1 u) \\
\dot{K}_6 &= \frac{-K_6}{K_4}\,\dot{K}_4 + \frac{F\,r^2}{m\Delta(x)K_4}\left[\lambda_2\lambda_3 2Q_3\tan\theta - v\sin\theta\cos\theta\,\lambda_3^2 - K_3\lambda_3 - \rho_2\lambda_2\right] \\
\dot{K}_7 &= 0 \\
\dot{K}_1 &= \frac{F}{m\Delta(x)}\left[-\lambda_1\left(\frac{2}{r}(v\,Q_2 + Q_3\lambda_3) + \rho_1\right) - \lambda_2(-ur\,\lambda_2 - 2Q_3\lambda_3\tan Q_5 + \rho_2)\right. \\
&\quad \left. + \lambda_3(-\lambda_3\cos Q_5\,(v\sin Q_5 - ur\cos Q_5) - \rho_3)\right] + \frac{F\sigma}{m^2}\,\Delta(x)\ .
\end{aligned} \tag{78}
$$

We next record the L's:

$$
\begin{aligned}
L_1 &= t + \frac{uQ_4}{K_5} - \frac{k}{K_5^{3/2}}\cos^{-1}\frac{k - K_5 Q_4}{\sqrt{k^2 - K_4^2 K_5}} \\
L_2 &= \frac{Q_7}{\sigma} + t - L_1 \\
L_3 &= Q_6 - \sin^{-1}\frac{Q_3\tan Q_5}{\sqrt{K_4^2 - Q_3^2}} + \cos^{-1}\frac{K_4K_6 + K_3Q_3}{\sqrt{K_4^2 - Q_3^2}\sqrt{K_7^2 - K_3^2}} \\
L_4 &= \frac{(K_1 - K_2)K_4(K_4^2 - kQ_4)}{u\,Q_4(k^2 - K_4^2K_5)} + \frac{Q_1K_4}{u\,Q_4^2} - Q_2\,\frac{K_4}{v} + \frac{K_3K_4Q_3\tan Q_5}{v(K_4^2 - Q_3^2)} \\
&\quad + K_6\left[\frac{+(K_4^2K_5 + K_5kQ_4 - 2k^2)}{u\,Q_4\,(k^2 - K_4^2K_5)} + \frac{Q_3^2\tan Q_5}{v(K_4^2 - Q_3^2)}\right] \\
&\quad - \frac{\sqrt{K_4^2(K_7^2 - K_6^2 - K_3^2) - 2K_4K_3K_6Q_3 - K_7^2Q_3^2}\;\;Q_3}{K_4(K_4^2 - Q_3^2)}
\end{aligned} \tag{79}
$$

$$L_4 = Q_1 \frac{K_4^3\, u}{Q_4^2\,(k^2 - K_4^2\, K_5)} - \rho_1 \frac{K_4\,(K_4^2 - k\, Q_4)}{Q_4\,(k^2 - K_4^2\, K_5)} - \frac{K_6\,(K_4^2 + k\, Q_4)\, u}{Q_4\,(k^2 - K_4^2\, K_5)} - \rho_2 \frac{Q_3^2 \tan Q_5}{K_4\,(K_4^2 - Q_3^2)}$$

$$- Q_2 \frac{v\,(K_4^2 + Q_3^2 \tan^2 Q_5)}{K_4\,(K_4^2 - Q_3^2)} + \frac{K_3\, Q_3 \tan Q_5\, v}{K_4\,(K_4^2 - Q_3^2)} - \frac{\sqrt{(K_7^2 - K_3^2)\,(K_7^2 - K_6^2)}\;\; Q_3 \cos L_7}{K_7\,(K_4^2 - Q_3^2)}$$

$$L_5 = \left\{ Q_1 \left[\frac{-3(t - L_1)(K_4^2 - kQ_4)}{Q_4^3} + \frac{K_4^4\, u}{Q_4^2\,(k^2 - K_4^2\, K_5)} - u \right] - \rho_1 \left[3u(t - L_1) - 2Q_4 + \frac{K_4^2(K_4^2 - kQ_4)}{Q_4\,(k^2 - K_4^2\, K_5)} \right] \right.$$

$$\left. + K_4\, K_6 \left[\frac{-\,(K_4^2 + kQ_4)\, u}{Q_4(k^2 - K_4^2\, K_5)} + \frac{3(t - L_1)}{Q_4^2} \right] \right\} \frac{1}{2K_5}$$

$$L_5 = \frac{(K_1 - K_2)}{K_5} \left\{ \frac{3}{2}\,(t - L_1) - \frac{Q_4}{u} + \frac{K_4^2(K_4^2 - kQ_4)}{2u\, Q_4(k^2 - K_4^2 K_5)} \right\} + \frac{Q_1}{2u} - \frac{K_4 K_6\,(K_4^2 - kQ_4)}{2uQ_4\,(k^2 - K_4^2\, K_5)}$$

$$L_6 = \cos^{-1} \frac{K_4^2 - kQ_4}{Q_4 \sqrt{k^2 - K_4^2\, K_5}} - \sin^{-1} \frac{K_4 \sin Q_5}{\sqrt{K_4^2 - Q_3^2}} + \cos^{-1} \frac{K_3 K_4 + K_6 Q_3}{\sqrt{K_4^2 - Q_3^2}\, \sqrt{K_7^2 - K_6^2}}$$

$$L_7 = \sin^{-1} \frac{K_7^2\, Q_3 + K_4 K_3 K_6}{K_4 \sqrt{K_7^2 - K_3^2}\, \sqrt{K_7^2 - K_6^2}}$$

Note that these relationships were used:

$$\cos^{-1} \frac{K_3 K_4 + K_6 Q_3}{\sqrt{K_4^2 - Q_3}\sqrt{K_7^2 - K_6^2}} = \sin^{-1} \frac{K_4\sqrt{K_7^2 - K_3^2}\cos L_7}{K_7\sqrt{K_4^2 - Q_3^2}}$$

$$\cos^{-1} \frac{K_4 K_6 + K_3 Q_3}{\sqrt{K_4^2 - Q_3^2}\sqrt{K_7^2 - K_3^2}} = \sin^{-1} \frac{K_4\sqrt{K_7^2 - K_6^2}\cos L_7}{K_7\sqrt{K_4^2 - Q_3^2}}$$

For this problem

$$\dot{\overline{Q}} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{Fr^2 \lambda_3 \cos^2\theta}{m\Delta(x)} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Again, we may write

$$\dot{L}_i = \frac{\partial^2 S}{\partial K_i \, \partial \overline{K}} \dot{\overline{K}} + \frac{\partial^2 S}{\partial K_i \, \partial \overline{Q}} \dot{\overline{Q}} + \frac{\partial^2 S}{\partial K_i \, \partial t} \qquad (80)$$

where the

$$\frac{\partial^2 S}{\partial K_i \partial q}$$

is obtained by FORMAC and $\dot{\overline{K}}$ and $\dot{\overline{Q}}$ are given before.

Next we will develop the inversions needed to define $\Delta(x)$:

$$Q_1 = 2L_5 u - \frac{(K_1 - K_2)}{K_5}\left\{3u(t - L_1) - 2Q_4 + \frac{K_4^2 (K_4^2 - kQ_4)}{Q_4(k^2 - K_4^2 K_5)}\right\} + \frac{K_4 K_6 (K_4^2 - kQ_4)}{Q_4(k^2 - K_4^2 K_5)}$$

$$Q_2 = -\frac{v}{K_4} L_4 + \frac{K_3 Q_3 \tan Q_5}{(K_4^2 - Q_3^2)} + \frac{2vL_5}{Q_4^2} + K_6 \left[-\frac{uv\,(K_4^2 + kQ_4)}{K_4 Q_4 (k^2 - K_4^2 K_5)} + \frac{Q_3^2 \tan Q_5}{K_4\,(K_4^2 - Q_3^2)} \right]$$

$$- Q_3 v \frac{\sqrt{K_4^2\,(K_7^2 - K_3^2 - K_6^2) - 2\,K_3 K_4 K_6 Q_3 - K_7^2 Q_3^2}}{K_4^2\,(K_4^2 - Q_3^2)} + \frac{v\,(K_1 - K_2)}{Q_4 K_5} \left[-\frac{3\,(t - L_1)}{Q_4} + \frac{(K_4^2 + kQ_4)\,u}{(k^2 - K_4^2 K_5)} \right]$$

$$\lambda_3 = Q_2 \frac{Q_3 \sec^2 Q_5}{v} - \frac{\tan Q_5 K_4 (K_4 K_3 + K_6 Q_3)}{v\,(K_4^2 - Q_3^2)} + \frac{\sqrt{K_7^2\,(K_4^2 - Q_3^2) - 2\,K_3 K_4 K_6 Q_3 - K_4^2\,(K_6^2 + K_3^2)}}{(K_4^2 - Q_3^2)}$$

From our equations for L's, we have:

$$\sin L_7 = \frac{K_7^2 Q_3 + K_4 K_3 K_6}{K_4 \sqrt{K_7^2 - K_3^2}\ \sqrt{K_7^2 - K_6^2}} \tag{82}$$

$$\sin\,(f - L_6) = \frac{\cos Q_5}{\sqrt{K_7^2 - K_6^2}} \left[\frac{K_4 \tan Q_5\,(K_4 K_3 + K_6 Q_3) - \sqrt{P}\ v}{(K_4^2 - Q_3^2)} \right]$$

where

$$\sqrt{P} = \sqrt{K_7^2\,(K_4^2 - Q_3^2) - 2\,K_3 K_4 K_6 Q_3 - K_4^2\,(K_6^2 + K_3^2)}$$

$$\cos f = \frac{K_4^2 - kQ_4}{Q_4 \sqrt{k^2 - K_4^2 K_5}}$$

Let

$$G_2 = -\frac{L_4}{K_4} + \frac{(K_1 - K_2)\,(K_4^2 - kQ_4)}{uQ_4\,(k^2 - K_4^2 K_5)} + \frac{K_6\,(K_4^2 K_5 + K_5 kQ_4 - 2k^2)}{K_4\,uQ_4\,(k^2 - K_4^2 K_5)} + \frac{Q_1}{uQ_4^2} \tag{83}$$

Then:

$$Q_2 = vG_2 + \frac{Q_3 \sec Q_5}{K_4^2}\sqrt{K_7^2 - K_6^2}\,\sin\,(f - L_6)$$

$$\cos Q_5 \lambda_3 = Q_3 \sec Q_5 G_2 - \frac{v}{K_4^2}\sqrt{K_7^2 - K_6^2}\,\sin\,(f - L_6)$$

$$Q_2^2 + \cos^2 Q_5\, \lambda_3^2 = G_2^2\, K_4^2 + \frac{(K_7^2 - K_6^2)}{K_4^2}\sin^2\,(f - L_6) \tag{84}$$

where

$$\sin\,(f - L_6) = \frac{K_4\, uQ_4 \cos L_6 - (K_4^2 - kQ_4) \sin L_6}{Q_4 \sqrt{k^2 - K_4^2\, K_5}}$$

One notes on the inspections of G_2, Q_1 and $\sin\,(f - L_6)$ that these are not functions of L_7, $\dot{L}_3$, or K_3. So that $F/\dot{m}\,\left[\Delta\,(x)\right]$ and H_o do not contain these variables. This implies that $\dot{K}_7 = 0$, $\dot{K}_3 = 0$, and $L_3 = 0$ as desired.

Formally we may write

$$\Delta\,(x)^2 = Q_1^2 + Q_4^2\left[G_2^2\, K_4^2 + \frac{K_7^2 - K_6^2}{K_4^2}\sin^2\,(f - L_6)\right] \tag{85}$$

We now proceed to write the modified Delaunay variables.

Let:

$$K_1 = -\alpha_1 \frac{\alpha_5^{3/2}}{k} \qquad K_4 = \alpha_6$$

$$K_2 = \alpha_2 \frac{\alpha_5^{3/2}}{k} \qquad K_3 = \alpha_3$$

$$K_5 = \alpha_5 \qquad K_6 = \alpha_4$$

$$K_7 = \alpha_7$$

Then we will have β_1, β_2, and β_5 as before with β_3 and β_4 similar to α_3 and α_4 before. The Delaunay-like variables will be β_1, α_5, α_6, β_3, and β_4. The relationship for w is given through β_7. The new generating function becomes:

$$S = -\frac{\alpha_5^{3/2}}{k}\alpha_1 t + \frac{\alpha_5^{3/2}}{k}\frac{\alpha_2 Q_7}{\sigma} + (\alpha_1 + \alpha_2)\left[-\frac{uQ_4\alpha_5^{1/2}}{k} + \cos^{-1}\frac{k - \alpha_5 Q_4}{\sqrt{k^2 - \alpha_6^2\alpha_5}}\right]$$

$$- Q_1 u - Q_2 v + \alpha_3\left[Q_6 - \sin^{-1}\frac{Q_3 \tan Q_5}{\sqrt{\alpha_6^2 - Q_3^2}} + \cos^{-1}\frac{\alpha_4\alpha_6 + \alpha_3 Q_3}{\sqrt{\alpha_6^2 - Q_3^2}\sqrt{\alpha_7^2 - \alpha_3^2}}\right]$$

$$+ \alpha_4\left[\cos^{-1}\frac{\alpha_6^2 - kQ_4}{Q_4\sqrt{k^2 - \alpha_5\alpha_6^2}} - \sin^{-1}\frac{\alpha_6 \sin Q_5}{\sqrt{\alpha_6^2 - Q_3^2}} + \cos^{-1}\frac{\alpha_3\alpha_6 + \alpha_4 Q_3}{\sqrt{\alpha_6^2 - Q_3^2}\sqrt{\alpha_7^2 - \alpha_4^2}}\right]$$

$$+ \alpha_7 \sin^{-1}\frac{\alpha_7^2 Q_3 + \alpha_6\alpha_3\alpha_4}{\alpha_6\sqrt{\alpha_7^2 - \alpha_3^2}\sqrt{\alpha_7^2 - \alpha_4^2}} \quad . \qquad (87)$$

Our α's are now defined as

$$\alpha_1 = \frac{k}{\alpha_5^{3/2}} H_o$$

$$\alpha_2 = \frac{k}{\alpha_5^{3/2}} \sigma \lambda_7$$

$$\alpha_3 = \rho_3$$

$$\alpha_4 = \frac{1}{\alpha_6}(\lambda_2 Q_3^2 \sec^2 Q_5 \tan Q_5 - \alpha_3 Q_3 \sec^2 Q_5 - \rho_2 v) \qquad (88)$$

$$\alpha_5 = \frac{2k}{Q_4} - \frac{\alpha_6^2}{Q_4^2} - u^2$$

$$\alpha_6^2 = v^2 + Q_3^2 \sec^2 Q_5$$

$$\alpha_7^2 = (\lambda_3 v - Q_2 Q_3 \sec^2 Q_5 + \alpha_3 \tan Q_5)^2 + (\rho_2 - \lambda_3 Q_3 \tan Q_5)^2 + \alpha_3^2$$

where the equations for α_5 and α_6 define u and v of the generating function.

Next we record the β's:

$$\beta_1 = -\frac{\alpha_5^{3/2}}{k} t - \frac{uQ_4\alpha_5^{1/2}}{k} + \cos^{-1} \frac{k - \alpha_5 Q_4}{\sqrt{k^2 - \alpha_5\alpha_6^2}}$$

$$\beta_2 = \frac{\alpha_5^{3/2}}{k\sigma} Q_7 + \frac{\alpha_5^{3/2}}{k} t + \beta_1$$

$$\beta_3 = Q_6 - \sin^{-1} \frac{Q_3 \tan Q_5}{\sqrt{\alpha_6^2 - Q_3^2}} + \cos^{-1} \frac{\alpha_4\alpha_6 + \alpha_3 Q_3}{\sqrt{\alpha_6^2 - Q_3^2}\sqrt{\alpha_7^2 - \alpha_3^2}}$$

$$\beta_4 = \cos^{-1} \frac{\alpha_6^2 - kQ_4}{Q_4\sqrt{k_2 - \alpha_5\alpha_6^2}} - \sin^{-1} \frac{\alpha_6 \sin Q_5}{\sqrt{\alpha_6^2 - Q_3^2}} + \cos^{-1} \frac{\alpha_3\alpha_6 + \alpha_4 Q_3}{\sqrt{\alpha_6^2 - Q_3^2}\sqrt{\alpha_7^2 - \alpha_4^2}}$$

$$\beta_5 = -\frac{3\alpha_5^{1/2}}{2k}\left(\alpha_1 t - \frac{\alpha_2 Q_7}{\sigma}\right) + \frac{(\alpha_1 + \alpha_2)\alpha_5^{1/2}}{ku}\left(Q_4 - \frac{\alpha_6^2(\alpha_6^2 - kr)}{2Q_4(k^2 - \alpha_5\alpha_6^2)}\right)$$

$$+ \frac{Q_1}{2u} - \frac{\alpha_4\alpha_6(\alpha_6^2 - kQ_4)}{2uQ_4(k^2 - \alpha_5\alpha_6^2)}$$

$$\beta_6 = -\frac{\alpha_5^{3/2}(\alpha_1+\alpha_2)\alpha_6(\alpha_6^2 - kQ_4)}{kuQ_4(k^2 - \alpha_5\alpha_6^2)} + \frac{Q_1\alpha_6}{uQ_4^2} - \frac{Q_2\alpha_6}{v}$$

$$+ \frac{Q_3(\alpha_6\alpha_3 + \alpha_4 Q_3)\tan Q_5}{v(\alpha_6^2 - Q_3^2)} + \frac{\alpha_4(\alpha_5\alpha_6^2 + \alpha_5 kQ_4 - 2k^2)}{uQ_4(k^2 - \alpha_5\alpha_6^2)} - \frac{Q_3\sqrt{P}}{\alpha_6(\alpha_6^2 - Q_3^2)}$$

$$\beta_7 = \sin^{-1}\frac{\alpha_7^2 Q_3 + \alpha_6\alpha_3\alpha_4}{\alpha_6\sqrt{\alpha_7^2 - \alpha_3^2}\sqrt{\alpha_7^2 - \alpha_4^2}}$$

where

$$P = \alpha_7^2(\alpha_6^2 - Q_3^2) - 2\alpha_3\alpha_4\alpha_6 Q_3 - \alpha_6^2(\alpha_4^2 + \alpha_3^2) \quad .$$

We proceed as before to determine the λ's needed for $\Delta(x)$:

$$Q_1 = 2u\beta_5 + \frac{3u\alpha_5^{1/2}}{k}\left(\alpha_1 t - \frac{\alpha_2 Q_7}{\sigma}\right) - \frac{2(\alpha_1+\alpha_2)\alpha_5^{1/2}}{k}\left(Q_4 - \frac{\alpha_6^2(\alpha_6^2 - kQ_4)}{2Q_4(k^2 - \alpha_5\alpha_6^2)}\right)$$

$$- \frac{\alpha_4\alpha_6(\alpha_6^2 - kQ_4)}{Q_4(k^2 - \alpha_5\alpha_6^2)}$$

$$Q_2 = vG_2 + \frac{Q_3}{\alpha_6^2}\left(\frac{\alpha_6\tan Q_5(\alpha_6\alpha_3 + \alpha_4 Q_3)}{(\alpha_6^2 - Q_3^2)} - \frac{v\sqrt{P}}{(\alpha_6^2 - Q_3^2)}\right)$$

$$G_2 = -\frac{\beta_6}{\alpha_6} - \frac{\alpha_5^{3/2}(\alpha_1+\alpha_2)(\alpha_6^2 - kQ_4)}{kuQ_4(k^2 - \alpha_5\alpha_6^2)} + \frac{Q_1}{uQ_4^2} + \frac{\alpha_4(\alpha_5\alpha_6^2 + \alpha_5 kQ_4 - 2k^2)}{\alpha_6 uQ_4(k^2 - \alpha_5\alpha_6^2)}$$

$$\cos Q_5 \lambda_3 = Q_3 \sec Q_5 G_2 - \frac{v \cos Q_5}{\alpha_6^2} \left[\frac{\alpha_6 \tan Q_5 (\alpha_6 \alpha_3 + \alpha_4 Q_3) - v \sqrt{P}}{\alpha_6^2 - Q_3^2} \right]$$

$$\sin \beta_7 = \frac{\alpha_7^2 Q_3 + \alpha_3 \alpha_4 \alpha_6}{\alpha_6 \sqrt{\alpha_7^2 - \alpha_3^2} \sqrt{\alpha_7^2 - \alpha_4^2}}$$

$$\cos^{-1} \frac{\alpha_6^2 - kQ_4}{Q_4 \sqrt{k^2 - \alpha_5 \alpha_6^2}} = f$$

$$\sin (f - \beta_4) = \frac{(\alpha_3 \alpha_6 + \alpha_4 Q_3) \alpha_6 \sin Q_5 - v \cos Q_5 \sqrt{P}}{(\alpha_6^2 - Q_3^2) \sqrt{\alpha_7^2 - \alpha_4^2}}$$

so that

$$\cos Q_5 \lambda_3 = Q_3 \sec Q_5 G_2 - \frac{v}{\alpha_6^2} \sqrt{\alpha_7^2 - \alpha_4^2} \sin (f - \beta_4)$$

$$Q_2 = v G_2 + \frac{Q_3 \sec Q_5}{\alpha_6^2} \sqrt{\alpha_7^2 - \alpha_4^2} \sin (f - \beta_4) \tag{91}$$

and

$$Q_2^2 + \cos^2 Q_5 \lambda_3^2 = \alpha_6^2 G_2^2 + \frac{\alpha_7^2 - \alpha_4^2}{\alpha_6^2} \sin^2 (f - \beta_4)$$

$$\sin (f - \beta_4) = \frac{\alpha_6 u Q_4 \cos \beta_4 - (\alpha_6^2 - k Q_4) \sin \beta_4}{Q_4 \sqrt{k^2 - \alpha_5 \alpha_6^2}}$$

Again we note that Q_1 and $Q_2^2 + \cos^2 Q_5 \lambda_3^2$ do not contain α_3, β_7, and β_3. Therefore, $\dot{\beta}_3 = 0$, $\dot{\alpha}_3 = 0$, and $\dot{\alpha}_7 = 0$, as desired.

We formally write:

$$\Delta^2(x) = Q_1^2 + Q_4^2 \left[G_2^2 \alpha_6^2 + \frac{\alpha_7^2 - \alpha_4^2}{\alpha_6^2} \sin^2 (f - \beta_4) \right] \tag{93}$$

The same Q's are used in this problem and the $\dot{\bar{Q}}$ is the same as previously cited. Below are listed the $\dot{\bar{\alpha}}$'s:

$$\dot{\alpha}_3 = 0$$

$$\dot{\alpha}_7 = 0$$

$$\dot{\alpha}_6 = \frac{F Q_4^2 (v Q_2 + Q_3 \lambda_3)}{m \Delta(x) \alpha_6} \tag{94}$$

$$\dot{\alpha}_5 = -2 \frac{F}{m\Delta(x)} (v Q_2 + Q_3 \lambda_3 + u Q_1)$$

$$\dot{\alpha}_2 = \frac{k\sigma F}{\alpha_5^{3/2} m^2} \Delta(x) + \frac{3\alpha_2 F}{\alpha_5 m \Delta(x)} (v Q_2 + Q_3 \lambda_3 + u Q_1)$$

$$\dot{\alpha}_4 = \frac{F Q_4^2}{m \Delta(x) \alpha_6} \left[2 Q_3 \lambda_3 \lambda_2 \tan Q_5 - \lambda_3 \alpha_3 - \rho_2 \lambda_2 - v \lambda_3^2 \sin Q_5 \cos Q_5 \right.$$
$$\left. - \frac{\alpha_4}{\alpha_6} (v Q_2 + Q_3 \lambda_3) \right]$$

$$\dot{\alpha}_1 = \frac{3 \alpha_1 F}{\alpha_5 m \Delta(x)} (v Q_2 + Q_3 \lambda_3 + u Q_1) + \frac{Fk}{m \Delta(x) \alpha_5^{3/2}} \left(\lambda_1 \left[\frac{2}{Q_4} (Q_2 v + Q_3 \lambda_3) + \rho_1 \right] \right.$$
$$\left. + \lambda_2 \left[-u Q_4 \lambda_2 - 2 Q_3 \lambda_3 \tan Q_5 + \rho_2 \right] + \lambda_3 \left[\lambda_3 \cos Q_5 (-u Q_4 \cos Q_5 + v \sin Q_5) + \rho_3 \right] \right)$$
$$- \frac{Fk\sigma}{\alpha_5^{3/2} m^2} \Delta(x)$$

We have now presented the information to proceed as before.

We develop the Poincaré variables in two steps. First, we consider the transformations of section II and note that we need the following transformations:

$$
\begin{aligned}
K_1 &= -\frac{K_5^{*3/2}}{k} K_1^* \\
K_2 &= \frac{K_5^{*3/2}}{k} K_2^* \\
K_6 &= -K_1^* + K_6^* - K_2^* \\
K_i &= K_i^* \qquad i = 3,\ 4,\ 5,\ 7\ .
\end{aligned}
\tag{95}
$$

The K's with the asterisks are the new constants. Our S function then becomes:

$$
\begin{aligned}
S = &+\frac{K_5^{*3/2}}{k}\left(-K_1^* t + K_2^* \frac{Q_7}{\sigma}\right) + (K_1^* + K_2^*)\left[-\frac{u\,Q_4\,K_5^{*1/2}}{k} + \cos^{-1}\frac{k - K_5^* Q_4}{\sqrt{k^2 - K_4^{*2} K_5^*}}\right. \\
&\left. - \cos^{-1}\frac{K_4^{*2} - k\,Q_4}{Q_4\sqrt{k^2 - K_4^{*2} K_5^*}} + \sin^{-1}\frac{K_4^* \sin Q_5}{\sqrt{K_4^{*2} - Q_3^{2}}}\right] \\
&- Q_1 u - Q_2 v + K_3^* \left[Q_6 - \sin^{-1}\frac{Q_3 \tan Q_5}{\sqrt{K_4^{*2} - Q_3^2}}\right] \\
&+ K_6^*\left(\cos^{-1}\frac{K_4^{*2} - k\,Q_4}{Q_4\sqrt{k^2 - K_4^{*2} K_5^*}} - \sin^{-1}\frac{K_4^* \sin Q_5}{\sqrt{K_4^{*2} - Q_3^2}}\right) \\
&+ K_7^* \sin^{-1}\frac{K_7^{*2}\,Q_3 + K_3^*\,K_4^*\,(K_6^* - K_1^* - K_2^*)}{K_4^*\sqrt{K_7^{*2} - K_3^{*2}}\sqrt{K_7^{*2} - (K_6^* - K_1^* - K_2^*)^2}} \\
&+ (K_6^* - K_1^* - K_2^*)\cos^{-1}\frac{K_3^* K_4^* + Q_3\,(K_6^* - K_1^* - K_2^*)}{\sqrt{K_4^{*2} - Q_3^2}\sqrt{K_7^{*2} - (K_6^* - K_1^* - K_2^*)^2}} \\
&+ K_3^* \cos^{-1}\frac{K_4^*\,(K_6^* - K_1^* - K_2^*) + K_3^* Q_3}{\sqrt{K_4^{*2} - Q_3^2}\sqrt{K_7^{*2} - K_3^{*2}}}
\end{aligned}
\tag{96}
$$

Our K*'s are now defined:

$$K_1^* = \frac{k}{K_5^{*3/2}} H_o$$

$$K_2^* = \frac{k}{K_5^{*3/2}} \sigma \lambda_7$$

$$K_3^* = \rho_3 \tag{97}$$

$$K_4^{*2} = v^2 + Q_3^2 \sec^2 Q_5$$

$$K_5^* = \frac{2k}{Q_4} - \frac{K_4^2}{Q_4^2} - u^2$$

$$K_6^* = - \frac{k}{K_5^{3/2}}(K_1 - K_2) + \frac{1}{K_4} (\lambda_2 Q_3^2 \sec^2 Q_5 \tan Q_5 - \rho_3 Q_3 \sec^2 Q_5 - \rho_2 v)$$

$$K_7^{*2} = (\lambda_3 v - \lambda_2 Q_3 \sec^2 Q_5 + \rho_3 \tan Q_5)^2 + (\rho_2 - \lambda_3 Q_3 \tan Q_5)^2 + K_3^{*2} \quad .$$

We next record the L's:

$$L_1^* = - \frac{K_5^{*3/2}}{k} t - \frac{u Q_4 K_5^{*1/2}}{k} + \sin^{-1} \frac{K_4^* \sin Q_5}{\sqrt{K_4^{*2} - Q_3^2}} \tag{98}$$

$$+ \cos^{-1} \left(1 - \frac{u^2 Q_4}{k + K_4^* K_5^{*1/2}} \right) - \cos^{-1} \frac{K_4^* K_3^* + (K_6^* - K_1^* - K_2^*) Q_3}{\sqrt{K_4^{*2} - Q_3^2} \sqrt{K_7^{*2} - (K_6^* - K_1^* - K_2^*)^2}}$$

$$L_2^* = \frac{K_5^{*3/2}}{k} \quad \frac{Q_7}{\sigma} + L_1^* + \frac{K_5^{*3/2}}{k} t$$

$$L_3^* = Q_6 - \sin^{-1} \frac{Q_3 \tan Q_5}{\sqrt{K_4^{*2} - Q_3^2}} + \cos^{-1} \frac{K_4^* (K_6^* - K_1^* - K_2^*) + K_3^* Q_3}{\sqrt{K_4^{*2} - Q_3^2} \sqrt{K_7^{*2} - K_3^{*2}}}$$

$$L_6^* = \cos^{-1} \frac{K_4^{*2} - k Q_4}{Q_4 \sqrt{k^2 - K_4^{*2} K_5^*}} - \sin^{-1} \frac{K_4^* \sin Q_5}{\sqrt{K_4^{*2} - Q_3^2}} + \cos^{-1} \frac{K_3^* K_4^* + (K_6^* - K_1^* - K_2^*) Q_3}{\sqrt{K_4^{*2} - Q_3^2} \sqrt{K_7^{*2} - (K_6^* - K_1^* - K_2^*)^2}}$$

$$L_7^* = \sin^{-1} \frac{K_7^{*2} Q_3 + K_4^* K_3^* (K_6^* - K_1^* - K_2^*)}{K_4^* \sqrt{\left[K_7^{*2} - (K_6^* - K_1^* - K_2^*)^2\right] \left[K_7^{*2} - K_3^{*2}\right]}}$$

$$L_5^* = \frac{3K_5^{*1/2}}{2k} \left(K_2^* \frac{Q_7}{\sigma} - K_1^* t\right) + (K_1^* + K_2^*) \left[- \frac{uQ_4}{2k K_5^{*1/2}} + \frac{K_5^{*1/2} Q_4}{2uk} \right.$$

$$\left. - \frac{K_4^{*2} - 2k Q_4 - K_4^* K_5^{*1/2} Q_4}{2 K_5^{*1/2} u Q_4 (k + K_4^* K_5^{*1/2})} \right] - \frac{K_6^* K_4^* (K_4^{*2} - k Q_4)}{2u Q_4 (k^2 - K_4^{*2} K_5^*)} + \frac{Q_1}{2u}$$

$$L_4^* = (K_1^* + K_2^*) \left[\frac{k + K_4^* K_5^{*1/2}}{k u Q_4} + \frac{\left(k - K_5^* Q_4\right)}{u Q_4 (k + K_4^* K_5^{*1/2})} - \frac{Q_3^2 \tan Q_5}{v (K_4^{*2} - Q_3^2)} \right]$$

$$+ \frac{Q_1 K_4^*}{u Q_4^2} - \frac{Q_2 K_4^*}{v} + \frac{K_3^* K_4^* Q_3 \tan Q_5}{v (K_4^{*2} - Q_3^2)} - K_6^* \left[\frac{k (k - K_5^* Q_4)}{u Q_4 (k^2 - K_4^{*2} K_5^*)} + \frac{1}{u Q_4} - \frac{Q_3^2 \tan Q_5}{v (K_4^{*2} - Q_3^2)} \right]$$

$$- \frac{\sqrt{K_4^{*2} \left[K_7^{*2} - (K_6^* - K_2^* - K_1^*)^2 - K_3^{*2}\right] - 2 K_4^* K_3^* Q_3 (K_6^* - K_2^* - K_1^*) - K_7^{*2} Q_3^2} \quad Q_3}{K_4^* (K_4^{*2} - Q_3^2)}$$

$$L_4^* = - \frac{Q_1 (k^2 + K_4^{*2} K_5^* + k K_4^* K_5^{*1/2}) u}{(Q_4^2 K_5^{*3/2} - K_4^* k) (k + K_4^* K_5^{*1/2})} - \frac{K_6^* K_5^{*3/2} (K_4^{*2} + k Q_4) u Q_4}{(K_5^{*3/2} Q_4^2 - k K_4^*) (k^2 - K_4^{*2} K_5^*)}$$

$$+ \frac{\rho_1 Q_4}{(K_5^{*3/2} Q_4^2 - K_4^* k)} \left[k + K_4^* K_5^{*1/2} + \frac{k (k - K_5^* Q_4)}{k + K_4^* K_5^{*1/2}} \right] - \frac{\rho_2 Q_3^2 \tan Q_5}{K_4^* (K_4^{*2} - Q_3^2)}$$

$$+ \frac{K_3^* v Q_3 \tan Q_5}{K_4^* (K_4^{*2} - Q_3^2)} - \frac{Q_2 (K_4^{*2} + Q_3^2 \tan^2 Q_5) v}{K_4^* (K_4^{*2} - Q_3^2)}$$

$$- \frac{Q_3 \sqrt{K_4^{*2} \left[K_7^{*2} - (K_6^* - K_1^* - K_2^*)^2 - K_3^{*2}\right] - 2 K_4^* K_3^* Q_3 (K_6^* - K_1^* - K_2^*) - K_7^{*2} Q_3^2}}{K_4^* (K_4^{*2} - Q_3^2)}$$

$$L_5^* = \frac{3K_5^{*1/2}}{2k} \left(K_2^* \frac{Q_7}{\sigma} - K_1^* t\right) - \frac{K_6^* K_4^* u Q_4 K_5^{*1/2} (K_4^{*2} + k Q_4)}{2 (k^2 - K_4^{*2} K_5^*) (K_5^{*3/2} Q_4^2 - K_4^* k)}$$

$$+ \frac{u Q_1}{2 K_5^{*1/2} (K_5^{*3/2} Q_4^2 - K_4^* k)} \left(- K_5^* Q_4^2 - K_4^{*2} + \frac{K_4^{*2} k}{(k + K_4^* K_5^{*1/2})} \right)$$

$$+ \frac{\rho_1 Q_4^2}{2K_5^{*1/2} (K_5^{*3/2} Q_4^2 - K_4^* k)} \left(-u^2 Q_4 + K_5^* Q_4 + k - \frac{(K_4^{*2} - k Q_4) k}{Q_4 (k + K_4^* K_5^{*1/2})} \right)$$

One notes that the factor $k^2 - K_4^{*2} K_5^*$ still remains in L_6^*, L_5^*, and L_4^*. To remove these we need two transformations. One we rename L_6 and K_6. The other will affect Eqs. (46) of section II. This will not be done directly because of the needed constants. These transformations, which constitute the second step, are

$$S = L_6^* L_6' + \text{identity}$$

$$\frac{\partial S}{\partial L_6^*} = K_6^* = L_6' \qquad \frac{\partial S}{\partial L_6'} = -K_6' = L_6^*$$

$$S = -\sqrt{k - K_4' K_5'^{1/2}} \sin K_6' \beta_6 + \sqrt{k - K_4' K_5'^{1/2}} \cos K_6' \beta_4$$

$$- K_5' \beta_5 - K_i' \beta_i \qquad i = 1, 2, 3, 7 \tag{99}$$

$$\frac{\partial S}{\partial K_4'} = - L_4' = \frac{+ K_5'^{1/2} \left[+ \sin K_6' \beta_6 - \cos K_6' \beta_4\right]}{2 \sqrt{k - K_4' K_5'^{1/2}}}$$

$$\frac{\partial S}{\partial K_5'} = - L_5' = \frac{+ K_4 \left[\sin K_6' \beta_6 - \cos K_6' \beta_4\right]}{4 K_5'^{1/2} \sqrt{k - K_4' K_5'^{1/2}}} - \beta_5$$

$$\frac{\partial S}{\partial K_6'} = - L_6' = - \sqrt{k - K_4' K_5'^{1/2}} (\cos K_6' \beta_6 + \sin K_6' \beta_4) = - K_6^*$$

$$\frac{\partial S}{\partial \beta_4} = - \alpha_4 = \sqrt{k - K_4' K_5'^{1/2}} \cos K_6'$$

$$\frac{\partial S}{\partial \beta_6} = -\alpha_6 = -\sqrt{k - K_4' K_5'^{1/2}} \sin K_6'$$

$$\frac{\partial S}{\partial \beta_5} = -\alpha_5 = -K_5'$$

The other transformations result in identities. Our transformations are:

$$K_i^* = \alpha_i \qquad\qquad i = 1, 2, 3, 7$$

$$L_i^* = \beta_i$$

$$K_5^* = \alpha_5$$

$$K_4^* = \frac{k - \alpha_4^2 - \alpha_6^2}{\alpha_5^{1/2}} \qquad\qquad \alpha_4 = -\sqrt{k - K_4^* K_5^{*\,1/2}} \cos L_6^*$$

$$K_6^* = -\alpha_4 \beta_6 + \alpha_6 \beta_4 \qquad\qquad \alpha_6 = -\sqrt{k - K_4^* K_5^{*\,1/2}} \sin L_6^*$$

$$L_4^* = -\frac{\alpha_5^{1/2} (\alpha_6 \beta_6 + \alpha_4 \beta_4)}{2(\alpha_4^2 + \alpha_6^2)} \tag{100}$$

$$L_5^* = -\frac{(k - \alpha_4^2 - \alpha_6^2)\ (\alpha_6 \beta_6 + \alpha_4 \beta_4)}{4\alpha_5\ (\alpha_6^2 + \alpha_4^2)} + \beta_5$$

$$L_6^* = \tan^{-1} \frac{\alpha_6}{\alpha_4}$$

$$\beta_4 = \frac{2 L_4^* \sqrt{k - K_4^* K_5^{*\,1/2}} \cos L_6^*}{K_5^{*\,1/2}} - \frac{K_6^* \sin L_6^*}{\sqrt{k - K_4^* K_5^{*\,1/2}}}$$

$$\beta_6 = \frac{+2\, L_4^* \sqrt{k - K_4^* K_5^{*\,1/2}} \sin L_6^*}{K_5^{*\,1/2}} + \frac{K_6^* \cos L_6^*}{\sqrt{k - K_4^* K_5^{*\,1/2}}}$$

$$\beta_5 = L_5^* - \frac{K_4^*}{2\, K_5^*} L_4^*$$

Next we will rewrite our equations in the new terms:

$$\alpha_1 = \frac{k}{\alpha_5^{3/2}} \quad H_o \qquad\qquad \alpha_3 = \rho_3$$

$$\alpha_2 = \frac{k}{\alpha_5^{3/2}} \quad \lambda_7\, \sigma \qquad\qquad \alpha_5 = \frac{2k}{Q_4} - u^2 - \frac{v^2 + Q_3^2 \sec^2 Q_5}{Q_4^2}$$

$$\alpha_7^2 = (\lambda_3\, v - \lambda_2 Q_3 \sec^2 Q_5 + \rho_3 \tan Q_5)^2 + (\rho_2 - \lambda_3\, Q_3 \tan Q_5)^2 + \alpha_3^2$$

$$\sin\beta_7 = \frac{\alpha_7^2\, Q_3 + \rho_3\, (\lambda_2\, Q_3^2 \sec^2 Q_5 \tan Q_5 - \rho_3\, Q_3 \sec^2 Q_5 - \rho_2\, v)}{\sqrt{\alpha_7^2 - \alpha_3^2}\,\sqrt{\alpha_7^2\, (v^2 + Q_3^2 \sec^2 Q_5) - (\lambda_2\, Q_3^2 \sec^2 Q_5 \tan Q_5 - \rho_3\, Q_3 \sec^2 Q_5 - \rho_2\, v)^2}} \tag{101}$$

$$\frac{v^2 + Q_3^2 \sec^2\theta - k\,Q_4}{Q_4\sqrt{k + \alpha_5^{1/2}\sqrt{v^2 + Q_3^2 \sec^2 Q_5}}} = \sqrt{k - \alpha_5^{1/2}\sqrt{v^2 + Q_3^2 \sec^2 Q_5}} \quad \cos f$$

$$\sin \text{Я} = \frac{\sqrt{v^2 + Q_3^2 \sec^2 Q_5}\ \sqrt{\alpha_7^2 - \alpha_3^2} \quad \cos\beta_7}{\alpha_7 \sqrt{v^2 + Q_3^2 \tan^2 Q_5}}$$

$$\cos \text{Я} = \frac{(v^2 + Q_3^2 \sec^2 Q_5)\alpha_3 + Q_3\, (\lambda_2\, Q_3^2 \sec^2 Q_5 \tan Q_5 - \rho_3\, Q_3 \sec^2 Q_5 - \rho_2\, v)}{\sqrt{v^2 + Q_3^2 \tan^2 Q_5}\sqrt{\alpha_7^2\, (v^2 + Q_3^2 \sec^2 Q_5) - (\lambda_2\, Q_3^2 \sec^2 Q_5 \tan Q_5 - \rho_3\, Q_3 \sec^2 Q_5 - \rho_2 v)^2}}$$

$$\begin{bmatrix} \alpha_4 \\ \alpha_6 \end{bmatrix} = \left\{ - \frac{v^2 + Q_3^2 \sec^2 Q_5 - kQ_4}{Q_4} \begin{bmatrix} +\cos \text{Я} & +\sin \text{Я} \\ \sin \text{Я} & -\cos \text{Я} \end{bmatrix} - \sqrt{v^2 + Q_3^2 \sec^2 Q_5}\ \ u \begin{bmatrix} -\sin \text{Я} & \cos \text{Я} \\ \cos \text{Я} & \sin \text{Я} \end{bmatrix} \right\}$$

$$\frac{\begin{bmatrix} v \cos Q_5 \\ \sqrt{v^2 + Q_3^2 \sec^2 Q_5} \sin Q_5 \end{bmatrix}}{\sqrt{k + \alpha_5^{1/2}\sqrt{v^2 + Q_3^2 \sec^2 Q_5}}\ \sqrt{v^2 + Q_3^2 \tan^2 Q_5}}$$

$$\beta_1 = -\frac{\alpha_5^{3/2}}{k} t - \frac{u\, Q_4\, \alpha_5^{1/2}}{k} + \cos^{-1}\left(1 - \frac{u^2 Q_4}{2k - \alpha_6^2 - \alpha_4^2}\right)$$

$$+ \sin^{-1} \frac{(k - \alpha_6^2 - \alpha_4^2)\sin Q_5}{\alpha_5^{1/2}\sqrt{v^2 + Q_3^2 \tan^2 Q_5}} - \sin^{-1} \frac{(k - \alpha_6^2 - \alpha_4^2)\sqrt{\alpha_7^2 - \alpha_3^2} \quad \cos\beta_7}{\alpha_5^{1/2}\, \alpha_7 \sqrt{v^2 + Q_3^2 \tan^2 Q_5}}$$

$$\beta_2 = \frac{\alpha_5^{3/2}}{k} \quad \frac{Q_7}{\sigma} + \beta_1 + \frac{\alpha_5^{3/2}}{k} \; t$$

$$\beta_3 = Q_6 - \sin^{-1} \frac{Q_3 \tan Q_5}{\sqrt{v^2 + Q_3^2 \tan^2 Q_5}} + \sin^{-1} \sqrt{\frac{\frac{(k - \alpha_6^2 - \alpha_4^2)^2}{\alpha_5} \alpha_7^2 - K_6^2 K_4^2}{\alpha_7 \sqrt{v^2 + Q_3^2 \tan^2 Q_5}}} \cos \beta_7$$

where

$$K_6 K_4 = Q_2 Q_3^2 \sec^2 Q_5 \tan Q_5 - \alpha_3 Q_3 \sec^2 Q_5 - \rho_2 v$$

$$= \frac{k - \alpha_4^2 - \alpha_6^2}{\alpha_5^{1/2}} \Big[(\alpha_6 \beta_4 - \alpha_4 \beta_6) - (\alpha_1 + \alpha_2) \Big]$$

$$\begin{bmatrix} \beta_4 \\ \\ \beta_6 \end{bmatrix} = \begin{bmatrix} \alpha_6 & \alpha_4 \\ \\ -\alpha_4 & \alpha_6 \end{bmatrix} \begin{bmatrix} \dfrac{K_6^*}{\alpha_4^2 + \alpha_6^2} \\ \\ \dfrac{-2 L_4^*}{\alpha_5^{1/2}} \end{bmatrix} \tag{102}$$

where K_6^* and two forms of L_4^* are given in Eqs. (97) and (98).

$$\beta_5 = \frac{3\alpha_5^{1/2}}{2k} \left(\alpha_2 \; \frac{Q_7}{\sigma} - \alpha_1 t\right) - \frac{Q_1 u}{2\alpha_5} + \frac{\rho_1 Q_4}{\alpha_5} \tag{103}$$

$$+ \frac{1}{2\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]} \Bigg\{ \rho_2 Q_3^2 \tan Q_5 - \alpha_3 v Q_3 \tan Q_5$$

$$+ Q_2 v \left(\frac{(k - \alpha_6^2 - \alpha_4^2)^2}{\alpha 5} + Q_3^2 \tan^2 Q_5 \right)$$

$$+ \frac{Q_3}{\alpha_5^{1/2}} \sqrt{\Big[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\Big]\Big[\alpha_7^2 - (\alpha_6 \beta_4 - \alpha_4 \beta_6 - \alpha_1 - \alpha_2)^2\Big] - \Big[\alpha_3 (k - \alpha_6^2 - \alpha_4^2) + Q_3 \alpha_5^{1/2} (\alpha_6 \beta_4 - \alpha_4 \beta_6 - \alpha_1 - \alpha_2)\Big]^2} \Bigg\}$$

$$\beta_5 = \frac{3\alpha_5^{1/2}}{2k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) + \frac{1}{2\alpha_5}\left\{ Q_1 \frac{\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_4^2)^2}{\alpha_5 u Q_4^2} + Q_2 \frac{(k - \alpha_6^2 - \alpha_4^2)^2}{\alpha_5 v} \right.$$

$$- \frac{\alpha_3 (k - \alpha_6^2 - \alpha_4^2)^2 Q_3 \tan Q_5}{v\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]} + \frac{(\alpha_6\beta_4 - \alpha_4\beta_6)(k - \alpha_6^2 - \alpha_4^2)}{\alpha_5^{1/2}}\left[\frac{2}{uQ_4} - \frac{Q_3^2 \alpha_5 \tan Q_5}{v\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}\right]$$

$$+ (\alpha_1 + \alpha_2)\left[\frac{2(\alpha_5^2 Q_4^2 - k(k - \alpha_4^2 - \alpha_6^2)}{k u Q_4 \alpha_5^{1/2}} + \frac{(k - \alpha_6^2 - \alpha_4^2)\alpha_5^{1/2} Q_3^2 \tan Q_5}{v\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}\right]$$

$$\frac{+ Q_3 \alpha_5^{1/2} \sqrt{\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]\left[\alpha_7^2 - (\alpha_6\beta_4 - \alpha_4\beta_6 - \alpha_1 - \alpha_2)^2\right] - \left[\alpha_3 (k - \alpha_6^2 - \alpha_4^2) + Q_3 \alpha_5^{1/2} (\beta_4\alpha_6 - \alpha_4\beta_6 - \alpha_1 - \alpha_2)\right]^2}}{\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}$$

One obtains the following time derivatives:

$$\dot{\alpha}_5 = -\frac{2F}{m\Delta(x)}(u\lambda_1 + v\lambda_2 + Q_3\lambda_3)$$

$$\dot{\alpha}_2 = -\frac{3k}{2\alpha_5^{5/2}}\sigma\lambda_7\dot{\alpha}_5 + \frac{k\sigma}{\alpha_5^{3/2}}\frac{F}{m^2}\Delta(x)$$

$$\dot{\alpha}_3 = 0$$

$$\dot{\alpha}_7 = 0$$

$$\dot{\beta}_7 = \frac{F Q_4^2}{\cos\beta_7 \, m\Delta(x)\sqrt{\alpha_7^2 - \alpha_3^2}}\left\{ \frac{\lambda_3\left[\alpha_7^2 \cos^2 Q_5 + \rho_3 (2\lambda_2 Q_3 \tan Q_5 - \rho_3 - v\lambda_3 \sin Q_5 \cos Q_5)\right] - \rho_2 \rho_3 \lambda_2}{\sqrt{\alpha_7^2 (v^2 + Q_3^2 \sec^2 Q_5) - \left[(\lambda_2 Q_3^2 \tan Q_5 - \rho_3 Q_3)\sec^2 Q_5 - \rho_2 v\right]^2}} \right.$$

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$$\left. - \frac{\sin\beta_7 \sqrt{\alpha_7^2 - \alpha_3^2}\left\{\alpha_7^2 (v\lambda_2 + Q_3\lambda_3) - \left[(\lambda_2 Q_3 \tan Q_5 - \rho_3) Q_3 \sec^2 Q_5 - \rho_2 v\right]\left[\lambda_3 (2\lambda_2 Q_3 \tan Q_5 + \rho_3 - v\lambda_3 \sin Q_5 \cos Q_5) - \rho_2 \lambda_2\right]\right\}}{\left\{\alpha_7^2 (v^2 + Q_3^2 \sec^2 Q_5) - \left[(\lambda_2 Q_3 \tan Q_5 - \rho_3) Q_3 \sec^2 Q_5 - \rho_2 v\right]^2\right\}} \right\}$$

$$
\dot{\text{Я}} = \frac{F\, Q_4^2 \sqrt{\alpha_7^2 - \alpha_3^2}\, \cos\beta_7\, v\, Q_3\, (\lambda_3\, v \cos^2 Q_5 - \lambda_2 Q_3)}{m\Delta\,(x)\,\alpha_7 (v^2 + Q_3^2 \tan^2 Q_5)^{3/2} \sqrt{v^2 + Q_3^2 \sec^2 Q_5}\, \cos \text{Я}} - \tan\text{Я} \tan\beta_7 \dot{\beta}_7
$$

$$
\begin{bmatrix} \dot{\alpha}_4 \\ \dot{\alpha}_6 \end{bmatrix} = \frac{1}{\sqrt{k + \alpha_5^{1/2} \sqrt{v^2 + Q_3^2 \sec^2 Q_5}}\ \sqrt{v^2 + Q_3^2 \tan^2 Q_5}} \left\{ \frac{F\, Q_4^2}{m\,\Delta\,(x)} \left(- \frac{v^2 + Q_3^2 \sec^2 Q_5 - kQ_4}{Q_4} \begin{bmatrix} \cos\text{Я} & \sin\text{Я} \\ \sin\text{Я} & -\cos\text{Я} \end{bmatrix} \right.\right.
$$

$$
\left. - \sqrt{v^2 + Q_3^2 \sec^2 Q_5}\; u \begin{bmatrix} -\sin\text{Я} & \cos\text{Я} \\ \cos\text{Я} & \sin\text{Я} \end{bmatrix} \right) \begin{bmatrix} \lambda_2 \cos Q_5 \\ \dfrac{(v\,\lambda_2 + Q_3\,\lambda_3) \sin Q_5}{\sqrt{v^2 + Q_3^2 \sec^2 Q_5}} \end{bmatrix}
$$

$$
+ \left[\left(\frac{-2\, F\, Q_4\, (\lambda_2\, v + \lambda_3\, Q_3)}{m\Delta\,(x)} + \sqrt{v^2 + Q_3^2 \sec^2 Q_5}\; u\, \dot{\text{Я}} \right) \begin{bmatrix} \cos\text{Я} & \sin\text{Я} \\ \sin\text{Я} & -\cos\text{Я} \end{bmatrix} \right.
$$

$$
+ \left\{ - \frac{v^2 + Q_3^2 \sec^2 Q_5 - kQ_4}{Q_4} \dot{\text{Я}} - \frac{F \left[u\, Q_4^2 \,(v\lambda_2 + Q_3\,\lambda_3) + \lambda_1\,(v^2 + Q_3^2 \sec^2 Q_5) \right]}{m\Delta\,(x) \sqrt{v^2 + Q_3^2 \sec^2 Q_5}} \right\}
$$

$$
\left. \left. \begin{bmatrix} -\sin\text{Я} & \cos\text{Я} \\ \cos\text{Я} & \sin\text{Я} \end{bmatrix} \right] \begin{bmatrix} v \cos Q_5 \\ \sqrt{v^2 + Q_3^2 \sec^2 Q_5}\, \sin Q_5 \end{bmatrix} \right\}
$$

$$
- \left(\frac{FQ_4^2\,(v\lambda_2 + Q_3\,\lambda_3 \sin^2 Q_5) \left[\sqrt{v^2 + Q_3^2 \sec^2 Q_5} \left(k + \alpha_5^{1/2} \sqrt{v^2 + Q_3^2 \sec^2 Q_5} \right) 2\alpha_5^{1/2} + (v^2 + Q_3^2 \tan^2 Q_5)\, \alpha_5 \right]}{m\Delta\,(x)\; 2\alpha_5^{1/2} \sqrt{v^2 + Q_3^2 \sec^2 Q_5} \left(k + \alpha_5^{1/2} \sqrt{v^2 + Q_3^2 \sec^2 Q_5} \right) (v^2 + Q_3^2 \tan^2 Q_5)} \right.
$$

$$
\left. + \frac{\dot{\alpha}_5 \sqrt{v^2 + Q_3^2 \sec^2 Q_5}}{4\, \alpha_5^{1/2} \left(k + \alpha_5^{1/2} \sqrt{v^2 + Q_3^2 \sec^2 Q_5} \right)} \right) \begin{bmatrix} \alpha_4 \\ \alpha_6 \end{bmatrix}
$$

Define

$$
\sin i = \frac{\alpha_5^{1/2} \sqrt{v^2 + Q_3^2 \tan^2 Q_5}}{k - \alpha_6^2 - \alpha_4^2}
$$

$$\dot{\beta}_1 = \frac{2\,Q_4}{(\alpha_5 Q_4 + k - \alpha_6^2 - \alpha_4^2)}\left\{\frac{F\lambda_1 \alpha_5^{1/2}}{m\Delta(x)} + \frac{(\alpha_6\dot{\alpha}_6 + \alpha_4\dot{\alpha}_4)\,u}{(2k - \alpha_6^2 - \alpha_4^2)}\right\} + \frac{FQ_4\left[(u^2 - \alpha_5)\lambda_1 + (v\lambda_2 + Q_3\lambda_3)\,u\right]}{m\Delta(\chi)\,\alpha_5^{1/2}\,k}$$

$$+ \frac{FQ_4^2(-\cos i\,\lambda_2 + \sqrt{\sin^2 i - \sin^2 Q_5}\,\lambda_3 \cos Q_5)\tan Q_5 \cos i}{m\Delta(x)\sin i\sqrt{v^2 + Q_3^2\tan^2 Q_5}} - \dot{\alpha}_5 \frac{3\alpha_5^{1/2} t}{2k}$$

$$+ \frac{\sqrt{\alpha_7^2 - \alpha_3^2}\left\{2(\alpha_6\dot{\alpha}_6 + \alpha_4\dot{\alpha}_4)\cos\beta_7 + (k - \alpha_6^2 - \alpha_4^2)\left[\sin\beta_7\dot{\beta}_7 + \cos\beta_7\left(\frac{\dot{\alpha}_5}{2\alpha_5} + \frac{FQ_4^2(v\lambda_2 + Q_3\lambda_3\sin^2 Q_5)}{m\Delta(\chi)\,(v^2 + Q_3^2\tan^2 Q_5)}\right)\right]\right\}}{\sqrt{(k - \alpha_6^2 - \alpha_4^2)^2\,(\alpha_7^2\sin^2\beta_7 + \alpha_3^2\cos^2\beta_7) - \alpha_5\alpha_7^2 Q_3^2}}$$

$$\dot{\beta}_2 = \frac{3\alpha_5^{1/2}\dot{\alpha}_5}{2k}\left(\frac{Q_7}{\sigma} + t\right) + \dot{\beta}_1$$

$$\dot{\beta}_3 = 0$$

$$\dot{K}_6^* = \dot{\alpha}_1 + \dot{\alpha}_2 - \frac{(K_6^* - \alpha_1 - \alpha_2)FQ_4^2\,(v\,Q_2 + Q_3\lambda_3)}{m\Delta(X)\,(v^2 + Q_3^2\sec^2 Q_5)}$$

$$+ \frac{FQ_4^2\left[(2Q_2\,Q_3\tan Q_5 - \alpha_3 - \lambda_3 v\sin Q_5\cos Q_5)\,\lambda_3 - \rho_2 Q_2\right]}{m\Delta(X)\sqrt{v^2 + Q_3^2\sec^2 Q_5}}$$

$$\dot{\alpha}_1 = -\frac{3\alpha_1}{2\alpha_5}\dot{\alpha}_5 - \frac{k}{\alpha_5^{3/2}}\left\{\frac{F}{m\Delta(\chi)}\left[-\lambda_1\left(\frac{2}{Q_4}(v\,Q_2 + Q_3\lambda_3) + \rho_1\right)\right.\right.$$

$$\left. - \lambda_2(-uQ_4\lambda_2 - 2Q_3\lambda_3\tan Q_5 + \rho_2) + \lambda_3(-\lambda_3\cos Q_5\,(v\sin Q_5 - u\,Q_4\cos Q_5) - \rho_3)\right]$$

$$\left. + \frac{F\sigma}{m^2}\,\Delta(\chi)\right\}$$

The time derivatives of $\dot{L}_4^*$ and $\dot{\beta}_5$ may be obtained in a similar manner. They will not be presented here because of their length.

The following equations give information to define $\Delta(x)$:

$$Q_1 = 2u\left\{\beta_5 - \frac{(k-\alpha_4^2-\alpha_6^2)\,(\alpha_6\beta_6+\alpha_4\beta_4)}{4\alpha_5\,(\alpha_6^2+\alpha_4^2)}\right\} - \frac{3\alpha_5^{1/2}u}{k}\left(\alpha_2\frac{Q_7}{\sigma}-\alpha_1 t\right)$$

$$+\,(\alpha_1+\alpha_2)\left[\frac{u^2 Q_4}{k\alpha_5^{1/2}} - \frac{\alpha_5^{1/2}Q_4}{k} + \frac{\frac{(k-\alpha_6^2-\alpha_4^2)^2}{\alpha_5} - 2kQ_4 - (k-\alpha_6^2-\alpha_4^2)\,Q_4}{\alpha_5^{1/2}Q_4\,(2k-\alpha_6^2-\alpha_4^2)}\right] \qquad (105)$$

$$+\,\frac{K_6^* K_4^*\left(\frac{(k-\alpha_6^2-\alpha_4^2)^2}{\alpha_5} - kQ_4\right)}{Q_4\,(\alpha_6^2+\alpha_4^2)\,(2k-\alpha_6^2-\alpha_4^2)}$$

where

$$\frac{K_4^*\,K_6^*}{(\alpha_6^2+\alpha_4^2)}$$

is well defined except at $\alpha_6 = \alpha_4 = 0$.

There:

$$\frac{K_4^* K_6^*}{(\alpha_6^2+\alpha_4^2)} = -\,\frac{\alpha_1+\alpha_2}{\alpha_5^{1/2}}\ . \qquad (106)$$

$$Q_2 = v\left\{\frac{2\beta_5}{Q_4^2} - \frac{L_4^*\left[\alpha_5^2 Q_4^2 - (k-\alpha_6^2-\alpha_4^2)^2\right]}{Q_4^2\alpha_5^{3/2}\,(k-\alpha_6^2-\alpha_4^2)}\right\}$$

$$+\,\frac{(\alpha_1+\alpha_2)\,v\,u\left[(k-\alpha_6^2-\alpha_4^2)^2+\alpha_5 k Q_4\right]}{\alpha_5^{1/2}Q_4 k\,(k-\alpha_6^2-\alpha_4^2)\,(2k-\alpha_6^2-\alpha_4^2)} - \frac{3\alpha_5^{1/2}v}{Q_4^2 k}\left(\alpha_2\frac{Q_7}{\sigma}-\alpha_1 t\right) \qquad (107)$$

$$-\,\frac{K_6^* K_4^* v u\left[(k-\alpha_6^2-\alpha_4^2)^2+\alpha_5 k Q_4\right]}{(k-\alpha_2^2-\alpha_4^2)^2\,Q_4\,(\alpha_6^2+\alpha_4^2)\,(2k-\alpha_6^2-\alpha_4^2)} - \frac{\sqrt{N}\;v\,\alpha_5^{3/2}\,Q_3}{(k-\alpha_6^2-\alpha_4^2)^2\left[(k-\alpha_6^2-\alpha_4^2)^2-\alpha_5 Q_3^2\right]}$$

$$+\,\frac{Q_3\tan Q_5\,\alpha_5^{3/2}}{(k-\alpha_6^2-\alpha_4^2)\left[(k-\alpha_6^2-\alpha_4^2)^2-\alpha_5 Q_3^2\right]}\left\{\frac{\alpha_3\,(k-\alpha_6^2-\alpha_4^2)}{\alpha_5^{1/2}} + Q_3\,(K_6^*-\alpha_1-\alpha_2)\right\}$$

where

$$N = \left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]\left[\alpha_7^2 - (K_6^* - \alpha_1 - \alpha_2)^2\right] - \left[\alpha_3 (k - \alpha_6^2 - \alpha_4^2) + Q_3 \alpha_5^{1/2} (K_6^* - \alpha_1 - \alpha_2)\right]^2$$

$$\rho_2 = Q_3^2 \sec^2 Q_5 \tan Q_5 \left\{ \frac{2\beta_5}{Q_4^2} - \frac{L_4^*\left[\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_4^2)^2\right]}{Q_4^2 \alpha_5^{3/2} (k - \alpha_6^2 - \alpha_4^2)} \right.$$

$$- \frac{3\alpha_5^{1/2}}{Q_4^2 k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) + \frac{(\alpha_1 + \alpha_2)\, u \left[(k - \alpha_6^2 - \alpha_4^2)^2 + \alpha_5 k Q_4\right]}{\alpha_5^{1/2} (k - \alpha_6^2 - \alpha_4^2)\, Q_4 k\, (2k - \alpha_6^2 - \alpha_4^2)}$$

$$\left. - \frac{K_4^* K_6^* u \left[(k - \alpha_6^2 - \alpha_4^2)^2 + \alpha_5 k Q_4\right]}{(k - \alpha_6^2 - \alpha_4^2)\, Q_4 (\alpha_6^2 + \alpha_4^2)(2k - \alpha_6^2 - \alpha_4^2)} - \frac{\sqrt{N}\, \alpha_5^{3/2} Q_3}{(k - \alpha_5^2 - \alpha_4^2)^2 \left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]} \right\}$$

$$- \frac{\alpha_3 Q_3 \sec^2 Q_5\, v\, \alpha_5}{\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]} - \frac{(K_6^* - \alpha_1 - \alpha_2)\left[(k - \alpha_6^2 - \alpha_4^2)^2 + \alpha_5 Q_3^2 \tan^2 Q_5\right] v\, \alpha_5^{1/2}}{(k - \alpha_6^2 - \alpha_4^2)\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}$$

$$\rho_1 = - \frac{(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 k Q_4}{\alpha_5 Q_4^3} \left\{ 2\beta_5 + \frac{(k - \alpha_6^2 - \alpha_4^2)}{\alpha_5^{3/2}} L_4^* - \frac{3\alpha_5^{1/2}}{k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) \right\}$$

$$- (\alpha_1 + \alpha_2)\, u \left[\frac{\alpha_5^{1/2}}{k} + \frac{(k - \alpha_4^2 - \alpha_6^2)^3}{k \alpha_5^{3/2} Q_4^2 (2k - \alpha_4^2 - \alpha_6^2)} \right] + \frac{K_6^* K_4^* u (k - \alpha_6^2 - \alpha_4^2)^2}{\alpha_5 Q_4^2 (\alpha_6^2 + \alpha_4^2)(2k - \alpha_6^2 - \alpha_4^2)}$$

$$\lambda_3 = \left\{ \frac{2\beta_5}{Q_4^2} - \frac{L_4^*\left[\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_4^2)^2\right]}{Q_4^2 \alpha_5^{3/2} (k - \alpha_6^2 - \alpha_4^2)} - \frac{3\alpha_5^{1/2}}{Q_4^2 k}\left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right) \right.$$

$$+ \frac{(\alpha_1 + \alpha_2)\, u \left[(k - \alpha_6^2 - \alpha_4^2)^2 + \alpha_5 k Q_4\right]}{\alpha_5^{1/2} Q_4 k (k - \alpha_6^2 - \alpha_4^2)(2k - \alpha_6^2 - \alpha_4^2)} - \frac{\sqrt{N}\, \alpha_5^{3/2} Q_3}{(k - \alpha_6^2 - \alpha_4^2)^2 \left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}$$

$$\left. - \frac{K_6^* K_4^*\, u \left[(k - \alpha_6^2 - \alpha_4^2)^2 + \alpha_5 k Q_4\right]}{Q_4 (\alpha_6^2 + \alpha_4^2)(k - \alpha_6^2 - \alpha_4^2)^2 (2k - \alpha_6^2 - \alpha_4^2)} \right\} Q_3 \sec^2 Q_5 + \frac{\sqrt{N}\, \alpha_5^{1/2}}{\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}$$

$$- \frac{\tan Q_5 \left[\alpha_3 (k - \alpha_6^2 - \alpha_4^2) + \alpha_5^{1/2} Q_3 (K_6^* - \alpha_1 - \alpha_2)\right] v\alpha_5}{(k - \alpha_6^2 - \alpha_4^2)\left[(k - \alpha_6^2 - \alpha_4^2)^2 - \alpha_5 Q_3^2\right]}$$

If we proceed as before we write:

$$\Delta^2(X) = Q_1^2 + Q_4^2 \left\{ \left[\frac{2(k - \alpha_6^2 - \alpha_4^2)\beta_5}{\alpha_5^{1/2} Q_4^2} - \frac{L_4^* \left[\alpha_5^2 Q_4^2 - (k - \alpha_6^2 - \alpha_4^2)^2\right]}{Q_4^2 \alpha_5^2} \right. \right.$$

$$+ \frac{u\left[(k - \alpha_6^2 - \alpha_4^2)^2 + \alpha_5 k Q_4\right]\left[(\alpha_1 + \alpha_2)(k - \alpha_6^2 - \alpha_4^2) - k\alpha_5^{1/2} \frac{K_4^* K_6^*}{(\alpha_6^2 + \alpha_4^2)}\right]}{\alpha_5 Q_4 k (2k - \alpha_6^2 - \alpha_4^2)(k - \alpha_6^2 - \alpha_4^2)} \tag{108}$$

$$\left. - \frac{3}{k Q_4^2} \left(\alpha_2 \frac{Q_7}{\sigma} - \alpha_1 t\right)(k - \alpha_6^2 - \alpha_4^2) \right]^2$$

$$\left. + \left[\sin\left(f - \tan^{-1} \frac{\alpha_6}{\alpha_4}\right) \alpha_5^{1/2} \frac{\sqrt{\alpha_7^2 - (K_6^* - \alpha_1 - \alpha_2)^2}}{k - \alpha_6^2 - \alpha_4^2} \right]^2 \right\}$$

It can be seen, from the equations for β_1, β_3, and Q_4 that $\Delta(x)$ does not contain β_3, β_7, or α_3.

Again we write:

$$H = \frac{\alpha_5^{3/2}}{k} \alpha_1 + \frac{F\Delta(x)}{\sigma\left[\frac{k}{\alpha_5^{3/2}}(\beta_2 - \beta_1) - t\right]}$$

and we obtain

$$\dot{\alpha}_3 = 0$$

$$\dot{\alpha}_7 = 0$$

$$\dot{\beta}_3 = 0 \; .$$

APPENDIX

The following tables of equations were found useful in developing the equations presented in the text and in developing computer programs for numerically evaluating the ordinary differential equations.

Only equations for the initial constants (K, L) of section II(K-D-P-variables) and some tables for the Poincaré variables are presented. Tables for the other variables may be deduced from these tables or derived in a manner indicated by the tables.

In short, these tables are presented only to aid the interested reader in understanding the results and extending them for his special purposes.

TABLE FOR K-D-P VARIABLES

$$Z = -Q_2 \frac{K_4 \sin i \cos (K_7 - f)}{\sqrt{1-\sin^2 i \sin^2 (K_7-f)}} + Q_5 \sin^{-1} \left[\sin i \sin (K_7-f)\right]$$

$$+ Q_6 \sin^{-1} \frac{\cos i \sin (K_7-f)}{\sqrt{1-\sin^2 i \sin^2(K_7-f)}}$$

$$\sin i = \frac{\sqrt{K_4^2-K_3^2}}{K_4}$$

$$\frac{\partial i}{\partial K_3} = -\frac{1}{K_4 \sin i}, \quad \frac{\partial i}{\partial K_4} = \frac{\cos i}{K_4 \sin i}, \quad \frac{\partial^2 i}{K_3 \partial K_4} = \frac{1}{K_4^2 \sin^3 i}$$

$$\frac{\partial^2 i}{\partial K_3^2} = -\frac{\cos i}{K_4^2 \sin^3 i} \qquad \frac{\partial^2 i}{\partial K_4^2} = -\frac{\cos i\,(2-\cos^2 i)}{K_4^2 \sin^3 i}$$

$$\cos f = \frac{K_4^2 - kQ_4}{Q_4\sqrt{k^2-K_5K_4^2}}$$

$$\frac{\partial f}{\partial K_4} = -\frac{2k^2-K_5K_4^2-K_5kQ_4}{uQ_4\,(k^2-K_5K_4^2)}, \quad \frac{\partial f}{\partial K_5} = \frac{-K_4\,(K_4^2-kQ_4)}{2uQ_4(k^2-K_5K_4^2)}$$

$$\frac{\partial f}{\partial Q_4} = \frac{K_4}{uQ_4^2}, \quad \frac{\partial^2 f}{\partial K_4^2} = \frac{-2K_4K_5k(k-K_5Q_4)}{uQ_4(k^2-K_5K_4^2)^2} - \frac{K_4(2k^2-K_5K_4^2-K_5kQ_4)}{u^3Q_4^3(k^2-K_5K_4^2)}$$

$$\frac{\partial^2 f}{\partial K_5^2} = \frac{-K_4^3(K_4^2-kQ_4)}{2u\,Q_4(k^2-K_4^2K_5)^2} - \frac{K_4(K_4^2-kQ_4)}{4u^3\,Q_4(k^2-K_5K_4^2)}$$

$$\frac{\partial^2 f}{\partial K_4 \partial K_5} = -\frac{k^2(K_4^2-kQ_4)}{uQ_4(k^2-K_4^2K_5)^2} - \frac{2k^2-K_5K_4^2-K_5kQ_4}{2u^3Q_4(k^2-K_5K_4^2)}$$

$$\frac{\partial Z}{\partial i} = - Q_2 \frac{K_4 \cos i \cos (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} + Q_5 \frac{\cos i \sin (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$- Q_6 \frac{\sin i \sin (K_7 - f) \cos (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]}$$

$$\frac{\partial^2 Z}{\partial i^2} = Q_2 \frac{K_4 \sin i \cos (K_7 - f) \left[\cos^2 (K_7 - f) - 2 \cos^2 i \sin^2 (K_7 - f)\right]}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{5/2}}$$

$$- Q_5 \frac{\sin (K_7 - f) \cos^2 (K_7 - f) \sin i}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} - Q_6 \frac{\sin (K_7 - f) \cos (K_7 - f) \cos i \left[1 + \sin^2 i \sin^2 (K_7 - f)\right]}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^2}$$

$$\frac{\partial^2 Z}{\partial i \partial (K_7 - f)} = + Q_2 \frac{K_4 \cos i \sin (K_7 - f) \left[\cos^2 i - 2 \sin^2 i \cos^2 (K_7 - f)\right]}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{5/2}}$$

$$+ Q_5 \frac{\cos i \cos (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} - Q_6 \frac{\sin i \left[\cos^2 (K_7 - f) - \sin^2 (K_7 - f) \cos^2 i\right]}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^2}$$

$$\frac{\partial^2 Z}{\partial i \partial K_4} = - Q_2 \frac{\cos i \cos (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}}$$

$$\frac{\partial Z}{\partial (K_7 - f)} = + Q_2 \frac{K_4 \sin i \cos^2 i \sin (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} + Q_5 \frac{\sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$+ Q_6 \frac{\cos i}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]}$$

$$\frac{\partial^2 Z}{\partial (K_7 - f)^2} = Q_2 \frac{K_4 \sin i \cos^2 i \cos (K_7 - f) \left[1 + 2 \sin^2 i \sin^2 (K_7 - f)\right]}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{5/2}}$$

$$- Q_5 \frac{\sin i \cos^2 i \sin (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}} + Q_6 \frac{2 \sin^2 i \cos i \sin (K_7 - f) \cos (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^2}$$

$$\frac{\partial^2 Z}{\partial (K_7 - f)\, \partial K_4} = Q_2 \frac{\sin i \cos^2 i \sin (K_7 - f)}{\left[1 - \sin^2 i \sin^2 (K_7 - f)\right]^{3/2}}$$

$$\frac{\partial Z}{\partial K_4} = -Q_2 \frac{\sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

We note:

$$\frac{\partial Z}{\partial q_j} = \frac{\partial Z}{\partial i} \frac{\partial i}{\partial q_j} + \frac{\partial Z}{\partial K_4} \frac{\partial K_4}{\partial q_j} + \frac{\partial Z}{\partial (K_7 - f)} \frac{\partial (K_7 - f)}{\partial q_j}$$

$$\frac{\partial^2 Z}{\partial q_j \partial q_k} = \frac{\partial^2 Z}{\partial i^2} \frac{\partial i}{\partial q_j} \frac{\partial i}{\partial q_k} + \frac{\partial^2 Z}{\partial i \partial K_4} \left(\frac{\partial i}{\partial q_j} \frac{\partial K_4}{\partial q_k} + \frac{\partial i}{\partial q_k} \frac{\partial K_4}{\partial q_j} \right) + \frac{\partial Z}{\partial i} \frac{\partial^2 i}{\partial q_j \partial q_k}$$

$$+ \frac{\partial^2 Z}{\partial i\, \partial (K_7 - f)} \left(\frac{\partial i}{\partial q_j} \frac{\partial (K_7 - f)}{\partial q_k} + \frac{\partial i}{\partial q_k} \frac{\partial (K_7 - f)}{\partial q_j} \right) + \frac{\partial^2 Z}{\partial K_4 \partial (K_7 - f)} \left(\frac{\partial (K_7 - f)}{\partial q_j} \frac{\partial K_4}{\partial q_k} + \frac{\partial (K_7 - f)}{\partial q_k} \frac{\partial K_4}{\partial q_j} \right)$$

$$+ \frac{\partial^2 Z}{\partial (K_7 - f)^2} \frac{\partial (K_7 - f)}{\partial q_j} \frac{\partial (K_7 - f)}{\partial q_k} + \frac{\partial Z}{\partial (K_7 - f)} \frac{\partial^2 (K_7 - f)}{\partial q_j \partial q_k}$$

where q_j and q_k are Q_4, K_3, K_4, K_5 and K_7.

$$Z(3) = \cos^{-1} \frac{k - K_5 Q_4}{\sqrt{k^2 - K_4^2 K_5}}$$

$$\frac{\partial Z(3)}{\partial K_4} = -\frac{K_4(k - K_5 Q_4) K_5^{1/2}}{u Q_4 (k^2 - K_4^2 K_5)}$$

$$\frac{\partial Z(3)}{\partial K_5} = -\frac{K_4^2 K_5 Q_4 + k K_4^2 - 2k^2 Q_4}{2K_5^{1/2} u Q_4 (k^2 - K_4^2 K_5)}$$

$$\frac{\partial Z(3)}{\partial Q_4} = \frac{K_5^{1/2}}{u Q_4}$$

$$\frac{\partial^2 Z(3)}{\partial^2 K_4} = \frac{-K_5^{1/2}(k - K_5 Q_4)\left[u^2 Q_4^2 (k^2 + K_5 K_4^2) + K_4^2 (k^2 - K_4^2 K_5)\right]}{u^3 Q_4^3 (k^2 - K_4^2 K_5)^2}$$

$$\frac{\partial^2 Z(3)}{\partial K_5 \partial K_4} = -\frac{k^2 K_4 (k - K_5 Q_4)}{K_5^{1/2} u Q_4 (k^2 - K_4^2 K_5)^2} - \frac{K_4 (K_5 K_4^2 Q_4 + k K_4^2 - 2k^2 Q_4)}{2u^3 Q_4^3 K_5^{1/2} (k^2 - K_4^2 K_5)}$$

$$\frac{\partial^2 Z(3)}{\partial Q_4 \partial K_4} = \frac{K_4 K_5^{1/2}}{u^3 Q_4^3}$$

$$\frac{\partial^2 Z(3)}{\partial^2 K_5} = \frac{-(K_4^2 - k Q_4)k + Q_4(k^2 - K_5 K_4^2)}{4K_5^{1/2} u^3 Q_4 (k^2 - K_5 K_4^2)} + \frac{k(K_4^2 - k Q_4)(k^2 - 3K_4^2 K_5)}{4K_5^{3/2} u Q_4 (k^2 - K_5 K_4^2)^2} - \frac{1}{4K_5^{3/2} u}$$

$$\frac{\partial^2 Z(3)}{\partial K_5 \partial Q_4} = \frac{2k Q_4 - K_4^2}{2K_5^{1/2} u^3 Q_4^3}$$

TABLE OF POINCARÉ VARIABLES

$$\sin i = \frac{\alpha_5^{1/2}\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}{k - \alpha_6^2 - \alpha_7^2}$$

$$\frac{\partial i}{\partial \alpha_3} = \frac{2\alpha_3}{\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2\,\frac{k - \alpha_2^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}$$

$$\frac{\partial i}{\partial \alpha_4} = \frac{2\alpha_4}{\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}$$

$$\frac{\partial i}{\partial \alpha_6} = \frac{2\alpha_6\,(\alpha_3^2 + \alpha_4^2)}{(k - \alpha_6^2 - \alpha_7^2)\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}$$

$$\frac{\partial i}{\partial \alpha_7} = \frac{2\,\alpha_7(\alpha_3^2 + \alpha_4^2)}{(k - \alpha_6^2 - \alpha_7^2)\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}$$

$$\frac{\partial i}{\partial \alpha_5} = \frac{(\alpha_3^2 + \alpha_4^2)}{2\alpha_5\sqrt{(\alpha_3^2 + \alpha_4^2)\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)}}$$

$$\frac{\partial K_7}{\partial \alpha_3} = \frac{\alpha_4}{\alpha_3^2 + \alpha_4^2}, \quad \frac{\partial K_7}{\partial \alpha_4} = \frac{-\alpha_3}{\alpha_3^2 + \alpha_4^2}, \quad \frac{\partial K_7}{\partial \alpha_6} = \frac{-\alpha_7}{\alpha_6^2 + \alpha_7^2}$$

$$\frac{\partial K_7}{\partial \alpha_7} = \frac{\alpha_6}{\alpha_6^2 + \alpha_7^2}, \quad \frac{\partial^2 K_7}{\partial \alpha_3^2} = \frac{-2\alpha_3\alpha_4}{(\alpha_3^2 + \alpha_4^2)^2}, \quad \frac{\partial^2 K_7}{\partial \alpha_3 \partial \alpha_4} = \frac{\alpha_3^2 - \alpha_4^2}{(\alpha_3^2 + \alpha_4^2)^2}$$

$$\frac{\partial^2 K_7}{\partial \alpha_4^2} = \frac{2\alpha_3\alpha_4}{(\alpha_3^2 + \alpha_4^2)^2}, \quad \frac{\partial^2 K_7}{\partial \alpha_6^2} = \frac{2\alpha_6\alpha_7}{(\alpha_6^2 + \alpha_7^2)^2}, \quad \frac{\partial^2 K_7}{\partial \alpha_6 \, \partial \alpha_7} = \frac{\alpha_7^2 - \alpha_6^2}{(\alpha_6^2 + \alpha_7^2)^2}$$

$$\frac{\partial^2 K_7}{\partial \alpha_7^2} = \frac{-2\alpha_6\alpha_7}{(\alpha_6^2 + \alpha_7^2)^2}$$

$$\cos f = \frac{(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5 k Q_4}{\alpha_5 Q_4 \sqrt{(\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}}$$

$$\frac{\partial f}{\partial \alpha_6} = \frac{+2\alpha_6 \left[k(k - \alpha_5 Q_4) + (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2) \right]}{\alpha_5^{1/2} u Q_4 (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}$$

$$\frac{\partial f}{\partial \alpha_7} = \frac{+2\alpha_7 \left[k(k - \alpha_5 Q_4) + (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2) \right]}{\alpha_5^{1/2} u Q_4 (\alpha_6^2 + \alpha_7^2)(2k - \alpha_6^2 - \alpha_7^2)}$$

$$\frac{\partial f}{\partial \alpha_5} = \frac{+(k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{3/2} u Q_4}$$

$$\frac{\partial^2 f}{\partial \alpha_5^2} = \frac{-3(k - \alpha_6^2 - \alpha_7^2)}{2 u Q_4 \alpha_5^{5/2}} - \frac{(k - \alpha_6^2 - \alpha_7^2)\left[(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5^2 Q_4^2 \right]}{2\alpha_5^{7/2} u^3 Q_4^3}$$

$$\frac{\partial^2 f}{\partial \alpha_6 \, \partial \alpha_5} = \frac{-2\alpha_6}{\alpha_5^{3/2} u Q_4} - \frac{2\alpha_6 (k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5^{5/2} u^3 Q_4^3}$$

$$\frac{\partial^2 f}{\partial \; _7 \partial \alpha_5} = \frac{-2\alpha_7}{\alpha_5^{3/2} u Q_4} - \frac{2\alpha_7 (k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5^{5/2} u^3 Q_4^3}$$

$$\frac{\partial^2 f}{\partial \alpha_6^2} = \frac{+2k(k - \alpha_5 Q_4)}{\alpha_5^{1/2}} \left[\frac{2k(\alpha_7^2 - \alpha_6^2) + (\alpha_6^2 + \alpha_7^2)\ (3\alpha_6^2 - \alpha_7^2)}{uQ_4(\alpha_6^2 + \alpha_7^2)^2 (2k - \alpha_6^2 - \alpha_7^2)^2} - \frac{2\alpha_6^2 (k - \alpha_6^2 - \alpha_7^2)}{u^3 Q_4^3\ (\alpha_6^2 + \alpha_7^2)\ \alpha_5\ (2k - \alpha_6^2 - \alpha_7^2)} \right] + \frac{2}{\alpha_5^{1/2}\ uQ_4} - \frac{4\alpha_6^2 (k - \alpha_6^2 - \alpha_7^2)}{\alpha_5^{3/2} u^3 Q_4^3}$$

$$\sin (K_7 - f) = \frac{-\sin (\phi - K_6)}{\sqrt{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}}$$

$$\cos (K_7 - f) = \frac{\cos i \cos (\phi - K_6)}{\sqrt{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}}$$

$$1 - \sin^2 i \sin^2 (K_7 - f) = \frac{\cos^2 i}{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}$$

$$\frac{\partial Z}{\partial i} = -Q_2 \frac{(k - \alpha_6^2 - \alpha_7^2) \cos (\phi - K_6) \left[\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)\right]}{\alpha_5^{1/2} \cos i}$$

$$-Q_5 \sin (\phi - K_6) + Q_6 \tan i \cos (\phi - K_6) \sin (\phi - K_6)$$

$$\frac{\partial Z}{\partial (K_7 - f)} = -Q_2 \frac{(k - \alpha_6^2 - \alpha_7^2) \sin i \sin (\phi - K_6) \left[\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)\right]}{\alpha_5^{1/2} \cos i}$$

$$+Q_5 \sin i \cos (\phi - K_6) + Q_6 \frac{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}{\cos i}$$

$$\frac{\partial Z}{\partial K_4} = -Q_2 \sin i \cos (\phi - K_6)$$

$$\frac{\partial^2 Z}{\partial (K_7 - f)^2} = \frac{\left[\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)\right]}{\cos^2 i} \left[Q_2 K_4 \cos (\phi - K_6) \left[\cos^2 i + 3 \sin^2 i \sin^2 (\phi - K_6)\right]\right.$$

$$\left. + Q_5 \cos i \sin (\phi - K_6) - 2 Q_6 \sin i \cos (\phi - K_6) \sin i (\phi - K_6)\right] \sin i$$

$$\frac{\partial^2 Z}{\partial (K_7 - f) \partial K_4} = - Q_2 \frac{\sin i \sin (\phi - K_6) (\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}{\cos i}$$

$$\frac{\partial^2 Z}{\partial (K_7 - f) \partial i} = \frac{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}{\cos^2 i} \left\{Q_2 K_4 \sin (\phi - K_6) \left[3 \sin^2 i \cos^2 (\phi - K_6) - 1\right]\right.$$

$$\left. + Q_5 \cos i \cos (\phi - K_6) - Q_6 \sin i \left[1 - 2 \sin^2 (\phi - K_6)\right]\right\}$$

$$\frac{\partial^2 Z}{\partial i^2} = Q_2 \frac{(k - \alpha_6^2 - \alpha_7^2) \sin i \cos (\phi - K_6) \left[\cos^2 (\phi - K_6) - 2 \sin^2 (\phi - K_6)\right] (\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}{\alpha_5^{1/2} \cos^2 i}$$

$$+ Q_5 \frac{\sin (\phi - K_6) \cos^2 (\phi - K_6) \sin i}{\cos i} + Q_6 \frac{\sin (\phi - K_6) \cos (\phi - K_6) \left[\cos^2 i + 2 \sin^2 i \sin^2 (\phi - K_6)\right]}{\cos^2 i}$$

TABLE OF NOTATIONS

Mixed Variables

$$\sin \theta = -\sin (K_7 - f) \sin i$$

$$\cos \theta = \sqrt{1 - \sin^2 (K_7 - f) \sin^2 i}$$

$$\sin \phi = \frac{\sin K_6 \cos (K_7 - f) - \sin (K_7 - f) \cos i \cos K_6}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$\cos \phi = \frac{\sin K_6 \cos i \sin (K_7 - f) + \cos K_6 \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$w = K_3$$

$$v = \frac{K_4 \sin i \cos (K_7 - f)}{\sqrt{1 - \sin^2 i \sin^2 (K_7 - f)}}$$

$$\sin \theta = \frac{+ \sin (\phi - K_6) \sin i}{\sqrt{\cos^2 i + \sin^2 (\phi - K_6) \sin^2 i}}$$

$$\cos \theta = \frac{\cos i}{\sqrt{\cos^2 i + \sin^2 (\phi - K_6) \sin^2 i}}$$

$$\tan \theta = \tan i \sin (\phi - K_6)$$

$$Q_4 = r = \frac{K_4^2 \sqrt{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}}{k \sqrt{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)} + \sqrt{k^2 - K_4^2 K_5} \left\{ \cos i \cos K_7 \cos (\phi - K_6) - \sin K_7 \sin (\phi - K_6) \right\}}$$

$$v = K_4 \sin i \cos (\phi - K_6)$$

$$u = \frac{\sqrt{k^2 - K_5 K_4^2}}{K_4} \quad \frac{\sin K_7 \cos i \cos (\phi - K_6) + \cos K_7 \sin (\phi - K_6)}{\sqrt{\cos^2 i + \sin^2 i \sin^2 (\phi - K_6)}}$$

Poincaré Variables

$$\sin i = \sqrt{\frac{\alpha_5 (\alpha_3^2 + \alpha_4^2) (2 \frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2)}{k - \alpha_6^2 - \alpha_7^2}}$$

$$\cos i = 1 - \frac{(\alpha_3^2 + \alpha_4^2) \alpha_5^{1/2}}{k - \alpha_6^2 - \alpha_7^2}$$

$$\sin K_7 = \frac{\alpha_3 \alpha_7 - \alpha_4 \alpha_6}{\sqrt{(\alpha_3^2 + \alpha_4^2)(\alpha_6^2 + \alpha_7^2)}}$$

$$\cos K_7 = \frac{\alpha_4 \alpha_7 + \alpha_3 \alpha_6}{\sqrt{(\alpha_3^2 + \alpha_4^2)(\alpha_6^2 + \alpha_7^2)}}$$

$$\sin(\phi - K_6) = \frac{\alpha_4 \sin\phi - \alpha_3 \cos\phi}{\sqrt{\alpha_3^2 + \alpha_4^2}}$$

$$\cos(\phi - K_6) = \frac{\alpha_4 \cos\phi + \alpha_3 \sin\phi}{\sqrt{\alpha_3^2 + \alpha_4^2}}$$

$$v = \sqrt{2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2} \quad (\alpha_4 \cos\phi + \alpha_3 \sin\phi)$$

$$\tan\theta = \frac{\sqrt{2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}}{\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2} \quad (\alpha_4 \sin\phi - \alpha_3 \cos\phi)$$

$$\cos\theta = \frac{\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2}{\sqrt{\frac{(k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5} - \left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_3 \sin\phi + \alpha_4 \cos\phi)^2}}$$

$$\sin\theta = \frac{\sqrt{\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)} \quad (\alpha_4 \sin\phi - \alpha_3 \cos\phi)}{\sqrt{\frac{(k - \alpha_6^2 - \alpha_7^2)^2}{\alpha_5} - \left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_3 \sin\phi + \alpha_4 \cos\phi)^2}}$$

$$\sin(K_7 - f) = \frac{-(\alpha_4 \sin\phi - \alpha_3 \cos\phi)\,(k - \alpha_6^2 - \alpha_7^2)}{\sqrt{(\alpha_3^2 + \alpha_4^2)\left[(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)^2\right]}}$$

$$\cos(K_7 - f) = \frac{\alpha_5^{1/2}\left(\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)}{\sqrt{(\alpha_3^2 + \alpha_4^2)\left[(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)^2\right]}}$$

$$u = \frac{\alpha_5^{1/2}\sqrt{2k - \alpha_6^2 - \alpha_7^2}\left[(k - \alpha_6^2 - \alpha_7^2)(\alpha_7 \sin\phi - \alpha_6 \cos\phi) - \alpha_5^{1/2}(\alpha_3\alpha_7 - \alpha_4\alpha_6)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)\right]}{(k - \alpha_6^2 - \alpha_7^2)\sqrt{(k - \alpha_6^2 - \alpha_7^2)^2 - \alpha_5\left(2\,\frac{k - \alpha_6^2 - \alpha_7^2}{\alpha_5^{1/2}} - \alpha_3^2 - \alpha_4^2\right)(\alpha_4 \cos\phi + \alpha_3 \sin\phi)^2}}$$

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